Lezione 5 -

Recap

↳ We have introduced random variables, absolutely continuous and discrete random variables.

Definition of Discrete random variables

↳ A random variable which assumes values in a finite subset of $\mathbb{R}$ called S

$$P(X \in S) = 1$$

Discrete density → probability X is equal to any x

- $p(x) = 0$ if $x \in S$
- $p_X(x) > 0$ for all $x \in S$, and $\sum_{x_i} p(x_i) = 1$

Theorem → density → distribution function.

1. $F_X(x) = P(X \leq x) = \sum_{i: x_i \leq x} p_X(x_i)$, $x \in \mathbb{R}$

2. if $x_1 \leq x_2 \leq x_3 \leq \dots$ Distribution function → Probability Density.

$$p_X(x_k) = F_X(x_k) - F_X(x_{k-1}) \text{ if } k \geq 2 \quad p_X(x_1) = F_X(x_1)$$

Example : X = n° of bits received wrong [over 4 transmitted]

transmission $n = 4$ bits each ⟨ ok / wrong

$X \in S = \{0,1,2,3,4\}$

$p_X(0) = P(X=0) = 0{,}6561$

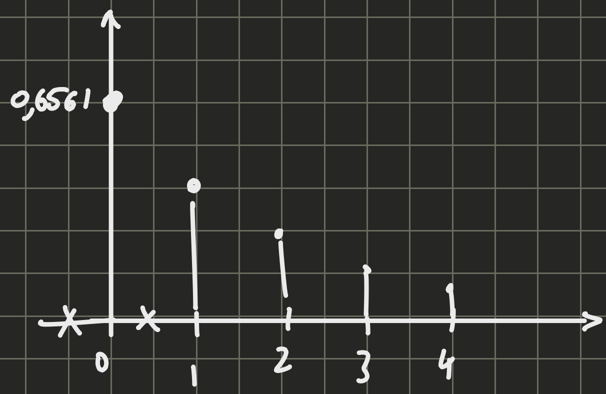
$p_X(1) = P(X=1) = .2916$

$p_X(2) = \quad = .0486$

$p_X(3) = \quad = 0{,}0036$

$p_X(4) = \quad = 0{,}0001$

Not to scale



$$F_X(x) = \sum_{i:x_i \leq x} p_X(x_i)$$

if $x < 0$ $\quad F_X(x) = 0$

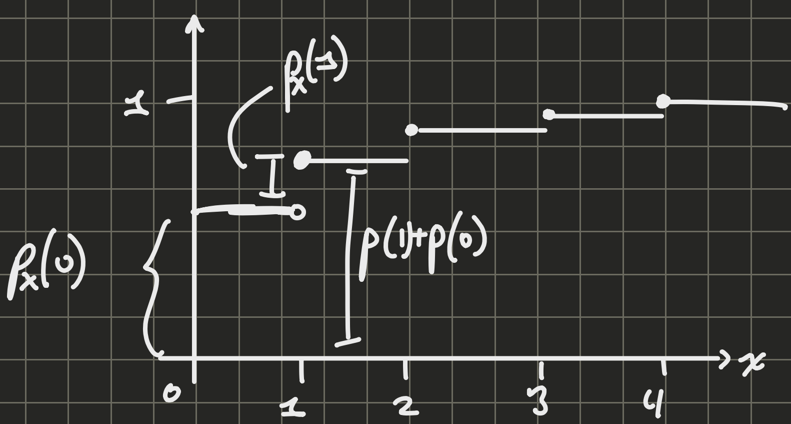
if $0 \leq x < 1$ $\quad F_X(x) = P(X<1) = p_X(0) = .6561$

$\hookrightarrow F_X(0)$

if $1 \leq x < 2$ $\quad F_X(x) = P(X<2) = p_X(0) + p_X(1) = .9477$

if $x \geq 2$ $\quad F_X(x) = P(X \leq 4) = p_X(0) + p_X(1) + p_X(2) + p_X(3) + p_X(4) = 1$

$\Rightarrow$ Plot of Distribution function



The step/jump is the density in the point.

