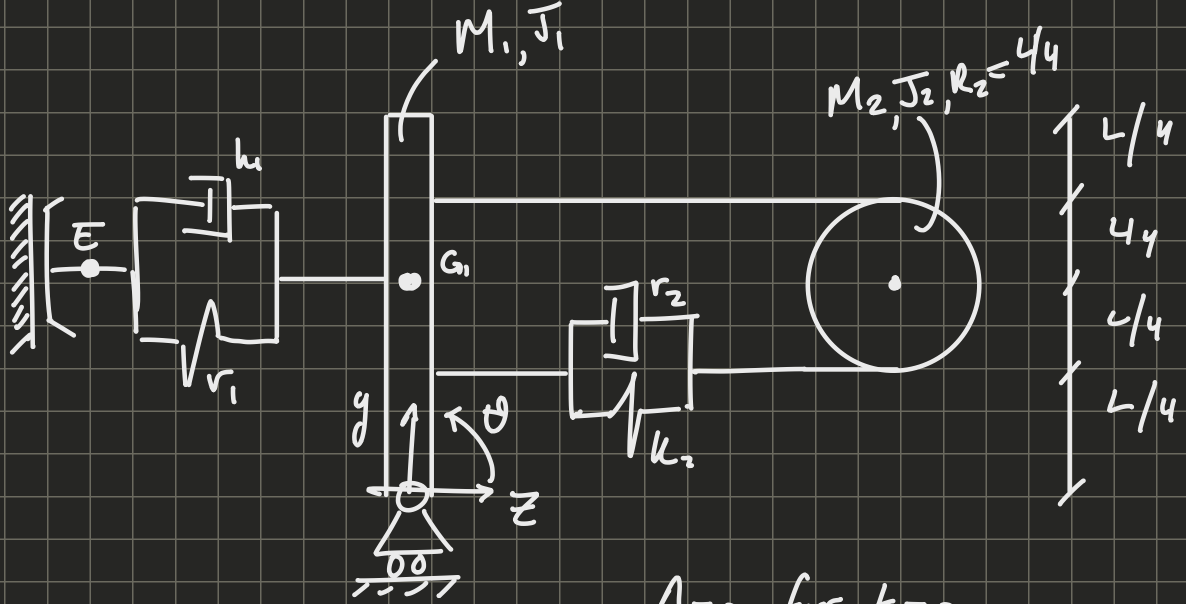
Esercitazione 6 - Matrici 30/4/25 → io ho mancato alcune di esercitazioni



$A \equiv 0$ for $t=0$

- EOM2 → We will go through the NLEOM this time but we will never really need to.

- $\Delta l_{01} = \Delta l_{02} = 0$

$$\bar{X}_n = \begin{Bmatrix} \theta \\ z \end{Bmatrix}$$

- $z = z_0 \cos(\Omega t)$

Step 1

2 bodies	$2 \times 3 = 6$ dof	→ 2 dof
1 cart	$-1 \times 1 = -1$	
1 rope	$-1 \times 1 = -1$	
1 hinge	$-2 \times 1 = -2$	

$$\bar{X}_n = \begin{Bmatrix} \theta \\ z \end{Bmatrix}$$

There is one since rope without a spring in the middle which can constrain our system.

Step 2

- $E_C = \frac{1}{2}\, \dot{\underline{y}}_M^T \begin{bmatrix} M_1 \end{bmatrix} \dot{\underline{y}}_M$ with $\dot{\underline{y}}_M = \begin{Bmatrix} v_1 \\ \omega_1 \\ \omega_2 \end{Bmatrix}$

$$[M_1] = diag(M_1, J_1, J_2)$$

- $D = \frac{1}{2}\, \dot{\underline{\Delta l}}^T \begin{bmatrix} r \end{bmatrix} \dot{\underline{\Delta l}}$ with $\dot{\underline{\Delta l}} = \begin{Bmatrix} \dot{\Delta l}_1 \\ \dot{\Delta l}_2 \end{Bmatrix}$

$$[r_1] = diag(r_1, r_2)$$

- $V_k = \frac{1}{2}\, \underline{\Delta l}^T \begin{bmatrix} k_1 \end{bmatrix} \underline{\Delta l}$ $\quad \underline{\Delta l} = \begin{Bmatrix} \Delta l_1 \\ \Delta l_2 \end{Bmatrix}$

