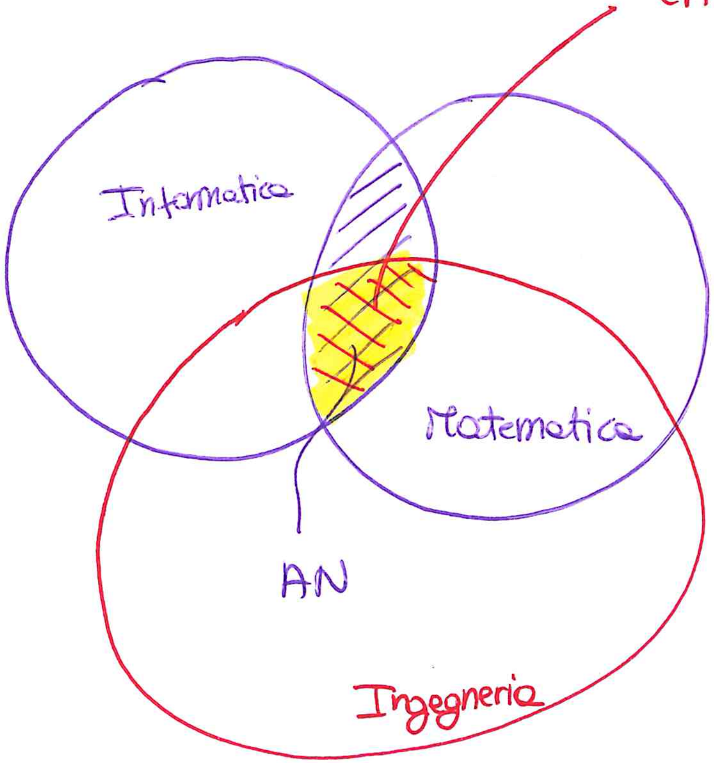
ANALISI NUMERICA (AN)

CALCOLO SCIENTIFICO

CALCOLO SCIENTIFICO (CS)



$CS \subset AN$

$$\frac{\text{Modellistica}}{\text{Metodi}}$$

mox.polimi.it

seconda parte del corso

MODELLI

CS

errore di modello

$e_m = x_F - x$

PROBLEMA FISICO x_F

legge di Ampere

MODELLO MATEMATICO

$I(f) = \int_a^b f(x)dx$ x

errore computazionale

$e_c = x - \tilde{x}$

$\tilde{x}$ PC

VALIDAZIONE

METODI

errore di troncamento

$e_t = x - x_N$

PROBLEMA NUMERICO

$I^*(f) = \sum_{i=1}^{N} f(x_i)\alpha_i$

x_N

errore di arrotondamento (roundoff)

$e_r = x_N - \tilde{x}$

AN

$e_c = e_t + e_r = x - \cancel{x_N} + \cancel{x_N} - \tilde{x} = x - \tilde{x}$

Prima parte del corso

STIMATORI DELL'ERRORE

errore assoluto

$$|x-\tilde{x}|$$

errore relativo → più utile

$$\frac{|x-\tilde{x}|}{|x|} \qquad x \neq 0$$

$x = 100$

$\tilde{x} = 100.1$

$|x-\tilde{x}| = 0.1$

$$\frac{|x-\tilde{x}|}{|x|} = \frac{0.1}{100} = 10^{-3} = 0.1\%$$

stesso assoluto

relativo diverso

$x = 0.2$

$\tilde{x} = 0.1$

$|x-\tilde{x}| = 0.1$

$$\frac{|x-\tilde{x}|}{|x|} = \frac{0.1}{0.2} = 0.5 = 50\%$$

floating point → errore dalla struttura dei floats, perché poi i computer non riescono a rappresentare numeri infiniti, quindi arrotondano ad un certo punto

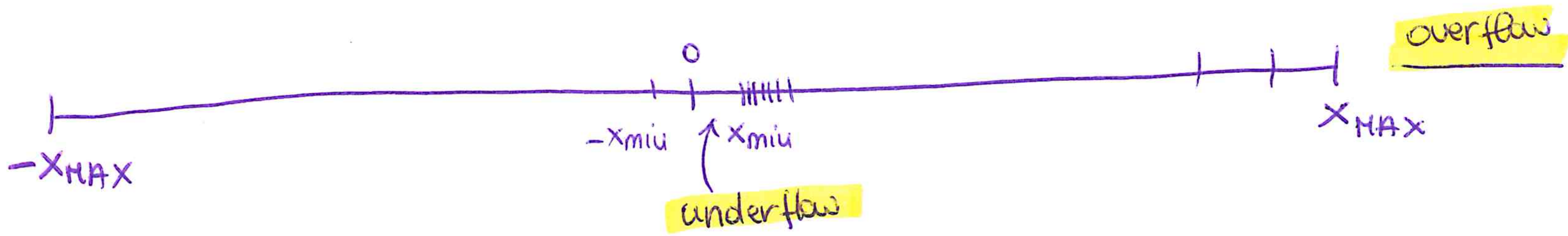
$\mathbb{R}$

arrotondamento

$$\forall x \in \mathbb{R} \rightsquigarrow fl(x) \in \mathbb{F}$$

$$x \neq fl(x)$$

$$\mathbb{F} \subset \mathbb{R}$$



$$\mathbb{R} \ni x \simeq fl(x) = (-1)^s \, (0.\underbrace{a_1 a_2 \dots a_t}_{\text{mantissa}}) \, (\underset{\text{base }(2)}{\beta})^{\overset{\text{esponente}}{e}}$$

→ struttura dei float

$\Uparrow$

$\mathbb{F}$

$s \nearrow 0$ positivi

$s \searrow 1$ negativi

$$0 \le a_i \le \beta - 1 \qquad i = 1, \dots, t$$

$$L \le e \le U$$

$\boxed{a_1 \neq 0}$ RAPPRESENTAZIONE NORMALIZZATA

↓

unicità di rappresentazione

0.003

$0.03 \cdot 10^{-1}$

$0.0003 \cdot 10$

$\boxed{0.3 \cdot 10^{-2}}$

$0.03 \longrightarrow 0.3 \cdot 10^{-1}$

$0 \in \mathbb{F}$

$\mathbb{F}(\beta, t, L, U)$

$\mathbb{F}(2, 53, -1021, 1024)$

cardinalità di $\mathbb{F} = \# \mathbb{F}$

$$= 2(\beta - 1)\beta^{t-1}(U - L + 1) + 1$$

$2 \to$ >0, <0

$(\beta - 1) \to a_1$

$\beta^{t-1} \to a_2, \dots a_t$

$(U - L + 1) \to e$

SISTEMI LINEARI

$Ax = b$

$A \in \mathbb{R}^{n \times n}$

$b, x \in \mathbb{R}^n$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n \end{cases}$$

A non singolare $\iff \exists!\, x \rightarrow$ esiste soluzione al sistema.

$\boxed{\text{CRAMER} + \text{LAPLACE}}$

$3(n+1)!$

$$x_i = \frac{\det(A_i)}{\det(A)} \qquad i = 1, \dots, n \qquad (n+1) \text{ determinanti}$$

$$\det(A) = \begin{cases} a_{11} & n = 1 \\ \sum_{j=1}^{n} a_{ij} \Delta_{ij} & n > 1 \end{cases}$$

$\Delta_{ij} \downarrow (-1)^{i+j} \det(A_{ij})$, $A_{ij} \in \mathbb{R}^{(n-1)\times(n-1)}$

$\rightarrow$ processo molto costoso computazionalmente

$n = 15$ sec

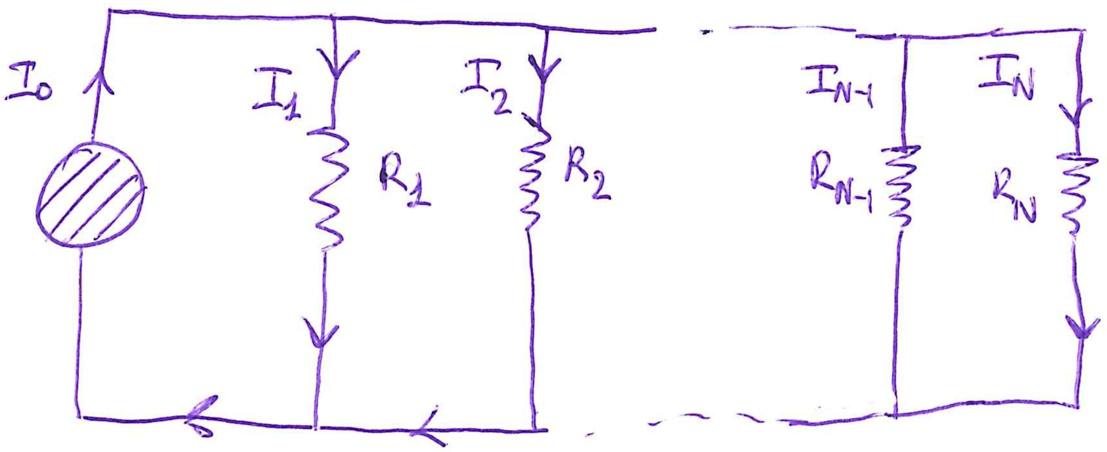
$n = 20$ giorni

$n = 25$ anni

Strassen $n^{3.8}$

$O(n!)$ è <u>molto</u> grande

$\Rightarrow$ servono altre soluzioni meno costose



I_0 nota

$I_1, I_2, \dots, I_N$? N incognite

$\hookrightarrow \vec{i} = [I_1, I_2, \dots, I_N]^T$

$V = R_k I_k$

$\boxed{\text{KIRCHHOFF}}$

$I_0 = I_1 + I_2 + \dots + I_N$ 1 equazione

$R_{k-1} I_{k-1} = R_k I_k \quad k = 2, \dots, N$ (N-1) equazioni

N equazioni

$\boxed{A\vec{i} = \vec{f}}$

$$\underbrace{\begin{bmatrix} 1 & 1 & \dots & \dots & 1 & 1 \\ R_1 & -R_2 & 0 & & & 0 \\ 0 & R_2 & -R_3 & 0 & \dots & 0 \\ & & & & & \\ 0 & & & 0 & +R_{N-1} & -R_N \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ \vdots \\ I_{N-1} \\ I_N \end{bmatrix}}_{\vec{i}} = \underbrace{\begin{bmatrix} I_0 \\ 0 \\ 0 \\ \\ 0 \\ 0 \end{bmatrix}}_{\vec{f}}$$

r × c = i

$\boxed{\text{diretti}}$ $\quad$ $\boxed{\text{iterativi}}$

$I_0 = I_1 + \dots + I_n$

$R_1 I_1 - R_2 I_2 = 0 \qquad R_1 I_1 = R_2 I_2$

$R_2 I_2 = R_3 I_3$

$I_1 = \frac{R_2}{R_1} \cdot \frac{R_3}{R_2} \cdot \ldots \cdot \frac{R_n}{R_4} I_n$

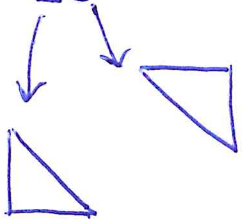
$I_2 = \frac{R_1}{R_2} \cdot I_1 - \frac{R_n}{R_2}$

Fattorizzazione LU

→ metodo per trovare x_i

$A \in \mathbb{R}^{n \times n}$ non singolare ($Ax = b$) → per aver soluzione

$A = LU$



$L, U \in \mathbb{R}^{n \times n}$

$0 \neq \det(A) = \det(LU) = \underbrace{\det(L)}_{l_{11} l_{22} \dots l_{nn}} \underbrace{\det(U)}_{u_{11} u_{22} \dots u_{nn}}$

$\Longleftrightarrow$

$l_{ii} \neq 0$ $\quad i = 1, \dots, n$

$u_{ii} \neq 0$

se no det() sarebbe 0

$Ax = b \qquad A = LU$

$L\underbrace{Ux}_{y} = b \Longleftrightarrow \begin{cases} Ly = b & (1) \\ Ux = y & (2) \end{cases}$

"sparsa"

$O(n) = a$ $\quad a = Cn$

$O(100)$

$A \in \mathbb{R}^{n \times n}$

n^2

$O(n^2)$

$O(n)$ sparsa

$n = 10^{\cancel{?}}; 10^2$

(sparse)

$Ly = b \qquad n = 3$

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$l_{11} y_1 = b_1 \rightarrow y_1 = \frac{b_1}{\underset{\neq 0}{l_{11}}}$$

$$l_{21} y_1 + l_{22} y_2 = b_2 \rightarrow y_2 = \frac{1}{\underset{\neq 0}{l_{22}}} \left[b_2 - l_{21} y_1 \right]$$

$$l_{31} y_1 + l_{32} y_2 + l_{33} y_3 = b_3 \rightarrow y_3 = \frac{1}{\underset{\neq 0}{l_{33}}} \left[b_3 - l_{31} y_1 - l_{32} y_2 \right]$$

$\boxed{\text{array}}$

$L \in \mathbb{R}^{n \times n}$

$$\boxed{\begin{aligned} y_1 &= \frac{b_1}{l_{11}} \\ y_i &= \frac{1}{\underset{\neq 0}{l_{ii}}} \left[b_i - \sum_{j=1}^{i-1} l_{ij} y_j \right] \qquad i = 2, 3, \dots, n \end{aligned}}$$

algoritmo delle sostituzioni in avanti

$\forall i \quad$ 1 divisione $\quad (i-1)$ sottrazioni $\quad (i-1)$ moltiplicazioni

$$\underbrace{\sum_{i=1}^{n} 1}_{n} + 2\sum_{i=1}^{n} (i-1) = n + 2\sum_{i=1}^{n} i - 2\underbrace{\sum_{i=1}^{n} 1}_{n} = -n + \cancel{2}\underbrace{\sum_{i=1}^{n} i}_{\frac{n(n+1)}{\cancel{2}}} = \cancel{-n} + n^2 + \cancel{n} = n^2$$

$$Ux = y$$

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$u_{33} x_3 = y_3 \rightarrow x_3 = \frac{y_3}{u_{33}} \quad u_{33} \neq 0$$

$$u_{22} x_2 + u_{23} x_3 = y_2 \rightarrow x_2 = \frac{1}{\underset{\neq 0}{u_{22}}} \left[y_2 - u_{23} x_3 \right]$$

$$u_{11} x_1 + u_{12} x_2 + u_{13} x_3 = y_1 \longrightarrow x_1 = \frac{1}{\underset{\neq 0}{u_{11}}} \left[y_1 - u_{12} x_2 - u_{13} x_3 \right]$$

$U \in \mathbb{R}^{n \times n}$

y_n

$$\boxed{\begin{aligned} x_n &= \frac{y_n}{u_{nn}} \\ x_i &= \frac{1}{u_{ii}} \left[y_i - \sum_{j=i+1}^{n} u_{ij} x_j \right] \qquad i = n-1, \dots, 1 \end{aligned}}$$

algoritmo delle sostituzioni all'indietro

n^2

pattern di sparsità (spy) → Matlab

$$\begin{bmatrix} \times & \times & \times & \cdots & \times \\ & \times & \times & \cdots & \times \\ & & \times & & \vdots \\ & & & \ddots & \\ & & & & \times \end{bmatrix}$$

$Ax = b$

$A = LU$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix} = \begin{bmatrix} \times & \times \\ \times & \times \end{bmatrix}$$

poniamo le diagonali come 1 per avere lo stesso numero di incognite che equazioni

$l_{11} = l_{22} = 1$

$$\begin{cases} a_{11} = \overset{1}{\boxed{l_{11}}}\, u_{11} \\ a_{12} = \overset{1}{\boxed{l_{11}}}\, u_{12} \\ a_{21} = l_{21}\, u_{11} \\ a_{22} = l_{21} u_{12} + \underbrace{l_{22}}_{1} u_{22} \end{cases}$$

a_{11}, a_{12}: R1(U)

a_{21}: C1(L)

a_{22}: R2(U)

4 eq.

~~6~~ 4 incognite

$$A = \underbrace{\begin{bmatrix} 1 & & 0 \\ \times & \ddots & \\ & \times & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} \times & \cdots & \times \\ & \ddots & \\ 0 & & \times \end{bmatrix}}_{U}$$

$$n\begin{bmatrix} & A & \end{bmatrix}_{n} = \begin{bmatrix} L & 0 \end{bmatrix} \begin{bmatrix} 0 & U \end{bmatrix}$$

n^2 equazioni

$\frac{n(n+1)}{2} + \frac{(n+1)n}{2}$ incognite $= n^2 + \cancel{n}$ incognite

$l_{ii} = 1 \quad i = 1, \dots, n$

In forma generale togliamo n incognite dalla diagonale di L, tutte 1 perché non possono esser 0, se no non c'è soluzione.

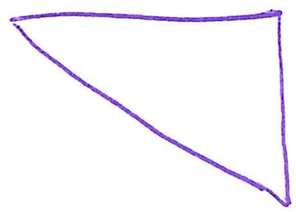
Eliminazione Gaussiana $(Ax = b)$

X, Y

$a, b \in \mathbb{R}$

$aX + bY$

$$A = A^{(1)} = \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} \\ \boxed{a_{21}^{(1)}} & a_{22}^{(1)} & a_{23}^{(1)} \\ \boxed{a_{31}^{(1)}} & a_{32}^{(1)} & a_{33}^{(1)} \end{bmatrix} \begin{matrix} b_1^{(1)} \\ b_2^{(1)} \\ b_3^{(1)} \end{matrix} \longrightarrow$$



STEP 1

Troviamo i moltiplicatori che eliminano ogni riga

$$\ell_{21} = \frac{a_{21}^{(1)}}{a_{11}^{(1)}} \quad a_{11}^{(1)} \neq 0 \qquad \ell_{31} = \frac{a_{31}^{(1)}}{a_{11}^{(1)}} \quad a_{11}^{(1)} \neq 0$$

E poi li applichiamo.

$$A^{(1)} \longrightarrow A^{(2)} = \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} \\ 0 & \boxed{a_{32}^{(2)}} & a_{33}^{(2)} \end{bmatrix} \begin{matrix} b_1^{(1)} \\ b_2^{(2)} \\ b_3^{(2)} \end{matrix} \xrightarrow{\text{STEP 2}} A^{(3)} = \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} \\ 0 & 0 & a_{33}^{(3)} \end{bmatrix} \begin{matrix} b_1^{(1)} \\ b_2^{(2)} \\ b_3^{(3)} \end{matrix} = U$$

$$b_2^{(2)} = b_2^{(1)} - \ell_{21} b_1^{(1)}$$
$$b_3^{(2)} = b_3^{(1)} - \ell_{21} b_1^{(1)}$$

$$b_3^{(3)} = b_3^{(2)} - \ell_{32} b_2^{(2)}$$

$$\boxed{R2_{new} = R2_{old} - \ell_{21}\, R1_{old}}$$

$$a_{21}^{(1)} - \ell_{21} a_{11}^{(1)} = a_{21}^{(1)} - \frac{a_{21}^{(1)}}{\cancel{a_{11}^{(1)}}} \cancel{a_{11}^{(1)}} = 0$$

$$a_{22}^{(2)} = a_{22}^{(1)} - \ell_{21} a_{12}^{(1)}$$
$$a_{23}^{(2)} = a_{23}^{(1)} - \ell_{21} a_{13}^{(1)}$$

$$\boxed{R3_{new} = R3_{old} - \ell_{31}\, R1_{old}}$$

$$a_{31}^{(2)} = a_{31}^{(1)} - \ell_{31} a_{11}^{(1)} = a_{31}^{(1)} - \frac{a_{31}^{(1)}}{\cancel{a_{11}^{(1)}}} \cancel{a_{11}^{(1)}} = 0$$

$$a_{32}^{(2)} = a_{32}^{(1)} - \ell_{31} a_{12}^{(1)}$$
$$a_{33}^{(2)} = a_{33}^{(1)} - \ell_{31} a_{13}^{(1)}$$

$$\ell_{32} = \frac{a_{32}^{(2)}}{\underset{\substack{\neq \\ 0}}{a_{22}^{(2)}}}$$

$$R3_{new} = R3_{old} - \ell_{32} \, R_{2\,old}$$

$$a_{32}^{(3)} = a_{32}^{(2)} - \ell_{32} \, a_{22}^{(2)} = \cancel{a_{32}^{(2)}} - \frac{\cancel{a_{32}^{(2)}}}{\cancel{a_{22}^{(2)}}} \cancel{a_{22}^{(2)}} = 0$$

$$a_{33}^{(3)} = a_{33}^{(2)} - \ell_{32} \, a_{23}^{(2)}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix} \qquad L A^{(n)} = A^{(1)}$$

I moltiplicatori compongono gli elementi non diagonali di L

Esempio:

$$A = A^{(1)} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & -1 \\ -1 & 1 & 5 \end{bmatrix} \longrightarrow A^{(2)} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & -3 \\ 0 & 3 & 6 \end{bmatrix} \longrightarrow A^{(3)} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & -3 \\ 0 & 0 & 15/4 \end{bmatrix} = U$$

$\ell_{21} = 2$

$\ell_{31} = -1$

$\ell_{32} = -\frac{3}{4}$

$2 - 2 \cdot 1 = 0$
$0 - 2 \cdot 2 = -4$
$-1 - 2 \cdot 1 = -3$

$-1 + (+1) \cdot 1 = 0$
$1 + (+1) \cdot 2 = 3$
$5 + (+1) \cdot 1 = 6$

$3 + \frac{3}{\cancel{4}} \cdot (-\cancel{4}) = 0$
$6 + \frac{3}{4} \cdot (-3) = 6 - \frac{9}{4} = \frac{15}{4}$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -3/4 & 1 \end{bmatrix} \quad \text{t.c.} \quad \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -3/4 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & -3 \\ 0 & 0 & 15/4 \end{bmatrix}}_{U} \overset{?}{=} \underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & -1 \\ -1 & 1 & 5 \end{bmatrix}}_{A}$$

È la matrice originale, quello che volevamo.

$$-1 \cdot 2 + \frac{3}{\cancel{4}} \cdot \cancel{4} = 1$$

$$-1 \cdot 1 + \frac{3}{4} \cdot 3 + \frac{15}{4} = -1 + \frac{9}{4} + \frac{15}{4} = \frac{20}{4} = 5$$

lu $\quad$ [L,U] = lu(A)

inv(A)

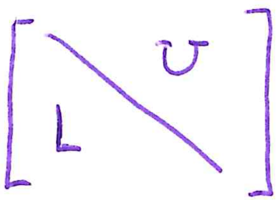
\ $\quad$ x = A\b $\quad$ sparse

1	(1,1)
1	(2,2)
⋮	
1	(10,10)

$\det(A) = u_{11} \cdots u_{nn}$

tril

triu



$$A = LU$$

$$\det(A) = \det(LU)$$
$$= \det(L)\det(U)$$
$$= \underbrace{(l_{11} \cdots l_{nn})}_{1}(u_{11} \cdots u_{nn})$$

$\frac{2}{3} n^3$ $\quad O(n^3)$

MEG = Metodo di Eliminazione Gaussiana

$$\boxed{Ax = b}$$

STRADA 1

* MEG
* $Ly = b$
* $Ux = y$

$$\boxed{\frac{2}{3} n^3 + 2n^2}$$

STRADA 2

$[A \mid b]$

* MEG

$[U \mid b^*]$

$b^* = [b_1^{(1)}, b_2^{(2)}, b_3^{(3)}]^T$

* $Ux = b^*$

$$\boxed{\left(> \frac{2}{3} n^3\right) + n^2}$$

solo 1 perché ne dobbiamo fare solo indietro e non avanti.

perché aggiungiamo la colonna dei b, aumenterà minimamente il costo.

STRADA 3

$x = A^{-1} b$

* A^{-1} $\quad \frac{8}{3} n^3$
* $x = A^{-1} b$ $\quad n^2$

$$\boxed{\frac{8}{3} n^3 + n^2}$$

$n = 10$

$n^2 = 10^2$

$n^3 = 10^3$

$$A x = \tilde{b}_1$$

$$A x = \tilde{b}_2$$

$$\vdots$$

$$A x = \tilde{b}_q$$

$$\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_q \in \mathbb{R}^n$$

$\boxed{\text{I}^a \text{ strada}}$

$*$ LU di A (1! volta)

$$A x = \tilde{b}_1 \quad \begin{cases} Ly = \tilde{b}_1 \\ Ux = y \end{cases}$$

$$A x = \tilde{b}_2 \quad \begin{cases} Ly = \tilde{b}_2 \\ Ux = y \end{cases}$$

$$\vdots$$

$$A x = \tilde{b}_q \quad \begin{cases} Ly = \tilde{b}_q \\ Ux = y \end{cases}$$

$$\boxed{\frac{2}{3} n^3 + q\,(2n^2)}$$

$\boxed{\text{II}^a \text{ strada}}$

$$[A \mid \tilde{b}_1] + Ux = \tilde{b}_1^*$$

$$[A \mid \tilde{b}_2] + Ux = \tilde{b}_2^*$$

$$\vdots$$

$$[A \mid \tilde{b}_q] + Ux = \tilde{b}_q^*$$

$$q\left(\frac{2}{3} n^3 + n^2\right)$$

la magnitudine è molto più grande

$$\Longrightarrow \quad A^{-1}$$

$$\begin{array}{cc} & \begin{array}{cccc} c_1 & c_2 & & c_n \end{array} \\ \begin{bmatrix} A \end{bmatrix} & \begin{bmatrix} & A^{-1} & \end{bmatrix} \end{array} = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots\ 1 \end{bmatrix}$$

$$A c_1 = [1, 0, \dots, 0]^T$$

$$A c_2 = [0, 1, 0, \dots 0]^T$$

$$\vdots$$

$$A c_n = [0, 0, \dots, 0, 1]^T$$

$$\frac{2}{3} n^3 + n\,(2n^2) = \frac{2}{3} n^3 + 2n^3$$

$$= \frac{8}{3} n^3$$

questa è la ragione perché trovare l'inversa costa, devi controllare ogni colonna e controllare che i termini funzionino.

$$A^{(1)} = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 2 & 2 \\ 3 & 6 & 4 \end{bmatrix} \longrightarrow A^{(2)} = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 0 & -4 \\ 0 & 3 & -5 \end{bmatrix} \longrightarrow A^{(3)} = U = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 0 & -4 \\ & & \end{bmatrix}$$

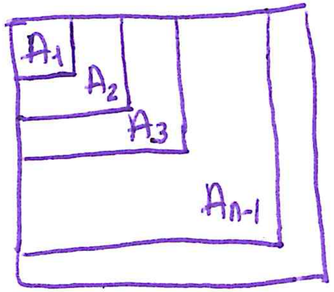
$\ell_{21} = 2$	$2 - 2 \cdot 1$	$3 - 3 \cdot 1$	$\cancel{\ell_{32} = \frac{3}{0}}$ $\not\exists$!
$\ell_{31} = 3$	$2 - 2 \cdot 3$	$6 - 3 \cdot 1$	$a_{22}^{(2)} = 0$
		$4 - 3 \cdot 3$	

$\boxed{a_{kk}^{(k)} \neq 0}$ pivot

$\boxed{\exists!\ LU}$ → NS $\Longleftrightarrow$ Tutte le sottomatrici sono non-singolari

→ S $\Longrightarrow$
1. Dominanza diagonale stretta per colonne
2. " " " righe
3. A è simmetrica definita positiva.

$\boxed{\text{NS}}$ Sia $A \in \mathbb{R}^{n \times n}$. Allora $\exists!$ LU di $A \Longleftrightarrow$ le sottomatrici principali di A denotate con A_i con $i = 1, \dots, n-1$ sono non singolari.

$A =$ 

↪ non richiedono, significa che potrebbero esistere L e U, ma se det(A) ↗ A stessa $= 0$, allora il sistema non ha soluzione.

$$A_1 = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$ singolare!

$$A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \not\exists\ LU$$

$$A_3 = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ \beta & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 2-\beta \end{bmatrix}}_{U} \quad \cancel{!}\ LU \quad \beta \in \mathbb{R}$$

5

1) matrici a dominanza diagonale stretta per righe

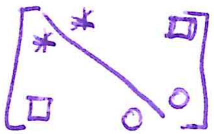
$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \qquad i = 1, \dots, n \qquad \begin{bmatrix} 4 & 0 & -1 \\ 3 & -7 & 2 \\ -2 & 1 & 9 \end{bmatrix} = A$$

2) matrici a dominanza diagonale stretta per colonne

$$|a_{jj}| > \sum_{\substack{i=1 \\ i \neq j}}^{n} |a_{ij}| \qquad j = 1, \dots, n$$

3) matrici simmetriche e definite positive (sdp)

$$A = A^T \qquad a_{ij} = a_{ji}$$



(A==A') (simmetria)

$$x^T A x > 0 \quad \forall x \in \mathbb{R}^n \text{ con } x \neq 0$$ (definita positiva)

$(1 \times n)(n \times n)(n \times 1)$

CHECK PRATICO

1) simmetria $(Av = \lambda v)$ (λ, v), $\lambda \in \mathbb{R}$, $v \in \mathbb{R}^n$

(↳ eig(A) "$\in \mathbb{R}$")

2) d.p.

eig(A) > 0 | tutti autovalori > 0

$Ax = b$

LU MEG

1) $\exists!$ $\boxed{LU \text{ di } A}$ anche se A è singolare (NS e S)

2) $\boxed{Ax = b}$ A non singolare

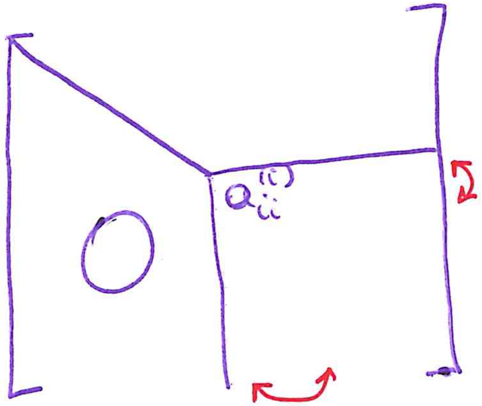
3) $\exists!$ $\boxed{LU \text{ di } A}$ anche quando NS e S non sono verificate se A non singolare

$$A = \tilde{A}^{(1)} = \begin{bmatrix} 1 & 1 & 3 \\ 3 & 6 & 4 \\ 2 & 2 & 2 \end{bmatrix} \longrightarrow \tilde{A}^{(2)} = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 3 & -5 \\ 0 & 0 & -4 \end{bmatrix} = U \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$l_{21} = 3$

$l_{31} = 2$

$l_{32} = 0$



$\boxed{a_{ii}^{(i)} = 0}$

pivoting → lo scambio tra righe o colonne in una matrice

(per righe, per colonne)

i) quando permutare

ii) quali righe scambiare ?

se $a_{i+1,i}^{(i)} \neq 0$ $\quad i \leftrightarrow i+1$

P = matrici di permutazione

$$PA^{(1)} = \tilde{A}^{(1)}$$

ortogonale $\left[PP^T = P^T P = I \right]$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

se P è prima, si scambiano le righe

$$PA^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 2 & 2 & 2 \\ 3 & 6 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 3 \\ 3 & 6 & 4 \\ 2 & 2 & 2 \end{bmatrix} = \tilde{A}^{(1)}$$

$$A^{(1)}P = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 2 & 2 \\ 3 & 6 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 6 \end{bmatrix}$$

se P è dopo, si scambiano le colonne

$I = P$ inizio

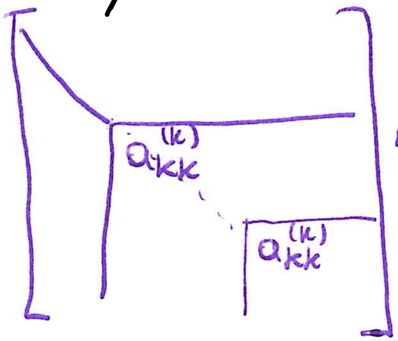
P inizia come I, poi ad ogni scambio si scambiano anche in P, agisce come un diario di tutti i cambi.

$n = 100$

$a_{kk}^{(k)} = 0 \quad k = 10, 80$

$10 \leftrightarrow 11$

$80 \leftrightarrow 81$



$\updownarrow = U$

$P \equiv I$	no pivoting
$P \not\equiv I$	pivoting

$$Ax = b \qquad PAx = Pb$$

$$\boxed{PA = LU} \qquad \underbrace{LU}_{y}x = Pb$$

$$\begin{cases} Ly = \boxed{Pb} \\ Ux = y \end{cases}$$

Bisogna ricordarsi di far questo per avere una soluzione coerente al sistema.

$$a_{kk}^{(k)} = 10^{-15}$$

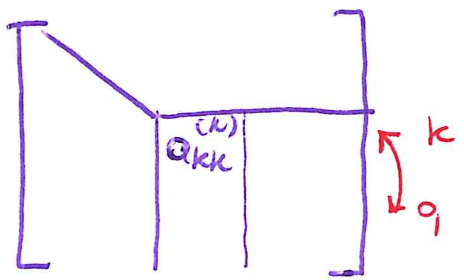
$$\ell_{k+1,k} = a_{k+1,k}^{(k)} / a_{kk}^{(k)}$$

$$\ell_{k+2,k} = a_{k+2,k}^{(k)} / a_{kk}^{(k)}$$

$$\vdots$$

$$\ell_{n,k} = a_{n,k}^{(k)} / a_{kk}^{(k)}$$

valori molto grandi



$$\max_{i=k+1,\dots,n} |a_{ik}^{(k)}| \qquad q$$

$$A = \begin{bmatrix} 1 & 1+0.5\cdot10^{-15} & 3 \\ 2 & 2 & 20 \\ 3 & 6 & 4 \end{bmatrix}$$

$$A - LU = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

→ se non utilizziamo il pivoting allora l'errore peggiora più va avanti il calcolo, perché la magnitudine dei moltiplicatori sarà immensa.

~~$[\tilde{L}, U] = lu(A);$~~ $A = \tilde{L} U$

$[L, U, P] = lu(A);$ $\longleftrightarrow$ $\boxed{PA = LU}$

$A = \underbrace{P^{-1} L}_{\tilde{L}} U$

questo è quello che dobbiamo fare se non ci esce $\tilde{L}$

(spy)

$Ax = b$

$PAx = Pb$

$PA = LU$

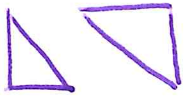
$P^{-1}LUx = P^{-1}Pb$

$P^{-1}L \underbrace{Ux}_{y} = b$

$$\begin{cases} P^{-1}Ly = b \\ Ux = y \end{cases}$$

O si può fare così ma non aiuta la risoluzione del sist

① A sdp $A = R^T R$ Cholesky $\frac{n^3}{3}$



L U

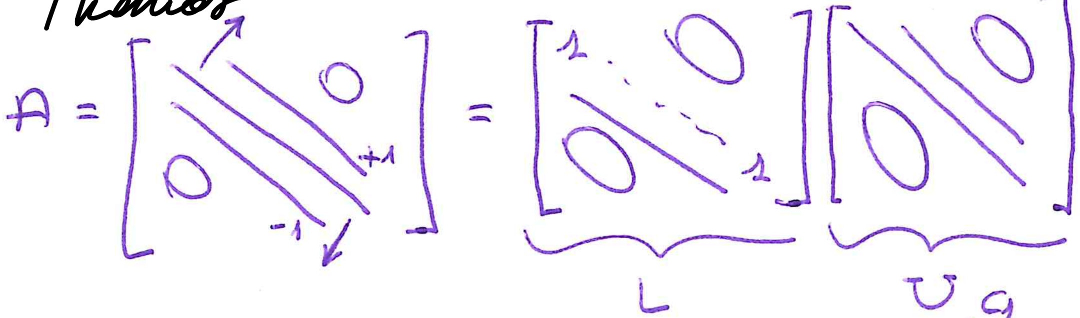
$r_{ii} \geq 0$

Cholesky

Algoritmi Specifico

Thomas

②



$$A = \begin{bmatrix} a_1 & c_1 & 0 \\ e_2 & a_2 & c_2 \\ 0 & e_3 & a_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ \beta_2 & 1 & 0 \\ 0 & \beta_3 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} \alpha_1 & \gamma_1 & 0 \\ 0 & \alpha_2 & \gamma_2 \\ 0 & 0 & \alpha_3 \end{bmatrix}}_{U}$$

$\gamma_1 = c_1$, $\gamma_2 = c_2$

$\cancel{\beta_2}, \cancel{\beta_3}$

$\cancel{\alpha_1}, \cancel{\alpha_2}, \alpha_3$

γ_1, γ_2
$\| \quad \|$
$c_1 \quad c_2$

[R1 * C2]
[R2 * C3]

$\boxed{\alpha_1}$ $\quad R1 * C1 \quad$ $\alpha_1 = a_1$

$\boxed{\beta_2}$ $\quad R2 * C1 \quad$ $\beta_2 \alpha_1 = e_2 \longrightarrow \beta_2 = \dfrac{e_2}{\alpha_1}$

$\boxed{\alpha_2}$ $\quad R2 * C2 \quad$ $\beta_2 \gamma_1 + \alpha_2 = a_2 \longrightarrow \alpha_2 = a_2 - c_1 \beta_2$

$\boxed{\beta_3}$ $\quad R3 * C2 \quad$ $\beta_3 \alpha_2 = e_3 \longrightarrow \beta_3 = \dfrac{e_3}{\alpha_2}$

$\boxed{\alpha_3}$ $\quad R3 * C3 \quad$ $\beta_3 \gamma_2 + \alpha_3 = a_3 \longrightarrow \alpha_3 = a_3 - c_2 \beta_3$

$$\boxed{\begin{array}{c} \alpha_1 = a_1 \qquad \beta_i = \dfrac{e_i}{\alpha_{i-1}} \qquad i = 2, \dots, n \\ \alpha_i = a_i - c_{i-1} \beta_i \end{array}}$$

$3(n-1)$

$\beta_i \quad 1 \quad \forall i$

$\alpha_i \quad 2 \quad$ 3 operazioni

$Ax = b \qquad LUx = b \qquad \begin{cases} Ly = b \\ Ux = y \end{cases}$

L, U bidiagonali

$$\begin{bmatrix} 1 & 0 & 0 \\ \beta_2 & 1 & 0 \\ 0 & \beta_3 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$y_1 = b_1$$

$$\beta_2 y_1 + y_2 = b_2 \rightarrow y_2 = b_2 - \beta_2 y_1$$

$$\beta_3 y_2 + y_3 = b_3 \rightarrow y_3 = b_3 - \beta_3 y_2$$

$$\boxed{\begin{aligned} & y_1 = b_1 \\ & y_i = b_i - \beta_i y_{i-1} \\ & \quad i = 2, \dots, n \end{aligned}} \qquad y_i \quad 2$$

$$2(n-1)$$

$$\begin{bmatrix} \alpha_1 & c_1 & 0 \\ 0 & \alpha_2 & c_2 \\ 0 & 0 & \alpha_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\alpha_3 x_3 = y_3 \rightarrow x_3 = \frac{y_3}{\alpha_3}$$

$$\alpha_2 x_2 + c_2 x_3 = y_2 \rightarrow x_2 = \frac{1}{\alpha_2}\left[y_2 - c_2 x_3\right]$$

$$\alpha_1 x_1 + c_1 x_2 = y_1 \rightarrow x_1 = \frac{1}{\alpha_1}\left[y_1 - c_2 x_2\right]$$

$$\boxed{\begin{aligned} & x_n = \frac{y_n}{\alpha_n} \\ & x_i = \left[y_i - c_i x_{i+1}\right]\frac{1}{\alpha_i} \\ & \quad i = n-1, \dots, 1 \end{aligned}} \qquad \begin{matrix} 1 & x_n \\ 3 & x_i \end{matrix}$$

$$3(n-1)+1$$

Non ci viene chiesto, però

Thomas

$$\underbrace{3(n-1)}_{LU} + \underbrace{2(n-1)}_{Ly=b} + \underbrace{3(n-1)+1}_{Ux=y} = 8(n-1)+1 = \boxed{8n-7}$$

MEG + pivoting $\Longrightarrow$ LU accurata $\overset{?}{\Longrightarrow}$ $Ax = b$ x accurata

$LU - PA = 0$ se

$A_n x_n = b_n$ Hiebert $n \geq 1$ $a_{ij} = \dfrac{1}{i+j-1}$

$$\begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 & \dots \\ 1/2 & 1/3 & 1/4 & \dots & \\ 1/3 & 1/4 & \dots & & \\ & & & & \end{bmatrix}$$

sdp

b_n : $x_n = [1, \dots, 1]^T$

$\tilde{x}_n$

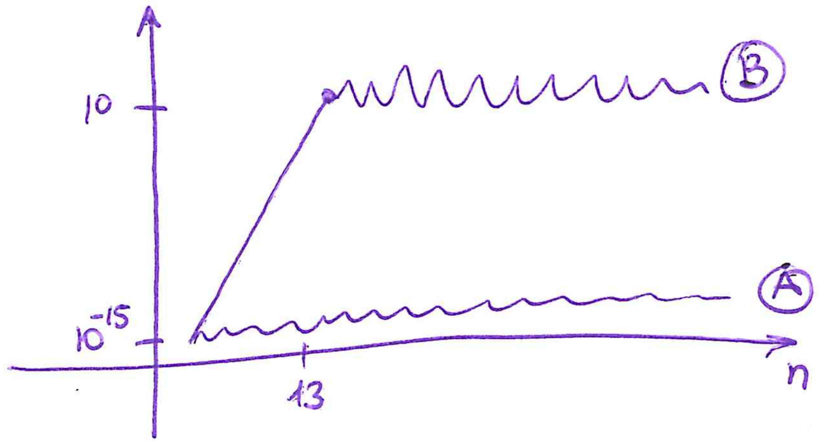
$R_n = L_n U_n - P_n A_n \in \mathbb{R}^{n \times n}$

A $\boxed{\max_{i,j} |r_{ij}|}$

$$\boxed{E_n = \frac{\|x_n - \hat{x}_n\|}{\|x_n\|}}$$ B

$n \geq 13$ $E_n \geq 10$

$\boxed{1000\ \%}$



$Ax = b$

perturb. soluzione

realtà

$(A+\delta A)(x+\delta x) = b+\delta b$ $\quad A, \delta A \in \mathbb{R}^{n\times n} \quad b, \delta b \in \mathbb{R}^n$ (noti)

$\tilde{x}$ $\quad x, \delta x \in \mathbb{R}^n$ (incognite)

perturbazione sui dati $\quad$ "

$$\frac{\|\delta A\|}{\|A\|} \quad \boxed{\frac{\|\delta b\|}{\|b\|}} \quad \boxed{\frac{\|\delta x\|}{\|x\|}}$$

$k(A)$ numero di condizionamento

$\delta A = 0$

$$\frac{\|\delta x\|}{\|x\|} \leq \underbrace{\sqrt{\frac{\lambda_{max}(A^TA)}{\lambda_{min}(A^TA)}}}_{k_2(A)} \frac{\|\delta b\|}{\|b\|}$$

A^TA

$\lambda_{max}, \lambda_{min}$

$k(A) = 1$ ideale

$k(A) = 100$

$$\frac{\|\delta b\|}{\|b\|} \simeq 10^{-5}$$

$$\frac{\|\delta x\|}{\|x\|} \leq 10^{-3}$$

$k(A) \geq 1$

$k(A) = \|A\| \, \|A^{-1}\|$

$$\boxed{\begin{aligned} k_1(A) &= \|A\|_1 \, \|A^{-1}\|_1 \\ k_\infty(A) &= \|A\|_\infty \, \|A^{-1}\|_\infty \\ k_2(A) &= \|A\|_2 \, \|A^{-1}\|_2 \end{aligned}}$$

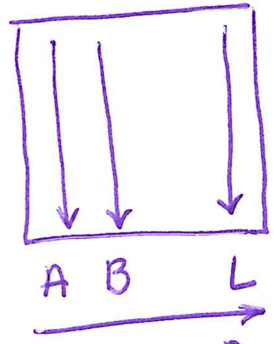
$$k_2(A) = \sqrt{\frac{\lambda_{max}(A^T A)}{\lambda_{min}(A^T A)}} = \|A\|_2 \, \|A^{-1}\|_2$$

$$\|A^{-1}\|_2 = \frac{1}{\sqrt{\lambda_{min}(A^T A)}}$$

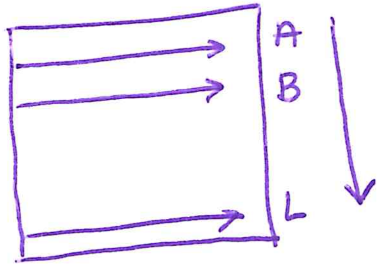
cond(A) $\|\cdot\|_2$

condest(A) $\|\cdot\|_1$ sparsa

$\|A\|_1 =$ max somma x colonne $= \max\limits_{j} \sum\limits_{i=1} |a_{ij}|$



$\|A\|_\infty =$ max somma x righe $= \max\limits_{i} \sum\limits_{j=1}^{n} |a_{ij}|$



$\|A\|_2 =$ norma spettrale $= \sqrt{\lambda_{max}(A^T A)}$

se A è simmetrica

$$\lambda_{max}(A^T A) = [\lambda_{max}(A)]^2$$

$$\|A\|_2 = \lambda_{max}(A)$$

$$\|A^{-1}\|_2 = [\lambda_{min}(A)]^{-1}$$

$$k_2(A) = \frac{\lambda_{max}(A)}{\lambda_{min}(A)}$$

$\delta A = 0$

$$\frac{\|\delta x\|}{\|x\|} \leq k(A) \frac{\|\delta b\|}{\|b\|} \quad (**)$$

Hilbert

$k(A_n) \nearrow$

$k(A_4) > 15.000$

$Ax = b$ ben condizionato $k(A)$ è piccolo

mal " $k(A)$ è grande

$k(A) \geq 1$

$\delta A \neq 0$ Se $\|\delta A\| \, \|A^{-1}\| < 1$ allora

$$\frac{\|\delta x\|}{\|x\|} \leq \frac{k(A)}{1 - k(A) \frac{\|\delta A\|}{\|A\|}} \left[\frac{\|\delta A\|}{\|A\|} + \frac{\|\delta b\|}{\|b\|} \right]$$

Oss: • Se $\|\delta A\| = 0$ trovo $(**)$

• $\|\delta A\| \, \|A^{-1}\| < 1$

$$\frac{\|\delta A\|}{\|A\|} < \frac{1}{\underbrace{\|A^{-1}\| \, \|A\|}_{k(A)}} \qquad \frac{\|\delta A\|}{\|A\|} < \frac{1}{k(A)}$$

$$1 - k(A) \frac{\|\delta A\|}{\|A\|} > 0$$

Metodi iterativi

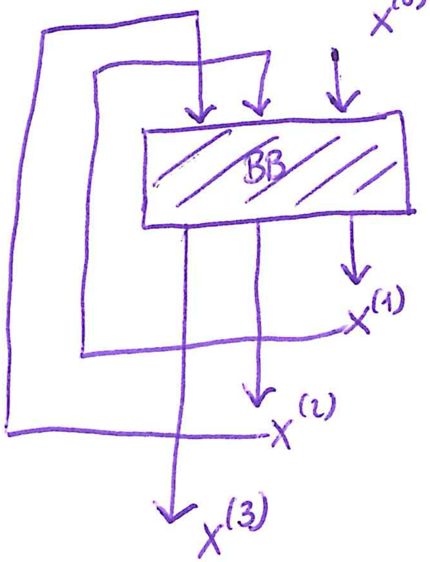
$Ax = b$

obbiettivo dei metodi iterativi

$\lim_{k \to +\infty} x^{(k)} = x$ convergenza

$\left[e^{(k)} = x - x^{(k)} \right.$

$\left. \lim_{k \to +\infty} e^{(k)} = \underset{\mathbb{R}^n}{0} \right]$



guess iniziale

Insieme di tutte le approssimazioni

$x^{(k)} \in \mathbb{R}^n$

$\{x^{(k)}\}$

$x^{(k)} \simeq x$

STOPPING CRITERIA

1) $k^* \in \mathbb{N}$ t.c.

