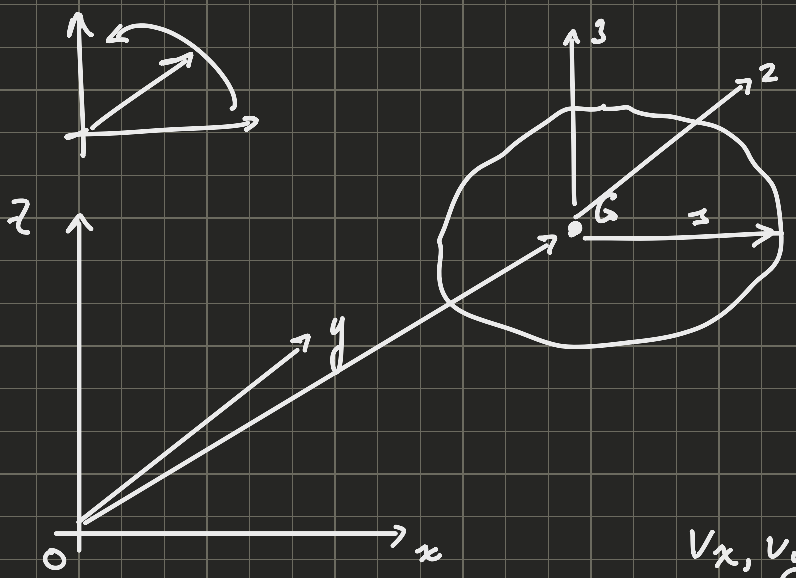
Lezione 10 -

Explaination of why we constructed that table.



There is an order in which we give the rotation, since changing the order will lead to a different position.

v_x, v_y, v_z

$\omega_1, \omega_2, \omega_3$

$$\begin{bmatrix} m & & & & & \\ & m & & & & \\ & & m & & & \\ & & & J_1 & & \\ & & & & J_2 & \\ & & & & & J_3 \end{bmatrix}$$

↳ Tensor of inertia considering the 1,2,3 reference system

If we write:

$$E_c = \frac{1}{2} m v_x^2 + \frac{1}{2} m v_y^2 + \frac{1}{2} m v_z^2 + \frac{1}{2} J_1 \omega_1^2 + \frac{1}{2} J_2 \omega_2^2 + \frac{1}{2} J_3 \omega_3^2$$

$n_c \rightarrow$ number of bodies in system

↳ We have to do this for every body in the system:

$$E_c = \sum_{j=1}^{n_c} \sum_{i=1}^{6} \frac{1}{2} M_{ij} \dot{y}_{M ij}^2$$

Appropriate ↳ Velocity component

We are going to be building a vector:

$$\underline{\dot{y}_M} = \{\dot{y}_{M1}, \dot{y}_{M2}, \dots, \dot{y}_{M n_c}\}^T \quad (6 \cdot n_c \times 1)$$

↳ Super long vector.

where each small vector is:

$$\dot{y}_{Mj} = \{v_{xj}, v_{yj}, v_{zj}, \omega_{1j}, \omega_{2j}, \omega_{3j}\}^T \quad 6 \times 1$$

The idea is to write E_c in matrix form rather than the "$\sum$" form:

$$E_c = \frac{1}{2} \underbrace{\dot{y}_M^T}_{1 \times 6n_c} \underbrace{\begin{bmatrix} \diagdown & & \\ & m & \\ & & \diagdown \end{bmatrix}}_{6n_c \times 6n_c} \underbrace{\dot{y}_M}_{6n_c \times 1}$$

$6n_c \times 6n_c$ → diagonal is all inertial terms of every body.

$$\dot{y} = \begin{Bmatrix} v_{1,1} \\ \vdots \\ v_{6,n_c} \end{Bmatrix}$$

$$\begin{bmatrix} \diagdown & & \\ & m & \\ & & \diagdown \end{bmatrix} = \begin{bmatrix} m_{1,1} & & & & \\ & m_{1,2} & & & \\ & & m_{1,3} & & \\ & & & \ddots & \\ & & & & m_{6,n_c} \end{bmatrix}$$

We are trying to find the matrix form of what we have already done in the scalar form.

