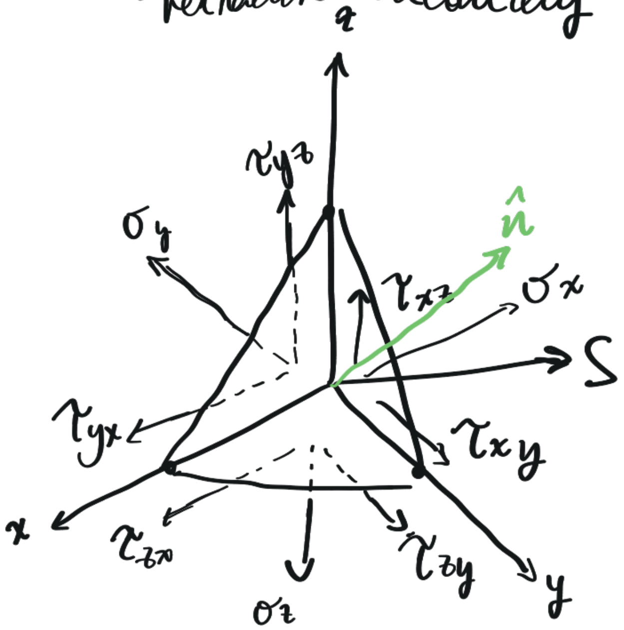
$\bar{\bar{\sigma}} \to$ tensore di sforzo

$$\bar{\bar{\sigma}} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$ matrice simmetrica

$\tau_{xy} \to$ sforzo che gia superficie di normale x e verso a y

Tetraedro di Cauchy



S sforzo qualsiasi

$\hat{n}$ = versore normale alla superficie

$$\hat{n} = \begin{bmatrix} \cos\alpha \\ \cos\beta \\ \cos\gamma \end{bmatrix}$$

Facendo l'equilibrio:

$$S_x\, dA = \sigma_x \underbrace{dA\cos\alpha}_{\text{proiezione di A lungo x}} + \tau_{yx}\, dA\cos\beta + \tau_{zx}\, dA\cos\gamma$$

$$S_x = \sigma_x \cos\alpha + \tau_{yx}\cos\beta + \tau_{zx}\cos\gamma$$

$$S_y = \tau_{yx}$$

Per tutti e tre:

$$\begin{cases} S_x = \sigma_x \cos\alpha + \tau_{yx}\cos\beta + \tau_{zx}\cos\gamma \\ S_y = \tau_{yx}\cos\gamma + \sigma_y\cos\beta + \tau_{zy}\cos\gamma \\ S_z = \tau_{xz}\cos\alpha + \tau_{yz}\cos\beta + \sigma_z\cos\gamma \end{cases}$$

$$\bar{S} = \begin{Bmatrix} S_x \\ S_y \\ S_z \end{Bmatrix} = \overset{\bar{\bar{\sigma}}}{\underbrace{\begin{bmatrix} \sigma_x & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_y & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix}}_{\text{Per soluzioni non banali}}} \cdot \overset{\hat{n}}{\begin{bmatrix} \cos\alpha \\ \cos\beta \\ \cos\gamma \end{bmatrix}}$$

$\neq 0$

$$\bar{S} = \bar{\bar{\sigma}} \cdot \hat{n}$$

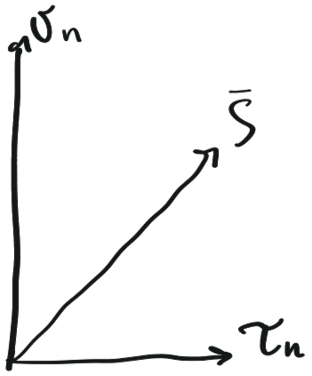
Tornando al tetraedro, proiettiamo S lungo a $\hat{n}$

$$\sigma_n = \bar{S}^T \cdot \hat{n} = \bar{S} \cdot \hat{n}^T$$

$$= S_x\cos\alpha + S_y\cos\beta + S_z\cos\gamma$$

$$\sigma_n = (\bar{\bar{\sigma}} \cdot \hat{n})^T \cdot \hat{n} = \hat{n}^T \bar{\bar{\sigma}} \cdot \hat{n}$$

↳ Sforzo normal lungo una superfice qualsiasi



$\tau_n = \sqrt{|\bar{S}|^2 - \sigma_n^2}$ Questa roba non esce all'esame

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \text{ è diagonalizzabile } \begin{bmatrix} \sigma_{x'} & 0 & 0 \\ 0 & \sigma_{y'} & 0 \\ 0 & 0 & \sigma_{z'} \end{bmatrix}$$

