Esercitazione 9 – Two population hypothesis testing

Exercise 1

Two analysis tests used to determine the level of impurities in a substance.

Specimen	1	2	3	4	5	6	7	8
Test X	1.2	1.3	1.5	1.4	1.7	1.8	1.4	1.3
Test Y	1.4	1.7	1.5	1.3	2.0	2.1	1.7	1.6

→ On the same unit so they are paired data

Assume N

a) Two-sided $CI_{0.99}$ for the mean difference

$$W_i = X_i - Y_i \qquad i = 1, \dots, 8$$

$$W_1, \dots, W_8 \sim N(\mu_X - \mu_Y, \sigma_W^2)$$

$$CI_{1-\alpha}(\mu_X - \mu_Y) = \left(\bar{w} - t_{1-\frac{\alpha}{2}}(n-1)\frac{S_w}{\sqrt{n}}, \bar{w} + t_{1-\frac{\alpha}{2}}(n-1)\frac{S_w}{\sqrt{n}} \right)$$

$$\bar{w} = \bar{x} - \bar{y} = -0.2125$$

$$S_w^2 = \frac{1}{n-1}\left(\sum_{i=1}^{8} w_i^2 - n\bar{w}\right) = 0.0298$$

$$\alpha = 0.01 \Rightarrow t_{0.995}(7) = 3.4995$$

$$= (-0.4261, 0.0011)$$

b) At 1% is there evidence to say that $\mu_X = \mu_Y$

$H_0: \mu_X = \mu_Y \qquad H_1: \mu_X \neq \mu_Y$

$H_0: \mu_\omega = 0 \qquad H_1: \mu_\omega \neq 0$

Since $0 \in CI_{0.99}(\mu_\omega) \Rightarrow$ We cannot reject H_0

c) $H_0: \mu_\omega \leq -0.16$ vs. $H_1: \mu_\omega > -0.16$

We reject H_0 if:

$$\underbrace{\frac{\bar{\omega} - (-0.16)}{S_\omega/\sqrt{n}}}_{-0.862} > \underbrace{t_{1-\alpha}(n-1)}_{2.9979}$$

We cannot reject H_0 at level 1%

Exercise 3

Output of 2 processes, sampled independent (not paired)

Process X: $n = 64$, $\bar{x} = 12.5$ $\quad \sigma_x = 2.1$

Process Y: $m = 100$, $\bar{y} = 11.9$ $\quad \sigma_Y = 2.2$

a) At 5%, are the means different

$H_0: \mu_X = \mu_Y$ vs. $H_1: \mu_X \neq \mu_Y$

Large sample case, two population, known (but different) variance

$$z_0 = \frac{\bar{x} - \bar{y} - 0}{\sqrt{\frac{\sigma_x^2}{n} + \frac{\sigma_y^2}{m}}} = 1.75$$

Reject H_0 if $|z_0| > z_{1-\frac{\alpha}{2}} \rightarrow z_{0.975} = 1.96$

$\hookrightarrow \alpha = 0.05$

$\Rightarrow$ we cannot reject H_0

b) $\mathbb{P}\{\underline{\underline{\text{not}}}$ committing type II error$\}$?

Type II error: error of wrongly accepting H_0

$\beta = \mathbb{P}\{$ reject H_0 when H_0 is false $\}$ = Power of Test

$$= \mathbb{P}_{H_1}(\text{reject } H_0) = \mathbb{P}_{H_1}\{\underline{X} \in CR\} =$$

$\hookrightarrow$ Critical Region.

$$= \mathbb{P}_{H_1}\left(\left|\frac{\bar{X}-\bar{Y}}{\sqrt{\frac{\sigma_X^2}{n}+\frac{\sigma_Y^2}{m}}}\right| > z_{1-\frac{\alpha}{2}}\right\}$$

$$= 1-\mathbb{P}\left\{-z_{1-\frac{\alpha}{2}}\sqrt{\dots} < \bar{X}-\bar{Y} < z_{1-\frac{\alpha}{2}}\sqrt{\dots}\right.$$

$$\frac{\bar{X}-\bar{Y}-(\mu_X-\mu_Y)}{\sqrt{\frac{\sigma_X^2}{n}+\frac{\sigma_Y^2}{m}}} \overset{\text{approx}}{\sim} N(0,1)$$

