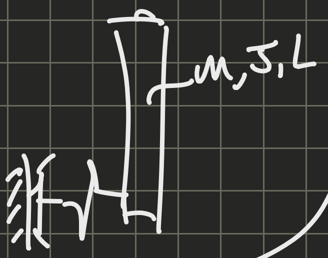
Lezione 3 -

In the last lesson we wrote the equation of motion of the pendulum.



$$\left(m\frac{L^2}{4}+J\right)\ddot{\theta}+k\,l^2\sin\theta\cos\theta+mg\,\frac{l}{2}\sin\theta=0$$

We are not interested in the whole motion, but small motions.

We can linearise

There is a bit of loose motion, allowing for small rotations

If we wanted to not have it we would have to bolt the sides.

It basically acts like the hinge in our system, with some flexural stiffness

If we wanted to solve it can only solve it by numerical integration and get the whole motion

But since we are interested in the small motion we can linearise the motion:

Non linear $\left(m\frac{L^2}{4}\right)\ddot{\theta} +$

$$\tilde{J}\ddot{\theta} + kl^2\theta + mg\frac{l}{2}\theta = 0$$

$\sin\theta \simeq \theta$
$\cos\theta \simeq 1$

$\underbrace{\tilde{J}\ddot{\theta}}_{\text{Was already linear}}$

$$\left[\tilde{J}\ddot{\theta} + \left(kl^2 + \frac{mgl}{2}\right)\theta = 0\right] \rightarrow$$ This is what we want

$$\underbrace{\tilde{J}\ddot{\theta}}_{\text{Mass/moment of inertia}} + \underbrace{k_T\theta}_{\text{Stiffness}} = 0$$

$\propto \ddot{\theta}$ $\quad$ $\propto \theta$

prop

Is this the smartest way to get the vibration around equilibrium?
↳ We have to do a lot of eliminations and interpretation

We want to go directly to the linear equation

$$E_c = \frac{1}{2}\left(m\frac{L^2}{4} + J\right)\dot{\theta}^2$$ → since it not dependent on θ we know we will always get the $\ddot{\theta}$

We start from here:

0-order Taylor term

0, we don't have kinetic energy at equilibrium

$$E_c = \frac{1}{2}m^*(q)\dot{q}^2 \underset{1}{\simeq} \frac{1}{2}\cancel{m^*(q)\dot{q}^2}\Big|_{\substack{q=q_0\\ \dot{q}=0}} + \frac{1}{2}\frac{\partial m^*(q)}{\partial q}\dot{q}^2\Big|_{\substack{q=q_0\\ \dot{q}=0}}$$

where $q = q_0$ is the equilibrium position, in which we also get $\dot{q}=0$ and $\ddot{q}=0$ also 0

→ We apply taylor to derive the work

$$\simeq \underbrace{\frac{1}{2}\cancel{m^*(q)\dot{q}^2}\Big|_{\substack{q=q_0\\ \dot{q}=0}}}_{\text{0-order Taylor term; }\to 0\text{, we don't have kinetic energy at equilibrium}} + \frac{1}{2}\cancel{\frac{\partial m^*(q)}{\partial q}\dot{q}^2}\Big|_{\substack{q=q_0\\ \dot{q}=0}} + \frac{1}{2}\cdot 2\,\cancel{m^*(q)\,\dot{q}}\Big|_{\substack{q=q_0\\ \dot{q}=0}}\dot{q} +$$

1st order terms are 0 null

$$+ \frac{1}{2}\cdot\frac{1}{2}\cancel{\frac{\partial^2 m^*(q)}{\partial q^2}\dot{q}^2}\Big|_{\substack{q=q_0\\ \dot{q}=0}}(q-q_0)^2 + 2\cdot\frac{1}{2}\cancel{\frac{\partial m^*(q)}{\partial q}\dot{q}}\Big|_{\substack{q=q_0\\ \dot{q}=0}}\dot{q}(q-q_0)$$

$$+ \underbrace{\frac{1}{2}m^*(q)\Big|_{\substack{q=q_0\\ \dot{q}=0}}\dot{q}^2}_{\neq 0\text{, since }\dot{q}^2\text{ is out of the evaluation}}$$

We apply $\frac{d}{dt}\left(\frac{\partial E_c}{\partial \dot{q}}\right) - \frac{\partial E_c}{\partial q} \Rightarrow = \underline{m^*(q_0)\,\ddot{q}}$

The reduced mass at the equilibrium position.

The kinetic energy is independent of the position, we don't care about the evaluation

This simplifies our kinematics, by non allowing dependence and making things linear.

Going to linear solutions makes solutions much easier to get to.

