Lezione 3 -

Note on notation

$$I_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases}$$

The expectation is the weighted center

The variance is the dispersion of the random variable.

Some times we are interested in some function which manipulates the random variable

$$y = g(X) \Rightarrow y = X^2,\ y = 1/X,\ y = \log(X) \text{ if } X > 0$$

↳ In this course we will be looking a cases where y remains a random variable.

We can calculate the expectation of y, knowing X, as

$$E[y] = \int_{\mathbb{R}} g(X) \cdot f_x(X)\, dx$$

$$E[X^n] = \int_{\mathbb{R}} x^n \cdot f_x(X)\, dx \rightarrow n^{th} \text{ moment of } x$$

$$Var(X) = \underbrace{E(X^2)}_{\text{Second moment}} - \underbrace{E(X)^2}_{\text{Square of first moment}}$$

Exercise

Let X be a absolutely ~~contant~~ continuous random variable with Density

$$f_x = \begin{cases} cx(1-x) & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases} = cx(1-x)\, I_{(0,1)}(x)$$

1. Find c such that f_X is a density of a absolutely continuous random variable

Positivity $\hookrightarrow$ i) $cx(1-x) \geq 0 \quad \forall x \in (0,1) \Rightarrow c \geq 0$

Integral to 1 $\hookrightarrow$ ii) $1 = \int_{\mathbb{R}} f_X(x)\,dx = c\int_0^1 x(1-x)\,dx = c\int_0^1 x - x^2\,dx$

$= c\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{c}{6} \Rightarrow c = 6 \geq 0$ ✓

2. $EX = \int_{\mathbb{R}} x \cdot f_X(x)\,dx = \int_0^1 x \cdot 6 \cdot x(1-x)\,dx = 6\int_0^1 (x^2 - x^3)\,dx$

$= 6\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{1}{2}$

$$Var(X) = E(X^2) - E(X)^2 = \frac{3}{10} - \left(\frac{1}{2}\right)^2 = \frac{3}{10} - \frac{1}{4} = \frac{1}{20}$$

$$E(X^2) = \int_0^1 x^2 \cdot 6x(1-x)\,dx = 6\int_0^1 x^3 - x^4 = 6\left[\frac{x^4}{4} - \frac{x^5}{5}\right]_0^1$$

$= \frac{3}{10}$

$$sd(X) = \sqrt{Var(X)} = \frac{1}{\sqrt{20}}$$

Definition : Median

The median m of an absolutely continuous random variable with d.f. F_X is $\hookrightarrow$ the value such that:

$$P(X < m) = P(X \leq m) = \frac{1}{2}$$

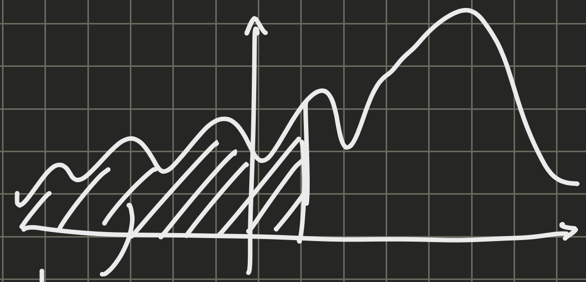
or equivalently

$$m = F_X^{-1}\left(\frac{1}{2}\right) \iff F(m) = \frac{1}{2}$$

We can also define an α-quantile, we need a real value x_α

$$P(X \le x_\alpha) = \alpha$$

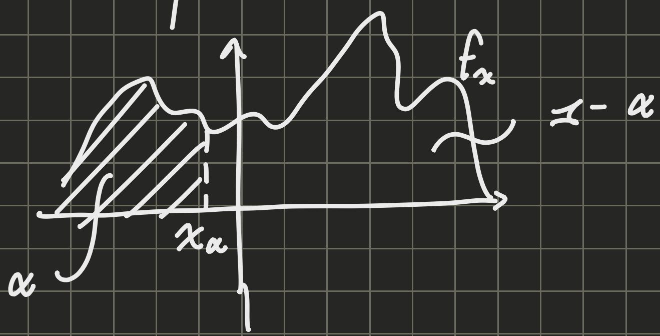
X, f_x



area is $\frac{1}{2}$

if $\alpha = \frac{1}{2} \Rightarrow \alpha$-quantile is the median

$$F_x(x_\alpha) = \alpha$$

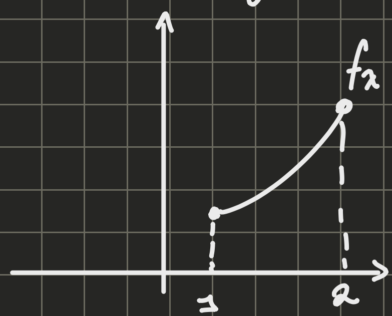


Exercise

Let X be abs. continuous r.v, with density $f_x(x) = \frac{3}{7} x^2 I_{[1,2]}(x)$

$$F_x(X) = \begin{cases} 0 & x < 1 \\ \frac{x^3 - 1}{7} & 1 \le x < 2 \\ 1 & x \ge 2 \end{cases}$$

