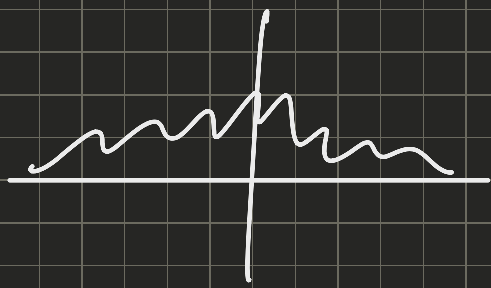
Lesione 4 -

Absolutely Continuous Distributions – Symmetric Densities

Let $X \sim f_X$ where $f_X(x)$ is symmetric with a such that

$$f_X(x-a) = f_X(a-x)$$

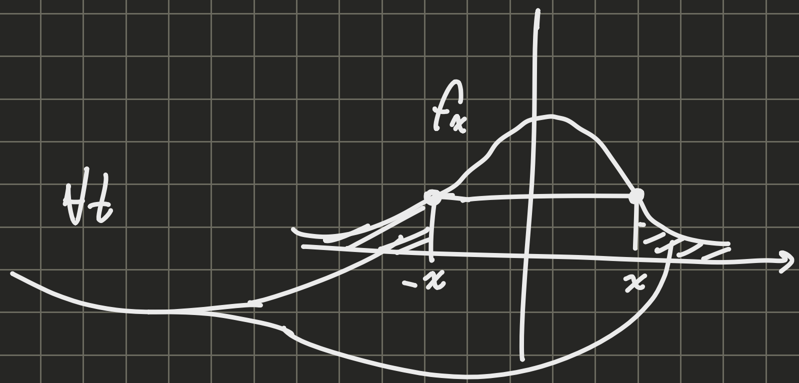
The median is equal is a



if EX exists $\Rightarrow EX = a$

Properties: (for $a=0$)

1) $f_X(x) = f_X(-x) \quad \forall x$



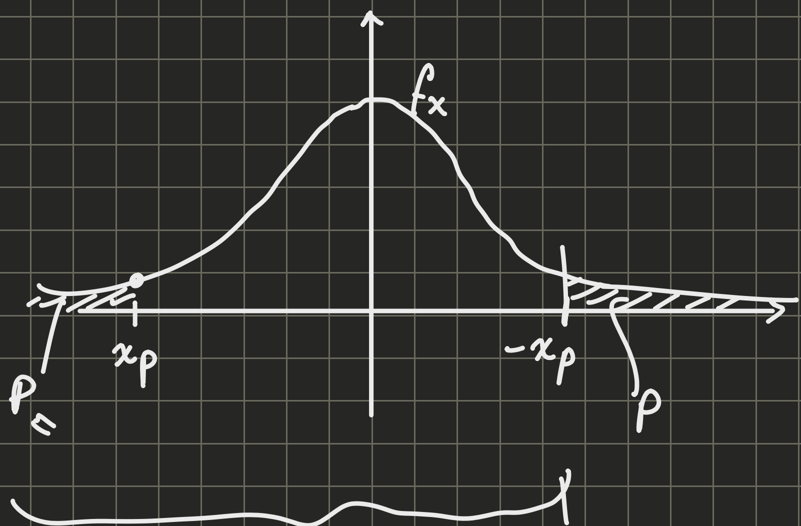
2) $P(X \leq -t) = P(X \geq t) \quad \forall t$

$$\begin{array}{ccc} P(X \leq -t) & = & P(X \geq t) \\ \| & & \| \\ & & P(X > t) \\ & & \| \\ F_X(-t) & = & 1 - F_X(t) \end{array}$$

$$\Rightarrow F_X(-t) = 1 - F_X(t)$$

3) $-x_p = x_{1-p}$

x_p: quantile of the distribution of order $p \in (0,1)$



Area before is $1-p$

$$\Rightarrow -x_p = x_{1-p}$$

Standardization of random variable of X

Yesterday we introduced $Y=aX+b$ where $a,b\in\mathbb{R}$, $a\neq 0$, this is common when we need to change the scale of our random variable.

Definition

Let us consider any random variable X such EX exists and $Var(X)>0$

Let's consider a new random variable $Z = \frac{X-EX}{\sqrt{Var(X)}}$

therefore $a=\frac{1}{\sqrt{Var(x)}}$ and $b = -\frac{EX}{\sqrt{Var}}$

This is a specific type of linear transformation called standardization.

