Esercitazione 3 – Discrete Random Variables

3.1

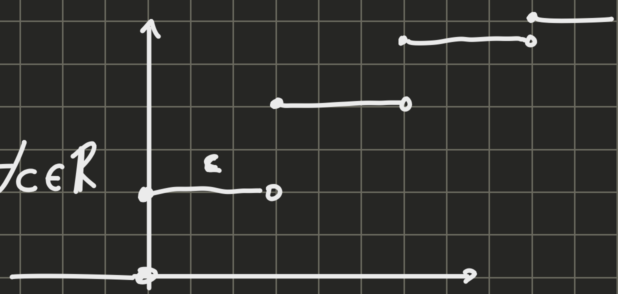
$$F(x)=\begin{cases} 0 & \text{if } x<0 \\ c & \text{if } 0\le x<1 \\ 2/3 & \text{if } 1\le x<2 \\ 11/12 & \text{if } 2\le x<3 \\ 1 & \text{if } x\ge 3 \end{cases}$$

a) c?

1) F non-decreasing $[0,2/3]$

2) $\lim_{x\to-\infty} F(x)=0$ & $\lim_{x\to\infty} F(x)=1$ ✓ $\forall c\in\mathbb{R}$

3) <u>Right</u>-handed continuity $= \forall c\in\mathbb{R}$



$\Rightarrow c\in[0,2/3]$

b) $P(X\ge 1/2)$? $P(2<X\le 4)$? $P(2\le X<4)$? $P(X>3)$? $c=2/3$

$$P\left(X>\tfrac{1}{2}\right)=1-P\left(X\le\tfrac{1}{2}\right)=1-P(X=0)=1-p_X(0)=1-c=1/3$$

$$p_X(k)=\begin{cases} F_X(k) & k=0 \\ F_X(k^+)-F_X(k^-) & k\ge 1 \end{cases}$$ → in this case

$$P(2<X\le 4)=P(X=3)+P(X=4)=p_X(3)+p_X(4)=$$

$$=\left(F_X(3^+)-F_X(3^-)\right)+\left(F_X(4^+)-F_X(4^-)\right)$$

$$=1-\frac{11}{12}=\frac{1}{12}$$

$$P(2 \le x \le 4) = \frac{1}{12} + P(x=2) = \frac{1}{12} + p_x(2^+) - p_x(2^-) = \frac{1}{12} + \frac{11-8}{12} = \frac{1}{3}$$

$$P(x<3) = P(x=0) + P(x=1) + P(x=2) = p_x(0) + p_x(1) + p_x(2)$$

$$= \frac{2}{3} + 0 + \frac{1}{4} = \frac{8+3}{12} = \frac{11}{12} \checkmark$$

d) $c = 2/3$

$E(x)$? $Var(x)$?

$$E(x) = \sum_{k \in S} k\, p_x(k) = 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{12} = \frac{3}{4} \Rightarrow E(x)^2 = \frac{9}{16}$$

$$p_x = \begin{cases} c & , \text{if } k = 0 \\ 0 & , \text{if } k = 1 \\ 1/4 & \text{if } k = 2 \\ 11/12 & \text{if } k \le 3 \end{cases}$$

$$Var(x) = E(x^2) - E(x)^2$$

$$E(x^2) = \sum_{k \in S} k^2 p_x(k) = 4 \cdot \frac{1}{4} + 9 \cdot \frac{1}{12} = \frac{28}{16}$$

$$Var(x) = \frac{19}{16}$$

e) $y = \sqrt{x}$ $E(x)$? $Var(x)$?

$$E(Y) = E(\sqrt{x}) = \sum_{k \in S} \sqrt{k}\, p_x(k) = \sqrt{2} \cdot \frac{1}{4} + \sqrt{3} \cdot \frac{1}{12} = \frac{3\sqrt{2} + \sqrt{3}}{12}$$

$$Var(Y) = \underbrace{E(Y^2)}_{E(x)} - E(Y)^2 = \frac{3}{4} - \left(\frac{3\sqrt{2} + \sqrt{3}}{12}\right)^2 = \frac{29 - 2\sqrt{6}}{48}$$

Skipped 3.3

3.4

$n = 10$ questions, 4 possible answers 1 correct. Independent & random

X = number of correct answers

a) p_x?

