Gears

Mechanical power transmission can be accomplished though different means.

- Constant torque
  - Rigid
    - Friction wheel drive
    - Gear Drive
  - Flexible
    - Belt drive
    - Chain drive
    - Rope drive
- Variable torque
  - → Mechanical
  - → Electrical
  - → Hydraulic

In these lessons we are going to focus on gears.
↳ Gears have an ample array of design solution which we can choose from to solve the same problem, we are going to focus specifically on spur, helical and worm gears.

Definitions and terminology
↳ Cogwheel vs. gear → gears are made of cogwheels/cogs/gearwheels; gears are composed of 2 cogwheels.

Power goes from the driving wheel, or <u>pinion</u>, to the driven wheel, or crown.

Transmission Ratio: $\tau = \frac{\omega_2}{\omega_1} = \frac{d_{p1}}{d_{p2}} = \frac{z_1}{z_2}$

$\omega_1 \rightarrow$ input

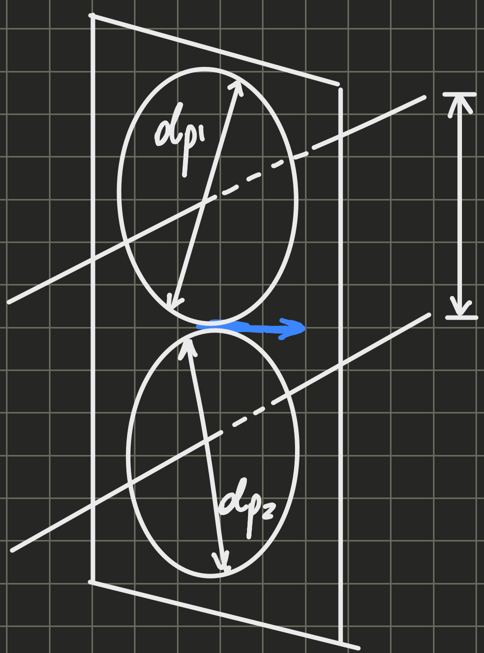
$\omega_2 \rightarrow$ output

Reciprocal

$\rightarrow$ Gear Ratio: $u = \frac{1}{\tau} = \frac{\omega_1}{\omega_2} = \frac{d_{p2}}{d_{p1}} = \frac{z_2}{z_1}$

Straight tooth spur gear

- $\rightarrow$ Cylindrical Body
- $\rightarrow$ Power transmission between parallel shafts.
- $\rightarrow$ Straight teeth $\rightarrow$ parallel to axis



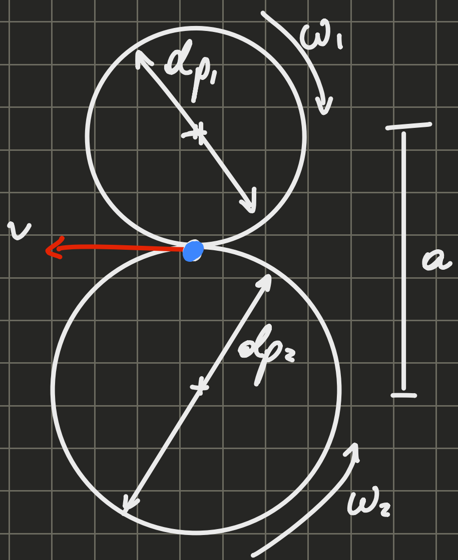
The most functional shape to transfer torque between two shafts is the circle.

The peripheral speed has to be the same

be the Same.

$d_{p1} \rightarrow$ pitch diameter of 1

$d_{p2} \rightarrow$ " " " 2



$v_1 = v_2 = v$

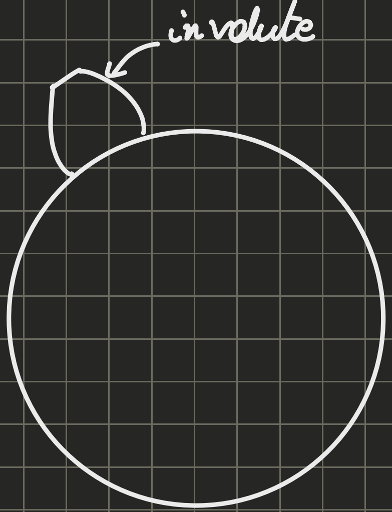
$$\omega_1 \frac{d_{p1}}{\cancel{2}} = \omega_2 \frac{d_{p2}}{\cancel{2}} \rightarrow \underbrace{\frac{\omega_2}{\omega_1}}_{\tau} = \frac{d_{p1}}{d_{p2}}$$

$$a = \frac{d_{p1} + d_{p2}}{2}$$

These are friction wheels, not gears, their mode of power transfer is through force, particularly friction. The problem with this approach is that it is limited in it's power transmission capabilities when compared to gears or other modes of transfer.