Can we do the same with the potential energy:

$V = V_k + V_g$

constant

0 → since we are at equilibrium

$$V(q) \simeq \cancel{V(q_0)} + \cancel{\left.\frac{\partial V}{\partial q}\right|_{q=q_0}(q-q_0)} + \frac{1}{2}\underbrace{\left.\frac{\partial^2 V}{\partial q^2}\right|_{q=q_0}}(q-q_0)^2$$

Taylor and $q=q_0$

Always constant

$$\boxed{\frac{1}{2}K(q-q_0)^2}$$

$$\frac{\partial V}{\partial q} = k(q-q_0) \qquad k = \left.\frac{\partial^2 V}{\partial q^2}\right|_{q=q_0}$$

We want to find this explicitly

Starting from $V = V_k + V_g$

$$V = \frac{1}{2}k\left(\Delta l_0 + \Delta l_d(q)\right)^2 + mg\,h(q)$$

$$\frac{\partial V}{\partial q} = k\left(\Delta l_0 + \Delta l_d(q)\right)\frac{\partial \Delta l_d(q)}{\partial q} + mg\frac{\partial h(q)}{\partial q}$$

Our target, then we evaluate in $q=q_0$

$$\frac{\partial^2 V}{\partial q^2} = \underbrace{k\Delta l_0 \cdot \overbrace{\frac{\partial^2 \Delta l_d(q)}{\partial q^2}}^{0 \text{ if } \Delta l_d \text{ is non-linear}}}_{k''} + k\overbrace{\left(\frac{\partial \Delta l_d(q)}{\partial q}\right)^2}^{k'} + k\cancel{\Delta l_d(q)}\cdot\frac{\partial^2 \Delta l_d(q)}{\partial q^2} + mg\underbrace{\frac{\partial^2 h(q)}{\partial q^2}}_{k''}$$

Constant that is $\neq 0$ if there is a preload

0 → the dynamic extension of the static equilibrium position

We always have this, since it's the kinematic relation, the elongation as a function of the position

Go back to these notes.

k' for  is $\Delta l = l\theta$

$$V \simeq V_0 + \frac{1}{2}(k_I + k_{II} + k_{III})(q-q_0)^2$$

$$k_I = k\left(\frac{\partial \Delta l_d(q)}{\partial q}\right)^2$$ → Generalized Stiffness

$$k_{II} = mg \underbrace{\frac{\partial^2 h(q)}{\partial q^2}}\Big|_{q=q_0}$$ → Gravitational Stiffness

→ Like projection of the gravity to counter the rotational forces.

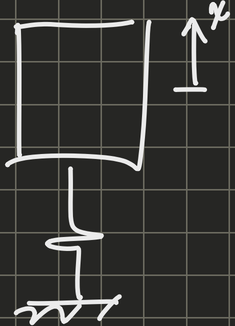
$$k_{III} = \underbrace{k\Delta l_0}\ \underbrace{\frac{\partial^2 \Delta l_d(q)}{\partial q^2}}\Big|_{q=q_0}$$ Souplesse

If it moves vertically, if $h_G(q)$ is linear, then h'' is 0.

If $\Delta l_0 = 0$, $k_{III} = 0$

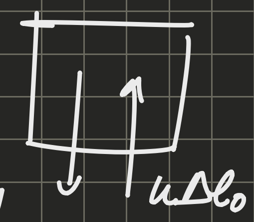
If $\Delta l_d(q)$ is like $h'' = 0$

↳ How the preload affects the lateral stiffness.



$\Rightarrow \Delta l_d = x \Rightarrow k_{III} = 0$

$h_G = x \Rightarrow k_{II} = 0$



They are in equilibrium at all positions so we don't care.

Always the initial example

$\Delta l = l\sin\theta$ $\qquad \frac{\partial \Delta l}{\partial \theta} = L\cos\theta$

$h_G = \frac{-L}{2}\cos\theta$ $\qquad - \quad \frac{\partial h_G}{\partial \theta} = \frac{L}{2}\sin\theta$

