Esercitasione 3 -

Continuation of the last esercitasione

$$\frac{d}{dt}\left(\frac{\partial E_c}{\partial \dot\theta}\right) - \cancel{\frac{\partial E_c}{\partial \theta}} + \cancel{\frac{\partial D}{\partial \dot\theta}} + \frac{\partial V}{\partial \theta} = \cancel{Q}$$

$0 \quad 0 \quad 0$

$$\frac{d}{dt}\left(\frac{\partial E_c}{\partial \dot\theta}\right) - \frac{\partial E_c}{\partial \theta} = J^*(\theta)\cdot\ddot\theta + \frac{1}{2}\frac{\partial J^*(\theta)}{\partial \theta}\cdot\dot\theta^2 = J^*\ddot\theta$$

$$\frac{\partial V}{\partial \theta} = \frac{\partial V_g}{\partial \theta} + \frac{\partial V_k}{\partial \theta}$$

$$\frac{\partial V_g}{\partial \theta} = M_A g \frac{\partial h_s}{\partial \theta} = M_A g \frac{\partial}{\partial \theta}(2L\sin\theta - L) = M_A g\, 2L\cos\theta$$

$$\frac{\partial V_k}{\partial \theta} = k\Delta l \frac{\partial \Delta l}{\partial \theta} = k\left[\Delta l_0 + \left(4L\sin\left(\frac{\theta}{2}-\frac{\pi}{4}\right) - 2\sqrt{2}L\right)\right]\cdot$$
$$\cdot 2L\cos\left(\frac{\theta}{2}-\frac{\pi}{4}\right)$$

Step 6 $\rightarrow$ NL EOM

$$J^*\ddot\theta + k\left(\Delta l_0 + 4l\sin\left(\frac{\theta}{2}-\frac{\pi}{4}\right) - 2\sqrt{2}L\right)\cdot 2L\cos\left(\frac{\theta}{2}-\frac{\pi}{4}\right) +$$
$$M_A g\, 2L\cos\theta = 0$$

$$J\ddot\theta + k\left(\Delta l_0 + 2\sqrt{2}L\sqrt{1-\sin\theta} - 2\sqrt{2}L\right)\cdot\left(\frac{-L\sqrt{2}\cos\theta}{\sqrt{1-\sin\theta}}\right) + M_A g\, 2L\cos\theta = 0$$

$\rightarrow$ different equations based on the Δl we go, the second equation is valid for all θ

Step 7 ($\Delta l_0 = ?$) $\quad \theta_0 = \pi$

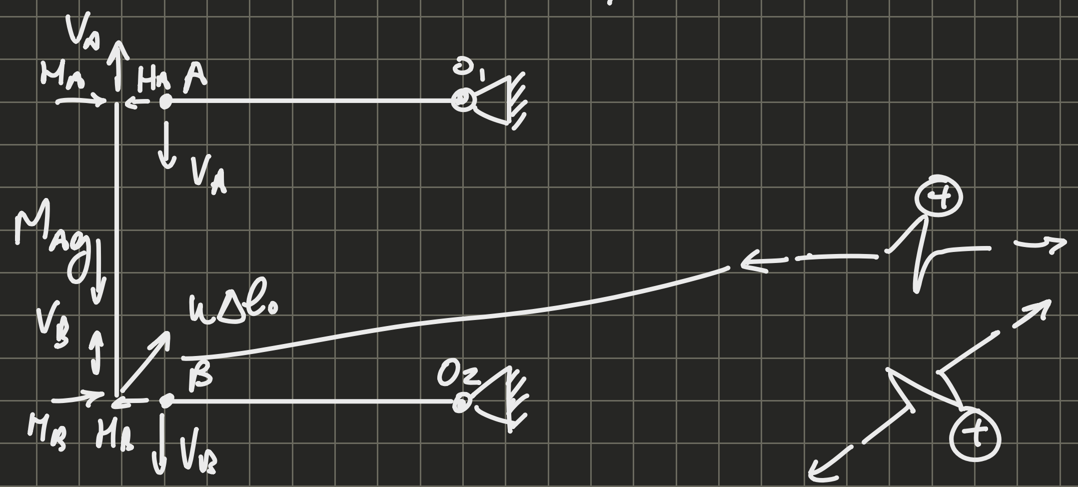
Method 1 $\left.\frac{\partial V}{\partial \theta}\right|_{\theta_0} = \cancel{\sum_i Q_i}$ $\to 0$