Gears, fix the avoid the issues of the force based method and utilise a shape based method, in which the power transmission is limited by the shape and strength of the teeth.

The best shape to transmit is the <u>involute curve of a circle</u>, since the shape is the best at avoiding sliding between teeth.



The contact point of two teeth, evolves by following line; this line is tangent to the pitch circle of both the gear wheels. This line is called the line of action.

The pitch point is the only point where we have rolling without sliding, in the rest of the points rolling occurs.

The base circle is used to draw the involute, rather than the pitch circle, the base circle is useful for the manufacturing and the kinematics.

The relationship between the base and pitch diameter is:

$$\left\{ \frac{d_b}{2} = \frac{d_p}{2} \cos\alpha \right.$$

$\rightarrow$ α is the angle of the line of action, called pressure angle.

THE fundamental geometrical quantity, is the module, m, of the cogwheel:

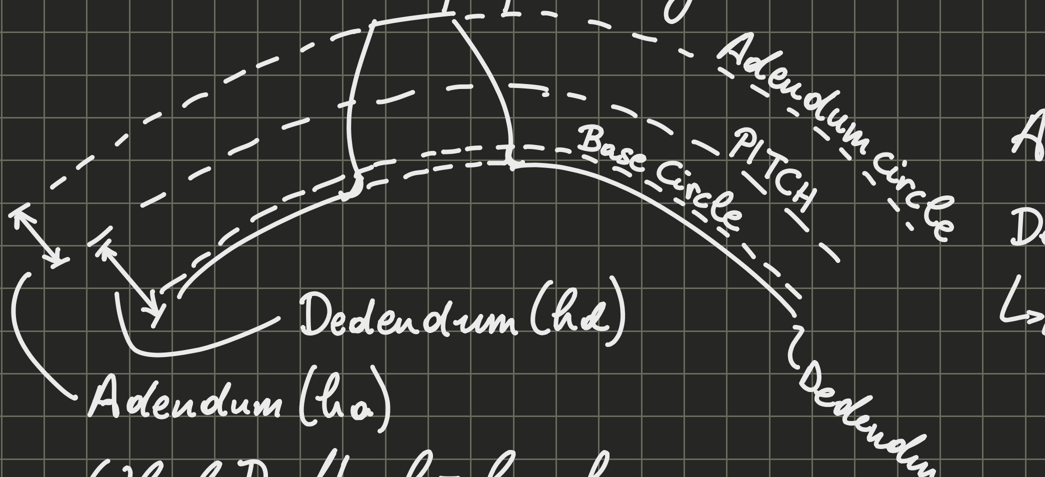
$$\boxed{m = \frac{d_p}{z} = \frac{p}{\pi}} \longrightarrow p \text{ is the circular pitch}$$

m is not geometrically derived, it is an abstract quantity. Having m, we can define almost all the proprieties of a gear. The one drawback of defining everything through m, is that we can only engage with the same module.

With the module we can derive some of the equations we defined before for the gear ratio:

$$u = \frac{\omega_1}{\omega_2} = \frac{d_{p2}}{d_{p1}} = \frac{m z_2}{m z_1} = \frac{z_2}{z_1}$$

Geometric proportioning with module:



Whole Depth: $h = h_a + h_d$

Adendum: $h_a = m$

Dedendum: $h_d = 1{,}25\, m$

↳ Larger to include fillet radius.

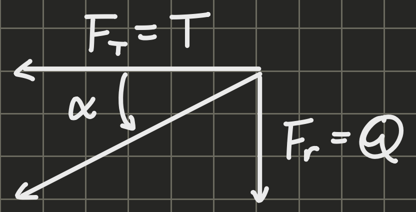
Length of action (g_α) = distance from start and end of interaction.

Contact Ratio: $\varepsilon_\alpha = \frac{g_\alpha}{d_{pB}}$

↳ On average, how many teeth are simultaneously engaged.

If $\varepsilon_\alpha < 1 \Rightarrow$ the movement is non-continuous, it's intermittent.

The thrusts are distributed differently depending on the point, as such we don't consider the thrust directly, and tend to decompose it into it's components.