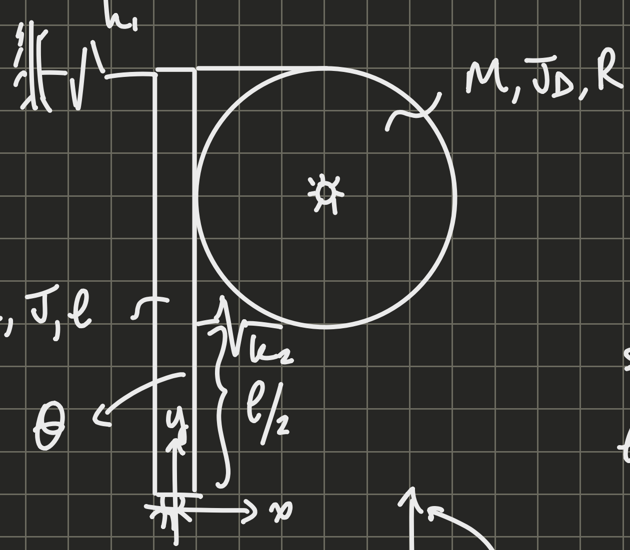
$$\tilde{J}\ddot{\theta} + \underbrace{kL^2\sin\theta\cos\theta} + mg\frac{L}{2}\sin\theta = 0$$

$$\Delta \ell = L\theta \qquad V = \frac{1}{2}(k_I + k_{II})\,\theta^2$$

$$\left.\frac{\partial^2 h}{\partial \theta^2}\right|_{\theta=0} = \frac{L}{2}$$

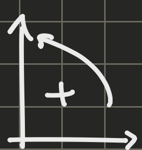
$$\Rightarrow J\ddot{\theta} + \Big(\underbrace{k L^2}_{k'} + \underbrace{mg\,\frac{L}{2}}_{k_3}\Big)\theta = 0$$

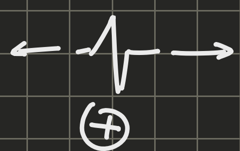
<u>Another example</u>



$2R = \frac{\ell}{2}$

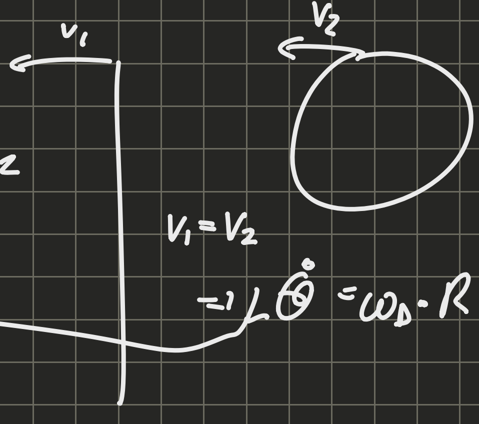
The springs are not preloaded since the system would then be hyperstatic.





$$E_c = \frac{1}{2} m v_G^2 + \frac{1}{2} J \omega_P^2 + \frac{1}{2} J_D \omega_D^2$$

We want to write $J^*(\theta_0)$



$$v_G^2 = \frac{L}{4}\dot{\theta}^2 \qquad \omega_G^2 = \dot{\theta}^2 \qquad \omega_D^2 = \frac{L^2}{R^2}\dot{\theta}^2$$

$v_1 = v_2$

$\Rightarrow L\dot{\theta} = \omega_D \cdot R$

$$\Rightarrow \omega_D = \frac{L}{R}\dot{\theta}$$

Passing through the linear terms reduces the required calculations by a lot.

$$E_c = \frac{1}{2}\underbrace{\left(m\frac{L^2}{4} + J + \frac{L^2}{R^2}J_D\right)}_{\hat{J}}\dot{\theta}^2$$

$$V_k = \frac{1}{2} k \Delta l_1^2 + \frac{1}{2} k_2 \Delta l_2^2$$

	θ
Δl_1	$-L$
Δl_2	$\frac{3}{2}L$

$$V_k = \frac{1}{2}\underset{\uparrow}{k_1 L^2}\theta^2 + \frac{1}{2}\underset{\uparrow}{k_2 \frac{9}{2}L^2}\theta^2$$

Generalized Stiffness

$$h_G = \frac{L}{2}\cos\theta$$

$$\frac{\partial h_G}{\partial\theta} = -\frac{L}{2}\sin\theta$$

$$\frac{\partial^2 h_G}{\partial\theta^2} = -\frac{L}{2}\cos\theta\Big|_{\theta=0} = -\frac{L}{2}$$

before it was helping the stiffness like another spring.

$$\hat{J}\ddot{\theta} + k_1 L^2\theta + k_2\frac{9}{4}L^2\theta - mg\frac{L}{2}\theta$$

$$\hat{J}\ddot{\theta} + \Big(\underbrace{k_1 L^2 + k_2\frac{9}{4}L^2}_{k_1} - \overbrace{mg\frac{L}{2}}^{k_2}\Big)\theta = 0$$

$k_2 \to$ Gravitation Stiffness
↳ reduces the stiffness of the system

k_1 → General Stiffness

↳ Negative because it tends to go down, and the equilibrium position is not stable

Looking at the exercise it will be a minus, so even if we make an error and get a plus we

Instability reduces our stiffness.

put a minus, if we
leave the sign it means
we have not understood
the system.