$y_{mij} = y_{mj}\,(x_1, \dots, x_n) \rightarrow$ every position is a function of the degrees of freedom

$\hookrightarrow$ no dot, since position

$\underline{x} = \{x_1, \dots, x_n\} \rightarrow$ Degrees of Freedom

$\Rightarrow y_m = y_m(\underline{x}) = y_m(\underline{x}(t))$

To have $\dot{y}$ we need to derive y_m with respect to time

$$\dot{y}_{mij} = \frac{\partial y_{mij}}{\partial x_1} \cdot \dot{x}_1 + \dots + \frac{\partial y_{mij}}{\partial x_n} \cdot \dot{x}_n$$

$$\underline{\dot{y}_m} = \underbrace{\left[\frac{\partial y_m}{\partial \underline{x}}\right]}_{\text{This is the Jacobian matrix}} \underline{\dot{x}} \qquad \left[\frac{\partial y_m}{\partial \underline{x}}\right] = \begin{bmatrix} \frac{\partial y_{11}}{\partial x_1} & \dots & \frac{\partial y_{11}}{\partial x_n} \\ & & \vdots \\ & & \frac{\partial y_{m\,6nc \times 6nc}}{\partial x_n} \end{bmatrix}$$

This is the Jacobian matrix, it defines the relationship between a position of our system and our degrees of freedom.

$\Rightarrow$ we can write E_c in the form:

$$E_c = \frac{1}{2}\,\underline{\dot{x}}^T \underbrace{\left[\frac{\partial y_m}{\partial \underline{x}}\right]^T \begin{bmatrix} \ddots & & \\ & m & \\ & & \ddots \end{bmatrix} \left[\frac{\partial y_m}{\partial \underline{x}}\right]} \underline{\dot{x}}$$

Mass matrix of the system, which is a function of $\underline{x}$.

Like : $\underbrace{\frac{1}{2} m(q)}\dot{q}^2$

↳ It is this, but in matrix form.

Long story short:

$$E_c = \frac{1}{2}\underline{\dot{x}}^T\left[M(\underline{x})\right]\underline{\dot{x}}$$

Where $\left[M(\underline{x})\right] = \left[\Lambda_m(\underline{x})\right]^T\left[{}^{\diagdown}{}_{m}{}_{\diagdown}\right]\left[\Lambda_m(\underline{x})\right]$

We can do the Taylor expansion to the second order like in scalar form, at the equilibrium position to linearise:

$$\begin{cases} \underline{x} = \underline{x}_0 \\ \underline{\dot{x}} = \underline{0} \end{cases}$$

$$\left[\Lambda_m(x_0)\right]\left[{}^{\diagdown}{}_{m}{}_{\diagdown}\right]\left[\Lambda_m(x_0)\right] = [M]$$

Potential Energy:

n_k = number of springs

n_c = number of rigid bodies $\qquad p_j = m_j g$

$$V = \sum_{i=1}^{n_k}\frac{1}{2}k_i\Delta l_i^2 + \sum_{j=1}^{n_c} p_j h_{Gj} = \frac{1}{2}\underline{\Delta l}^T\left[{}^{\diagdown}{}_{k}{}_{\diagdown}\right]\underline{\Delta l} + \underline{p}^T\underline{h}_G$$

$$\underline{\Delta l} = \begin{bmatrix} \Delta l_1 \\ \Delta l_2 \\ \vdots \\ \Delta l_{n_u} \end{bmatrix} \qquad \underline{P} = \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_{n_c} \end{bmatrix} \qquad \underline{h}_G = \begin{bmatrix} h_{G1} \\ h_{G2} \\ \vdots \\ h_{G n_c} \end{bmatrix}$$

$$\begin{bmatrix} \diagdown_{k} \end{bmatrix} = \begin{bmatrix} k_1 & & & \\ & k_2 & & \\ & & \ddots & \\ & & & k_{n_u} \end{bmatrix}$$

constant

$\Rightarrow$ Only $\underline{\Delta l} = \underline{\Delta l}(\underline{x})$ and $\underline{h}_G = \underline{h}_G(\underline{x})$

