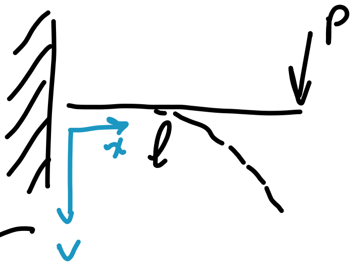
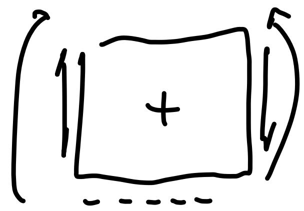
Esercitazione 8

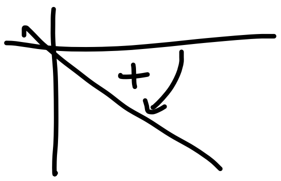
ES 1)

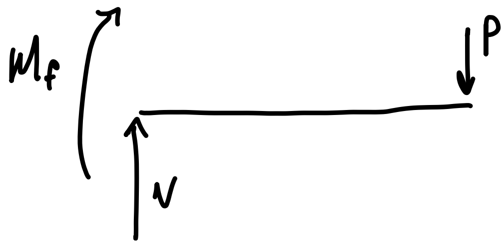


Linea elastica: fondamentale definire convenzioni



Sistema di riferimento

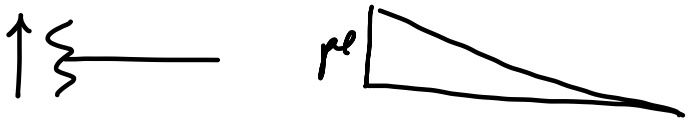




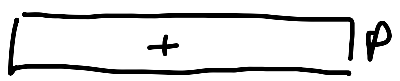
$$\sum_i F_y = 0 = \Rightarrow V = P$$

$$\sum_i M_p = -\underbrace{V}_{=P} \cdot l - M_f \Rightarrow M_f = -Pl$$

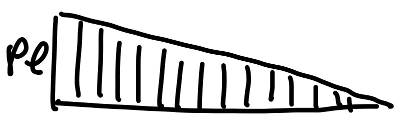
Azioni Interne



T

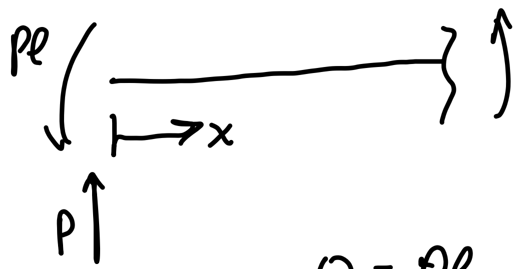


M



Linea Elastica

$$M = -EJ \frac{d^2v}{dx^2}$$ → importante x crescente, quindi veniamo dall'incastro



$$0 = Pl - Px + M \Rightarrow \underbrace{M = P(x-l)}$$

torna con il diagramma sopra

$$M = -EJ \frac{d^2v}{dx^2} = P(x-l)$$

Vogliamo sapere P

$$\frac{dv}{dx} = \frac{P}{-EJ}\left(lx - \frac{x^2}{x}\right) + C_1$$

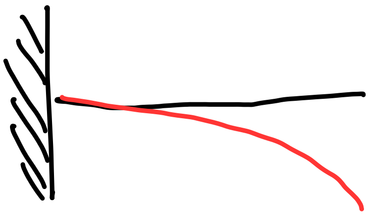
$$v(x) = \frac{P}{-EJ}\left(l\frac{x^2}{2} - \frac{x^3}{6} + C_1x + C_2\right)$$

2 costante bisogna imporre due condizioni al contorno (azioni interne o vincoli)

$$v(0) = 0 \Rightarrow C_2 = 0$$

$$v'(0) = 0 \Rightarrow C_1 = 0$$ → incastro blocca rotazioni

$\Rightarrow v(x) = \frac{P}{EJ}\left(\ell\frac{x^2}{2} - \frac{x^3}{6}\right) \rightarrow$ facciamo la derivata per trovare x per cui v è massimo, in questo caso è ovvio



$$v_{MAX} = \delta = \frac{P}{EJ}\left(\frac{\ell^2}{2} - \frac{\ell^3}{6}\right)$$

Approccio per isostatiche

~~Approccio~~ iperstatico applicato a isostato

Per iperstatiche dobbiamo usare questo approccio:

$$\frac{d^4v}{dx^4} = q(x) = 0$$

$$\frac{d^3v}{dx^3} = C_1$$ ← Taglio (derivata del momento)

$$\frac{d^2v}{dx^2} = C_1x + C_2$$ ← Momento flettente

$$\frac{dv}{dx} = C_1\frac{x^2}{2} + C_2x + C_3$$

$$v(x) = C_1\frac{x^3}{6} + C_2\frac{x^2}{2} + C_3x + C_4$$

Condizioni di contorno che di solito imponiamo

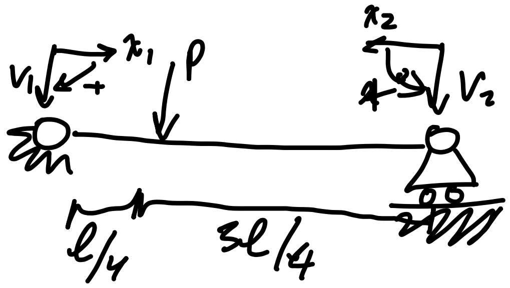
$v(0)=0 \leftarrow$ Perché a vincolo

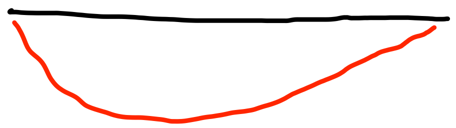
$v'(0)=0$

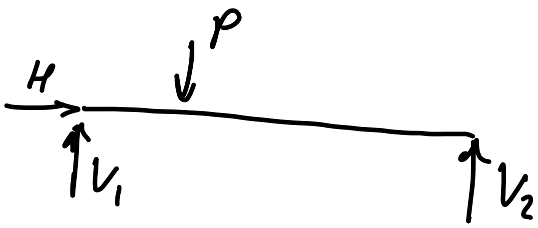
$v''(\ell)=0 \leftarrow$ Perché estremo libero

$$-EJv'''(\ell) = P$$

Esercizio 2







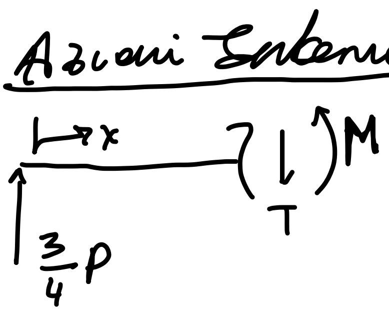
$$H = 0$$

$$\sum M_\ell = 0 = -V_1\ell + P\frac{3}{4}\ell \Rightarrow V_1 = \frac{3}{4}P$$

sul carrello

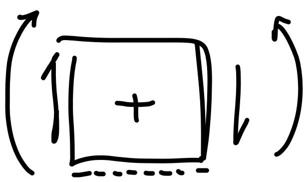
$$\sum F_y = 0 \Rightarrow V_2 = \frac{1}{4}P$$