$\sigma_{x'}, \sigma_{y'}, \sigma_{z'}$ sono autovalori per cui dobbiamo trovare gli autovalori che sono i vettori di base che possono descrivere un sistema completamente

$$\det\left(\bar{\bar{\sigma}} - \sigma_p \bar{\bar{I}}\right) = 0$$

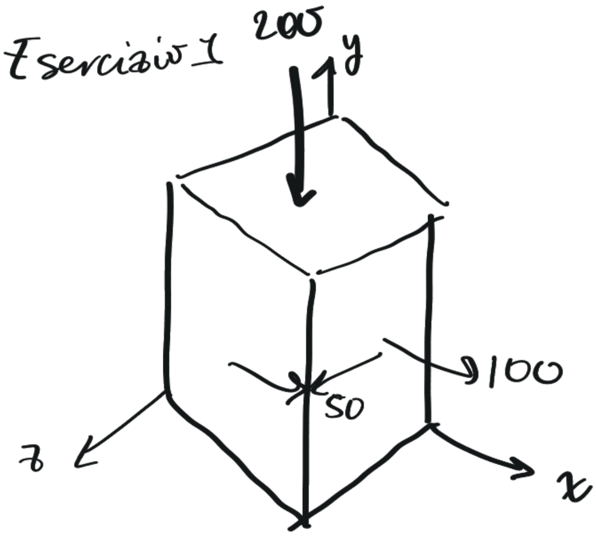
$$\forall \sigma_p \quad \underbrace{\left(\bar{\bar{\sigma}} - \sigma_p \bar{\bar{I}}\right)}\cdot \hat{n} = \underline{0}$$

Stress principale

↳ Vogliamo trovare le nostre direzioni principali dagli stress principali

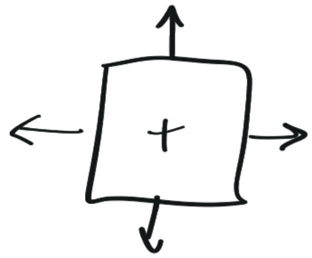
$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

Esercizio 1

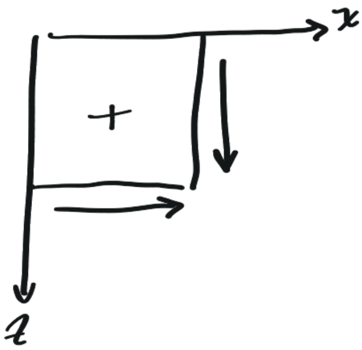


Convenzione

Sforzo Normali:



Sforzo Tangenziale:



$$\bar{\bar{\sigma}} = \begin{bmatrix} 100 & 0 & 50 \\ 0 & -200 & 0 \\ 50 & 0 & 0 \end{bmatrix}$$

y direzione principale, non ha τ_{yz} o τ_{yx}

Ma già tutti gli sforzi tangenziali nulli

$$\left.\begin{array}{l} ① \ [1, 0, 0] \\ ② \ [0, 0, 1] \\ ③ \ [1, 1, 1] \\ ④ \ [0{,}9238, 0, 0{,}3827] \end{array}\right\} \text{Direzioni lungo cui dobbiamo calcolare gli sforzi}$$

$$\bar{S} = \bar{\bar{\sigma}} \cdot \hat{n} = \begin{bmatrix} 100 & 0 & 50 \\ 0 & -200 & 0 \\ 50 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \\ 50 \end{bmatrix}$$

$$\sigma_n = S^{-T} \cdot \hat{n} = [100, 0, 50] \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 100\,MPa$$

↳ Sforzo normale alla superficie normale a x è $100\,MPa$

$$\tau_n = \sqrt{100^2 + 50^2 - 100^2} = 50\,MPa$$

Seconda Direzione

$$\begin{bmatrix} 100 & 0 & 50 \\ 0 & -200 & 0 \\ 50 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 50 \\ 0 \\ 0 \end{bmatrix}$$

$$\sigma_n = S^{-T} \cdot \hat{n} = \{50, 0, 0\} \cdot \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} = 0\,MPa$$

Non c'è σ_z ha senso

$$\tau_n = \sqrt{|S|^2 - \sigma_n^2} = 50\,MPa$$

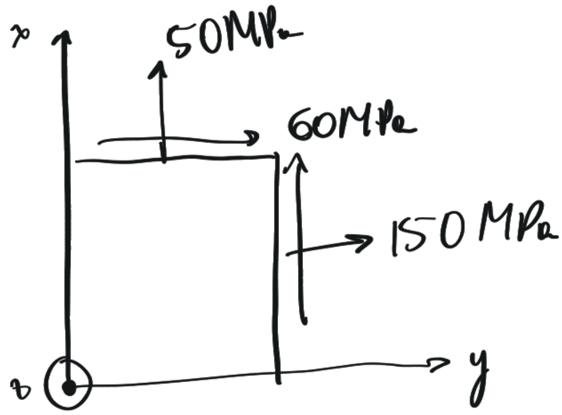
↳ Sforzo Taglio associato alla direzione z

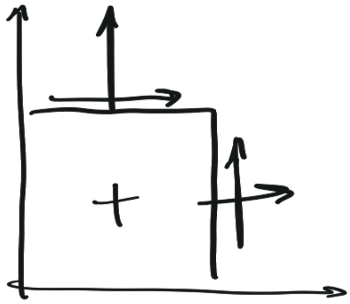
$$\hat{n} = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \frac{1}{\sqrt{3}}$$