We find that cause of the α, the two components are related. Therefore while evaluating, when T is increased by 1,6 then Q will also increase by 1,6. This relationship also tells that if we know one, we know the other.

$$F_t = \frac{2C_1}{d_{p_1}} = \frac{2C_2}{d_{p_2}} \quad \text{and} \quad F_r = F_t \tan\alpha$$

Helical Gears

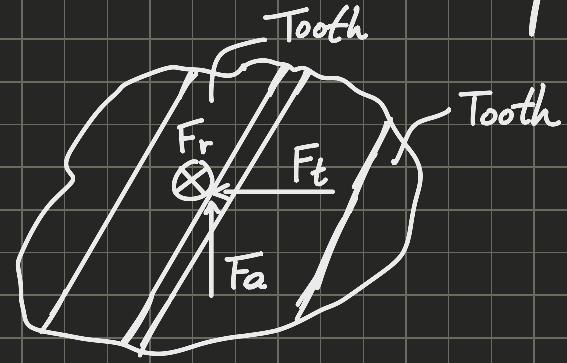
↳ Similor to spur gears but the teeth are inclined.

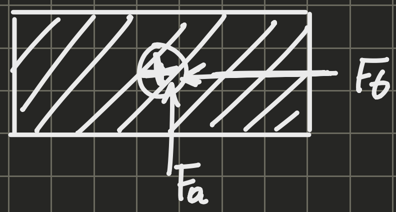
Advantages:

- Can transmit power since the greater area is able to bear more load.
- Quieter
- Less losses to sliding

Disadvantages:

→ An axial thrust will develop due to the inclination





→ To resist axial motion, we will need to add constraints.

→ Transmission ratios up to 8:1 (low maximum)

⇒ we'll need to use gear trains to increase overall ratio.

We have 2 planes, the transversal and normal ones.
Because of this the equations will change to reflect it.

$$p_n = \pi \cdot m_n \quad \Rightarrow \quad p_t = \frac{p_n}{\cos\beta} \qquad m_t = \frac{p_t}{\pi} = \frac{m_n}{\cos\beta} \qquad d_p = m_t \cdot z = \frac{m_n}{\cos\beta} \cdot z$$

$$\tan\alpha_t = \frac{\tan\alpha_n}{\cos\beta} \qquad \sin\beta_b = \sin\beta \cdot \cos\alpha_n \quad \text{On the base cylinder}$$

$$p_{bn} = \pi \cdot m_n \cdot \cos\alpha_n \quad \Rightarrow \quad p_{bt} = \pi \cdot m_n \cdot \frac{\cos\alpha_n}{\cos\beta_b} \qquad d_b = d_p \cdot \cos\alpha_t = z \cdot m_n \cdot \frac{\cos\alpha_n}{\cos\beta_b}$$

We will have two modules, one for each plane.
The meaning of every equation is the same, just now in more directions

Radial Contact Ratio:

$$\varepsilon_\alpha = \frac{g_\alpha}{p_{bt}}$$

Overlap Contact Ratio:

$$\varepsilon_\beta = \frac{b \cdot \tan\beta_b}{p_{bt}} = \frac{b\tan\beta}{p_t}$$

Total Contact Ratio: $\varepsilon_\gamma = \varepsilon_\alpha + \varepsilon_\beta$

$$\left.\begin{aligned} F_t &= \text{Same as before} \\ F_r &= F_t \cdot \tan\alpha_t = F_t \cdot \frac{\tan\alpha_n}{\cos\beta} \\ F_a &= F_t \cdot \tan\beta \end{aligned}\right\} \quad F = \sqrt{F_t^2 + F_r^2 + F_a^2}$$

Worm Gear

→ Unlike the other two, this type of gear works for shafts that are inclined with respect to eachother. Additionally, unlike the other two, the worm gear functions solely due to pure sliding, there is no rolling involved.

→ The "worm" is a long cylindrical gear, similar to a helical gear.

→ It is impossible for the worm to be pushed by it's companion gear wheel. This is a consequence of the kinematics being derived from sliding rather than roll. Another consequence of the pure sliding is that they produce more heat, and so are less efficient.

Advantages:
- High gear ratios
- Compact
- Silent
- Low vibrations

Disadvantages:
- Low efficiency
- Heat produced requires lubricant to dissipate.
- Limited power transmission

Worm gears are basically an extreme case of a helical gear with high angle, and a spur gear. Due to the high angle, the worm gear tends to not have many teeth /starts. Having less starts the efficiency drops, but the gear ratio is greater. The inverse is true with more starts.

In the geometry there are 4 independent quantities:
- number of starts (teeth for worms) of the worm. (z_1)
- number of teeth in non-worm gear. (z_2)
- diametral quotient ($q = d_1/m$) → d_1 → pitch diameter of worm
- module m

From these all the kinematics and geometries can be derived.

When looking at the thrusts, their name changes based on whether they are on the worm or the gear.
In worm gears, to the nominal diameters we need to add the components that are a result of the friction, which can also be derived.

Interference and undercutting

When we manufacture teeth, undercutting can occur. Undercutting occurs when a machine accidentally cuts the base of the teeth, weakening them.