Azioni Interne



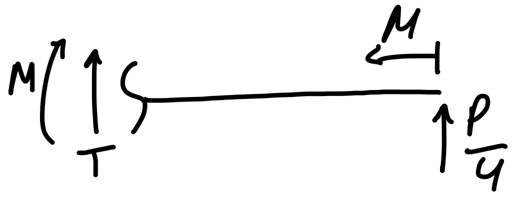
$$T = \frac{3}{4}P$$

$$-\frac{3}{4}Px + M = 0 \Rightarrow M = \frac{3}{4}Px$$



$$0 \leq x \leq \frac{\ell}{4}$$

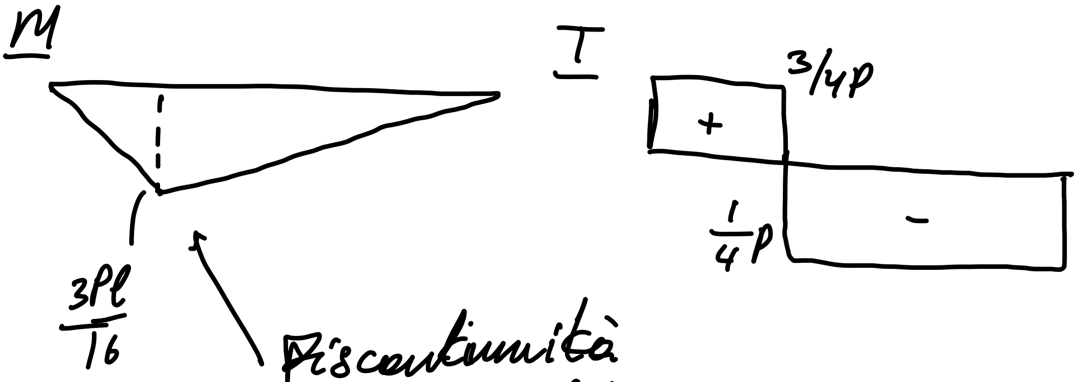
Cambiando sistema di riferimento al carrello:



$$T = \frac{-P}{4}$$

$$0 = \frac{P}{4}x - M \Rightarrow M = \frac{P}{4}x_2$$

$$0 \leq x_2 \leq \frac{3}{4}l$$



Discontinuità quindi dobbiamo usare 2 sistemi di riferimento

$C_r \Rightarrow$ 2 calcoli per V_{MAX}

Linea Elastica

$$\begin{cases} -EJ \dfrac{d^2 v_1}{d^2 x_1} = M_{\ell} \overset{left}{=} \dfrac{3}{4} P x_1 \\ -EJ \dfrac{d^2 v_2}{d x_2^2} = M_r \overset{right}{=} \dfrac{P}{4} x_2 \end{cases}$$

$$\begin{cases} \dfrac{d v_1}{d x_1} = \dfrac{-P}{EJ} \cdot \left(\dfrac{3}{8} x_1^2 + C_1\right) \\ \dfrac{d v_2}{d x_2} = \dfrac{-P}{EJ} \cdot \left(\dfrac{x_2^2}{8} + C_3\right) \end{cases}$$

$$\begin{cases} v_1(x) = \dfrac{-P}{EJ} \left(\dfrac{1}{8} x_1^3 + C_1 x_1 + C_2\right) \\ v_2(x) = \dfrac{-P}{EJ} \left(\dfrac{x_2^3}{24} + C_3 x_2 + C_4\right) \end{cases}$$

Condizioni al contorno

$v_1(0) = 0$

$v_1\left(\frac{l}{4}\right) = v_2\left(\frac{3l}{4}\right)$

$v_2(0) = 0$

$v_1'\left(\frac{l}{4}\right) = -v_2'\left(\frac{3}{4}l\right)$

v'è la mutazione

$v_1(0)=0 \Rightarrow C_2=0$

$v_2(0)=0 \Rightarrow C_4=0$

$$\frac{-P}{EJ}\left(\frac{3}{24}\left(\frac{\ell}{4}\right)^3+C_1\cdot\frac{\ell}{4}\right)=-\frac{P}{EJ}\left(\frac{1}{24}\left(\frac{3}{4}\ell\right)^3+C_3\cdot\frac{3}{4}\ell\right)$$

$$\frac{-P}{EJ}\left(\frac{3}{8}\left(\frac{\ell}{4}\right)^2+C_1\right)=\frac{P}{EJ}\left(\frac{1}{8}\left(\frac{3}{4}\ell\right)^2+C_3\right)$$

$C_1=\frac{-7}{128}P\ell^2$

$C_3=\frac{-5}{128}P\ell^2$

$$v_1(x_1)=\frac{-P}{EJ}\left(\frac{3}{24}x_1^3-\frac{7}{128}\ell^2 x\right)$$

$$v_2(x_2)=\frac{-P}{EJ}\left(\frac{x_2^3}{24}-\frac{5}{128}\ell^2 x\right)$$

<u>Trovare freccia massima:</u>

$v_1'(x_1)=0 \Rightarrow x_{1MAX}=0{,}38\ell$ fuori dal dominio $0\le x_1\le\frac{\ell}{4}$

$\nexists\, x_{1MAX}$

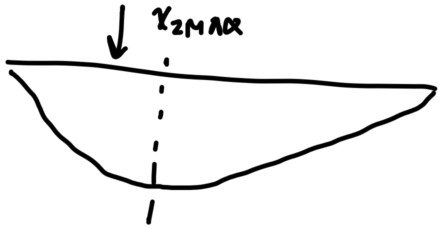
$v_2'(x_2)=0 \Rightarrow x_{2MAX}=0{,}559\ell$ $\quad 0\le x_2\le 0{,}75\ell$