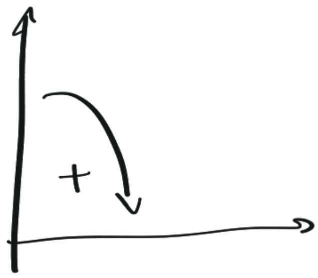
$$\begin{bmatrix} 100 & 0 & 50 \\ 0 & -200 & 0 \\ 50 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{3}} = \begin{Bmatrix} 150/\sqrt{3} \\ -200/\sqrt{3} \\ 50/\sqrt{3} \end{Bmatrix}$$

$$\sigma_n = \{150, -200, 50\} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \frac{1}{3} = \frac{150 - 200 + 50}{3} = 0\,MPa$$

$$\tau_n = \sqrt{|S^2| - (\sigma_n)^2} = \frac{50}{3}\sqrt{26}$$







$$\begin{bmatrix} 50 & 60 & 0 \\ 60 & 150 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\det\left(\bar{\bar{\sigma}} - \sigma_p \bar{\bar{I}}\right) = 0$$

$$\begin{vmatrix} 50-\sigma_p & 60 & 0 \\ 0 & 150-\sigma_p & 0 \\ 0 & 0 & -\sigma_p \end{vmatrix} = 0$$

$$-\sigma_p\left((50-\sigma_p)(150-\sigma_p) - 60^2\right) = 0$$

$$\left.\begin{aligned} \sigma_{p_1} &= 0 \\ \sigma_{p_2} &= 21{,}9\ MPa \\ \sigma_{p_3} &= 178{,}1\ MPa \end{aligned}\right\} \text{3 autovalori}$$

Convenzione Sforzi Principali

$\sigma_1 > \sigma_2 > \sigma_3$

In questo caso:

$\sigma_1 = 178{,}1\ MPa$, $\sigma_2 = 21{,}9\ MPa$, $\sigma_3 = 0\ MPa$

Imponiamo $\sigma_p = \sigma_3$

$$\begin{bmatrix} 50-0 & 60 & 0 \\ 60 & 150-0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \underbrace{\begin{bmatrix} \cos\alpha \\ \cos\beta \\ \cos\gamma \end{bmatrix}}_{\hat{n}} = 0$$

Perchè vogliamo trovare α, β, γ

$$\begin{cases} 50\cos\alpha + 60\cos\beta = 0 \\ 60\cos\alpha + 150\cos\beta = 0 \\ 0 = 0 \end{cases}$$

$\alpha =$

$\beta =$

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\cos\alpha = \frac{-60}{50}\cos\beta$

$\cos\beta\left(\frac{-60^2}{50} + 150\right) = 0 \Rightarrow \beta = \pm 90° \Rightarrow \alpha = \pm 90°$

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\cos^2\gamma = 1 \Rightarrow \gamma = 0° \rightarrow \sigma_3$ verso asse z, direzione principale

σ_3 inclinata di 90° rispetto all'asse x

" " " " " y

" " 0 " " z

<u>Imponendo</u> σ_2

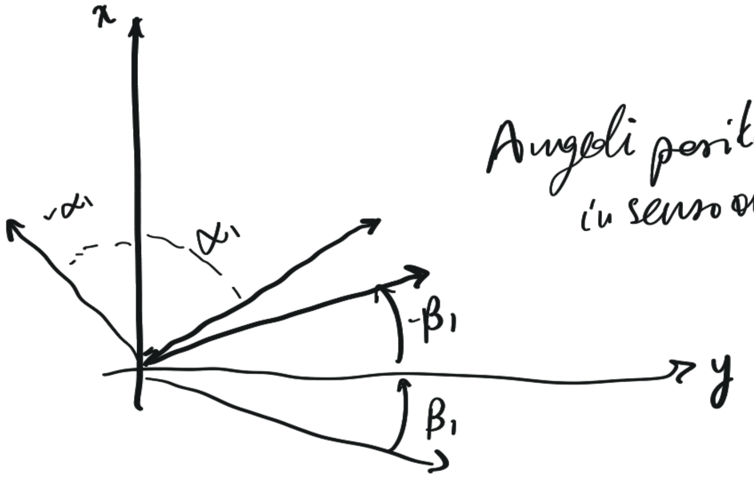
$$\begin{bmatrix} -128,1 & 60 & 0 \\ 60 & -28,1 & 0 \\ 0 & 0 & -178,1 \end{bmatrix} \begin{bmatrix} \cos\alpha \\ \cos\beta \\ \cos\gamma \end{bmatrix} = 0$$

$$\begin{cases} -128,1 \cos\alpha + 60\cos\beta = 0 \\ 60\cos\alpha - 28,1\cos\beta = 0 \\ -178,1\cos\gamma = 0 \Rightarrow \gamma = \pm 90° \end{cases}$$

$$128\cos\alpha + 60\cos\beta = 0$$
$$\cos^2\alpha + \cos^2\beta + \cancel{\cos^2\gamma} = 1 \quad \Rightarrow$$
$$\gamma = \pm 90°$$

$$\overbrace{\alpha_1 = \pm 64,9 \quad \alpha_2 = \pm 115,1°}^{180°}$$
$$\underbrace{\beta_1 = \pm 25,1° \quad \beta_2 = \pm 154,9°}_{180°}$$
$$\gamma =$$

β_1 angolo tra asse y e direzione principale



Angoli positivi in senso orario

$$\begin{Bmatrix} 64,9° \\ -25,1° \\ 90° \end{Bmatrix}$$

Si vogliono angoli che coincidono in questo caso

α_1 e $-\beta_1$ coincidono quindi è la direzione σ_2

Imponiamo $\sigma_p = \sigma_1$

$$\begin{bmatrix} 28{,}1 & 60 & 0 \\ 60 & 128{,}1 & 0 \\ 0 & 0 & -21{,}9 \end{bmatrix} \begin{bmatrix} \cos\alpha \\ \cos\beta \\ \cos\gamma \end{bmatrix} = 0$$

