Esercitazione 4 – More than one variable

$X \setminus Y$	-2	-1	0	3	p_X
0	$0,1$	0	$0,4$	$0,2$	$0,7$
1	$\underline{0,05}$	$\underline{0,1}$	0	$\underline{0,15}$	$0,3$
p_Y	$0,15$	$0,1$	$0,4$	$0,35$	1

a) $a = P(X=0, Y=3)$?

$$p_X(k) = \sum_{y \in S} p_{X,Y}(x,y)$$

$$p_X(0) = \sum_{y \in S} p_{X,Y}(0,y) = 0,1 + 0 + 0,4 + a = 0,7 \Rightarrow a = 0,2$$

$b = P(X=1, Y=2)$

$$p_X(1) = 0,3 = \sum_{y \in S} p_{X,Y}(1,y) = b + 0,1 + 0 + 0,15 \Rightarrow b = 0,05$$

b) p_X? p_Y? $E[X]$? $E[Y]$? $Var(X)$? $Var(Y)$?

$$p_X(k) = \begin{cases} 0,7 & \text{if } k=0 \\ 0,3 & \text{if } k=1 \end{cases} \Rightarrow X \sim Be(0,3)$$

$$p_Y(k) = \begin{cases} 0,15 & k=-2 \\ 0.1 & k=-1 \\ 0.4 & k=0 \\ 0,35 & k=1 \end{cases}$$

$X \rightarrow E[X] = p = 0,3$

$\hookrightarrow Var(X) = p(1-p) = 0,3 \cdot 0,7 = 0,21$

$$E[Y] = \sum_{y \in S_y} y\, p_X(y) = -2 \cdot 0{,}15 + -1 \cdot 0{,}1 + 3 \cdot 0{,}35 = 0{,}65$$

$$Var[Y] = E[Y^2] - E[Y]^2 = 3{,}4275$$

$$E[Y^2] = 4 \cdot 0{,}15 + 0{,}1 + 9 \cdot 0{,}35$$

c) $\underbrace{X \perp\!\!\!\perp Y}_{\text{X independent from Y}}$?

$$X \perp\!\!\!\perp Y \Leftrightarrow p_{X,Y}(x,y) = p_X(x) \cdot p_Y(y) \quad \forall x,y \in \mathbb{R}$$

$x = 0, y = -2 \qquad 0{,}1 \neq 0{,}7 \cdot 0{,}15 \Rightarrow X \not\perp\!\!\!\perp Y$

$Var(X-Y)$?

$$Var\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i^2 Var(X_i) + 2 \sum_{\substack{i=1,n \\ j<i}} a_i a_j \, cov(X_i, X_j)$$

$a_i \to 1$, $a_j \to -1$

$$\Rightarrow Var(X) + Var(Y) - 2Cov(X,Y) = Var(X-Y)$$

$$Cov(X,Y) = E[XY] - E[X]E[Y] = 0{,}25 - 0{,}3 \cdot 0{,}65 = 0{,}055$$

$$E[XY] = \sum_{\substack{x \in S_X \\ y \in S_Y}} xy\, p_{X,Y}(x,y) = -2 \cdot 0{,}05 - 0{,}1 + 3 \cdot 0{,}15 = 0{,}25$$

↳ not considering null values of x and y

$$\to Var(X-Y) = 0{,}21 + 3{,}4275 - 2 \cdot 0{,}055 = 3{,}5275$$

d) $\rho_{X,Y}$?

$$\rho_{X,Y} = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} \in (-1,1)$$

→ indicator strength of linear relationship between X & Y.

$$= \frac{0{,}055}{\sqrt{0{,}21 \cdot 3{,}42}} = 0{,}0648 > 0$$

↳ weakly positively correlated.