$\exists\, x_{2MAX}$

v_{MAX} sarà in corrispondenza di questo

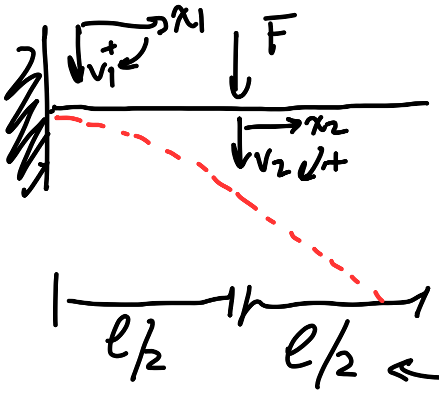
$$v_{MAX}=\delta=v_2(x_{2,MAX})=\frac{-P}{EJ}\left(\frac{1}{24}\left(\frac{8\cdot5}{128}\right)^{3/2}\ell^3-\frac{5}{128}\sqrt{\frac{8\cdot5}{128}}\,\ell^3\right)$$

Il punto di v_{MAX} non è al punto di applicazione ma vicino, perché elasticità



$v_2' = 0$ è il punto dove c'è v_{MAX}

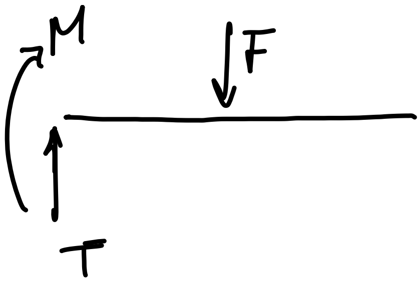
Esercizio 3



Dopo F, $T=0$ e $M=0$, non ci sono forze interne

dopo il carico la pendenza è come a $l/2$

Qui è come esercizio 1

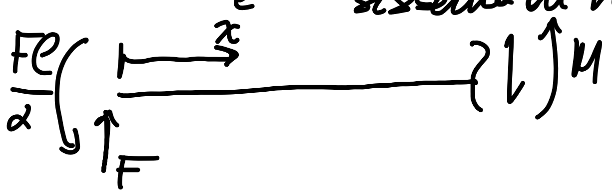


$$V = F$$

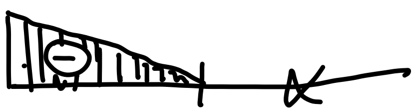
$$\sum M = 0 = -\frac{Fl}{2} - M \Rightarrow M = -\frac{Fl}{2}$$

Già scelto un sistema di riferimento

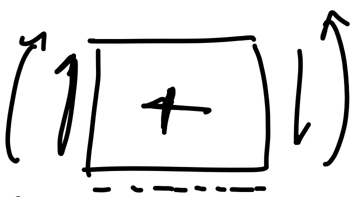
T



M



$$T = F$$



$$\sum M = 0 = -Fx + \frac{Fl}{2} + M = F\left(x - \frac{l}{2}\right)$$

$$-EJ\,\frac{d^2v_1}{dx_2^2} = Fx_1 - \frac{F\ell}{2}$$

$$-EJ\,\frac{d^2v_2}{dx_2^2} = 0$$

$$\begin{cases} \dfrac{dv_1}{dx_1} = \dfrac{-F}{EJ}\left(\dfrac{x^2}{2} - \dfrac{\ell}{2}x + C_1\right) \\ \dfrac{dv_2}{dx_2} = C_2 \end{cases}$$

$$\begin{cases} v_1(x) = \dfrac{-F}{EJ}\left(\dfrac{x_1^3}{6} - \dfrac{\ell}{4}x_1^2 + C_1x_1 + C_3\right) \\ v_2(x) = C_2x_2 + C_4 \end{cases}$$

Condizioni

$v_1(0) = 0 \Rightarrow C_3 = 0$

$v_1' = (0) = 0 \Rightarrow C_1 = 0$

$v_1\left(\frac{\ell}{2}\right) = v_2(0)$ (2)

$v_1'\left(\frac{\ell}{2}\right) = v_2'(0)$ (1)

(1) $\Rightarrow C_2 = \frac{-F}{EJ}\left(\frac{1}{2}\frac{\ell^2}{4} - \frac{\ell^4}{4}\right)$

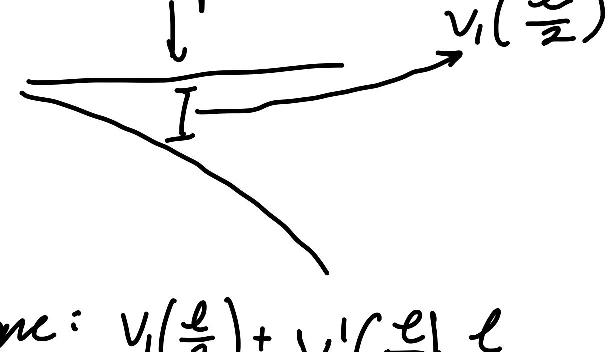
(2) $\Rightarrow C_2 \cdot 0 + C_4 = \frac{-F}{EJ}\left(\frac{1}{6}\cdot\left(\frac{\ell}{2}\right)^3 - \frac{\ell}{4}\left(\frac{\ell}{2}\right)^2\right)$

$C_2 = -\frac{1}{8}F\frac{\ell^2}{4}/EJ$

$C_4 = \frac{-1}{24}F\ell^3/EJ$

Freccia Massima

$v_2\left(\frac{\ell}{2}\right)$



Si può fare questo

Si può fare: $v_1\left(\frac{\ell}{2}\right) + v_1'\left(\frac{\ell}{2}\right)\frac{\ell}{2}$

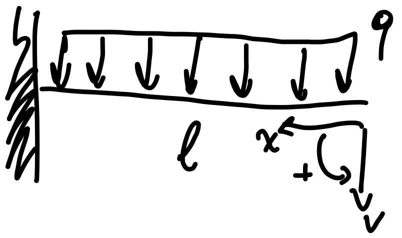
Per la pendenza non

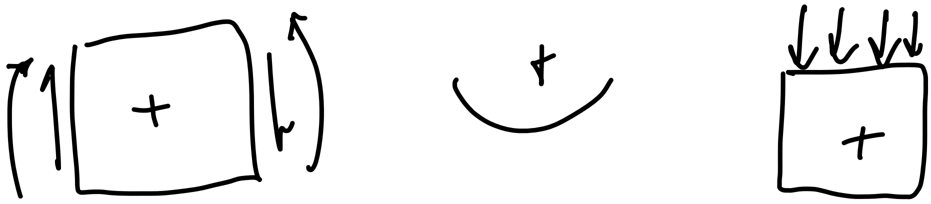
cambia

Esercizio 4 → Da linea elastica a azioni interne

(stesso metodo per iperstatico)

