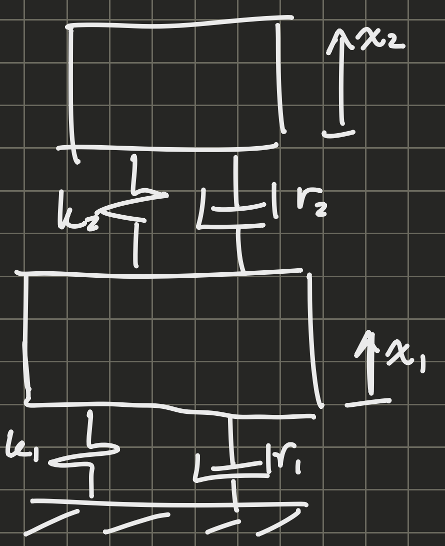
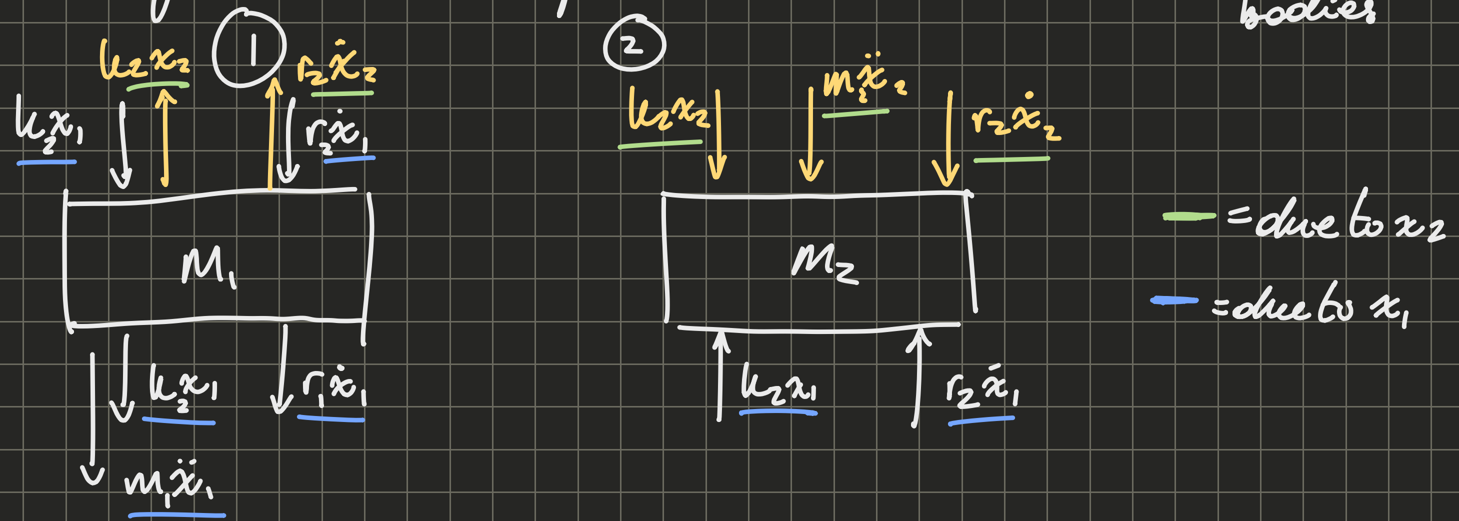
Lezione 8 - Systems with multiple degrees of freedom

Similar but more complex
We use a similar paths to find the equations of motion and then the added theory.

Simplest multiple degree of freedom system:



We can use linear superposition, freezing all but one degree, looking at that, and then one by one doing the same and adding together.
The system are not independent, some forces will act on both bodies



Eventhough we have frozen x_2, forces will still be generated on M_2.

The movement of the first mass generate forces which act on the second mass.

We freeze the motion (x_2), not k_2 and r_2.

The system is coupled through the k_2 and r_2, the x_1 and x_2 are connected through them.

