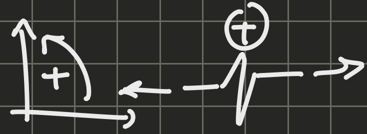
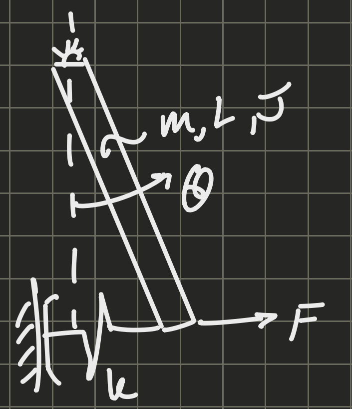
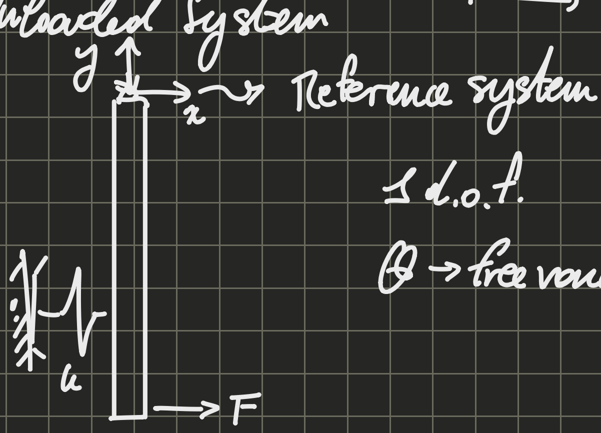
Lezione 2-

Start of Systematic approach to writing the equation of motion

Starting with 1 d.o.f:







1 d.o.f.

$\theta \rightarrow$ free variable

First step is usually counting the d.o.f and choosing the proper (simplest ones). In this case it's trivial.

We also need to place our system of reference and the convention of the motion.

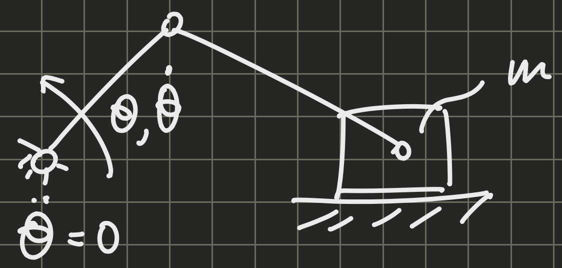
For Lagrange we need to fix 3 quantities:

$E_c, V, \delta^* \mathcal{L}$, also $q, \dot{q}, \ddot{q}$

$E_c = \frac{1}{2} m v_G^2 + \frac{1}{2} J_G \omega^2 \rightarrow$ König Theorem $\rightarrow$ general formula for each body.

v_G and ω are physical quantities, we need to figure out how to write them as function of our free variables. θ in this case

$$= \frac{1}{2}\, \underbrace{m^*(q)}_{\text{Reduced mass which is a function of the position}}\, \underbrace{\dot{q}^2}_{\text{Derivative of the degree of freedom}}$$



$$E_c = \frac{1}{2} m(\theta)\, \dot{\theta}^2$$

$m^*(\theta)$ is the reduced mass as a function of θ

$m^*(q)$ is the reduced mass / or $\underbrace{\text{moment of inertia}}_{\text{if } q = \theta}$, related to the physical system.

$E_c = \frac{1}{2} m^*(\theta)\, \dot{q}^2$, general way of writing the kinetic energy.

In the original problem m is not a function of θ, but in the piston it is as θ and $\dot{\theta}$ are related.

$$\underbrace{\frac{d}{dt}\left(\frac{\partial E_c}{\partial \dot{q}}\right) - \frac{\partial E_c}{\partial q}} + \frac{\partial V}{\partial q} = Q$$

$$\frac{\partial E_c}{\partial \dot{q}} = m^*(q)\, \dot{q} \quad \rightarrow \frac{d}{dt}\left(\frac{\partial E_c}{\partial \dot{q}}\right) = m^*(q)\, \ddot{q} + \frac{\partial m^*(q)}{\partial q} \cdot \frac{dq}{dt}\, \dot{q}$$

$$= m^*(q)\, \ddot{q} + \frac{\partial m^*(q)}{\partial q} \cdot \dot{q}^2$$

$$\frac{\partial E_c}{\partial q} = \frac{1}{2}\frac{\partial m^*(q)}{\partial q}\cdot\dot{q}^2$$ For each body we write this

$$\frac{d}{dt}\left(\frac{\partial E_c}{\partial \dot{q}}\right) - \frac{\partial E_c}{\partial q} = m^*(q)\ddot{q} + \frac{1}{2}\frac{\partial m^*(q)}{\partial q}\dot{q}^2$$

$\Rightarrow$ instead doing many derivates, we only need to do one

If m is not dependent on q the only the first part is needed.