$\|x - x^{(k^*)}\| \leq TOL$ (10^{-5})

controllo accuratezza

2) N_{max}

$\|x - x^{(k^*)}\| \leq \boxed{S \leq TOL}$

computabile

Forma generica

BB $\boxed{x^{(k+1)} = Bx^{(k)} + g}$ $k \geq 0$

$B \in \mathbb{R}^{n \times n}$

$g \in \mathbb{R}^n$

$(\forall B \ \forall g \ 1 \text{ metodo})$

consistenza

$x = Bx + g$

$\left[(I - B)x = g \right.$

$\left. (I - B)A^{-1}b = g \right]$

Obbiettivi di metodi numerici:

1. convergenza
2. consistenza
3. stabilità

$$\underbrace{x - x^{(k+1)}}_{e^{(k+1)}} = Bx + \cancel{g} - Bx^{(k)} - \cancel{g} = B(\underbrace{x - x^{(k)}}_{e^{(k)}})$$

$$\boxed{e^{(k+1)} = Be^{(k)} \quad \forall k \geq 0}$$

$$e^{(k)} = B e^{(k-1)}$$

$$\|e^{(k)}\| = \|Be^{(k-1)}\| \leq \|B\|_2 \|e^{(k-1)}\| \simeq \underbrace{\rho(B)}_{\text{raggio spettrale di } B} \|e^{(k-1)}\|$$

raggio spettrale di B max(abs(eig(B)))

L'autovalore più grande in termini assoluti.

se B è sdp max(eig(B))

$$\boxed{\|e^{(k)}\| \lesssim \rho(B) \|e^{(k-1)}\| \quad \forall k > 0}$$

$$\|e^{(k-1)}\| \lesssim \rho(B) \|e^{(k-2)}\|$$

$$\|e^{(k-2)}\| \lesssim \rho(B) \|e^{(k-3)}\|$$

$$\|e^{(k)}\| \lesssim \rho(B) \|e^{(k-1)}\| \lesssim [\rho(B)]^2 \|e^{(k-2)}\| \lesssim [\rho(B)]^3 \|e^{(k-3)}\| \ldots \lesssim [\rho(B)]^k \|e^{(0)}\|$$

$$\|e^{(k)}\| \lesssim [\rho(B)]^k \|e^{(0)}\|$$

$$\lim_{k \to +\infty} e^{(k)} = 0 \qquad \boxed{\rho(B) < 1} \quad \forall x^{(0)} \in \mathbb{R}^n$$

→ tale che ogni possibile $x^{(0)}$ converga, serve $\rho(B) < 1$

Teorema: Se lo schema $x^{(k+1)} = Bx^{(k)} + g$ è consistente allora lo schema è convergente $\forall x^{(0)} \in \mathbb{R}^n \iff \rho(B) < 1$. Più $\rho(B)$ è piccolo, più la convergenza è veloce.

$$\boxed{\begin{array}{c} B_1, g_1 \\ \rho(B_1) = 0.23 \end{array}} \qquad \begin{array}{c} B_2, g_2 \\ \rho(B_2) = 0.87 \end{array} \qquad \begin{array}{c} B_3, g_3 \\ \rho(B_3) = 1.0003 \end{array}$$

$$\|e^{(k)}\| \leq [\rho(B)]^k \|e^{(0)}\|$$

? k_{min} t.c. $\dfrac{\|e^{(k_{min})}\|}{\|e^{(0)}\|} \leq \boxed{[\rho(B)]^{k_{min}} \leq \varepsilon = (10^{-5})}$ $\Rightarrow K_{min} \geq \log_{\rho(B)} \varepsilon$

$\downarrow$

schema iterativo di Richardson

Schema di Richardson

$Ax = b \quad (Ax - b = 0) \quad \alpha_k \in \mathbb{R}$

$\alpha_k A x = \alpha_k b$

$\alpha_k A = P - (P - \alpha_k A)$

$\forall P \in \mathbb{R}^{n \times n}$ precondizionatore, invertibile ($\exists P^{-1}$)

non è la matrice di scambio

$Px - (P - \alpha_k A)x = \alpha_k b$

$Px = (P - \alpha_k A)x + \alpha_k b$

Consistente per costruzione, perché Richardson è derivata da una funzione già in se consistente

$x^{(k+1)} \quad x^{(k)}$

$x^{(k+1)} = Bx^{(k)} + g$

$Px^{(k+1)} = (P - \alpha_k A)x^{(k)} + \alpha_k b$

RICHARDSON

$$\boxed{x^{(k+1)} = \underbrace{(I - \alpha_k P^{-1} A)}_{B_{\alpha_k}} x^{(k)} + \underbrace{\alpha_k P^{-1} b}_{g_{\alpha_k}}}$$

$Px^{(k+1)} = \alpha_k (\underbrace{b - Ax^{(k)}}_{r^{(k)} \text{ (residuo)}}) + Px^{(k)}$

$x^{(k+1)} = x^{(k)} + \underbrace{\alpha_k \underbrace{P^{-1} r^{(k)}}_{z^{(k)}}}_{\text{correzione}} \quad k \geq 0$

$z^{(k)}$ (residuo precondizionato)

Forma più nota

correzione da approssimazione

$z^{(k)} = P^{-1} r^{(k)}$

$Pz^{(k)} = r^{(k)}$

P "semplice"

stazionario: $\alpha_k = \alpha \in \mathbb{R} \ \forall k$

↪ i.e $\alpha_k = \text{cost}$

dinamico: $\alpha_k \ \forall k$

↪ i.e $\alpha_k \neq \text{cost}$

Metodo di Jacobi

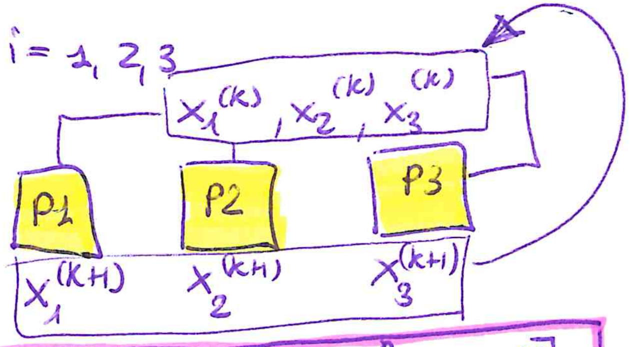
$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \longrightarrow x_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \longrightarrow x_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \longrightarrow x_3 \end{cases}$$

$$x_1^{(k+1)} = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)}\right] \quad P1$$

$$x_2^{(k+1)} = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1^{(k)} - a_{23}x_3^{(k)}\right] \quad P2$$

$$x_3^{(k+1)} = \frac{1}{a_{33}}\left[b_3 - a_{31}x_1^{(k)} - a_{32}x_2^{(k)}\right] \quad P3$$

$a_{ii} \neq 0 \quad i = 1, 2, 3$



A $\quad x^{(k+1)} = B_J x^{(k)} + g_J$

B $\quad x^{(k+1)} = x^{(k)} + \alpha_{k,J} P_J^{-1} r^{(k)}$

$$\boxed{x_i^{(k+1)} = \frac{1}{a_{ii}}\left[b_i - \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij}x_j^{(k)}\right] \quad i = 1, \dots, n}$$

$$x^{(k+1)} = D^{-1}b - D^{-1}(A-D)x^{(k)}$$

A $\quad x^{(k+1)} = \underbrace{(I - D^{-1}A)}_{B_J}x^{(k)} + \underbrace{D^{-1}b}_{g_J}$

$$x^{(k+1)} = x^{(k)} + D^{-1}b - D^{-1}Ax^{(k)}$$

$$x^{(k+1)} = x^{(k)} + D^{-1}\underbrace{(b - Ax^{(k)})}_{r^{(k)}}$$

B $\quad x^{(k+1)} = x^{(k)} + D^{-1}r^{(k)}$

$\alpha_{k,J} = \alpha_J = 1$

$P_J = D$

$$D = \begin{bmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{bmatrix} \qquad x^{(k)} = [x_1^{(k)}, \dots, x_n^{(k)}]^T \qquad x^{(k+1)} = [x_1^{(k+1)}, \dots, x_n^{(k+1)}]^T$$

$a_{ii}x_i^{(k+1)} = [Dx^{(k+1)}]_i \qquad b = [b_1, \dots, b_n]^T$

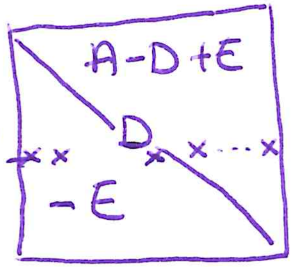
$$\boxed{Dx^{(k+1)} = b - (A-D)x^{(k)}}$$

Metodo di Gauss-Seidel

$$\boxed{x_i^{(k+1)} = \frac{1}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1} a_{ij}\,x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}\,x_j^{(k)}\right] \qquad \begin{array}{l} i=1,\dots,n \\ k\geq 0 \\ a_{ii}\neq 0 \end{array}}$$

Ⓐ $x^{(k+1)} = B_{GS}\,x^{(k)} + g_{GS}$

Ⓑ $x^{(k+1)} = x^{(k)} + \alpha_{k,GS}\,P_{GS}^{-1}\,r^{(k)}$

$A =$ 

$$-E = \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ a_{21} & \ddots & & \vdots \\ a_{31} & a_{32} & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{n\times n}$$

$$Dx^{(k+1)} = b + E\,x^{(k+1)} - (A-D+E)x^{(k)}$$

$$Dx^{(k+1)} = b + Ex^{(k+1)} + (D-E-A)x^{(k)}$$

$$\boxed{(D-E)x^{(k+1)} = b + (D-E-A)x^{(k)}}$$

$$x^{(k+1)} = (D-E)^{-1}b + (D-E)^{-1}((D-E)-A)x^{(k)}$$

Ⓐ $x^{(k+1)} = \underbrace{(I-(D-E)^{-1}A)}_{B_{GS}}x^{(k)} + \underbrace{(D-E)^{-1}b}_{g_{GS}}$

$$x^{(k+1)} = x^{(k)} + (D-E)^{-1}b - (D-E)^{-1}Ax^{(k)}$$

$$x^{(k+1)} = x^{(k)} + (D-E)^{-1}\underbrace{(b-Ax^{(k)})}_{r^{(k)}}$$

Ⓑ $x^{(k+1)} = x^{(k)} + (D-E)^{-1}r^{(k)}$

$\alpha_{k,GS} = \alpha_{GS} = 1$

$P_{GS} = D-E$

CONVERGENZA JACOBI e GAUSS-SEIDEL

JACOBI

CNS consistenza (x costruzione)

$\rho(B_J) < 1$ con $B_J = I - D^{-1}A$

CS

1) A dominanza diagonale stretta x righe

2) A dominanza " stretta x colonne

GAUSS-SEIDEL

CNS consistenza (x costruzione)

$\rho(B_{GS}) < 1$ $B_{GS} = I - (D-E)^{-1}A$

CS

1) A dominanza diagonale stretta x righe

2) A " " " x colonne

3) A sdp

Proposizione: Sia $A \in \mathbb{R}^{n \times n}$ sia tridiagonale, non singolare, con $a_{ii} \neq 0$ per $i = 1, \dots, n$.
Allora J e GS sono o entrambi convergenti o entrambi divergenti.
Se entrambi convergenti allora

$$\rho(B_{GS}) = [\rho(B_J)]^2.$$

ε

$\rho(B_J) = \frac{1}{4}$ $\qquad \left(\frac{1}{4}\right)^{k_{min}} \leq \varepsilon$ $\qquad \frac{1}{\varepsilon} \leq 4^{k_{min}}$ $\qquad \log_4 \frac{1}{\varepsilon} \leq k_{min}$

$\rho(B_{GS}) = \frac{1}{4^2}$ $\qquad \left(\frac{1}{4^2}\right)^{k_{min}} \leq \varepsilon$ $\qquad \frac{1}{\varepsilon} \leq 4^{2k_{min}}$ $\qquad \log_4 \frac{1}{\varepsilon} \leq 2k_{min}$

$$\frac{1}{2}\left[\log_4 \frac{1}{\varepsilon}\right] \leq k_{min}$$

Ex

$$A = \begin{bmatrix} 3 & -1 & 0 & \cdots & 0 \\ -2 & \ddots & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -2 & 3 \end{bmatrix}$$

$Ax = b$

b t.c. $x = [1, \dots, 1]^T$

$\rho(B_J) < 1$

$\rho(B_{GS}) < 1$

J $\qquad$ GS

$x^{(0)} = 0 \qquad \varepsilon = 10^{-12}$

J $\quad$ 277 iterazioni

GS $\quad$ 143 iterazioni

Proposizione: siano A e P sdp. Allora Richardson stazionario converge $\forall x^{(0)} \in \mathbb{R}^n$ se e solo se

$$0 < \alpha < \frac{2}{\lambda_{max}}$$ dove $\lambda_{max} = \max_i \lambda_i$ con λ_i autovalore di $P^{-1}A$.

La scelta ottimale per α è $$\alpha_{opt} = \frac{2}{\lambda_{max} + \lambda_{min}}$$ con λ_{min} autovalore minimo di $P^{-1}A$.

Inoltre

$$\|\underbrace{e^{(k)}}_{x - x^{(k)}}\|_A \le \underbrace{\left(\frac{k(P^{-1}A) - 1}{k(P^{-1}A) + 1}\right)}_{<1}^k \|\underbrace{e^{(0)}}_{x - x^{(0)}}\|_A$$

dove $\|w\|_A = \sqrt{w^T A w}$ $\forall w \in \mathbb{R}^n$ (norma in energia)

Dimostrazione:

$\rho(B_\alpha) < 1$ $\quad B_\alpha = I - \alpha P^{-1}A$

λ_i autovalori di $P^{-1}A$ (>0)

μ_i " di B_α

$\mu_i = 1 - \alpha\lambda_i$ $\quad |1 - \alpha\lambda_i| < 1$ {Dobbiamo garantire per avere $\rho(B) < 1$ $\forall i$

$$\underbrace{\underbrace{-1 < 1 - \alpha\lambda_i}_{\alpha\lambda_i < 2} < 1}_{-\alpha\lambda_i < 0}$$

$-\alpha\lambda_i < 0 \rightarrow \boxed{\alpha > 0}$

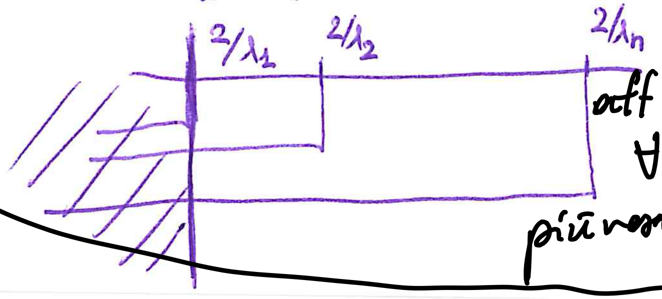
$\alpha\lambda_i < 2$

$\alpha < \frac{2}{\lambda_i}$ $\forall i$ $\rightarrow \boxed{\alpha < \frac{2}{\lambda_{max}}}$

affinché valga $\forall i$, la condizione più restrittiva è $\lambda_i = \lambda_{max}$

se vale per λ_{max}, vale $\forall \lambda_i$

$\lambda_1 > \lambda_2 > \dots > \lambda_n$



α ottimo t.c. $\left[\begin{array}{l}\text{velocità di convergenza è max} \\ \rho(B_\alpha) \text{ è minimo}\end{array}\right.$

$B_\alpha = I - \alpha P^{-1}A$

$1 - \alpha\lambda_i \qquad \lambda_i \quad P^{-1}A$

$\lambda_1 > \lambda_2 > \lambda_3$

$\boxed{P^{-1}A}$

$|1-\alpha\lambda_1|$

$|1-\alpha\lambda_2|$

$|1-\alpha\lambda_3|$

$\boxed{B_\alpha}$

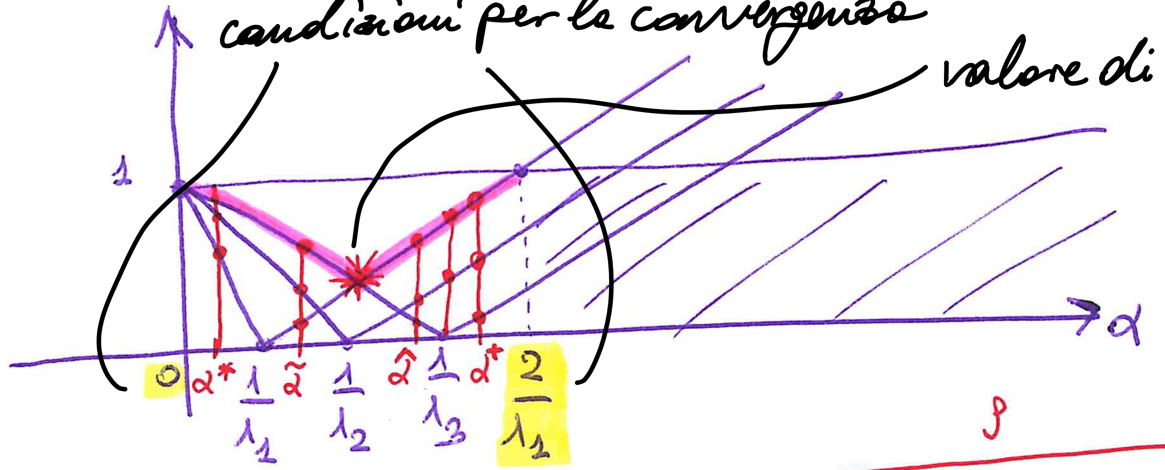
$\frac{1}{\lambda_1} < \frac{1}{\lambda_2} < \frac{1}{\lambda_3}$

$\alpha = 0 \qquad (0,1)$

$\alpha = \frac{1}{\lambda_i}$

condizioni per la convergenza

valore di α che tiene $\rho(B)$ il minimo possibile



α che minimizza il $\overbrace{\text{massimo autovalore in modulo}}^{\rho}$

$\cancel{\alpha\lambda_3 - 1} \qquad 1 - \alpha\lambda_3$

$\alpha\lambda_1 - 1 \qquad \cancel{1-\alpha\lambda_1}$

$\alpha\lambda_1 - 1 = 1 - \alpha\lambda_3$

$\alpha(\lambda_1 + \lambda_3) = 2$

$\alpha_{opt} = \frac{2}{\lambda_1+\lambda_3} \rightarrow \boxed{\alpha_{opt} = \frac{2}{\lambda_{max}+\lambda_{min}}}$

$\boxed{\rho\left(B_{\alpha_{opt}}\right)} = 1 - \alpha_{opt}\lambda_{min} = 1 - \frac{2\lambda_{min}}{\lambda_{max}+\lambda_{min}} = \boxed{\frac{\lambda_{max}-\lambda_{min}}{\lambda_{max}+\lambda_{min}}}$

$$\begin{cases} P \text{ non singolare} \\ P \text{ "facile"} \\ P \quad \underbrace{k(P^{-1}A)}_{\text{matrice A precondizionata}} \text{ più piccolo possibile} \end{cases} \qquad P z^{(k)} = r^{(k)}$$

$$\underbrace{\frac{\overbrace{k(P^{-1}A)}^{10^{..}} - 1}{k(P^{-1}A) + 1}}_{<1} \sim 1$$

$$\underbrace{\frac{\overbrace{k(P^{-1}A)}^{10} - 1}{k(P^{-1}A) + 1}}_{<1} = \frac{9}{11} < 1$$

Proposizione: siano A e P sdp. Allora Richardson dinamico converge $\forall x^{(0)} \in \mathbb{R}^n$ se

$$\alpha_k = \alpha_{opt,k} = \frac{(z^{(k)})^T r^{(k)}}{(z^{(k)})^T A z^{(k)}} \qquad k \geq 0 \qquad [P z^{(k)} = r^{(k)}]$$

Inoltre vale

$$\|e^{(k)}\|_A \leq \left(\frac{k(P^{-1}A) - 1}{k(P^{-1}A) + 1}\right)^k \|e^{(0)}\|_A \qquad k \geq 0$$

metodo del gradiente precondizionato

($P \equiv I$ metodo del gradiente

$\alpha_k = \alpha_{opt,k} = \dfrac{(r^{(k)})^T r^{(k)}}{(r^{(k)})^T A r^{(k)}}$)

4 passi $\quad x^{(0)} \longrightarrow r^{(0)} = b - Ax^{(0)}$

GP

G (3 passi)

Richardson stazionario (3 passi)

$k = 0, 1, \dots$

1) $\boxed{Pz^{(k)} = r^{(k)}} \longrightarrow z^{(k)}$

MEG + LU per determinare $z^{(k)}$

LU di P 1!

Non avevo letto

2) $\alpha_k = \dfrac{(z^{(k)})^T r^{(k)}}{(z^{(k)})^T A z^{(k)}} \longrightarrow \alpha_k$

3) $x^{(k+1)} = x^{(k)} + \alpha_k z^{(k)} \longrightarrow x^{(k+1)}$

4) $r^{(k+1)} = b - Ax^{(k+1)} = b - A(x^{(k)} + \alpha_k z^{(k)}) = \underbrace{b - Ax^{(k)}}_{r^{(k)}} - \alpha_k A z^{(k)} = r^{(k)} - \alpha_k A z^{(k)} \longrightarrow r^{(k+1)}$

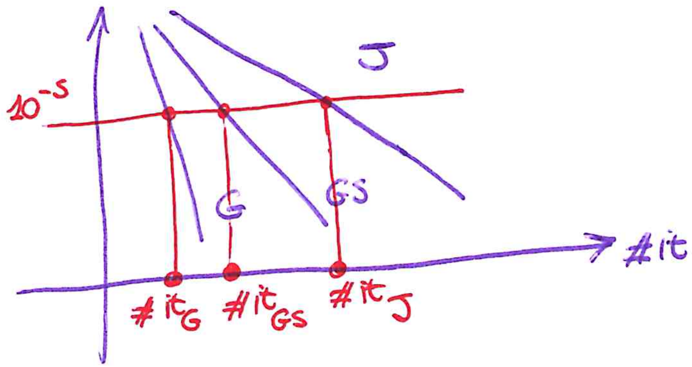
STOPPING CRITERIA

J, GS, G

$$\begin{cases} 2x_1 + x_2 = 1 \\ x_1 + 3x_2 = 0 \end{cases}$$

$$x = \left(\frac{3}{5}, -\frac{1}{5}\right)^T$$

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \text{ sdp}$$

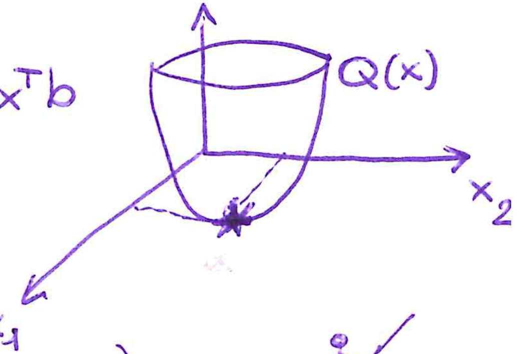


$$\left|\begin{aligned} x^{(k+1)} &= x^{(k)} + \alpha_k r^{(k)} \\ \alpha_k &= \frac{(r^{(k)})^T r^{(k)}}{(r^{(k)})^T A r^{(k)}} \end{aligned}\right.$$

A sdp

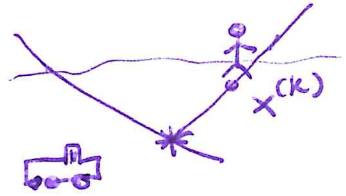
$Ax = b \iff$ minimizzare $Q(x) = \frac{1}{2} x^T A x - x^T b$

$x = [x_1, x_2]^T$



$\nabla Q(x) = Ax - b$

$\nabla Q(x) = 0 \qquad Ax - b = 0 \qquad Ax = b$



$x^{(k+1)} = x^{(k)} + \alpha_k d^{(k)}$

steepest descent method

$d^{(k)} = -\nabla Q(x^{(k)}) = -[Ax^{(k)} - b] = b - Ax^{(k)} = r^{(k)}$

$x^{(k+1)} = x^{(k)} + \alpha_k r^{(k)} \qquad ?\ \alpha_k$

$Q(x^{(k)} + \alpha_k r^{(k)}) = \tilde{Q}(\alpha_k) \qquad \frac{d\tilde{Q}}{d\alpha_k} = 0$

$$\frac{1}{2}\left(x^{(k)} + \alpha_k r^{(k)}\right)^T A \left(x^{(k)} + \alpha_k r^{(k)}\right) - \left(x^{(k)} + \alpha_k r^{(k)}\right)^T b = \tilde{Q}(\alpha_k)$$

$$\frac{d\tilde{Q}}{d\alpha_k} = r^{(k)T} A x^{(k)} + \alpha_k \underbrace{r^{(k)T} A r^{(k)}}_{(1\times n)(n\times n)(n\times 1)} - r^{(k)T} b = 0 \rightarrow \alpha_k = \frac{r^{(k)T} b - r^{(k)T} A x^{(k)}}{r^{(k)T} A r^{(k)}} = \frac{r^{(k)T} r^{(k)}}{r^{(k)T} A r^{(k)}}$$

$r^{(k)T}(b - Ax^{(k)})$

Metodo del gradiente coniugato (precondizionato)

A sdp $\quad x^{(k+1)} = x^{(k)} + \tilde{\alpha}_k \, p^{(k)}$ $\quad \{p^{(k)}\}$ A-coniugate (A-ortogonale)

$p^{(k)} \neq r^{(k)}$ $\quad p^{(i)}, p^{(j)} \quad p^{(i)T} p^{(j)} = 0$

$p^{(k+1)}$ t.c. $(Ap^{(i)})^T p^{(k+1)} = 0 \quad \forall i = 0, \dots, k$

$$(Ap^{(i)})^T p^{(j)} = 0 \iff \left[p^{(i)T} A^T p^{(j)} = 0 \right]$$

Algoritmo a 5 passi: $x^{(0)} \rightarrow r^{(0)} = b - Ax^{(0)}$; $p^{(0)} = r^{(0)}$

1) $\tilde{\alpha}_k = \dfrac{(p^{(k)})^T r^{(k)}}{(p^{(k)})^T A p^{(k)}} \longrightarrow \tilde{\alpha}_k$

2) $x^{(k+1)} = x^{(k)} + \tilde{\alpha}_k p^{(k)} \longrightarrow x^{(k+1)}$

3) $r^{(k+1)} = r^{(k)} - \tilde{\alpha}_k A p^{(k)} \longrightarrow r^{(k+1)} \longrightarrow z^{(k+1)}$

4) $\beta_k = \dfrac{(Ap^{(k)})^T r^{(k+1)}}{(Ap^{(k)})^T p^{(k)}} \longrightarrow \beta_k$

5) $p^{(k+1)} = r^{(k+1)} - \beta_k p^{(k)} \longrightarrow p^{(k+1)}$

$$\|e^{(k)}\|_A \leq \left[c^* \right]^k \|e^{(0)}\|_A$$

$$c^* = c^*\left(\sqrt{k(A)}\right)$$

Hilbert
sdp

n	$K(H_n)$	$\backslash$	PG		PGC	
4	$O(10^{4})$	$O(10^{-13})$	$O(10^{-3})$	985	$O(10^{-2})$	3
6	$O(10^{6})$	$O(10^{-10})$	$O(10^{-3})$	1813	$O(10^{-3})$	4
8	⋮	⋮	⋮	⋮	⋮	4
10	⋮	⋮	⋮	⋮	⋮	
12	⋮	⋮	⋮	⋮	⋮	
14	$O(10^{17})$	$O(10)$	$O(10^{-3})$	1379	$O(10^{-3})$	5

$$P \equiv D = \begin{bmatrix} a_{11} & & & 0 \\ & a_{22} & & \\ & & \ddots & \\ 0 & & & a_{nn} \end{bmatrix} \quad ; \quad x^{(0)} = 0, \quad TOL = 10^{-6}$$

CRITERI D'ARRESTO

- N_{max}
- accuratezza

$\|x - x^{(k)}\| \leq \boxed{\delta \leq TOL}$ (10^{-9})

δ → stimatore (computabile)
- → residuo $r^{(k)}$
- → incremento $x^{(k+1)} - x^{(k)}$

affidabilità

Residuo $r^{(k)} = b - Ax^{(k)} \in \mathbb{R}^n$

residuo relativo

$$? \; k_{min} \quad \text{t.c.} \quad \boxed{\frac{\|r^{(kmin)}\|}{\|b\|}} \leq TOL$$

$$\left(x^{(0)} = 0 \quad r^{(0)} = b - Ax^{(0)} = b \right)$$

$$\frac{\|e^{(kmin)}\|}{\|x\|} \leq \underbrace{C}_{10^2} \boxed{\frac{\|r^{(kmin)}\|}{\|b\|} \leq \underbrace{TOL}_{10^{-6}}} \qquad \boxed{C = k(A)}$$

$0.5 \; 10^{-6}$

$\delta x = \tilde{x} - x$

$$(\delta A = 0) \quad \boxed{\frac{\|\delta x\|}{\|x\|} \leq k(A) \frac{\|\delta b\|}{\|b\|}}$$

$Ax = b$

$(A + \delta A)(x + \delta x) = b + \delta b$

$A(x + \delta x) = b + \delta b$

$\tilde{x} = x + \delta x$

$\boxed{\tilde{x} = x^{(kmin)}}$

$$A\underbrace{(x + \delta x)}_{\tilde{x}} = b + \delta b$$

$$\delta b = A\tilde{x} - b = -\underbrace{(b - A\tilde{x})}_{\tilde{r}} = -\tilde{r}$$

$$\boxed{\frac{\|\delta x\|}{\|x\|} \leq k(A) \frac{\|\tilde{r}\|}{\|b\|}}$$

$$\boxed{\frac{\|x^{(kmin)} - x\|}{\|x\|}} \leq k(A) \frac{\|b - Ax^{(kmin)}\|}{\|b\|} = \boxed{k(A) \frac{\|r^{(kmin)}\|}{\|b\|}}$$

Incremento

$$x^{(k+1)} - x^{(k)} = \delta^{(k)}$$

? kmin t.c. $\|e^{(kmin)}\| \leq C \boxed{\|\delta^{(kmin)}\| \leq TOL \; (10^{-5})}$

B sdp $\quad x^{(k+1)} = Bx^{(k)} + g$

$$e^{(kmin)} = x - x^{(kmin)} = \underbrace{x - x^{(kmin+1)}}_{e^{(kmin+1)}} + \underbrace{x^{(kmin+1)} - x^{(kmin)}}_{\delta^{(kmin)}}$$

$$\|e^{(kmin)}\| = \|x - x^{(kmin+1)} + \delta^{(kmin)}\|$$

$$\leq \underbrace{\|e^{(kmin+1)}\|} + \|\delta^{(kmin)}\|$$

$$\leq \rho(B)\|e^{(kmin)}\| + \|\delta^{(kmin)}\|$$

$$\|e^{(kmin)}\| \leq \overbrace{\boxed{\frac{1}{1-\rho(B)}}}^{C} \|\delta^{(kmin)}\|$$

$$\left[\; \|e^{(k+1)}\| \leq \|B\|_2 \|e^{(k)}\| \simeq \rho(B) \quad \middle| \quad B \text{ sdp} \quad \middle| \quad \|B\|_2 = \sqrt{\lambda_{max}(B^TB)} \;\right]$$

$$\|B\|_2 = \sqrt{\lambda_{max}(B^2)} = \cancel{\sqrt{[\lambda_{max}(B)]^2}} \equiv \rho(B)$$

Approssimazione di autovalori e autovettori

$A \in \mathbb{C}^{n\times n}$? $\lambda \in \mathbb{C}\ (\mathbb{R})$ e $x \in \mathbb{C}^n\ (\mathbb{R}^n)$ t.c. $Ax = \lambda x$

$x \neq 0$ $\quad (\alpha x \quad \forall \alpha \in \mathbb{C},\ \alpha \neq 0)$

Quoziente di Rayleigh: noto x, l'autovalore λ associato è

$$\lambda = \frac{x^H A x}{\|x\|^2}$$

x^H $\quad x^T$ $\quad (x_i = a+ib \longrightarrow x_i^H = a-ib)$

trasposto complesso coniugato

$$\|x\|^2 = x^T x \longrightarrow \|x\|^2 = x^H x$$

Polinomio caratteristico: $p_A(\lambda) = \det(A - \lambda I)$

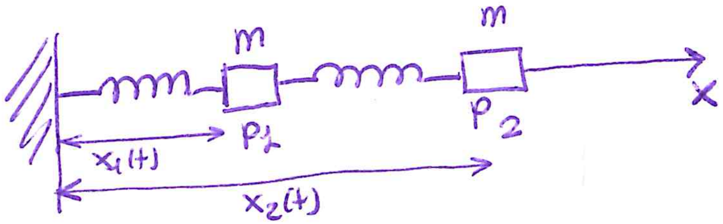
Matrice diagonalizzabile: A si dice diagonalizzabile se $\exists\ U \in \mathbb{C}^{n\times n}$ invertibile e t.c.

$$U^{-1} A U = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$$

colonne $\equiv$ autovettori di A

Spettro: $\{\lambda_i\}_{i=1}^n$ $\quad \rho(A) = \max_i |\lambda_i|$

$\boxed{\text{Sistema di 2 molle}}$



$x_i(t)$ posizione occupata da P_i all'istante t

Ho 2 corpi puntiformi P_1 e P_2 di massa m ; 2 molle con stesso coefficiente k di elasticità ; liberi di muoversi lungo la direzione x

Per la seconda legge della dinamica

$$m\ddot{x}_1 = k(x_2-x_1) - kx_1$$

$$m\ddot{x}_2 = k(x_1-x_2)$$

$$m\begin{bmatrix}\ddot{x}_1\\ \ddot{x}_2\end{bmatrix} = \begin{bmatrix}-2k & k\\ k & -k\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

siamo interessati ad oscillazioni libere

Ansatz $\quad x = \begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}A_1\\ A_2\end{bmatrix}e^{i\omega t}$

$$\dot{x} = \begin{bmatrix}A_1\\ A_2\end{bmatrix} i\omega e^{i\omega t} \qquad \ddot{x} = \begin{bmatrix}A_1\\ A_2\end{bmatrix}(-\omega^2)e^{i\omega t}$$

$$-\omega^2 m\begin{bmatrix}A_1\\ A_2\end{bmatrix}\cancel{e^{i\omega t}} = \begin{bmatrix}-2k & k\\ k & -k\end{bmatrix}\begin{bmatrix}A_1\\ A_2\end{bmatrix}\cancel{e^{i\omega t}}$$

$$\begin{bmatrix}2k & -1k\\ -1k & 1k\end{bmatrix}\begin{bmatrix}A_1\\ A_2\end{bmatrix} = \omega^2 m\begin{bmatrix}A_1\\ A_2\end{bmatrix}$$

