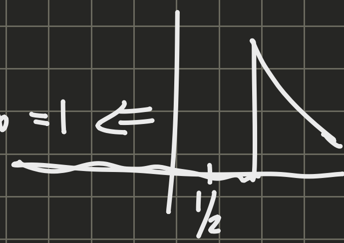
Esercitazione 2

2.1

X abs. cont. r.v. $f(x)=\begin{cases}2e^{-2(x-1)} & x\geq 1\\ 0 & ,\text{otherwise}\end{cases}$

$y=e^x$

a) $P(y>\sqrt{e})=P(e^x>\sqrt{e})=P(x>\frac{1}{2})=\int_{1/2}^{\infty} f(x)\,dx=1 \Leftarrow$



b) $E(Y)=\int_{\mathbb{R}} e^x f(x)\,dx=\int_1^{\infty} e^x\, 2\exp(-2(x-1))\,dx=\int_1^{\infty} 2e^{-x+2}=-2\left[e^{-x+2}\right]_1^{\infty}$

$=-2(0-e)=2e$

$y=g(x)=e^x$ $\quad E(g(x))=\int_{\mathbb{R}} g(x)f(x)\,dx$

c) f_y? → It's easier to find F_y and then differentiate.

$F_Y(y)=P(Y\leq y)=P(e^x\leq y)=P(x\leq \log(y))=\int_{-\infty}^{\log(y)} f_x\,dx$

if $\log(y<1)$, $y<e$, $F_y(y)=0$

if $\log(y>1)$, $y\geq e$, $F_y(y)=\int_1^{\log} 2e^{-2(x-1)}\,dx=-\left[e^{-2(x+1)}\right]_1^{\log(y)}=1-\frac{e^2}{y^2}$

$F_y(y)=\begin{cases}0, & y<e\\ 1-\frac{e^2}{y^2}, & y\geq e\end{cases}$ → $f_x=\frac{2e^2}{y^3}\,\mathbb{I}_{(e,\infty)}$

2.2

$E(X)=10h$ $\quad x\sim Exp(\lambda)$

↳ lifetime of AAA

y lifetime bbb → $y=x^2$

a) $P(X\leq 15)$

$E(X)=\frac{1}{\lambda}=10\Rightarrow \lambda=\frac{1}{10}$

$$F_X(x) = \left(1 - e^{-\frac{1}{10}x}\right) \mathbb{I}_{(0,\infty)}(x)$$

$$P(X \le 15) = F(15) = 1 - e^{-1,5} = 0,7769$$

b) $\mathbb{E}(Y) = ? = \mathbb{E}(X^2) =$

$\hookrightarrow \mathbb{E}(g(x)) = \int_{\mathbb{R}} g(x) f_X(x)\, dx$

$\hookrightarrow$ Instead of calculating we can go:

$$Var(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 \Rightarrow \mathbb{E}(X^2) = 100 + 100 = 200$$

$\hookrightarrow$ we know for $X \sim Exp(\lambda) = \frac{1}{\lambda^2} = 100$

2.3

$$X \to \overset{\text{cdf}}{F_X(x)} = \left(1 - e^{-2(x-1)}\right) \mathbb{I}_{(1,\infty)}(x)$$

$y = x - 1$

like before $F_Y \to f_y$

$$F_Y(y) = P(Y \le y) = P(X - 1 \le y) = P(X \le y-1) = F_X(y-1)$$

$$\Rightarrow F_Y(y) = \left(1 - e^{-2y}\right) \underbrace{\mathbb{I}_{(1,\infty)}}\,(y+1)$$

$$\overset{!}{=} \left(1 - e^{-2y}\right) \mathbb{I}_{(0,\infty)}(y) \to \begin{cases} 1 & \text{if } y+1 \ge 1 \\ 0 & y+1 < 1 \end{cases} = \begin{matrix} 1, & y \ge 0 \\ 0, & y < 0 \end{matrix}$$

$$= \mathbb{I}_{[0,\infty]}(y)$$

$$f_y(y) = 2e^{-2y}\, \mathbb{I}_{[0,\infty)}(y)$$

$$\Rightarrow Y \sim Exp(2)$$

$$X \sim Exp(\lambda) = \lambda e^{-\lambda x}$$

b) $\mathbb{E}(X)$? $Var(X)$?

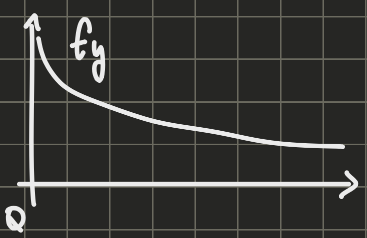
$$\mathbb{E}(x) = \int_{\mathbb{R}} x f_x(x)\,dx$$

Since $Y \sim Exp(2)$ we know $E(Y)$ and $Var(Y)$ and X is a linear transformation of Y, so instead of calculating we can use

$$\mathbb{E}(Y) + 1 = \frac{3}{2}$$
$$\Leftrightarrow 1/\lambda = \frac{1}{2}$$
$$Var(X) = Var(y+1) = var(Y) = \frac{1}{\lambda^2} = \frac{1}{4}$$

To reduce how much we calculate, it's much easier to use known values to find unknown values through manipulations.

