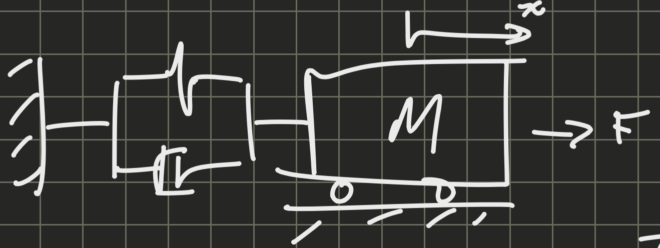
Lezione 5 - Forced Motion

We will look at what happens when a 1 degree system is forced to move.

A force is working, putting energy into the system.

This means that we are interested in the particular solution to the differential equation, unlike the general one we looked at last time



$m\ddot{x} + r\dot{x} + kx = F$ → Time dependent $F(t)$ → Today

→ State dependent $F(x,\dot{x})$ → Tomorrow

$F(t)$ $\begin{cases} \text{CONSTANT} \\ \text{SINUSOIDAL} \\ \text{PERIODIC} \\ \text{APERIODIC} \\ \text{RANDOM} \end{cases}$ (e.g. if we turn the system 90°

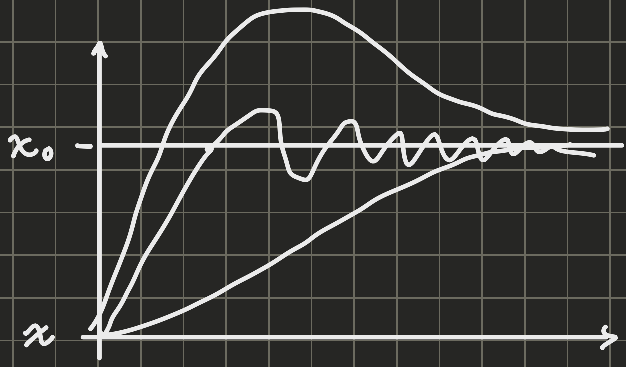
$x(t) = x_g(t) + x_p(t)$

$x_p(t)$ ↳ These two lessons

$x_g(t)$ → last time

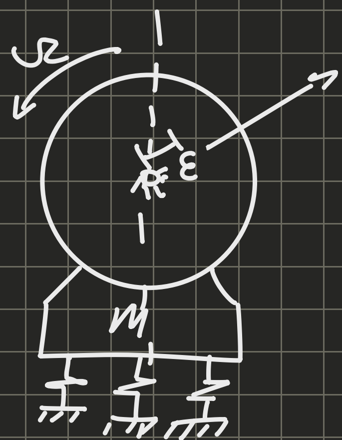
↳ After a while $x_g(t) = 0$ (with damping)

$\Rightarrow x(t) = x_p(t)$ for $t \to \infty$



With constant forces.
3 ways to do to the solution, which depend on the damping

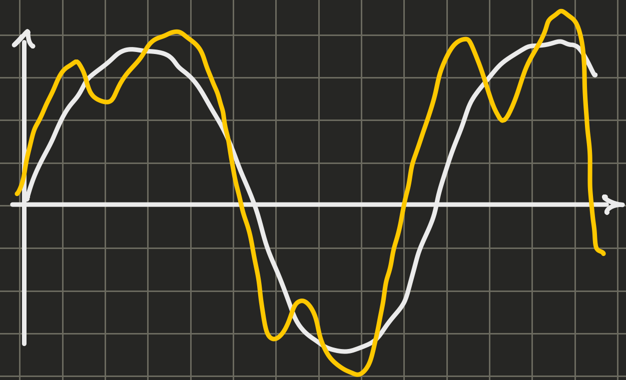
We are not interested in constant forces since it only means a new equilibrium position.



→ sinusoidal

ε is present if the density isn't homogeneous, so the center of gravity is not on the center.