Dove non sappiamo come analizzare le azioni interne





$$T = -\frac{dM}{dx}$$

$$q = \frac{dT}{dx}$$

derivata 4^a

$$EJ\, v^{(4)} = q$$

Taglio

$$EJ\, v''' = qx + C_1$$

Momento

$$EJ\, v'' = q\frac{x^2}{2} + C_1 x + C_2$$

$$EJ\, v' = q\frac{x^3}{6} + C_1\frac{x^2}{2} + C_2 x + C_3$$

$$EJ\, v = q\frac{x^4}{24} + C_1\frac{x^3}{6} + C_2\frac{x^2}{2} + C_3 x + C_4$$

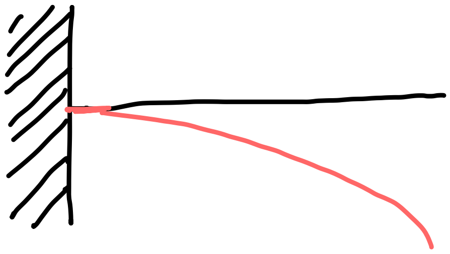
$$\begin{cases} v(l) = 0 \\ v'(l) = 0 \\ EJ\, v''(0) = 0 \\ -EJ\, v'''(0) = 0 \end{cases}$$

$$EJ\, v'''(0) - C_1 = 0$$

$$-EJ\, v''(0) = C_2 = 0$$

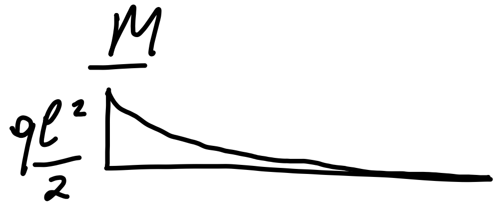
$$v'(l) = \frac{1}{EJ}\left(q\frac{l^3}{6} + C_3\right) = 0 \Longrightarrow \frac{-q}{EJ}\,\frac{l^3}{6}$$

$$v(\ell) = \frac{1}{EJ}\left(q\frac{\ell^3}{24} + C_3\ell + C_4\right) = 0 \Rightarrow C_4 = \frac{q}{EJ}\,\frac{\ell^4}{8}$$

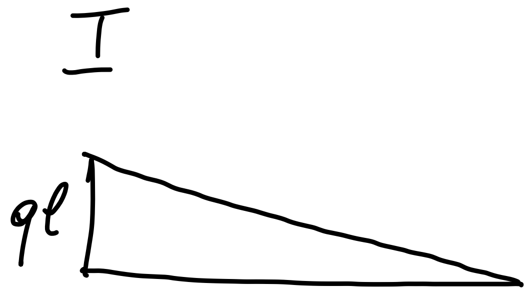


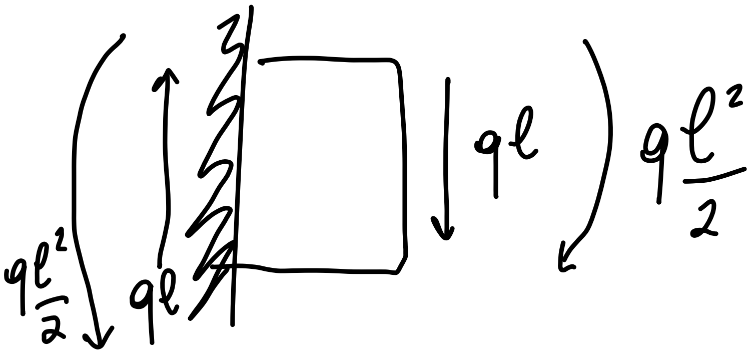
$$v(x) = \frac{1}{EJ}\left(q\frac{x^4}{24} - q\frac{\ell^3}{6}x + q\frac{\ell^4}{8}\right)$$

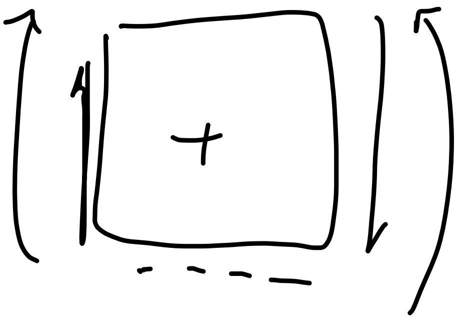
$$M = -EJ\frac{d^2v}{dx^2} = -q\left(\frac{x^2}{2}\right)$$



$$T = \frac{dM}{dx} = EJ\frac{d^3v}{dx^3} = qx$$

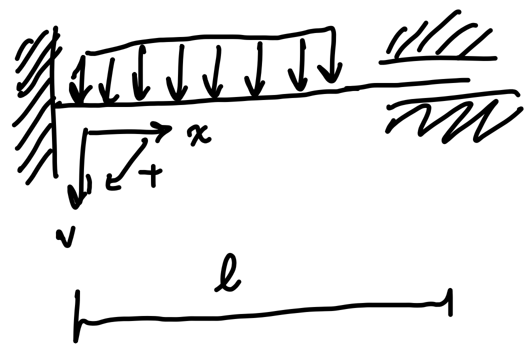


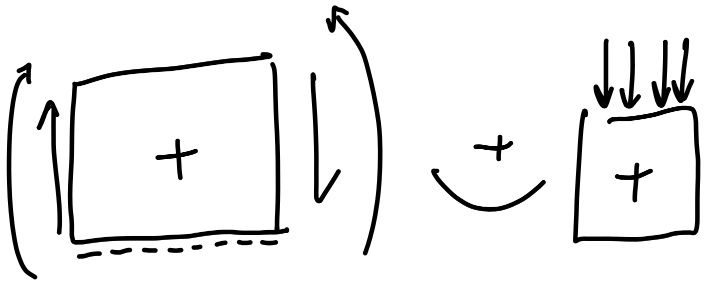




→ Stesso metodo usato per le iperstatiche

Esercizio 5 → Caso iperstatico → esercizio come prima ma veramente iperstatico





$$T = \frac{dM}{dx} \qquad q = -\frac{dT}{dx}$$ Cambio segno perché cambio di sistema di riferimento

↳ Fine della lezione di ieri.