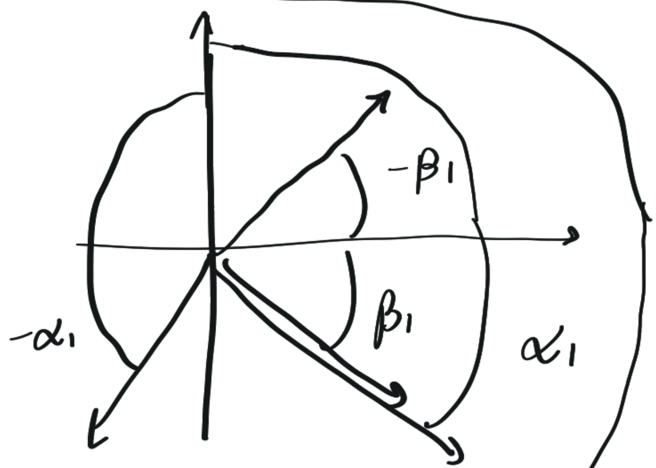
$$\begin{cases} 28{,}1 \cos\alpha + 60 \cos\beta = 0 \\ 60 \cos\alpha + 128{,}1 \cos\beta = 0 \\ \gamma = \pm 90^\circ \end{cases}$$

$$\begin{cases} 28{,}1 \cos\alpha + 60 \cos\beta = 0 \\ \cos^2\alpha + \cos^2\beta + \underbrace{\cos^2\gamma}_{\to 0} = 1 \\ \gamma \pm 90^\circ \end{cases}$$

$\alpha_1 \pm (180^\circ - 25{,}1^\circ)$ $\quad \alpha_2 = \pm 25{,}1^\circ$

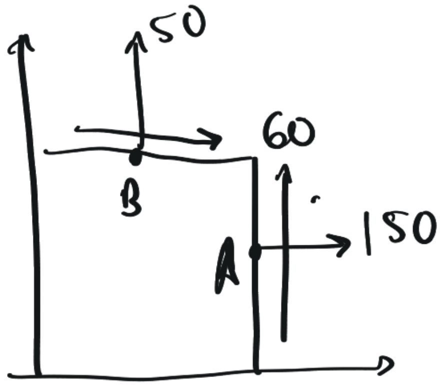
$\beta_1 = \pm 64{,}9^\circ$ $\quad \beta_2 = \pm(180 - 64{,}9^\circ)$

$\gamma = 90$



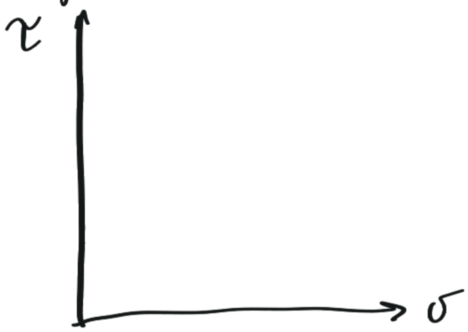
$$\left\{ \begin{matrix} 180 - 25{,}1^\circ \\ 64{,}9^\circ \\ 90^\circ \end{matrix} \right\}$$ sono sfasati di 90 da

Metodi Cerchi di Mohr

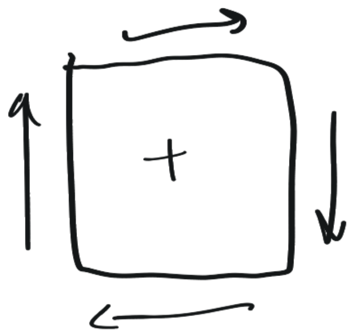


Vogliamo portare lo stato di sforzo al piano di Mohr

piano di Mohr



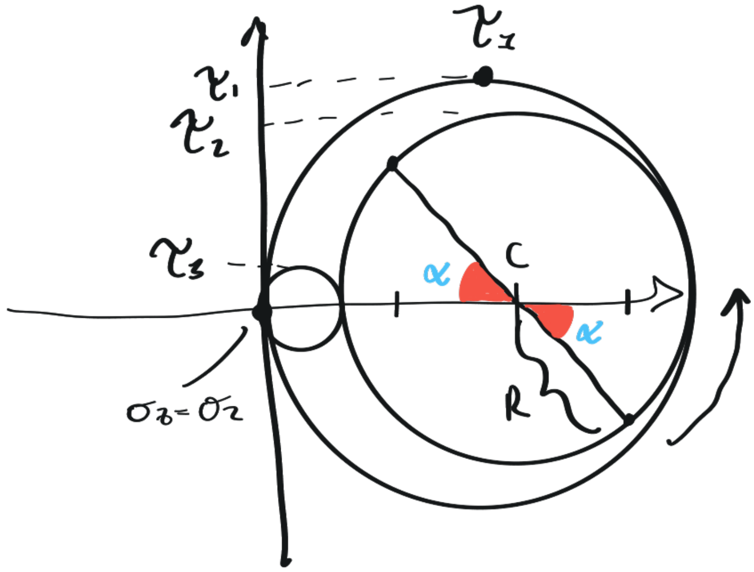
Gli sforzi e punti sono le coordinate sul piano di Mohr



Positivo: Orario

$$A = \begin{pmatrix} \overset{\sigma}{150}, & \overset{\tau}{-60} \end{pmatrix}$$

$$B = (50, 60)$$



A e B giacciono su 2 punti opposti di una circonferenza di Mohr

Gli angoli su piano di Mohr sono il doppio dell'angolo di Mohr, dovremo compensare.