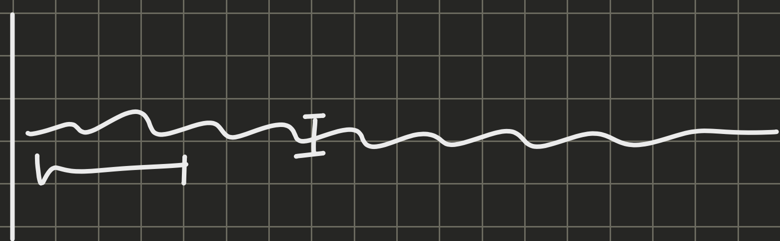
$F \cos \Omega t$



— = periodic (more than one harmonic)
still periodic but not sinusoidal since it's a sum of more sinusoidal function.

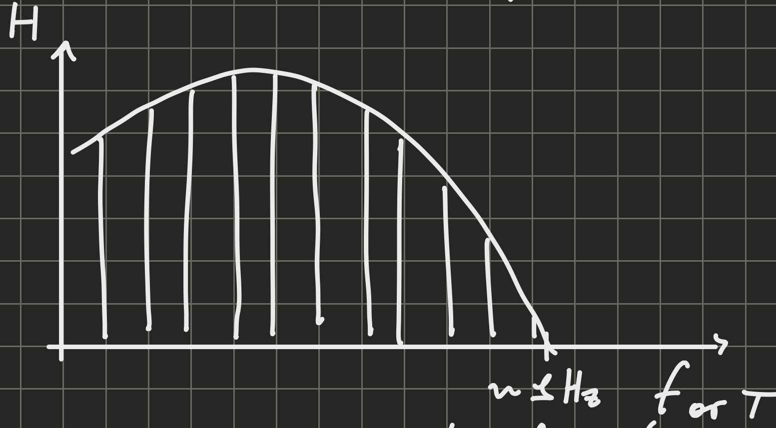
Random forces consider cases in which there is irregularity of surfaces.

All these things can be defined by spectra.



The randomness can be described by the length and heights of waves.

We can describe the irregularity using a spectra



We can use a Fourier transform to isolate all the frequencies and their amplitudes.

For the road the spectra changes shape based on the velocity with which we travel over it.

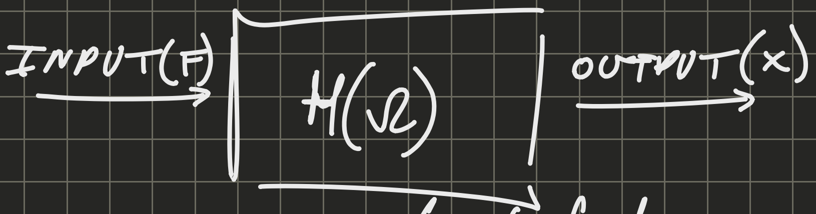
The wind will change based on the area in which we look at it. The wind also interacts with the soil, so it will have a form as it goes up, the speed with which the speed increases also changes based on the area.
Tall building, more turbulence but lower velocity at lower heights. Flat lands, less turbulence but higher velocity gradient.

We want to return to the sinusoidal function

If we can solve the linear system for one sinusoidal, we can define a transfer function.

Starting from: $m\ddot{x} + r\dot{x} + kx = F_0 e^{i\Omega t}$

We want a relationship between input and output.



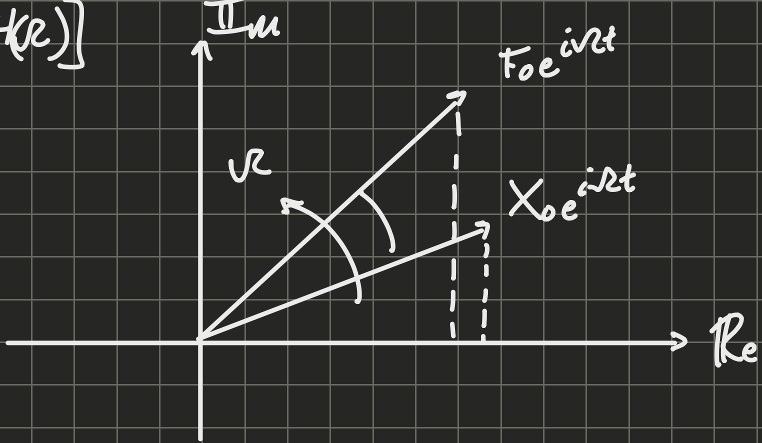
transfer function

If we define this, then we can use any sinusoidal force and get an output. Since we're not solving one case but for a general F_0 and Ω.