The expectation of Z will be:

$$E(Z) = E\left(\frac{X-EX}{\sqrt{Var(X)}}\right) = \frac{EX-EX}{\sqrt{Var(X)}} = 0$$

This is like $E[aX+b]$ which we have already done

$$Var(Z) = Var\left(\frac{X-EX}{\sqrt{Var(X)}}\right) = \left(\frac{1}{\sqrt{Var X}}\right)^2 \cdot Var(X) = 1$$

We can used this transformation to compare two distributions, even if they have different scales, we can still compare them.

Z represents the distance of X from it's mean, in terms of

It's standard deviation.

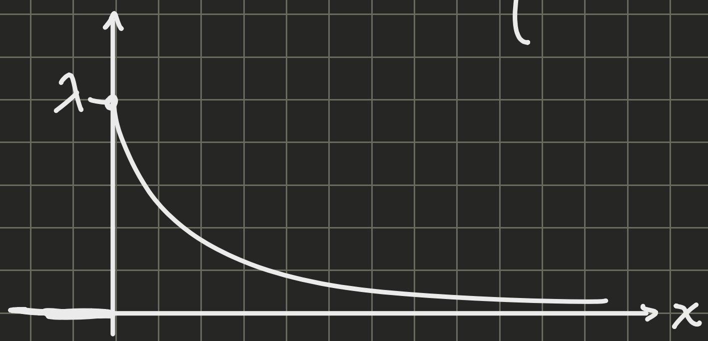
Distribution: Exponential with parameter λ

↳ This is a very useful distribution for engineering.

Let's take the parameter $\lambda > 0$

$X \sim Exp(\lambda)$ → $\sim$ ⇒ X distributed as (Distribution)

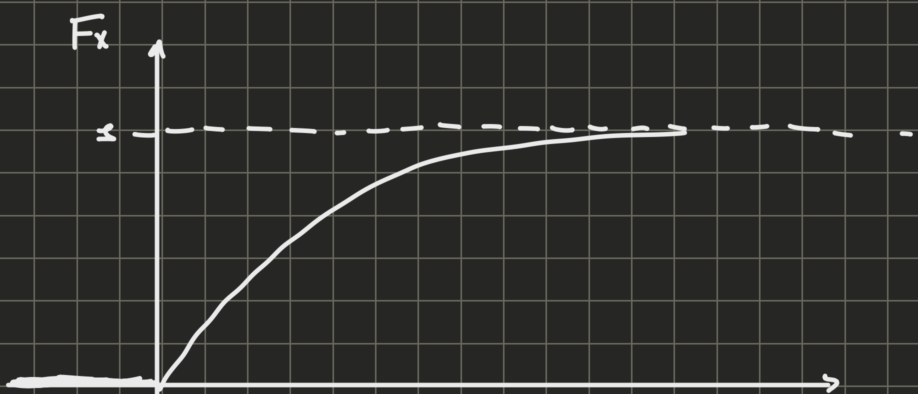
The density of X, $f_X = \begin{cases} 0 & \text{if } x \leq 0 \\ \lambda e^{-\lambda x} & \text{if } x \geq 0 \end{cases}$



The distribution function of X is:

$$F_X = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 - e^{-\lambda x} & \text{if } x \geq 0 \end{cases}$$

$$\int_0^x \lambda e^{-\lambda u}\, du = \left[-e^{-\lambda u} \right]_{u=0}^{u=x} = 1 - e^{-\lambda x}$$