Dato che il piano normale e θ non ha τ sappiamo che σ_θ è sforzo principale

$$C = \frac{\sigma_A + \sigma_B}{2} = 100 MPa$$

$$R = \sqrt{(C - \sigma_B^2) + \tau_B^2} = 78,1 MPa$$

Sforzi principale punti di intersezione tra le circonferenze e asse x. Perchè a quei punti $\tau = 0$

$\sigma_1 = C + R = 178,1 MPa$ } punto di intersezione più a destra

$\sigma_3 = 0 \Rightarrow \sigma_\theta = \sigma$ } punto di intersezione più a sinistra.

$\sigma_2 = C - R = 21,9 MPa$

τ_{max} è il punto più in alto, quindi è nient'altro che R

$$\tau_1 = \frac{\sigma_1 - \sigma_3}{2} = R$$ raggio più grande

$$\tau_2 = \frac{\sigma_1 - \sigma_2}{2} =$$

$$\tau_3 = \frac{\sigma_2 - \sigma_3}{2} =$$

Troviamo α per portare agli angoli principale

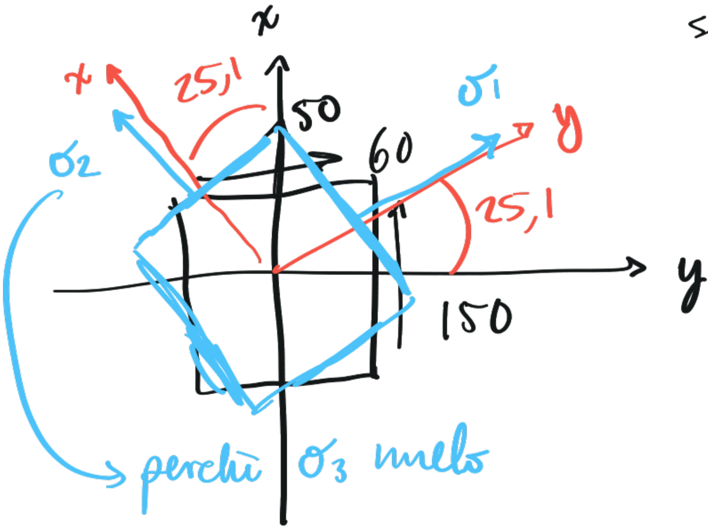
$$\tau = R \sin \alpha_M$$

$$\Rightarrow \alpha_M = \arcsin\left(\frac{\tau}{R}\right) = 2\alpha_R \Rightarrow \alpha_R = \frac{1}{2}\arcsin\left(\frac{\tau}{R}\right)$$

$\uparrow$ α di mohr

$\uparrow$ α reale

Il segno di α_p è dipendente se A è sopra ($\alpha_R > 0$) o se A è sotto ($\alpha_R < 0$), se si gira in senso orario su Mohr si gira in senso orario su x-y

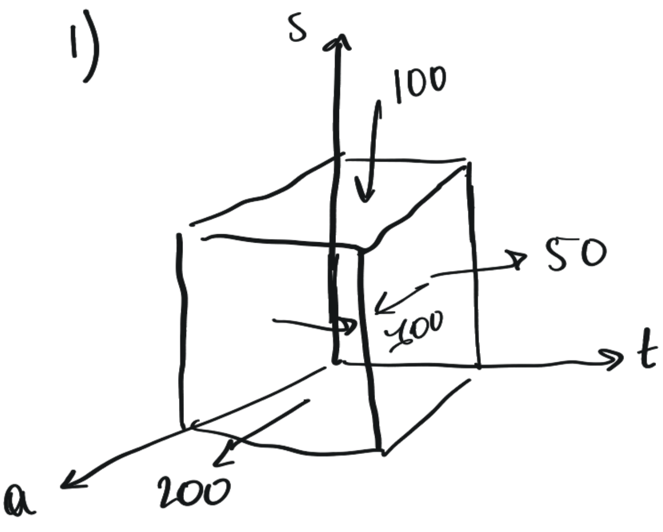


Da ricordare

C, R, $\alpha_M = 2\alpha_R$, mantenere segno di α_M con α_R

Esercizio con Mohr

1)



$$\begin{bmatrix} 200 & 100 & 0 \\ 100 & 50 & 0 \\ 0 & 0 & -100 \end{bmatrix} = \bar{\bar{\sigma}}$$

→ concende con le assi

$$\det \begin{vmatrix} 200-\sigma_p & 100 & 0 \\ 100 & 50-\sigma_p & 0 \\ 0 & 0 & -100-\sigma_p \end{vmatrix} = 0$$