$$EJ\, v^{(4)} = q$$

$$\begin{cases} v(0) = 0 \\ v'(0) = 0 \\ v'(l) = 0 \\ v(l) = 0 \end{cases}$$

$$EJ\, v''' = qx + C_1$$

$$EJ\, v'' = q\frac{x^2}{2} + C_1 x + C_2$$

$$EJ\, v' = q\frac{x^3}{6} + C_1\frac{x^2}{2} + C_2 x + C_3$$

$$EJ\, v = q\frac{x^4}{24} + C_1\frac{x^3}{6} + C_2\frac{x^2}{2} + C_3 x + C_4$$

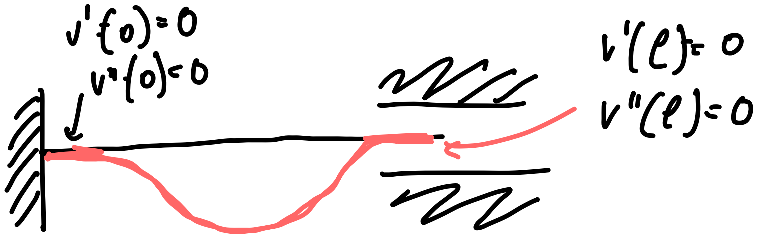
$$v(0) = \frac{C_4}{EJ} = 0 \Rightarrow C_4 = 0$$

$$v'(0) = 0 = \frac{C_3}{EJ} \Rightarrow C_3 = 0$$

$$v'(l) = \frac{q}{EJ}\cdot\frac{l^3}{6} + \frac{C_1}{EJ}\frac{l^2}{2} + \frac{C_2 l}{EJ} = 0 \qquad \Rightarrow C_2 = q\frac{l^2}{12}$$

$$v(l) = 0 = \frac{q}{EJ}\frac{l^4}{24} + \frac{C_1 l^3}{EJ6} + \frac{C_2}{EJ}\frac{l^2}{2} \qquad \Rightarrow C_1 =$$

$$v(x) = \frac{1}{EJ}\left(q\frac{x^4}{24} - \frac{ql}{12}x^3 + \frac{ql}{24}x^2\right)$$



$$M = EJ\frac{d^2v}{dx^2}$$

$$T = EJ\frac{d^3v}{dx^3}$$

$$M = -q\frac{x^2}{2} + q\frac{lx}{2} - q\frac{l^2}{12}$$

$$T = -qx + q\frac{l}{2}$$

$$M(0) = -q\frac{l^2}{12}$$

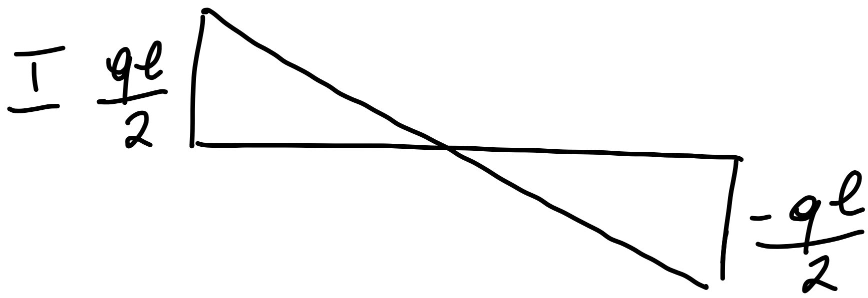
$$M\left(\frac{l}{2}\right) = ql^2/24$$

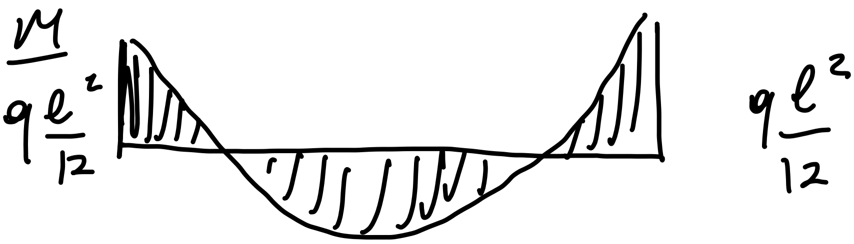
$$M(l) = q\frac{l^2}{12}$$

$$T(0) = \frac{ql}{2}$$

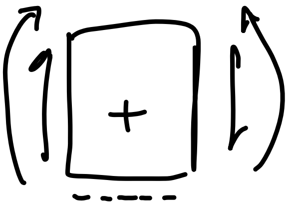
$$T\left(\frac{l}{2}\right) = 0$$

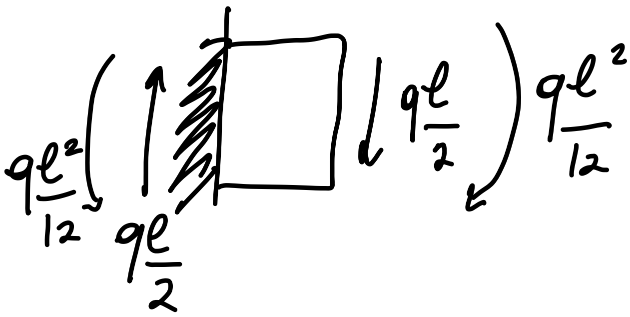
$$T(l) = -\frac{ql}{2}$$

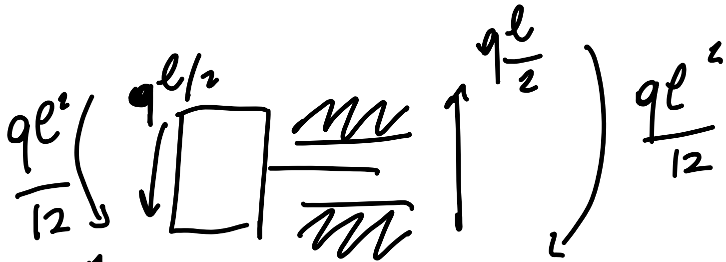




Razioni Vincolari

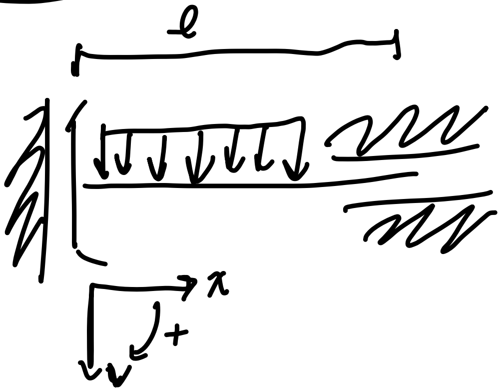


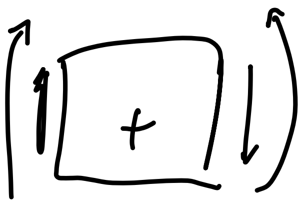


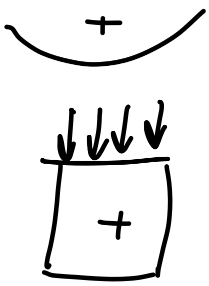


Invertito perché negativo

Esercizio 6







$$M = -EJ\frac{d^2v}{dx^2}$$

$$T = \frac{dM}{dx}$$

$$q = \frac{dT}{dx}$$

$$EJv^{(4)} = q$$

$$EJv''' = qx + C_1$$

$$EJv'' = q\frac{x^2}{2} + C_1x + C_2$$

$$\begin{cases} v(l) = 0 \\ v'(l) = 0 \\ T(0) = 0 = -EJv''' \\ v'(0) = 0 \end{cases}$$

$$EJv' = q\frac{x^3}{6} + C_1\frac{x^2}{2} + C_2 x + C_3$$

$$EJv' = q\frac{x^4}{24} + C_1\frac{x^3}{6} + C_2\frac{x^2}{2} + C_3 x + C_4$$

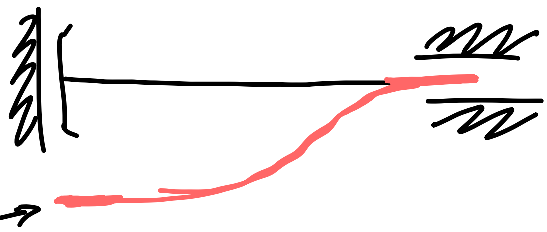
$C_1 = 0$

$C_2 = -q\frac{\ell^2}{6}$

$C_3 = 0$

$C_4 = q\frac{\ell^4}{24}$

$$v(x) = \frac{q}{EJ}\left(\frac{x^4}{24} - \frac{\ell^2 x^2}{12} + \frac{\ell^4}{24}\right)$$



