Esercitazione 1

1.1

D: probability of TV (years)

$P\{D \leq 1\} = 0{,}05$

$P\{D \leq 5\} = 0{,}45$

a) $P\{D > 5\} = 1 - P\{D \leq 5\} = 0{,}55$

b) $P\{1 < D < 5\} = P(D < 5) - P(D < 1) = 0{,}45 - 0{,}05 = 0{,}4$

c) $P(D > 1) = 1 - P(D < 1) = 1 - 0{,}05 = 0{,}95$

1.2

W: Weight of each a batch (kg)

$P(W < 30) = 0{,}13$
$P(W < 33{,}5) = .15$

a) $P(30 < W < 35) = P(W < 35) - P(W < 30) = .15 - .13 = 0{,}02$

b) $P(W > 35) = 1 - P(W < 35) = .85$

1.3

Recall: definition of a probability density

Let X be a random variable, with density f_X

Properties of f_X:

1) $f_X(x) \geq 0 \quad \forall x \in \mathbb{R}$

2) $\int_{\mathbb{R}} f_X(x)\, dx = 1$

a) R is random variable, radius

$$f_R(x) = \begin{cases} cxe^{-x^2} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases} \qquad c \in \mathbb{R}$$

Prop 1. $f_R(x) \geq 0 \quad \forall x \in \mathbb{R}$

if $x \leq 0 \quad f_R(x) = 0 \quad \forall c \in \mathbb{R}$

if $x > 0 \quad f_R(x) = cxe^{-x^2} \geq 0 \iff c \geq 0$

Prop 2. $\int_{\mathbb{R}} f_R(x)\, dx = 1 \qquad \int_0^\infty cxe^{-x^2} dx = 1 \Rightarrow \frac{c}{-2} \int_0^\infty -2xe^{-x^2} dx -$

$$= -\frac{c}{2}\left[e^{-x^2}\right]_0^\infty = -\frac{c}{2}(0-1) = \frac{c}{2} = 1 \Rightarrow c = 2$$

b) $F_R(x) = \int_{-\infty}^{x} f_R(t)\, dt$

$\to$ if $x \leq 0 \to F_R = 0$

$\to$ if $x > 0 \Rightarrow = \int_0^x 2te^{-t^2} dt = -\left[e^{-t^2}\right] = 1 - e^{-x^2}$

$$\Rightarrow F_R(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 - e^{-x^2}, & \text{if } x > 0 \end{cases}$$

Recall $\mathbb{I}_{(a,b)} = \begin{cases} 1, & x \in [a, b] \\ 0 & \text{otherwise} \end{cases} \Longrightarrow F_R(x) = (1 - e^{-x^2})\, \mathbb{I}_{(a,b)}(x)$

c) $P(R > 2)$

Knowing $F_X(x) = P(X \leq x)$

$$\Rightarrow P(R>2) = 1 - P(R \leq 2) = 1 - F_R(2) = e^{-2^2} = e^{-4} \simeq 0{,}018$$

1.4

$$f(x) = \begin{cases} \frac{1}{3}x + k & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{where } k \in \mathbb{R}$$

a)

Prop 1. $f(x) \geq 0 \quad \forall x \in \mathbb{R} \Rightarrow \forall x \in [0,1], \ \frac{1}{3}x + k \geq 0$

$\hookrightarrow k \geq -\frac{1}{3}x \quad \forall x \in [0,1]$

$k \geq -\frac{1}{3}$, $\boxed{k \geq 0}$

$\hookrightarrow$ More strict.