Equilibrium has to be write for each mass:

$$\begin{cases} m_1\ddot{x}_1 + r_2\dot{x}_1 + r_2\dot{x}_1 - r_2\dot{x}_2 - k_2x_2 + k_1x_1 + k_2x_1 = 0 \\ \\ m_2\ddot{x}_2 + r_2\dot{x}_2 + k_2x_2 - k_2x_1 - r_2\dot{x}_2 = 0 \end{cases}$$

↳ we are going to write it in a matrix form:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \cdot \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{bmatrix} r_1+r_2 & -r_2 \\ -r_2 & r_2 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \underline{0}$$

It can be also written as:

$$[M]\underline{\ddot{x}} + [R]\underline{\dot{x}} + [k]\underline{x} = 0 \rightarrow \text{resembles our equilibrium}$$

→ Diagonal ⇒ Inertial forces are independent on other degrees of freedom, the are only affected by one.

generalized stiffness on dof 1 when we move dof 1.

$$\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$$

$-k_2$ → effect of the second dof on the first one

k_2 → only the second spring is activable how first degree acts on second.

when we move the second dof, so only that spring can generate forces on our second mass.

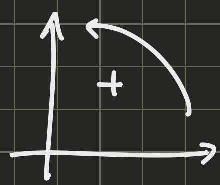
$$\begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} \rightarrow \text{same stories with each element}$$

rows = equations (equilibrium on each degree of freedom)

columns = degree of freedom which is generating the effect on the other dofs.

Let's write the equation of motion through Lagrange:

$$E_c = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$



$$V = V_k + V_g = \frac{1}{2} k_1 \Delta l_1^2 + \frac{1}{2} k_2 \Delta l_2^2$$

$$D = \frac{1}{2} r_1 \Delta \dot{l}_1^2 + \frac{1}{2} r_2 \Delta \dot{l}_2^2$$

Coefficients of kinematic relations between dofs and Δl

	x_1	x_2
Δl_1	1	0
Δl_2	-1	1

Since springs and dampers are in parallel, this table (matrix) will be the same, for $\Delta \dot{l}$

$$V = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 (x_2 - x_1)^2$$

we had written $\Delta l_2 = x_1 - x_2$ or $\Delta l_2 = x_2$ which was a mistake.

$$D = \frac{1}{2} r_1 \dot{x}_1^2 + \frac{1}{2} r_2 (\dot{x}_2 - \dot{x}_1)^2$$

To write the EOM we need to derive with respect to all the dot.

x_1

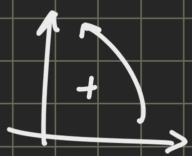
$$\underbrace{m_1 \ddot{x}_1}_{\frac{\partial}{\partial t}\frac{\partial E_c}{\partial x_1}} + \underbrace{k_2 (x_2 - x_1)(-1) + k_1 x_1}_{\frac{\partial V}{\partial x_1}} + \underbrace{r_1 x_1 + r_2 (\dot{x}_2 - \dot{x}_1)(-1)}_{\frac{\partial D}{\partial \dot{x}_1}} = 0$$

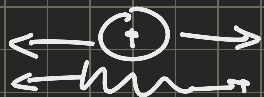
We made an error (now recalculated), which was $\Delta l = x_1 - x_2 \Rightarrow k_1(x_1 - x_2)$, this is a mistake since this implies that the coupling is due to k_1, rather than k_2.

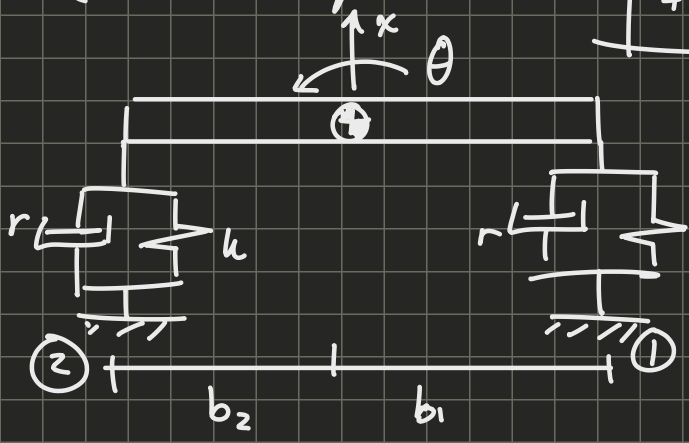
x_2

$$\underbrace{m_2 \ddot{x}_2}_{\frac{\partial}{\partial t}\frac{\partial E_c}{\partial \dot{x}_2}} + \underbrace{k_2 (x_2 - x_1)}_{\frac{\partial V}{\partial x_2}} + \underbrace{r_2 (\dot{x}_2 - \dot{x}_1)}_{\frac{\partial D}{\partial \dot{x}_2}} = 0$$

