Lesione 2 -

A random experiment is an experiment whose outcome is determinable with absolute certainty before the experiment.

The distribution associated between E and P function of X.

Assigning the probability function is difficult since we need to define them for all E.

Definition: Cumulative Distribution function

$X$ r.v , $\{ P(X \in E,\ E \in \mathcal{F})$

We call distribution of $X$ the following function:

$$F_X(x) := P(X \le x) \quad \forall x \in \mathbb{R}$$
$$= P(X \in (-\infty, x])$$

$0 \le F_X(x) \le 1$

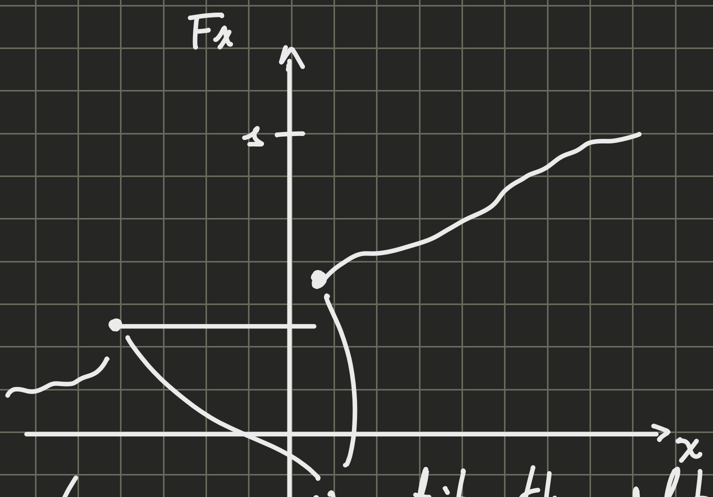
Properties of $F_X$ distribution function of $X$

1) $F_X$ is a non-decreasing function $\quad \forall x_1, x_2 \in \mathbb{R}$ with $x_1 \le x_2$
$\Rightarrow F_X(x_1) \le F_X(x_2)$
$\quad \| \qquad\qquad \|$
$P(X \in (-\infty, x_1]) \le P(X \in (-\infty, x_2])$

2) $F_X$ is a right-continuous function
↳ means that $\lim_{h \to 0} F_X(x+h) = F_X(x) \quad \forall x$
$h \to 0$

3) $\lim_{x \to -\infty} F_X(x) = 0 \rightarrow$ because for $x \to -\infty$ $E \to \emptyset$

$\lim_{x \to \infty} F_X(x) = 1 \rightarrow$ because for $x \to \infty \to E \to \mathbb{R}$




We put it on the right because of right continuity.

Theorem

Suppose you assign: $F: \mathbb{R} \to \mathbb{R}$ such that:

i. F is non-decreasing
ii. F is right-continuous
iii. $\lim_{x \to -\infty} F(x) = 0 \quad \lim_{x \to \infty} F(x) = 1$

then $\Rightarrow \exists$ some r.v. $X$ such that $F_X(x) = \underbrace{F(x)}_{\text{given by us.}} \ \forall x \in \mathbb{R}$

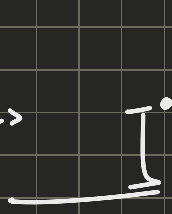
Given that we have a distribution function, there exists a random variable which has the same distribution function so we can treat our fenomenon as random.

REMARK $\quad a < b \qquad P(a < X \leq b) = P(X \leq b) - P(X \leq a)$

$\Longrightarrow \overset{!}{=} F_X(b) - F_X(a)$

because of how we defined $F_X$

$$P(X=x) = F_X(x^+) - F_X(x^-) = F_X(x) - F_X(x^-)$$



It's the jump.

If $F_X$ is continuous at $\Rightarrow P(X=x)=0$ !!

(We will work with discrete rather than continuous function, so we will look at intervals around $x$)

We will look at Gaussian distributions and discrete random variables.

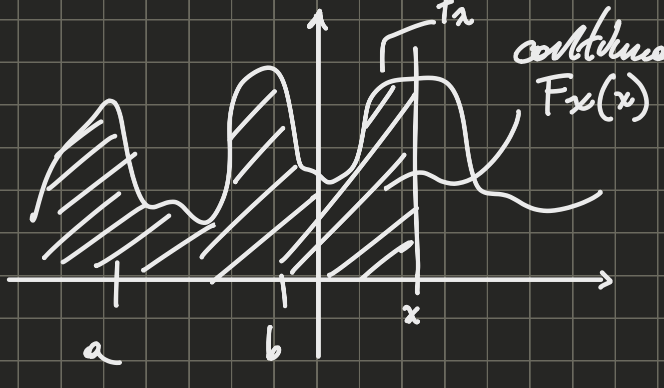
Definition: Absolutely Continuous Random Variable

$X$ is r.v. with distribution function (d.f) $F_X$.