$$\Rightarrow k\Big[\underbrace{\Delta l_0}_{\text{Static preload}} + \underbrace{4\sin\left(\frac{\theta}{2}-\frac{\pi}{4}\right) - 2\sqrt{2}L}_{\Delta l d \;\hookrightarrow\; \text{at } \theta_0 = 0}\Big]2L\cos\left(\frac{\theta}{2}-\frac{\pi}{4}\right) + 2M_A gL\cos\theta = 0$$

$$k\Delta l_0 \, 2L\frac{\sqrt{2}}{2} - 2M_A g L = 0 \to \Delta l_0 = \frac{M_A g\sqrt{2}L}{kL} = \frac{M_A g\sqrt{2}}{k}$$

↳ Always check sign.

Method 2 → Static Equilibrium



$M_{O_1} \Rightarrow V_A = 0$

$M_{O_2} \Rightarrow V_B = 0$

$$\sum_i F_V(A,B) = 0 \to -M_A g + k\Delta l_0 \sin\left(\frac{\pi}{4}\right)$$

$$\Rightarrow \Delta l_0 = \frac{M_A g\sqrt{2}}{k}$$

Step 8 → Linearisation

↳ choose an equilibrium point → $\theta_0 = \pi$

↳ Apply change of variables

$$\begin{cases} \tilde{\theta} = \theta - \theta_0 \\ \dot{\tilde{\theta}} = \dot{\theta} \\ \ddot{\tilde{\theta}} = \ddot{\theta} \end{cases}$$

$\cdot \ J^*(\theta) = J_0^*$

$\cdot \ V = V_0 + \cancel{\left.\frac{\partial V}{\partial \theta}\right|_{\theta_0}} \tilde{\theta} + \frac{1}{2} \left.\frac{\partial V}{\partial \theta^2}\right|_{\theta_0} \cdot \tilde{\theta}^2$

$\cancel{\left.\frac{\partial V}{\partial \theta}\right|_{\theta_0}} = \cancel{Q_s} \to 0$

$\left.\frac{\partial V}{\partial \theta^2}\right|_{\theta_0} \to k_I + k_{II} + k_{III}$

$$k_I = k\left(\left.\frac{\partial \Delta l}{\partial \theta}\right|_{\theta_0}\right)^2 = 2kL^2 \ \boxed{>0}$$

$$k_{II} = k\Delta l_0 \left(\left.\frac{\partial \Delta l^2}{\partial \theta^2}\right|_{\theta_0}\right) = -M_a g L \ \boxed{<0}$$

$$k_{III} = M_a g \left(\left.\frac{\partial h_G}{\partial \theta}\right|_{\theta_0}\right) = 0$$

→ in this case the gravity does not act like a spring because it does not act like a restoring force

$$l_m(\theta) = 4L\sin\left(\frac{\theta}{2} - \frac{\pi}{4}\right) = \Delta l$$

$$\frac{\partial l_m}{\partial \theta} = 2L\cos\left(\frac{\theta}{2} - \frac{\pi}{4}\right)$$

$$\frac{\partial^2 l_m}{\partial \theta^2} = -L\sin\left(\frac{\theta}{2} - \frac{\pi}{4}\right)$$