Other example







$$E_c = \frac{1}{2} m v_G^2 + \frac{1}{2} J \omega^2 = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} J \dot{\theta}^2$$

$$V = \frac{1}{2} k_1 \Delta l_1^2 + \frac{1}{2} k_2 \Delta l_2^2 = \frac{1}{2} k_1 (x + b_1 \theta)^2 + \frac{1}{2} k_2 (x - b_2 \theta)^2$$

$$D = \frac{1}{2} r_1 \dot{\Delta l_1}^2 + \frac{1}{2} r_2 \dot{\Delta l_2}^2 = \frac{1}{2} r_1 (\dot{x} + b_1\dot{\theta})^2 + \frac{1}{2} k_2 (\dot{x} - b_2\dot{\theta})^2$$

We expet [m] to be diagonal since the motion is different eventhough it's the same body.

[R] and [k] cannot be diagonal since the effect of one dof with cause forces for the other.

	x	θ
Δl_1	1	b_1
Δl_2	1	$-b_2$

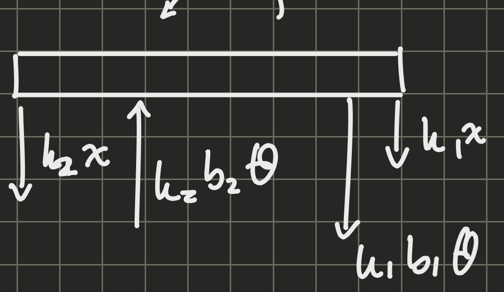
x: $m\ddot{x} + k_1(x + b_1\theta) + k_2(x - b_2\theta) + r_1(\dot{x} + b_1\dot{\theta}) + r_2(\dot{x} - b_2\dot{\theta}) = 0$

θ: $J\ddot{\theta} + k_1(x + b_1\theta)\, b_1 + k_2(x - b_2\theta)\cdot(-b_2) + r_1(\dot{x} + b_1\dot{\theta})\cdot b_1 + r_2(\dot{x} - b_2\dot{\theta})\cdot(-b_2) = 0$

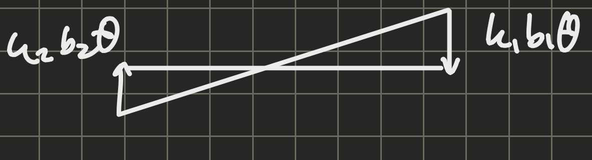
N·m   Nm

$$\begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} r_1 + r_2 & r_1 b_1 - r_2 b_2 \\ r_1 b_1 - r_2 b_2 & r_1 b_1^2 + r_2 b_2^2 \end{bmatrix} \begin{Bmatrix} \dot{x} \\ \dot{\theta} \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & k_1 b_1 - k_2 b_2 \\ k_1 b_1 - k_2 b_2 & k_1 b_1^2 + k_2 b_2^2 \end{bmatrix} \begin{Bmatrix} x \\ \theta \end{Bmatrix} = \underline{0}$$

Nm·m   Nm²



Units when multiplied by x and θ, one is a force balance, the other is a moment balance

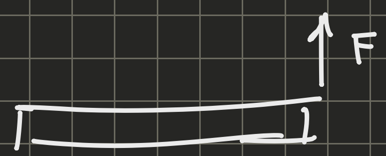


we need to be able to understand the units to be able to check if we have made errors.

rows: dofs that are feeling the action
columns: dofs that are generating the action.

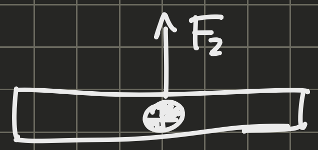
$$\delta^* \mathcal{L} = \vec{F}\,\delta\vec{y}_F^{\,*} = F\,\delta x^* + F b\,\delta\theta^*$$

	δx	$\delta\theta$
δy_F	1	b_1



$\Rightarrow$ if we add F

then our system = $\begin{Bmatrix} F \\ F\cdot b \end{Bmatrix}$

If instead F_2 where to act like 

$\Rightarrow \delta\mathcal{L} = \begin{Bmatrix} F_2 \\ 0 \end{Bmatrix}$, it does not act on θ.

The energy of F is going to the system through x.

Even though energy is going the system through x, since $[k]$ and $[R]$ are not diagonal the system is coupled, the motion of x influences θ, so even though we are not operating on θ, it will move.

F acts on one dof, but activates all of them through coupling.