$X$ is absolutely continuous if $\exists\ f_X : \mathbb{R} \to \mathbb{R}^+$ such that it's distribution function can be written as:

$$F_X(x) = \int_{-\infty}^{x} f_X(u)\, du \quad \Rightarrow F_X \text{ is a continuous function}$$

$f_X$ is the density of the absolutely continuous random variable $X$.



$$P(a < X \leq b) = F_X(b) - F_X(a) = \int_{-\infty}^{b} f_X(u)\,du - \int_{-\infty}^{a} f_X(u)\,du$$

$a \leq b$

$$= \int_{a}^{b} f_X(u)\,du$$

$$P(X = a) = 0 = \int_{a}^{a} f_X(u)\,du$$

$a \leftarrow b$

$\Rightarrow P(a \leq X) = P(a < X)$ or $P(X < b) = P(X \leq b)$

because we are only taking absolutely continuous functions.

<u>Theorem</u>

We introduce $f: \mathbb{R} \to \mathbb{R}$ with the properties:

i) $f(x) \geq 0 \quad \forall x \in \mathbb{R}$

ii) $\int_{\mathbb{R}} f(x)\,dx = 1$

$\Rightarrow \exists$ r.v $X$ which is absolutely continuous.

whose density $f_X(x) = f(x)$

$\underbrace{\forall x \in \mathbb{R}}_{\text{Not alway true}}$

<u>Example</u>

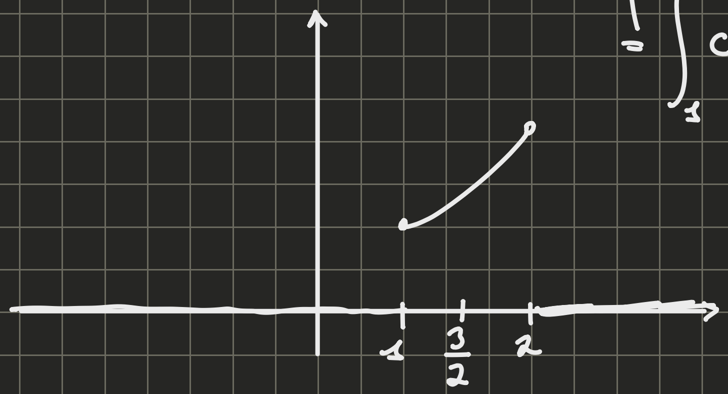
Let $f(x) = \begin{cases} cx^2 & \text{if } 1 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$

1) Find $c$ such that $f(x)$ is a density ($\rightarrow$ it follows the two properties)

i) $f(x) \geq 0 \ \forall x \Rightarrow cx^2 \geq 0 \quad \forall x \in [1,2] \Leftrightarrow c \geq 0$

ii) $1 = \int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{1} \cancelto{0}{f(x)}dx + \int_{1}^{2} f(x)dx + \int_{2}^{\infty} \cancelto{0}{f(x)}dx$

$= \int_{1}^{2} cx^2\, dx = \left[ c\frac{x^3}{3} \right]_1^2 = \frac{c}{3}(8-1) = \frac{7}{3}c$



$\Rightarrow \frac{7}{3}c = 1$

$\Rightarrow c = \frac{3}{7} \geq 0$

satisfied first condition.

2) Computer $P\left(x > \frac{3}{2}\right) = 1 - \overbrace{P\left(X \leq \frac{3}{2}\right)}^{=F_X\left(\frac{3}{2}\right)}$ → Not the only way we could have computed it, in general

$= 1 - \int_{-\infty}^{3/2} f(x)\, dx$

$= 1 - \int_{1}^{3/2} \frac{3}{7}x^2\, dx$

$$P\left(x > \frac{3}{2}\right) = 1 - \cancel{\frac{3}{7}}\left[\frac{x^3}{\cancel{3}}\right]_1^{3/2} = 1 - \frac{1}{7}\left(\frac{28}{8} - 1\right)$$

If X is absolutely continuous with density f

$$P(X \in E) \ \forall E \in \mathcal{F} \Rightarrow = \int_E f(x)dx$$

So we could have written it as

$$P\left(X > \frac{3}{2}\right) \Rightarrow E = \left(\frac{3}{2}, \infty\right) = \int_{3/2}^{\infty} f(x)dx = \int_{3/2}^{2} \frac{3}{7}x^2 dx$$

REMARK

Having fixed $f_X$ absolutely continuous $\to F_X(x) = \int_{-\infty}^{x} f_X(u)\,du$

X absolutely continuous can we $F_X \to f_X$ density

Yes, $\Rightarrow$ it's density $f_X = F_X'(x)$ in all X where $F_X$ is differentiable

There will be at most a finite number of points in which it's not differentiable.

That means we can choose the density at the point since it has no affect on the probability.