- $V_g = \underline{p}^T \cdot \underline{h}$ with $\underline{p} = \begin{bmatrix} m_1 g \\ m_2 g \end{bmatrix}$ $\quad \underline{h} = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$

- $\delta^* \mathcal{L} = \underline{F}^T \cdot \delta \underline{y}_F$ with $\underline{F} = R_2$ $\quad \delta \underline{y}_F = [y_A]$

$\hookrightarrow$ reaction force to oscillation

Recap

$$\underline{y}_m = \begin{bmatrix} y_1(q) \\ y_2(q) \\ y_3(q) \end{bmatrix} \longrightarrow$$

$$\dot{y}_1(q) = \frac{d}{dt}\left(y_1(q)\right) = \boxed{\frac{\partial y_1}{\partial q_1}} \cdot \dot{q}_1 + \boxed{\frac{\partial y_1}{\partial q_2}} \dot{q}_2 + \boxed{\frac{\partial y_1}{\partial q_3}} \dot{q}_3 + \dots + \frac{\partial y_1}{\partial q_n} \dot{q}_n$$

coefficients

$$\dot{y}_2(q) = \dots$$

$\vdots$

q = independent variables

y = physical variables

$\dot{y}_m(q)$

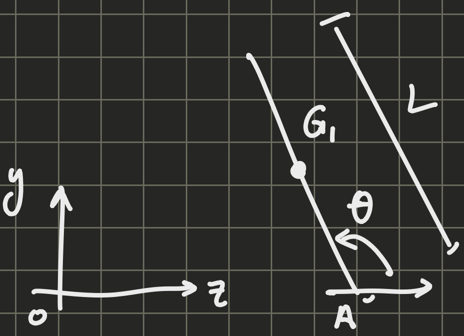
Thus is better represented as a matrix:

$$\dot{y}(q) = \underbrace{\begin{bmatrix} \frac{\partial y_1}{\partial q_1} & \frac{\partial y_1}{\partial q_2} & \cdots & \frac{\partial y_1}{\partial q_n} \\ \vdots & \ddots & & \\ \frac{\partial y_m}{\partial q_1} & \cdots & & \frac{\partial y_m}{\partial q_n} \end{bmatrix}}_{\left[\Lambda_m(q)\right]} \cdot \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

size (y) is not necessarily dim (q), it can be longer.

<u>Step 3</u>

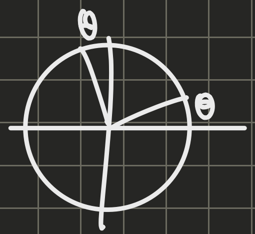
$V_{G_1} = ?$



$$(G_1 - O) = (G_1 - A) + (A - O) = \left[\frac{L}{2}\cos\left(\frac{\pi}{2}+\theta\right)\hat{\imath} + \frac{L}{2}\sin\left(\frac{\pi}{2}+\theta\right)\hat{\jmath}\right] + z\,\hat{\imath}$$

↳ This is non-linear, but it's just for recap, at the end we will write the linear relationship directly.