Y_j = outcome of question j

$$Y_j = \begin{cases} 1, & \text{correct answer} \\ 0 \end{cases} \qquad Y_j \sim Be\left(\frac{1}{4}\right)$$

$$X = \sum_{j=1}^{10} Y_j \sim Bin\left(10, \frac{1}{4}\right)$$

$$p_X(k) = \binom{10}{k}\left(\frac{1}{4}\right)^k\left(\frac{3}{4}\right)^{10-k} \qquad k = 0,1,2,\dots,10$$

b) $P(x \geq 5) = 1 - P(x \leq 4) = 1 - (p_X(0) + p_X(1) + p_X(2) + p_X(3) + p_X(4))$

$= 0{,}0781$

c) $\mathbb{E}(X)$? $Var(x)$?

$$X \sim Bin\left(\underset{n}{10}, \underset{p}{1/4}\right) \Rightarrow \mathbb{E}[X] = np = 2{,}5$$

$$Var[X] = np(1-p) = 1{,}875$$

<u>3,5</u>

$$f_X(x) = \frac{6}{(x+6)^2}\, \mathbb{I}_{(0,\infty)}(x)$$

$x \Rightarrow$ a.c.r.v

$$\mathbb{P}(X \leq 12) = \int_{-\infty}^{12} \frac{6}{(x+6)^2}\,dx = \int_0^{12} \frac{6}{(x+6)^2}\,dx = -6\left[\frac{1}{x+6}\right]_0^{12} = -6\left(\frac{1}{18} - \frac{1}{6}\right)$$

$$= -6\left(\frac{1-3}{18}\right) = \frac{12}{18} = \frac{2}{3}$$

Device with 5 batteries

X_j = lifetime of battery j

Y_j = status of battery j within first 12 hours

$$Y_j = \begin{cases} 1 & X_j \le 12 \\ 0, & \text{--} \end{cases}$$

$Y_j \sim Be(p)$, $p = \mathbb{P}(X_j \le 12) = \frac{2}{3}$

$\Rightarrow Y_j \sim Be(\frac{2}{3})$

Y = number of batteries that need to be replaced within 12 hours.

$$= \sum_{j=1}^{5} Y_j \sim Bin(5, 2/3)$$

$$\mathbb{E}[Y] = np = \frac{10}{3}$$

c) $P(Y=2) = ? = \binom{5}{2}\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^3 = 0{,}1646$

3,6

Device detecting neutronos in 10^{-9} s

X = number of neutronics detected in 10^{-9} s

$X \sim Poi(\lambda)$ to model a number of events in a period of time

$\mathbb{E}[X] = \lambda = 5 \Rightarrow X \sim Poi(5)$

b) $P(X=0)$? $P(X=6)$? $P(X<10)$?

$$P(X=0) = p_X(0) = e^{-5}$$

$$p_X(k) = \frac{\lambda^k}{k!} e^{-\lambda} = \frac{5^k}{k!} e^{-5}$$

$P(X=6) = \frac{5^6}{6!} e^{-5} = 0{,}1462$ → different from text because text was an bigwor and was read as > 6, in the unting

$P(X<10) = p_X(0) + \dots + p_X(9) = .9682$

of the answers.

3,7

$X_1, \dots, X_n$

$$f(x) = \begin{cases} \frac{2}{x^3} & \text{if } x > 1 \\ 0 & \text{otherwise} \end{cases}$$

a) $P(X_1 \geq 23) = \int_{23}^{\infty} f(x)\,dx = \int_{23}^{\infty} \frac{2}{x^3}\,dx = -\left[\frac{1}{x^2}\right]_{23}^{\infty} = -\left(0 - \frac{1}{23^2}\right) = \frac{1}{23^2} = 0{,}001890$

b) $Y_j = \begin{cases} 1 & \text{if } X_j \geq 23 \\ 0 & \text{otherwise} \end{cases}$ $\quad Y_j \sim Be(p),\ p = P(Y_j = 1) = P(X_j \geq 23) = 0{,}001890$

$$Y = \sum_{j=1}^{10} Y_j \sim Bin(10, p)$$

$$P(Y < 3) = P(Y=0) + P(Y=1) + P(Y=2) = 0{,}9999991979$$

c) $np \leq 10$, $n \geq 50$ $\Rightarrow Bin(n,p) \approx Poi(\lambda) \quad \lambda = np$

$$Y \simeq Poi(np) = Poi(1000 \cdot 0{,}001890) = Poi(1{,}89)$$

$$P(Y < 3) = p(0) + p(1) + p(2) = \frac{\lambda^0}{0!} e^{-\lambda} + \dots = 0{,}7064$$