Si può spostare ma non può ruotare

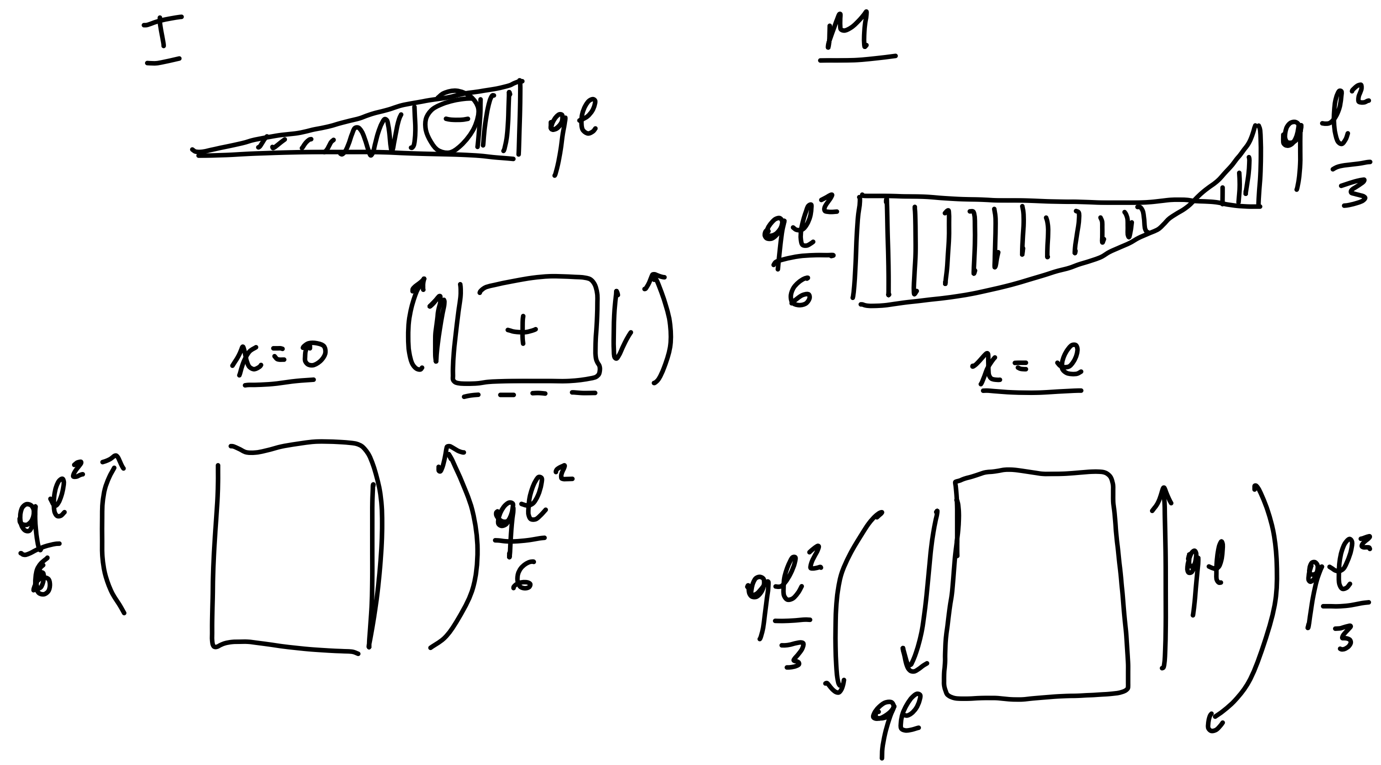
$$M(x) = -EJ\,v''(x) = -\frac{qx^2}{2} + q\frac{\ell^2}{16}$$

$$T(x) = -EJ\,v'''(x) = -qx$$

$T(0) = 0$ $\quad T(\ell) = -q\ell$

$$M(0) = q\frac{\ell^2}{6}$$

$$M(\ell) = -\frac{q\ell^2}{3}$$



Geometria delle Aree

Consideriamo un sistema di massa possiamo scrivere le coordinate del baricentro come:

$$x_G = \frac{\sum_i m_i x_i}{m}$$

$$y_G = \frac{\sum_i m_i y_i}{m}$$

Media pesata delle masse

$$z_G = \frac{\sum_i m_i z_i}{m}$$

$$x_G = \frac{\int_V \rho\, x \, dV}{m} \qquad y_G = \frac{\int_V \rho\, y \, dV}{m} \qquad z_G = \frac{\int_V \rho\, z \, dV}{m}$$

$$= \frac{\cancel{\rho} \int_V x \, dV}{\cancel{\rho} V} \qquad = \frac{\int_V y \, dV}{V} \qquad = \frac{\int_V z \, dV}{V}$$