To avoid undercutting we can find the smallest number of teeth necessary, with the equation:

$$z_{min} = \frac{2}{\sin^2 \alpha} \longrightarrow \text{For } \alpha = 20°,\ z_{min} = 17$$

To avoid undercutting while designing:
- Recalculate transmission to consider gear with $> z_{min}$ teeth.
- Adapt profile-shifted teeth which avoids undercutting but changes the shape of the teeth.

Day 2

Gearwheel Failure Modes

The methology for gearwheel sizing is based on in-service failure modes.

Gears has the highest number of possible failure points, we are going to have to do more than one check.

The 4 main failure modes are:

↳ fatigue fracture due to bending (bending) } Most common

↳ surface damage due to contact fatigue (pitting) }

↳ adhesive wear (scuffing/consumative)

↳ wear → removal of material cycle by cycle.

Bending causes the highest stresses at the bottom of the tooth.

Pitting is surface corrossion, it doesn't occur deeper.

↳ overtime the corrosion will coalesce and cause failure

↳

Scuffing is a type of wear that includes plasticity.

↳ ⇒ wear + change in material proprieties

Design of gearwheels → ISO 6336

- → 2 steps:
  - → Pre-sizing in simplified conditions/constraints
  - → Checks with complete condition
    - → Check on bending
    - → Check on contact fatigue
    - → Other checks based on condition set by conditions

We are looking at the checks first, then the pre-sizing, even though in normal operations they are done the other way around.

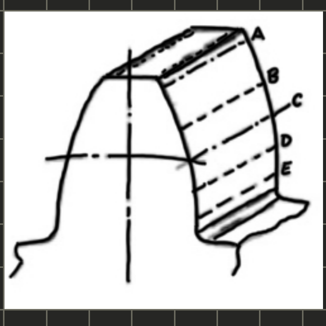
Checks

Bending Check

The tooth can be seen as a cantilever beam with fatigue load pulsating at 0 ⇒ $R=0$

We always calculate considering the average teeth in contact.

As an example we take a contact ratio between 1 and 2



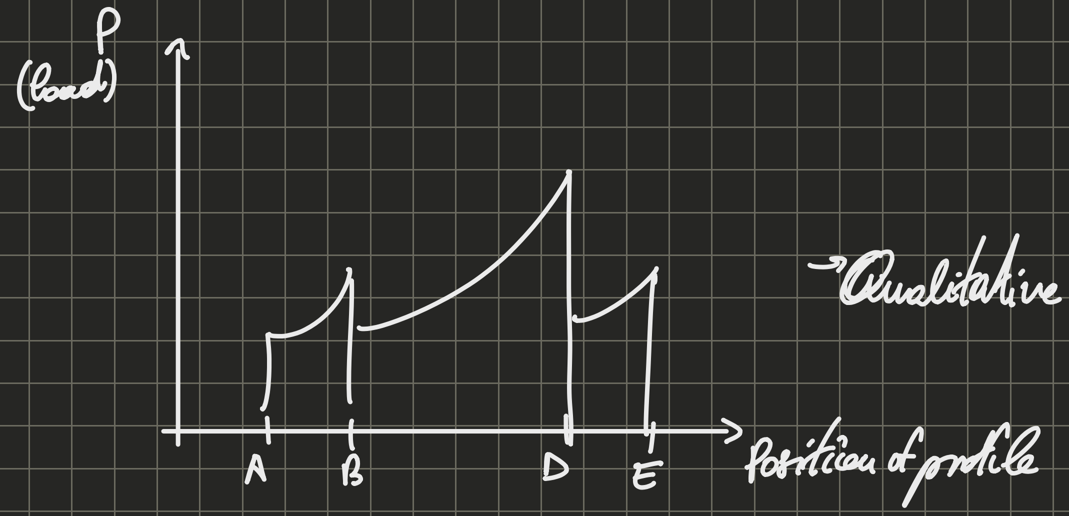
A is not a critical point because when the contact is at this point, there are 2 points of contact, so the stress is distributed between the two teeth in contact.

B is the highest point of contact, and this since tooth engagement continues until D, at which point we have 2 teeth in contact again, and so the stress is shared again.

Is B or D worse?

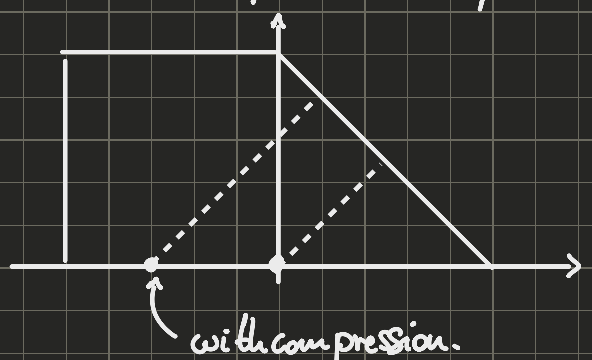
This isn't an easy problem since it's not easier.