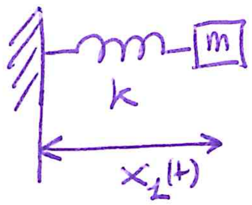
matrice di stiffness

$$\underbrace{\begin{bmatrix}2k & -k\\ -k & k\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}A_1\\ A_2\end{bmatrix}}_{V} = \underbrace{\omega^2 m}_{\lambda}\underbrace{\begin{bmatrix}A_1\\ A_2\end{bmatrix}}_{V}$$

problema agli autovalori

$\boxed{\text{EX2}}$

$$\boxed{\text{Oscillatore armonico semplice}}$$



$$m\ddot{x}_1 = -k x_1$$

$$\ddot{x}_1 = -\frac{k}{m} x_1$$

$$\begin{cases} \dot{x}_1 = y \\ \dot{y} = -\frac{k}{m} x_1 \end{cases} \qquad \begin{bmatrix} x_1 \\ y \end{bmatrix} = X \qquad \dot{X} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ y \end{bmatrix}$$

$$x_1 = A_1 e^{i\omega t}$$

$$y = A_2 e^{i\omega t}$$

A_1 e A_2 costanti

$$i\omega A_1 \cancel{e^{i\omega t}} = A_2 \cancel{e^{i\omega t}}$$

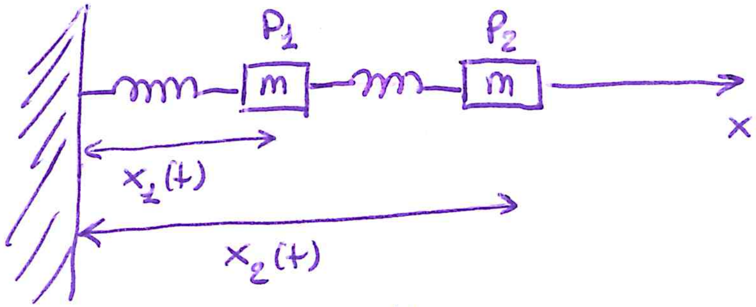
$$i\omega A_2 \cancel{e^{i\omega t}} = -\frac{k}{m} A_1 \cancel{e^{i\omega t}}$$

$$\underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}}_{v} = \underbrace{i\omega}_{\lambda} \underbrace{\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}}_{v}$$

problema agli autovalori

$\boxed{\text{Ex 1}}$

Sistema di 2 molle



$x_i(t)$ $\quad P_i \quad t$

$$\begin{cases} m\ddot{x}_1 = k(x_2-x_1)-kx_1 \\ m\ddot{x}_2 = k(x_1-x_2) \end{cases} \qquad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$m\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} -2k & k \\ k & -k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} e^{i\omega t}$$

$$\dot{x} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} i\omega e^{i\omega t}; \quad \ddot{x} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}(-\omega^2)e^{i\omega t}$$

$$-\omega^2 m\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}\cancel{e^{i\omega t}} = \begin{bmatrix} -2k & k \\ k & -k \end{bmatrix}\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}\cancel{e^{i\omega t}}$$

$$\underbrace{\begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}}_{v} = \underbrace{\omega^2 m}_{\lambda}\underbrace{\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}}_{v}$$

TACOMA

METODI NUMERICI $\rightarrow$ λ, x specifici

METODI NUMERICI $\rightarrow$ spettro (QR)

Metodo delle potenze (λ_{max}, x)

$A \in \mathbb{R}^{n \times n}$ (H1) $|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \dots \geq |\lambda_n|$ (λ_1, x_1)

$\downarrow$ autovalore dominante $\qquad$ $\downarrow$ autovettore unitario

(H2) autovettori di A linearmente indipendenti

$x^{(0)} \in \mathbb{C}^n$ (guess iniziale), sia $y^{(0)} = \dfrac{x^{(0)}}{\|x^{(0)}\|}$

$$\boxed{\begin{aligned} & k = 1, 2, \dots \\ & x^{(k)} = A y^{(k-1)} \\ & y^{(k)} = \frac{x^{(k)}}{\|x^{(k)}\|} \simeq x_1 \\ & \lambda^{(k)} = (y^{(k)})^H A y^{(k)} \simeq \lambda_1 \end{aligned}}$$

(A) N_{MAX}

(B) $|\lambda^{(k)} - \lambda^{(k-1)}| < \varepsilon |\lambda^{(k)}|$

$k = 1$ $\quad \left[x^{(1)} = A y^{(0)} \quad y^{(1)} = \dfrac{A y^{(0)}}{\|x^{(1)}\|} \quad \lambda^{(1)} = (y^{(1)})^H A y^{(1)} \right.$

$k = 2$ $\quad \left[\begin{aligned} & x^{(2)} = A y^{(1)} \quad y^{(2)} = \frac{A y^{(1)}}{\|x^{(2)}\|} = \frac{A^2 y^{(0)}}{\|x^{(1)}\| \, \|x^{(2)}\|} = \beta^{(2)} A^2 y^{(0)} \\ & \beta^{(2)} = \left(\frac{1}{\|x^{(1)}\| \, \|x^{(2)}\|} \right) \quad \lambda^{(2)} = (y^{(2)})^H A y^{(2)} \end{aligned} \right.$

$\vdots$

k $\quad \left[\begin{aligned} & x^{(k)} = A y^{(k-1)} \quad y^{(k)} = A^k y^{(0)} \beta^{(k)} \quad \beta^{(k)} = \left(\prod_{i=1}^{k} \|x^{(i)}\| \right)^{-1} \\ & \lambda^{(k)} = (y^{(k)})^H A y^{(k)} \end{aligned} \right.$

Analisi di convergenza

$\{x_i\}$ linearmente indipendenti $\Rightarrow$ base per $\mathbb{C}^n$

$$x^{(0)} = \sum_{i=1}^{n} \underset{\substack{\Uparrow \\ \mathbb{C}}}{\alpha_i} x_i \qquad y^{(0)} = \gamma^{(0)} \sum_{i=1}^{n} \alpha_i x_i \qquad \gamma^{(0)} = \frac{1}{\|x^{(0)}\|}$$

$k=1$

$$x^{(1)} = Ay^{(0)} = \gamma^{(0)} \sum_{i=1}^{n} \alpha_i \underbrace{Ax_i}_{\lambda_i x_i} = \gamma^{(0)} \sum_{i=1}^{n} \alpha_i \lambda_i x_i$$

$$y^{(1)} = \gamma^{(1)} \sum_{i=1}^{n} \alpha_i \lambda_i x_i \qquad \gamma^{(1)} = \frac{1}{\|x^{(0)}\| \|x^{(1)}\|}$$

$k=2$

$$x^{(2)} = Ay^{(1)} = \gamma^{(1)} \sum_{i=1}^{n} \alpha_i \lambda_i \underbrace{Ax_i}_{\lambda_i x_i} = \gamma^{(1)} \sum_{i=1}^{n} \alpha_i \lambda_i^2 x_i$$

$$y^{(2)} = \gamma^{(2)} \sum_{i=1}^{n} \alpha_i \lambda_i^2 x_i \qquad \gamma^{(2)} = \frac{1}{\|x^{(0)}\| \|x^{(1)}\| \|x^{(2)}\|}$$

generalizzando

$$y^{(k)} = \gamma^{(k)} \sum_{i=1}^{n} \alpha_i \lambda_i^k x_i \qquad \gamma^{(k)} = \frac{1}{\|x^{(0)}\| \|x^{(1)}\| \dots \|x^{(k)}\|}$$

$$= \gamma^{(k)} \left[\alpha_1 \lambda_1^k x_1 + \sum_{i=2}^{n} \alpha_i \lambda_i^k x_i \right]$$

$$= \gamma^{(k)} \alpha_1 \lambda_1^k \left[x_1 + \underbrace{\sum_{i=2}^{n} \frac{\alpha_i}{\alpha_1} \left(\frac{\lambda_i}{\lambda_1}\right)^k x_i}_{<1} \right]$$

$\hookrightarrow 0 \quad k \to +\infty$

$\alpha_1 \neq 0$

$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \dots \geq |\lambda_n|$

1) $y^{(k)}$ si allinea a x_1 per $k \to +\infty$ se $\boxed{\alpha_1 \neq 0}$

2) se $y^{(k)}$ si allinea a x_1 $\Longrightarrow$ $\lambda^{(k)} \longrightarrow \lambda_1$ per $k \to +\infty$

3) velocità di convergenza

$$\| x_1 - y^{(k)} \|$$

$$\left\| \cancel{x_1} - \cancel{x_1} - \sum_{i=2}^{n} \frac{\alpha_i}{\alpha_1}\left(\frac{\lambda_i}{\lambda_1}\right)^k x_i \right\| = \left\| \sum_{i=2}^{n} \frac{\alpha_i}{\alpha_1}\left(\frac{\lambda_i}{\lambda_1}\right)^k x_i \right\|$$

norma unitaria

$$= \left[\sum_{i=2}^{n} \left[\frac{\alpha_i}{\alpha_1}\right]^2 \left(\frac{\lambda_i}{\lambda_1}\right)^{2k} \right]^{1/2}$$

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \dots \geq |\lambda_n|$$

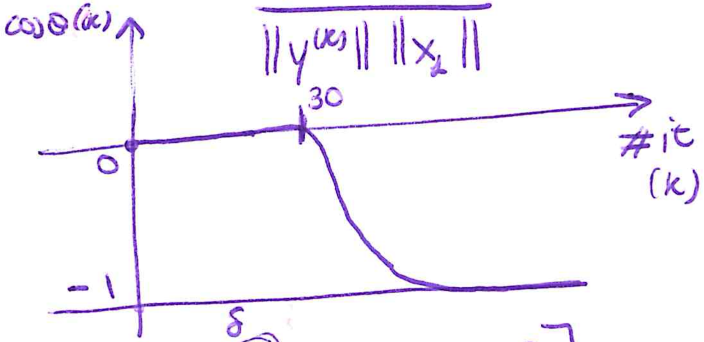
$$\leq \underbrace{\left|\frac{\lambda_2}{\lambda_1}\right|^k}_{<1} \underbrace{\left(\sum_{i=2}^{n} \left[\frac{\alpha_i}{\alpha_1}\right]^2 \right)^{1/2}}_{\text{quantità fissata}}$$

$\hookrightarrow 0 \quad k \to +\infty$ (separabilità degli autovalori)

Ex:

$x^{(0)} \perp x_1$

$$\cos\theta^{(k)} = \frac{(y^{(k)})^T x_1}{\|y^{(k)}\| \, \|x_1\|}$$



$$\begin{array}{c} \delta \\ \begin{bmatrix} 16 & 2 & 3 & 13 \\ 5 & 11 & 10 & 8 \\ 9 & 7 & 6 & 12 \\ 4 & 14 & 15 & 1 \end{bmatrix} \end{array}$$

Ex: $\delta = 30$ $\longrightarrow$ 22 iterazioni

$\lambda_1 = 39.396 \quad \lambda_2 = 17.821$

$\delta = -30 \quad \lambda_1 = -30.643 \quad \lambda_2 = 29.736$

$\boxed{\varepsilon = 10^{-10}}$ $\longrightarrow$ 708 iterazioni

4) A Hermitiana $A^H = A$

Se A è Hermitiana, $$\|x_1 - y^{(k)}\| \leq \left|\frac{\lambda_2}{\lambda_1}\right|^{2k} \left[\sum_{i=2}^{n} \left[\frac{\alpha_i}{\alpha_1}\right]^2\right]^{1/2}.$$

5) Potenze converge anche se λ_1 è una radice multipla di $p_A(\lambda)$.

6) Se $\exists$ 2 autovalori distinti ma entrambi di modulo massimo, potenze non converge.

Generalizzazioni

① Metodo delle potenze inverse: approssima autovalore in modulo minimo (+ corrispondente autovettore)

A non singolare ($\exists A^{-1}$)

$$Ax = \lambda x$$
$$A^{-1}Ax = \lambda A^{-1}x$$
$$x = \lambda A^{-1}x$$
$$A^{-1}x = \frac{1}{\lambda}x$$

A	A^{-1}
$\{\lambda, x\}$	$\{\frac{1}{\lambda}, x\}$

$|\lambda_1| > |\lambda_2| \geq \dots \geq |\lambda_n|$

$\left|\frac{1}{\lambda_1}\right| < \left|\frac{1}{\lambda_2}\right| \leq \quad \leq \left|\frac{1}{\lambda_n}\right|$

(λ_n, x_n) A $\qquad$ $\left(\frac{1}{\lambda_n} (\rightsquigarrow \lambda_n), x_n\right)$ A^{-1}

$x^{(0)} \in \mathbb{C}^n$, $y^{(0)} = \frac{x^{(0)}}{\|x^{(0)}\|}$

$k = 1, 2, \dots$

$$x^{(k)} = A^{-1}y^{(k-1)}$$
$$y^{(k)} = \frac{x^{(k)}}{\|x^{(k)}\|} \simeq x_n$$
$$\mu^{(k)} = (y^{(k)})^H A^{-1} y^{(k)} \simeq \frac{1}{\lambda_n}$$

1) $Ax^{(k)} = y^{(k-1)}$

2) $\mu^{(k)} = (y^{(k)})^H A y^{(k)} \simeq \lambda_n$

Potenze inverse con shift: approssimare l'autovalore di A più vicino a $\mu \in \mathbb{C}$.

λ_μ $\mu \to$ SHIFT

$$A_\mu = A - \mu I$$

$$\lambda(A_\mu) = \lambda(A) - \mu$$

$\lambda_{min}(A_\mu)$ potenze inverse A_μ

$\hookrightarrow \lambda_\mu = \lambda_{min}(A_\mu) + \mu$

	x_1	x_2		x_n	
	$\lvert\lambda_1\rvert >$	$\lvert\lambda_2\rvert \geq$	$\dots$	$> \lvert\lambda_n\rvert$	A
	$\lambda_1 - \mu$	$\lambda_2 - \mu$		$\lambda_n - \mu$	A_μ
$\lambda_{min}(A_\mu)$	$\lambda_i - \mu$			x_i	

A	A_μ
$\{\lambda, x\}$	$\{\lambda - \mu, x\}$

$$Ax = \lambda x$$

$$Ax - \mu I x = \lambda x - \mu x$$

$$\underbrace{(A - \mu I)}_{A_\mu} x = (\lambda - \mu) x$$

$$x^{(k)} = A_\mu^{-1} y^{(k-1)}$$

$$y^{(k)} = \frac{x^{(k)}}{\|x^{(k)}\|} \simeq x_\mu \quad (x_i)$$

$$\mu^{(k)} = (y^{(k)})^H A_\mu^{-1} y^{(k)} \simeq \frac{1}{\lambda_{min}(A_\mu)}$$

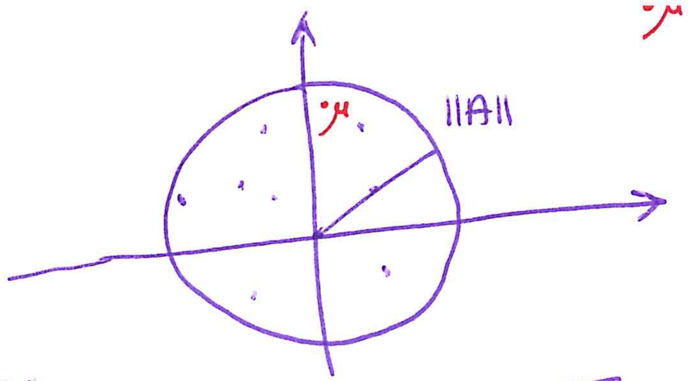
$$\overset{A}{\{\lambda_i, x_i\}} \qquad \overset{A_\mu}{\{\lambda_i - \mu, x_i\}} \qquad \overset{A_\mu}{\left\{\frac{1}{\lambda_i - \mu}, x_i\right\}}$$

$$x_i^H A x_i = \lambda_i$$

$$x_i^H A_\mu x_i = \lambda_i - \mu \longrightarrow$$

$$x_i^H A_\mu^{-1} x_i = \frac{1}{\lambda_i - \mu} \longrightarrow$$

Scelta dello shift



1)

$$\lambda v = A v$$

$$|\lambda| \cancel{\|v\|} = \|\lambda v\| = \|A v\| \le \|A\| \cancel{\|v\|} \longrightarrow |\lambda| \le \|A\|$$

$$\|A\|_2 = \sqrt{\lambda_{max}(A^T A)}$$

2) cerchi di Gershgorin:

$$C_i^r = \left\{ z \in \mathbb{C} \text{ t.c. } |z - a_{ii}| \le \sum_{\substack{j=1 \\ j \ne i}}^{n} |a_{ij}| \right\} \qquad i = 1, \dots, n$$

$$C_i^c = \left\{ z \in \mathbb{C} \text{ t.c. } |z - a_{ii}| \le \sum_{\substack{j=1 \\ j \ne i}}^{n} |a_{ji}| \right\} \qquad i = 1, \dots, n$$

<u>Proposizione</u>: gli autovalori di $A \in \mathbb{C}^{n \times n}$ appartengono alla regione di $\mathbb{C}$ identificata da

$$\left[\bigcup_{i=1}^{n} C_i^r \right] \cap \left[\bigcup_{i=1}^{n} C_i^c \right].$$

Inoltre se $\exists$ m cerchi riga (colonna) $1 \le m \le n$ sono disgiunti (sconnessi) dall'unione dei restanti $n-m$, allora la loro unione contiene esattamente m autovalori.

Equazioni non lineari

f non lineare $\quad \mathbb{R}$ $\quad \mathbb{P}_r = \{ a_0 + a_1 x + a_2 x^2 + \ldots + a_r x^r \}$ $\quad ax+b$

sin, cos, tanh
exp, log,
$\sqrt[s]{}$

$$\boxed{?\ \alpha \in \mathbb{R} \text{ t.c. } f(\alpha)=0}$$

? V gas T, p

$$\left[p + a\left(\frac{N}{V}\right)^2\right](V - Nb) = kNT$$

Abel
$\mathbb{P}_r \quad r > 4$

? V t.c. $f(V)=0$

$$\left[p + a\left(\frac{N}{V}\right)^2\right](V - Nb) - kNT = f(V)$$

Metodi di punto fisso

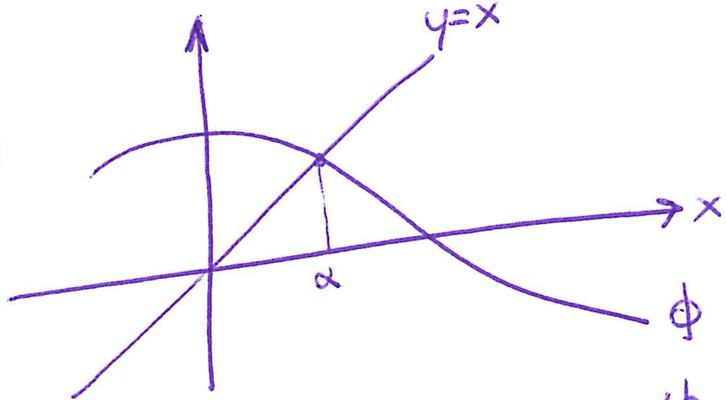
$1 \quad \cos$ —— $\alpha^* = 0.73908513$

$\cos \alpha^* = \alpha^*$

$\Phi : [a,b] \subset \mathbb{R} \to \mathbb{R}, \quad \Phi \in C^0([a,b])$

α^* è punto fisso di Φ se $\Phi(\alpha^*) = \alpha^*$

$\boxed{x^{(k+1)} = \phi(x^{(k)})}$



$x^{(0)} = 1$

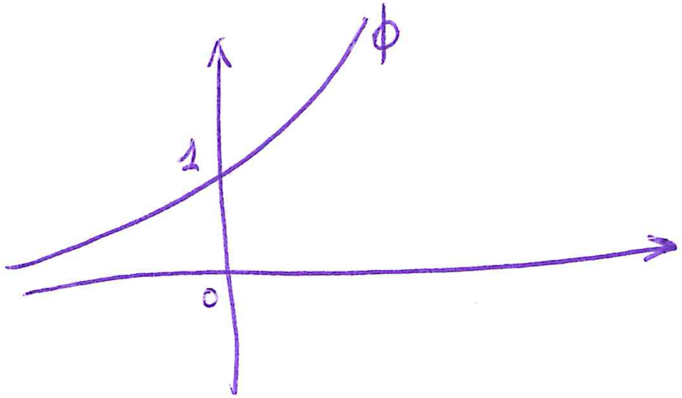
$x^{(1)} = \cos(x^{(0)})$

$x^{(2)} = \cos(x^{(1)})$

$x^{(3)} = \cos(x^{(2)}) \quad \to \quad x^{(k+1)} = \cos(x^{(k)})$

$\vdots$

$\phi(x) = \cos(x)$



$? \alpha : f(\alpha) = 0$ $\rightleftarrows$ $? \alpha : \phi(\alpha) = \alpha$

$$f(x) = 0 \longleftrightarrow x = \phi(x) \qquad x^{(k+1)} = \phi(x^{(k)})$$

$$\underbrace{f(x) + x}_{\phi(x)} = x$$

$\boxed{Hp}$ $f(\alpha) = 0$ $\boxed{\text{Tesi}}$ $\phi(\alpha) = \alpha$

$$\phi(\alpha) = \underbrace{f(\alpha)}_{=0} + \alpha = \alpha$$

$$\boxed{f(x) + x = \phi(x)}$$

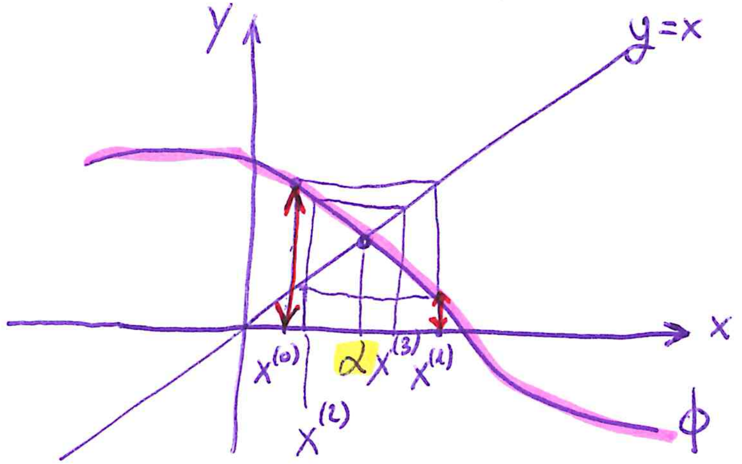
$\boxed{Hp}$ $\alpha = \phi(\alpha)$ Tesi $f(\alpha) = 0$

$$\underbrace{\phi(\alpha) = f(\alpha) + \cancel{\alpha} = \cancel{\alpha}}_{f(\alpha) = 0}$$

$$f(x) = 0$$

$$f(x) + 3x = 3x \qquad \overset{\phi(x)}{\overset{\parallel}{\boxed{\frac{f(x) + 3x}{3}}}} = x$$

$$x^{(k+1)} = \phi(x^{(k)})$$



α attrattore

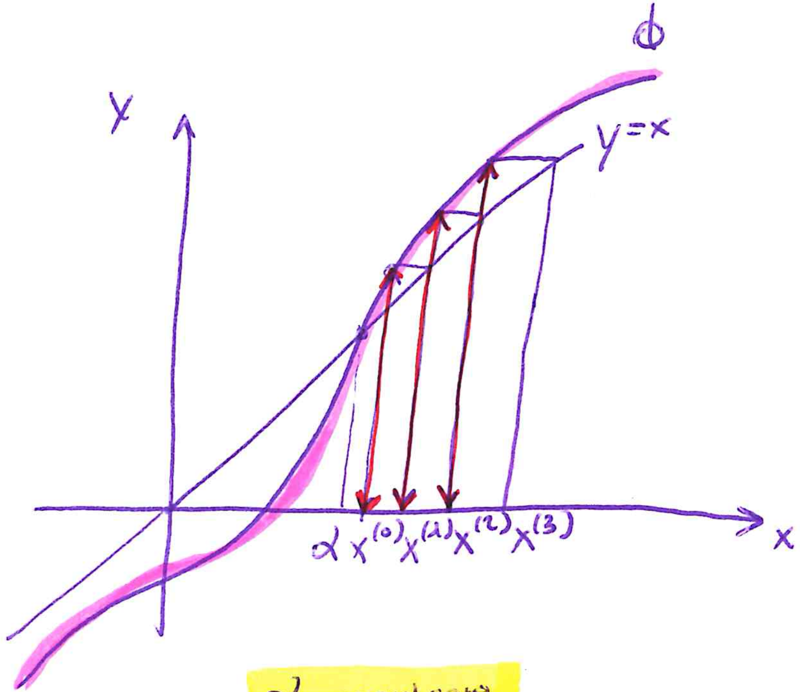
$x^{(0)}$

$x^{(1)} = \phi(x^{(0)})$

$x^{(2)} = \phi(x^{(1)})$

$x^{(3)} = \phi(x^{(2)})$

$\vdots$



$\boxed{\text{convergenza condizionata}}$

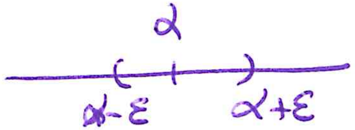
α repulsore

ϕ'

Ostrowski (locale) : $\exists\, \alpha$ punto fisso di ϕ in I_α ; $\phi \in C^1(\bar{I}_\alpha)$. Se $|\phi'(\alpha)| < 1 \implies \exists\, \delta > 0$ t.c.
se $x^{(0)}$: $|x^{(0)} - \alpha| < \delta$, $\{x^{(k)}\}$ converge a α e vale la relazione

$$\lim_{k \to +\infty} \frac{x^{(k+1)} - \alpha}{x^{(k)} - \alpha} = \phi'(\alpha)$$

$p = 1$ (convergenza almeno lineare)
$C = \phi'(\alpha)$

[I_α = intorno di α  $I_\alpha = (\alpha - \varepsilon, \alpha + \varepsilon)$; $\bar{I}_\alpha = [\alpha - \varepsilon, \alpha + \varepsilon]$]

Ordine di convergenza : $\{x^{(k)}\}$ $\quad \alpha \quad$ ordine di convergenza $= p$ se

$$\frac{|x^{(k+1)} - \alpha|}{|x^{(k)} - \alpha|^p} \leq \underbrace{C}_{\text{indipendente da } k} \qquad \forall k \geq k_0$$

Se $p = 1$ allora $C < 1$

($p=1$ lineare, $p=2$ quadratica ....)

Ex $\quad p=2 \quad |x^{(k+1)} - \alpha| \leq \underset{10}{C^*} \underbrace{\underbrace{\underbrace{|x^{(k)} - \alpha|^2}_{10^{-1}}}_{10^{-2}}}_{10^{-4}}$

$p=3 \quad |x^{(k+1)} - \alpha| \leq \tilde{C} \underbrace{\underbrace{\underbrace{|x^{(k)} - \alpha|^3}_{10^{-1}}}_{10^{-3}}}_{10^{-9}}$

$p=1 \quad |x^{(k+1)} - \alpha| \leq C \underbrace{|x^{(k)} - \alpha|}_{10^{-1}}$

C: $\cancel{1.1}$ $\quad 0.5$

$x = \phi_1(x)$ t.c. $|\phi_1'(\alpha_1)| = 0.32$ $\qquad$ $x = \phi_2(x)$ t.c. $|\Phi_2'(\alpha_2)| = 0.99$

$f(x) = 0$

$|\phi'(\alpha)| < 1$ convergenza

$|\phi'(\alpha)| > 1$ divergenza

$|\phi'(\alpha)| = 1$ /

non unicità di ϕ → $\exists \alpha$
→ convergenza o meno
→ ordine diverso o velocità di convergenza $\neq$

<u>Ex</u>: 1) $\phi(x) = \cos(x)$

$\phi'(x) = -\sin(x)$ $\quad$ ok

$|\phi'(\alpha)| = |\sin(\alpha)| < 1$ → ordine di convergenza almeno 1

2) $\phi(x) = x^2 - 1$

$x = \phi(x) \qquad x = x^2 - 1 \qquad x^2 - x - 1 = 0$

$\alpha_\pm = \dfrac{1 \pm \sqrt{5}}{2}$

$\phi'(x) = 2x$

$|\phi'(\alpha_\pm)| = |1 \pm \sqrt{5}| > 1$ $\quad$ KO

3) $f(x) = \log(x) - \gamma = 0 \qquad \gamma \in \mathbb{R}$

$\phi_1 \qquad \underbrace{\log(x) - \gamma + x}_{\Phi_1(x)} = x$

$\phi_2 \qquad x\log(x) - \gamma x = 0 \qquad x \log x = \gamma x \qquad \overbrace{\dfrac{x \log x}{\gamma}}^{\phi_2(x)} = x \qquad \gamma \neq 0$

$\phi_3 \equiv \phi_N$

$\gamma = -2 \qquad \Phi_1$ KO $\qquad \Phi_2$ ok $\qquad \Phi_N$ ok

$? \; x^{(0)}$ **bisezione** $(a,b) \ni \alpha$

$I^{(0)}, I^{(1)}, \dots, I^{(k)}$

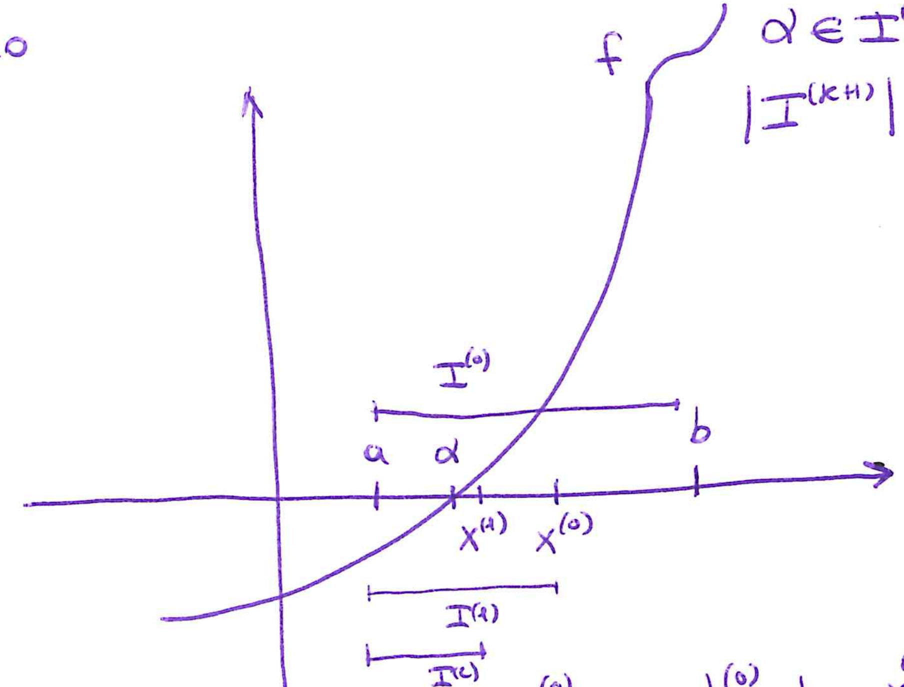
$I^{(0)} \supset I^{(1)} \supset I^{(2)} \supset \dots \supset I^{(k)}$ intervalli incapsulati

Ip: $f \in C^0([a,b])$

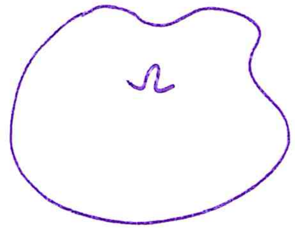
$f(a)\, f(b) < 0$

$\alpha \in I^{(k)} \quad \forall k$

$$|I^{(k+1)}| = \frac{|I^{(k)}|}{2}$$



$$\boxed{\begin{gathered} f(x) = 0 \\ ?\, \alpha : f(\alpha) = 0 \end{gathered}}$$



$|\Omega|$ = misura di Ω

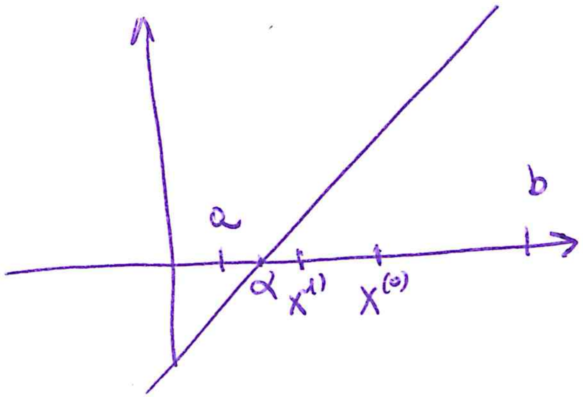
$|I|$ = lunghezza di I

$I^{(0)} = [a,b]; \quad a^{(0)} = a, \; b^{(0)} = b; \; x^{(0)} = \dfrac{a+b}{2}$

$I^{(1)} = [a^{(1)}, b^{(1)}]; \quad a^{(1)} = a^{(0)}, \; b^{(1)} = x^{(0)}; \; x^{(1)} = \dfrac{a^{(1)} + b^{(1)}}{2}$

$y = a_0 + a_1 x$

$$\alpha = -\frac{a_0}{a_1}$$



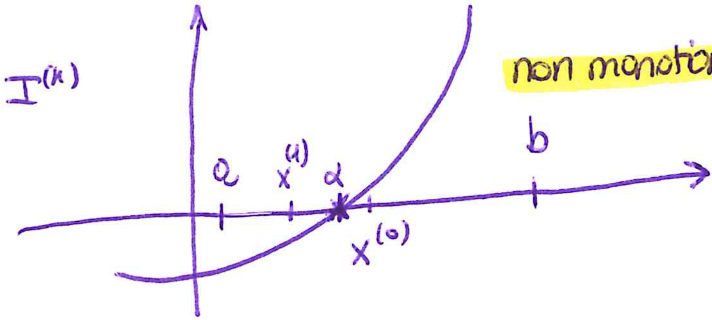
$\{x^{(k)}\}$

$\downarrow$

punti medi $I^{(k)}$

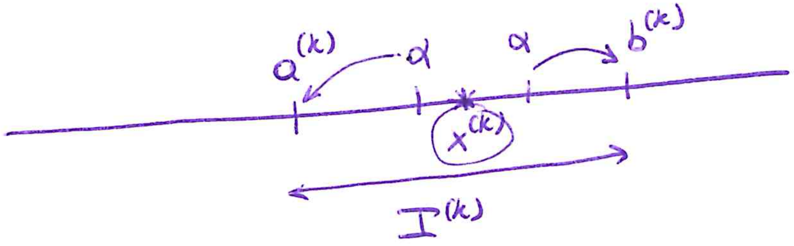
convergenza incondizionata

non monotonia



bisezione $\left(x^{(0)}\right)$ PREDICTOR-CORRECTOR punto fisso (Newton)

$$|e^{(k)}| = |\alpha - x^{(k)}| \leq \frac{1}{2}|I^{(k)}| = \frac{1}{2}\frac{|I^{(0)}|}{2^k} = \frac{b-a}{2^{k+1}} = (b-a)\underbrace{\left(\frac{1}{2}\right)^{k+1}}_{\to 0,\ k\to+\infty}$$



$$|I^{(0)}|$$

$$|I^{(1)}| = \frac{|I^{(0)}|}{2}$$

$$|I^{(2)}| = \frac{|I^{(1)}|}{2} = \frac{|I^{(0)}|}{2^2}$$

$\vdots$

$$\boxed{|I^{(k)}| = \frac{|I^{(0)}|}{2^k}}$$

$$|e^{(k)}| \leq \boxed{(b-a)\left(\frac{1}{2}\right)^{k+1} < \underset{10^{-5}}{TOL}}$$

? k_{min} $\quad |e^{(k_{min})}| < TOL$

$$\frac{(b-a)}{TOL} < 2^{k+1}$$

$$\log_2\left(\frac{b-a}{TOL}\right) < k+1$$

$$\hookrightarrow k > \underbrace{\log_2\left(\frac{b-a}{TOL}\right) - 1}_{N_{max}}$$

Teorema (locale): Sia α un punto fisso per ϕ; $\Phi \in C^p(I_\alpha)$. $p \in \mathbb{N}$. Se $\phi^{(i)}(\alpha) = 0 \quad 1 \le i \le p-1$

e $\phi^{(p)}(\alpha) \neq 0 \implies \exists\, \delta$ t.c. $|x^{(0)} - \alpha| < \delta \quad \{x^{(k)}\}$ converge ad α e

vale

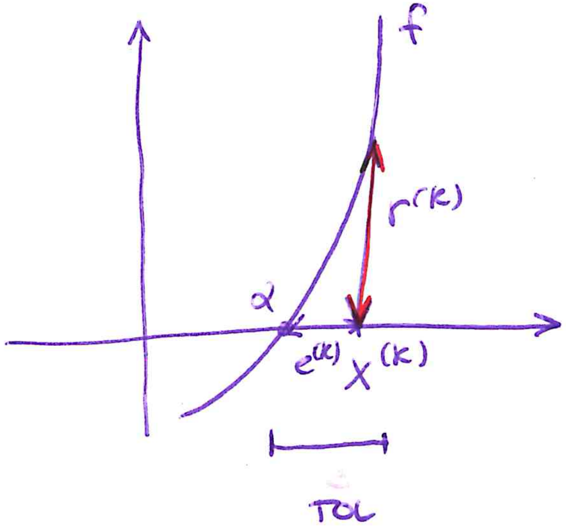
$$\lim_{k \to +\infty} \frac{x^{(k+1)} - \alpha}{[x^{(k)} - \alpha]^p} = \underbrace{\frac{\Phi^{(p)}(\alpha)}{p!}}_{C}$$

$\boxed{\text{ordine } p}$

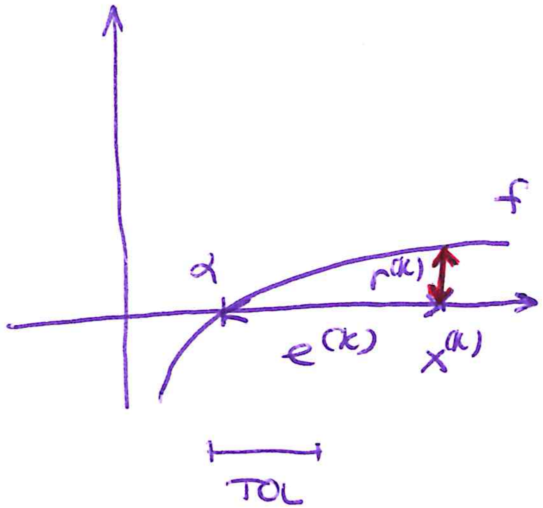
$[N_{MAX}]$ accuratezza $\rightarrow$ stimatore $\rightarrow$ residuo / incremento

$S_1 \qquad r^{(k)} = f(x^{(k)}) \qquad x^{(k)}$

affidabilità di $S_1 \longrightarrow |f'(\alpha)| \simeq 1$



SOVRASTIMA



SOTTOSTIMA

S_2 $\qquad x^{(k+1)} - x^{(k)}$ $\qquad$ affidabilità

$\alpha - x^{(k)}$ $\qquad S_2$

$$\alpha - x^{(k)} = \alpha - x^{(k+1)} + \underbrace{x^{(k+1)} - x^{(k)}}_{S_2}$$

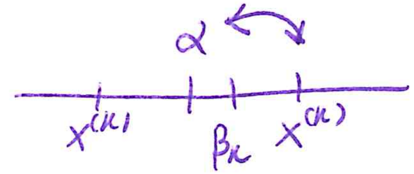
$$\alpha - x^{(k+1)} = \phi(\alpha) - \phi(x^{(k)}) = \phi'(\beta_k)(\alpha - x^{(k)})$$

$$\alpha - x^{(k)} = \underbrace{\phi'(\beta_k)(\alpha - x^{(k)})} + x^{(k+1)} - x^{(k)}$$

$$[1 - \phi'(\beta_k)](\alpha - x^{(k)}) = x^{(k+1)} - x^{(k)}$$

$$\alpha - x^{(k)} = \underbrace{\frac{1}{1-\phi'(\beta_k)}}(x^{(k+1)} - x^{(k)})$$

k sufficientemente grande $\quad \boxed{\dfrac{1}{1-\phi'(\alpha)}}$



CASO IDEALE $\phi'(\alpha) = 0$ $\quad p \geq 2$

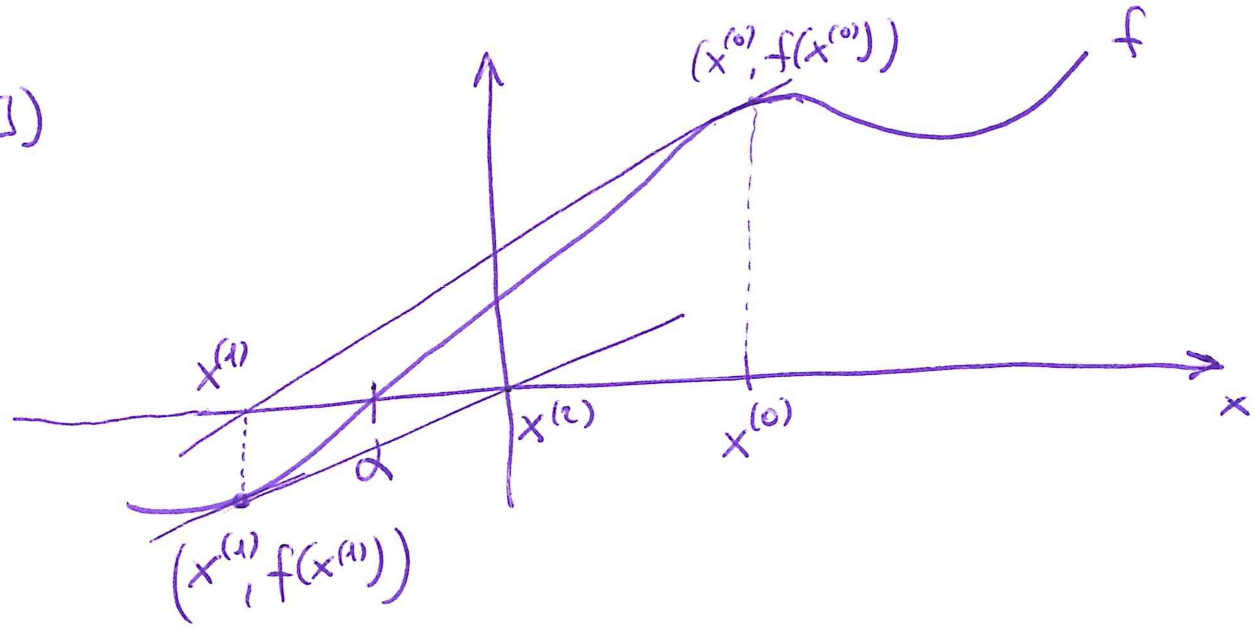
AFFIDABILITA' $\phi'(\alpha)$ vicino a zero