$$EX = \int_0^{\infty} x \cdot \lambda e^{-\lambda x}\, dx = \frac{1}{\lambda}$$

Integration by parts

$$Var(X) = \frac{1}{\lambda^2}$$

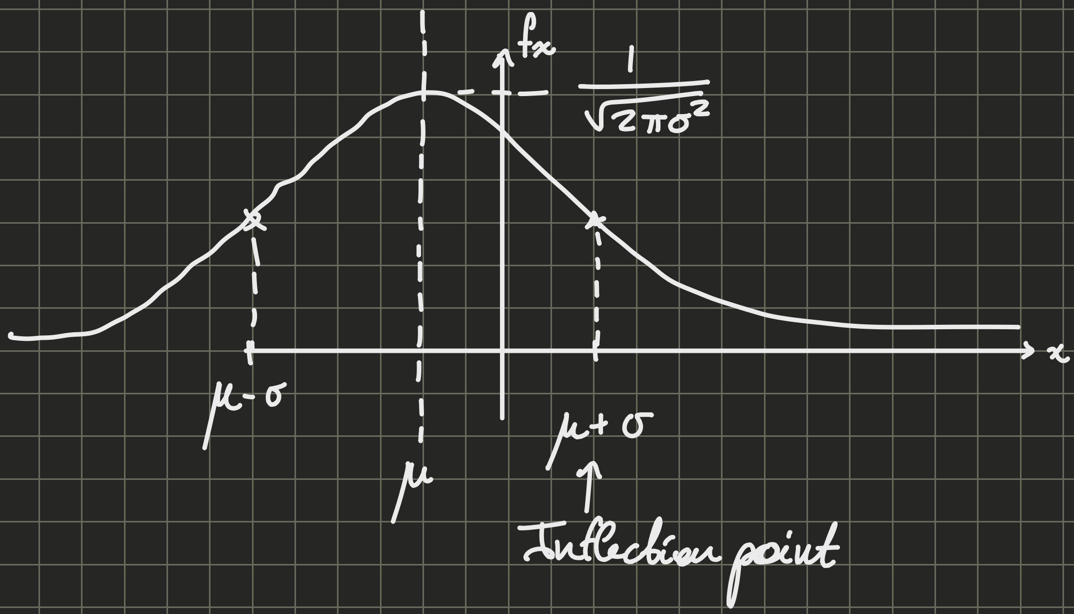
λ = hazard/failure rate/istantaneous propensity to failure.

Distribution: The Gaussian/Normal of the rand variable

$f_X(x, \mu, \sigma^2)$ with $\mu \in \mathbb{R}$, $\sigma^2 > 0 \rightarrow \sigma = \sqrt{\sigma^2}$

$\Rightarrow$ simmetric to μ

$$f_x(x, \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$



$EX = \mu$

$Var(X) = \sigma^2$

The two parameters of the distribution describe out distribution

$X \sim N(\mu, \sigma^2) \rightarrow$ X distributed like the gaussian/normal calligraphic

$$F_X = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(s-\mu)^2}{2\sigma^2}} ds$$

We can say that: $P(\mu - 3\sigma < X < \mu + 3\sigma) = .9973 \approx 1$

Standard Gaussian/Normal $(\mu = 0, \sigma^2 = 1)$ of the r.v X

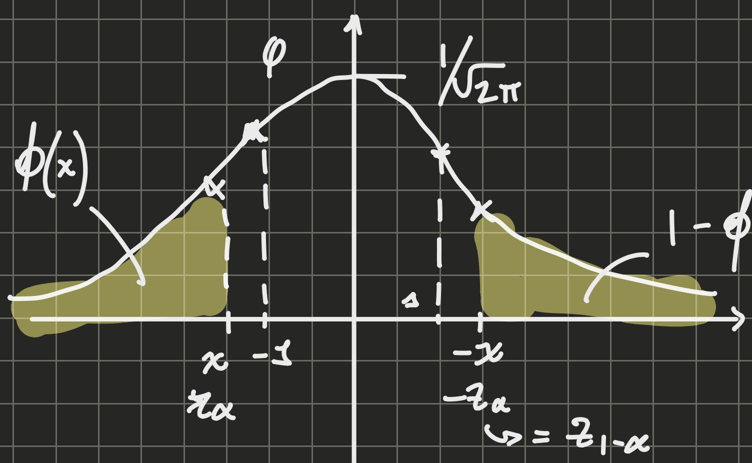
$$X \sim N(0,1)$$

The density will be:

$\rightarrow$ standard notation for this specific case

$$f_x(x, \mu=0, \sigma^2=1) = \varphi(x)$$

$$F_X(x, \mu=0, \sigma^2) = \Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{\frac{-s^2}{2}} ds$$



$\rightarrow$ Disegnato male

Since it's simmetric around 0:

$$\phi(x) = 1 - \phi(-x) \quad \forall x \in \mathbb{R}$$

α - quantile of $X \sim N(0,1)$

$$\phi(z_\alpha) = \alpha \overset{\text{symmetry}}{\Rightarrow} z_\alpha = -z_{1-\alpha}$$

If $X \sim N(\mu, \sigma^2)$, the standardized Z is:

$$Z = \frac{X-\mu}{\sqrt{\sigma^2}}, \quad EZ = 0, \; Var Z = 1$$

Z will still be a standard distribution and in fact it is:

$$\Rightarrow Z \sim N(0,1)$$

When we standardise a $X \sim N(\mu, \sigma^2)$, the standardised variable Z will be $Z \sim N(0,1)$.