Tentative solution $= X_0 e^{i\Omega t}$

In forced motion, our system vibrates at the same frequency of the force.

$$\Rightarrow \underbrace{\left(-\Omega^2 m + i\Omega r + k\right)}_{[H(\Omega)]} X_0 \cancel{e^{i\Omega t}} = F_0 \cancel{e^{i\Omega t}}$$



The transfer function defines how big the X_0 is and how large the phase difference is.

$$X_0 = \frac{F_0}{-\Omega^2 m + i\Omega r + k} = F_0 \cdot \underbrace{\frac{1}{-\Omega^2 m + i\Omega r + k}}_{\text{Mechanical impedance of the system}}$$

This is the transfer function, but we want it dimensionless to define a general solution.

$$X_0 = \frac{F_0/k}{-\Omega^2 \frac{m}{k} + i\Omega \frac{r}{k} + 1}$$

F_0/k → static deflection of the system

$$= \frac{X_{static}}{1 - \frac{\Omega^2}{\omega_0^2} + \underbrace{i\Omega \frac{2mr}{2mk}}}$$

$$\to i\Omega \frac{r}{2m\omega_0^2} = i\frac{\Omega}{\omega_0} \cdot \frac{2r}{2m\omega_0} = i\,a\,2h$$

We define:

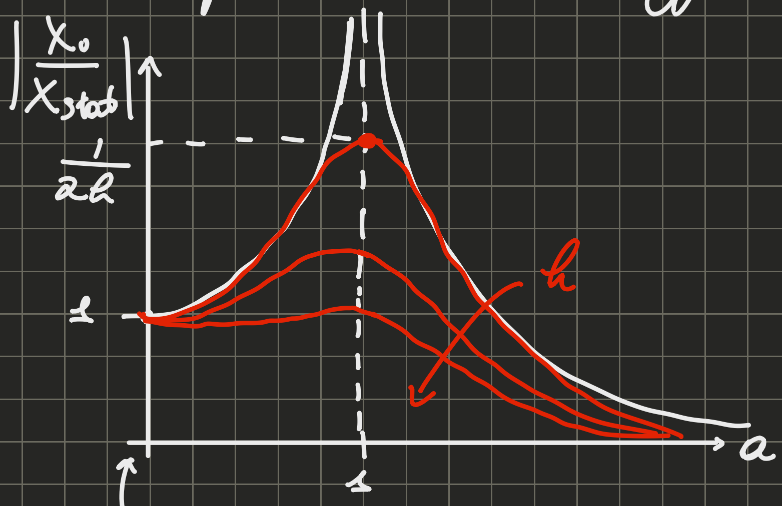
$$a = \frac{\Omega}{\omega_0}$$

$$h = \frac{r}{2m\omega_0}$$

$$\frac{X_0}{X_{static}} = \frac{1}{\underbrace{1 - a^2 + 2ahi}}$$

Complex multiplier which is a function of a and h.

This amplification can be bigger and smaller than 1.