$$(-100-\sigma_p)\left((200-\sigma_p)(50-\sigma_p) - 100^2\right) = 0$$

$\sigma_{p_1} = -100$ MPa

$\sigma_{p_2} = 0$ MPa

$\sigma_{p_3} = 250$ MPa

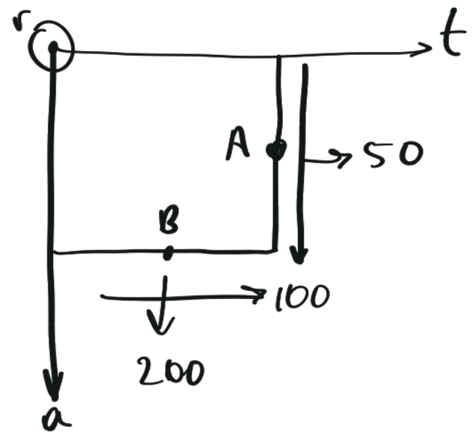
$\sigma_1 > \sigma_2 > \sigma_3$

$\sigma_1 = 250$ MPa

$\sigma_2 = 0$ MPa

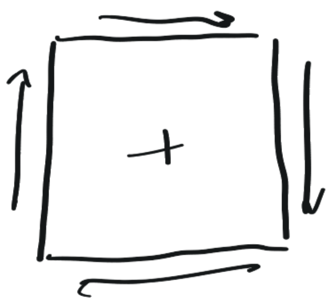
$\sigma_3 = -100$ MPa

$$\tau_{max} = \tau_y = \frac{\sigma_1 - \sigma_2}{2} = 175 \text{ MPa}$$

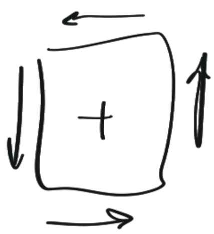


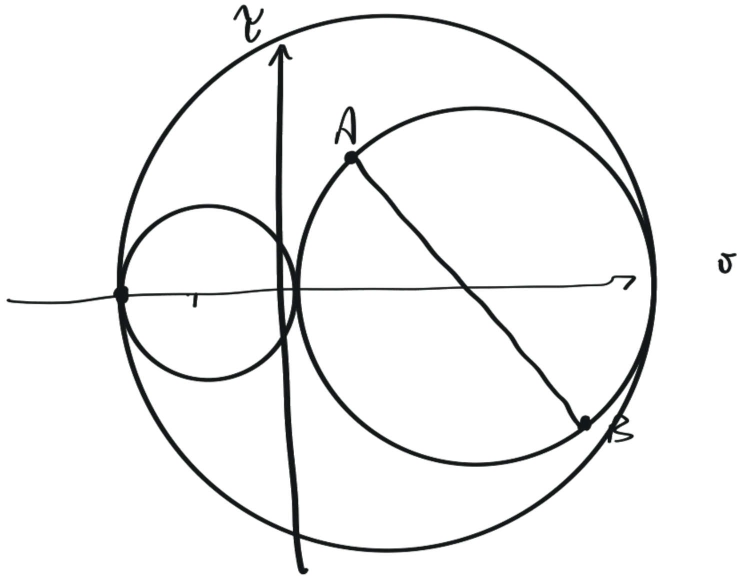
A = (50, 100)

B = (200, -100)



$\alpha_R = \frac{1}{2} \alpha_M$

Se  allora $\alpha_R = \frac{-1}{2} \alpha_M$



$\sigma_3 = -100\,MPa$
$\sigma_1 = 250\,MPa$

$$C = \frac{\sigma_A + \sigma_B}{2} = 125$$

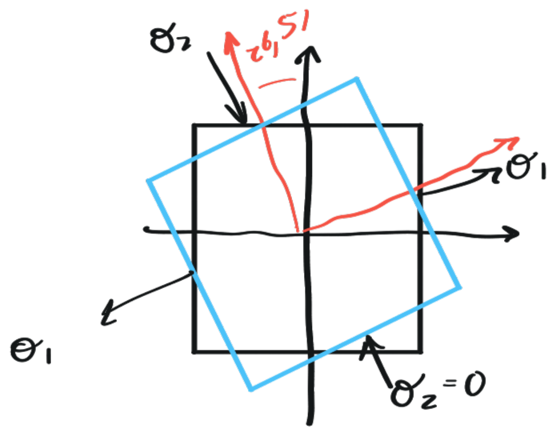
$$R = \sqrt{(\sigma_A - C)^2 + \tau_A^2} = 125$$