In general these are non-linear, so we go directly to the linear, by doing the Taylor expansion:

$$V = V(\underline{x}_0) + \left\{ \frac{\partial V}{\partial \underline{x}} \right\}_{\underline{x}=\underline{x}_0} \bar{\underline{x}} + \frac{1}{2} \bar{\underline{x}}^T \underbrace{\left[\frac{\partial}{\partial x} \left[\frac{\partial V}{\partial x} \right] \right]_{\underline{x}=\underline{x}_0} \bar{\underline{x}}}_{\text{what we are interested in}}$$

$$\underline{x} - \underline{x}_0 = \bar{\underline{x}}$$

$$\rightarrow \underline{V} \simeq \underline{V}_0 + \frac{1}{2} \bar{\underline{x}}^T \left[\frac{\partial}{\partial \underline{x}} \left[\frac{\partial V}{\partial \underline{x}} \right] \right] \bar{\underline{x}}$$

$$\left\{ \frac{\partial V}{\partial \underline{x}} \right\} = \underline{\Delta l}^T \begin{bmatrix} \diagdown_{k} \end{bmatrix} \left[\frac{\partial \underline{\Delta l}}{\partial \underline{x}} \right]_{\underline{x}=\underline{x}_0} + \underline{P}^T \left[\frac{\partial \underline{h}_G}{\partial \underline{x}} \right]_{\underline{x}=\underline{x}_0} = (*)$$

$\hookrightarrow$ we are not interested, we want the second derivative to put

Since deriving based on $\underline{x}$ is impractical since it adds another dimension, we re-write and the derive in $\Sigma\Sigma$ form

$$\textcircled{*} = \sum_{j=1}^{n_u} \Delta l_j \, k_j \, \frac{\partial \Delta l_j}{\partial \underline{x}} + \sum_{j=1}^{n_c} \rho_j \, \frac{\partial h_{g j}}{\partial \underline{x}} \Bigg|_{\underline{x} = \underline{x}_0}$$

Taking the second derivative now:

General Stiffness in more dimensions

$$\left[\frac{\partial}{\partial \underline{x}} \left[\frac{\partial V}{\partial \underline{x}} \right] \right] = \left[\sum_{j=1}^{n_u} \left\{ \frac{\partial \Delta l_j}{\partial \underline{x}} \right\} k_j \left\{ \frac{\partial \Delta l_j}{\partial \underline{x}} \right\}_{\underline{x} = \underline{x}_0} + \right.$$

$$+ \sum_{j=1}^{n_u} k_j \, \Delta l_j \Big|_{\underline{x} = \underline{x}_0} \frac{\partial}{\partial \underline{x}} \left\{ \frac{\partial \Delta l_j}{\partial \underline{x}} \right\} \Bigg|_{\underline{x} = \underline{x}_0}$$

Souplesse in more + degrees of freedom

static preload of system

$$+ \sum_{j=1}^{n_c} \rho_j \, \frac{\partial}{\partial \underline{x}} \left\{ \frac{\partial h_{g j}}{\partial \underline{x}} \right\}_{\underline{x} = \underline{x}_0}$$

Again, the same as gravitational stiffness but with more bodies.

We can re-write it as:

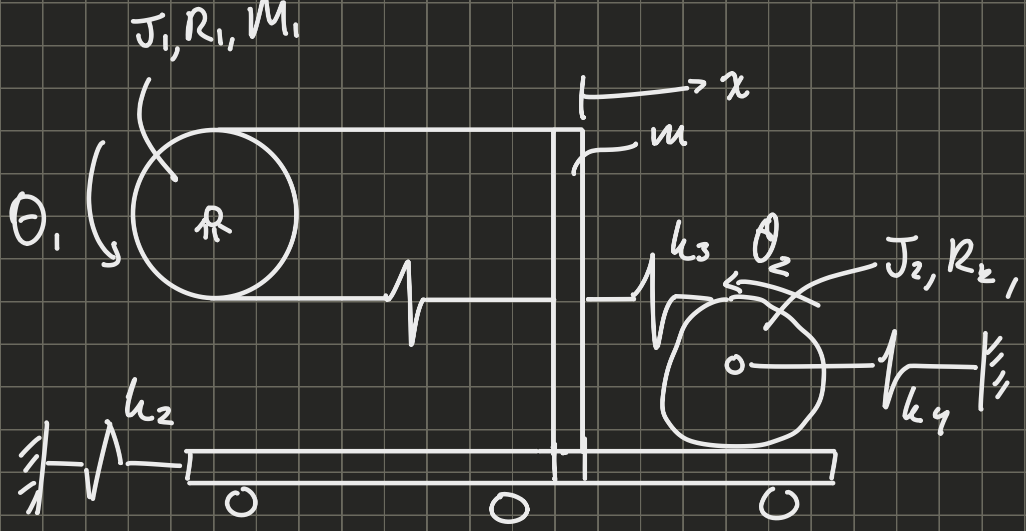
$$\left[k_1 \right] = \left[\frac{\partial \Delta \underline{l}}{\partial \underline{x}} \right]^T_{\underline{x} = \underline{x}_0} \left[\ddots_{k} \ddots \right] \left[\frac{\partial \Delta \underline{l}}{\partial \underline{x}} \right]_{\underline{x} = \underline{x}_0}$$