Exercise 2

X\Y	-1	0	2	6	p_X
-2	1/9	1/27	1/27	1/9	
1	2/9	0	1/9	1/9	
3	0	0	1/9	4/27	
p_Y					

a) $P(X>0, Y\geq 0)$? $P(|XY|\geq 2)$? $P(x\geq y)$?

$$P(X>0, Y\geq 0) = 0 + 1/9 + 1/9 + 1/9 + 0 + 1/9 + 4/27 = \frac{13}{27}$$

$$P(|XY|\geq 2) = \frac{1}{9} + \frac{1}{27} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + 0 + \frac{1}{9} + \frac{4}{27} = \frac{20}{27}$$

$$P(X\geq Y) = p_{X,Y}(1,-1) + p_{X,Y}(1,0) + p(3,-1) + p(3,0) + p(3,2)$$
$$= \frac{2}{9} + 0 + 0 + 0 + \frac{1}{9} = \frac{1}{3}$$

Exercise 4.3

X\Y	-1	0	1	p_X
-15	0	2/36	0	2/36
-1	4/36	2/36	0	6/36
0	1/36	26/36	1/36	28/36
p_Y	5/36	30/36	1/36	

a) $Cor(X,Y)$? Are they correlated?

$$Cov(X,Y) = E[(X-EX)(Y-EY)] = E[XY] - E[X]\cdot E[Y]$$

$$E[XY] = \sum_{\substack{x\in S_x \\ y\in S_y}} xy\, p_{X,Y}(x,y) = (-1)(-1)\cdot\frac{4}{36} = \frac{4}{36}$$

$$E[X] = \sum_{x\in S_x} x\, p_X(x) = -15\cdot\frac{2}{36} + -1\cdot\frac{6}{36} = -1$$

$$E[Y] = \sum_{y\in S_y} y\, p_Y(y) = -1\cdot\frac{5}{36} + 1\cdot\frac{1}{36} = \frac{-4}{36}$$

$Cov(X,Y) = \frac{4}{36} - \frac{4}{36} = 0 \Rightarrow X,Y$ are uncorrelated.

b) $X \perp\!\!\!\perp Y$?

$X \perp\!\!\!\perp Y \Rightarrow cov(X,Y) = 0$

$\not\Leftarrow$

$X \perp\!\!\!\perp Y \Leftrightarrow \forall x,y \; p_{X,Y} = p_X \cdot p_Y$

In this case this is not true so they are not independent.

if $x=-15, y=-1$ $\quad 0 \neq \frac{2}{36}\cdot\frac{5}{36} \Rightarrow X \not\perp\!\!\!\perp Y$

Exercise 4

$X,Y: \sigma_x^2 = \sigma_y^2 = 20$, $\quad X \perp\!\!\!\perp y$

$u = 0{,}5x + 0{,}1y \qquad V = bx + y + c$

a) $Var(X+Y)$? $Var(X-Y)$?

$Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y) = Var(X) + Var(Y) = 40$

$X \perp\!\!\!\perp Y \Rightarrow Cov(X,Y) = 0$

$Var(X-Y) = Var(X) + Var(Y) - 2Cov(X,Y) = Var(X) + Var(Y) = 40$

b) $Var(U)$? $Var(V)$?

$$Var(U) = Var(0{,}5X + 0{,}1Y) = 0{,}5^2 Var(X) + 0{,}1^2 Var(Y)$$
$$\overset{!}{=} \frac{1}{4}\cdot 20 + \frac{1}{100}\cdot 20 = \frac{26}{5} = 5{,}2$$

Remember

$$Var\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i^2 Var(X_i) + 2\sum_{\substack{i,j=1 \\ j<i}}^{n} a_i a_j \, cov(X_i, X_j)$$

$$Var(V) = Var(bX + Y + c) = b^2 Var(X) + Var(y) + \cancel{Var(c)}$$
$$\overset{!}{+} \cancel{2b\,cov(X,Y)} + \cancel{2b\,cov(x,c)} + \cancel{2\,cov(y,c)}$$
$$= b^2 \cdot 20 + 20 = 20(1 + b^2)$$

c) $Cor(U,V)$?

$$cov(0{,}5X + 0{,}1y, bX + y + c) = cov(0{,}5x, bx) + \overset{\perp\!\!\!\perp}{\cancel{cov(0{,}5x, y)}}$$
$$+ \cancel{cov(0{,}5, c)} + \overset{\perp\!\!\!\perp}{\cancel{cov(0{,}1y, bx)}} + cov(0{,}1y, y)$$
$$+ \cancel{cov(0{,}1y, c)} = \frac{b}{2}\underbrace{cov(x,x)}_{=Var(X)} + 0{,}1\underbrace{cov(y,y)}_{Var(y)}$$
$$= \frac{b}{2} Var(x) + 0{,}1 Var(y) = 10 \cdot b + 2$$