$m^*(q)$ can be whatever cos, sin, etc.

From this we are getting the inertia forces, we see that much of this is forces (or torques if the variables are θ)

All these should be forces or torques, therefore from Lagrange we write dynamic equilibria.

The potential energy V are from gravitational forces and forces from springs.

We can split V in two components:

$$V = V_k + V_g = \underbrace{\frac{1}{2}k\,\Delta\ell^2}_{\text{Potential Energy of the spring}} + mg\,h_G = \underbrace{\frac{1}{2}k\,\Delta\ell^2(q) + mg\,h_G(q)}_{\text{For each spring or mass we write these}}$$

We can split Δl in two parts:

$$V_k = \frac{1}{2}k\Delta l^2(q) = \frac{1}{2}k\Big(\underbrace{\Delta l_0}_{\text{Preload of the spring}} + \overset{\text{dynamic}}{\Delta l_d(q)}\Big)^2$$

↳ Difference in natural length and length in equilibrium position.

$\Delta l_0 \neq 0$, otherwise we have motion.

$$\frac{\partial V_k}{\partial q} = \underbrace{k\Delta l_0}\,\frac{\partial \Delta l_d(q)}{\partial q} + k\,\Delta l_d(q)\,\underbrace{\frac{\partial \Delta l_d(q)}{\partial q}}$$

↳ Physically the variation of the length of the spring when we vary the d.o.f.

It is the relation between the length and the d.o.f.

The terms from E_k are inertia

These terms are the stiffness of the system.

→ Tension of the spring in the preloaded condition

We can walk on a tight rope, even though they don't have flexional stiffness, because we preload the rope, allowing us to stay up.

$$k\Delta l_0 \,\frac{\partial \Delta l_d(q)}{\partial q}$$

→ tension in the rope

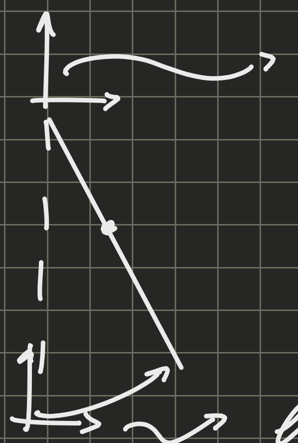
↳ This is the kinematic relation, which

what is keeping us up because of the tension we have pre-loaded into the system.

$$V_g = mg\, h_G(q)$$

$$\frac{\partial V_g}{\partial q} = mg \frac{\partial h_G(q)}{\partial q}$$

Kinematic relation between vertical position of center of gravity and the d.o.f.



$$h_G = -\frac{L}{2} \sin\theta$$

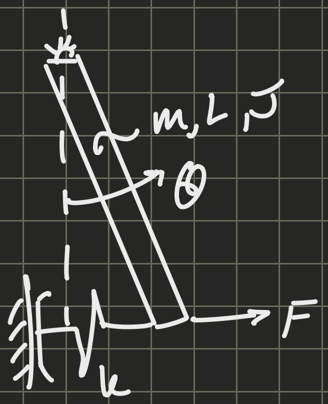
$$h_G = L - \frac{L}{2} \sin\theta$$

Different system have different h_G, but the derivative is the same.
Since the positional difference doesn't change anything.

With only 3 derivatives we get the equation of motion.

We get a non-linear differential equation.

This is very abstract but once we apply it it becomes easier.



$$E_c = \frac{1}{2} m \underline{v_G}^2 + \frac{1}{2} J \underline{\omega}^2$$

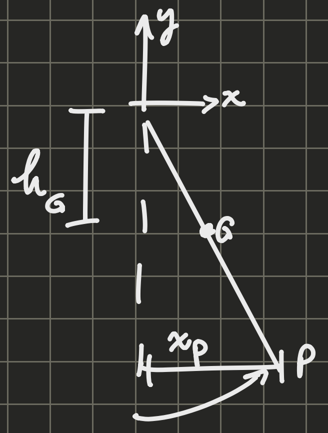
$$V = V_k + V_g = \frac{1}{2} k \underline{\Delta l}^2 + mg \underline{h}_G$$