$\boxed{\text{Newton}}$ (metodo delle tangenti)

? α $\quad f(\alpha)=0 \quad$ f non lineare

$\downarrow$

f, f' $\qquad f \in C^1([a,b])$



$$P_k = \left(x^{(k)}, f(x^{(k)})\right) \qquad x^{(k+1)} = \text{"}y=0\text{"} \cap \text{"tg a } f \text{ in } P_k\text{"}$$

$$\begin{cases} y = 0 \\ y(x) = f(x^{(k)}) + f'(x^{(k)})(x - x^{(k)}) \end{cases}$$

$$0 = f(x^{(k)}) + f'(x^{(k)})(x^{(k+1)} - x^{(k)})$$

$f(x) = 0$

$$\boxed{\phi_N(x) = x - \frac{f(x)}{f'(x)}} \qquad x^{(k+1)} = \phi_N(x^{(k)})$$

$$\boxed{x^{(k+1)} = x^{(k)} - \frac{f(x^{(k)})}{f'(x^{(k)})}} \quad \text{Newton}$$

$f'(x^{(k)}) \neq 0$

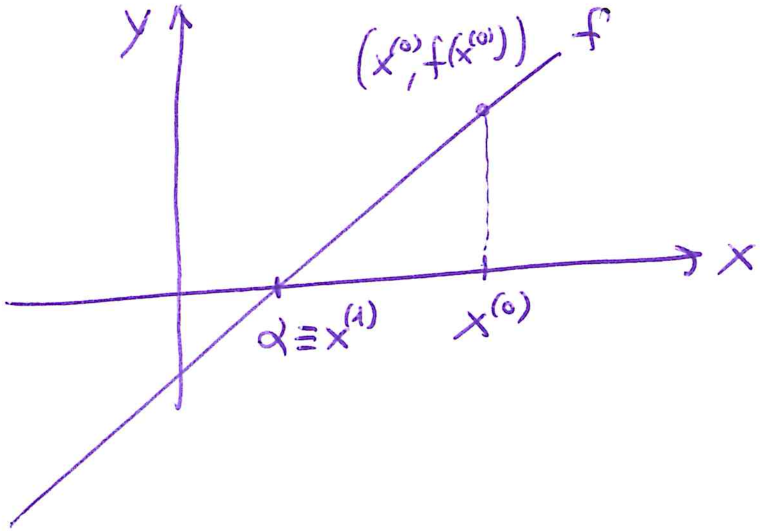
$$\boxed{\text{Newton} \simeq \text{Taylor}}$$

$f \in C^2([a,b])$ $\qquad x^{(k)} \qquad x^{(k+1)} \qquad$ II° ordine

$$0 = f(\alpha) \underset{\substack{k \text{ suff.} \\ \text{grande}}}{\simeq} f(x^{(k+1)}) \underset{\simeq}{=} f(x^{(k)}) + (x^{(k+1)} - x^{(k)}) f'(x^{(k)}) + \cancel{O([x^{(k+1)} - x^{(k)}]^2)}$$

$$0 = f(x^{(k)}) + (x^{(k+1)} - x^{(k)}) f'(x^{(k)})$$

$$f(x) = a_0 + a_1 x \qquad \alpha = -\frac{a_0}{a_1}$$



$$x^{(1)} = x^{(0)} - \frac{f(x^{(0)})}{f'(x^{(0)})}$$

$$= x^{(0)} - \frac{a_0 + a_1 x^{(0)}}{a_1}$$

$$= \frac{\cancel{a_1 x^{(0)}} - a_0 - \cancel{a_1 x^{(0)}}}{a_1} = \frac{-a_0}{a_1}$$

Convergenza (Newton)

H1) $x^{(0)}$ sufficientemente vicino ad α (BISEZIONE/NEWTON)

H2) α zero semplice di f $[f(\alpha)=0$ MA $f'(\alpha)\neq 0]$ $\cancel{(x-1)^3}$

Newton converge

H3) $f \in C^2([a,b])$

$$\Rightarrow \lim_{k\to+\infty} \frac{x^{(k+1)}-\alpha}{[x^{(k)}-\alpha]^2} = \underbrace{\frac{f''(\alpha)}{2f'(\alpha)}}_{C}$$

$p=2$
(convergenza quadratica)

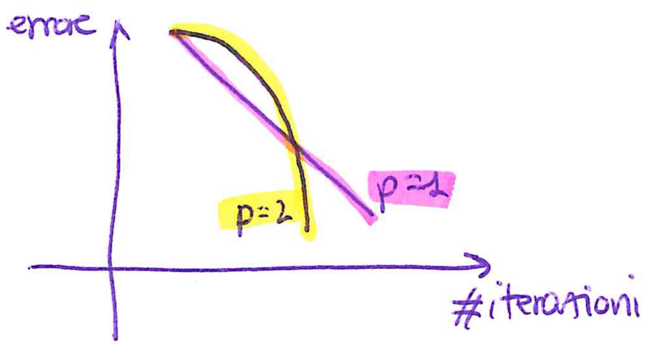
$$\boxed{\text{Newton modificato}}$$

$$x^{(k+1)} = x^{(k)} - m\,\frac{f(x^{(k)})}{f'(x^{(k)})} \qquad k \geq 0$$

molteplicità d'α

$f'(x^{(k)}) \neq 0$

$f(x) = (x-1)\log(x)$ $\quad \alpha = 1$, $m = 2$



H1) + H3) converge quadraticamente ($p=2$)

Scalare

$f(x) = 0$

$$x^{(k+1)} = x^{(k)} - \frac{f(x^{(k)})}{f'(x^{(k)})}$$

$$x^{(k+1)} = x^{(k)} + \delta x^{(k)} \quad \text{dove} \quad \delta x^{(k)} = -\frac{f(x^{(k)})}{f'(x^{(k)})}$$

$$f'(x^{(k)})\, \delta x^{(k)} = -f(x^{(k)})$$

Vettoriale

$$\begin{cases} f_1(x_1, x_2, \dots, x_n) = 0 \\ f_2(x_1, x_2, \dots, x_n) = 0 \\ \vdots \\ f_n(\boxed{x_1, x_2, \dots, x_n}) = 0 \end{cases}$$

$\vec{x}^T \in \mathbb{R}^n \ni \vec{0}$

Es $n=2$

$$\begin{cases} x_1^2 + x_2^2 - \pi = 0 \\ \operatorname{tg}\left(\frac{\pi}{3} x_1\right) + \sqrt{x_2} + 1 = 0 \end{cases}$$

$\vec{f} = [f_1, f_2, \dots, f_n]^T$

$$\boxed{\vec{f}(\vec{x}) = \vec{0}}$$

$\vec{x}^{(k)} = (x_1^{(k)}, x_2^{(k)}, \dots, x_n^{(k)})^T \in \mathbb{R}^n$

$\vec{x}^{(k+1)} = (x_1^{(k+1)}, x_2^{(k+1)}, \dots, x_n^{(k+1)})^T \in \mathbb{R}^n$

$$\vec{x}^{(k+1)} = \vec{x}^{(k)} + \overrightarrow{\delta x}^{(k)}$$

$$(J_F)_{ij}(\vec{x}) = \frac{\partial f_i}{\partial x_j}(\vec{x}) \quad i,j = 1, \dots, n$$

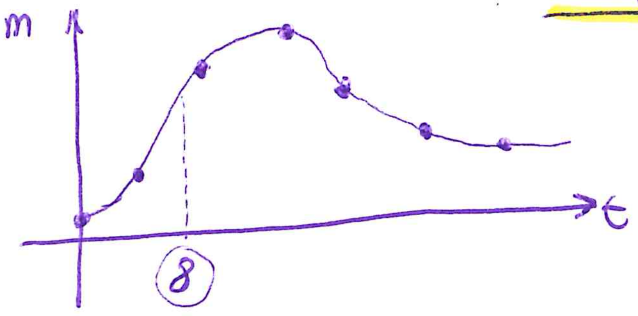
$$\underbrace{J_F(\vec{x}^{(k)})}_{A} \underbrace{\boxed{\overrightarrow{\delta x}^{(k)}}}_{x} = \underbrace{-\vec{f}(\vec{x}^{(k)})}_{b}$$

Approssimazione di dati e funzioni

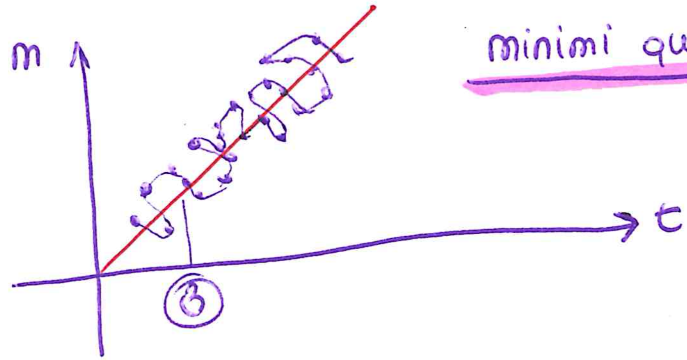
$f \in C^0([a,b])$

$(t_i, m_i) \quad i=0,\dots,n$

interpolazione



minimi quadrati



f

$$I(f) = \int_a^b f(x)\,dx \simeq \int_a^b \tilde{f}(x)\,dx$$

$$\uparrow$$
$$\tilde{f}$$

Taylor

$f(x) \quad x_0$

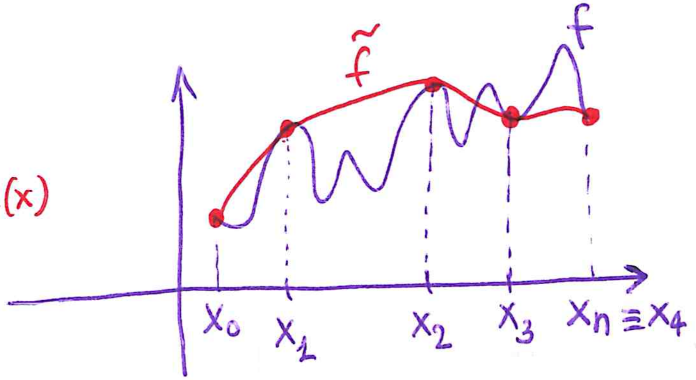
$\frac{1}{x} \quad x_0 = 1$

$e^x \quad x_0 = 0$

$\tilde{f}(x) = \Pi_4 f(x)$

$\uparrow$

$\mathbb{P}_4$



Interpolazione

$\{(x_i, y_i)\}_{i=0}^{n} \quad (n+1)$

$\downarrow$ nodi $\quad \downarrow$ valori

$x_i \neq$

dati $\quad y_i =$ misurazione

$x_i = t_i$

funzione $\quad y_i = f(x_i)$

$? \tilde{f}$ t.c. $\boxed{\tilde{f}(x_i) = y_i \quad i=0,\dots,n}$

condizioni di interpolazione

$\tilde{f}$
- polinomio: $\tilde{f} \in \mathbb{P}_n = \{ p_n(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n,\ a_i \in \mathbb{K} \}$
- trigonometrica: $\tilde{f}(x) = \sum_k a_k e^{ikx}$ (Fourier)
- razionale: $\tilde{f}(x) = \dfrac{p_k(x)}{q_s(x)} = \dfrac{a_0 + a_1 x + \dots + a_k x^k}{b_0 + b_1 x + \dots + b_s x^s}$ $\quad k, s \in \mathbb{N}$, $\quad a_i, b_i \in \mathbb{R}$

Interpolazione polinomiale Lagrangiana

Proposizione: $\{(x_i, y_i)\}_{i=0}^{n}$ con $x_i \neq$. $\exists!$ Π_n polinomio di grado $\leq n$ t.c. $\Pi_n(x_i) = y_i$ $\quad i = 0, \dots, n$

Π_n polinomio interpolatore

$\tilde{f} = \Pi_n$ dati $\quad y_i = m_i$

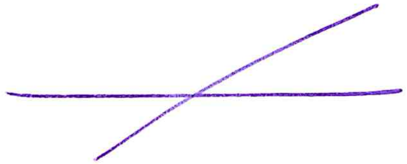
$\tilde{f} = \Pi_n f$ funzione $\quad y_i = f(x_i) \quad f \in C^0([a,b])$

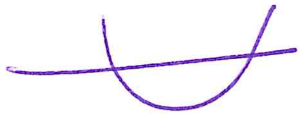
Dim: unicità

$\Pi_n \in \mathbb{P}_n \quad \Pi_n(x_i) = y_i \quad i = 0, \dots, n$

$\neq$

$\Pi_n^* \in \mathbb{P}_n \quad \Pi_n^*(x_i) = y_i \quad i = 0, \dots, n$





$r_n(x) = \Pi_n(x) - \Pi_n^*(x) \in \mathbb{P}_n$

$r_n(x_i) = \Pi_n(x_i) - \Pi_n^*(x_i) = y_i - y_i = 0 \quad i = 0, \dots, n$

$r_n \in \mathbb{P}_n$ MA ha $(n+1)$ zeri $\rightarrow$ $\boxed{r_n(x) = 0 \quad \forall x \in \mathbb{R}}$

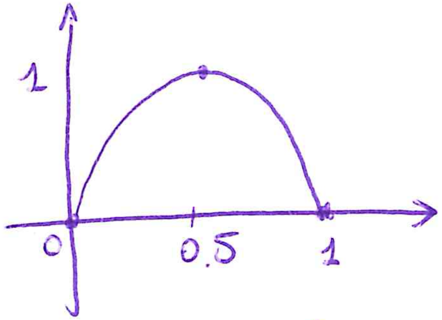
$\Pi_n(x) - \Pi_n^*(x) = 0 \quad \forall x \in \mathbb{R}$

$\Pi_n(x) = \Pi_n^*(x)$ $\quad$ $\boxed{\text{ASSURDO}}$

$\exists \; \Pi_n \qquad \{(x_i, y_i)\}_{i=0}^{2}$

$x_0 = 0 \qquad x_1 = 0.5 \qquad x_2 = 1$

$y_0 = 0 \qquad y_1 = 1 \qquad y_2 = 0$



$\varphi_1 \in \mathbb{P}_2$

$$\left.\begin{array}{ll} \boxed{\text{I}} & \varphi_1(0) = 0 \\ \boxed{\text{II}} & \varphi_1(0.5) = 1 \\ \boxed{\text{III}} & \varphi_1(1) = 0 \end{array}\right] \text{condizioni di interpolazione}$$

$\boxed{\text{I}}$ e $\boxed{\text{III}}$ $\qquad x(x-1)$

$a \in \mathbb{R} \qquad (x-0)(x-1) \in \mathbb{P}_2$

$(x-a)$

$\boxed{\text{II}}$

$$\frac{(x-0)(x-1)}{(0.5-0)(0.5-1)} = \frac{x(x-1)}{\frac{1}{2}\left(-\frac{1}{2}\right)} = -4(x-1)x = -4(x^2-x)$$

$x_0 \quad x_1 \quad \dots \quad x_{k-1} \quad x_k \quad x_{k+1} \quad \dots \quad x_n$

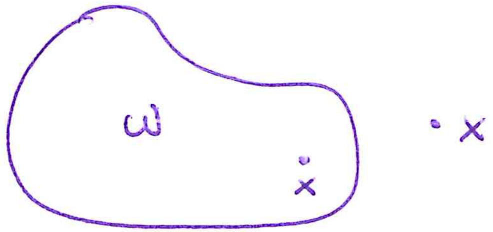
$y_0 = 0 \quad y_1 = 0 \quad \dots \quad y_{k-1} = 0 \quad y_k = 1 \quad y_{k+1} = 0 \quad \dots \quad y_n = 0$

polinomi caratteristici di Lagrange

$$\varphi_k(x_j) = \delta_{jk} = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases} : \mathbb{P}_n \ni \varphi_k(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_0)(x_k-x_1)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)} = \prod_{\substack{j=0 \\ j\neq k}}^{n} \frac{(x-x_j)}{(x_k-x_j)}$$

delta di Kronecker

$$\chi_\omega(x) = \begin{cases} 0 & x \notin \omega \\ 1 & x \in \omega \end{cases}$$



$x_0 \quad x_1 \quad x_2$

$y_0 \quad y_1 \quad y_2$

$\varphi_0, \quad \varphi_1, \quad \varphi_2, \in \mathbb{P}_2$

$\Pi_2 \in \mathbb{P}_2 \qquad \Pi_2(x_0) = y_0$

$\Pi_2(x_1) = y_1$

$\Pi_2(x_2) = y_2$

$\varphi_0(x_0) = 1 \quad \varphi_0(x_1) = 0 \quad \varphi_0(x_2) = 0$

$\varphi_1(x_0) = 0 \quad \varphi_1(x_1) = 1 \quad \varphi_1(x_2) = 0$

$\varphi_2(x_0) = 0 \quad \varphi_2(x_1) = 0 \quad \varphi_2(x_2) = 1$

$\Pi_2(x) = a\varphi_0(x) + b\varphi_1(x) + c\varphi_2(x)$

$? \; a, b, c \in \mathbb{R}$

$\Pi_2 \in \mathbb{P}_2$

$$\Pi_2(x_0) = a\underbrace{\varphi_0(x_0)}_{1} + b\underbrace{\varphi_1(x_0)}_{0} + c\underbrace{\varphi_2(x_0)}_{0} = y_0$$

$a = y_0$

$$\Pi_2(x_1) = a\underbrace{\varphi_0(x_1)}_{0} + b\underbrace{\varphi_1(x_1)}_{1} + c\underbrace{\varphi_2(x_1)}_{0} = y_1$$

$b = y_1$

$$\Pi_2(x_2) = a\underbrace{\varphi_0(x_2)}_{0} + b\underbrace{\varphi_1(x_2)}_{0} + c\underbrace{\varphi_2(x_2)}_{1} = y_2$$

$c = y_2$

$\Pi_2(x) = y_0\varphi_0(x) + y_1\varphi_1(x) + y_2\varphi_2(x)$

$x_0 \quad x_1 \quad \cdots \quad x_n$

$y_0 \quad y_1 \quad \cdots \quad y_n$

$$\Pi_n(x) = y_0\varphi_0(x) + y_1\varphi_1(x) + \dots + y_n\varphi_n(x)$$

$\cap$

$\mathbb{P}_n$

$$\boxed{\Pi_n(x) = \sum_{k=0}^{n} y_k \varphi_k(x) \qquad \varphi_k(x) = \prod_{\substack{j=0 \\ j\neq k}}^{n} \frac{x-x_j}{x_k-x_j}}$$

forma di Lagrange del polinomio interpolatore

$[ax+by,\ a,b\in\mathbb{R}]$

$$\Pi_n(x_i) \overset{?}{=} y_i$$

$$\Pi_n(x_i) = \sum_{k=0}^{n} y_k \underbrace{\varphi_k(x_i)}_{=1 \text{ per } k=i} = y_i\varphi_i(x_i) = y_i$$

polyfit

polyval

$c = \text{polyfit}(x, y, n)$, $c \in \mathbb{R}^{n+1}$; $x \to \{x_i\}$, $y \to \{y_i\}$, $n \to$ grado

$c(1)x^n + c(2)x^{n-1} + \dots + c(n)x + c(n+1) = \Pi_n(x)$

$d = \text{polyval}(c, z)$: $d \in \mathbb{R}$ / $d \in \mathbb{R}^q$; $z \in \mathbb{R}$ / $z \in \mathbb{R}^q$

$\Pi_n(z) = d$

$q = 2 \qquad z = [z_1, z_2]^T$

$$d = \begin{bmatrix} \Pi_n(z_1) \\ \Pi_n(z_2) \end{bmatrix}$$

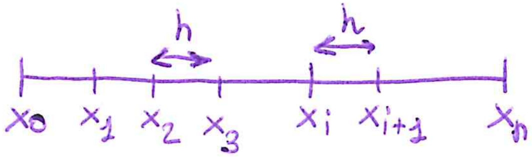
Errore Sia $I = [x_0, x_n]$; $\{(x_i, y_i)\}_{i=0}^{n}$ x_i distinti. Se $f \in C^{n+1}(I)$, allora $\forall x \in I$ $\exists\, \alpha = \alpha(x) \in I$ t.c.

$$E_n f(x) = f(x) - \underbrace{\Pi_n f(x)}_{\tilde{f}} = \frac{f^{(n+1)}(\alpha(x))}{(n+1)!} \prod_{i=0}^{n} (x - x_i)$$

$E_n f(x)$: errore di interpolazione

$E_n f(x_i) = 0$, $i = 0, \dots, n$

$\boxed{x_i \text{ uniformi}}$



$h = \dfrac{x_n - x_0}{n} = \dfrac{|I|}{n}$ passo di discretizzazioni

$x_k = x_{k-1} + h \qquad k = 1, \dots, n$

$[x_k = x_0 + kh \qquad k = 1, \dots, n]$

Se $\{x_i\}$ sono uniformi

$$\left| \prod_{i=0}^{n} (x - x_i) \right| \leq n! \, \frac{h^{n+1}}{4} \qquad \forall x \in I$$

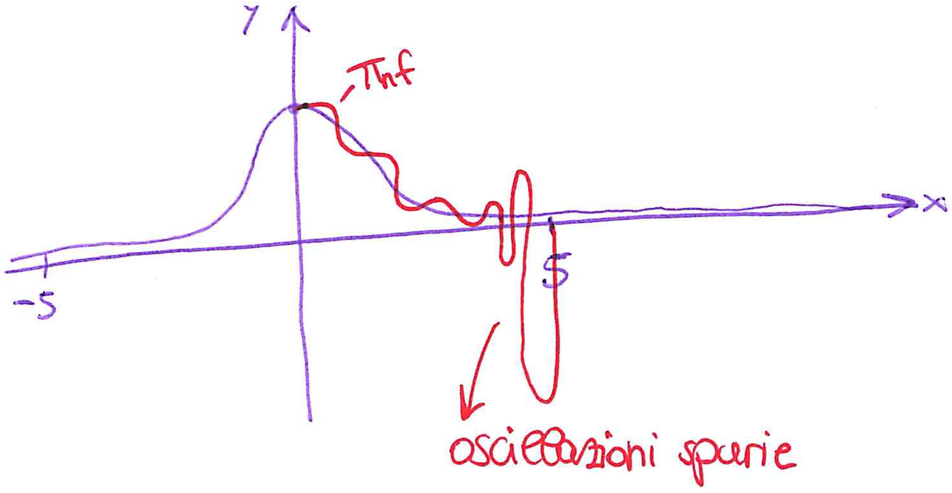
$$\max_{x \in I} |E_n f(x)| \leq \max_{x \in I} \frac{|f^{(n+1)}(x)|}{(n+1)!} \left| \prod_{i=0}^{n} (x - x_i) \right| \leq \max_{x \in I} \frac{|f^{(n+1)}(x)|}{\cancel{(n+1)!}_{(n+1)}} \frac{\cancel{n!}\, h^{n+1}}{4}$$

$$\max_{x \in I} |E_n f(x)| \leq \underbrace{\boxed{\frac{h^{n+1}}{4(n+1)}}}_{①} \; \underbrace{\boxed{\max_{x \in I} |f^{(n+1)}(x)|}}_{②}$$

① $\xrightarrow{n \to +\infty} 0$

② $\to 0$ ok ; $\to C$ ok ; $\to +\infty$ → ok / ko

$$f(x) = \frac{1}{1+x^2} \qquad [-5,5]$$



	$\boxed{1}$	$\boxed{2}$	$\boxed{1}\cdot\boxed{2}$
$n=3$	$O(10)$	$O(10^0)$	$O(10^1)$
$n=9$	$O(10^{-2})$	$O(10^5)$	$O(10^4)$
$n=15$	$O(10^{-5})$	$O(10^{12})$	$O(10^7)$
$n=21$	$O(10^{-10})$	$O(10^{19})$	$O(10^{10})$
$\downarrow n\to+\infty$	0	$+\infty$	$+\infty$

FENOMENO DI RUNGE

Soluzioni al fenomeno di Runge

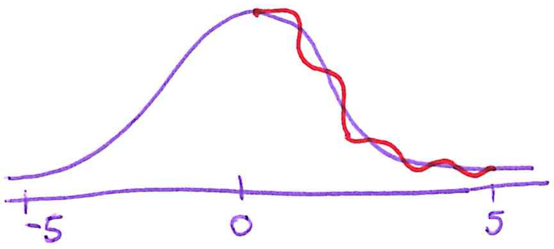
1) nodi di Chebyshev ; nodi di Chebyshev - Gauss - Lobatto

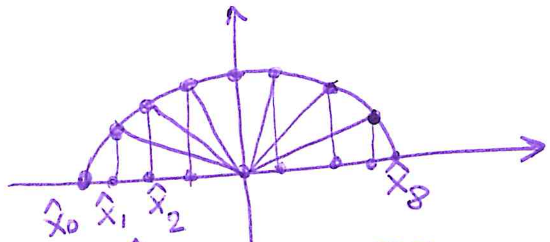
$$I = [a,b]$$

$$\boxed{x_i = \frac{b+a}{2} + \frac{b-a}{2}\hat{x}_i} \qquad [a,b]$$

$$\hat{x}_i = -\cos\left(\frac{\pi i}{n}\right) \quad i = 0,\dots,n \qquad [-1,1]$$

Se $f \in C^1([a,b]) \Rightarrow \Pi_n f \to f \quad n \to +\infty$





Ex: $n=8 \quad \hat{x}_0 = -\cos\left(\frac{\pi 0}{8}\right) = -1$

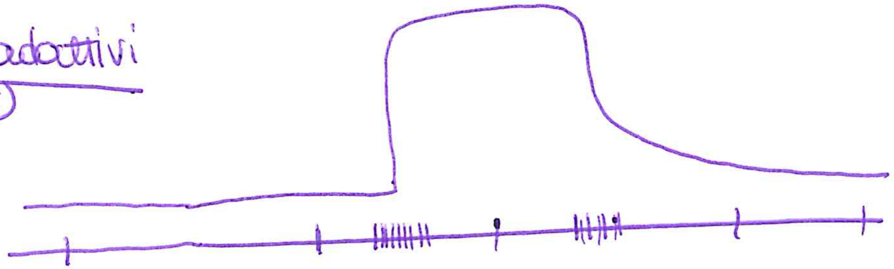
$\hat{x}_1 = -\cos\left(\frac{\pi}{8}\right)$

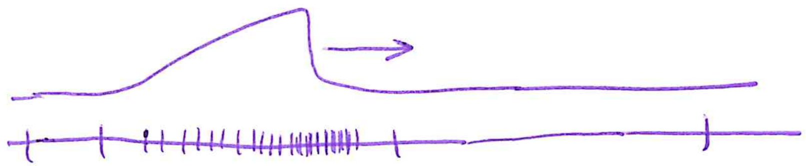
$\hat{x}_2 = -\cos\left(\frac{\pi}{4}\right)$

$\vdots$

$\hat{x}_8 = -\cos(\pi)$

② Nodi adattivi (plot)





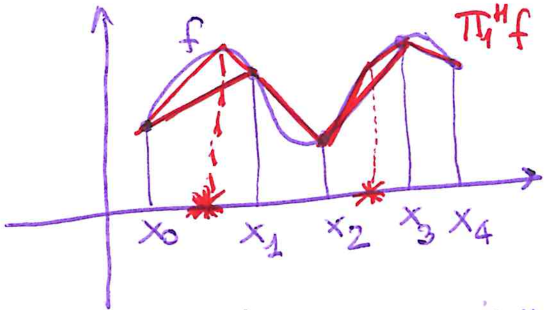
③ Interpolazione lineare a tratti

$\Pi_1^H f \in C^0(I)$, $I = [x_0, x_n]$

$\Pi_1^H f \big|_{I_i} \in \mathbb{P}_1(I_i)$, $\Pi_1^H f(x_i) = y_i$ $\quad i = 0, \dots, n$

$$\Pi_1^H f(x) \big|_{I_i} = f(x_i) + \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}(x - x_i)$$

$I_i \quad (x_i, f(x_i)), (x_{i+1}, f(x_{i+1}))$

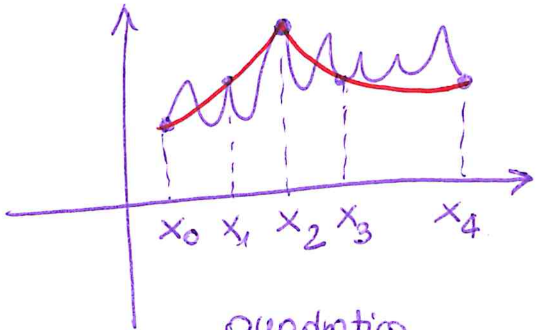


x_i non necessariamente uniformi

$I_i = [x_i, x_{i+1}]$ $\quad i = 0, \dots, n-1$

$h_i = |I_i| = x_{i+1} - x_i$

$H = \max_i h_i$

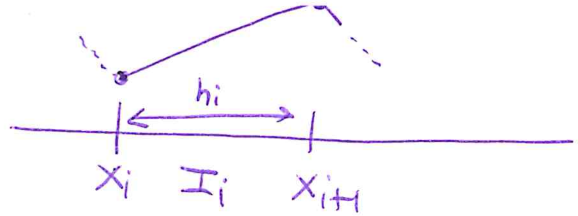


quadratica a tratti

$\Pi_2^H f$

$n=1$

$I_i \quad \max_{x\in I_i} \left| E_n f(x)\big|_{I_i} \right| \leq \frac{h_i^2}{8} \max_{x\in I_i} |f''(x)|$



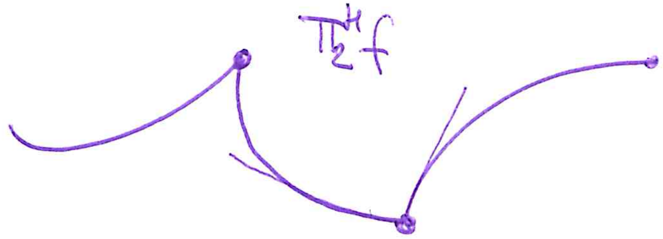
$I \quad \max_{x\in I} |E_n f(x)| \leq \underbrace{\frac{H^2}{8}}_{\substack{\downarrow\, n\to+\infty \\ 0}} \underbrace{\max_{x\in I} |f''(x)|}_{\text{costante}}$

$f \in C^2(\bar{I})$

EF (elementi finiti)

$d = \text{interp1}(x, y, z)$

d: $\mathbb{R}$, $\mathbb{R}^q$ — x: nodi — y: valori — z: $\mathbb{R}$, $\mathbb{R}^q$



$n=2$

$\frac{H^3}{12} \max_{x\in I} |f'''(x)|$

$f \in C^3(\bar{I})$

4

CG

CAD

C^1

C^2

Interpolazione cubica spline

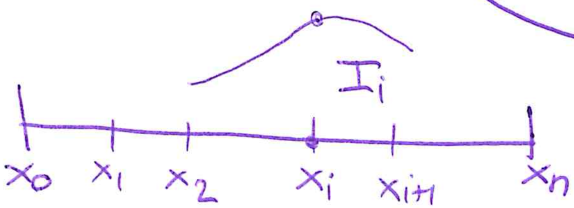
s_3 t.c. $s_3\big|_{I_i} \in \mathbb{P}_3(I_i)$

$\boxed{A}$ $s_3(x_i) = f(x_i) \quad i=0,\dots,n$

$\boxed{B}$ $s_3 \in C^2(\bar{I})$ (smooth)

I_i 4 incognite

$4n$ incognite



$\boxed{A}$ $n+1$ condizioni

$\boxed{B}$ $s_3 \in C^2(\bar{I})$ $\rightarrow$ $s_3 \in C^0(\bar{I})$ $\boxed{B1}$; $s_3' \in C^0(\bar{I})$ $\boxed{B2}$; $s_3'' \in C^0(\bar{I})$ $\boxed{B3}$

$\boxed{B1}$ $s_3^-(x_i) = s_3^+(x_i)$ $\quad i=1,\dots,n-1$ $\quad (n-1)$ condizioni

$\boxed{B2}$ $[s_3'(x_i)]^- = [s_3'(x_i)]^+$ $\quad i=1,\dots,n-1$ $\quad (n-1)$ condizioni

$\boxed{B3}$ $[s_3''(x_i)]^- = [s_3''(x_i)]^+$ $\quad i=1,\dots,n-1$ $\quad (n-1)$ condizioni

---

$(n+1) + 3(n-1) = 4n-2$ condizioni

$4n$ incognite

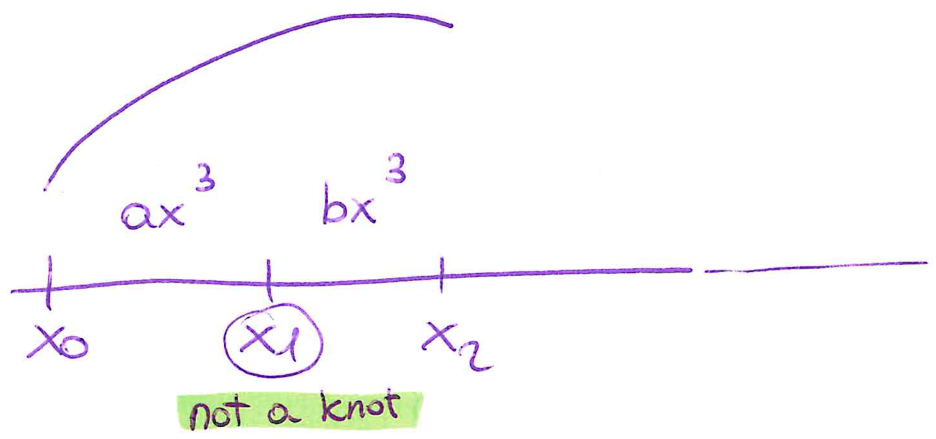
MANCANO 2 condizioni

1) $s_3''(x_0) = s_3''(x_n) = 0$ (naturale)

2) s_3''' continua in x_1 e x_{n-1} (not-a-knot)

$\mathbb{R} \ni$, $\mathbb{R}^q \ni$ } d = spline (x, y, z)

$\in \mathbb{R}$ $\in \mathbb{R}^q$



$$\left[S_3'''(x_1)\right]^- = \left[S_3'''(x_1)\right]^+ \longrightarrow a = b$$

S_3'	$3ax^2$	$3bx^2$
S_3''	$6ax$	$6bx$
S_3'''	$6a$	$6b$

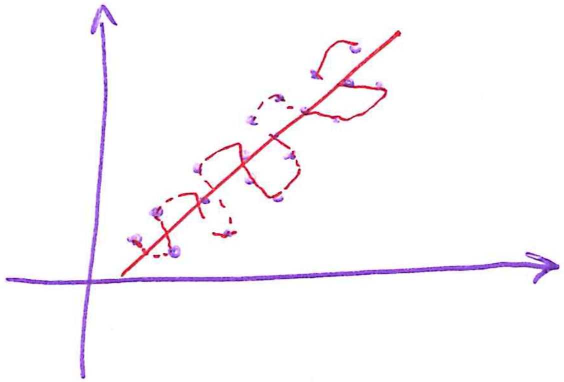
$$\begin{array}{cccc}
\boxed{S_3 \simeq f} & H^4 & C_0 \\
 & & \wedge\!\!\!| \\
S_3' \simeq f' & H^3 & C_1 \\
 & & \wedge\!\!\!| \\
S_3'' \simeq f'' & H^2 & C_2
\end{array}$$

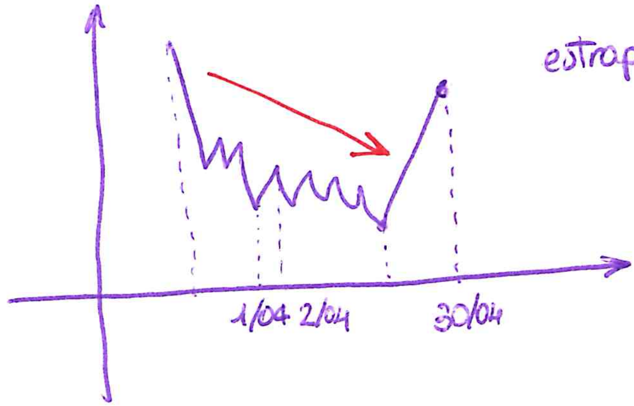
$$\max_{x \in I} \left| S_3^{(q)}(x) - f^{(q)}(x) \right| \leq C_q H^{4-q} \max_{x \in I} |f^{(4)}(x)| , \quad q = 0, 1, 2$$

$$f \in C^4(\bar{I})$$

Minimi quadrati

$\{(x_i, y_i)\}_{i=0}^{n}$





~~interpolatione~~

$\{(x_i, y_i)\}_{i=0}^{n}$ $\quad x_i \neq$

$y_i = f(x_i)$

$$\boxed{\begin{array}{l} ?\ \tilde{f} \in \mathbb{P}_m \quad m \ll n \quad \text{t.c.} \\ \displaystyle\sum_{i=0}^{n} [y_i - \hat{f}(x_i)]^2 \leq \sum_{i=0}^{n} [y_i - p_m(x_i)]^2 \quad \forall p_m \in \mathbb{P}_m \end{array}}$$

- m sceeto da utente
- DATA FITTING
  - se $\hat{f}$ $\exists$, $\hat{f}$ approssimatione nel senso dei minimi quadrati ($m=1$, $\hat{f}$ retta di regressione)
- $m = n$, $\tilde{f} \equiv \Pi_n f$
- rinuncio alle $(n+1)$ condizioni ~~$\hat{f}(x_i) = y_i$~~ $i=0,\dots,n$

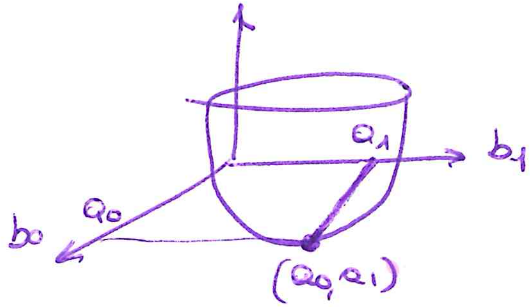
? $\tilde{f}(x) = a_0 + a_1 x + \dots + a_m x^m$ $\qquad$ $p_m(x) = b_0 + b_1 x + \dots + b_m x^m$

? $a_0, a_1, \dots, a_m$

$$\Phi(b_0, b_1, \dots, b_m) = \sum_{i=0}^{n} \left[y_i - (b_0 + b_1 x_i + b_2 x_i^2 + \dots + b_m x_i^m) \right]^2 \qquad \text{D}$$

$$\Phi(a_0, a_1, \dots, a_m) = \sum_{i=0}^{n} \left[y_i - (a_0 + a_1 x_i + a_2 x_i^2 + \dots + a_m x_i^m) \right]^2 \qquad \text{S}$$

$$\boxed{\Phi(a_0, a_1, \dots, a_m) = \min_{b_0, b_1, \dots, b_m} \Phi(b_0, b_1, \dots, b_m)}$$



$m = 1$ $\qquad$ $\tilde{f}(x) = a_0 + a_1 x$ $\qquad$? a_0, a_1 $\qquad$ $p_1(x) = b_0 + b_1 x$

$$\Phi(a_0, a_1) = \min_{b_0, b_1} \Phi(b_0, b_1) \quad \text{con} \quad \Phi(b_0, b_1) = \sum_{i=0}^{n} \left[y_i - b_0 - b_1 x_i \right]^2$$

$$= \sum_{i=0}^{n} \left[y_i^2 + b_0^2 + b_1^2 x_i^2 - 2 y_i b_0 - 2 y_i b_1 x_i + 2 b_0 b_1 x_i \right]$$

$$\begin{cases} \dfrac{\partial \Phi}{\partial b_0}(a_0, a_1) = 0 \quad (1) \\[2ex] \dfrac{\partial \Phi}{\partial b_1}(a_0, a_1) = 0 \quad (2) \end{cases}$$

$$\frac{\partial \Phi}{\partial b_0}(b_0, b_1) = \sum_{i=0}^{n} \left[2 b_0 - 2 y_i + 2 b_1 x_i \right]$$

$$\frac{\partial \Phi}{\partial b_0}(a_0, a_1) = \sum_{i=0}^{n} \left[2 a_0 - 2 y_i + 2 a_1 x_i \right] = 0$$

$$\sum_{i=0}^{n} \left[a_0 - y_i + a_1 x_i \right] = 0 \quad (1)$$

$$\frac{\partial \Phi}{\partial b_1}(b_0, b_1) = \sum_{i=0}^{n} \left[2b_1 x_i^2 - 2y_i x_i + 2b_0 x_i \right]$$

$$\frac{\partial \Phi}{\partial b_1}(a_0, a_1) = \sum_{i=0}^{n} \left[2a_1 x_i^2 - 2y_i x_i + 2a_0 x_i \right] = 0$$

$$\sum_{i=0}^{n} \left[a_1 x_i^2 - y_i x_i + a_0 x_i \right] = 0 \qquad (2)$$

$$\begin{cases} \sum_{i=0}^{n} \left[a_0 - y_i + a_1 x_i \right] = 0 & (1) \\ \sum_{i=0}^{n} \left[a_1 x_i^2 - y_i x_i + a_0 x_i \right] = 0 & (2) \end{cases}$$

$$\begin{cases} a_0 \underbrace{\boxed{\sum_{i=0}^{n} 1}}_{B_{11} \,=\, (n+1)} + a_1 \underbrace{\boxed{\sum_{i=0}^{n} x_i}}_{B_{12}} = \sum_{i=0}^{n} y_i \\ a_0 \underbrace{\boxed{\sum_{i=0}^{n} x_i}}_{B_{21}} + a_1 \underbrace{\boxed{\sum_{i=0}^{n} x_i^2}}_{B_{22}} = \sum_{i=0}^{n} x_i y_i \end{cases}$$

$$\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} \ \\ \ \end{bmatrix}$$

simmetrica

$\hat{f} \in \mathbb{P}_m \quad ?\, a_0, a_1, \dots, a_m$

$$\begin{cases} a_0 (n+1) + a_1 \sum x_i + a_2 \sum x_i^2 \dots + a_m \sum x_i^m = \sum y_i \\ a_0 \sum x_i + a_1 \sum x_i^2 + a_2 \sum x_i^3 \dots + a_m \sum x_i^{m+1} = \sum x_i y_i \\ \vdots \\ a_0 \sum x_i^m + a_1 \sum x_i^{m+1} + a_2 \sum x_i^{m+2} + \dots + a_m \sum x_i^{2m} = \sum x_i^m y_i \end{cases}$$

equazioni normali

c = polyfit(x, y, m)

d = polyval(c, z)