(Cumulative) Distribution Function



$$\int_1^x \frac{3}{7} u^2 \, du = \frac{3}{7} \left[\frac{u^3}{3} \right]_1^x = \frac{1}{7}(x^3 - 1)$$

$$F_x(m) = \frac{1}{2} \Leftrightarrow \frac{m^3 - 1}{7} = \frac{1}{2} \Rightarrow m = \sqrt[3]{\frac{7}{2} + 1} = \sqrt[3]{\frac{9}{2}}$$

Homework: Compute the 25% quantile of X.

Density of $Y = g(X)$ where X is an abs. continuous, r.v and density f_X

→ Another abs. cont...

Y is absolutely continuous, when g and g^{-1} are differentiable

Step 1: we compute the distribution function of y:

$$F_Y(y) = P(Y \leq y) = P(\underbrace{g(X)}\leq y) = P(X \in g^{-1}((-\infty, y]))$$

$g(X) \in (-\infty, y]$

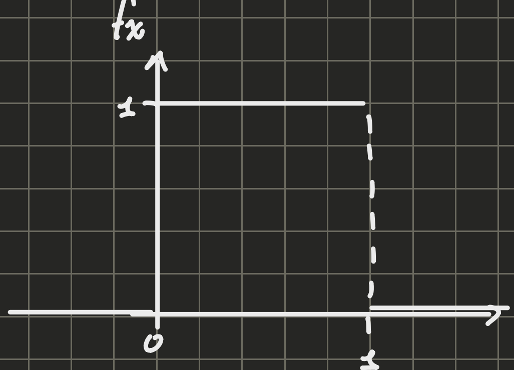
We need to find the intervals in which $g(X) \leq y$

→ Basically solving this inequality.

Step 2: $F_Y'(y)$ in y where the derivative of $F(y)$ exists

$$F_Y'(y) = f_Y(y)$$

Example 1



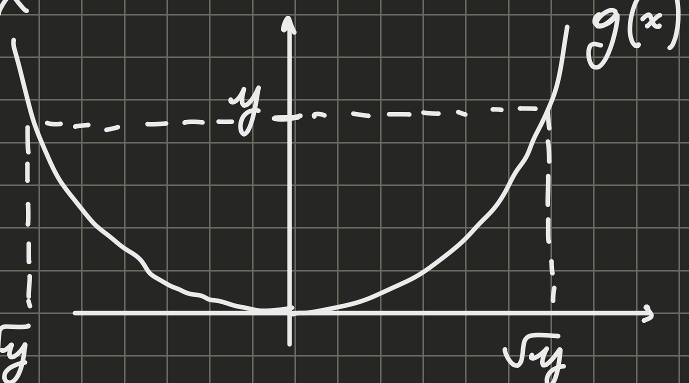
X, $f_X(x) = I_{(0,1)}(x)$

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$

It's a density because it's always positive and the area is 1.

Let us consider $Y = X^2$

$g(x) = x^2$



Step 1

for $y<0$ $F_Y(y)=0$

$$F_Y(y) = P(Y \leq y) = P(X^2 \leq y) =$$
$y>0$

$$X^2 \leq y \iff -\sqrt{y} \leq X \leq \sqrt{y}$$
when $y>0$

for how we defined f_X

$$= P(-\sqrt{y} \leq X \leq \sqrt{y}) = P(0 \leq X \leq \sqrt{y}) = F_X(\sqrt{y})$$

$$F_Y(y) = \begin{cases} 0 & \text{if } y=0 \\ \sqrt{y} & \text{if } 0 \leq \sqrt{y} \leq 1 \\ 1 & \text{if } \sqrt{y} \geq 1 \end{cases} = \begin{cases} 0 & \text{if } y<0 \\ \sqrt{y} & \text{if } 0 \leq y < 1 \\ 1 & \text{if } y \geq 1 \end{cases}$$

Step 2

$$F_Y'(y) = \begin{cases} 0 & \text{if } y<0, y \geq 1 \\ \frac{1}{2\sqrt{y}} & \text{if } 0 \leq y < 1 \end{cases} \Rightarrow f_Y(y) = \begin{cases} \frac{1}{2\sqrt{y}} & 0<y<1 \\ 0 & \text{if } y \leq 0, y \geq 1 \end{cases}$$