$$= \left(z - \frac{L}{2}\sin\theta\right)\hat{\imath} + \left(\frac{L}{2}\cos\theta\right)\hat{\jmath}$$

considering 

$$V_{G_1} = \frac{d}{dt}(G_1 - O) = \dot{z} - \frac{L}{2}\cos\theta \cdot \dot{\theta}\,\hat{i} - \frac{L}{2}\sin\theta \cdot \dot{\theta}\,\hat{j}$$

$$\frac{\partial y_1}{\partial \theta} = -\frac{L}{2}\cos\theta\,\hat{i} - \frac{L}{2}\sin\theta\,\hat{j} \longrightarrow = \frac{-L}{2}$$

$$\frac{\partial y_1}{\partial z} = +1\,\hat{i} \longrightarrow = 1$$

We linearize around the equilibrium position $(\theta_0 = 0,\ z_0 = 0)$

$$\Lambda(1,:) = \left[\frac{-L}{2},\ 1\right]$$

We can directly find the linearised coefficients by locking all the dof and unfreezing only one at a time

- $\boxed{z}$ $\quad \theta = 0 \longrightarrow V_{G_1} = \dot{z}\,\hat{i}$
- $\boxed{\theta}$ $\quad z = 0 \longrightarrow V_{G_1} = -\frac{L}{2}\dot{\theta}\,\hat{i}$

$$\vec{\omega}_1 = \dot{\theta}_1\hat{k} \longrightarrow \frac{\partial \theta_1}{\partial \theta} = 1 \quad ; \quad \frac{\partial \theta_1}{\partial z} = 0$$

$\vec{\omega}_2 = ?$

- p_1 $\boxed{\theta}$ $\quad z = 0$ $\quad \frac{3}{4}\dot{\theta}$  $\quad \frac{3}{4}L\dot{\theta} = \omega_2 R$

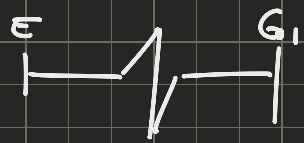
- p_2 $\boxed{z}$ $\quad \theta = 0$  $\quad -\dot{z} = \omega_2 R$

only the rope imposes a velocity

λ_m	$\dot\theta$	$\dot z$
V_{G_1}	$-L/2$	1
ω_1	1	0
ω_L	3	$-1/R$

$\Delta l_1 = ?$

$$l_m = (G_1 - E) = x_{G_1} - x_E$$



$$x_E = \text{const}$$

$$x_{G_1} = z - \frac{L}{2}\sin\theta$$

$$\Delta l = l_m(\theta) - l_m(\theta_0) \qquad \Delta\ l_m(\theta_0) = 0$$

$$= \left(z - \frac{L}{2}\sin\theta + \cancel{c}\right) - \left(0 + \cancel{c}\right)$$

$$\left.\frac{\partial l_1}{\partial\theta}\right|_{q_0} = -\frac{L}{2}$$

$$\left.\frac{\partial l_1}{\partial z}\right|_{q_0} = 1$$

$\Delta l_2 = ?$

$$l_m = x_p - x_q$$

$$x_p = \theta_2 R + C_1 = -z + \frac{3}{4}L\theta + c_2 + C_1$$

$$x_Q = z - \frac{L}{4}\theta$$

$$l_{m_2} = (x_p - x_q) = \frac{3}{4}L\theta - z + C - \left(z - \frac{L}{4}\theta\right) = L\theta - 2z + C$$

$$\Delta l_2 = \ln(q) - \ln(q_0) = L\theta - 2z$$

$\lambda_u(\theta_0)$	θ	z
Δl_1	$-L/2$	1
Δl_2	L	-2

- $h_2 = ?$ $= \text{const}$
- $h_1 = ?$ $h_1 = \frac{L}{2}\cos\theta$

$$\longrightarrow H_{h_1}(q) = \begin{bmatrix} \frac{\partial h_1^2}{\partial \theta^2} & \frac{\partial h_1^2}{\partial \theta \partial z} \\ \frac{\partial h_1^2}{\partial z \partial \theta} & \frac{\partial h_1^2}{\partial z^2} \end{bmatrix}$$