$\sigma_1 = C + R = 250\,MPa$

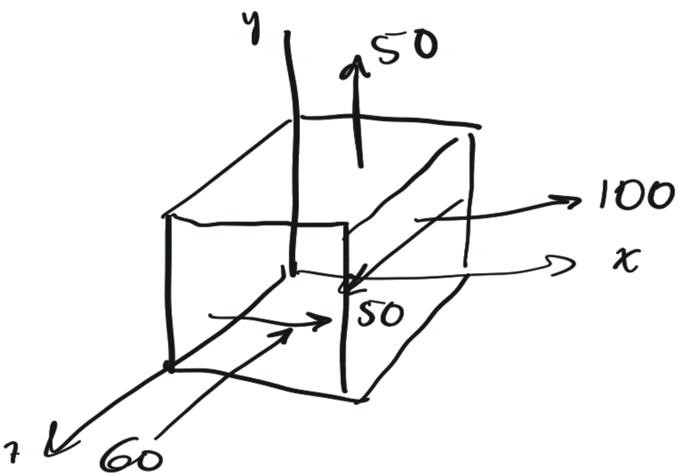
$\sigma_2 = C - R = 0\,MPa$

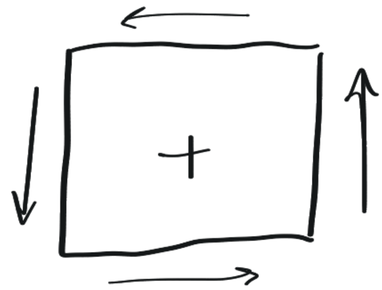
$\sigma_3 = -100\,MPa$

$$\alpha_R = \frac{1}{2}\arcsin\left(\frac{\tau_A}{R}\right) = 26{,}57°$$

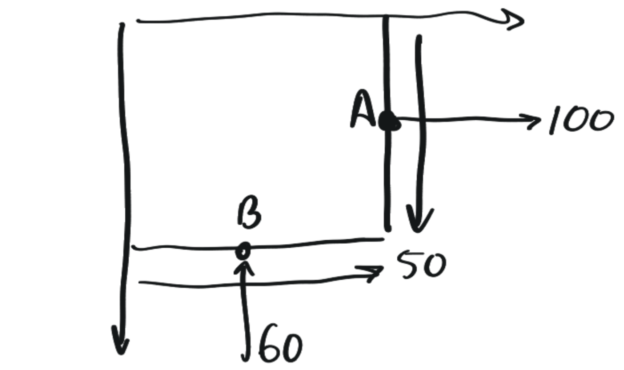


Esercizio Mohr 2)



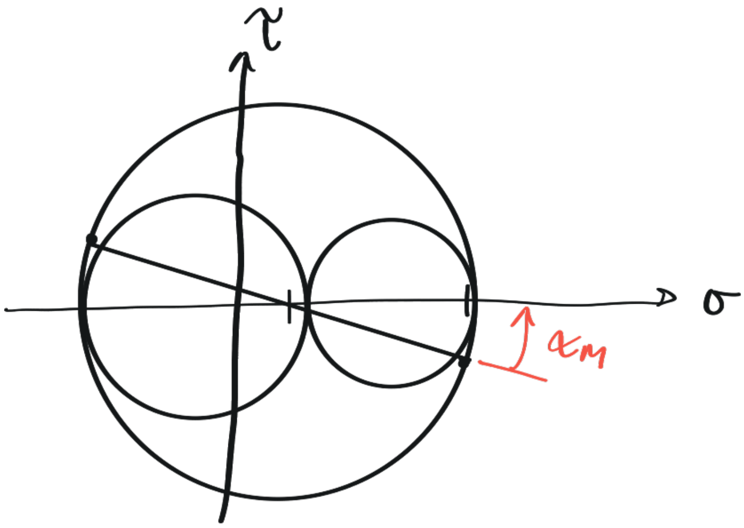


$\alpha_R = \frac{1}{2}\alpha_M$



$A = (100, -50)$

$B = (-60, 50)$



$$C = \frac{\sigma_A + \sigma_B}{2} = 20\,MPa$$

$$R = \sqrt{(\sigma_A - C)^2 + \tau_A^2} = 94\,MPa$$

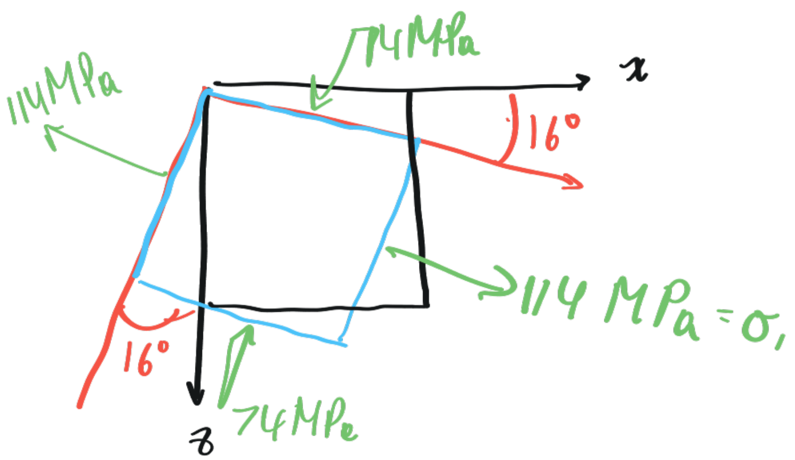
$$\sigma_1 = C + R = 114\,MPa = \sigma_x$$

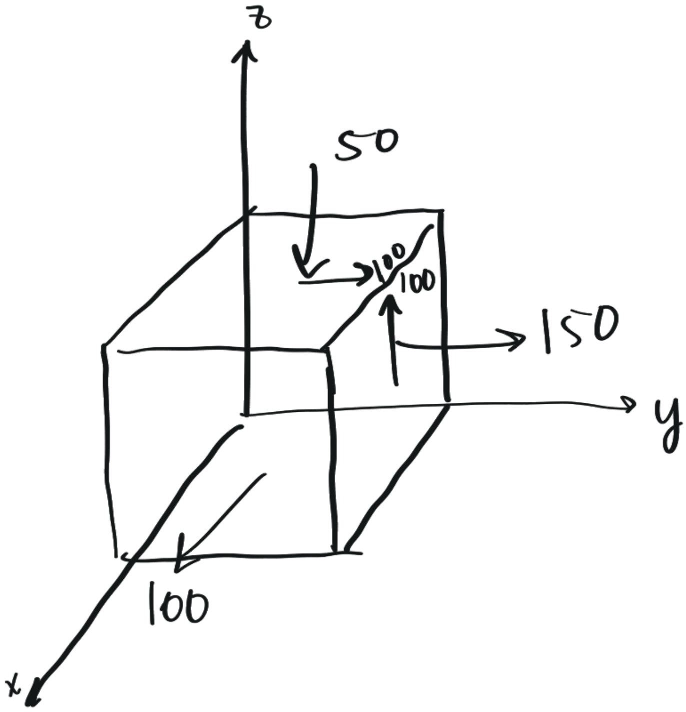
$$\sigma_3 = C - R = -74\,MPa = \sigma_z$$