Prop 2. $\int_{\mathbb{R}} f(x)\,dx = 1 \Rightarrow \int_0^1 \left(\frac{1}{3}x + k\right) dx$

$$= \left[\frac{x^2}{6} + kx\right]_0^1 = \frac{1}{6} + k - 0 \Rightarrow k = \frac{5}{6}$$

b) $$\mathbb{E}[X] = \int_{\mathbb{R}} x f(x)\,dx = \int_0^1 x\left(\frac{1}{3}x + \frac{5}{6}\right) dx = \int_0^1 \frac{1}{3}x^2 + \frac{5}{6}x\,dx$$

$$= \left[\frac{x^3}{9} + \frac{5}{12}x^2\right]_0^1 = \frac{1}{9} + \frac{5}{12} = \frac{19}{36}$$

$$\mathrm{Var}(X) = \int_{\mathbb{R}} (x - \mathbb{E}X)^2 f(x)\,dx$$

$$= \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \frac{18}{36} - \left(\frac{19}{36}\right)^2 = \frac{107}{1296}$$

$$\mathbb{E}(g(X)) = \int g(x) f(x)\,dx \Rightarrow \mathbb{E}(X^2) = \int_{\mathbb{R}} x^2 f(x)\,dx = \int x^2\left(\frac{1}{3}x + \frac{5}{6}\right) dx$$

$$= \int_0^1 \left(\frac{1}{3}x^3 + \frac{5}{6}x^2\right) dx$$

$$= \left[\frac{x^4}{12} + \frac{5}{18}x^3\right]_0^1 = \frac{18}{36}$$

c) Find $q_{1/3}$ for which $P(X \leq q_{1/3}) = \frac{1}{3}$

$$q = q_{1/3}$$

if $q_{1/3} < 0 \Rightarrow P(x \leq q_{1/3}) = \int_{-\infty}^{q} f(x)\,dx = 0 \neq \frac{1}{3}$

if $q > 1 \Rightarrow P(x \leq q) = 1 \neq 1/3$

if $0 \leq q \leq 1 \Rightarrow P(x \leq q) = \int_{-\infty}^{q} \frac{1}{3}x + \frac{5}{6}\,dx = \frac{1}{3} = \int_{0}^{q} \frac{1}{3}x + \frac{5}{6}\,dx + \cancel{\int_{-\infty}^{0} \frac{1}{3}x + \frac{5}{6}\,dx}$ 0

$$\left[\frac{x^2}{6} + \frac{5}{6}x\right]_0^q \qquad \frac{q^2}{\cancel{6}} + \frac{5}{\cancel{6}}q = \cancel{\tfrac{1}{3}}\,2$$

$$q^2 + 5q = 2$$

since $q < 0$

$$q^2 + 5q - 2 = 0 \Rightarrow q \to \cancel{\frac{-5-\sqrt{33}}{2}}, \quad \to \frac{-5+\sqrt{33}}{2}$$

$\hookrightarrow > 0$ ✓
< 1

$\Rightarrow q \simeq 0{,}372$

1,5

$$F_x = \begin{cases} 0 & \text{if } x > 0 \\ \alpha x + \frac{x^2}{6} & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$

Proprieties of Fx

- F_x is non-decreasing
- $\lim_{x \to -\infty} F_x = 0$
- $\lim_{x \to \infty} F_x = 1$
- F_x continuous

a) 1) F_X is non decreasing $\forall x < 0$, $\forall x \geq 1$ F_X is continuous $\Rightarrow$ non decreasing $\forall \alpha \in \mathbb{R}$

$\forall\ 0 \leq x \leq 1$ $F_X(x) = \alpha x + \frac{x^2}{2}$

$$\Rightarrow \frac{d}{dx} F_X(x) = \alpha + \frac{x}{3} \geq 0 \Rightarrow \alpha \geq -\frac{x}{3} \quad \forall x \in [0,1]$$

$$\Rightarrow \boxed{\alpha \geq 0} \text{ or } \alpha \geq -\frac{1}{3}$$

2) $\lim_{x \to -\infty} F_X = 0 = 0$ ✓

$\lim_{x \to \infty} F_X = 1 = 1$ ✓

3) $\forall x < 0$ F_X const $\Rightarrow$ cont

$x \in (0,1)$, F_X const

$\forall x \geq 1$, F_X const

$$\lim_{x \to 0^+} F_X(x) = \lim_{x \to 0^-} F_X(0) \Rightarrow \lim_{x \to 0^+} \left(\alpha x + \frac{x^2}{6}\right) = 0 = 0 \Rightarrow \forall \alpha \in \mathbb{R}$$

$$\lim_{x \to 1^+} F_X(1) = \lim_{x \to 1^-} F_X(x) \Rightarrow 1 = \alpha + \frac{1}{6} \Rightarrow \alpha = \frac{5}{6}$$

b) $$F_X = \begin{cases} 0, & x < 0 \\ \frac{5}{6}x + \frac{x^2}{6}, & 0 \leq x \leq 1 \\ 1, & x \geq 1 \end{cases}$$