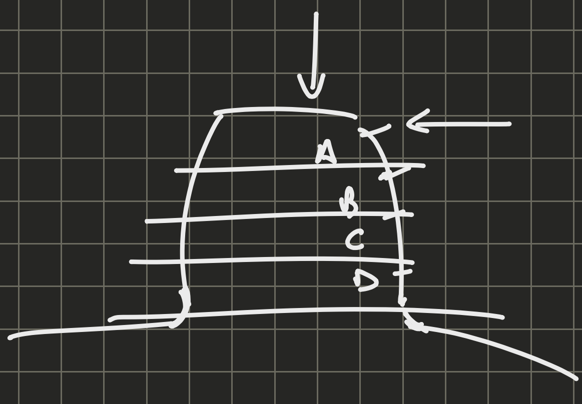
Diagram of distribution of load:



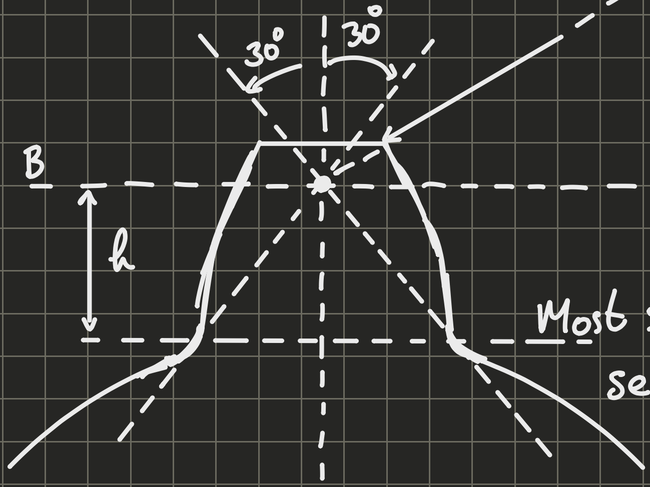
But the area is not linear so it's not immediately easy to determine.

Looking at the standard, from a safety perspective, we are told to take biggest load (that in B) and place in A.
To not be too safety we are told to add an adjustment coefficient.
We can also decompose the force in its horizontal and vertical directions.
The horizontal causes bending, while the one in the vertical direction causes compression, which can only help us.
Compression helps us because it increases the fatigue limit.





How do we find the critical section?



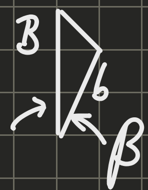
This is an exact approach but it is close enough that we are ok with it.

The most stressed section is the section with the most bending moment.

For this section we can calculate

$\measuredangle \alpha$ (pressure angle)

$$\rightarrow \sigma_{MAX} = \frac{M_f}{W_f} = \frac{F_{bn} \cdot \cos\alpha_n \cdot h_{Fa}}{\frac{1}{6} \cdot \frac{b}{\cos\beta} \cdot s_{Fn}^2}$$



→ For helical gears $B = \frac{b}{\cos\beta}$ ⇝ for $\beta = 0 \Rightarrow$ spur gear

→ Considering the kinematics of helical gears, we can write:
(but the result is also, with adjustments, valid for spur gears:

$$F_{bn} = \frac{F_t}{\cos\beta \cdot \cos\alpha_n} \Longrightarrow \frac{F_{bn}}{b/\cos\beta} = \frac{F_t}{b\cos\alpha_n}$$

$$\Longrightarrow \sigma_{MAX} = \frac{6F_t\cos\alpha_{an} \cdot h_{Fa}}{b \cdot \cos\alpha_n \cdot s_{Fn}^2}$$

Normalizing by m_n^2 :

$$\sigma_{MAX} = \frac{F_t}{b \cdot m_n} \cdot \underbrace{\frac{6\left(\frac{h_{Fa}}{m_n}\right) \cdot \cos\alpha_{an}}{\left(\frac{S_{Fn}}{m_n}\right)^2 \cdot \cos\alpha_n}}_{Y_{Fa}} = \frac{F_t}{b \cdot m_n} \cdot Y_{Fa}$$

↳ Form factor, plotted in ISO 6336

The check on our system is:

$$\sigma_F \leq \sigma_{FP} \rightarrow \text{Permissible}$$

Where, after applying additional adjustment coefficients:

$$\sigma_F = Y_{Fa} \, Y_{sa} \, Y_\varepsilon \, Y_\beta \cdot \frac{F_t}{b \cdot m_n} \cdot \left(K_A \cdot K_v \cdot K_{F\alpha} \cdot K_{F\beta} \right)$$

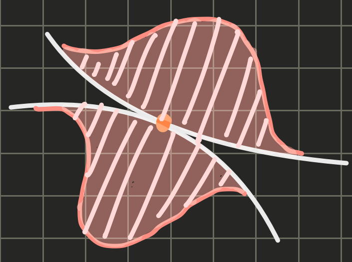
$$\sigma_{FP} = \frac{\sigma_{Flim} \cdot Y_{st} \cdot Y_{Nt}}{S_{Fmin}} \cdot Y_{srelT} \cdot Y_{RrelT} \cdot Y_x$$

While these equations help us resolve the system, they occupy too much time to solve, so we are not going to use them.