Se corpo omogeneo e sezione piccola

$$x_G = \frac{\int_A x\,dA}{A}$$

$\int_A x A \leftarrow$ Momento statico, o momento del primo ordine

$$y_G = \frac{\int_A y\,dA}{A}$$

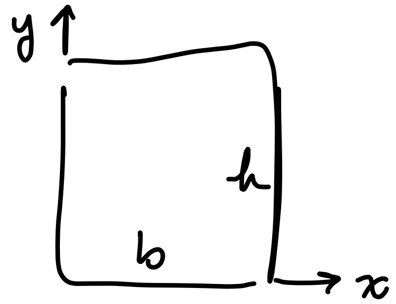
$$z_G = \frac{\int_A z\,dA}{A}$$

Caso piano:

$$S_y = \int_A x\,dA \qquad S_x = \int_A y\,dA$$

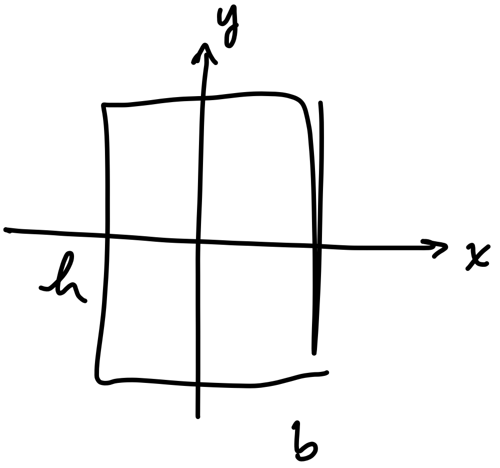
$$[L^3]$$

<u>Esempio</u> Sistema con riferimento baricentro Omogeneo ha momento statico nullo



$$S_y = \int_A x\,dA = \int_0^h \int_0^b x\,dx\,dy =$$

$$= \int_0^h \left[\frac{x^2}{2}\right]_0^b dy = \frac{hb^2}{2}$$

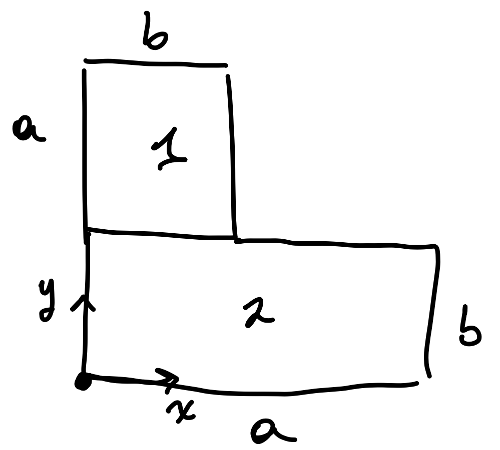


$$S_y = \int_A x\,dA = \int_{-\frac{h}{2}}^{h/2} \int_{-b/2}^{b/2} x\,dx\,dy$$

$$= \int_{-h/2}^{h/2} \overset{\to 0}{\cancel{\left[\frac{x^2}{2}\right]_{-b/2}^{b/2}}}\, dy = 0$$

$$\frac{b^2}{8} - \frac{b^2}{8}$$

Sezioni Composte ⇝ 2 rettangoli



$$S_y = S_y^1 + S_y^2$$

$$x_{G1} = \frac{1}{A_1} S_y^1 = \int_{A_1} x\,dy \cdot \frac{1}{A_1}$$

$$= \frac{1}{ab} \int_0^b \left(\int_0^a x\,dx \right) dy$$

$$= \frac{1}{ab} \cdot \frac{a^2 b}{2} = \frac{a}{2}$$

$$x_{G2} = \frac{1}{A_2} \int_A x\,dA = \frac{1}{ab} \int_b^{a+b} \int_0^b x\,dx\,dy = \frac{b}{2}$$

$$x_G = \frac{\overbrace{\frac{b}{2}}^{x_{G2}} \cdot A_2 + \overbrace{\frac{a}{2}}^{x_{G1}} \cdot A_1}{A_1 + A_2} = \frac{a+b}{2}$$

$$x_G = \frac{x_{G1} A_1 + \dots}{\sum A}$$

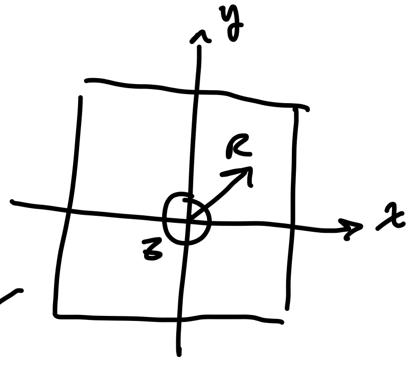
Momento d'Inerzia

↳ La resistenza di una sezione alla rotazione

$$J_x = \int_A y^2\,dA \qquad [L^4]$$

Momento del secondo ordine rispetto a x

$$J_y = \int_A x^2\,dA$$



$J_{xy} = \int_A xy \, dA$ $\qquad R = \sqrt{x^2+y^2}$

↳ momento centrifugo

$$J_p = J_\oplus = \int_A r^2 dA = \int_A (x^2+y^2)\, dA = \underbrace{\int_A x^2 dA}_{J_x} + \underbrace{\int_A y^2 dA}_{J_y}$$

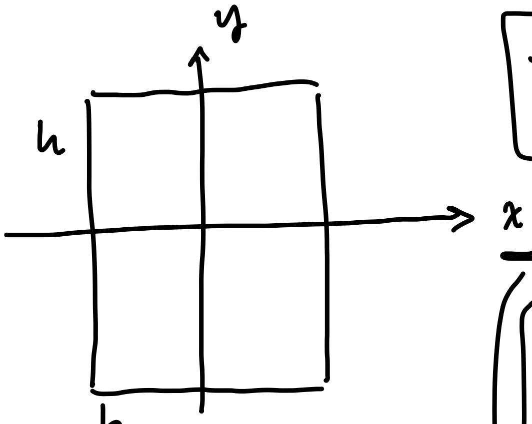
Momento di inerzia polare → Attitudine alla torsione

↳ Quando vettore di rotazione uscente dal piano

→ Cambiamo da cartesiano a polare

$$J_p = J_x + J_y$$

Momento di inerzia per sezione rettangolare

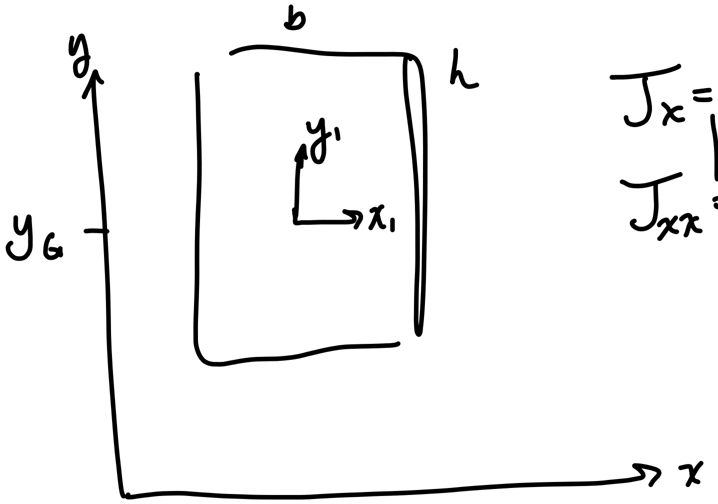


$$\boxed{J_x = \int_A y^2 dA = \int_{-b/2}^{b/2} \int_{-h/2}^{h/2} y^2 \, dy \, dx = \frac{bh^2}{12}}$$