Compute f_X

We know $F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt \Rightarrow f_X = F_X' = \begin{cases} 0 & \text{if } x < 0 \\ \frac{5}{6} + \frac{x}{3} & 0 \leq x \leq 1 \\ 0 & \text{if } x \geq 1 \end{cases}$

$$f_X = \left(\frac{5}{6} + \frac{x}{3}\right) \mathbb{I}_{(0,1)}(x)$$

c) $$\mathbb{E}(X) = \int_{\mathbb{R}} x f_X(x)\,dx = \int_0^1 \left(\frac{5}{6}x + \frac{x^2}{3}\right) dx = \left[\frac{5}{12}x^2 + \frac{x^3}{9}\right]_0^1 = \frac{5}{12} + \frac{1}{9} = \frac{19}{36}$$

$$Var(X) = E(X^2) - E(X)^2 = \frac{107}{1296}$$

$$\hookrightarrow = \int_0^1 \frac{5}{6}x^2 + \frac{x^3}{3}\,dx = \frac{13}{36}$$

d) $q_{0,3} \to q = P(X < q) = 0,3$

$$q < 0 \Rightarrow \int^q f_X(x)\,dx = 0 \neq 0,3$$

$$q > 1 \Rightarrow \int_{-\infty}^q f_X(x)\,dx = 1 \neq 0,3$$

$$0 < q < 1 \Rightarrow \int_0^q f_X(x)\,dx = 0,3 = \int_0^q \left(\frac{5}{6} + \frac{x}{3}\right) = \left[\frac{5}{6}x + \frac{x^2}{6}\right]_0^q = \frac{5}{6}q + \frac{q^2}{6} = 0,3$$

$$\Rightarrow q \begin{cases} \nearrow \frac{-25+\sqrt{805}}{10} \in (0,1) \checkmark \\ \searrow \frac{-25-\sqrt{805}}{10} < 0 \times \text{ impossible.} \end{cases}$$

$$q = \frac{-25+\sqrt{805}}{10}$$

e) $Y = 2X$, $E[Y]$? $Var(Y)$? $\sigma(Y)$?

$$E(aX+b) = aE(X) + b$$

$$\Rightarrow E(Y) = 2 \cdot \frac{19}{36} + 0 = \frac{19}{18}$$

$$Var(aX+b) = a^2 Var(X) \Rightarrow Var(Y) = Var(2X) = 4Var(X) = \frac{107}{324}$$

$$\sigma(Y) = \sqrt{Var(Y)} = \frac{\sqrt{107}}{18} \simeq 0,575$$

Coefficient of Variation: $C(X) = \frac{\sigma(X)}{|E(X)|}$ $\to$ measure of relative distribution.

$$\Rightarrow C(Y) = \frac{\sigma(Y)}{|\mathbb{E}(Y)|} = \frac{\sqrt{107}/18}{19/18} = \frac{\sqrt{107}}{19}$$

1,6

$$f_X(x) = \begin{cases} \frac{e^{\frac{-x}{3000}}}{3000} & , x > 0 \\ 0 & , x \leq 0 \end{cases} \qquad X \sim f$$

$$P(X > 1000) = F(1000) = \int_{1000}^{\infty} f_X(x)\,dx = \int_{1000}^{\infty} \frac{e^{\frac{-x}{3000}}}{3000}\,dx = -\left[e^{\frac{-x}{3000}}\right]_{0}^{\infty}$$

$$\cong 0,71$$