The distribution of a discrete random variable is a step function. The jumps are at the support points. The jump in the step are the probabilities of the individual support points.

We can recover the density by looking at all the steps.

$$p_X(x_k) = F_X(x_k) - F_X(x_{k-1}$$

Theorem

To assign a discrete random variable is sufficient to assign its density.

Mean of a discrete random variable

Discrete random variable with density p_X and $S = \{x_1, x_2, \dots\}$

$$E(X) = \sum_{i=1}^{\infty} x_i \, p_X(x_i)$$

The variance is:

$$Var(X) = \sum_{i=1}^{\infty} (x_j - E(X)) \cdot p_X(x_i)$$

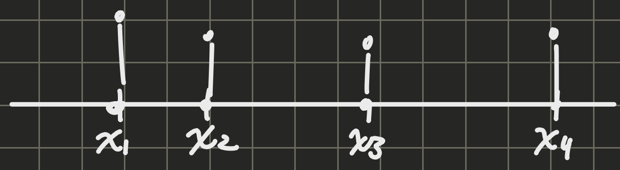
The standard deviation:

$$\sigma(X) = \sqrt{Var(X)}$$

$$P(X \in B) = \sum_{i,\, x_i \in B} p_X(x_i)$$

$B \subset \mathbb{R}$

$S \subset \mathbb{R}$



$EX =$ is the center of gravity of the random variable

$$Var(X) = \sum_{i=1}^{\infty} (x_i - EX)^2 p_X(x_i)$$ → indicates the variability around the mean.

Proprieties:

$Var(X) \geq 0$

If $Var(X) = 0 \Leftrightarrow x =$ constant almost surely.

$\Rightarrow P(x = c) = 1$ $\quad$ $c =$ some constant

Let X be a discrete random variable with $S_X = \{x_1, x_2, \dots\}$ and p_X

Let's consider $g : \mathbb{R} \to \mathbb{R}$ such that $Y = g(X)$

e.g. $Y = \log(X)$, $Y = X^3$, $Y = \sqrt{X}$

$$E[g(X)] = \sum_{i=1}^{\infty} g(x_i) \cdot p_X(x_i)$$

$$E[X^2] = \sum_{i=1}^{\infty} x_i^2 \, p_X(x_i)$$ $\qquad$ $E[X^n] = n^{th}$ moment of X

↳ still second moment

Proprietics of E and Var (Analogous to discrete)

↳ $E[aX+b] = aEX+b$

↳ $Var(aX+b) = a^2 Var(X)$

↳ $Var(X) = E(X^2) - E(X)^2$

↳ $Var(X) = 0$ only if $X = c$ almost surely.

some constant

Degenerate Random Variable

↳ Random variable $X = c$ almost surely, where c is a real-valued constant

Bernoulli Random Variable.

$X \sim Be(p)$ if:

$$X = \begin{cases} 1 \text{ with probability } p \\ 0 \text{ with probability } 1-p \end{cases}$$

$p \in (0,1)$

↳ $EX = 1 \cdot p_X(1) + 0 \cdot p_X(0) = p \cdot 1 + 0 \cdot (1-p) = p$

$Var(X) = E(X^2) - E(X)^2 = p - p^2 = p(1-p)$

$E(X^2) = p \quad E(X)^2 = p^2$

X is a uniform discrete random if $S = \{x_1, x_2, \dots, x_n\}$

and $p_X(x_i) = p = \frac{1}{n} \quad \forall i$

↳ # of elements in S

↳ $1^2 \cdot p_X(1) + 0^2 p_X(0) = 1 \cdot p + 0 \cdot (1-p) = p$

Sequence of n Bernoulli Trials

Let's introduce a sequence of Bernoulli trials.