$$\Delta l = \Delta l_d + \cancel{\Delta l_0}$$

$$l = l_m(\theta) - \cancel{l_m(\theta_0)}$$

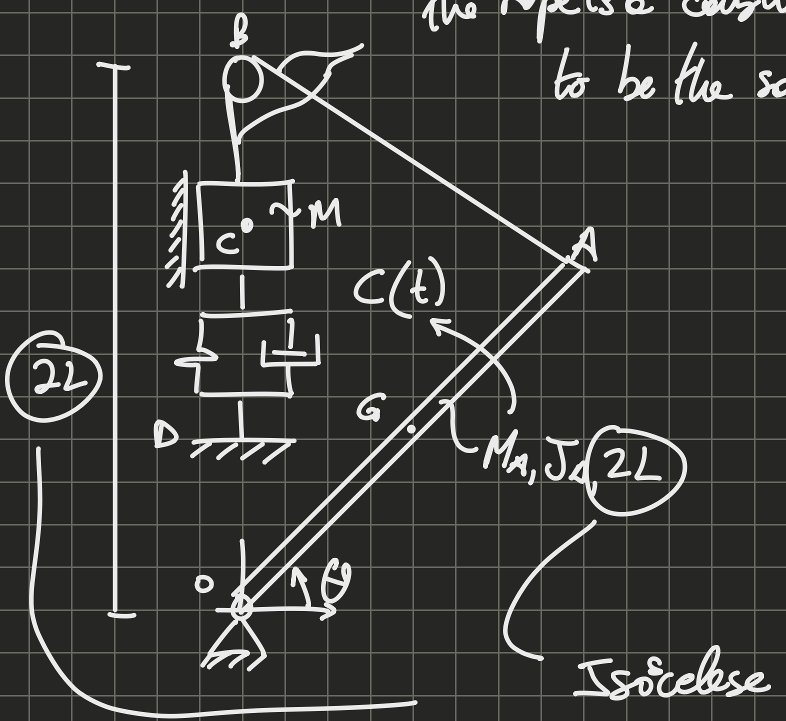
L EOM

$$J^* \ddot{\tilde{\theta}} + \cancel{r^* \dot{\tilde{\theta}}} + k^* \tilde{\theta} = 0$$

$r^* \dot{\tilde{\theta}} \to 0$

Exercise 3

the rope is a constraint, since both have to be the same velocities



① NL EOM

② $\Delta l_0 = ?$ $\quad \theta_0 = \frac{\pi}{2}$

③ L EOM

Step 1

2 bodies	3x2	= 6 dof	
1 hinge	-2x1 =	2 doc	1 dof ⇒ θ
1 slider	-2x1 =	2 doc	
1 rope	-1x1 =	1 doc	

Step 2

$$E_c = \frac{1}{2} m_A V_A^2 + \frac{1}{2} J_A \omega_A^2 + \frac{1}{2} M V_c^2 + \cancel{\frac{1}{2} J \omega_c^2}$$

$$D = \frac{1}{2} r \dot{\Delta l}^2$$

$$V = V_g + V_k = M_A g h_G + M_c g h_c + \frac{1}{2} k \Delta l^2$$

$$\mathcal{L}^* = \vec{C}(t) \cdot \delta \vec{S}_F$$ → displacement of the point of application of force

$$C(t) = C_0 \cos(\Omega t)$$

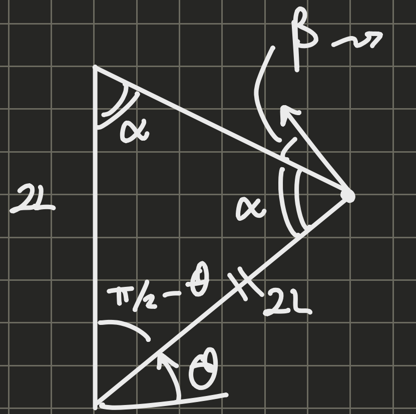
Step 3

$\omega_A = \dot{\theta}\,\hat{k}$

$\vec{v}_G = \dot{\theta}\,\hat{k} \times (L\cos\theta\,\hat{i} + L\sin\theta\,\hat{j}) = \dot{\theta}L\cos\theta\,\hat{j} - \dot{\theta}L\sin\theta\,\hat{i}$

$= \dot{\theta}L(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})$

$\vec{V}_C = ?$



we need to find this angle since while V_C is always tangential to the rod, the α is not, so we have to use that angle to find V_C, by also projecting V_C onto the side of the triangle.

So to calculate V_C we are going to use a geometric approach.