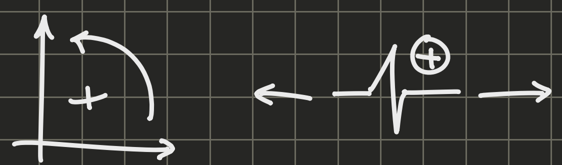
$$= \left[\Delta_k \right]^T \left[\ddots_{k} \ddots \right] \left[\underline{\Delta}_k \right]$$

Different $\hookrightarrow$ Jacobian matrix at equilibrium

We can evaluate the stiffness without passing through the non-linear form of our equation.

$$\frac{\partial}{\partial x}\left[\frac{\partial \Delta l_j}{\partial \underline{x}}\right] = \begin{bmatrix} \frac{\partial^2 \Delta l_j}{\partial x_1 x_1} & \cdots & \frac{\partial^2 \Delta l_j}{\partial x_1 x_{nu}} \\ & \ddots & \vdots \\ & & \frac{\partial^2 \Delta l_j}{\partial x_{nu} \partial x_{nu}} \end{bmatrix}_{\underline{x} = \underline{x}_0}$$





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3 degrees of freedom $= \{x, \theta_1, \theta_2\}$

$$\Rightarrow y_m = \{v_{G1}, \omega_1, v_{G2}, \omega_2\}^T \qquad \underline{x} = \{x, \theta_1, \theta_2\}^T$$

$$\begin{bmatrix} \diagdown & & \\ & m & \\ & & \diagdown \end{bmatrix} = \begin{bmatrix} m & 0 & 0 & 0 \\ 0 & J_1 & 0 & 0 \\ 0 & 0 & M_2 & 0 \\ 0 & 0 & 0 & J_2 \end{bmatrix}$$

$$T_c = \frac{1}{2}\dot{y}_m^T \left[\diagdown_m \diagdown\right] \dot{y}_m = \frac{1}{2}\dot{\underline{x}}^T \left[\Lambda_m\right]^T \left[\diagdown_m \diagdown\right] \left[\Lambda_m\right] \dot{\underline{x}}$$

To find the different elements of Λ_m, we freeze each dof and unfreeze them one at a time:

$$\begin{array}{c|ccc} & x & \dot{\theta}_1 & \dot{\theta}_2 \\ \hline V_{G1} & 1 & 0 & 0 \\ \omega_1 & 0 & 1 & 0 \\ V_{G2} & 1 & 0 & -R_2 \\ \omega_2 & 0 & 0 & 1 \end{array}$$

From here we get the mass matrix $[M]$: with:

$$[\Lambda_m]^T [m] [\Lambda_m] = [M]$$

$$[M] = \begin{bmatrix} m + M_2 & 0 & -R_2 M_2 \\ 0 & J_1 & 0 \\ -R_2 M_2 & 0 & J_2 + M_2 R_2^2 \end{bmatrix}$$

$$V_k = \frac{1}{2} \underline{\Delta l}^T \begin{bmatrix} \ddots & & \\ & k & \\ & & \ddots \end{bmatrix} \underline{\Delta l} = \frac{1}{2} \underline{x}^T [\Lambda_k]^T \begin{bmatrix} \ddots & & \\ & k & \\ & & \ddots \end{bmatrix} [\Lambda_k] \underline{x}$$

$$\Lambda_k = \begin{array}{c|ccc} & x & \theta_1 & \theta_2 \\ \hline \Delta l_1 & 1 & -R_1 & 0 \\ \Delta l_2 & 1 & 0 & 0 \\ \Delta l_3 & 0 & 0 & -2R_2 \\ \Delta l_4 & -1 & 0 & R_2 \\ \Delta l_5 & 1 & R_2 & 0 \end{array}$$

Motion in 2-n gdl system

Like in the 1 gdl case, we can have different types of motion which describe our system.