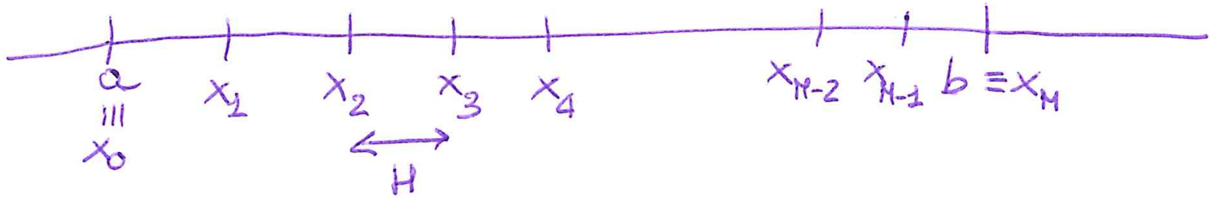
Approssimazione di integrali definiti

$$I(f) = \int_a^b f(x)\,dx \qquad f \in C^0([a,b])$$

$$\wr\wr \qquad \downarrow$$

$$\tilde{I}(f) = \int_a^b \tilde{f}(x)\,dx \qquad \tilde{f} \in \mathbb{P}_q$$

formula di quadratura



$$H = \frac{b-a}{M} \;; \quad I_k = [x_{k-1}\, x_k] \;; \quad x_k = x_{k-1} + H, \quad x_k = x_0 + kH$$

partizione uniforme

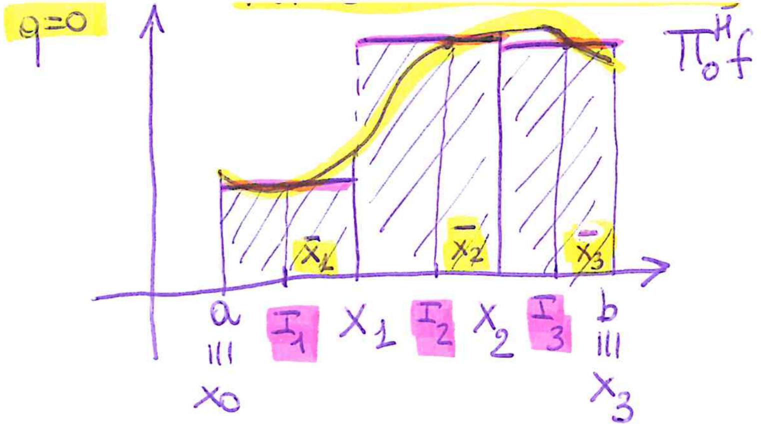
$$I(f) = \int_a^b f(x)\,dx = \sum_{k=1}^{M} \int_{I_k} f(x)\,dx \simeq \sum_{k=1}^{M} \int_{I_k} \tilde{f}(x)\,dx = \tilde{I}(f)$$

$$\tilde{f}(x) \in \mathbb{P}_q(I_k)$$

$q = 0$ rettangolo

$q = 1$ trapezio

$q = 2$ Simpson



$\bar{x}_k = \frac{x_{k-1}+x_k}{2}$; $I_k = [x_{k-1}, x_k]$

$$\tilde{I}^c_{PM}(f) = \sum_{k=1}^{M} H f(\bar{x}_k) = H \sum_{k=1}^{M} f(\bar{x}_k)$$

$$\tilde{I}_{PM}(f) = (b-a) f\left(\frac{a+b}{2}\right)$$

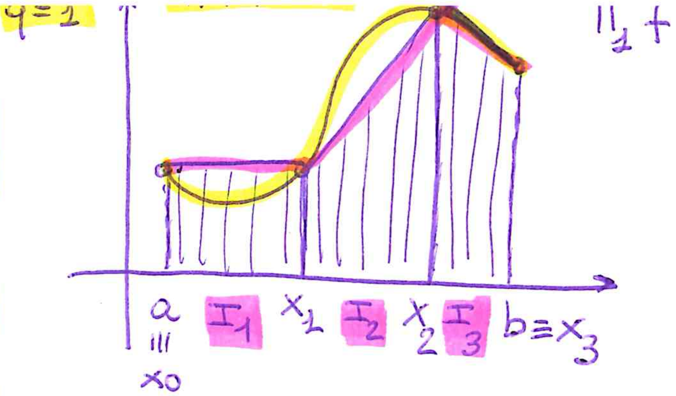
$$I(f) - \tilde{I}_{PM}(f) = \frac{1}{24}(b-a)^3 f''(\beta) \quad f \in C^2([a,b])$$

$\alpha, \beta \in [a,b]$

$$I(f) - \tilde{I}^c_{PM}(f) = \frac{1}{24}(b-a) H^2 f''(\alpha) \quad f \in C^2([a,b])$$

oda: 2

gde: 1



$$\tilde{I}^c_T(f) = \frac{H}{2} \sum_{k=1}^{M} [f(x_{k-1}) + f(x_k)]$$

$$= \frac{H}{2}[f(a)+f(b)] + H \sum_{k=1}^{M-1} f(x_k)$$

$$\tilde{I}_T(f) = \frac{(b-a)}{2}[f(a)+f(b)]$$

$$I(f) - \tilde{I}_T(f) = \frac{-1}{12}(b-a)^3 f''(\sigma)$$

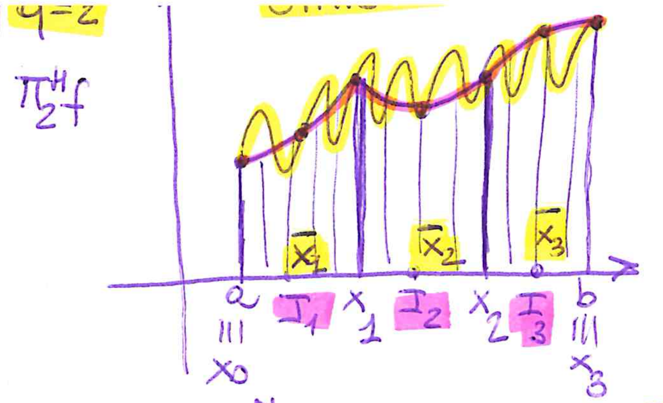
$\eta, \sigma \in [a,b]$

$f \in C^2([a,b])$

$$I(f) - \tilde{I}^c_T(f) = \frac{-1}{12}(b-a) H^2 f''(\eta)$$

oda: 2

gde: 1



$$\tilde{I}^c_S(f) = \frac{H}{6} \sum_{k=1}^{M} [f(x_{k-1}) + 4f(\bar{x}_k) + f(x_k)]$$

$$= \frac{H}{6}[f(a)+f(b)] + \frac{H}{3}\sum_{k=1}^{M-1} f(x_k) + \frac{2}{3} H \sum_{k=1}^{M} f(\bar{x}_k)$$

$$\tilde{I}_S(f) = \frac{(b-a)}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

$$I(f) - \tilde{I}_S(f) = \frac{-1}{2880}(b-a)^5 f^{(4)}(\nu)$$

$\delta, \nu \in [a,b]$

$f \in C^4([a,b])$

$$I(f) - \tilde{I}^c_S(f) = \frac{-1}{2880}(b-a) H^4 f^{(4)}(\delta)$$

oda: 4

gde: 3

Errore punto medio

$$f(\bar{x})(b-a)$$

$$\underbrace{I(f) - \tilde{I}_{PM}(f)}_{E_{PM}(f)} = \int_a^b f(x)\,dx - \overbrace{\int_a^b f(\bar{x})\,dx} = \int_a^b [f(x) - f(\bar{x})]\,dx = \circledast$$

$$\bar{x} = \frac{a+b}{2}$$

Taylor $f \in C^2([a,b])$

$$f(x) = f(\bar{x}) + f'(\bar{x})(x-\bar{x}) + \frac{f''(\alpha(x))}{2}(x-\bar{x})^2$$

$$\circledast = \cancel{\int_a^b f'(\bar{x})(x-\bar{x})\,dx} + \frac{1}{2}\int_a^b f''(\alpha(x))(x-\bar{x})^2\,dx = \frac{1}{2} f''(\underset{[a,b]}{\beta}) \int_a^b (x-\bar{x})^2\,dx = \frac{1}{2} f''(\beta) \cdot \frac{1}{12}(b-a)^3$$

$$\int_a^b (x-\bar{x})\,dx = \left.\frac{(x-\bar{x})^2}{2}\right|_a^b = \frac{1}{2}\left[\left(b - \frac{a+b}{2}\right)^2 - \left(a - \frac{a+b}{2}\right)^2\right] = \frac{1}{2}\left[\left(\frac{b-a}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2\right] = 0$$

$$\int_a^b (x-\bar{x})^2\,dx = \left.\frac{(x-\bar{x})^3}{3}\right|_a^b = \frac{1}{3}\left[\left(\frac{b-a}{2}\right)^3 - \left(\frac{a-b}{2}\right)^3\right] = \frac{1}{3}\cdot\frac{1}{\cancel{8}_{\,4}}\left[\cancel{2}(b-a)^3\right] = \frac{1}{12}(b-a)^3$$

$$(b-a)^2 = (a-b)^2$$

$$(b-a)^3 = -(a-b)^3$$

$$\boxed{I(f) - \tilde{I}_{PM}(f) = \frac{1}{24}(b-a)^3 f''(\beta)}$$

$$I(f) - \tilde{I}^{c}_{PM}(f) = \sum_{k=1}^{M} \underbrace{\left[\int_{I_k} f(x)dx - I_{PM}(f|_{I_k}) \right]}_{\frac{1}{24} H^3 f''(\beta_k),\ \beta_k \in I_k} = \frac{H^3}{24} \sum_{k=1}^{M} 1 \cdot f''(\beta_k) = \frac{H^3}{24} f''(\alpha) \underbrace{\sum_{k=1}^{M} 1}_{M}, \quad \alpha \in [a,b]$$

$$= \frac{H^{\cancel{3}\,2}}{24} M f''(\alpha) = \frac{1}{24} (b-a) H^2 f''(\alpha) \qquad f \in C^2([a,b])$$

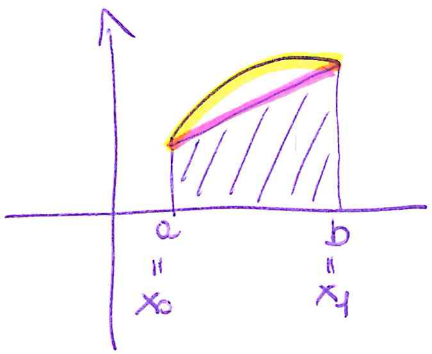
$$M = \frac{b-a}{\cancel{H}}$$

Errore trapezio

$$I(f) - \tilde{I}_T(f) = \int_a^b \underbrace{E_1 f(x)}\, dx = \int_a^b \frac{f''(\xi(x))}{2} (x-a)(x-b)\, dx$$

$$= \frac{1}{2} f''(\sigma) \int_a^b (x-a)(x-b)\, dx, \quad \sigma \in [a,b]$$

$$= \frac{1}{2} f''(\sigma) \left(-\frac{1}{6}\right)(b-a)^3 = -\frac{1}{12} (b-a)^3 f''(\sigma)$$



$$E_n f(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \prod_{i=0}^{n} (x - x_i)$$

$$\boxed{n=1}$$

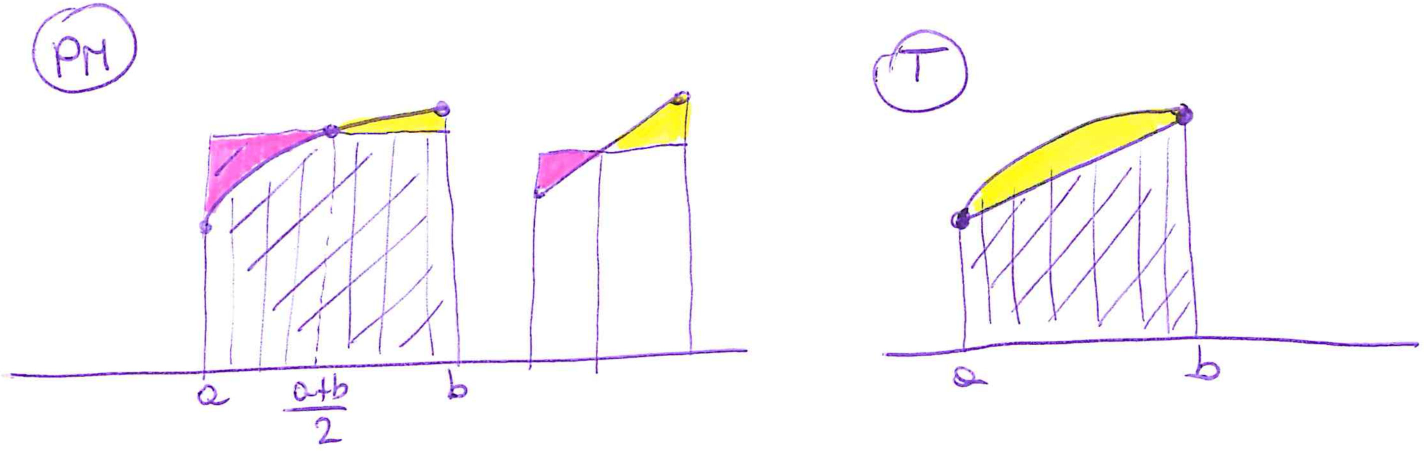
$$\int_a^b (x-a)(x-b)\,dx = -\int_a^b \frac{(x-a)^2}{2} dx + \cancel{\left.\frac{(x-a)^2}{2}(x-b)\right|_a^b} = -\frac{1}{2}\int_a^b (x-a)^2 dx$$

x parti

$$= -\frac{1}{2} \left.\frac{(x-a)^3}{3}\right|_a^b = -\frac{1}{6} (b-a)^3$$

$$\max_{x\in[a,b]} |I(f)-\tilde{I}(f)| \leq C\,(b-a)^{\rho} \max_{x\in[a,b]} |f^{(e)}(x)|$$

$$\max_{x\in[a,b]} |I(f)-\tilde{I}^{c}(f)| \leq C\,(b-a)\,H^{\mu} \underbrace{\max_{x\in[a,b]} |f^{(m)}(x)|} \leq TOL\ (10^{-5}) \rightarrow H$$



__grado di esatezza__ : il max grado dei polinomi __integrati esattamente__ dalle formule di quadratura

__ordine di accuratezza__ : velocità con cui l'errore và a zero (formule composite)

$I(f) \neq \tilde{I}(f)$

$f \in \mathbb{P}_n \quad I(f) \equiv \tilde{I}(f)$

$\mathbb{P}_0$	1
$\mathbb{P}_1$	x
$\mathbb{P}_2$	x^2
$\mathbb{P}_3$	x^3

$\int_a^b 1\,dx \overset{?}{=} \tilde{I}(1)$

$\int_a^b x\,dx \overset{?}{=} \hat{I}(x)$

$\vdots$

gde=1 $\leftarrow$

$f\in\mathbb{P}_0 \quad ?\ I(f)\equiv\hat{I}(f)$ ✓

$f\in\mathbb{P}_1 \quad ?\ I(f)\equiv\hat{I}(f)$ ✓

$f\in\mathbb{P}_2 \quad ?\ I(f)\equiv\hat{I}(f)$ ✗

$f\in\mathbb{P}_3 \quad ?\ I(f)\equiv\hat{I}(f)$ ✓

Formule di quadratura interpolatorie

- punto medio ($J=1$; $\eta_1 = \left(\frac{[illegible]}{2}\right)$ [illegible]
- trapezi ($J=2$; $\eta_1 = a$; $\eta_2 = b$; $\alpha_1 = \alpha_2 = \frac{(b-a)}{2}$)
- (*) Simpson ($J=3$; $\eta_1 = a$; $\eta_2 = \frac{a+b}{2}$; $\eta_3 = b$; $\alpha_1 = \alpha_3 = \frac{b-a}{6}$. $\alpha_2 = \frac{2}{3}(b-a)$)

Newton-Cotes (*) | Gaussiane

$$I(f) \simeq \tilde{I}(f) = \sum_{j=1}^{J} f(\eta_j)\,\alpha_j$$

η_j: nodi di quadratura

α_j: pesi di quadratura

$\alpha_j \in \mathbb{R}$

$\eta_j \in [a,b]$

$\boxed{gde = 0}$

$$\underbrace{I(1)}_{\int_a^b 1\,dx = \boxed{(b-a)}} = \tilde{I}(1) = \boxed{\sum_{j=1}^{J} \alpha_j}$$

Formule di Gaussiane

$J=2$ $\quad \eta_1, \eta_2, \alpha_1, \alpha_2$ (4 incognite) t.c. gde sia massimo

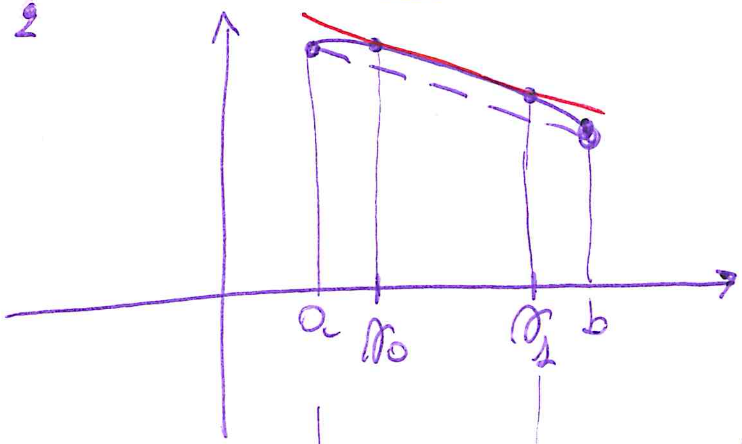
$$\begin{cases}
(b-a) = \alpha_1 + \alpha_2 & \text{condizione 1} \quad gde = 0 \\
\int_a^b x\,dx = \frac{b^2}{2} - \frac{a^2}{2} = \tilde{I}(x) = \sum_{j=1}^{2} \eta_j \alpha_j = \eta_1\alpha_1 + \eta_2\alpha_2 & \text{"} \quad 2 \quad gde = 1 \\
\int_a^b x^2\,dx = \frac{b^3}{3} - \frac{a^3}{3} = \tilde{I}(x^2) = \eta_1^2\alpha_1 + \eta_2^2\alpha_2 & \text{"} \quad 3 \quad gde = 2 \\
\int_a^b x^3\,dx = \frac{b^4}{4} - \frac{a^4}{4} = \tilde{I}(x^3) = \eta_1^3\alpha_1 + \eta_2^3\alpha_2 & \text{"} \quad 4 \quad gde = 3
\end{cases}$$

$$\int_a^b f(x)\,dx \simeq \int_a^b \tilde{f}(x)\,dx \qquad \tilde{f} \in \mathbb{P}_q$$

$$\tilde{I}_G(f) = \frac{b-a}{2}\left[f(x_0) + f(x_1)\right]$$

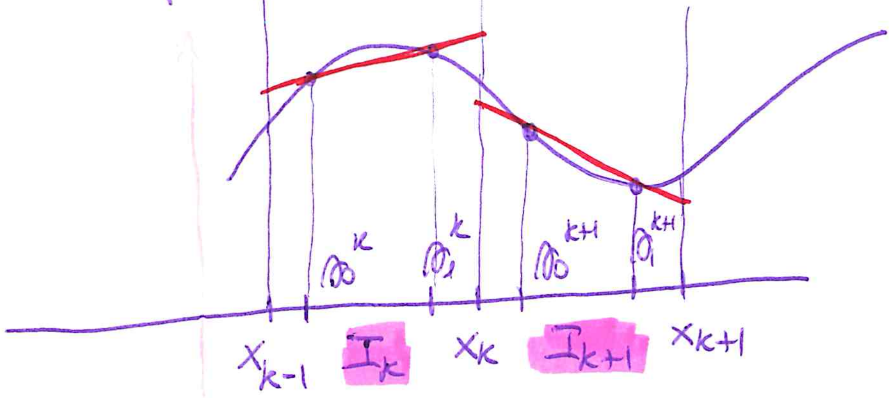
$$\eta_1 = x_0 = a + \underbrace{\left(1 - \frac{1}{\sqrt{3}}\right)\left(\frac{b-a}{2}\right)}_{\simeq \frac{1}{5}} \; ; \; \underset{\eta_2}{\underset{\shortparallel}{x_1}} = a + \overbrace{\left(1 + \frac{1}{\sqrt{3}}\right)\left(\frac{b-a}{2}\right)}^{\simeq 4/5}$$

$$\alpha_1 = \alpha_2 = \frac{b-a}{2}$$



semplice → composita

$$\tilde{I}_G^c(f) = \frac{H}{2}\sum_{k=1}^{M}\left[f(x_0^k) + f(x_1^k)\right]$$



$$I(f) - \tilde{I}_G(f) = \frac{1}{4320}(b-a)^5 f^{(4)}(\eta)$$

$\xi, \eta \in [a,b]$

$f \in C^4([a,b])$

$$I(f) - \tilde{I}_G^c(f) = \frac{1}{4320}(b-a)H^4 f^{(4)}(\xi)$$

oda: 4

gde: 3

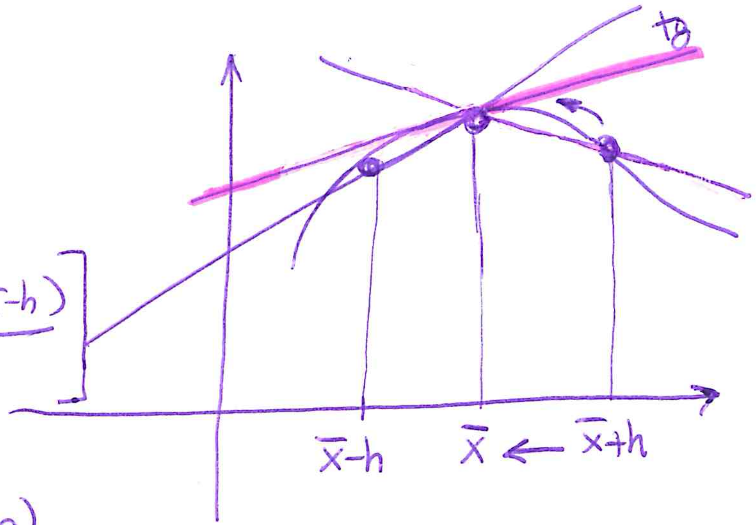
Approssimazione di derivate

$f'(\bar{x})$

$f : [a,b] \subset \mathbb{R} \to \mathbb{R}$

$f \in C^1([a,b])$

$\bar{x} \in [a,b]$



$$\lim_{h\to 0} \frac{f(\bar{x}+h)-f(\bar{x})}{h} = f'(\bar{x}) \left[= \lim_{h \to 0} \frac{f(\bar{x})-f(\bar{x}-h)}{h} \right]$$

$$f'(\bar{x}) \simeq \delta_+ f(\bar{x}) = \frac{f(\bar{x}+h)-f(\bar{x})}{h} \quad (h \text{ piccolo})$$

differenza finita in avanti

$f'(\bar{x}) - \delta_+ f(\bar{x})$

Taylor $f \in C^2([a,b])$

$$f(\bar{x}+h) = f(\bar{x}) + h f'(\bar{x}) + \frac{h^2}{2} f''(\alpha) \qquad \alpha \in [\bar{x}, \bar{x}+h]$$

$$\cancel{h} f'(\bar{x}) - \frac{f(\bar{x}+h)}{h} + \frac{f(\bar{x})}{h} = -\frac{h^{\cancel{2}}}{2} f''(\alpha)$$

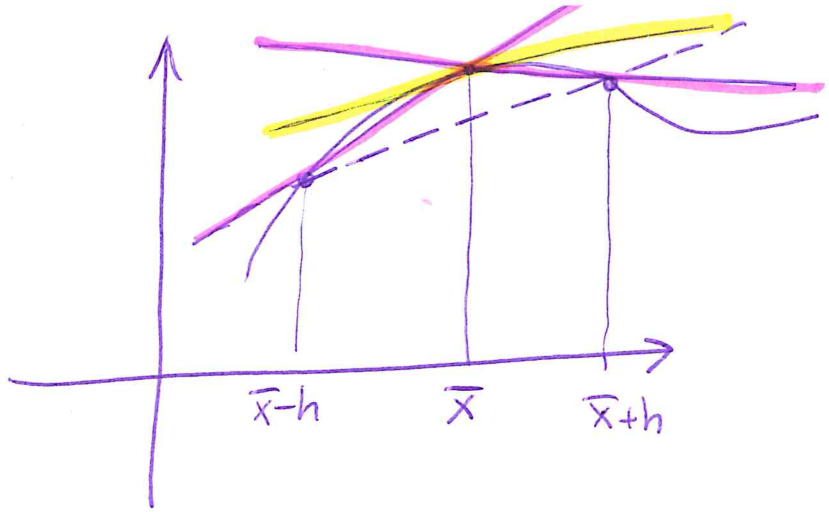
$$f'(\bar{x}) - \underbrace{\frac{f(\bar{x}+h)-f(\bar{x})}{h}}_{\delta_+ f(\bar{x})} = -\frac{h}{2} f''(\alpha)$$

ACCURATA AL PRIM'ORDINE RISPETTO AD h

$$f'(\bar{x}) \simeq \delta_- f(\bar{x}) = \frac{f(\bar{x}) - f(\bar{x}-h)}{h} \quad (h \text{ piccolo})$$

$\downarrow$

DIFFERENZA FINITA ALL'INDIETRO



$$f'(\bar{x}) - \delta_- f(\bar{x})$$

Taylor

$f \in C^2([a,b])$

$$f(\bar{x}-h) = f(\bar{x}) - h f'(\bar{x}) + \frac{h^2}{2} f''(\delta) \qquad \delta \in [\bar{x}-h, \bar{x}]$$

$$\cancel{h} f'(\bar{x}) + \frac{f(\bar{x}-h)}{h} - \frac{f(\bar{x})}{h} = \frac{h^{\cancel{2}}}{2} f''(\delta)$$

$$\underbrace{f'(\bar{x}) - \frac{f(\bar{x}) - f(\bar{x}-h)}{h}}_{\delta_- f(\bar{x})} = \frac{h}{2} f''(\delta)$$ Accurata al 1° ordine rispetto ad h

<u>Esercizio</u>

1) RETTA che interpola f nei punti $(\bar{x}-h, f(\bar{x}-h)), (\bar{x}, f(\bar{x}))$ $\Big] \equiv \delta f(\bar{x})$

2) DERIVATA

$$f'(\bar{x}) = \lim_{h\to 0} \frac{f(\bar{x}+h)-f(\bar{x}-h)}{2h}$$

$$f'(\bar{x}) \simeq \delta f(\bar{x}) = \frac{f(\bar{x}+h)-f(\bar{x}-h)}{2h}$$

$\downarrow$

DIFFERENZA FINITA CENTRATA

$\sigma \in [\bar{x}, \bar{x}+h]$

$\mu \in [\bar{x}-h, \bar{x}]$

$f'(\bar{x}) - \delta f(\bar{x})$

$f \in C^3([a,b])$

$$\ominus \quad \begin{aligned} f(\bar{x}+h) &= \cancel{f(\bar{x})} + h f'(\bar{x}) + \cancel{\frac{h^2}{2} f''(\bar{x})} + \frac{h^3}{6} f'''(\sigma) \\ f(\bar{x}-h) &= \cancel{f(\bar{x})} - h f'(\bar{x}) + \cancel{\frac{h^2}{2} f''(\bar{x})} - \frac{h^3}{6} f'''(\mu) \end{aligned}$$

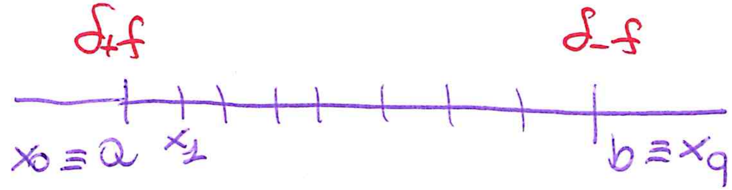
$$f(\bar{x}+h) - f(\bar{x}-h) = 2h f'(\bar{x}) + \frac{h^3}{6}\{f'''(\sigma) + f'''(\mu)\}$$

$$\cancel{2h} f'(\bar{x}) - \frac{f(\bar{x}+h)}{2h} + \frac{f(\bar{x}-h)}{2h} = -\frac{h^{\cancel{3}\,2}}{6\cdot 2}\{f'''(\sigma) + f'''(\mu)\}$$

$$f'(\bar{x}) - \underbrace{\frac{f(\bar{x}+h)-f(\bar{x}-h)}{2h}}_{\delta f(\bar{x})} = -\frac{h^2}{12}\{f'''(\sigma) + f'''(\mu)\}$$

ACCURATA AL SECOND'ORDINE RISPETTO AD h

$$f'(\bar{x}) \begin{cases} \delta_+ f(\bar{x}) & (\mathrm{I}) \\ \delta_- f(\bar{x}) & (\mathrm{I}) \\ \delta f(\bar{x}) & (\mathrm{II}) \end{cases}$$



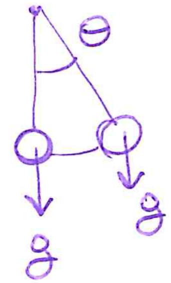
EQUAZIONI DIFFERENZIALI ORDINARIE (EDO)

ODE

① Legge di Newton $\quad m \dfrac{d^2 x}{dt^2}(t) = f(t)$

② $f = f(x(t))$ $\quad m \dfrac{d^2 x}{dt^2}(t) = -kx(t) \quad k>0$

③ pendolo semplice $\quad \ell \dfrac{d^2\theta}{dt^2}(t) + \alpha \dfrac{d\theta}{dt}(t) + g \sin\theta(t) = 0$



PROBLEMI DI CAUCHY :

$$F(t, y(t), y'(t), \dots, y^{(p)}(t)) \overset{\circledast}{=} 0 \qquad t \in I = [t_0, T]$$

$$\Updownarrow$$

$$y^{(p)}(t) \overset{\circledast}{=} f(t, y(t), y'(t), \dots, y^{(p-1)}(t)) \qquad t \in I$$

$y : I \to \mathbb{R}$
derivabile p volte
soddisfa ⊛

] SOLUZIONE ODE

ODE ordine 1 ($p=1$)

scalare $y'(t) = f(t, y(t)) \quad \forall t \in I$

$y: I \to \mathbb{R}$ $\quad$ f nota $f: I \times \mathbb{R} \to \mathbb{R}$

vettoriale $\vec{y}'(t) = \vec{f}(t, \vec{y}(t)) \quad \forall t \in I$

$\vec{f}$ nota

$\vec{y}(t) = (y_1(t), y_2(t), \dots, y_n(t))^T$

$\vec{f}: I \times \mathbb{R}^n \to \mathbb{R}^n$

$\vec{f}(t, \vec{y}(t)) = (f_1(t, \vec{y}(t)), f_2(t, \vec{y}(t)), \dots, f_n(t, \vec{y}(t)))^T$

$f_i: I \times \mathbb{R}^n \to \mathbb{R}$

$\boxed{\text{CONDIZIONI INIZIALI}}$

ODE $p=1$ scalare
$y(t_0) = y_0 \to$ noto

ODE $p=1$ vettoriale
$\vec{y}(t_0) = \vec{y}_0 \in \mathbb{R}^n \to$ noto

ODE $p=n$
$y(t_0) = y_0$
$y'(t_0) = y_{0,1}$
$\vdots$
$y^{(n-1)}(t_0) = y_{0,n-1}$
$\to$ noti

ODE ordine n ($p=n$)

$\Updownarrow$

sistema di n ODE ordine 1

$$\boxed{y^{(n)}(t) = f(t, y(t), y'(t), \dots, y^{(n-1)}(t)) \quad t \in I}$$

$y_1(t) = y(t)$

$y_2(t) = y'(t) = y_1'(t)$

$y_3(t) = y''(t) = y_2'(t)$

$\vdots$

$y_n(t) = y^{(n-1)}(t) = y_{n-1}'(t)$

$$\begin{cases} y_1'(t) = y_2(t) \\ y_2'(t) = y_3(t) \\ \vdots \\ y_{n-1}'(t) = y_n(t) \\ y_n'(t) = f(t, y_1(t), y_2(t), \dots, y_n(t)) \end{cases}$$

Problema di Cauchy

$$? \; y: I \subset \mathbb{R} \to \mathbb{R} \quad t.c. \quad \begin{cases} y'(t) = f(t, y(t)) & t \in I \\ y(t_0) = y_0 \end{cases}$$

dati

I

y_0

$f: I \times \mathbb{R} \to \mathbb{R}$

incognita

$y = y(t)$

Osserviamo: se f è continua rispetto alla prima variabile, allora

$$\int_{t_0}^{t} \frac{dy}{d\tau}(\tau)\, d\tau = \int_{t_0}^{t} f(\tau, y(\tau))\, d\tau \qquad t \in I$$

se $y \in C^1(\overline{I})$

$$y(t) = \underbrace{y(t_0)}_{y_0} + \int_{t_0}^{t} f(\tau, y(\tau))\, d\tau$$

Esempio: problema modello

$$\begin{cases} y'(t) = \lambda y(t) \\ y(t_0) = y_0 \end{cases} \qquad t \in I = [t_0, T], \quad \lambda \in \mathbb{R}^-$$

$$\boxed{y(t) = y_0 e^{\lambda(t-t_0)}}$$

problema di Cauchy

→ lineare : f è lineare rispetto a entrambi gli argomenti

$$\begin{cases} y'(t) = 3y(t) - 3t & t>0 \\ y(0) = 1 \end{cases}$$

→ non lineare

$$\begin{cases} y'(t) = \sqrt[3]{y(t)} & t>0 \\ y(0)=0 \end{cases}$$

→ con soluzione locale

$$\begin{cases} y'(t) = 1 + [y(t)]^2 & t>0 \\ y(0)=0 \end{cases}$$

$$y(t) = \tan(t) \qquad 0<t<\frac{\pi}{2}$$

Esistenza e unicità

BUONA POSIZIONE $\begin{cases} \exists! \\ \text{dipendenza continua dai dati (stabilità)} \end{cases}$

Osservazione: se f è continua rispetto ad entrambi gli argomenti, allora $\exists$ (almeno) una funzione y di classe C^1 (in un intorno di t_0) soluzione del problema di Cauchy MA non è necessariamente unica

$$\begin{cases} y'(t) = \frac{3}{2} y^{1/3}(t) & t>0 \\ y(0)=0 \end{cases} \qquad y(t)=0 \; ; \; y(t) = t^{3/2} \text{ etc...}$$

LIPSCHITZIANITA' uniforme : $f: I \times \mathbb{R} \to \mathbb{R}$ è lipschitziana rispetto ad y in I, uniformemente rispetto a t, se $\exists$ costante $L \geq 0$ t.c.

$$|f(t,y_1) - f(t,y_2)| \leq L|y_1 - y_2| \quad \forall y_1, y_2 \in \mathbb{R}, \; \forall t \in I$$

Teorema ($\exists!$ globale): Se $f: I\times\mathbb{R}\to\mathbb{R}$ è
i) continua rispetto ad entrambi gli argomenti
ii) Lipschitziana in I rispetto ad y, uniformemente rispetto a t,

allora $\exists!\ y: I\to\mathbb{R}$ soluzione $\begin{cases} y'(t)=f(t,y(t)) & t\in I \\ y(0)=y_0 \end{cases}$

ed è $y\in C^1(\bar{I})$.

Oss: una regolarità di tipo C^1 è maggiore della Lipschitz-continuità. Ovvero, se $f: I\times\mathbb{R}\to\mathbb{R}$ è continua di classe C^1 rispetto ad y, allora è anche Lipschitz-continua nel II° argomento

$$|f(t,y_1)-f(t,y_2)|\le \underbrace{\max_{\substack{t\in I\\ y\in\mathbb{R}}}\left|\frac{\partial f}{\partial y}(t,y)\right|}_{L>0}|y_1-y_2|$$

Esempi 1) $\begin{cases} y'(t)=3y(t)-3t & t>0 \\ y(0)=1 \end{cases}$

$f(t,y(t))=3y(t)-3t$ $\qquad |f(t,y_1)-f(t,y_2)|=|3y_1-\cancel{3t}-3y_2+\cancel{3t}|\le \underset{L}{\overset{\shortparallel}{3}}|y_1-y_2|$ $\quad\forall y_1,y_2\in\mathbb{R}$ $\quad\forall t>0$

$\hookrightarrow \exists!\ \left[y(t)=\left(1-\frac{1}{3}\right)e^{3t}+t+\frac{1}{3}\right]$

2) $f(t,y(t))=|y(t)|\notin C^1(\bar{I})$

$$\big||y_1|-|y_2|\big|\le \underset{L}{\overset{\shortparallel}{1}}|y_1-y_2| \qquad \forall y_1,y_2\in\mathbb{R} \quad \forall t>0$$

STABILITA' SECONDO LYAPUNOV :

$$(*)\ \begin{cases} y'(t)=f(t,y(t)) & t\in I \\ y(t_0)=y_0 \end{cases} \qquad \begin{cases} z'(t)=f(t,z(t))+\delta(t) & t\in I \\ z(t_0)=y_0+\delta_0 \end{cases}$$

$\delta: I\to\mathbb{R}$ perturbazione su f

$\delta_0\in\mathbb{R}$ " sul dato iniziale

$(*)$ si dice stabile (secondo Lyapunov) sull'intervallo I se, $\forall$ perturbazione $(\delta_0, \delta(t))$ t.c. $|\delta_0|<\varepsilon$ e $|\delta(t)|<\varepsilon$ $\forall t\in I$, con $\varepsilon>0$, e tale da garantire l'$\exists!$ della soluzione $z=z(t)$, $\exists$ una costante $C>0$ t.c.

$$|y(t)-z(t)|\leq C\varepsilon \quad \forall t\in I.$$

[asintoticamente stabile se per $\lim_{t\to+\infty}|\delta(t)|=0$ si ottiene $\lim_{t\to+\infty}|y(t)-z(t)|=0$]

In particolare, $\forall t\in I$ con $t>t_0$

$$|y(t)-z(t)|\leq \underset{\wedge\ \varepsilon}{|\delta_0|}+L\int_{t_0}^{t}|y(\tau)-z(\tau)|\,d\tau+\int_{t_0}^{t}\underset{\wedge\ \varepsilon}{|\delta(\tau)|}\,d\tau$$

Gronwall $\downarrow$

$$|y(t)-z(t)|\leq \underbrace{\varepsilon\left(1+|t-t_0|e^{L|t-t_0|}\right)}_{C} \quad \forall t\in I$$

La Trasformata di Laplace

$$\begin{tikzcd} \boxed{\text{ODE}} \arrow[r, bend left, "\mathcal{L}"] & \boxed{\text{P. ALGEBRICO}} \arrow[l, bend left, "\mathcal{L}^{-1}"] \end{tikzcd}$$

__Definizione__ : Data $f = f(t)$ con $t \geq 0$, la __trasformata di Laplace__ di f è

$$F(s) = \mathcal{L}[f(t)](s) = \int_0^{+\infty} f(t) e^{-st} dt \qquad s \in \mathbb{R}$$

Viceversa data $F = F(s)$, trovare $f = f(t)$ t.c. $\mathcal{L}[f(t)](s) = F(s)$ richiede il calcolo dell'__antitrasformata di Laplace__

$$\mathcal{L}^{-1}[F(s)](t) = f(t) \quad \text{se e solo se} \quad \mathcal{L}[f(t)](s) = F(s)$$

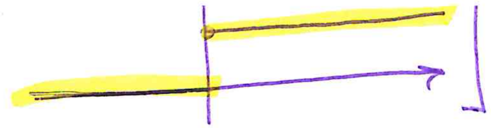
__Osservazione__ : $F(s)$ dipende da $f(t)$ solo per $t \geq 0$. $\Rightarrow$ $f(t) = 0$ per $t < 0$

__Osservazione__ : $F(s)$ esiste se A) $\exists$ primitiva su un generico intervallo limitato;

B) garantire la convergenza dell'integrale improprio.

Esempi 1) $f(t) = 1$

$\left[f(t) = H(t) = \text{Heaviside function} \begin{cases} 0 & t<0 \\ 1 & t \geq 0 \end{cases} \right]$



$$F(s) = \mathcal{L}[1](s) = \int_0^{+\infty} 1 \cdot e^{-st}\,dt = -\frac{1}{s} e^{-st} \Big|_0^{+\infty} = \lim_{b \to +\infty} \left[-\frac{1}{s} e^{-st} \right]_0^b = \lim_{b \to +\infty} \left[-\frac{1}{s} e^{-sb} + \frac{1}{s} \right] = \frac{1}{s}$$

$\boxed{s>0}$

2) $f(t) = e^{at}$

$$F(s) = \mathcal{L}[e^{at}](s) = \int_0^{+\infty} e^{at} e^{-st}\,dt = \int_0^{+\infty} e^{(a-s)t}\,dt = \lim_{b \to +\infty} \int_0^b e^{(a-s)t}\,dt$$

$$= \lim_{b \to +\infty} \left[\frac{e^{(a-s)t}}{(a-s)} \right]_0^b = \frac{-1}{a-s} = \frac{1}{s-a}$$

$a - s < 0$ $\quad \boxed{a<s}$

3) $f(t) = e^{t^2}$ $\quad \nexists\ \mathcal{L}[f(t)](s)$

$$F(s) = \mathcal{L}[e^{t^2}](s) = \lim_{b \to +\infty} \int_0^b e^{+t^2} e^{-st}\,dt = \lim_{b \to +\infty} \int_0^b e^{-st+t^2}\,dt$$

$$\int_0^b e^{-st+t^2}\,dt \geq \int_{|s|}^b e^{-st+t^2}\,dt \geq \int_{|s|}^b e^{0}\,dt = 1 \cdot (b - |s|) \xrightarrow[b \to +\infty]{} +\infty$$

$b > |s|$; $\quad t \geq |s| \Rightarrow t^2 \geq |s|t \Rightarrow t^2 - |s|t \geq 0$

$t^2 - st \geq t^2 - |s|t \geq 0$

Proprietà

1) **LINEARITA'**: siano $f(t)$ e $g(t)$ due funzioni che ammettono le trasformate di Laplace $F(s)$ e $G(s)$.