$y \rightarrow$



→ It will have a longer tail on the right.

2.4

$Z \rightarrow$ standardization of U

$U \sim \mathcal{U}(0,1)$

z? f_z?

Standardization: $$z = \frac{U - \mathbb{E}(U)}{\sqrt{Var(U)}} = \frac{U - 1/2}{\sqrt{1/12}} = 2\sqrt{3}\left(U - \frac{1}{2}\right) = \sqrt{3}(2U-1)$$

$$\mathbb{E}(U) = \frac{1}{2} = \frac{a+b}{2}$$
$$Var(U) = \frac{1}{10} = \frac{b-a}{12}$$

$\rightarrow F_z \rightarrow f_z$

$$F_Z(z) = P(Z \le z) = P(\sqrt{3}(2U-1) \le z) = P\left(U \le \frac{\left(\frac{z}{\sqrt{3}}+1\right)}{2}\right)$$
$$\overset{!}{=} F_U\left(\frac{1}{2}\left(\frac{z}{\sqrt{3}}+1\right)\right)$$

$$F_U(u) = \begin{cases} 0, & u<0 \\ u, & 0 \le u \le 1 \\ 1, & u \ge 1 \end{cases}$$

$$F_Z(z) = \begin{cases} 0, & \frac{1}{2}\left(\frac{z}{\sqrt{3}}+1\right) < 0 \\ \frac{1}{2}\left(\frac{z}{\sqrt{3}}+1\right), & 0 \le \frac{1}{2}\left(\frac{z}{\sqrt{3}}+1\right) < 1 \\ 1, & \frac{1}{2}\left(\frac{z}{\sqrt{3}}+1\right) > 1 \end{cases} = \begin{cases} 0, & z < -\sqrt{3} \\ \frac{z+\sqrt{3}}{2\sqrt{3}}, & -\sqrt{3} \le z < \sqrt{3} \\ 1, & z \ge \sqrt{3} \end{cases}$$

$$f_Z(z) = F_Z'(z) = \frac{1}{2\sqrt{3}} \mathbb{I}_{(-\sqrt{3},\sqrt{3})}(z) \Rightarrow Z \sim \mathcal{U}(-\sqrt{3}, \sqrt{3})$$

b) We are only doing the first, the rest can be homework

$$Y_2 = \left(U - \frac{1}{2}\right)^2$$

$$F_{Y_2} \to f_{Y_2} \to \mathbb{E}(Y_2)$$

$$F_{Y_2}(y) = P\left(\left(U-\frac{1}{2}\right)^2 \le y\right)$$

if $y<0$, $F_{Y_2}(y) = 0$

if $y \ge 0$ $F_{Y_2}(y) = P\left(\left|U - \frac{1}{2}\right| \le \sqrt{y}\right) = P\left(\frac{1}{2} - \sqrt{y} \le u \le \frac{1}{2} + \sqrt{y}\right)$

$$= P\left(U \le \frac{1}{2} + \sqrt{y}\right) - P\left(U \le \frac{1}{2} - y\right)$$

it's fine since they are abs. cont., it will not be with discrete ones.

$$= F_U\left(\frac{1}{2}+\sqrt{y}\right) - F_U\left(\frac{1}{2}-\sqrt{y}\right)$$

$$F_U\left(\frac{1}{2}+\sqrt{y}\right) = \begin{cases} 0, & \underbrace{\frac{1}{2}+\sqrt{y} \le 0}_{\text{impossible}} \\ \frac{1}{2}+\sqrt{y}, & 0 \le \frac{1}{2}+\sqrt{y} \le 1 \\ 1, & \frac{1}{2}+\sqrt{y} \ge 1 \end{cases} = \begin{cases} \frac{1}{2}+\sqrt{y}, & \frac{1}{2}+\sqrt{y} < 1 \\ 1, & \frac{1}{2}+\sqrt{y} \ge 1 \end{cases}$$

$$= \begin{cases} \frac{1}{2}+\sqrt{y}, & y < \frac{1}{4} \\ 1, & y \ge \frac{1}{4} \end{cases}$$