Pitting Check

Pitting is the result of contact (the problem immediately becomes non-linear).

The problem with contact is that there is a large concentration of stresses on what is typically a small area.



Since contact is an extremely local effect, we can no longer consider the material as isotropic or homogenous.

Due to its cyclic nature (being on a gear), the contact pressure induces fatigue. As the cycles go on, local issue with the material will develop and the nucleation of pits begins.

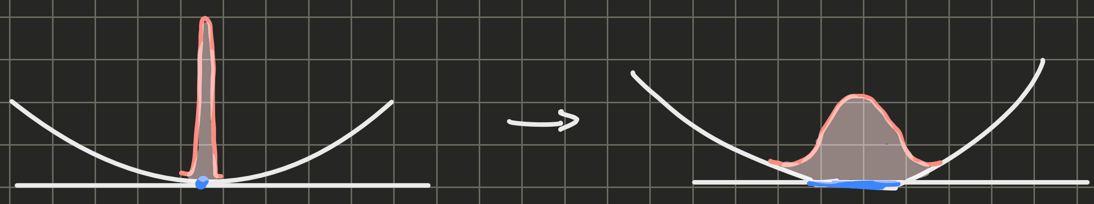
Contact pressure is a function of the external loads, the geometry and materials.

While there are many contact theories which attempt to describe the phenomena of contact, the most easily applicable in our case is the Hertz model of contact.

The Hertz model works off of a few theories:

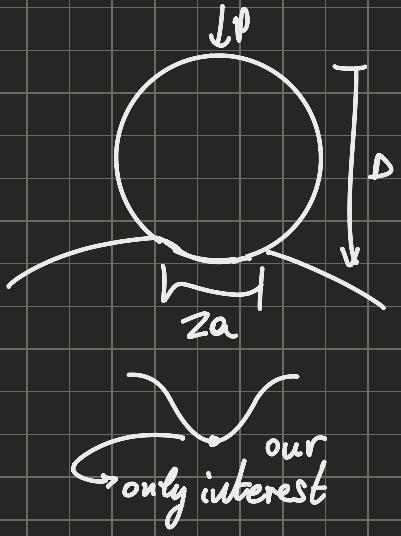
- → No plasticities considered
- → Each body is a half-space (⇒ the contact area is small relative to the body)
- → the surface is continuous and non-conforming (non-parallel planes and single contact point)
- → surfaces are frictionless

For contact, the body needs to be deformable, since if they are completely rigid the contact point is singular and the stress is unnatural.



Completely rigid → Deformable

According to Gauss we are able to describe any geometry with two curvatures on perpendicular planes. This also allows us to consider generic bodies, and to find that the contact surface between two spherical bodies is an ellipse.



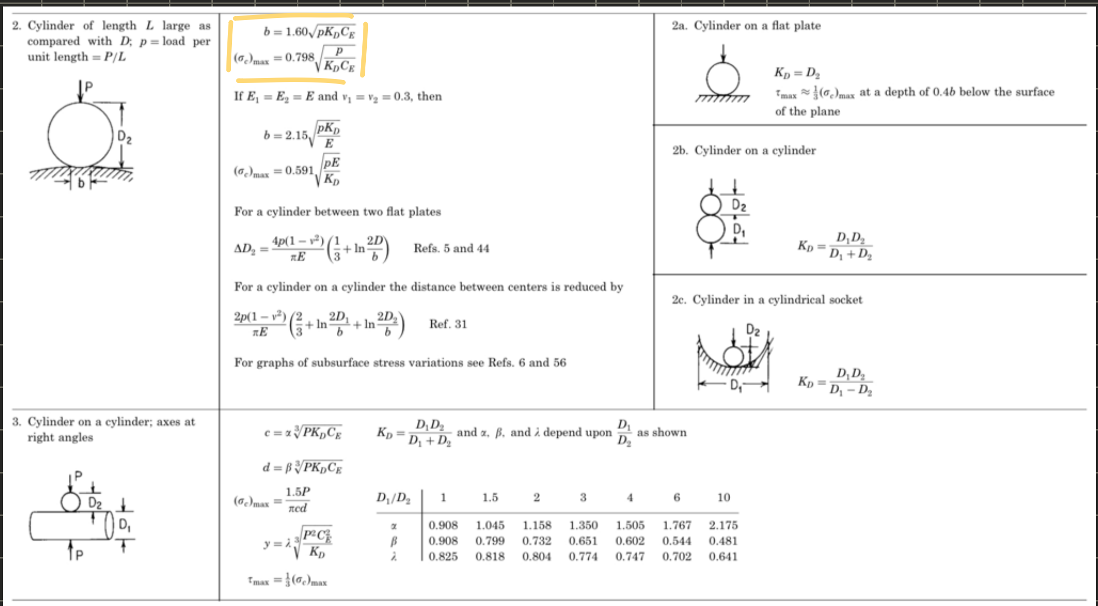
For spheres our stresses are:

$$a = 0{,}721 \sqrt[3]{P K_D C_E}$$

$$\sigma_{max} = \frac{1{,}5P}{\pi a^2} = 0{,}918 \sqrt[2]{\frac{P}{K_D^2 C_E^2}}$$

As we can see, contact is always non-linear.

For cylinders on the other hand, the contact region is rectangular.



2. Cylinder of length L large as compared with D; p = load per unit length = P/L

$$b = 1.60\sqrt{pK_DC_E}$$

$$(\sigma_c)_{max} = 0.798\sqrt{\frac{p}{K_DC_E}}$$

If $E_1 = E_2 = E$ and $\nu_1 = \nu_2 = 0.3$, then

$$b = 2.15\sqrt{\frac{pK_D}{E}}$$

$$(\sigma_c)_{max} = 0.591\sqrt{\frac{pE}{K_D}}$$

For a cylinder between two flat plates

$$\Delta D_2 = \frac{4p(1-\nu^2)}{\pi E}\left(\frac{1}{3} + \ln\frac{2D}{b}\right) \quad \text{Refs. 5 and 44}$$

For a cylinder on a cylinder the distance between centers is reduced by

$$\frac{2p(1-\nu^2)}{\pi E}\left(\frac{2}{3} + \ln\frac{2D_1}{b} + \ln\frac{2D_2}{b}\right) \quad \text{Ref. 31}$$

For graphs of subsurface stress variations see Refs. 6 and 56

2a. Cylinder on a flat plate

$K_D = D_2$

$\tau_{max} \approx \frac{1}{3}(\sigma_c)_{max}$ at a depth of $0.4b$ below the surface of the plane

2b. Cylinder on a cylinder

$$K_D = \frac{D_1D_2}{D_1 + D_2}$$

2c. Cylinder in a cylindrical socket

$$K_D = \frac{D_1D_2}{D_1 - D_2}$$

3. Cylinder on a cylinder; axes at right angles

$$c = \alpha\sqrt[3]{PK_DC_E} \qquad K_D = \frac{D_1D_2}{D_1+D_2} \text{ and } \alpha,\ \beta, \text{ and } \lambda \text{ depend upon } \frac{D_1}{D_2} \text{ as shown}$$

$$d = \beta\sqrt[3]{PK_DC_E}$$

$$(\sigma_c)_{max} = \frac{1.5P}{\pi cd}$$

$$y = \lambda\sqrt[3]{\frac{P^2C_E^2}{K_D}}$$

$$\tau_{max} = \frac{1}{3}(\sigma_c)_{max}$$

D_1/D_2	1	1.5	2	3	4	6	10
α	0.908	1.045	1.158	1.350	1.505	1.767	2.175
β	0.908	0.799	0.732	0.651	0.602	0.544	0.481
λ	0.825	0.818	0.804	0.774	0.747	0.702	0.641

There one such of the this table forevery type of geometry.

The cylindrical case is important to us as it is the one we will be using for spur gears. We can consider two cylinders of different radius to represent the local geometry.

We use oscillating cylinders, which have contact on the generatrix with force F_N.

The size of the contact patch can be found as:

$$a = \sqrt{\frac{4}{\pi} \cdot \frac{F_N}{b} \cdot \left(\frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{E_2} \right) \cdot \frac{R_1 - R_2}{R_1 + R_2}}$$

The maximum (Hertz) pressure from contact is:

$$p_H = \frac{2 \cdot F_N}{\pi \cdot a \cdot b}$$

The problem with using a and F_N is that the associated values change case by case.

→ Lubrication

Before continuing with contact, all contact problems mean that we have to consider lubrication.

Lubricants are needed to keep the part cool, so the elastic properties don't decrease. Lubricants also generally increase the size of the contact patch, which decreases stresses.

Both high and low viscosity lubricants exist.