Allora
$$\mathcal{L}[f(t)+g(t)](s) = F(s)+G(s)$$
$$\mathcal{L}[c f(t)](s) = c F(s) \qquad \forall c \in \mathbb{R}$$

$\forall s$ t.c. $F(s)$ e $G(s)$ siano definite. Risulta lineare anche l'antitrasformata, ovvero
$$\mathcal{L}^{-1}[F(s)+G(s)](t) = f(t)+g(t)$$
$$\mathcal{L}^{-1}[c F(s)](t) = c f(t) \qquad \forall c \in \mathbb{R}.$$

Ex: 1)a $f(t) = \sin(3t)+2$

$$F(s) = \mathcal{L}[f(t)](s) = \mathcal{L}[\sin(3t)](s) + \mathcal{L}[2](s)$$
$$= \frac{3}{s^2+9} + 2\mathcal{L}[1](s) = \frac{3}{s^2+9} + \frac{2}{s}$$

1)b $$F(s) = \frac{3}{s^2+16} - \frac{7}{(s-5)(s-12)} = \frac{3}{4}\,\frac{4}{s^2+16} - \frac{12-5}{(s-12)(s-5)}$$

$$f(t) = \frac{3}{4}\mathcal{L}^{-1}\left[\frac{4}{s^2+16}\right](t) - \mathcal{L}^{-1}\left[\frac{12-5}{(s-12)(s-5)}\right](t)$$
$$= \frac{3}{4}\sin(4t) - (e^{12t} - e^{5t}).$$

2) FORMULE DI SHIFT (O DI TRASLAZIONE):

$\mathcal{L}[1](s) = \frac{1}{s}$; $\mathcal{L}[e^{at}](s) = \frac{1}{s-a}$

i) moltiplicare $f(t)=1$ per e^{at} genera uno shift a dx di a della trasformata $F(s)$;

ii) sostituire nella trasformata s con s−a corrisponde a moltiplicare per e^{at} la funzione $f(t)$

$\boxed{\text{PRIMO TEOREMA DI SHIFT}}$: $\forall a \in \mathbb{R}$
$$\begin{cases} \mathcal{L}[e^{at} f(t)](s) = F(s-a) & [\text{i})] \\ \mathcal{L}^{-1}[F(s-a)](t) = e^{at} f(t) & [\text{ii})] \end{cases}$$

Ex: 2)a $\mathcal{L}[\cos(\alpha t)](s) = \frac{s}{s^2+\alpha^2} = F(s)$

$\hookrightarrow \mathcal{L}[e^{\beta t}\cos(\alpha t)](s) = F(s-\beta) = \frac{s-\beta}{(s-\beta)^2+\alpha^2}$

2)b Calcolare $\mathcal{L}^{-1}\left[\frac{4}{s^2+4s+20}\right](t)$.

$$\frac{4}{s^2+4s+20} = \frac{4}{(s+2)^2+16} = F(s+2) \qquad F(s) = \frac{4}{s^2+16}$$

$$\mathcal{L}^{-1}[F(s)](t) = \sin(4t) \implies \mathcal{L}^{-1}\left[\frac{4}{s^2+4s+20}\right](t) = \mathcal{L}^{-1}[F(s+2)](t)$$
$$= e^{-2t}\sin(4t)$$

3) TRASFORMATA DEL PRODOTTO DI CONVOLUZIONE

: prodotto di convoluzione di due funzioni $f(t)$ e $g(t)$ definite per $t \geq 0$ è data da

$$(f * g)(t) = \int_0^t f(t-u)\, g(u)\, du \quad \left(= \int_0^t f(u)\, g(t-u)\, du \right)$$

$\forall t \geq 0$ tale per cui l'integrale sia ben definito.

[VALE LA P. COMMUTATIVA $f*g = g*f$]

Siano $f(t)$ e $g(t)$ due funzioni con trasformata di Laplace $F(s)$ e $G(s)$, allora

$$\mathcal{L}[(f*g)(t)](s) = \mathcal{L}\left[\int_0^t f(t-u)g(u)\,du\right](s) = F(s)G(s)$$

Inoltre

$$\mathcal{L}^{-1}[F(s)G(s)](t) = (f*g)(t) = \int_0^t f(t-u)g(u)\,du.$$

Ex: $? \; \mathcal{L}^{-1}\left[\frac{1}{s^2(s+1)^2}\right](t)$; $\mathcal{L}^{-1}\left[\frac{1}{s^2}\right](t) = t \leftarrow g$

$\mathcal{L}^{-1}\left[\frac{1}{(s+1)^2}\right](t) = te^{-t} \leftarrow f$

$$\mathcal{L}^{-1}\left[\frac{1}{s^2(s+1)^2}\right](t) = \int_0^t (t-u)e^{-(t-u)}u\,du = te^{-t}\int_0^t e^u u\,du - e^{-t}\int_0^t u^2 e^u\,du$$

$= $ integrazione per parti

$= (t+2)e^{-t} + (t-2)$

4) TRASFORMATA DELLA DERIVATA :

$$\mathcal{L}[y'(t)](s) = \int_0^{+\infty} y'(t) e^{-st}\, dt = y(t) e^{-st} \Big|_0^{+\infty} - \int_0^{+\infty} y(t)(-s) e^{-st}\, dt$$

$$= -y(0) + s \underbrace{\int_0^{+\infty} y(t) e^{-st}\, dt}_{\mathcal{L}[y(t)](s)}$$

$$\boxed{\mathcal{L}[y'(t)](s) = s\mathcal{L}[y(t)](s) - y(0)}$$

$$\mathcal{L}[\underbrace{y''(t)}_{(y'(t))'}](s) = s\mathcal{L}[y'(t)](s) - y'(0) = s\left[s\mathcal{L}[y(t)](s) - y(0)\right] - y'(0)$$

$$\boxed{\mathcal{L}[y''(t)](s) = s^2\mathcal{L}[y(t)](s) - sy(0) - y'(0)}$$

PROBLEMA 1 :

$$\begin{cases} y'(t) = 4y(t) + 1 & t \geq 0 \\ y(0) = 1 \end{cases}$$

$$y'(t) - 4y(t) = 1$$

TRASFORMATA

$$\mathcal{L}[y'(t) - 4y(t)](s) = \mathcal{L}[y'(t)](s) - 4\mathcal{L}[y(t)](s) = s\mathcal{L}[y(t)](s) - \underbrace{y(0)}_{1} - 4\mathcal{L}[y(t)](s)$$

$$= \mathcal{L}[1](s) = \frac{1}{s}$$

$$s\mathcal{L}[y(t)](s) - 4\mathcal{L}[y(t)](s) - 1 = \frac{1}{s}$$

$$\mathcal{L}[y(t)](s) = Y(s) \qquad sY(s) - 4Y(s) - 1 = \frac{1}{s} \longleftarrow$$ PROBLEMA ALGEBRICO

$$Y(s) = \frac{1}{s-4}\left(1 + \frac{1}{s}\right) = \frac{1}{s-4} + \frac{1}{s(s-4)}$$

ANTI-TRASFORMATA

$$\mathcal{L}^{-1}\left[\frac{1}{s-4}\right] = e^{4t} \qquad \mathcal{L}^{-1}\left[\frac{1}{s(s-4)}\right] = \mathcal{L}^{-1}\left[-\frac{1}{4}\frac{-4}{(s-0)(s-4)}\right] = -\frac{1}{4}\left(e^{0t} - e^{4t}\right) = \frac{1}{4}\left(e^{4t} - 1\right)$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}[Y(s)](t) = \mathcal{L}^{-1}\left[\frac{1}{s-4} + \frac{1}{s(s-4)}\right](t) = \mathcal{L}^{-1}\left[\frac{1}{s-4}\right](t) + \mathcal{L}^{-1}\left[\frac{1}{s(s-4)}\right](t)$$

$$= e^{4t} + \frac{1}{4}e^{4t} - \frac{1}{4} = \frac{5}{4}e^{4t} - \frac{1}{4} = \frac{1}{4}\left(5e^{4t} - 1\right) \qquad t \geq 0$$

PROBLEMA 2 :

$$\begin{cases} y''(t) + 4y'(t) + 3y(t) = e^t & t \geq 0 \\ y(0) = 0 \\ y'(0) = 2 \end{cases}$$

$$\mathcal{L}[y''(t) + 4y'(t) + 3y(t)](s) = \mathcal{L}[y''(t)](s) + 4\mathcal{L}[y'(t)](s) + 3\mathcal{L}[y(t)](s)$$

$$= s^2\mathcal{L}[y(t)](s) - s\underbrace{y(0)}_{=0} - \underbrace{y'(0)}_{=2} + 4s\mathcal{L}[y(t)](s) - 4\underbrace{y(0)}_{=0} + 3\mathcal{L}[y(t)](s)$$

$$= \mathcal{L}[e^t](s) = \frac{1}{s-1}$$

$$\mathcal{L}[y(t)](s) = Y(s)$$

$$s^2 Y(s) - 2 + 4s\,Y(s) + 3\,Y(s) = \frac{1}{s-1} \qquad \boxed{\text{PROBLEMA ALGEBRICO}}$$

$$Y(s) = \frac{\frac{1}{s-1} + 2}{s^2 + 4s + 3} = \frac{2s-1}{(s-1)(s^2+4s+3)} \qquad (s+1)(s+3)$$

$$Y(s) = \frac{1}{8}\frac{1}{s-1} + \frac{3}{4}\frac{1}{s+1} - \frac{7}{8}\frac{1}{s+3}$$

$$y(t) = \mathcal{L}^{-1}[Y(s)](t) = \frac{1}{8}\mathcal{L}^{-1}\left[\frac{1}{s-1}\right](t) + \frac{3}{4}\mathcal{L}^{-1}\left[\frac{1}{s+1}\right](t) - \frac{7}{8}\mathcal{L}^{-1}\left[\frac{1}{s+3}\right](t)$$

$$y(t) = \frac{1}{8}e^t + \frac{3}{4}e^{-t} - \frac{7}{8}e^{-3t}$$

ODE: metodi numerici

$y = y(t)$

- esplicita
- implicita
- $\nexists$ soluzione esplicita/implicita

implicita:

$$y'(t) = \frac{y-t}{y+t}$$

$$\frac{1}{2}\ln(t^2+y^2) + \arctan\frac{y}{t} = c$$

$\nexists$ soluzione esplicita/implicita:

$$y'(t) = e^{-t^2}$$

$$y(t) = \sum_{k=0}^{+\infty} \dots$$

$$\begin{cases} y'(t) = f(t, y(t)) \\ y(0) = y_0 \end{cases} \qquad t \in (t_0, T) = I$$

tempo finale (T)

tempo iniziale (t_0)

y_0, f note

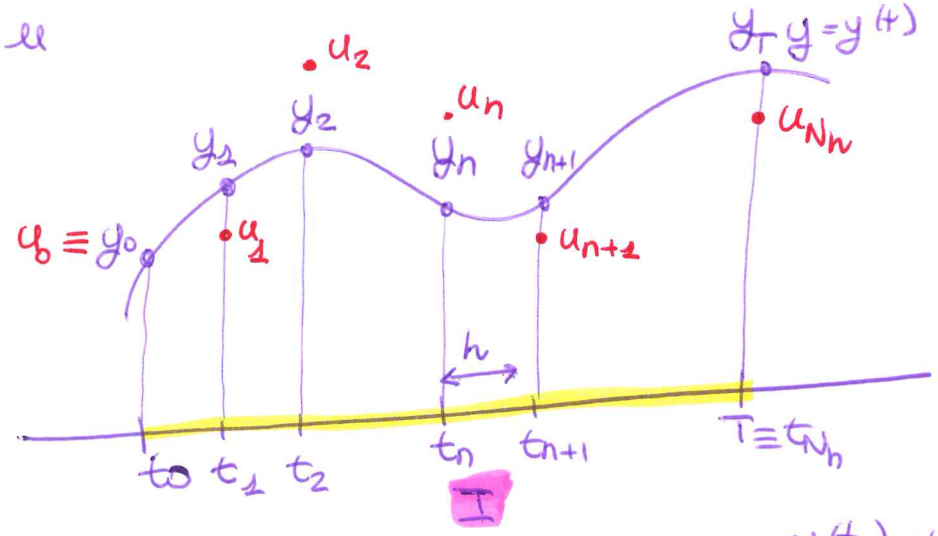
$y = y(t)$ incognita

$\{u_0 \equiv y_0, u_1, u_2, \dots, u_{N_h}\}$

$u_n \simeq y_n = y(t_n) \quad n = 0, \dots, N_h$

Approssimazione discreta di $y = y(t)$

$y = y(t)$ approssimata da u



N_h sottointervalli uniformi

$$t_0 < t_1 < t_2 \dots < t_{N_h} \equiv T$$

$$h = \text{passo di discretizzazione} = \frac{T - t_0}{N_h}$$

$t_n \quad 0 \le n \le N_h$ istanti

$$t_n = t_{n-1} + h \quad n = 1, \dots, N_h$$

$$\left[t_n = t_0 + nh \quad n = 0, \dots, N_h\right] \to t_{N_h} = t_0 + \cancel{N_h}\frac{T - t_0}{\cancel{N_h}} = T$$

valore esatto $y(t_n) = y_n$

$\forall n = 0, \dots, N_h \quad u_n \simeq y_n$

$$\begin{cases} y'(t) = f(t, y(t)) & t \in I \\ y(t_0) = y_0 \end{cases}$$

$t = t_n$ $\quad y'(t_n) = f(t_n, y(t_n))$ (collocazione)

$$\frac{y(t_{n+1}) - y(t_n)}{h} \simeq f(t_n, y(t_n))$$ (approssimazione della derivata)

$$\frac{y_{n+1} - y_n}{h} \simeq f(t_n, y_n)$$ (cambio notazione)

$$\frac{u_{n+1} - u_n}{h} = f(t_n, u_n)$$ (introduco l'approssimazione)

$$\boxed{\begin{array}{ll} u_{n+1} = u_n + h f(t_n, u_n) & 0 \le n \le N_h - 1 \\ u_0 = y_0 & \end{array}}$$

schema di Eulero in avanti
(" " " esplicito)

$u_0 = y_0$
$u_1 \simeq y_1$
$u_2 \simeq y_2$
⋮
$u_{N_h-1} \simeq y_{N_h-1}$
$u_{N_h} \simeq y_{N_h}$

metodo ad un passo
(metodo one-step)
metodo esplicito

EX:
[$u_{n+1} = F(u_n, u_{n-1}, u_{n-2})$
a 3 passi
one-step
multi-step]

esplicito $u_{n+1} = F(u_n)$ one-step

$u_{n+1} = F(u_n, u_{n-1}, u_{n-2}, \dots u_{n-p})$ multistep

implicito $u_{n+1} = F(u_{n+1}, u_n)$ one-step

$u_{n+1} = F(u_{n+1}, u_n, u_{n-1}, u_{n-2}, \dots u_{n-p})$ multi-step

$y'(t_{n+1}) = f(t_{n+1}, y(t_{n+1}))$ (collocazione)

(approssimazione derivata)

$\dfrac{y(t_{n+1}) - y(t_n)}{h} \simeq f(t_{n+1}, y(t_{n+1}))$

(cambio notazione)

$\dfrac{y_{n+1} - y_n}{h} \simeq f(t_{n+1}, y_{n+1})$

(approssimo)

$\dfrac{u_{n+1} - u_n}{h} = f(t_{n+1}, u_{n+1})$

$\hookrightarrow$ $\boxed{\begin{array}{ll} u_{n+1} = u_n + h f(t_{n+1}, u_{n+1}) & 0 \le n \le N_h - 1 \\ u_0 = y_0 & \end{array}}$

schema di Eulero all'indietro
(" " " implicito)

metodo one-step
metodo implicito

Esempio
$$\begin{cases} y'(t) = \underbrace{C y(t) \left(t - \frac{y(t)}{B}\right)}_{f(t, y(t))} \\ y(0) = y_0 \end{cases} \qquad B, C \in \mathbb{R}$$

$\boxed{EE}$
$$\begin{cases} u_{n+1} = u_n + h f(t_n, u_n) = u_n + h C u_n \left(t_n - \frac{u_n}{B}\right) \\ u_0 = y_0 \end{cases} \qquad 0 \le n \le N_h - 1$$

$\boxed{EI}$
$$\begin{cases} u_{n+1} = u_n + h f(t_{n+1}, u_{n+1}) = u_n + h C u_{n+1} \left(t_{n+1} - \frac{u_{n+1}}{B}\right) \\ u_0 = y_0 \end{cases} \qquad 0 \le n \le N_h - 1$$

$u_{n+1} \rightarrow x \qquad x = u_n + h C x \left(t_{n+1} - \frac{x}{B}\right)$

equazione non lineare

Schema di Crank-Nicolson

EE $\quad u_{n+1} = u_n + h f(t_n, u_n)$

EI $\quad u_{n+1} = u_n + h f(t_{n+1}, u_{n+1})$

$$2 u_{n+1} = 2 u_n + h \left[f(t_n, u_n) + f(t_{n+1}, u_{n+1}) \right]$$

$$\boxed{\begin{cases} u_{n+1} = u_n + \frac{h}{2} \left[f(t_n, u_n) + f(t_{n+1}, u_{n+1}) \right] \qquad 0 \le n \le N_h - 1 \\ u_0 = y_0 \end{cases}}$$

metodo one-step
metodo implicito

Analisi metodi one-step

$\rightarrow$ convergenza
$\rightarrow$ consistenza
$\rightarrow$ stabilità

Convergenza

$\forall n = 0, \dots, N_h \qquad |\underbrace{u_n - y_n}| \le C(h)$ con $C(h)$ infinitesimo rispetto ad h

e_n = errore associato all'istante t_n

$[e_0 = 0]$

$C(h) = C^* h^p \xrightarrow[h \to 0]{} 0$, con $C^* \in \mathbb{R}$

p = ordine di convergenza

Convergenza di EE

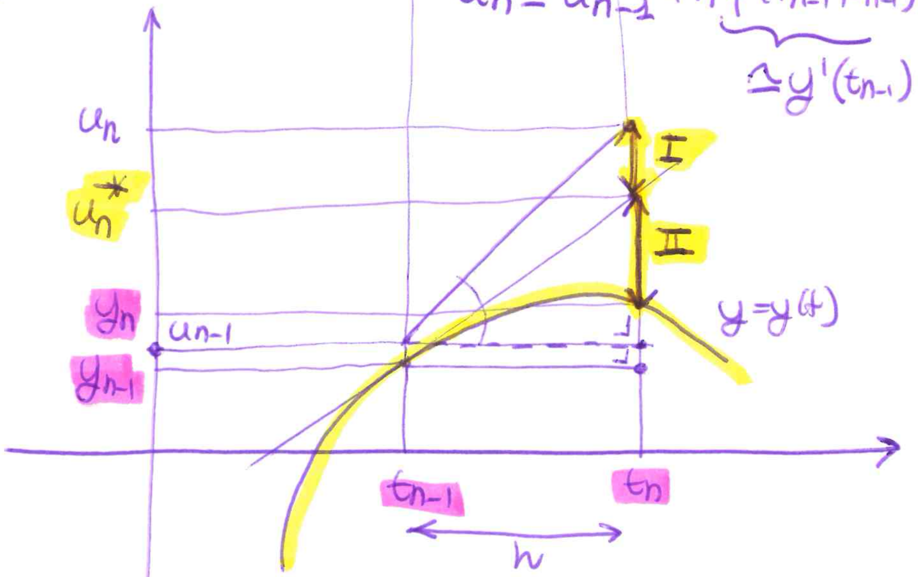
$$e_n = u_n - y_n = \underset{\text{I}}{(u_n - u_n^*)} + \underset{\text{II}}{(u_n^* - y_n)}$$

(I: accumulo errore; II: 1 passo)

$$u_n^* = y_{n-1} + h\underbrace{f(t_{n-1}, y_{n-1})}_{= y'(t_{n-1})}$$

(y_{n-1}: noto)

$$u_n = u_{n-1} + h\underbrace{f(t_{n-1}, u_{n-1})}_{\simeq y'(t_{n-1})}$$



II

$$u_n^* - y_n = y_{n-1} - y_n + h f(t_{n-1}, y_{n-1})$$
$$= y_{n-1} - y_n + h y'(t_{n-1}) = -\frac{h^2}{2} y''(\alpha_n) \xrightarrow{h \to 0} 0$$

Se $y \in C^2(I)$

$I = (t_0, T)$

$\bar{I} = [t_0, T]$

Taylor $\quad y_n = y_{n-1} + h y'(t_{n-1}) + \frac{h^2}{2} y''(\alpha_n)$

$\alpha_n \in (t_{n-1}, t_n)$

$\tau_n(h) = \dfrac{u_n^* - y_n}{h}$ errore di troncamento locale ; $\tau(h) = \max\limits_{n=0,\dots,N_h} |\tau_n(h)|$ errore di troncamento globale

$\boxed{EE}$ $\quad \tau_n(h) = -\dfrac{h}{2} \underbrace{y''(\alpha_n)}_{f'(\alpha_n, y(\alpha_n))} \Rightarrow \tau(h) = \dfrac{h}{2} M \qquad M = \max\limits_{t \in I} |f'(t, y(t))|$

↳ EE è consistente; $q = 1$ (ordine di consistenza 1).

Consistenza: uno schema ad un passo è consistente se $\lim\limits_{h \to 0} \tau(h) = 0$. Inoltre se $\tau(h) = O(h^q)$, lo schema ad un passo ha ordine di consistenza $= q$.

STABILITA'
- ZERO STABILITA' (intervalli limitati) $h \to 0$, $(N_h \to +\infty)$
- ASSOLUTA STABILITA' (" illimitati)
- ~~A-STABILITA' ($\mathbb{C}$)~~

EE $\begin{cases} u_{n+1} = u_n + h f(t_n, u_n) \\ u_0 = y_0 \end{cases}$

EE perturbato $\begin{cases} z_{n+1} = z_n + h[f(t_n, z_n) + \rho_{n+1}] \\ z_0 = y_0 + \rho_0 \end{cases}$

$I = [t_0, T]$

$\rho_0, \rho_{n+1} \in \mathbb{R}$ perturbazioni sui dati

EE è <u>zero stabile</u> se $\exists h_0 > 0$ e se $\exists C > 0$ t.c. $\forall h \leq h_0$ e $\forall \varepsilon > 0$ con $|\rho_n| \leq \varepsilon$ per $0 \leq n \leq N_h$

si ha
$$|u_n - z_n| \leq C \varepsilon \qquad 0 \leq n \leq N_h.$$

1) $\exists!$ $z = z(t)$

2) $C = C(\exp[(T-t_0)L]) \Longrightarrow I$ piccoli

<u>Oss</u>: per i metodi one-step <u>consistenti</u>, la zero-stabilità è una conseguenza di $\begin{cases} f \text{ continua rispetto ad entrambi gli argomenti} \\ f \text{. Lipschitz continua rispetto al II argomento} \end{cases}$

EE, EI, CN

<u>consistenza</u>, <u>stabilità</u>, <u>convergenza</u>

<u>Teorema di equivalenza di Lax-Richtmyer</u>

Ogni metodo consistente è convergente se e solo se è zero-stabile.

ASSOLUTA STABILITA'

$\cancel{I = [t_0, T]}$ $\qquad I = [t_0, +\infty)$

h fissato

$N_h \to +\infty$

$t_0 \quad t_1 \quad t_2$

problema modello

$$\begin{cases} y'(t) = \underbrace{\lambda y(t)}_{f(t, y(t))} & t \in (0, +\infty),\ \lambda \in \mathbb{R}^- \\ y(0) = 1 = y_0 \end{cases}$$

$$y(t) = e^{\lambda t} \xrightarrow[t \to +\infty]{} 0$$

$\boxed{EE}$

$$u_{n+1} = u_n + h f(t_n, u_n)$$
$$= u_n + h\lambda u_n = (1+h\lambda) u_n$$
$$= (1+h\lambda)^2 u_{n-1}$$
$$= (1+h\lambda)^3 u_{n-2}$$
$$\vdots$$
$$= (1+h\lambda)^{n+1} u_0, \qquad u_0 = y_0 = 1$$

$$u_n = u_{n-1} + h\lambda u_{n-1} = (1+h\lambda) u_{n-1}$$

$$u_{n-1} = (1+h\lambda) u_{n-2}$$

condizionatamente assolutamente stabile

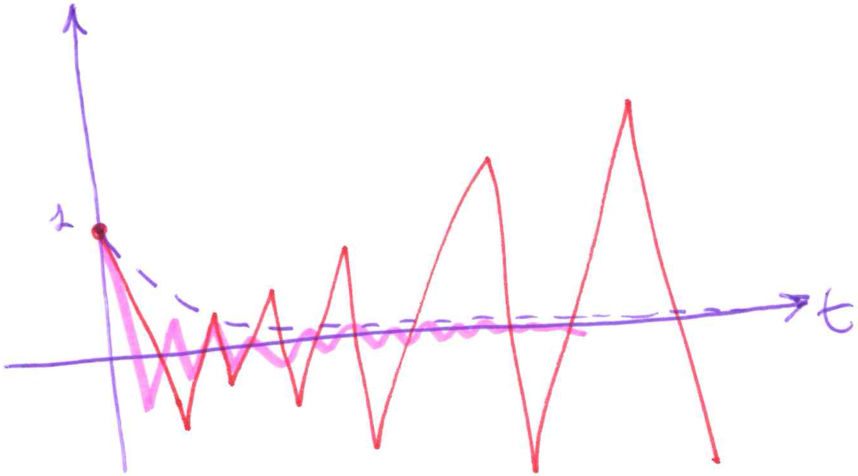
$$\boxed{u_{n+1} = (1+h\lambda)^{n+1}}$$

? $|u_{n+1}| \to 0$ per $n \to +\infty$

$$\underbrace{-1 < \underbrace{1 + h\lambda < 1}_{\text{vero}}}$$

$h|\lambda| = -h\lambda < 2$ condizione di stabilità

$\hookrightarrow \boxed{h < \frac{2}{|\lambda|}}$



$\lambda = -1 \qquad y(t) = e^{-t}$

$\boxed{h < 2}$

$h_1 = \frac{30}{14}\ (>2)\ ; \quad h_2 = \frac{30}{16}\ (<2)\ ; \quad h_3 = \frac{1}{2}$

$\boxed{EI}$

$$u_{n+1} = u_n + h\, f(t_{n+1}, u_{n+1}) = u_n + h\lambda u_{n+1}$$

$$(1-h\lambda)u_{n+1} = u_n \qquad u_{n+1} = \left(\frac{1}{1-h\lambda}\right)u_n$$

$$(1-h\lambda)u_n = u_{n-1} \rightarrow u_n = \left(\frac{1}{1-h\lambda}\right)u_{n-1} \qquad = \left(\frac{1}{1-h\lambda}\right)^2 u_{n-1}$$

$$(1-h\lambda)u_{n-1} = u_{n-2} \longrightarrow u_{n-1} = \left(\frac{1}{1-h\lambda}\right)u_{n-2} \qquad = \left(\frac{1}{1-h\lambda}\right)^3 u_{n-2}$$

$$= \quad \vdots$$

$$= \left(\frac{1}{1-h\lambda}\right)^{n+1} \underset{\substack{\| \\ y_0 = 1}}{u_0}$$

incondizionatamente assolutamente stabile

$$\boxed{u_{n+1} = \left(\frac{1}{1-h\lambda}\right)^{n+1}}$$

$? \ |u_{n+1}| \rightarrow 0, \ n \rightarrow +\infty \qquad \frac{1}{1-h\lambda} < 1 \quad \forall h$

CN

$$u_{n+1} = u_n + \frac{h}{2}\left[f(t_n, u_n) + f(t_{n+1}, u_{n+1})\right]$$

$$= u_n + \frac{h}{2}\left[\lambda u_n + \lambda u_{n+1}\right]$$

$$\left(1 - \frac{h}{2}\lambda\right) u_{n+1} = \left(1 + \frac{h}{2}\lambda\right) u_n$$

$$u_{n+1} = \left[\frac{1+\frac{h}{2}\lambda}{1-\frac{h}{2}\lambda}\right] u_n = \left[\frac{1+\frac{h}{2}\lambda}{1-\frac{h}{2}\lambda}\right]^2 u_{n-1} = \dots = \left[\frac{1+\frac{h}{2}\lambda}{1-\frac{h}{2}\lambda}\right]^{n+1} u_0$$

$$u_0 = y_0 = 1$$

$$\boxed{u_{n+1} = \left[\frac{1+\frac{h}{2}\lambda}{1-\frac{h}{2}\lambda}\right]^{n+1}}$$

$$? \quad |u_{n+1}| \to 0 \quad n \to +\infty$$

$\forall h$

incondizionatamente assolutamente stabile

Metodi predictor-corrector

$$\text{CN}\quad \begin{cases} u_{n+1} = u_n + \frac{h}{2}\left[f(t_n, u_n) + f(t_{n+1}, u_{n+1})\right] & n \geq 0 \\ u_0 = y_0 \end{cases}$$

$$u_{n+1}^{(0)} \longrightarrow \boxed{u_{n+1}^{(k+1)} = u_n + \underbrace{\frac{h}{2}\left[f(t_n, u_n) + f(t_{n+1}, u_{n+1}^{(k)})\right]}_{\Phi(u_{n+1}^{(k)})}} \quad k \geq 0$$

$u_{n+1}^{(k)} \simeq u_{n+1}$; $\{u_{n+1}^{(k)}\}$

$$\alpha = \phi(\alpha)$$
$$x^{(k+1)} = \phi(x^{(k)})$$

$\alpha \to u_{n+1}$

$x^{(k+1)} \to u_{n+1}^{(k+1)}$

Se $u_{n+1}^{(0)}$ è sufficientemente accurata $\Rightarrow u_{n+1}^{(1)} \simeq u_{n+1}$.

$\Updownarrow$

se lo schema implicito ha ordine p, allora $u_{n+1}^{(0)}$ è ottenuto da uno schema esplicito di ordine $\geq p-1$.

Lo schema predictor/corrector ha l'ordine di convergenza del corrector

$$\begin{cases} \text{implicito} & \text{CN} & p = 2 & \text{(corrector)} \\ \text{esplicito} & \text{EE} & p-1 = 1 & \text{(predictor)} \end{cases}$$

condizionatamente assolutamente stabile

Eulero migliorato (HEUN) ordine 2

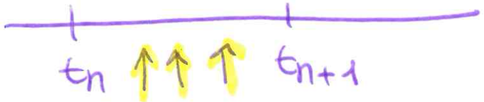
$$\begin{cases} u_{n+1}^{(0)} = u_n + hf(t_n, u_n) & \text{PREDIZIONE (EE)} \\ u_{n+1}^{(1)} = u_n + \frac{h}{2}\left[f(t_n, u_n) + f(t_{n+1}, u_{n+1}^{(0)})\right] & \text{CORREZIONE (CN)} \end{cases}$$

$\hookrightarrow u_{n+1}$

METODI DI ALTO ORDINE

"one-step" Runge-Kutta

multi-step (Adams, BDF)



Metodo di Runge-Kutta (RK) a s stadi

$$\begin{cases} u_{n+1} = u_n + h \sum_{i=1}^{s} b_i k_i & n \geq 0 \\ k_i = f\left(t_n + c_i h,\ u_n + h \sum_{j=1}^{s} a_{ij} k_j\right) & i = 1, \dots, s \\ u_0 = y_0 \end{cases}$$

array di Butcher

$$\begin{array}{c|c} \vec{c} & A \\ \hline & \vec{b} \end{array}$$

$$\begin{array}{c|cccc|l} c_1 & a_{11} & a_{12} & \dots & a_{1s} & \rightarrow k_1 \\ c_2 & a_{21} & a_{22} & \dots & a_{2s} & \rightarrow k_2 \\ \vdots & \vdots & & & & \\ c_s & a_{s1} & a_{s2} & \dots & a_{ss} & \rightarrow k_s \\ \hline & b_1 & b_2 & \dots & b_s & \end{array}$$

$[a_{ij}] = A \in \mathbb{R}^{s \times s}$

$\vec{c} = [c_1, c_2, \dots, c_s]^T \in \mathbb{R}^s$

$\vec{b} = [b_1, b_2, \dots, b_s]^T \in \mathbb{R}^s$

se le sole entrate non nulle sono per $j < i$, allora lo schema RK è esplicito.
Viceversa è implicito.
→ sistema di eq. non lineari

RK4

$$\begin{cases} u_{n+1} = u_n + \frac{h}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right] & n \geq 0 \\ u_0 = y_0 \end{cases}$$

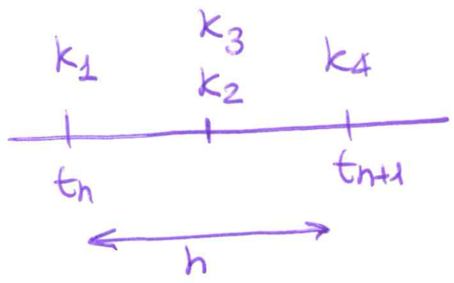
$s=4$

$$k_1 = f(t_n, u_n)$$

$$k_2 = f\left(t_n + \frac{h}{2}, u_n + \frac{h}{2}k_1\right)$$

$$k_3 = f\left(t_n + \frac{h}{2}, u_n + \frac{h}{2}k_2\right)$$

$$k_4 = f\left(t_{n+1}, u_n + h k_3\right)$$



$$\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ \hline & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{array}$$

→ esplicito

ordine di convergenza 4

EE $\equiv$ RK1

$$u_{n+1} = u_n + h\underbrace{f(t_n, u_n)}_{k_1}$$

$s=1$

$$\begin{array}{c|c} 0 & 0 \\ \hline & 1 \end{array}$$

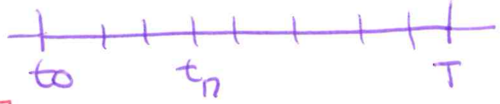
HEUN $\equiv$ RK2

$$\begin{cases} u_{n+1}^{(1)} = u_n + \frac{h}{2}\left[\underbrace{f(t_n, u_n)}_{k_1} + \underbrace{f(t_{n+1}, u^*_{n+1})}_{k_2}\right] \\ u^*_{n+1} = u_n + h\underbrace{f(t_n, u_n)}_{k_1} \end{cases}$$

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & \boxed{1} & 0 \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

Matlab $\begin{cases} \text{ode23} \\ \text{ode45} \end{cases}$

$y_i : [t_0, T] \longrightarrow \mathbb{R} \qquad i = 1, \dots, m \qquad y_1, y_2, \dots, y_m$



$$\left\{\begin{array}{l} \underbrace{\begin{array}{l} y_1'(t) \\ y_2'(t) \\ \vdots \\ y_m'(t) \end{array}}_{\vec{y}\,'(t)} \begin{array}{l} = \\ = \\ \\ = \end{array} \underbrace{\begin{array}{l} f_1(t, y_1(t), y_2(t), \dots, y_m(t)) \\ f_2(t, y_1(t), y_2(t), \dots, y_m(t)) \\ \vdots \\ f_m(t, \underbrace{y_1(t), y_2(t), \dots, y_m(t)}_{\vec{y}(t)}) \end{array}}_{\vec{F}(t, \vec{y}(t))} \end{array}\right. \qquad \underbrace{\begin{array}{l} y_1(t_0) \\ y_2(t_0) \\ \vdots \\ y_m(t_0) \end{array}}_{\vec{y}(t_0)} \begin{array}{l} = \\ = \\ \\ = \end{array} \underbrace{\begin{array}{l} y_{0,1} \\ y_{0,2} \\ \\ y_{0,m} \end{array}}_{\vec{y}_0}$$

$$\vec{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_m(t) \end{bmatrix} ; \qquad \vec{F}(t, \vec{y}(t)) = \begin{bmatrix} f_1(t, \vec{y}(t)) \\ f_2(t, \vec{y}(t)) \\ \vdots \\ f_m(t, \vec{y}(t)) \end{bmatrix} ; \qquad \vec{y}_0 = \begin{bmatrix} y_{0,1} \\ y_{0,2} \\ \vdots \\ y_{0,m} \end{bmatrix} \in \mathbb{R}^m$$

EE

$$\left\{\begin{array}{l} \vec{y}\,'(t) = \vec{F}(t, \vec{y}(t)) \\ \vec{y}(t_0) = \vec{y}_0 \end{array}\right. \longrightarrow \qquad \begin{array}{l} u_{n+1} = u_n + h f(t_n, u_n) \\ u_0 = y_0 \end{array}$$

$$\left[\begin{array}{l} \vec{u}_{n+1} = \vec{u}_n + h \vec{F}(t_n, \vec{u}_n) \\ \vec{u}_0 = \vec{y}_0 \end{array}\right.$$

$u_{n+1,i} \simeq y_i(t_{n+1})$ $\qquad$ $u_{n+1} \simeq y(t_{n+1})$

$i = 1, \dots, m$

$$
EE \left\{
\begin{array}{l}
\boxed{\begin{array}{l} u_{n+1,1} \\ u_{n+1,2} \\ \vdots \\ u_{n+1,m} \end{array}}
=
\boxed{\begin{array}{l} u_{n,1} \\ u_{n,2} \\ \\ u_{n,m} \end{array}}
+ h
\boxed{\begin{array}{l} f_1(t_n, u_{n,1}, u_{n,2}, \dots, u_{n,m}) \\ f_2(t_n, u_{n,1}, u_{n,2}, \dots, u_{n,m}) \\ \\ f_m(t_n, u_{n,1}, u_{n,2}, \dots, u_{n,m}) \end{array}}
\end{array}
\right. ;
\quad
\left\{
\begin{array}{l}
\boxed{\begin{array}{l} u_{0,1} \\ u_{0,2} \\ \vdots \\ u_{0,m} \end{array}}
=
\boxed{\begin{array}{l} y_{0,1} \\ y_{0,2} \\ \\ y_{0,m} \end{array}}
\end{array}
\right.
$$

$\vec{u}_{n+1} \in \mathbb{R}^m$ $\qquad$ $\vec{u}_n \in \mathbb{R}^m$ $\qquad$ $\vec{F}(t_n, \vec{u}_n)$ $\qquad$ $\vec{u}_0$ $\qquad$ $\vec{y}_0$

$$
\left\{
\begin{array}{l}
\vec{u}_{n+1} = \vec{u}_n + h\left[\theta \vec{F}(t_{n+1}, \vec{u}_{n+1}) + (1-\theta)\vec{F}(t_n, \vec{u}_n)\right] \\
\vec{u}_0 = \vec{y}_0
\end{array}
\right.
$$

θ - metodo

$0 \le \theta \le 1$

$\theta = 0$ EE

$\theta = 1$ EI

$\theta = \frac{1}{2}$ CN

$aF + bG$ $\qquad$ $a, b \in \mathbb{R}$

c.e. convessa

$$
\left[
\begin{array}{l}
a + b = 1 \\
\frac{1}{2} \quad \frac{1}{2} \\
\theta \; , \; 1-\theta
\end{array}
\right.
$$

$$\begin{cases} y^{(m)}(t) = f(t, y(t), y'(t), \dots, y^{(m-1)}(t)) & t \in [t_0, T] \\ y(t_0) = y_0, \quad y'(t_0) = y_1, \quad \dots, \quad y^{(m-1)}(t_0) = y_{m-1} \end{cases}$$

CAMBIO DI VARIABILE

$$w_1(t) = y(t), \quad w_2(t) = y'(t), \quad \dots, \quad w_m(t) = y^{(m-1)}(t)$$

$$\begin{cases} w_1'(t) = w_2(t) \\ w_2'(t) = w_3(t) \\ \vdots \\ w_m'(t) = f(t, w_1(t), w_2(t), \dots, w_m(t)) \end{cases} + \begin{cases} w_1(t_0) = y_0 \\ w_2(t_0) = y_1 \\ \vdots \\ w_m(t_0) = y_{m-1} \end{cases}$$

θ - metodo

Equazioni alle derivate parziali (EDP)

[Partial Differential Equations (PDEs)]

ODE

ODE	PDE
$y = y(t)$	$u = u(x,t)$

$$0 = F\Big(x, t, u, \underbrace{\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x_1}, \dots \frac{\partial u}{\partial x_d}}, \dots\dots, \frac{\partial^{p_1+\dots+p_d+p_t} u}{\partial x_1^{p_1}\dots\partial x_d^{p_d}\,\partial t^{p_t}}, g\Big)$$

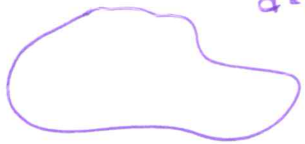
$\mathbb{R}$ $\quad$ $\mathbb{R}^d$

$x = [x_1, \dots, x_d]^T$

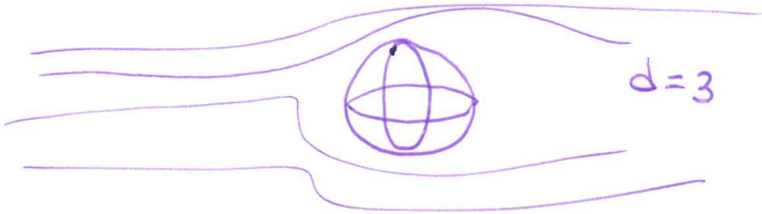
dati del problema

$d=1$

$d=2$



ordine $p = p_1 + \dots + p_d + p_t$



$d=3$

PDE → lineare

PDE → non lineare $\quad u\dfrac{\partial u}{\partial x} + 3u = f$

PDE → x stazionarie

PDE → x,t non stazionarie

Classificazione :

$$Lu = g$$

$$Lu = A\frac{\partial^2 u}{\partial x_1^2} + B\frac{\partial^2 u}{\partial x_1 \partial x_2} + C\frac{\partial^2 u}{\partial x_2^2} + D\frac{\partial u}{\partial x_1} + E\frac{\partial u}{\partial x_2} + Fu$$