Example 2 (Linear Transformation)

Let X be an abs. conti. r.v. with f_X density, and $a, b \in \mathbb{R}$ with $a \neq 0$

Let us take $Y = aX + b$

e.g. $X =$ temp. $C^\circ \Rightarrow Y =$ temp F°

$$Y = \frac{9}{5}X + 32 \qquad a = \frac{9}{5} \quad b = 32$$

<u>Step 1</u>

$$F_Y(y) = P(X \le y) = P(aX+b \le y) = P(aX \le y-b)$$

$$P\left(X \le \frac{y-b}{a}\right) \quad \text{if } a>0$$

$$= F_X\left(\frac{y-b}{a}\right)$$

$$P\left(X \ge \frac{y-b}{a}\right) \quad \text{if } a<0$$

$$= P\left(X > \frac{y-b}{a}\right) = 1 - F_X\left(\frac{y-b}{a}\right)$$

<u>Step 2</u>

$$F_Y'(y) = \begin{cases} \text{if } a>0 & f_X\left(\frac{y-b}{a}\right)\cdot\frac{1}{a} \\ \text{if } a<0 & -f_X\left(\frac{y-b}{a}\right)\cdot\frac{1}{a} = f_X\left(\frac{y-b}{a}\right)\cdot\left(\frac{-1}{a}\right) \end{cases}$$

If we wanted to write one single expression we can write the following:

$$f_Y(y) = f_X\left(\frac{y-b}{a}\right)\cdot\frac{1}{|a|}$$

$Y = aX + b$

Notable Absolutely Continuous Distributions

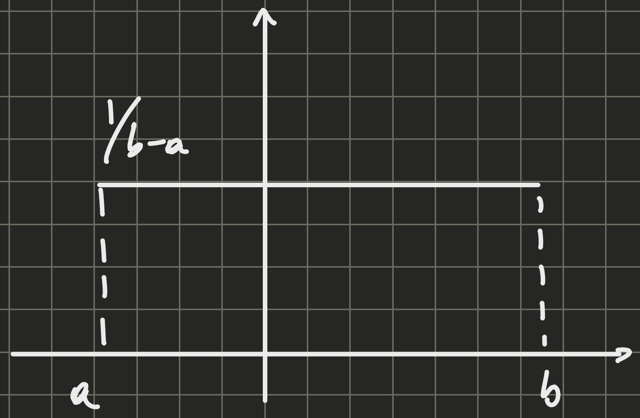
Uniform ~~on~~ Distribution on interval (a,b):

if $X \sim U((a,b))$ is r.v with density

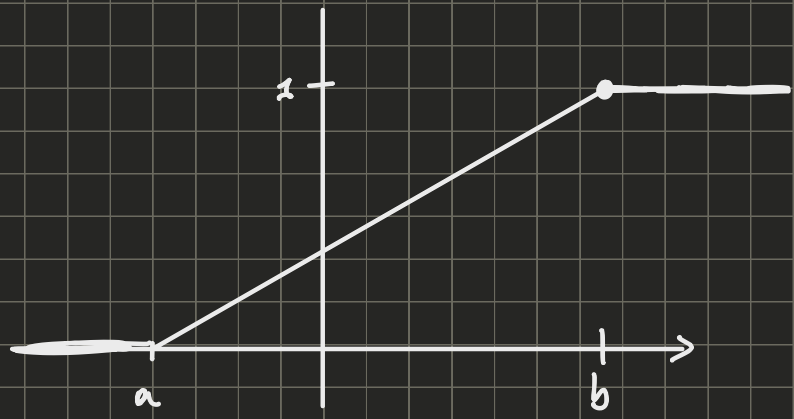
$$f_X(x) = \frac{1}{b-a}\, I_{(a,b)}(x)$$

Uniform on (a,b)

$\hookrightarrow f_X = \begin{cases} \frac{1}{b-a} & \text{if } a < x < b \\ 0 & \text{otherwise} \end{cases}$



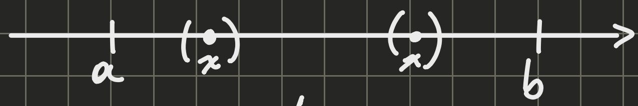
$$F_X(x) = \begin{cases} 0 & \text{if } x < a \\ \frac{x-a}{b-a} & \text{if } a \le x < b \\ 1 & \text{if } x \ge b \end{cases}$$



$$EX = \frac{a+b}{2}$$

$$\mathrm{Var}(X) = \frac{(b-a)^2}{12}$$

if $a=0$ & $b=1$,
the distribution is the same as before.



The probability of it being in an interval around an x is independent of x.