$$F_U\left(\tfrac{1}{2}-\sqrt{y}\right) = \begin{cases} 0, & \frac{1}{2}-\sqrt{y} \le 0 \\ \frac{1}{2}+\sqrt{y}, & 0 \le \frac{1}{2}-\sqrt{y} \le 1 \\ 1, & \underbrace{\frac{1}{2}-\sqrt{y} \ge 1}_{\text{impossible}} \end{cases} = \begin{cases} 0, & y < \frac{1}{4} \\ \frac{1}{2}-\sqrt{y}, & y \le \frac{1}{4} \end{cases}$$

$$\Rightarrow F_{Y_2} = \begin{cases} 2\sqrt{y}, & y < \frac{1}{2} \\ 1, & y \ge \frac{1}{4} \end{cases}$$

$$F_{Y_2} = \begin{cases} 0, & y < 0 \\ 2\sqrt{y}, & 0 \le y \le \frac{1}{4} \\ 1, & y \ge \frac{1}{4} \end{cases} \qquad f_{Y_2} = \frac{1}{\sqrt{y}}\, \mathbb{I}_{(0,1/4)}(y)$$

$$\mathbb{E}(Y_2) = E\left((U-1)^2\right) = E\left((U-E(U))^2\right)$$

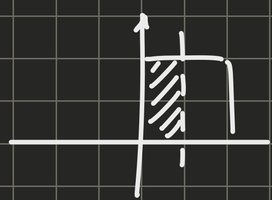
$$Var(U) = \frac{1}{12}$$

2,5

Timetable : 14:45 - 15:30

$X \to$ delay of the train (hours)

$X \sim \mathcal{U}(0,2)$



a) $P(0 \le X \le 1) = \frac{1}{2}$

b) $P(\frac{1}{4} \le X \le \frac{1}{3}) = F_X(\frac{1}{3}) - F(\frac{1}{4}) = \frac{1}{6} - \frac{1}{8} = \frac{1}{24}$

$$F_X = \begin{cases} 0, & x < \\ \frac{x-a}{b-a} & 0 \le x \le 2 \\ 1 & x \ge 2 \end{cases}$$

c) $\mathbb{E}(X) = ?$

$u? : P(x > u + E(x)) = \frac{1}{4}$

$\hookrightarrow$ like a quantile $q_\alpha = P(X \le q_\alpha) = \alpha$

$E(x) = \frac{a+b}{2} = 1$ hour

$P(x > u+1) = 1 - P(x \le u+1) = \frac{1}{4} \Rightarrow P(x \le u+1) \Rightarrow u+1 = q_{3,4}$

$\Rightarrow u = q-1$

$q<0$ and $q>2$ are impossible

$\Rightarrow \quad q \in [0,2]$

$$\int_0^q \underbrace{\frac{1}{2}}_{f_x} dx = \frac{3}{4} \Rightarrow q = \frac{3}{2} \Rightarrow u = \frac{1}{2}$$

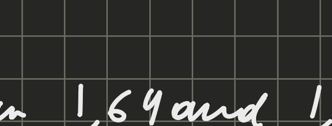
2.6

$Z \sim N(0,1)$

a) $P(z \le 0{,}2) = .57926$

$P(z > 0{,}2) = 1 - .57926 = .42074$

$\underbrace{P(z < -0{,}2)}_{\text{We can use simmetry}} = .42074$



$P(x_\alpha) = 1 - P(x_\alpha)$

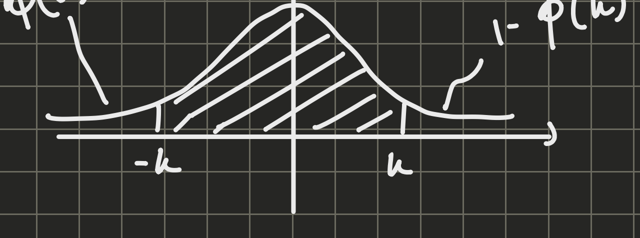
$P(-0{,}2 < z < 0{,}2) = .1586$

Questions are either to find probability or quantile

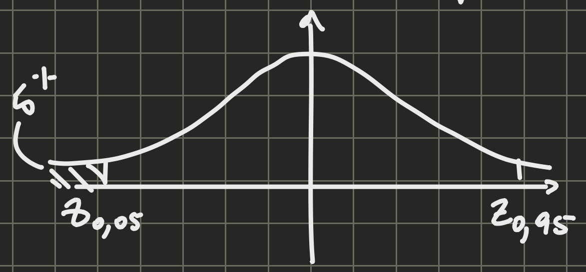
b) $z_{0,95}$? Between 1,64 and 1,65 $\rightarrow z_{0,95} = \frac{1{,}64+1{,}65}{2} = 1{,}645$

c) k? $P(-k \le z \le k) = 0{,}95 = 1 - 2\phi(k) \Rightarrow \phi(k) = .975$

$\Rightarrow k = 1{,}96$



d) $z_{0,05}$? → Exploit simmetry to find on table



$z_{0,05} = -z_{0,95} = -1,645$