$h = 0$

$h \neq 0$

Has to be 1, since static and therefore there is no dynamic amplification

seismographic case

when $a < 1$, quasi static behaviours

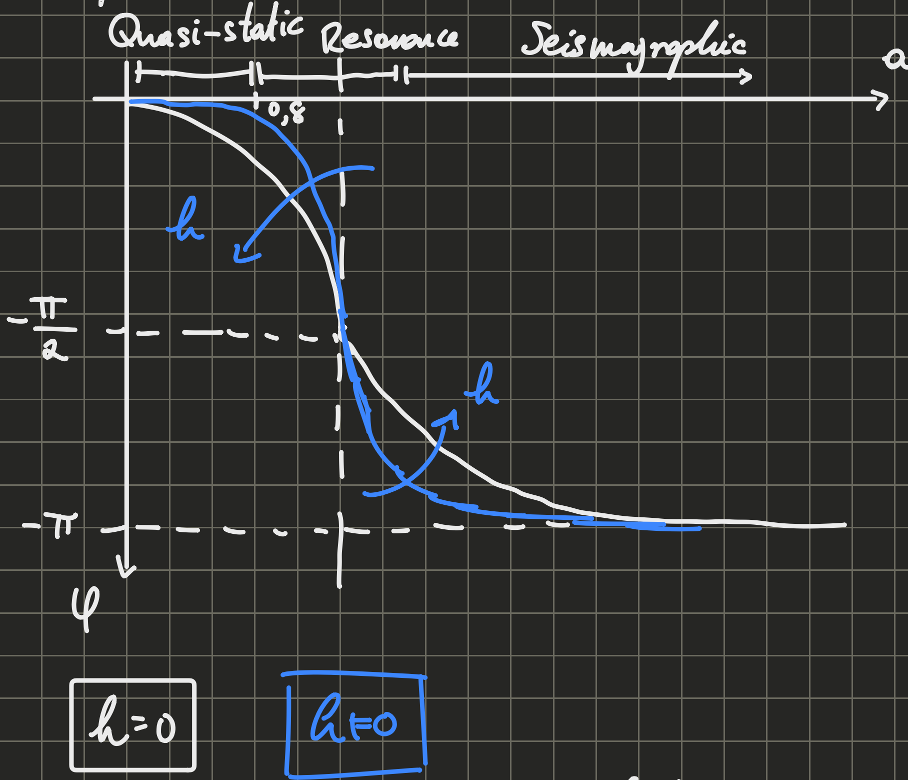
when $a > 1$, the displacement is smaller than the static case.

If $a \approx 1$ (resonance) we have huge dynamic amplification

When $a<1$, the mass is moving with the force as if we were applying statically, so no amplification

When $a>1$, the system is like a low pass filter, the system cannot follow the force, so it filters out the low frequency

$a \simeq 1$, the frequency is very close to the system's natural frequency. The system is prone to this movement, so a huge amplification occurs.



The phase angle being negative means that the force is before the movement, it makes sense physically

$\vec{F}_i + \vec{F}_v + \vec{F}_e + \vec{F}_a = 0$ ⟹ the has to be a balance in the forces.

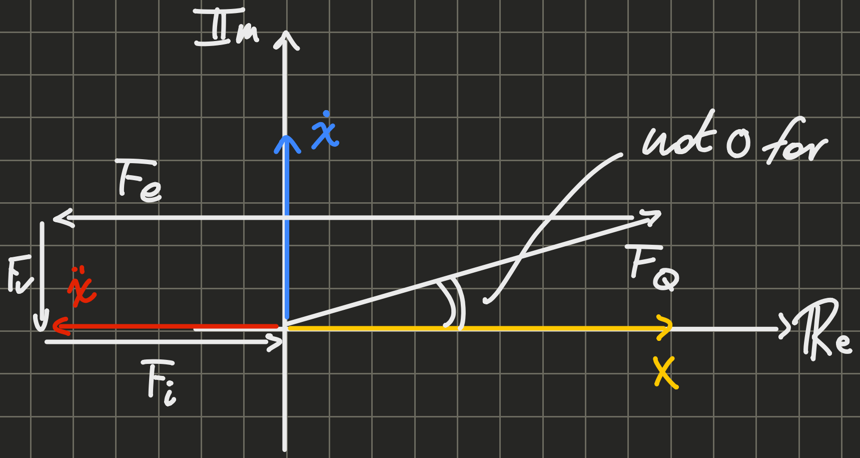
should be s

should be e

↳ inertial ↳ viscous ↳ elastic ↳ external

$X = X_0 e^{i\Omega t}$

$a \approx 0$



not 0 for visual purposes (from φ, a graph)