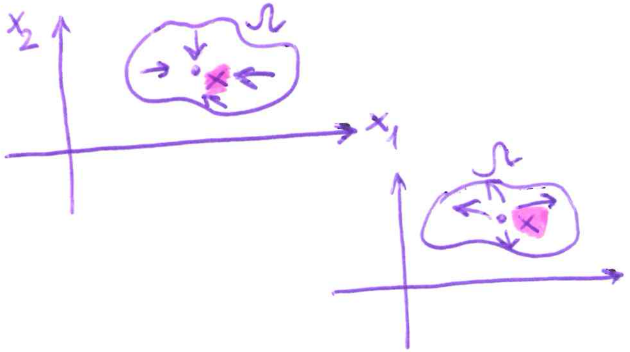
$x_1 = x$ $\qquad A, B, C, D, E, F \in \mathbb{R}$

$x_2 = y, t$

$$\Delta = B^2 - 4AC$$

$$\text{PDE} \to \begin{cases} \text{ellittiche} & \Delta < 0 \\ \text{paraboliche} & \Delta = 0 \\ \text{iperboliche} & \Delta > 0 \end{cases}$$

Esempi : equazione di Laplace



$d = 2,3$

$$\begin{cases} -\Delta u = 0 & \Omega \subset \mathbb{R}^d \\ u = g & \text{su } \partial\Omega \text{ (CB)} \end{cases}$$

$d = 1$

$$\begin{cases} -u'' = 0 & \Omega = I = [a,b] \\ u = g & \text{in } x = a,b \end{cases}$$

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \qquad \Delta u = \operatorname{div}(\nabla u)$$

$x_1 = x \qquad x_2 = y$

$A = -1 = C \qquad B = D = E = F = 0$

$\Delta = -4 < 0$ ellittico

$u : \Omega \longrightarrow \mathbb{R}$

$\nabla u \in \mathbb{R}^d$ $\qquad \nabla u = \begin{bmatrix} \frac{\partial u}{\partial x_1} \\ \vdots \\ \frac{\partial u}{\partial x_d} \end{bmatrix}$

$d=2$

$\nabla u = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix}$

∇ scalari $\longrightarrow$ vettori

div

$\nabla \cdot \begin{bmatrix} v_1 \\ \vdots \\ v_d \end{bmatrix} = \frac{\partial v_1}{\partial x_1} + \dots + \frac{\partial v_d}{\partial x_d}$

$d=2$

$\nabla \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2}$

$\nabla \cdot$ vettori $\longrightarrow$ scalari

$d=2$

$\nabla \cdot (\nabla u) = \nabla \cdot \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \Delta u$

Δ scalari $\longrightarrow$ scalari

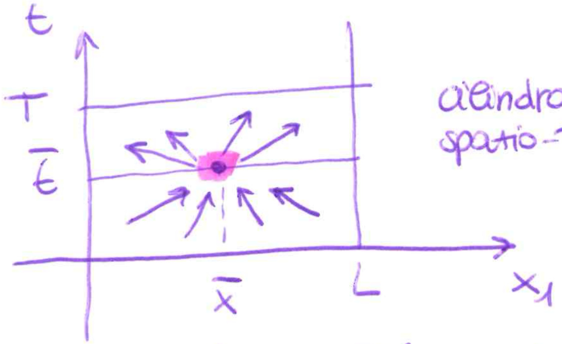
② equazione del calore $(d=1)$

$$\begin{cases} \dfrac{\partial u}{\partial t} - \mu \dfrac{\partial^2 u}{\partial x^2} = 0 & (0,L)\times(0,T) \\ + CB \text{ (condizione al bordo)} \\ + CI \text{ (condizione iniziale)} \end{cases}$$

$d=2,3$
$$\begin{cases} \dfrac{\partial u}{\partial t} - \mu \Delta u = 0 \\ + CB \\ + CI \end{cases}$$

$x_1 = x \quad x_2 = t \qquad A = -\mu \quad E = 1 \quad B = C = D = F = 0$

$\Delta = 0 \rightarrow$ parabolico



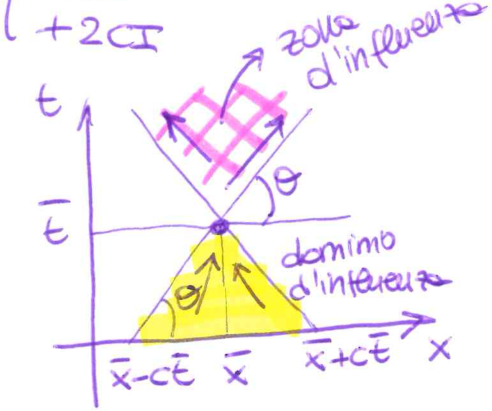
③ equazione delle onde $(d=1)$

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - c^2 \dfrac{\partial^2 u}{\partial x^2} = 0 & (0,L)\times(0,T) \\ + CB \\ + 2CI \end{cases}$$

$d=2,3$
$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - c^2 \Delta u = 0 \\ + CB \\ + 2CI \end{cases}$$

$x_1 = x \qquad x_2 = t \qquad A = -c^2 ; C = 1 ; B = D = E = F = 0$

$\Delta = 4c^2 > 0 \rightarrow$ iperbolico



$c = \dfrac{1}{tg\theta}$

Equazione di Poisson (Laplace)

2D/3D $\begin{cases} -\mu \Delta u = f \\ + CB \end{cases}$ $\qquad \left(2D/3D \begin{cases} -\mu \Delta u = 0 \\ + CB \end{cases} \right)$

$u = u(x) \quad x \in \Omega \subset \mathbb{R}^d \quad d \geq 1, \quad \mu$ costante

1D $\begin{cases} -\mu u'' = f \\ + CB \end{cases}$ $\qquad \left(1D \begin{cases} -\mu u'' = 0 \\ + CB \end{cases} \right)$

$\Big[-\underset{\substack{\downarrow \\ \text{costante}}}{\mu} \Delta u \qquad \Delta u = \nabla \cdot (\nabla u) \qquad -\nabla \cdot [\mu(x) \nabla u] \Big]$

Problema stazionario

(configurazione di equilibrio)

EX 1: termico

u temperatura

f sorgente calore

EX 2: meccanica

u spostamento

f carico

EX 3: fluidodinamica

u concentrazione

f sorgente

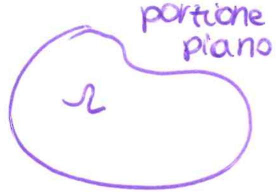
Glossario:

1) problemi ai limiti / ai valori al bordo

2) $\Omega \subset \mathbb{R}^d$ dominio limitato (insieme aperto e connesso)

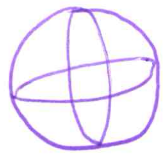
1D intervallo

2D



porzione piano

3D

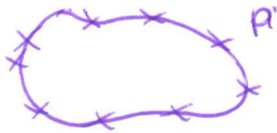


porzione spazio

3) bordo del dominio $\partial\Omega$

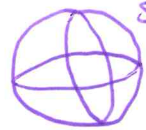
1D $x = a, x = b$

2D



curva nel piano

3D



superficie nello spazio

4) condizione al bordo (CB): configurazione dell'equilibrio al bordo

EX 2

CONDIZIONI DI **DIRICHLET** (u)

[fissiamo la posizione del bordo della membrana]

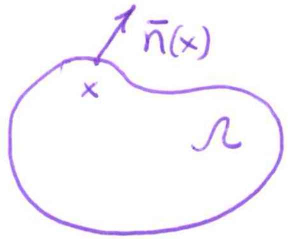
$$u(x) = g(x) \qquad x \in \partial\Omega$$

0 caso omogeneo

$\neq 0$ caso non omogeneo

CONDIZIONE DI **NEUMANN** (derivata di u)

[bordo libero di muoversi]



$\nabla u \cdot n = \left(\dfrac{\partial u}{\partial n}\right)$ derivata normale

1D

$$\mu u'(x) = h(x) \qquad x = a,\ x = b$$

0 omogeneo

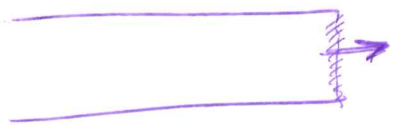
$\neq 0$ non omogeneo

2D/3D

$$\mu \nabla u(x) \cdot n(x) = h(x) \qquad x \in \partial\Omega$$

0 omogeneo

$\neq 0$ non omogeneo



CONDIZIONE DI **ROBIN** (u e derivata di u)

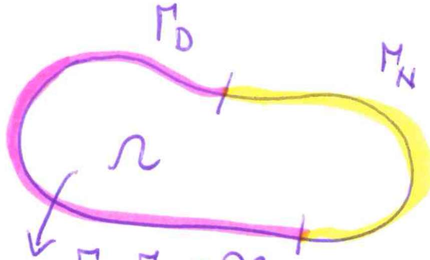
[attacco elastico]

$$\alpha(x)\underline{u(x)} + \underline{\mu(x)\nabla u(x) \cdot n(x)} = q(x) \qquad x \in \partial\Omega$$

$$\alpha(x) \in \mathbb{R}^+$$

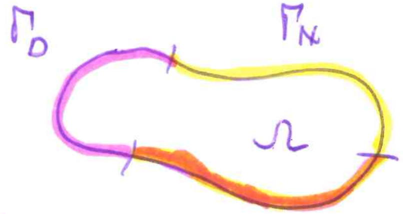
CONDIZIONI ~~MIS~~ **MISTE** (combinazione precedenti)

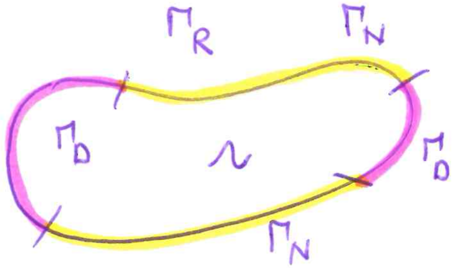
D/N
$$\begin{cases} u(x) = g(x) & x \in \Gamma_D \\ \mu(x)\nabla u(x) \cdot n(x) = h(x) & x \in \Gamma_N \end{cases}$$



$\Gamma_D, \Gamma_N \subset \partial\Omega$

$\Gamma_D \cap \Gamma_N = \emptyset$ e $\bar{\Gamma}_D \cup \bar{\Gamma}_N = \partial\Omega$





Analisi dell'equazione di Poisson

$\boxed{A}$ **Esistenza e unicità** : sia $\Omega \subset \mathbb{R}^d$ un dominio limitato e con frontiera regolare. Allora esiste al più una funzione di classe $C^2(\Omega) \cap C^1(\bar{\Omega})$ soluzione del problema di Poisson in Ω e tale che su $\partial\Omega$ vengono assegnate CB $\begin{cases} \text{Dirichlet} \\ \text{Robin} \\ \text{miste D/N} \end{cases}$ per dati sufficientemente regolari.

Nel caso di Neumann invece due soluzioni differiscono per una costante (è garantito solo $\exists$)

$\downarrow$

$$\begin{cases} -\mu \Delta u \overset{+\gamma u}{=} f & x \in \Omega \\ \mu \nabla u \cdot n = h & x \in \partial\Omega \end{cases}$$

$u + c$ con c costante

$$-\mu \Delta (u+c) = -\mu \Delta u \underbrace{-\mu \Delta c}_{=0} = -\mu \Delta u = f$$

$$\mu \nabla (u+c) \cdot n = \underbrace{\mu \nabla u \cdot n}_{h} + \underbrace{\mu \nabla c \cdot n}_{=0} = h$$

$$-\mu \Delta (u+c) + \gamma (u+c) =$$
$$= \underbrace{-\mu \Delta u}\ \underbrace{-\mu \Delta c}_{=0} \underbrace{+\gamma u} + \gamma c$$
$$= f + \gamma c$$

Osservazione : condizione di compatibilità

FORMULA DI GAUSS (teorema della divergenza)

$$\int_\Omega f(x)\,dx = -\int_\Omega \mu \Delta u(x)\,dx = -\int_\Omega \mu \nabla \cdot (\nabla u)(x)\,dx =$$

$$= -\int_{\partial\Omega} \mu \nabla u(s) \cdot n(s)\,ds = -\int_{\partial\Omega} h(s)\,ds$$

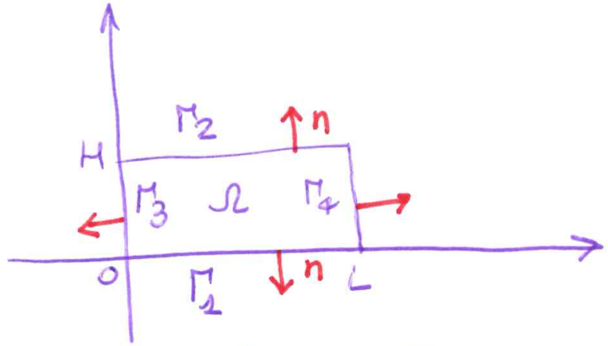
$$\boxed{\int_\Omega f(x)\,dx = -\int_{\partial\Omega} h(s)\,ds}$$

RIMEDI
1) ulteriore caratterizzazione fisica
2) aggiungere un termine di ordine zero (γu)

Ex:

$$\begin{cases} -\mu \Delta u(x) - f & \text{in } \Omega = (0,L)\times(0,H) \\ \mu \dfrac{\partial u}{\partial y}(x,0) = q_1 & x \in (0,L) \quad (\Gamma_1) \\ -\mu \dfrac{\partial u}{\partial y}(x,H) = q_2 & x \in (0,L) \quad (\Gamma_2) \\ \mu \dfrac{\partial u}{\partial x}(0,y) = q_3 & y \in (0,H) \quad (\Gamma_3) \\ -\mu \dfrac{\partial u}{\partial x}(L,y) = q_4 & y \in (0,H) \quad (\Gamma_4) \end{cases}$$

$f, q_i \in \mathbb{R} \qquad i = 1, \dots, 4$



$$-\mu \frac{\partial u}{\partial n} = -\mu \nabla u \cdot n = Q$$

$\Gamma_1 \quad n = (0,-1)^T$

$\Gamma_2 \quad n = (0,1)$

$\Gamma_3 \quad n = (-1,0)^T$

$\Gamma_4 \quad n = (1,0)^T$

Condizione di compatibilità:

$$\int_\Omega f(x)\,dx = f(HL) = -\sum_{i=1}^{4} \int_{\Gamma_i} q_i(s)\,ds$$

$$= -\sum_{i=1}^{4} q_i |\Gamma_i|$$

$$= -\left[(q_1+q_2)L + (q_3+q_4)H\right]$$

$$f = \frac{-\left[(q_1+q_2)L + (q_3+q_4)H\right]}{HL} = \frac{-[q_1+q_2]}{H} - \frac{[q_3+q_4]}{L}$$

Laplace

Definizione: una funzione u è **armonica** in un dominio $\Omega \subset \mathbb{R}^d$ se $u \in C^2(\Omega)$ e $\Delta u = 0$ in Ω.

Osservazione: la soluzione di $\Delta u = 0$ è una funzione armonica.

B **Teorema della media**: sia u una funzione armonica in $\Omega \subset \mathbb{R}^d$. Allora $\forall\, B_R(x) = \{y \in \mathbb{R}^d \,;\, \|y-x\| < R\}$ tale che $\overline{B_R}(x) \subset \Omega$ valgono le formule

$$u(x) = \frac{1}{|B_R(x)|} \int_{B_R(x)} u(y)\,dy \quad ; \quad u(x) = \frac{1}{|\partial B_R(x)|} \int_{\partial B_R(x)} u(s)\,ds$$

con $|\omega|$ misura di un generico insieme $\omega \subset \mathbb{R}^d$.

[2D $\quad |B_R(x)| = \pi R^2$ $\quad$ 3D $\quad |B_R(x)| = \frac{4}{3}\pi R^3$

$|\partial B_R(x)| = 2\pi R$ $\quad\quad |\partial B_R(x)| = 4\pi R^2$]

Osservazione: 1) la soluzione di $\Delta u = 0$ soddisfa la proprietà della media

2) **Teorema**: sia $u \in C^0(\Omega)$. Se $\forall x \in \Omega$ e $\forall\, B_R(x) \subset \Omega$ vale una delle formule della media, allora la u è armonica in Ω e $u \in C^\infty(\Omega)$.

C Principio del massimo:

Teorema: se $u \in C^0(\Omega)$ soddisfa la proprietà della media in $\Omega \subset \mathbb{R}^d$ e $p \in \Omega$ è un punto interno ad Ω ed è un punto estremo di u (ovvero è o un massimo o un minimo di u) globale, allora u è costante.

$\hookrightarrow$ massimi e minimi sono confinati al bordo

$$\max_{x \in \bar{\Omega}} u(x) = \max_{x \in \partial\Omega} u(x) \quad e \quad \min_{x \in \bar{\Omega}} u(x) = \min_{x \in \partial\Omega} u(x)$$

$$\left[\min_{x \in \partial\Omega} u(x) \leq u(x) \leq \max_{x \in \partial\Omega} u(x) \right]$$

Osservazioni: 1) una funzione armonica soddisfa il principio del massimo.

2) la soluzione di $\Delta u = 0$ soddisfa " " " " .

3) Teorema: data una funzione continua $v \in C^0(\bar{\Omega})$ con $\Omega \subset \mathbb{R}^d$. Se $\Delta v(x) > 0$ $\forall x \in \Omega$ allora

$$v(x) \leq \max_{x \in \partial\Omega} v(x);$$

viceversa se $\Delta v(x) < 0$ $\forall x \in \Omega$ allora

$$v(x) \geq \min_{x \in \partial\Omega} v(x).$$

$\boxed{-\Delta u = f}$

$\nearrow$ se $f(x) < 0$ $\forall x \in \Omega$

$\Rightarrow \Delta u > 0$ $\forall x \in \Omega$ $\Rightarrow$ u raggiunge il max sul bordo.

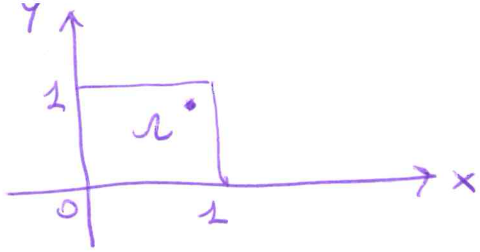
$\searrow$ se $f(x) > 0$ $\forall x \in \Omega$

$\Rightarrow \Delta u < 0$ $\forall x \in \Omega$ $\Rightarrow$ u raggiunge il min sul bordo.

Esempio $\quad -\Delta u = f \quad \text{in } \Omega = (0,1)^2$

$$\Omega = (0,1)^2 \Big\downarrow (0,1)\times(0,1)$$

$f = 1 + |x-y|$



CB Dirichlet non omogenee : $\underbrace{u(x,y)}_{x} = \underbrace{g(x,y)}_{x} = x + 2y$

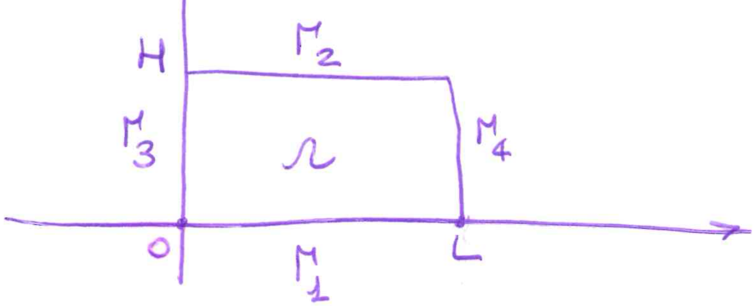
$f(x) > 0 \quad \forall x \in \Omega \quad \Rightarrow \quad \Delta u(x) < 0 \quad \forall x \in \Omega \Rightarrow \min\limits_{x\in\partial\Omega} u(x) \leq u(x) \quad \forall x \in \Omega$

$$0 = \min_{x\in\partial\Omega} g(x) = \min_{x\in\partial\Omega} u(x) \leq u(x) \quad \forall x \in \Omega$$

Separazione delle variabili in coordinate cartesiane

$$\begin{cases} -\Delta u(x,y) = 0 & x \in \Omega \\ u(x,0) = g_1(x) & x \in (0,L) \quad (\Gamma_1) \\ u(x,H) = 0 & x \in (0,L) \quad (\Gamma_2) \\ u(0,y) = u(L,y) = 0 & y \in (0,H) \quad (\Gamma_3, \Gamma_4) \end{cases}$$

$\Omega = (0,L)\times(0,H)$



$\partial\Omega = \bigcup\limits_{i=1}^{4} \Gamma_i \qquad g_1 : (0,L) \longrightarrow \mathbb{R}$

$\Gamma_1 = (0,L)\times\{0\}$
$\quad = \{(x,y)\in\mathbb{R}^2 ;\ x\in(0,L),\ y=0\}$

$\Gamma_2 = (0,L)\times\{H\}$;

$\Gamma_3 = \{0\}\times(0,H)$; $\Gamma_4 = \{L\}\times(0,H)$

$? \, U : \Omega \to \mathbb{R}$ t.c. $U(x,y) = X(x)\, Y(y)$

$X : (0,L) \to \mathbb{R}$; $Y : (0,H) \to \mathbb{R}$

PASSO 1 $\quad -\Delta U = 0$ in Ω

(1) $$-\Delta U(x,y) = -\frac{\partial^2 U}{\partial x^2}(x,y) - \frac{\partial^2 U}{\partial y^2}(x,y) = 0 \qquad \begin{array}{l} x \in (0,L) \\ y \in (0,H) \end{array}$$

$$\frac{\partial U}{\partial x}(x,y) = X'(x)\, Y(y) \qquad \frac{\partial U}{\partial y}(x,y) = X(x)\, Y'(y)$$

$$\frac{\partial^2 U}{\partial x^2}(x,y) = X''(x)\, Y(y) \qquad \frac{\partial^2 U}{\partial y^2}(x,y) = X(x)\, Y''(y)$$

$$+ X''(x)\, Y(y) + X(x)\, Y''(y) = 0 \qquad x \in (0,L) \quad y \in (0,H)$$

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \underbrace{k}_{\text{costante } \in \mathbb{R}}$$

2 problemi ai valori al bordo
$$\begin{cases} X''(x) - k X(x) = 0 & x \in (0,L) \quad (2) \\ Y''(y) + k Y(y) = 0 & y \in (0,H) \quad (3) \end{cases}$$

$\boxed{\text{PASSO 2}}$

$$U(0,y) = U(L,y) = 0 \qquad y \in (0,H)$$

$$\Updownarrow$$

$$X(0)Y(y) = X(L)Y(y) = 0 \qquad y \in (0,H)$$

$$\hookrightarrow X(0) = X(L) = 0 \quad \text{CB per } (2)$$

$$(2) \Leftrightarrow \begin{cases} +X''(x) - kX(x) = 0 & x \in (0,L) \\ \left.\begin{array}{l} X(0) = 0 \\ X(L) = 0 \end{array}\right\} \text{CB} \end{cases}$$

$\boxed{\text{CASO 1}}$ $k = \lambda^2$ con $\lambda \in \mathbb{R}$ MA $\lambda \neq 0$

$$X(x) = Ae^{\lambda x} + Be^{-\lambda x} \qquad A, B \in \mathbb{R}$$

$$\begin{cases} X(0) = A + B = 0 \\ X(L) = Ae^{\lambda L} + Be^{-\lambda L} = 0 \end{cases} \Rightarrow A = B = 0$$

$$\Downarrow$$

$$X(x) = 0 \quad \forall x \in [0,L]$$

$\boxed{\text{CASO 2}}$ $k = 0 \Rightarrow X(x) = A + Bx$, $A, B \in \mathbb{R}$

$$\begin{cases} X(0) = A = 0 \\ X(L) = BL = 0 \end{cases} \Rightarrow A = B = 0 \Rightarrow X(x) = 0 \quad \forall x \in [0,L]$$

CASO 3 $\quad k=-\lambda^2 \quad \lambda\in\mathbb{R}$ ma $\lambda\neq 0$

$$X(x)=A\cos(\lambda x)+B\sin(\lambda x) \qquad A,B\in\mathbb{R}$$

$$\begin{cases} X(0)=A\cos(\lambda 0)+B\sin(\lambda 0)=A=0 \\ X(L)=\cancel{A\cos(\lambda L)}+B\sin(\lambda L)=0 \end{cases} \Longrightarrow \begin{cases} A=0 \\ B\sin(\lambda L)=0 \end{cases}$$

$$\begin{cases} A=0 \\ B=0 \end{cases} \qquad \boxed{\begin{cases} A=0 \\ \lambda L=n\pi \quad n=1,2,\dots \end{cases} \quad \lambda_n=\frac{n\pi}{L}}$$

$X(x)=0$

$x\in[0,L]$

$$\boxed{X_n(x)=B_n\sin(\lambda_n x)=B_n\sin\left(\frac{n\pi}{L}x\right) \qquad n=1,2,\dots}$$

$$\{B_n\}_{n=1}^{\infty}$$

PASSO 3 $\quad k\to k_n=-\lambda_n^2=-\left(\frac{n\pi}{L}\right)^2$

$$Y''(y)-\left(\frac{n\pi}{L}\right)^2 Y(y)=0 \qquad y\in(0,H) \qquad n=1,2,\dots$$

$$Y_n(y)=D_{1,n}\,e^{\left(\frac{n\pi}{L}\right)y}+D_{2,n}\,e^{-\left(\frac{n\pi}{L}\right)y} \qquad n=1,2,\dots$$

$$\{D_{1,n}\}_{n=1}^{\infty} \qquad \{D_{2,n}\}_{n=1}^{\infty}$$

Soluzioni fondamentali

$$U_n(x,y) = X_n(x)\,Y_n(y) \qquad n=1,2,\dots$$

$$\begin{cases} -\Delta u = 0 \\ CB \quad \Gamma_3, \Gamma_4 \end{cases}$$

$$U(x,y) = \sum_{n=1}^{+\infty} U_n(x,y) = \sum_{n=1}^{+\infty} X_n(x)\,Y_n(y)$$

Γ_2

$$U(x,H)=0 \qquad x\in(0,L)$$

$$U_n(x,H) = X_n(x)\,Y_n(H) = 0 \qquad x\in(0,L), \quad n=1,2,\dots$$

$$\Longrightarrow \quad Y_n(H) = 0$$

$$Y_n(H) = D_{1,n}\, e^{\left(\frac{n\pi}{L}\right)H} + D_{2,n}\, e^{-\left(\frac{n\pi}{L}\right)H} = 0 \qquad n=1,2,\dots$$

$$\Longrightarrow \quad D_{2,n} = -\frac{D_{1,n}\, e^{\left(\frac{n\pi}{L}\right)H}}{e^{-\left(\frac{n\pi}{L}\right)H}} = -D_{1,n}\, e^{2\left(\frac{n\pi}{L}\right)H} \qquad n=1,2,\dots$$

$$Y_n(y) = D_{1,n}\left[e^{\left(\frac{n\pi}{L}\right)y} - e^{2\left(\frac{n\pi}{L}\right)H}\, e^{-\left(\frac{n\pi}{L}\right)y}\right] \qquad n=1,2,\dots$$

$$\sinh(z) = \frac{e^{z} - e^{-z}}{2} = -\sinh(-z) = -\frac{e^{-z} - e^{z}}{2}$$

$$Q = \frac{n\pi}{L} \qquad \left[e^{Qy} - e^{2QH} e^{-Qy}\right] \cdot \frac{e^{-QH}}{e^{-QH}} = \frac{e^{Q(y-H)} - e^{QH-Qy}}{2e^{-QH}} \cdot 2$$

$$Y_n(y) = D^{*}_{1,n} \sinh\left[\frac{n\pi}{L}(H-y)\right] \quad n = 1, \dots$$

$$D^{*}_{1,n} = D_{1,n}(-2)e^{QH}$$

$$= 2\,\frac{e^{\overbrace{Q(y-H)}^{z}} - e^{-Q(y-H)}}{2e^{-QH}}$$

$$= -2\,\frac{e^{Q(H-y)} - e^{-Q(H-y)}}{2e^{-QH}}$$

$$\frac{e^{Q(H-y)} - e^{-Q(H-y)}}{2} = \sinh\left[\frac{n\pi}{L}(H-y)\right]$$

$$U_n(x,y) = X_n(x)\,Y_n(y)$$

$$= B_n \sin\left(\frac{n\pi}{L}x\right) D^{*}_{1,n} \sinh\left[\frac{n\pi}{L}(H-y)\right]$$

$$U_n(x,y) = C_n \sin\left(\frac{n\pi}{L}x\right) \sinh\left[\frac{n\pi}{L}(H-y)\right] \quad n = 1, 2, \dots$$

$$C_n = B_n D^{*}_{1,n}$$

$$\hookrightarrow U(x,y) = \sum_{n=1}^{+\infty} C_n \sin\left(\frac{n\pi}{L}x\right) \sinh\left(\frac{n\pi}{L}(H-y)\right) \qquad \begin{cases} -\Delta u = 0 \\ \text{CB } \Gamma_2, \Gamma_3, \Gamma_4 \end{cases}$$

PASSO 4

$$U(x,0) = g_1(x) \qquad x \in (0,L)$$

$$\sum_{n=1}^{+\infty} C_n \underbrace{\sin\left(\frac{n\pi}{L}x\right)} \sinh\left(\frac{n\pi}{L}H\right) = g_1(x) \qquad x \in (0,L)$$

$\downarrow$

sufficientemente regolare

$(L^2(0,L))$

$$g_1(x) = \sum_{n=1}^{+\infty} A_{1,n} \underbrace{\sin\left(\frac{n\pi}{L}x\right)}$$

$\uparrow$ noti

$$C_n \sinh\left(\frac{n\pi}{L}H\right) = A_{1,n}$$

$$C_n = \frac{A_{1,n}}{\sinh\left(\frac{n\pi}{L}H\right)} \qquad A_{1,n} = \frac{2}{L}\int_0^L g_1(x) \sin\left(\frac{n\pi}{L}x\right)dx$$

$$\boxed{U(x,y) = \sum_{n=1}^{+\infty} \frac{A_{1,n}}{\sinh\left(\frac{n\pi}{L}H\right)} \sin\left(\frac{n\pi}{L}x\right) \sinh\left(\frac{n\pi}{L}(H-y)\right)}$$

$$\left\{\sin\left(\frac{n\pi}{L}x\right)\right\}_{n=1}^{+\infty}$$

$$\int_0^L \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)dx = \frac{L}{2}\delta_{mn}$$

$$\sum_{n=1}^{+\infty} A_{1,n}\underbrace{\int_0^L \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)dx}_{\frac{L}{2}\delta_{mn}} = \int_0^L g_1(x)\sin\left(\frac{m\pi}{L}x\right)dx$$

$$A_{1,m}\frac{L}{2} = \int_0^L g_1(x)\sin\left(\frac{m\pi}{L}x\right)dx$$

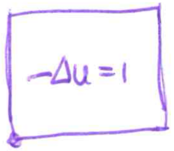
$$A_{1,m} = \frac{2}{L}\int_0^L g_1(x)\sin\left(\frac{m\pi}{L}x\right)dx$$

Necessità di avere una nuova formulazione

formulazione forte	formulazione debole
$-\mu \Delta u = f$ in Ω	
(differenziale)	(integrale)
$C^k(\Omega)$	spazi di Sobolev

$-\Delta u = f$ + DIRICHLET OMOGENEO

$\cancel{u \in C^2(\bar{\Omega})}$ $\qquad u \in C^2(\Omega) \cap C^0(\bar{\Omega})$



$-\Delta u = 0$

$-\Delta u = 1 \qquad (0,1)^2 = \Omega$

$f = 1 \qquad f \in C^0(\bar{\Omega})$

$-\Delta u(0,0) = -\frac{\partial^2 u}{\partial x^2}(0,0) - \frac{\partial^2 u}{\partial y^2}(0,0) = 0$

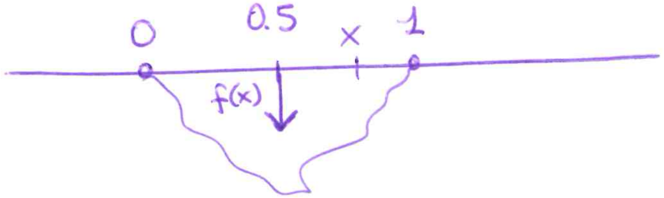
$u = 0 \qquad \partial\Omega$

$$\boxed{\text{se } f \in C^0(\bar{\Omega}) \Rightarrow u \in C^2(\Omega) \cap C^0(\bar{\Omega})}$$

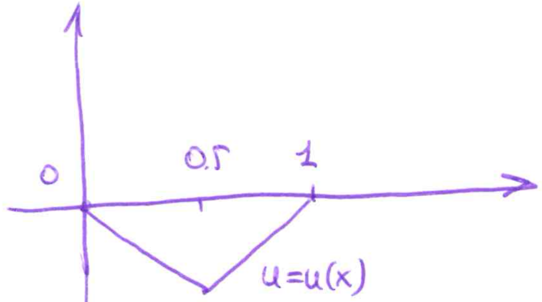
$\boxed{1D}$

$u = u(x)$

$\Omega = (0,1)$



$$\begin{cases} -u''(x) = f(x) & x \in \Omega = (0,1) \\ u(0) = u(1) = 0 \end{cases}$$



$u \in C^0(0,1)$

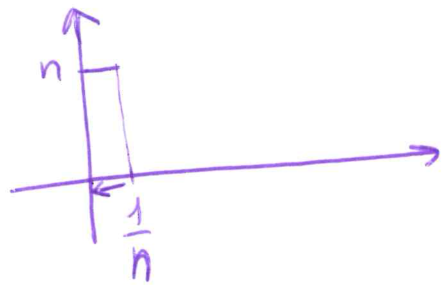
$F(x) = \int_0^x f(y)\,dy \qquad (0,x)$

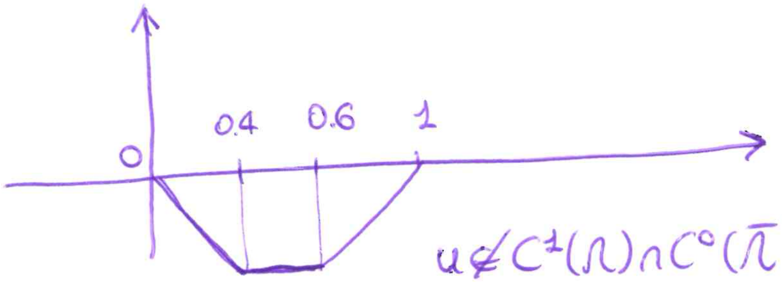
$f \qquad \delta$ di Dirac centrato in 0

$u \notin C^1(\Omega) \cap C^0(\bar{\Omega})$

$$\delta_0(x) = \lim_{n\to\infty} g_n(x)$$

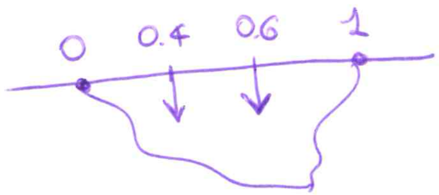
$$g_n(x) = \begin{cases} 0 & x < 0 \text{ e } x > \frac{1}{n} \\ n & 0 \le x \le \frac{1}{n} \end{cases}$$



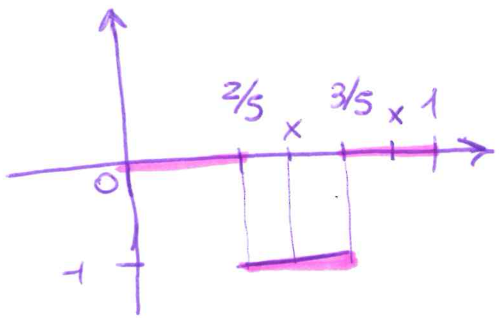


$u \notin C^1(\Omega) \cap C^0(\bar{\Omega})$

$u \in C^0(0,1)$



$$f(x) = \begin{cases} 0 & x \in \left(0, \frac{2}{5}\right) \cup \left[\frac{3}{5}, 1\right) \\ -1 & x \in \left[\frac{2}{5}, \frac{3}{5}\right) \end{cases}$$



$$u'(x) = \int_0^x u''(y)\,dy + c = -\underbrace{\int_0^x f(y)\,dy}_{F(x)} + c = -F(x) + c$$

$$F(x) = \int_0^x f(y)\,dy = \begin{cases} 0 & x \in \left(0, \frac{2}{5}\right) = I_1 \\ -x + \frac{2}{5} & x \in \left[\frac{2}{5}, \frac{3}{5}\right) = I_2 \\ -\frac{1}{5} & x \in \left[\frac{3}{5}, 1\right) = I_3 \end{cases} \Rightarrow u'(x) = \begin{cases} c & x \in I_1 \\ x - \frac{2}{5} + c & x \in I_2 \\ \frac{1}{5} + c & x \in I_3 \end{cases}$$

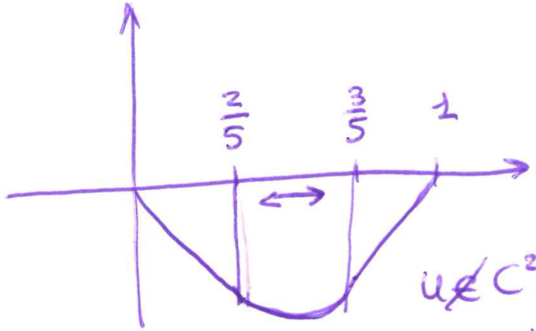
$$u(x) = \int_0^x u'(y)\,dy + c^*$$

$$u(x) = \begin{cases} Cx + D & x \in I_1 \\ \frac{x^2}{2} - \frac{2}{5}x + Cx + E & x \in I_2 \\ \frac{1}{5}x + Cx + F & x \in I_3 \end{cases}$$

$$\begin{cases} u(0) = 0 \\ u(1) = 0 \\ u\left(\frac{2}{5}\right)^- = u\left(\frac{2}{5}\right)^+ \\ u\left(\frac{3}{5}\right)^- = u\left(\frac{3}{5}\right)^+ \end{cases}$$

$\to C, D, E, F$

$$u(x) = \begin{cases} -\frac{x}{10} & x \in I_1 \\ \frac{x^2}{2} - \frac{x}{2} + \frac{2}{25} & x \in I_2 \\ \frac{x}{10} - \frac{1}{10} & x \in I_3 \end{cases}$$



$u \notin C^2(\Omega) \cap C^0(\bar{\Omega})$

$u \in C^1(0,1)$

$$\begin{cases} -u''(x) = f(x) & x \in (0,1) \\ u(0) = u(1) = 0 & \leftarrow \end{cases}$$

v funzione test (u trial, v test) $u, v \in V$

$$\underbrace{-\int_0^1 u''(x)v(x)\,dx} = \int_0^1 f(x)v(x)\,dx \qquad \forall\, v \in V$$

$$-\int_0^1 u''(x)v(x)\,dx = \int_0^1 u'(x)v'(x)\,dx - \cancel{\underbrace{u'(x)v(x)\Big|_0^1}_{u'(1)\underbrace{v(1)}_{=0} - u'(0)\underbrace{v(0)}_{=0}}} = \int_0^1 u'(x)v'(x)\,dx$$

PROPOSTA 1: $V \equiv C_0^1(0,1) = \{ v \in C^1(0,1) \text{ t.c. } v(0) = v(1) = 0 \}$

$$\boxed{?\, u \in V : \int_0^1 u'(x)v'(x)\,dx = \int_0^1 f(x)v(x)\,dx \quad \forall v \in V}$$

→ CB essenziali (D)

→ CB naturali (N, R)

Def 1 : se due funzioni $f, g : \Omega \to \mathbb{R}$ differiscono su un insieme di misura nulla, allora $f = g$ quasi ovunque in Ω e $\int_\Omega f \, d\Omega = \int_\Omega g \, d\Omega$

Def 2 : spazio $L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} \text{ t.c. } \int_\Omega |f|^p \, d\Omega < +\infty \right\}$

integrali di Lebesgue

LM

$$? \, u \in V : a(u,v) = F(v) \quad \forall v \in V$$

1) V Hilbert

2) $a(\cdot,\cdot)$ bilineare
continua *
coerciva *

3) $F(\cdot)$ lineare
limitato *

$$\begin{cases} -\mu_0 u''(x) + \sigma_0 u(x) = 0 & x \in (0,L) \\ \mu_0 u'(0) = q_1 \\ -\mu_0 u'(L) = q_2 \end{cases}$$

μ_0, σ_0 costanti positive $(\in \mathbb{R}^+)$, $q_1, q_2 \in \mathbb{R}$

non conservative

conservative

$-\mu u''$ $-(\mu u')'$

II° ordine diffusione

I° ordine convettivo bu', $(bu)'$

0 ordine reattivo

non conservativa   conservativa

$$\int_0^L -\mu u'' v = \cancel{\int_0^L u'(\mu v)' - \mu u' v \Big|_0^L}$$

$$-\int_0^L (\mu u')' v = \int_0^L \mu u' v' - \mu u' v \Big|_0^L$$

μ non costante

μ costante

$$\hookrightarrow \quad = \int_0^L \mu u' v' - \mu u' v \Big|_0^L$$

$$-\int_0^L \mu_0 u'' v \, dx + \int_0^L \sigma_0 u v = 0$$

$$\int_0^L \mu_0 u' v' \, dx \underbrace{- \mu_0 u' v \Big|_0^L}_{-\underbrace{\mu_0 u'(L)}_{q_2} v(L) + \underbrace{\mu_0 u'(0)}_{q_1} v(0)} + \int_0^L \sigma_0 u v = 0$$

$$\underbrace{\int_0^L \mu_0 u' v' + \int_0^L \sigma_0 u v}_{a(u,v)} = \underbrace{-q_1 v(0) - q_2 v(L)}_{F(v)}$$

$$\boxed{?\, u \in V : \quad a(u,v) = F(v) \quad \forall v \in V}$$

$H^k(\Omega) \subset C^0(\bar{\Omega})$ $\quad k > \frac{d}{2}$ $\quad d=1, k=1$ ok