Exeercise:

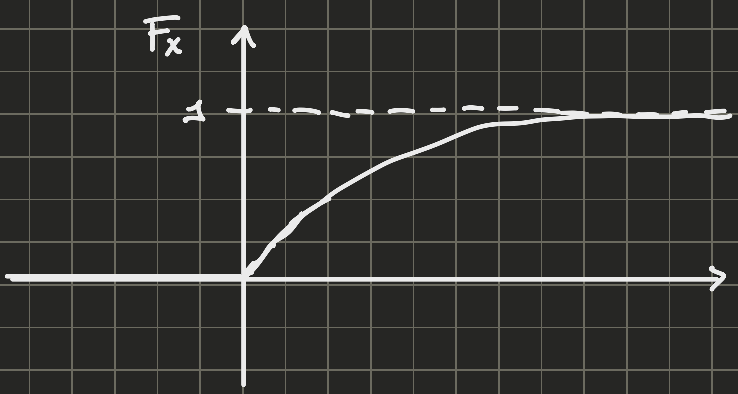
X is absolutely continuous with distribution function

$$F_X(x) = \begin{cases} \alpha & \text{if } x \le 0 \\ \dfrac{\beta x}{2x+120} & \text{if } x > 0 \end{cases} \qquad \alpha, \beta \in \mathbb{R} \text{ are unknown}$$

1) $\alpha, \beta$ such that $F_X$ is a distribution function

i) $\lim_{x \to -\infty} F_X(x) = \alpha = 0$

$\lim_{x \to \infty} F_X(x) = \frac{\beta}{2} = 1 \Rightarrow \beta = 2 \to \frac{x}{x+60}$



2) Compute the density of X

$$F_X'(x) = \begin{cases} 0 & \text{if } X<0 \\ \frac{\cancel{x}+60-\cancel{x}}{(x+60)^2} & \text{if } x>0 \end{cases}$$

This function is not differentiable in 0

$$f_X(x) = \begin{cases} 0 & \text{if } x<0 \\ \frac{60}{(x+60)^2} & \text{if } x>0 \end{cases}$$ → changing this to $x \leq 0$ does not change $F_X$

$f_X(0) = d \geq 0$, it doesn't matter because the distribution does not change due to a single point, whatever it's density.

3) $P(X \leq 20) = F_X(20) = \frac{20}{20+60} = \frac{1}{4}$

$P(X > 60) = 1 - F_X(60) = 1 - \frac{1}{2} = \frac{1}{2}$

Definition (Mean): X r.v. absolutely continuous with $f_X$ being its density

The <u>mean</u> (or expectation) of the random variable X, is:
or expected value

$$EX = \int_{\mathbb{R}} x \cdot f_X(x)\,dx$$

Similar to the barycenter of a stick with mass density of $f_X$

It's a location of the random variable

Definition : Variance

$X$ absolutely continuous, $f_X$ density, $EX$ expectation

$$Var(X) = \int_{\mathbb{R}} (x - EX)^2 \cdot f_X(x)\, dx$$

↳ Variance

Definition: Standard Deviation (s.d) of $X$

is $= \sqrt{Var(x)}$

Properties of expectation and variance → for absolutely continuous functions but also true for finite random variables

1) $a, b \in \mathbb{R}$ $\quad \underbrace{E}_{\text{expectation}}(aX+b) = a \cdot \underbrace{EX}_{\text{expectation of } X} + b$

↳ linearity

$$\int_{\mathbb{R}} (ax+b) \cdot f(x)\, dx = a \underbrace{\int_{\mathbb{R}} x \cdot f_X(x)\, dx}_{EX} + b \underbrace{\int f(x)\, dx}_{=1}$$

$= a(EX) + b$

2) $a, b \in \mathbb{R}$

$$\underbrace{Var(aX+b)} = a^2\, Var(X)$$

$$\int_{\mathbb{R}} \underbrace{((ax+b) - E(aX+b))}^2 \cdot f_X(x)\, dx = \int [a(x - EX)]^2 f_X(x)\, dx = a^2\, Var(X)$$

$$ax + b - \underbrace{E(aX+b)}_{aEX + b} = a(x - EX)$$

3) $Var(X) = \int_{\mathbb{R}} (x - EX)^2 \cdot f(x)\, dx = E(X^2) - (EX)^2$

$= \int_{\mathbb{R}} (x^2 + (EX)^2 - 2(EX) \cdot x) \cdot f_X(x)\, dx =$

$$= \int_{\mathbb{R}} x^2 f(x)\,dx + \int_{\mathbb{R}} EX^2 \cdot f(x)\,dx + \int_{\mathbb{R}} -2EX \cdot x\, f(x)\,dx$$

$$= \int_{\mathbb{R}} x^2 f(x)\,dx + (EX)^2 - 2EX \cdot EX = \underbrace{\int_{\mathbb{R}} x^2 f_x(x)}_{E(X^2)} - (EX)^2$$