Remark: If $X \sim N(\mu, \sigma^2)$, and we consider $a, b \in \mathbb{R}$ with $a \neq 0$

$$\Rightarrow Y = aX + b \underset{\text{is still}}{\sim} N(a \cdot \mu + b, a^2\sigma^2)$$

<u>Example</u>

Let the r.v $X \sim N(1,4)$, calculate $P(-1 < X < 4) = P(-1 < X \leq 4)$

$$= F_X(4) - F_X(-1)$$

The table gives us $\phi(x)$ for $x > 0 \in$ grid of points

it goes give us the negative x since $\phi(x) = 1 - \phi(-x)$

How do we calculate for the non-standardised equation; we have to standardise it:

$$-1 \le X \le 4 \iff \underbrace{-1-1}_{-\mu} < \underbrace{X - EX}_{\mu \text{ for whatever we're looking}} \le \underbrace{4-1}_{-\mu} \iff \frac{-2}{\sqrt{4}} \le \frac{X-EX}{\sqrt{Var(X)}} \le \frac{4-1}{\sqrt{4}}$$

$$\frac{X-EX}{\sqrt{Var(X)}} = Z \sim N(0,1)$$

$$\Rightarrow P(-1 \le X \le 4) = P\left(\frac{-2}{2} \le Z \le \frac{3}{2}\right) = \Phi(1,5) - \Phi(-1)$$

$$= \phi(1,5) - [1 - \phi(1)]$$

$$= \underbrace{\phi(1,5)}_{.93319} + \phi(1) - 1$$

$$.93319 + .84134 - 1 = \ldots$$

Other example:

X - score at the written test

with $X \sim N(21, 9)$

1) $P(X \ge 24) = 1 - P(X \le 24) = 1 - P\left(\frac{X-EX}{\sqrt{Var X}} \le \frac{24-21}{\sqrt{9}}\right)$, with $Z = \frac{X-EX}{\sqrt{Var X}}$

$$= 1 - P(Z \le 1) = 1 - \phi(1) = .15866$$

Standardized variable with standard gaussian

2) Homework $P(X < 18) = P(X \le 18)$

Errors from measurements will have gaussian distribution.

Discrete Random Variables → second type of random variables.

Definition:

A random variable is discrete if it assumes a finite number value with probability 1

Definition: Discrete Density

For a discrete random variable X with possible values

$S := (x_1, x_2, \dots)$ such that $P(X \in S) = 1$, the discrete density is defined as:

$$p_X(x) := P(X = x), \quad x \in \mathbb{R}$$

→ The support of X is finite.

Let X be a discrete r.v.

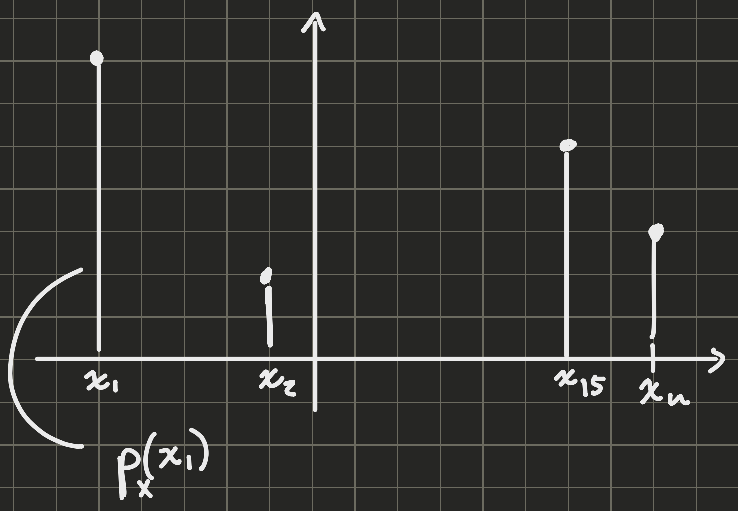
It will have a denumerable set $S = \{x_1, x_2, \dots\}$ such that $P(X \in S) = 1$

It's density is $\quad p_X(x) = P(X = x) \ \forall x$

if $x \notin S \Rightarrow p_X(x) = 0$

$$\sum_{x_i \in S} p_X(x_i) = 1$$

The plot of the density is a bar plot:



<u>Theorem</u>:

either finite or infinite

Let X be a discrete random variable with $S = \{x_1, x_2, \dots\}$,

with density p_X, F_X is the density function:

1) then $F_X(x) = \sum_{x_i \in S, x_i \le x} p_X(x_i) = 1$

2) suppose $x_1 < x_2 < x_3 < \dots$

We can say that:

$p_X(x_1) = F_X(x_1)$

$p_X(x_j) = F_X(x_j) - F_X(x_{j-1})$ for $j \geq 2$