$H^1 \subset C^0$

$d = 2,3$ $\quad H^2 \subset C^0$

$V \to H^1(\Omega)$ full N, full R

$V \to H^1_0(\Omega)$ full D

$V \to H^1_{\Gamma_D}(\Omega)$, $\Gamma_D \subset \partial\Omega$ misto D/N D/R

Dirichlet essenziali $\to$ spazio

Neumann, Robin naturale $\to$ formulazione

miste D/N D/R $\to$ spazio + formulazione

$V = H^1(0,L)$

$\Omega = (0,L)$

1) $H^1(\Omega)$ è di Hilbert

2) $a(\cdot,\cdot)$ bilineare

3) $a(\cdot,\cdot)$ continua $\quad$? $M>0$ t.c. $|a(u,v)| \le M \|u\|_V \|v\|_V$

$$\|u\|_V = \|u\|_{H^1(\Omega)} \qquad \|u\|^2_{H^1(\Omega)} = \|u\|^2_{L^2(\Omega)} + \underbrace{\|u'\|^2_{L^2(\Omega)}}_{|u|^2_{H^1(\Omega)}}$$

$|u|_{H^1}$ ↓ seminorma H^1

$$\|u\|_{L^2(\Omega)} \le \|u\|_{H^1(\Omega)}$$

(*)
$$|u|_{H^1(\Omega)} = \|u'\|_{L^2(\Omega)} \le \|u\|_{H^1(\Omega)}$$

$$|a(u,v)| = \left| \int_0^L \mu_0 u'v' + \int_0^L \sigma_0 uv \right| \le \mu_0 \left| \int_0^L u'v' \right| + \sigma_0 \left| \int_0^L uv \right|$$

Cauchy-Schwarz

$$\left| \int_\Omega fg \right| \le \|f\|_{L^2} \|g\|_{L^2} \quad \forall f,g \in L^2(\Omega)$$

$$\overset{CS}{\le} \mu_0 \|u'\|_{L^2} \|v'\|_{L^2} + \sigma_0 \|u\|_{L^2} \|v\|_{L^2}$$

$$\overset{(*)}{\le} \mu_0 \|u\|_{H^1} \|v\|_{H^1} + \sigma_0 \|u\|_{H^1} \|v\|_{H^1}$$

$$= \underline{(\mu_0 + \sigma_0)} \|u\|_V \|v\|_V$$

4) $a(\cdot,\cdot)$ coercivo $\quad ?\ \alpha > 0$ t.c. $a(v,v) \geq \alpha \|v\|_V^2$

$$a(v,v) = \int_0^L \mu_0 [v']^2 + \int_0^L \sigma_0 [v]^2$$

$$\geq \min(\mu_0,\sigma_0) \int_0^L [v']^2 + \min(\mu_0,\sigma_0) \int_0^L [v]^2$$

$$= \min(\mu_0,\sigma_0) \underbrace{\left[\underbrace{\int_0^L [v']^2}_{\|v'\|_{L^2}^2} + \underbrace{\int_0^L [v]^2}_{\|v\|_{L^2}^2} \right]}_{\|v\|_{H^1}^2} = \min(\mu_0,\sigma_0) \|v\|_V^2$$

$$a(v,v) \geq \underbrace{\min(\mu_0,\sigma_0)}_{\alpha} \|v\|_V^2$$

$\boxed{\text{Poincaré}}$

<u>Poincaré</u>: se $v \in H_0^1(\Omega)$ o $v \in H_{\Gamma_D}^1(\Omega) = \{v \in H^1(\Omega) \text{ t.c. } v|_{\Gamma_D} = 0\}$ con $\Gamma_D \subset \partial\Omega$, allora $\|v\|_{L^2} \leq C_p |v|_{H^1}$

$H_0^1(\Omega) = \{v \in H^1(\Omega) \text{ t.c. } v|_{\partial\Omega} = 0\}$

$|v|_{H^1} = \|v'\|_{L^2}$

$$\|v\|^2_{H^1} \overset{\text{def}}{=} \|v\|^2_{L^2} + \|v'\|^2_{L^2} \overset{P}{\leq} C_P^2 \|v'\|^2_{L^2} + \|v'\|^2_{L^2} = (C_P^2+1)\|v'\|^2_{L^2}$$

$$\|v\|^2_{H^1} \leq (C_P^2+1)\|v'\|^2_{L^2} = (C_P^2+1)|v|^2_{H^1}$$

$$(**) \quad \boxed{|v|^2_{H^1} \geq \frac{1}{C_P^2+1}\|v\|^2_{H^1}}$$

<u>Ex</u>:

$$\tilde{a}(v,v) = \mu_0 \underbrace{\int_0^L [v']^2}_{|v|^2_{H^1}} \geq \underbrace{\frac{\mu_0}{C_P^2+1}}_{\tilde{\alpha}} \|v\|^2_V$$

5) $F(\cdot)$ lineare

6) $F(\cdot)$ limitato $\quad \exists\, C>0$ t.c. $|F(v)| \leq C\|v\|_V$

$$|F(v)| = |q_1 v(0) + q_2 v(L)| \leq |q_1 v(0)| + |q_2 v(L)|$$
$$= |q_1||v(0)| + |q_2||v(L)| \leq C_T\left[|q_1|+|q_2|\right]\|v\|_{H^1}$$

<u>Stima di traccia</u>: $\Omega \subset \mathbb{R}^d$ aperto limitato con frontiera regolare. $\exists$ funzione lineare e <u>continua</u> $\gamma: H^k(\Omega) \to L^2(\partial\Omega)$, $k \geq 1$, $\gamma v = v|_{\partial\Omega}$ $\forall v \in H^k \cap C^0$.

$$\|\gamma v\|_{L^2(\partial\Omega)} \leq C_T \|v\|_{H^k}$$

$d=1$: $|v(a)|, |v(b)| \leq C_T \|v\|_{H^1}$

$\underline{\text{Disuguaglianza di Hölder}}$: $f \in L^p(\Omega)$, $g \in L^q(\Omega)$ t.c. $\frac{1}{p} + \frac{1}{q} = 1$ $\left(\begin{array}{l} p=q=2 \\ p=1,\ q=\infty \end{array}\right)$

$$\boxed{\|fg\|_{L^1} \le \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}}$$

$\underline{\text{Ex}}$:

$$\left| \int_0^L \underbrace{\mu(x)}_{f} \underbrace{u'(x)v'(x)}_{g} \right| \overset{H\ q=1\ p=\infty}{\le} \|\mu\|_{L^\infty(\Omega)} \|u'v'\|_{L^1(\Omega)} \overset{CS}{\le} \|\mu\|_{L^\infty} \|u'\|_{L^2} \|v'\|_{L^2} \overset{*}{\le} \|\mu\|_{L^\infty} \|u\|_V \|v\|_V$$

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a continuità — $*$ CS — Hölder

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a coercività — Poincaré — $**$

F limitatezza — $*$ CS — traccia

Corollario (LM): sotto le hp del lemma di LM, si ha che $\quad \|u\|_V \leq \frac{1}{\alpha} \|F\|_{V'}$

$\uparrow$ costante di coercività

[V' $\quad$ F funzionale $F: V \to \mathbb{R}$ lineare e limitato

$$\|F\|_{V'} = \sup_{\substack{v \in V \\ v \neq 0}} \frac{|F(v)|}{\|v\|_V} \quad \Rightarrow \quad \frac{|F(v)|}{\|v\|_V} \leq \|F\|_{V'}$$

$$|F(v)| \overset{\circledast}{\leq} \|F\|_{V'} \|v\|_V \quad \forall v \in V \quad]$$

Dim:

$$a(u,v) = F(v)$$

$$v = u$$

$$\alpha \|u\|_V^{\cancel{2}} \underset{\substack{\uparrow \\ \text{coercività}}}{\leq} a(u,u) = F(u) \leq |F(u)| \underset{\circledast}{\leq} \|F\|_{V'} \cancel{\|u\|_V}$$

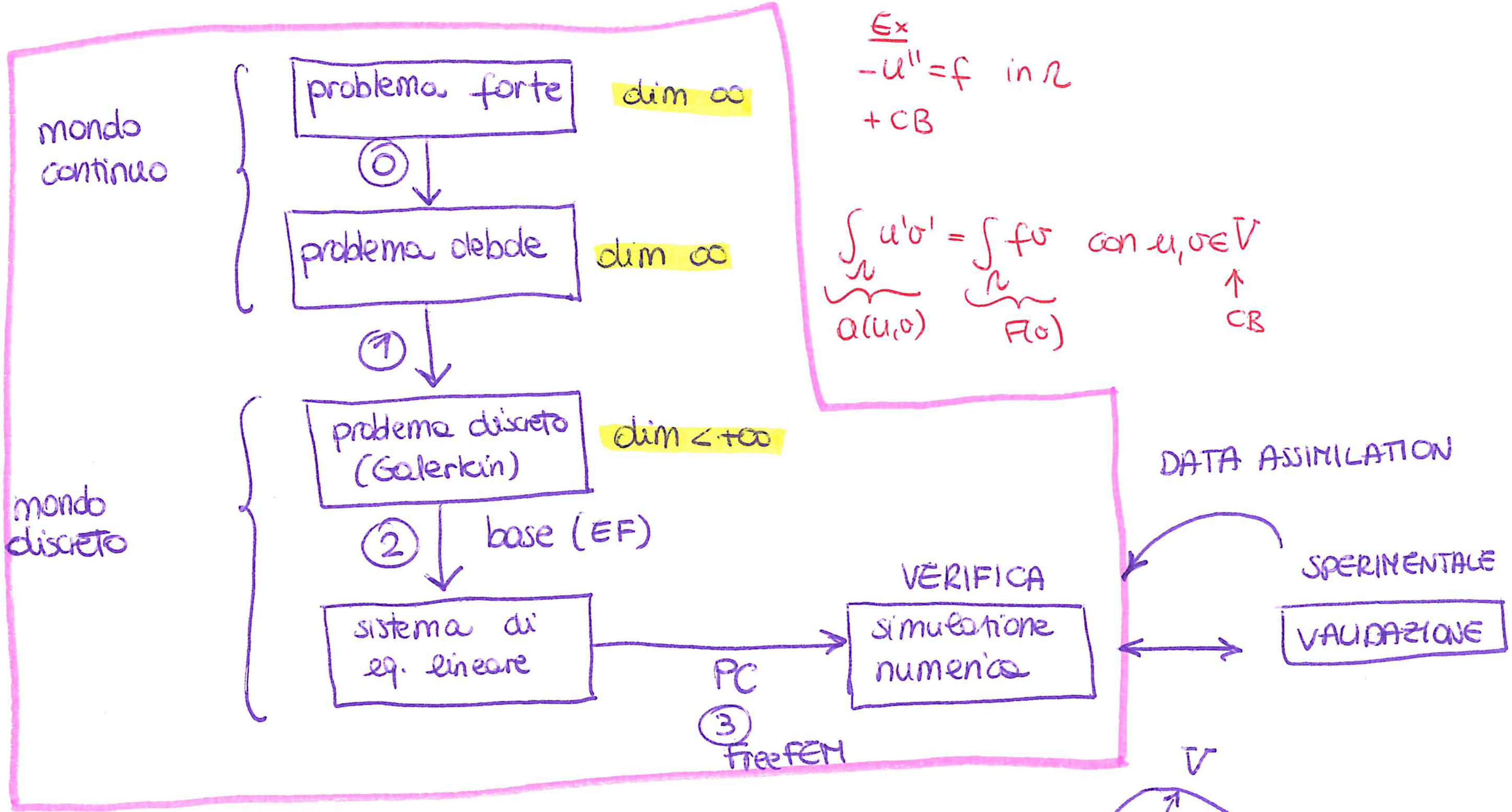
$$\|u\|_V \leq \frac{1}{\alpha} \|F\|_{V'} \qquad \blacksquare$$

Ex

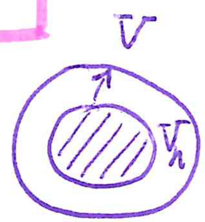
$-u'' = f$ in Ω

$+ CB$

$$\underbrace{\int_\Omega u'v'}_{a(u,v)} = \underbrace{\int_\Omega fv}_{F(v)} \quad \text{con } u, v \in \underset{\uparrow\, CB}{V}$$



$? \, u \in V : a(u,v) = F(v) \quad \forall v \in V$



$\{V_h\} \quad V_h \subset V \quad \begin{cases} h & \text{parametro di discretizzazione} \\ \dim V_h = N_h < +\infty \end{cases}$

1 <u>formulazione di Galerkin</u>: $? \, \underset{\text{soluzione discreta}}{u_h} \in V_h \subset V : a(u_h, \underset{\uparrow \text{ f. test discreta}}{v_h}) = F(v_h) \quad \forall v_h \in V_h$

② base V_h $\quad \{\varphi_j\}_{j=1}^{N_h}$ $\qquad v_h \leftarrow \varphi_i \quad i=1,\dots,N_h$

$$a(u_h, \varphi_i) = F(\varphi_i) \qquad i=1,\dots,N_h$$

$$V_h \ni u_h(x) = \sum_{j=1}^{N_h} u_j \varphi_j(x)$$

incognite

$$a\left(\sum_{j=1}^{N_h} u_j \varphi_j, \varphi_i\right) = F(\varphi_i) \qquad i=1,\dots,N_h$$

$$\sum_{j=1}^{N_h} u_j\, a(\varphi_j, \varphi_i) = F(\varphi_i) \qquad i=1,\dots,N_h$$

i-esima equazione

$$\boxed{A\vec{u} = \vec{f}}$$

$$\vec{u} = [u_1, u_2, \dots, u_{N_h}]^T \in \mathbb{R}^{N_h}$$

$$\vec{f} = [f_1, f_2, \dots, f_{N_h}]^T, \quad f_i = F(\varphi_i) \qquad i=1,\dots,N_h$$

$$A = [a_{ij}] \in \mathbb{R}^{N_h \times N_h}$$

matrice di rigidezza

$$a_{ij} = a(\varphi_j, \varphi_i) \qquad i,j=1,\dots,N_h$$

Ex $\quad -u'' = f$ in Ω $\qquad a_{ij} = \int_\Omega \varphi_j' \varphi_i'$

$$f_i = \int_\Omega f \varphi_i$$

$a(u,v) = a(v,u)$ simmetrica $\qquad$ Ex $\quad a(u,v) = \int_\Omega u'v'$ è simmetrica

$a(u,v) = \int_\Omega u'v$ non è simmetrica

$A\vec{u}=\vec{f}$

problem-dependent

① simmetria (Cholesky, PG, PCG)
(se $a(\cdot,\cdot)$ è simmetrico $\Longleftrightarrow$ A simmetr.)

② se $a(\cdot,\cdot)$ è coerciva $\Longrightarrow$ A dp.

$\boxed{\text{ANALISI DI GALERKIN}}$

base-dependent

① $K(A)$

② pattern di sparsità
$\downarrow$
(supporto delle φ_j)

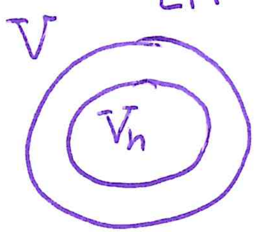
$$\text{supp}(\varphi_j) \subset \Omega$$
$$\| $$
$$\{x \in \Omega \text{ t.c. } \varphi_j(x) \neq 0\}$$

- $\exists!$
- consistenza
- stabilità
- ortogonalità di Galerkin
- convergenza

$\exists!$ u_h soluzione di Galerkin

LM

A dp $\Longrightarrow$ A è non singolare $\Longleftrightarrow$ $\exists A^{-1}$ $\Longleftrightarrow$ $Au=f$ ha $1!$ soluzione



stabilità

$$\|u_h\|_V \leq \frac{1}{\alpha} \|F\|_{V'}$$ (uniformemente rispetto ad h)

$$a(u_h, v_h) = F(v_h) \qquad = \int_\Omega f v_h \quad \forall v_h \in V_h$$

$$a(w_h, v_h) = G(v_h) \qquad = \int_\Omega (f+\rho) v_h \quad \forall v_h \in V_h$$

perturbazione sul dato

$$w_h = u_h + Z$$

perturbazione sulla soluzione

soluzione perturbata

$$a(u_h - w_h, v_h) = F(v_h) - G(v_h) \qquad \forall v_h \in V_h$$

$$v_h = u_h - w_h$$

$$\alpha \|u_h - w_h\|_V^2 \le a(u_h - w_h, u_h - w_h) = F(u_h - w_h) - G(u_h - w_h) = (F-G)(u_h - w_h)$$

$$\le |(F-G)(u_h - w_h)| \le \|F-G\|_{V'} \cancel{\|u_h - w_h\|_V}$$

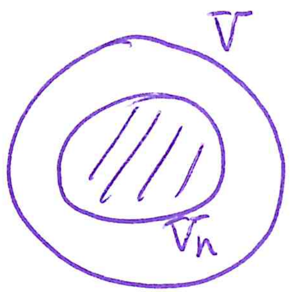
$$\|u_h - w_h\|_V \le \frac{1}{\alpha} \|F - G\|_{V'}$$

Ortogonalità di Galerkin: u_h soluzione del problema di Galerkin è t.c.

$$\underbrace{a(u-u_h}_{\text{errore di discretizzazione}}, v_h) = 0 \quad \forall v_h \in V_h .$$

Dim:

$a(u,v) = F(v) \quad \forall v \in V$



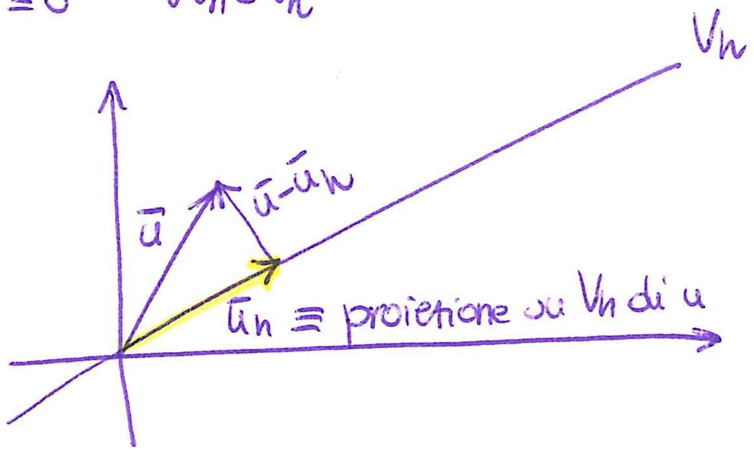
$a(u_h, v_h) = F(v_h) \quad \forall v_h \in V_h$

$v = v_h$

consistenza $\boxed{a(u, v_h) = F(v_h) \quad \forall v_h \in V_h}$

$a(u, v_h) - a(u_h, v_h) = F(v_h) - F(v_h) = 0 \quad \forall v_h \in V_h$

$a(u-u_h, v_h) = 0 \quad \forall v_h \in V_h$



Galerkin è un metodo di proiezione ortogonale

Se $a(\cdot,\cdot)$ è simmetrica, allora $a(\cdot,\cdot)$ definisce un prodotto scalare

$a(u,v) \implies$ norma in energia $\quad \|w_h\|_a = \sqrt{a(w_h, w_h)} \quad \forall w_h \in V_h$

$w_h(x) = \sum_{j=1}^{N_h} w_j \varphi_j(x) \quad \bar{w}_h = [w_1, w_2, \dots, w_{N_h}]^T \quad (\bar{w}_h)^T A (\bar{w}_h)$

$\|w\|_A = \sqrt{w^T A w}$

$$a(u-u_h, u-u_h) = a(u-u_h, u-v_h) + \underbrace{a(u-u_h, \overbrace{v_h-u_h}^{\in V_h})}_{\substack{=0 \\ \text{OG}}}$$

$$\pm\, a(u-u_h, v_h) \quad \forall v_h \in V_h$$

$$\alpha \|u-u_h\|^2_V \le a(u-u_h, u-u_h) = a(u-u_h, u-v_h) \le |a(u-u_h, u-v_h)| \le M \cancel{\|u-u_h\|_V} \|u-v_h\|_V$$

α: costante di coercività

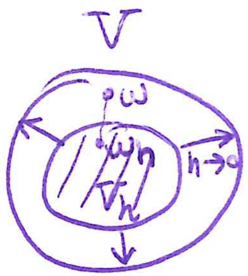
M: costante di continuità

$$\|u-u_h\|_V \le \frac{M}{\alpha} \|u-v_h\|_V \quad \forall v_h \in V_h$$

$$\inf_{w_h \in V_h} \|u-w_h\|_V \le \boxed{\|u-u_h\|_V \le \frac{M}{\alpha} \boxed{\inf_{w_h \in V_h} \|u-w_h\|_V}}$$

Lemma di Céa

errore di miglior approssimazione



$\boxed{\text{Hp}}$ $\displaystyle \lim_{h\to 0} \inf_{w_h \in V_h} \|w-w_h\|_V = 0 \quad \forall w \in V$ densità di V_h in V

↳ convergenza

$\|u-u_h\|_V \to 0$ per $h \to 0$

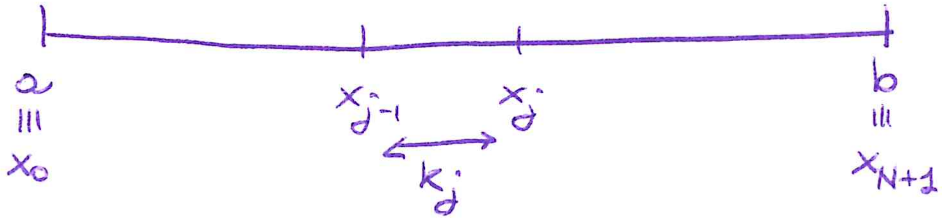
V_h = EF

$\{V_h\}$ $\quad V_h \subset V = H^1_0(\Omega)$

$$\begin{cases} -u''(x) = f(x) & x \in (a,b) = \Omega \\ u(a) = u(b) = 0 \end{cases}$$

1D

$T_h = \{K_j\}$



(N+1) sottointervalli

$K_j = (x_{j-1}, x_j)$

$\{x_j\}$ vertici della partizione

$X_h^r = \{ v_h \in C^0([a,b])$ t.c. $v_h|_{K_j} \in \mathbb{P}_r ,\ \forall K_j \in T_h \}$ EF space

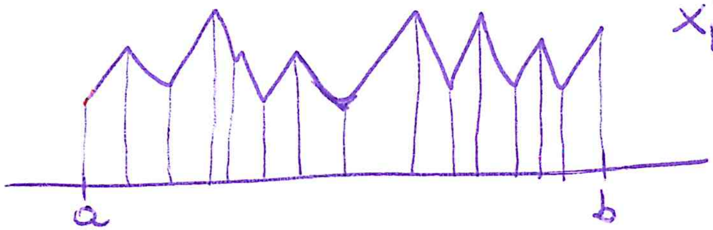
$h_j = x_j - x_{j-1}$

$h = \max_j h_j$

$r = 1, 2, \dots$

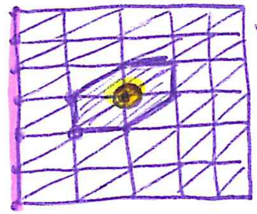
→ polinomi a tratti (piecewise polynomials)

$r = 1$



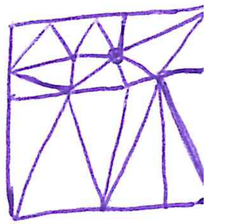
X_h^1 (EF lineari)

$\boxed{\Pi_H^r f}$

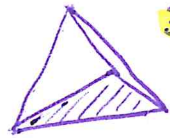


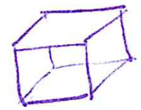
$\Omega \subset \mathbb{R}^d$

2D



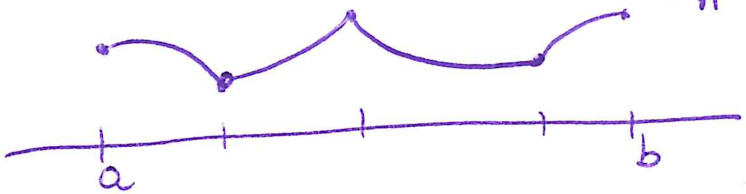
3D





(EF quadratici)

$r = 2$



X_h^2

1) $X_h^r \overset{?}{\subset} H^1(\Omega)$ SI

2) CB ? $\cancel{\boxed{V_h = X_h^r \cap V}}$ SI

→ $V_h = \{ v_h \in X_h^r$ t.c. $v_h(a) = v_h(b) = 0 \}$

Base per V_h : 1) funzioni facili da integrare

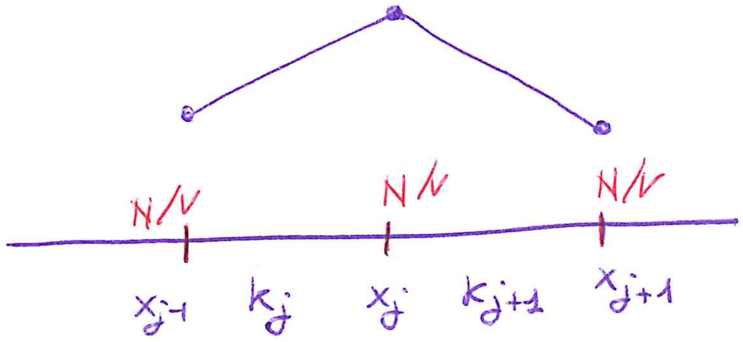
$\{\varphi_i\}$

2) supporto di φ_i sia piccolo ($\Longrightarrow$ sparsità di A)

3) base Lagrangiana $\varphi_i(x_j) = \delta_{ij} \begin{cases} \to 0 & i \neq j \\ \to 1 & i = j \end{cases}$

↑ nodo

Spazio X_h^1 $(\Pi_H^1 f)$



2 informazioni $\forall k_j$

$\left.\begin{array}{ll} \text{vertici} \quad (\text{geometrica}) & \text{"x"} \\ r=1 \to \text{|||} & \\ \text{nodi} & \text{"x"} \end{array}\right\}$

gradi di libertà "y" $\}$

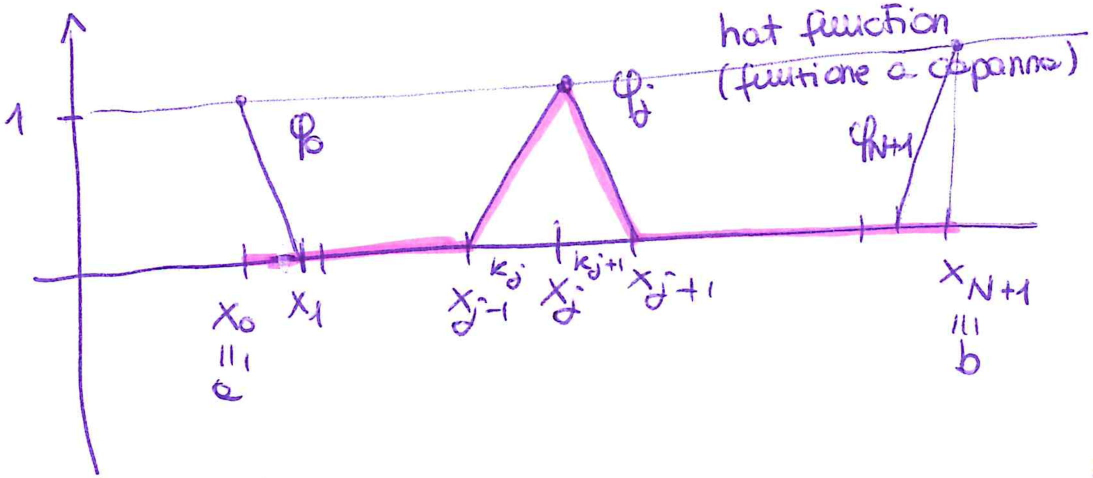
(degrees of freedom ; dofs)

$\{\text{nodi}\} \supseteq \{\text{vertici}\}$

(nodo, gdl)

1 funzione di base $\forall$ nodo

$\varphi_0, \varphi_1, \dots \varphi_{N+1}$ $(N+2)$ f. base



$$\begin{cases} \varphi_j \in X_h^1 & i,j = 0, \dots, N+1 \\ \varphi_j(x_i) = \delta_{ij} & \\ \operatorname{supp}(\varphi_j) \text{ piccolo} & \end{cases}$$

$$\operatorname{supp}(\varphi_j) = k_j \cup k_{j+1} = [x_{j-1}, x_{j+1}]$$

$$\operatorname{supp}(\varphi_0) = [x_0, x_1] = k_1$$

$$\operatorname{supp}(\varphi_{N+1}) = [x_N, x_{N+1}] = k_{N+1}$$

1) X_h^1 $\quad V_h \subset H_0^1(\Omega)$

$\{\varphi_j\} \quad j = 1, \dots, N$

$\nexists \ \varphi_0, \ \varphi_{N+1}$

[<u>Ex</u> $\quad V = H^1_{\Gamma_D}(\Omega) \quad \Gamma_D = \{b\} \quad \varphi_0, \varphi_1, \dots, \varphi_N \quad \nexists \ \varphi_{N+1}$

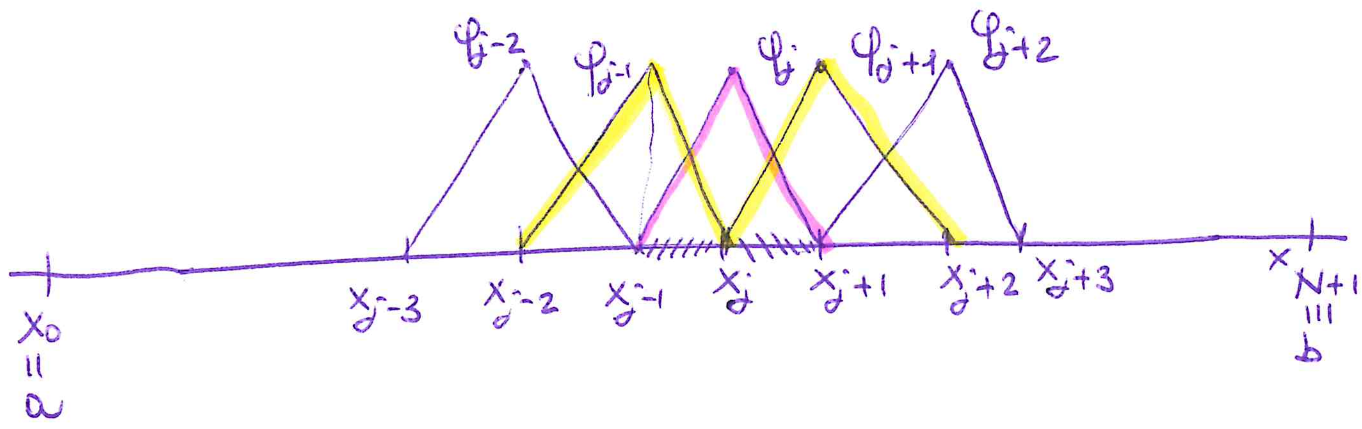
$V = H^1_{\Gamma_D}(\Omega) \quad \Gamma_D = \{a\} \quad \varphi_1, \varphi_2, \dots, \varphi_N, \varphi_{N+1} \quad \nexists \ \varphi_0$]

2) $$\varphi_j(x) = \begin{cases} \dfrac{x - x_{j-1}}{x_j - x_{j-1}} & x \in k_j \\ \dfrac{x_{j+1} - x}{x_{j+1} - x_j} & x \in k_{j+1} \\ 0 & \text{altrimenti} \end{cases}$$

3) $v_h \in V_h$ $\quad v_h(x) = \sum_j \tilde{v}_j \varphi_j(x) \quad \{\tilde{v}_j\} \subset \mathbb{R}$

$$v_h(x_i) = \sum_j \tilde{v}_j \underbrace{\varphi_j(x_i)}_{\delta_{ij}} = \tilde{v}_i \qquad \boxed{\tilde{v}_i = v_h(x_i)}$$

4) $a_{ij} = a(\varphi_j, \varphi_i) = \int_a^b \varphi_j' \varphi_i' \, dx \qquad i,j = 1, \dots, N$



$a_{ij} \neq 0 \qquad i = j-1, j, j+1 \longrightarrow$ <u>tridiagonale</u> (1D)

<u>partizione uniforme</u> $\qquad a_{j-1,j} \;,\; a_{j,j} \quad a_{j+1,j}$

$$a_{j-1,j} = \int_a^b \varphi_j' \varphi_{j-1}' \, dx = \int_{x_{j-1}}^{x_j} \underset{\left(\frac{1}{h}\right)}{\varphi_j'} \; \underset{\left(-\frac{1}{h}\right)}{\varphi_{j-1}'} \, dx = \int_{x_{j-1}}^{x_j} -\frac{1}{h^2} \, dx = -\frac{1}{h^2} h = -\frac{1}{h}$$

$$a_{j+1,j} = \int_a^b \varphi_j' \cdot \varphi_{j+1}' \, dx = \int_{x_j}^{x_{j+1}} \varphi_j' \varphi_{j+1}' \, dx = \int_{x_j}^{x_{j+1}} \left(-\frac{1}{h}\right)\left(\frac{1}{h}\right) dx = -\frac{1}{h^2} h = -\frac{1}{h}$$

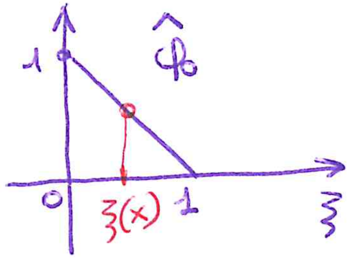
$$a_{jj} = \int_a^b [\varphi_j']^2 \, dx = \int_{x_{j-1}}^{x_j} \underbrace{[\varphi_j']^2}_{\frac{1}{h^2}} dx + \int_{x_j}^{x_{j+1}} \underbrace{[\varphi_j']^2}_{\frac{1}{h^2}} dx = \frac{1}{h^2} h + \frac{1}{h^2} h = \frac{2}{h}$$

$$A = \frac{1}{h}\begin{bmatrix} 2 & -1 & 0 & \cdots & 0\\ -1 & 2 & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0\\ \vdots & & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 2\end{bmatrix}$$

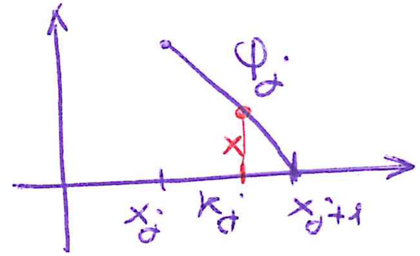
$\downarrow$

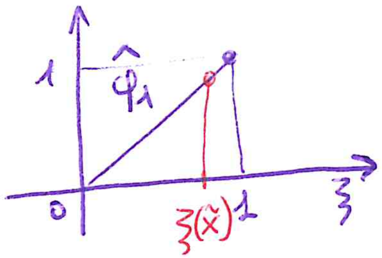
1D Poisson
+ CB Dirichlet
omogenee +
partizione uniforme
EF lineari

5) RIFERIMENTO $[0,1]$ $\xrightarrow{\gamma}$ FISICO K_j

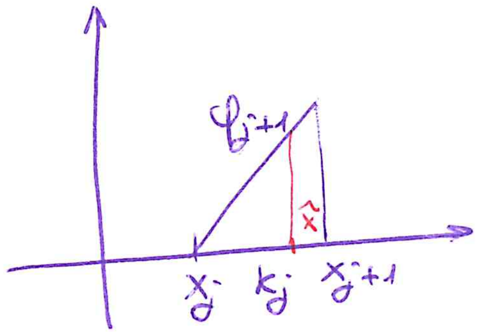


$\hat\varphi_0(\xi) = 1-\xi$ $\xrightarrow{\gamma}$





$\hat\varphi_1(\xi) = \xi$ $\xrightarrow{\gamma}$



$$\gamma : [0,1] \longrightarrow [x_j, x_{j+1}]$$

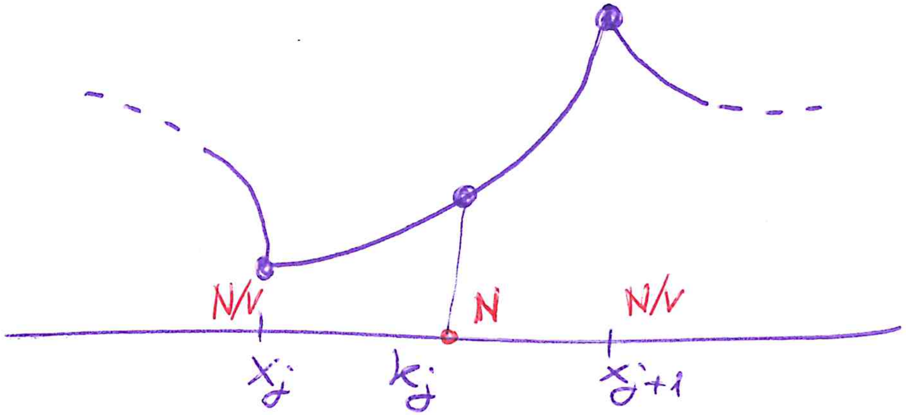
$$x = \gamma(\xi) = x_j + \xi(x_{j+1} - x_j)$$

$$\gamma(0) = x_j \; ; \; \gamma(1) = x_{j+1} \; ; \; \gamma\left(\tfrac{1}{2}\right) = \frac{x_j + x_{j+1}}{2}$$

$$\varphi_j(x) = \hat\varphi_0(\xi(x)) = \hat\varphi_0\left(\frac{x - x_j}{x_{j+1} - x_j}\right)$$

$$\varphi_{j+1}(\tilde x) = \hat\varphi_1(\xi(\tilde x)) = \hat\varphi_1\left(\frac{\tilde x - x_j}{x_{j+1} - x_j}\right)$$

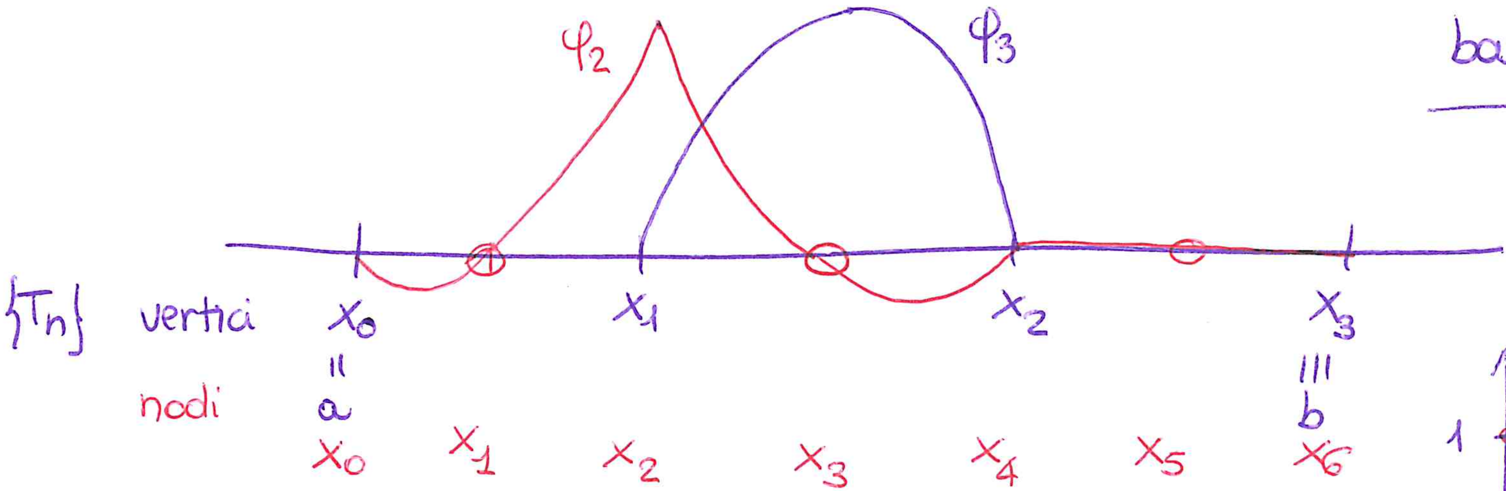
Spazio X_h^2 EF quadratici

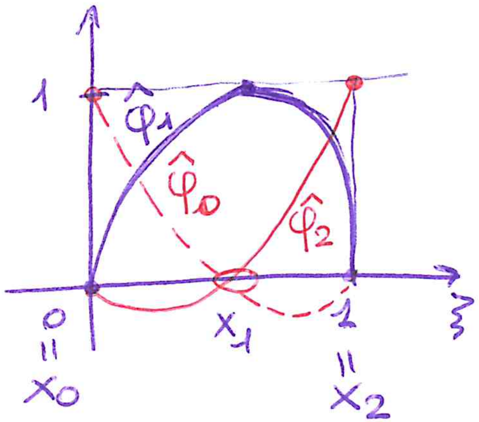


$\forall$ intervallo 3 dof (3 nodi), 2 vertici

nodi $\supset$ vertici

base Lagrangiana





$$\hat{\varphi}_0(\xi) = (1-\xi)(1-2\xi)$$

$$\hat{\varphi}_1(\xi) = 4\xi(1-\xi)$$

$$\hat{\varphi}_2(\xi) = \xi(2\xi-1)$$