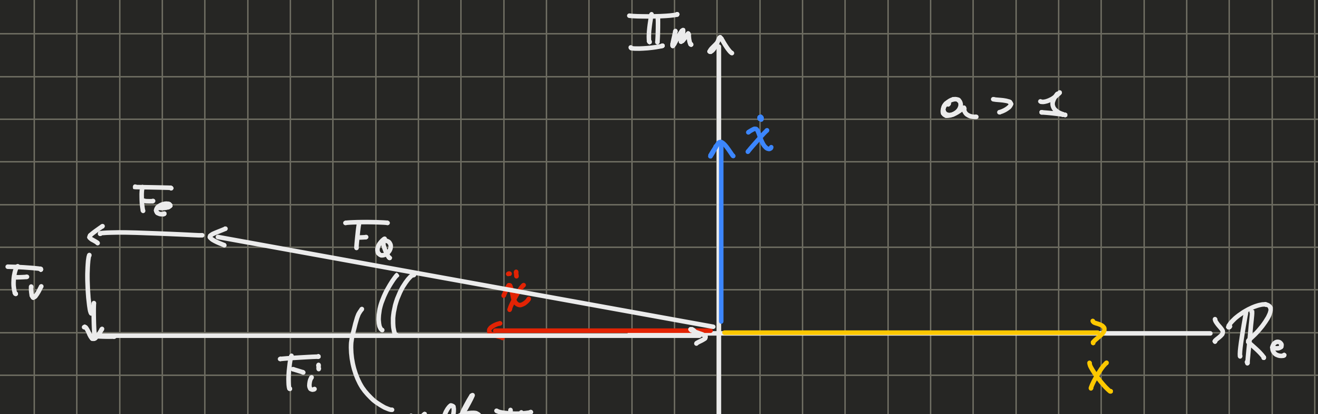
$$x = X_0 e^{i\Omega t}$$

$$\dot{x} = i\Omega X_0 e^{i\Omega t}$$

$$\ddot{x} = -\Omega^2 X_0 e^{i\Omega t}$$

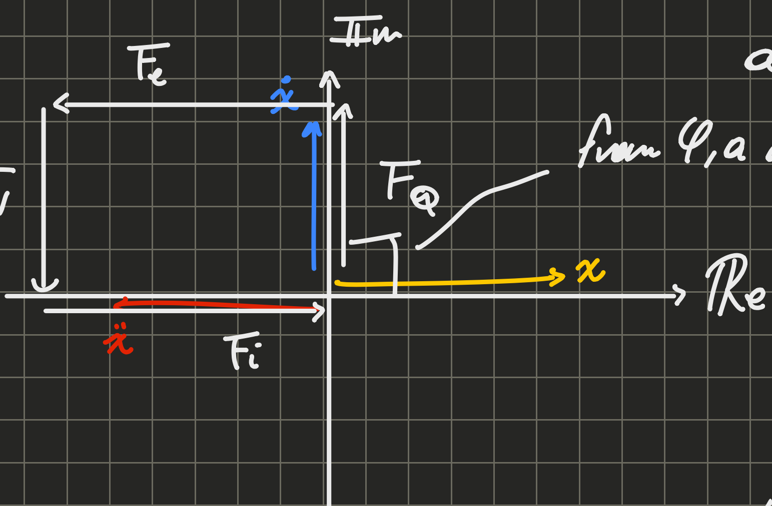
At $a<1$, all the external forces are mainly handled by the springs

$a > 1$



not $-\pi$ for visual purposes.(from φ, a diagram)

$a \approx 1$



from φ, a diagram.

$F_e + F_i = 0$

$\Rightarrow F_Q$ is balanced only by the F_v.

This is the reason why, when we have no damping, the dynamic amplification is ∞.