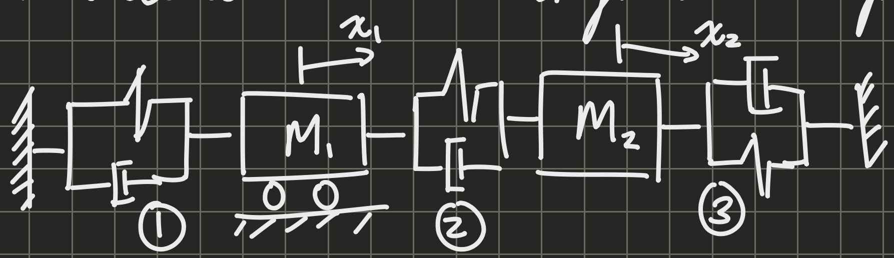
$$[M]\underline{\ddot{x}} + [R]\underline{\dot{x}} + [k]\underline{x} = \underline{F} \rightarrow \text{dampened-forced motion}$$

$$[M]\underline{\ddot{x}} + [R]\underline{\dot{x}} + [k]\underline{x} = \underline{0} \rightarrow \text{dampened free motion}$$

$$[M]\underline{\ddot{x}} + [k]\underline{x} = \underline{0} \rightarrow \text{undamped free motion.}$$

Undamped Free motion for a n degree system

We will look at this through an example:



$$\underline{x} = \{x_1, x_2\}^T$$

$$[M] = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \qquad [k] = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}$$

$\to$ not $[m]$, but $[M] = [\Lambda_m]^T [m] [\Lambda_m]$

We know: $[M]\ddot{\underline{x}} + [k]\underline{x} = \underline{0}$

We need to solve the system to find our vibration.

Let's suggest the tentative solution for $\underline{x}(t)$:

$$\underline{x} = \bar{\underline{x}} e^{i\omega t} \Rightarrow \underline{x}(t) = \bar{\underline{x}} e^{\lambda_i t} \qquad \lambda_i = \pm \omega i$$

$$\Rightarrow \bar{\underline{x}} = \begin{Bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{Bmatrix}$$

$\bar{x} \to$ static solution (I think) (Confirm with him)
$\hookrightarrow$ equivalent to x_0

$\bar{\underline{x}} \to$ vector of static solutions (I think)

$$\Rightarrow \left(-\omega^2 [M] + [K]\right) \bar{\underline{x}} \cancel{e^{i\omega t}} = \underline{0}$$

Two possible solutions:

- Static solution: $\bar{\underline{x}} = \underline{0} \to$ sub-case of dynamic study
- Dynamic equilibrium $\Rightarrow \det\left(-\omega^2[M] + [K]\right) = 0$

$$\det(\omega^2) = \begin{bmatrix} -\omega^2 m_{11} + k_{11} & -\omega^2 m_{12} + k_{12} \\ -\omega^2 m_{21} + k_{21} & -\omega^2 m_{22} + k_{22} \end{bmatrix} = 0$$

Determinat as a function of ω^2, not det of ω^2.

$$= (-\omega^2 m_{11} + k_{11})(-\omega^2 m_{22} + k_{22}) - (-\omega^2 m_{12} + k_{12})(-\omega^2 m_{21} + k_{21}) = 0$$

$$= a\omega^4 + b\omega^2 + c = 0$$

$$\omega^2_{I,II} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \begin{cases} \omega^2_I \\ \omega^2_{II} \end{cases}$$

Tutti e due, reali positivi.

↳ Same number of natural frequencies as degree of freedom (good check for us)

$$\omega_{1,3} = \pm\sqrt{\omega^2_I} \qquad \omega_{2,4} = \pm\sqrt{\omega^2_{II}}$$

Writing out the first equation of ours system only, because the solution to both equations relative to ω^2 is the same, so:

$$-(\omega^2_I m_{11} + k_{11})\,\bar{x}_1^{(1)} + (-\omega^2_I m_{12} + k_{12})\,\bar{x}_2^{(1)} = 0 \quad \text{with } \omega^2_I \text{ fixed}$$

$$-(\omega^2_{II} m_{11} + k_{11})\,\bar{x}_1^{(2)} + (-\omega^2_{II} m_{12} + k_{12})\,\bar{x}_2^{(1)} = 0 \quad \text{with } \omega^2_{II} \text{ fixed}$$