$cov(U,y) = 10b + 2$

d) b, c ? if U, V uncorrelated

U, V uncorrelated $\Longleftrightarrow cov(U,V) = 0 \Rightarrow b = -\frac{1}{5} \quad \forall c \in \mathbb{R}$

Exercise 4,5

X: INCORRECT BITS IN A

$X \sim Poi(\lambda_1): E[X] = 3 \Rightarrow \lambda = np = 3$

Y: " " IN B

$Y \sim Poi(\lambda_2): EX = 2 \Rightarrow \lambda = np = 2$

$X \perp\!\!\!\perp Y$

a) $P(X \geq 1, Y \leq 4)$?

$$P(X \geq 1, Y \leq 4) \underset{\text{since } \perp\!\!\!\perp}{=} P(X \geq 1) \cdot P(Y \leq 4) = (1 - p_X(0))(p_Y(0) + \dots + p_Y(4)) =$$

$$X \sim Poi(\lambda_1) \Rightarrow E[X] = \lambda_1 = 3 \Rightarrow X \sim Poi(3)$$

$$Y \sim Poi(2)$$

$$P(k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

$$= (1 - e^{-3}) e^{-2} \left(1 + 2 + \frac{4}{2} + \frac{8}{6} + \frac{16}{24}\right) = 0{,}90018$$

b) $Z = X + Y$, p_Z?

$X_1, \dots, X_n \sim Poi$ all $\perp\!\!\!\perp$ $\Rightarrow \sum_{i=1}^{n} X_i \sim Poi\left(\sum_{i=1}^{n} \lambda_i\right)$

$X_i \sim Poi(\lambda_i)$

$$\Rightarrow Z \sim Poi(5) \Rightarrow p_Z(x) = \frac{5^k}{k!} e^{-5} \quad k = 0, 1, \dots, n$$

d) $P(X = Z)$?

$$P(X = Z) = P(X = X + Y) = P(Y = 0) = p_Y(0) = \frac{2^0}{0!} e^{-2} = e^{-2}$$

$Y \sim Poi(2)$

<u>Exercise 4.6</u>

$Z_1, Z_2 \sim \mathcal{N}(0,1)$ $\quad Z_1 Z_2$ independent

$X_1 = 2Z_1 + Z_2$

$X_2 = 3Z_1 - 6Z_2 + 5$

a) $E[X_1], E[X_2], Var(X_1), Var(X_2)$?

$$E[X_1] = E[2Z_1 + Z_2] = 2\cancel{E[Z_1]} + \cancel{E[Z_2]} = 0$$

$$E[X_2] = E[3Z_1 - 6Z_2 + 5] = 3\cancel{E[Z_1]} - 6\cancel{E[Z_2]} + 5 = 5$$

$$Var(X_1) = Var(2Z_1 + Z_2) = 4 \cdot Var(Z_1) + Var(Z_2) = 5$$

$$Var(X_2) = Var(3Z_1 - 6Z_1 + 5) = 9\,Var(Z_1) + 36\,Var(Z_2) = 45$$

b) $E[Y], Var[Y], \rho_Y$? $\quad y = 5X_1 - 2X_2 = 4Z_1 + 17Z_2 - 10$

$$E[Y] = -10$$

$$Var(Y) = 16\,Var(Z_1) + 17^2\,Var(Z_2) = 305$$

$X_1, \dots, X_n : X_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$ independent

$$\sum_{i=1}^{n} X_i \sim \mathcal{N}\left(\sum_{i=1}^{n} \mu_i, \sum_{i=1}^{n} \sigma_i^2\right) \Rightarrow y \sim \mathcal{N}(-10, 305)$$

we get the same result.

c) $P(Y > 11)$?

$$P(Y > 11) = 1 - P(Y \leq 11) = 1 - P\left(\frac{Y+10}{\sqrt{305}} \leq \frac{11+10}{\sqrt{305}}\right) = 1 - \Phi(1{,}20) = 0{,}11507$$