$$\sigma_2 = -50\,MPa = \sigma_y$$

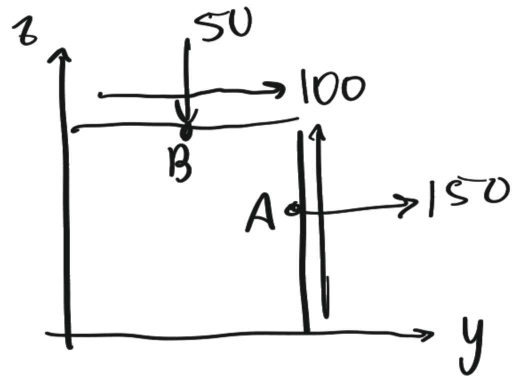
Abbiamo cambiato
la convenzione
quindi dobbiamo cambiare
il segno di α_R

$$\alpha_R = \frac{1}{2} \arcsin\left(\frac{\tau_A}{R}\right) = 16°$$





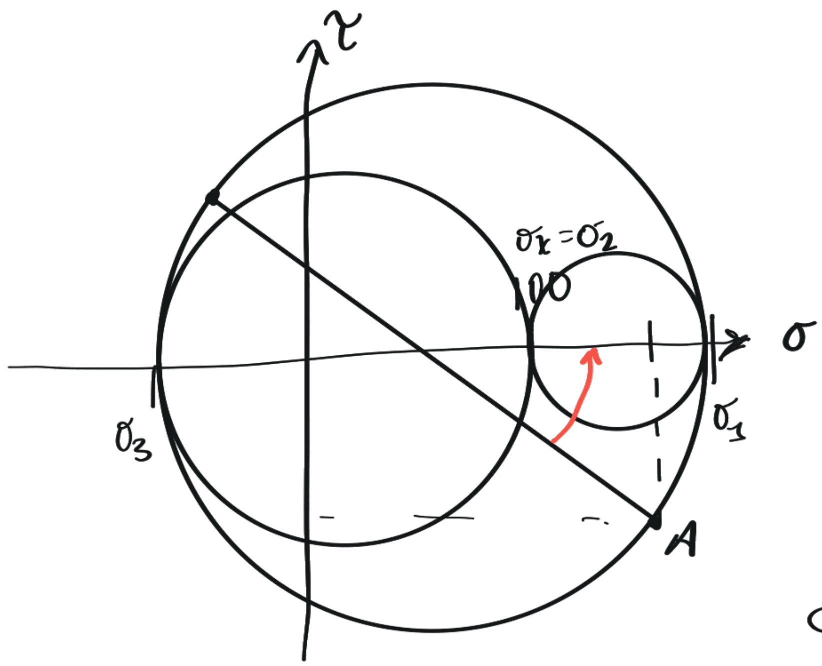
σ_x = direzione principale



$\alpha_R = \frac{1}{2}\alpha_M$ orario

$A = (150, -100)$

$B = (-50, 100)$



$$C = \frac{\sigma_A + \sigma_B}{2} = 50 \text{MP}$$

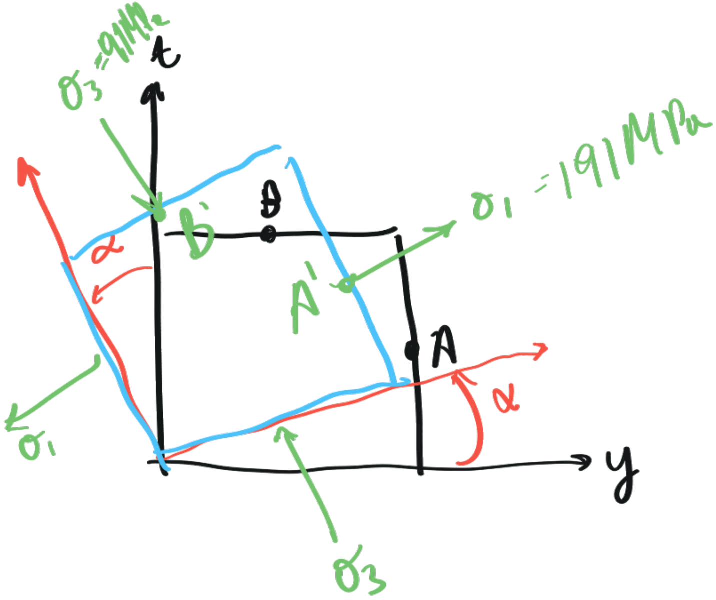
$$R = \sqrt{(\sigma_A - C)^2 + \tau_{xy}^2} = 141 \text{MP}$$

$\sigma_1 = C + R = 191 \text{MPa}$

$\sigma_2 = 100 \text{MPa}$

$\sigma_3 = C - R = -91 \text{MPa}$

$$\alpha_R = \frac{1}{2} \arcsin\left(\frac{\tau_A}{R}\right) = 22{,}5^\circ$$



σ3 = 91 MPa
t
σ1 = 191 MPa
B
B'
A'
A
α
α
σ1
σ1
y
σ3