First step: transform the underlined to a function of θ

$$v_G^2 = \frac{L^2}{4}\dot{\theta}^2$$

$$\omega^2 = \dot{\theta}^2$$

For Δl and h we can use something like this to focus solely on the geometry



$$h_G = y_G = \frac{-L}{2}\cos\theta$$

$$\Delta l = x_P = L\sin\theta$$

$$E_c = \frac{1}{2}\left(\frac{L}{4} m + J\right)\dot{\theta}^2 \qquad \frac{\partial m}{\partial q}$$

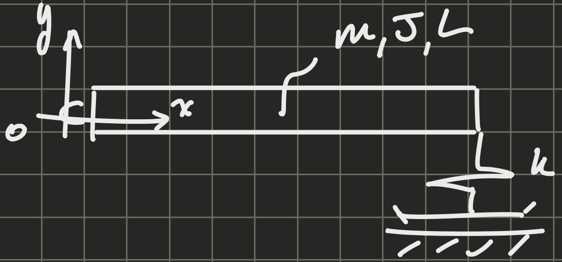
$$V_k = \frac{1}{2} k (L\sin\theta)^2 \qquad \frac{\partial \Delta l}{\partial q}$$

$$V_G = -mg\frac{L}{2}\cos\theta \qquad \frac{\partial h}{\partial q}$$

As we said we can use the derivates and plug them in as everything else will be the same

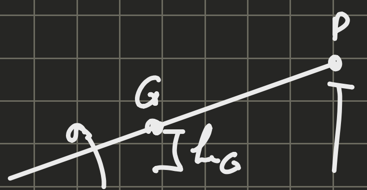
$$\Big(\underbrace{\frac{L}{4}m}_{\text{Inertial force}} + \underbrace{J}_{\text{Inertial term from moment of inertia}}\Big)\ddot{\theta} + \underbrace{kL^2\sin\theta\cos\theta}_{\frac{\partial V_k}{\partial q}\ \text{Spring Tension Force}} + \underbrace{mg\frac{L}{2}\sin\theta}_{\frac{\partial V_g}{\partial q}\ \text{Weight force}} = 0$$

Other example:



$E_c = \frac{1}{2}\left(m\frac{L^2}{4} + J\right)\dot{\theta}$ → The system is the same and kinetic energy is independent of position, for the same system.

$$V = V_k + V_g$$
$$= \frac{1}{2}k\left(\Delta l_0 + \Delta l_d(\theta)\right)^2 + mg\,h_G(\theta)$$



The same

$h_G = y_G = \frac{L}{2}\sin\theta$

$\Delta l_d(\theta) = x_p(\theta) = L\sin\theta$

↳ Not the same

$\frac{\partial h}{\partial q} = \frac{L}{2}\cos\theta$

$\frac{\partial \Delta l_d}{\partial q} = L\cos\theta$

Before $\Delta l_0 = 0$ now no

$\hat{J} = m\frac{L^2}{4} + J$

$$\hat{J}\ddot{\theta} + kL^2\sin\theta\cos\theta + \underbrace{k\Delta l_0 L\cos\theta}_{} + mg\frac{L}{2}\cos\theta = 0$$

New term which was not present before.

Lagrange does not give us a pure equation of motion, due to the presence of Δl_0, from the initial pre-load.

Since we don't know θ and Δl, we have find one of the two.

One approach to find Δl_0 is static equilibrium:

$$mg\frac{L}{2} - k\Delta l_0 L = 0 \rightarrow \frac{mg}{2k} = \Delta l_0$$

This approach doesn't allow us to find the sign without knowing the direction.

With this approach we find compression as positive since we are finding compression as positive when we said traction was positive.

This actually changes the result if we keep the positive sign. This means that we go from cancelling out to adding

The approach we can use is knowing that the derivative of the potential energy at the equilibrium is 0

Lagrange $\rightarrow E_c \rightarrow$ dynamic forces
$\hookrightarrow V \rightarrow$ static forces

$$\frac{\partial V}{\partial \theta} = 0$$

$$V(0) = kL^2 \cancel{\sin\theta}^{\,0} \cos\theta + k\Delta l_0 L\cos\theta + mg\frac{L}{2}\cos\theta$$

$$= k\Delta l_0 L + mg\frac{L}{2}$$

$$\Rightarrow \Delta l_0 = -\frac{mg}{2k}$$

We can evaluate V in the equilibrium position to find Δl_0

We find therefore that the general equation of motion is:

$$\boxed{\hat{J}\ddot{\theta} + kL^2\sin\theta\cos\theta = 0}$$

$\hookrightarrow$ This is non-linear, since we are studying vibrations, we can linearize the equation so we can study small oscillations.

It is possible to write the linear equation directly