Pitting begins from the pitch diameter, this is because it is the only point where rolling without sliding occurs, meaning that the lubricant is useless, making pitting more likely there, then it spreads out from there.

So we know where to consider the check:

$$R_1 = \frac{d_{p_1}}{2} \sin\alpha \qquad R_2 = \frac{d_{p_2}}{2} \sin\alpha$$

We get:

$$p_H = Z_E \cdot Z_H \cdot \sqrt{\frac{F_t}{b \cdot d_{p_1}} \cdot \frac{u+1}{u}}$$

Unlike the bending check there aren't many more coefficients.

Nominal Stress Factors

↳ quantities that and simplify the most important parameters of wheels.

Two stress factors:

- ↳ For bending
- ↳ For pitting

Stress Factor for Bending (U_L)

The full expression for fatigue has many factoring coefficients. So to define U_L we limit ourselves to the non-adjusted quantities.

$$U_L = \frac{F_t}{b \cdot m_n}$$

F_t → Tangential Force

m_n → normal module for helical gears (m for spur gears)

b ↳ face width

U_L is a stress $\Rightarrow$ [MPa]

The stress at the critical section is proportional to U_L.

U_L^* is the indicative limit, it can range from 40 to 170, usually 120 is the limit which we should not pass.

Lower values are generally used for simply quenched and tempered steels.

<u>Stress Factor for Pitting</u> (k)

$$K = \frac{F_t}{d_{p_1} \cdot b}\left(1 + \frac{z_1}{z_2}\right) = \frac{F_t}{d_{p_1} \cdot b}\left(\frac{u+1}{u}\right)$$

We have once again kept only the physical parameters.

k is also a stress [MPa]

The Hertz pressure is proportional to the square root of k.

$\hookrightarrow$ Non-linear.

limit of k $\rightarrow$ k^*, unlike U_L^*, is not a range, it is defined depending on the specific case we are studying.

Apart from lubrication we can change the surface hardness to reduce the pitting, the problem is that it becomes brittle. This is why all gears have surface hardening.

We can also can also improve surface finish.

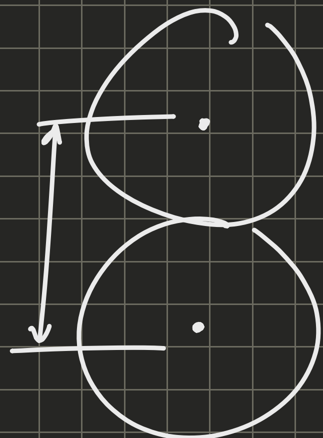
U_L^* and k^* depend on a lot of unknowns.

Protocol for pre-sizing

1. Determination of the general size of the wheel.
   - d_{p1}, d_{p2}, b (using k factor)
   - Cases: a) free center distance or given center distance

2) We get geometry of teeth, z_1, z_2, m_n (from U_L factor)

→ We only look at given center distance



$$\frac{d_{p1}}{2} + \frac{d_{p2}}{2} = a \qquad u = \frac{d_{p1}}{d_{p2}}$$

$$d_{p1} + u\, d_{p1} = a$$

$$(1+u)\, d_{p1} = a \rightarrow d_{p1} = \frac{2a}{u+1} \Rightarrow d_{p2} = u\, d_{p1}$$

We can get b as:

$$b = \frac{1}{k^*} \cdot \frac{2C_1}{d_{p1}^2} \cdot \frac{u+1}{u}$$

From k we can get the tooth number, and module, which are both standardized.

Spur Gear

$$U_L^* = \frac{F_t}{b \cdot m} \Rightarrow m = \frac{F_t}{b \cdot U_L^*}$$

We always try to use series 1 modules.

→ we find the closest value

$$\rightarrow z_1 = \frac{d_{p1}}{m} \rightarrow z_2 = u z_1$$

Helical Gears

$$U_L^* = \frac{F_t}{b \cdot m_n} \Rightarrow m_n = \frac{F_t}{b \cdot U_L^*} \longrightarrow m_t = \frac{m_n}{\cos\beta}$$

$$z_1 = \frac{d_{p1}}{m_t} \qquad z_2 = u z_1$$

Make sure to keep track.

The tooth numbers can only be integers.
It's best practice for z_1 and z_2 to be prime to each other.
This is so two teeth engage with every other tooth before they reengage, to reduce pitting and wear.

For the wheels with fewer teeth we always check for undercutting ($z \geq z_{min}$), if present we have to apply profile shift.
In many cases what we need will not perfectly match what we need, since we cannot relax the approximations so we have to accept some tollerances in certain values of the design. We might have to change some values to be able to adhere to stricter values.

On center distance we usually have 5-10% flexibility, while for gear ratio 2-3%.

Since we are using modular models with the values that we find we can find everything else.