Hessian matrix → one for each body

$$H_{h_1} = \begin{bmatrix} -\frac{L}{2}\cos\theta & 0 \\ 0 & 0 \end{bmatrix} \xrightarrow{\text{linearize}} H_{h_1}(q_0) = \begin{bmatrix} -\frac{L}{2} & 0 \\ 0 & 0 \end{bmatrix}$$

$$H_{h_2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$ → since h_2 const

k_I, k_{II}, k_{III}

$$[k_{III}] = \sum_{i=1}^{n_g} M_{ig}\left[H_{h_i}(q_0)\right]$$

n_g → number of bodies

$$[k_{II}] = \sum_{i=1}^{n_k} k_i \, \Delta l_{0i} \left[H \, \Delta l_i(q_0)\right]$$

↳ Hessian of Δl_i

$$\left[k_I \right] = \left[\Lambda_k(\bar{q}_0) \right]^T \begin{bmatrix} \ddots & & \\ & k_1 & \\ & & \ddots \end{bmatrix} \left[\Lambda_k(\bar{q}_0) \right]$$

$$\delta^* \mathcal{L} = ?$$

$$\boxed{R_3} \quad y_F(q) = (A-O) = z\,\hat{i} \qquad \frac{\partial (A-O)}{\partial \theta} = 0$$

$$\frac{\partial (A-O)}{\partial z} = 1$$

Λ_{F_0}	θ	z
$\delta\theta_A$	0	0
δx_A	0	1

→ This is useless in this case, but it's used for us to understand rotational displacements.

$$E_c = \frac{1}{2} q^T \underbrace{\left[\Lambda_m(q_0) \right]^T \begin{bmatrix} \ddots & & \\ & m_1 & \\ & & \ddots \end{bmatrix} \left[\Lambda_m(q_0) \right]}_{[M]} q$$

↳ Inertial matrix for the independent variables.

Example

$$\left[M \right] = \begin{bmatrix} -\frac{1}{2} & 1 & 3 \\ 1 & 0 & -1/R \end{bmatrix} \begin{bmatrix} M & & \\ & J_1 & \\ & & J_2 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & 1 \\ 1 & 0 \\ 3 & -\frac{1}{R} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{L}{2} M_1 & J_1 & 3J_2 \\ M_1 & 0 & -J_2/R \end{bmatrix} \begin{bmatrix} -\frac{L}{2} & 1 \\ 1 & 0 \\ 3 & -1/R \end{bmatrix}$$

$$= \begin{bmatrix} \frac{L^2}{4} M_1 + J_1 + 9J_2 & -\frac{L}{2} M + 0 - \frac{3J_2}{R} \\ -\frac{L}{2} + 0 - \frac{3J_2}{R} & M_1 + 0 + \frac{J_2}{R^2} \end{bmatrix}_{2\times 2} = [M]$$

$$\bullet\ D = \frac{1}{2} q^T \underbrace{[\Lambda_r(q_0)] \begin{bmatrix} \ddots & & \\ & r_i & \\ & & \ddots \end{bmatrix} [\Lambda_r(q_0)]}_{[R]} q$$

similar shape to $[K_I]$ because springs and dampers are in parallel.

$$[k_I] = [\Lambda_k(q_0)] \begin{bmatrix} \ddots & & \\ & k_i & \\ & & \ddots \end{bmatrix} [\Lambda_k(q_0)]$$

$$= \begin{bmatrix} k_1 \frac{L^2}{4} + k_2 L^2 - \frac{L}{2} M_1 g & -k_1 \frac{R}{2} - k_2\, 2L \\ -k_1 \frac{L}{2} - k_2 \cdot 2L & k_1 + 4k_2 \end{bmatrix}$$

$$\delta^* \mathcal{L} = F^T \cdot \delta y_F = F \cdot [\Lambda_F(\vec{q}_0)] \cdot \delta q$$

$$Q^T = F^T \cdot [\Lambda_F(q_0)] \longrightarrow [Q(q_0)]^T \cdot \delta q$$

$$\begin{bmatrix} Q \end{bmatrix}^T = \begin{bmatrix} 0 & R_v \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & R_v \end{bmatrix}$$