$l_{rope} = \overline{AB} + \overline{BC}$

$\overline{AB} = |(B-A)|$

$$\begin{cases} B = 0\,\hat{i} + 2L\,\hat{j} \\ A = 2L\cos\theta\,\hat{i} + 2L\sin\theta\,\hat{j} \end{cases}$$

$BA = -2L\cos\theta\,\hat{i} + (2L - 2L\sin\theta)\,\hat{j}$

$\|B-A\| = \sqrt{4L^2\cos^2\theta + 4L^2 + 4L^2\sin^2\theta - 8L^2\sin\theta}$

$= \sqrt{4L^2 + 4L^2 - 8\sin\theta} = 2\sqrt{2}L\sqrt{1-\sin\theta}$

$BC = 2L - h_C$

$\overline{AB} + \overline{BC} = l_{rope}$

$2\sqrt{2}L\sqrt{1-\sin\theta} + 2L - h_C = l_{rope}$

$$h_c = 2L + 2\sqrt{2}L\sqrt{1-\sin\theta} - l_{rope}$$

$$(C-O) = 0\hat{i} + h_c\hat{j}$$

$$\Rightarrow V_c = \frac{d}{dt}(C-O) = \frac{d}{dt}h_c = 2\sqrt{2}L(1-\sin\theta)^{-1/2} \cdot \frac{1}{2}(-\cos\theta)\cdot\dot{\theta} =$$

$$= \frac{\sqrt{2}L\cos\theta}{\sqrt{1-\sin\theta}}\dot{\theta}$$

$$h_G = L\sin\theta\,\hat{j}$$

$$\Delta l = l_m(\theta) - l_m(\text{UNLOADED})$$

$$l_m(\theta) = \|C-D\| = h_c - h_D = 2L + 2\sqrt{2}L\sqrt{1-\sin\theta} - l_{rope}) - h_D$$

$$\Delta l(\theta) = 2L + 2\sqrt{2}\sqrt{1-\sin\theta} - l_{ROPE} - h_D) - l_{UNLOADED}$$

$$= 2\sqrt{2}\sqrt{1-\sin\theta} - C_1 - l_{UNLOADED}$$

$$\rightarrow \Delta l\left(\frac{\pi}{2}\right) = 0 \rightarrow 2\sqrt{2}\cdot 0 - C_1 - l_{UNLOAD} = 0$$

$$\Rightarrow l_{UNLOAD} = -C_1$$

$$\Delta l = 2\sqrt{2}L\sqrt{1-\sin\theta}$$

$$\dot{\Delta l} = \frac{-\sqrt{2}L\cos\theta}{\sqrt{1-\sin\theta}}\dot{\theta}$$

$$\delta s_F = \frac{\delta s_F}{\delta\theta}\cdot\delta\theta = \frac{\delta\theta}{\delta\theta}\cdot\delta\theta = 1\cdot\delta\theta$$ $\Rightarrow$ the extenal force is work in the same direction of the independent variable!

Step 4

$$E_c = \frac{1}{2}\left[M_A L^2 + J_A + M_c \frac{2L^2\cos^2\theta}{1-\sin\theta}\right]\cdot\dot{\theta}^2 = \frac{1}{2}J^*(\theta)\cdot\dot{\theta}^2$$

$$D = \frac{1}{2} r\left[\frac{2L^2\cos^2\theta}{1-\sin\theta}\right]\cdot\dot{\theta}^2$$

$$V_g = M_A g\, L\sin\theta + M_c g\left[2L + 2\sqrt{2}L\sqrt{1-\sin\theta} - l_{rope}\right]$$

$$V_k = \frac{1}{2}k\left[2\sqrt{2}L\sqrt{1-\sin\theta}\right]^2$$

$$\delta \mathcal{L}^* = C(t)\cdot s\cdot\delta^*\theta \rightarrow \text{linear}$$

Step 5

$$\frac{d}{dt}\frac{\partial E_c}{\partial\dot{\theta}} - \frac{\partial E_c}{\partial\theta} \rightarrow J^*\ddot{\theta} + \frac{1}{2}\frac{\partial J^*(\theta)}{\partial\theta}\cdot\dot{\theta}^2$$

$$\cancel{(1-\sin\theta)}(1+\sin\theta)$$

$$\downarrow$$

$$J^*(\theta) = M_A L^2 + J_A + M\frac{2L^2\cos^2\theta}{1-\sin\theta} = M_A L^2 + J_A + 2M_c L^2\frac{1-\sin^2\theta}{\cancel{1-\sin\theta}}$$

$$= M_A L^2 + J_A + 2ML^2(1+\sin\theta)$$

$$\frac{\partial J^*}{\partial\theta} = 2ML^2\cos\theta$$