$$\boxed{J_y = \int_A x^2 dA = \int_{-h/2}^{h/2} \int_{-\frac{b}{2}}^{b/2} x^2 \, dx \, dy = \frac{hb^3}{12}}$$

$$\sigma_x = \frac{N}{A} + \frac{M}{J_y} y$$

→ Per sezione rettangolare (Da memorizzare)



$$J_x = \int_A y^2 \, dA = \int_A (y_G + y')^2 \, dA$$

$$J_{xx} = \int_{-b/2}^{b/2} \int_{-h/2}^{h/2} (y_G + y')^2 \, dy' \, dx'$$

$$J_x = \int_{-\frac{b}{2}}^{b/2} \int_{-h/2}^{h/2} (y_G{}^2 + y_G y' + y'^2) \, dx' \, dy'$$

$$= y_G^2 \underbrace{\int_{-\frac{b}{2}}^{b/2} \int_{-\frac{h}{2}}^{h/2} dx' \, dy'}_{A} + 2 \, y_G \underbrace{\cancel{\int_{-\frac{b}{2}}^{b/2} \int_{-\frac{h}{2}}^{h/2} y' \, dy' \, dx'}}_{\to 0}$$

visto prima

$$+ \underbrace{\int_{-\frac{b}{2}}^{b/2} \int_{-\frac{h}{2}}^{h/2} y'^2 \, dy' \, dx'}_{J_{x'}}$$

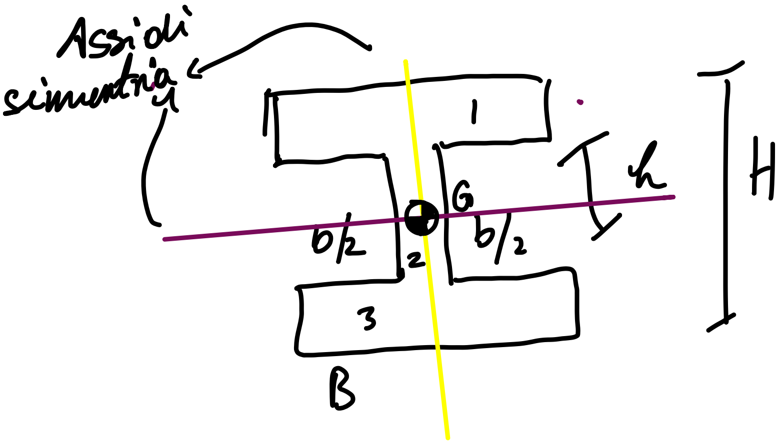
Momento di inerzia per asse non passante per il baricentro

$$J_x = J_{x'} + y_G^2 A$$

Più ci si allontana più resistenza alla rotazione

(

↳ Utile per travi a doppia T (I beam)



$$J_{x_1} = \frac{B\left(\frac{H-h}{2}\right)^3}{12}$$

$$J_{x_2} = \frac{(B-b)(h)^3}{12}$$

$$J_{x_3} = J_{x_1}$$

Momento di inerzia totale vs. Momento di inerzia sezione 2

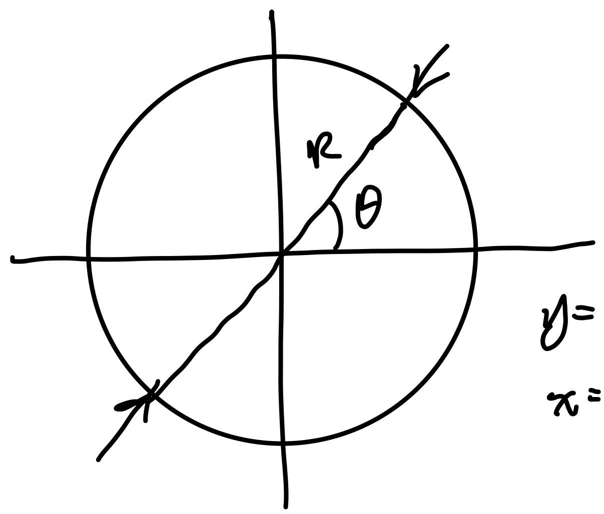
$$J_x = J_{x_2} + \left(J_{x_1} + A_1\, y_{G_1}^2\right) + \left(J_{x_3} + A_3\, y_{G_3}^2\right)$$

J_x ← Momento di inerzia totale

$$\sigma = \frac{N}{A} + \frac{M}{J}\, y$$

↖ Più grande J, meno sforzo

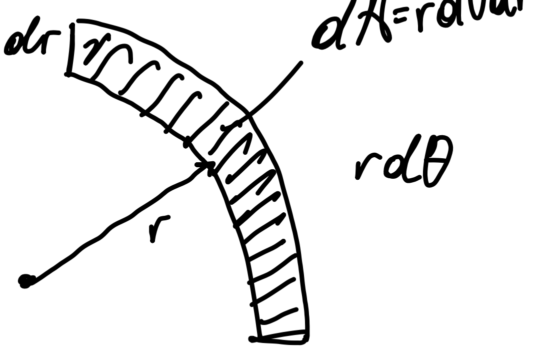
Momento di inerzia della sezione circolare



$$J_x = \int_A y^2\, dA = \int_A r^2 \sin^2\theta\, dA$$

$y = r\sin\theta$

$x = r\cos\theta$



$$J_x = \int_0^{D/2}\int_0^{2\pi} r^2 \sin^2\theta \cdot r \, d\theta \, dr = \int_0^{D/2} r^3 \int_0^{2\pi} \left(\frac{1-\cos 2\theta}{2}\right) d\theta \, dr$$

$$\sin^2\theta = 1 - \cos^2\theta$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

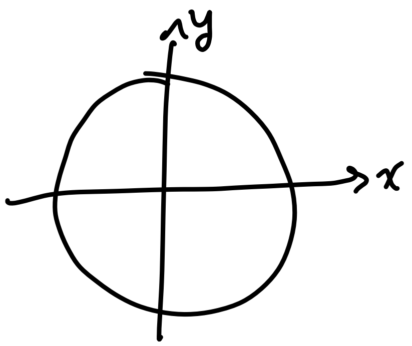
$$\left| \frac{\theta}{2} - \cancelto{0}{\frac{\sin 2\theta}{4}} \right|_0^{2\pi} = \pi$$

$$J_x = \pi \int_0^{D/2} r^3 \, dr = \pi \left[\frac{r^4}{4} \right]_0^{D/2}$$

$$\boxed{J_x = \frac{\pi D^4}{64}} = J_y$$

Momento di inerzia per sezione circolare (da memorizzare

→ Perchè simmetrica



$$J_p = J_x + J_y = \frac{\pi D^4}{32}$$

Per momento flettente J_x e J_y

Per momento torcente J_p