Hyppo. ① The outcome of single trial can only be

- SUCCESS – some result
- FAILURE – it's complement

Hypothesis ② The probability of "success" is constant, i.e. does not change with the trial.

$$p \in (0,1)$$

Hypothesis ③ The outcomes are independent, the outcome of one trial does not affect the outcome of another.

Let us consider X = number of successful result in n bernoulli trials.

p = probability of success in a trial $p \in (0,1)$

Then

$X \sim Bin(n,p)$ Binomial with parameters n and p.

$$p_X(x) = P(X=x) = \underbrace{\binom{n}{p}}_{n \text{ choose } p} p^x (1-p)^{n-x} \quad , x = 0,1,\dots,n \qquad S_X = \{0,1,\dots,n\}$$

x number of success in n Bernoulli trials.

$$\binom{n}{x} = \frac{n!}{x!(n-x)!} \qquad \binom{n}{0} := 1$$

come la availability di un servizio è un necessary.

$EX = n \cdot p$

$Var(X) = np(1-p)$

$X \sim Be(p) = X \sim Bin(1, p)$

A sequence of n Bernoulli is a sequence of independent trials with only two possible results.

Exercise:

Several microstructure, each has probability 5% to develop a crack

1) what is the probability that at least one crack micro structure over $n = 20$ tested

X_{20} = n° of cracked microstructure over the 20 tested

$X \sim Bin(20, 0{,}05)$

"success" = crack in micro structure

$$P(X_{20} \geq 1) = 1 - P(X_{20} < 1) = 1 - P(X_{20} = 0) = 1 - \binom{20}{0} p^0 (1-p)^{20}$$

$$= 1 - (1-p)^{20}$$

$$= 0{,}6415$$

Exercise 2

↳ Probability that there will be at most 3 cracked microstructure over $n = 100$ examined

$n = 100$ $p = 0{,}05$

X_{100} = n° of cracked microstructures over 100 tested

$X_{100} \sim Bin(100, 0{,}05)$

$$P(X_{100} \leq 3) = P(X_{100} = 0) + P(X_{100} = 1) + P(X_{100} = 2) + P(X_{100} = 3)$$

$$= \binom{100}{0} p^0 (1-p)^{100} + \binom{100}{1} p (1-p)^{99} + \binom{100}{2} p^2 (1-p)^{98} + \binom{100}{3} p^3 (1-p)^{97}$$

$$= 0{,}2578$$

Example:

Shooting a target, whose probability of hitting it is $p = 0{,}2$

1) $n = 2$, what is the probability of hitting the target at least once?

x_2 = number of times target hit over 2 trials $\sim Bin(2, .2)$

$$P(X_2 \geq 1) = 1 - P(X_2 = 0) = 1 - \binom{2}{0} p^0 (1-p)^2 \simeq .36$$

2) n to have more than 50% probability of hitting the target at least once

$$X_n \sim Bin(n, p) \rightarrow \underbrace{P(X_n \geq 1)}_{\text{what we want} > .5}$$

$0,5 < P(X_n \geq 1) = 1 - P(X_n = 0)$

$\Rightarrow \quad P(X_n = 0) < 0,5$

$\cancel{\binom{n}{0}} \cancel{p^0} (1-p)^n \Rightarrow (1-p)^n < \frac{1}{2}$

$n \ln(.8) < -\ln(2) \Rightarrow n = \frac{-\log(2)}{\log(0,8)} = 3,3$

$\Rightarrow n \geq 4$ to have probability of hitting target at least once

Poisson Distribution

Taking $X \sim Bin(n, p)$ but with $n \to \infty$, and p decreases such tha $np = \lambda = cost.$

Then

if $k = 0, \dots, n$, $P(X = k) = \binom{n}{k} p^k (1-p)^{n-k} \to e^{-\lambda} \frac{\lambda^k}{k!}$ $k = 0, 1.$