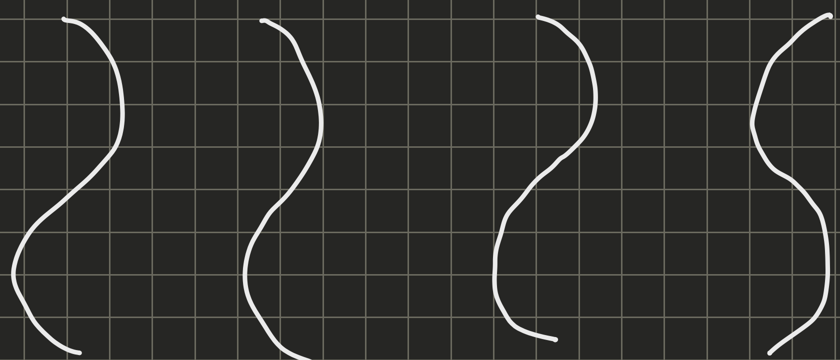
From these equations we can find the μ_i coefficients. These coefficients tell us how each mass vibrates relative to another when our system is operating at one of it's natural frequencies:

$$\mu_I = \frac{\bar{x}_2^{(1)}}{\bar{x}_1^{(1)}} = \frac{\left(-\omega_I^2 m_{12} + k_{12}\right)}{\left(-\omega_I^2 m_{11} + k_{11}\right)} \rightarrow \omega_I^2 \text{ fixed}$$

$$\mu_{II} = \frac{\bar{x}_2^{(2)}}{\bar{x}_1^{(2)}} = \frac{\left(-\omega_{II}^2 m_{12} + k_{12}\right)}{\left(-\omega_I^2 m_{11} + k_{11}\right)} \rightarrow \omega_{II}^2 \text{ fixed}$$

$$\underbrace{\underline{\bar{x}}^{(1)} = \begin{Bmatrix} 1 \\ 1/\mu_I \end{Bmatrix}} \qquad \underbrace{\underline{\bar{x}}^{(2)} = \begin{Bmatrix} 1 \\ 1/\mu_{II} \end{Bmatrix}}$$

First Mode of Vibrations — Second Mode of Vibration



m_1 m_2 (Examples) — m_1 m_2 (Examples)

We can put everything together in an equation to describe the motion of the full system:

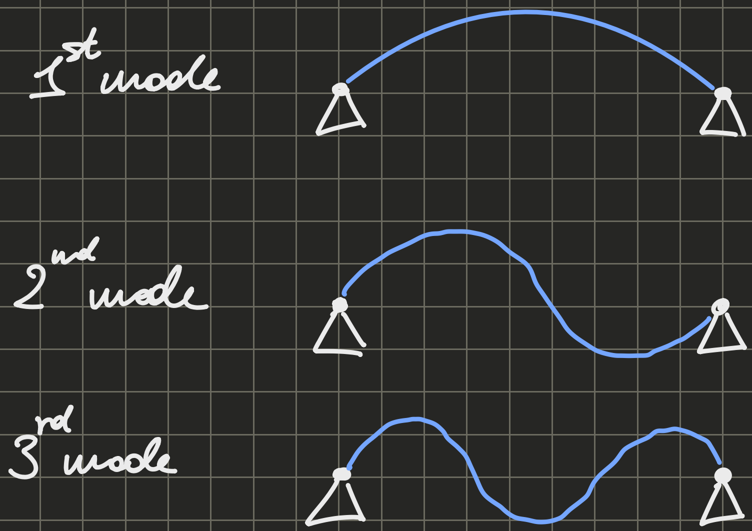
$$\underline{x}(t) = a_1 \underline{\bar{x}}^{(1)} e^{i\omega_1 t} + a_2 \underline{\bar{x}}^{(2)} e^{i\omega_2 t} + a_3 \underline{\bar{x}}^{(1)} e^{-i\omega_1 t} + a_2 \underline{\bar{x}}^{(2)} e^{-i\omega_2 t}$$

The coefficients represent the vibration the system is under.

To solve this equation we need initial conditions.

Going back to the modes of vibration.

A real system has infinite many modes of vibration, in general the can be represented like such:



All the modes occur as a response to a natural frequency of the system.

Note: μ is the coefficient of vibration of X_i relative to X_j as a response to the different, ω_k, natural frequencies of the system

$$\underline{\bar{X}}^{(k)} = \left\{ \begin{array}{c} \bar{X}_1/\bar{X}_1 \, (\omega_k) \\ \bar{X}_2/\bar{X}_1 \, (\omega_k) \\ \vdots \end{array} \right\}$$