$(\mu_X = 12.1,\ \mu_Y = 11.5 \Rightarrow \mu_X - \mu_Y = 0.6)$

$$\frac{\bar{X}-\bar{Y}-0.6}{\sqrt{}} \sim N(0,1)$$

standardise

$$= 1-\left\{\phi\left(z_{1-\frac{\alpha}{2}} - \frac{0.6}{\sqrt{\dots}}\right) - \phi\left(-z_{1-\frac{\alpha}{2}} - \frac{0.6}{\sqrt{\dots}}\right)\right\} = 1-\phi(0.21)+\phi(-3.71)$$

$= 0.4169$

Exercise 4

X : content of lead in hair in the past $(10^{-6} g)$
Y : content of lead in hair now $(10^{-6} g)$

	X	Y
sample size	50	100
sample mean	48.5	26.6
sample stdev.	14.5	12.3

a) At 1% has the lead content decreased by at least 16.5?
(Assume N, same variance)

$H_0: \mu_X = \mu_Y + 16.5$ vs. $H_1: \mu_X > \mu_Y + 16.5$

Two independent populations.

We reject H_0 if $\dfrac{\bar{x}_n - \bar{y}_m - \delta_0}{s_p \sqrt{\frac{1}{n} + \frac{1}{m}}} > z_{1-\alpha}$

$\delta_0 = 16.5 \quad n = 50, m = 100$
$\bar{x} = 48.5 \quad \bar{y} = 26.6$

$$s_p^2 = \frac{(n-1)s_x^2 + (m-1)s_Y^2}{n+m-2} = 170.8105$$

?
$z_0 = 2.39 > z_{0.99} = 2.325 \Rightarrow$ we can reject H_0.

b) p-value? (N, same variance case)

α^* : $t_0 = z_{1-\frac{\alpha}{2}} \rightarrow \alpha^* = 1 - \phi(t_0) = 1 - \phi(2.39) = 0.00842$

$\Rightarrow$ p-value = 8.42%

c) Remove the same variance clause

$$\frac{\bar{x} - \bar{y} - 16.5}{\sqrt{\frac{S_x^2}{n} + \frac{S_y^2}{m}}} \longrightarrow \text{test statistic}$$

$= 2.26$

$z_{0.99} = 2.34$

$\Rightarrow 2.26 = z_{1-\alpha^*} \Rightarrow \alpha^* = 1 - \phi(2.26) = 1.191\%$

Exercise 5

Two equally sized groups of tested men ($\underbrace{\text{1 group is placebo}}_{\text{control}}$, $\underbrace{\text{1 drug group}}_{\text{main}}$)

22000 (total number)

104/11000 of main group had a heart attack

189/11000 of control group " " "

a) Does the drug decrease the probability of having a heart attack?

X: # patient main group
Y: " " control group

$H_0: p_X \geq p_Y$ vs. $H_1: p_X < p_Y$

large sample case $(n = m = 11000)$

$$z_0 = \frac{\bar{x}_n - \bar{y}_m}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n}+\frac{1}{m}\right)}} \qquad \hat{p} = \frac{n\bar{x}_n + m\bar{y}_m}{n+m}$$

Now:

$$\bar{x}_n = \frac{104}{11000} = 0.00945$$

$$\bar{y}_m = \frac{189}{11000} = 0.01718 \implies z_0 = -5.00058$$

$$\hat{p} = 0.01332$$

Reject H_0 if: $z_0 < z_{1-\alpha}$ → we don't have α, so we use p-value.

p-value approach:

$$z_0 = z_{1-\alpha^*} \implies \alpha^* = 1 - \phi(5) \cong 0$$

p-value $\cong 0$ we have evidence to reject H_0.

b) 119 in the main group and 98 in the control group had a stroke

Does the drug increase the occurrence of a stroke?

X: # patients in the main group with a stroke

Y: 〃 〃 control 〃 〃 〃

$H_0: p_X \leq p_Y$ vs. $H_1: p_X > p_Y$

$$\bar{x} = \frac{119}{11000} \quad \bar{y} = \frac{98}{11000} \Rightarrow \hat{p} = 0.009864$$

$$z_0 = \frac{\bar{x} - \bar{y}}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n} + \frac{1}{m}\right)}} = 1.45$$

$$z_0 = z_{1-\alpha^*}$$

$$\alpha^* = 1 - \phi(1.45) = 7.353\%$$

There is not enough evidence to reject H_0.