Esercitazione 8 -

Exercise 1

X_i: lifetime of belt (1000 km)

$X \sim N(\mu, 51)$ $\quad X_1, \dots, X_{30}: \sum_i x_i = 1710$

a) Point Estimate and $CI_{0,95}(\mu)$?

Point Estimate for μ

$$\bar{x}_{30} = \frac{1}{30} \sum_i x_i = 57$$

We are in N with known variance case:

$$CI_{1-\alpha}(\mu) = \left(\bar{x}_n - z_{1-\frac{\alpha}{2}} \frac{\sqrt{51}}{\sqrt{30}}, \bar{x}_n + z_{1-\frac{\alpha}{2}} \frac{\sqrt{51}}{\sqrt{30}}\right)$$

$$= (54{,}44, 59{,}56)$$

$\hookrightarrow \alpha = 0{,}05, \quad \bar{x} = 57$

b) At $\alpha = 3\%$ is μ different from 58?

$H_0: \mu = 58$ vs. $H_1 \neq 58$ (at level 3%)

We construct the critical region:

$$\underbrace{\frac{|\bar{x}_n - 58|}{\sqrt{51}/\sqrt{30}}}_{z_0 = 0{,}77} > z_{1-\frac{\alpha}{2}}$$

$z_{.985} = 2.17$

Since $z_0 < z_{.985}$, we cannot reject H_0 at level 3%.

Since $\mu = 58 \in CI_{0.95}(\mu) \Rightarrow 58 \in CI_{0.97}(\mu)$

$\Rightarrow$ we cannot reject H_0 at 3%.

$\hookrightarrow$ Duplicity

c) $\mathbb{P}\{\text{lifetime} > 60000\}$? $\Rightarrow$ Estimate

since μ is unknown

$CI_{0.97}(p)$?

$\mathbb{P}\{X > 60\} \Rightarrow 1 - \mathbb{P}\{X \leq 60\}$

$X \sim N(\mu, 51) \Rightarrow p = 1 - \phi\left(\frac{60-\mu}{\sqrt{51}}\right) \xrightarrow{\text{plugin}} \hat{p} = 1 - \phi\left(\frac{60 - \bar{x}_n}{\sqrt{51}}\right)$

$\hookrightarrow$ estimator for p

Estimate for p:

$\bar{x}_n = 57 \rightarrow \hat{p} = 0.33724$

$CI_{0.97}(p) = ?$

$CI_{0.97}(\mu) = \left(\bar{x}_n - z_{0.985}\frac{\sqrt{51}}{\sqrt{30}}, \bar{x}_n + z_{0.985}\frac{\sqrt{51}}{\sqrt{30}}\right) = (54.17, 59.83)$

$\hookrightarrow$ N, σ^2 known

By plugin rule for interval estimators:

$CI_{0.97}(p) = \left(1 - \phi\left(\frac{60-54.17}{\sqrt{51}}\right), 1 - \phi\left(\frac{60-59.83}{\sqrt{51}}\right)\right)$

$\stackrel{!}{=} (0.20611, 0.49202)$

We want to test at level 3%

$H_0: p = 0.28$ vs. $H_1: p \neq 0.28$

Since $p = 0.28 \in C_{0.97}(p)$, then we cannot reject

H_0 at level 3%.

Exercise 2

X: pH of chemical solution

$X_1, \dots, X_{25} \overset{iid}{\sim} N(\mu, 0.02^2)$

Chemical process takes place only if $\mu > 8.2$

a) Write out the test

H_0 is the assumption such that if wrongly rejected (H_0 true but rejected), leads to the most severe consequences, since the error is a type I error which we can control

$H_0: \mu \leq 8.20$ vs. $H_1: \mu > 8.20$ → If this is wrong, the whole exercise is wrong.

Rejecting H_0 is a strong claim to H_1.

b) $CR = \{x : \bar{X}_{25} > \underline{8.2093}\}$

Find the confidence level of the test in a

We are in N with σ^2 known case:

$$CR = \left\{ x : \frac{\bar{X}_n - \mu}{\sqrt{\sigma/n}} > z_{1-\alpha} \right\} = \left\{ \bar{X}_n > 8.20 + \frac{0.02}{\sqrt{35}} z_{1-\alpha} \right\}$$

By imposing:

$$8.20 + \frac{0.02}{\sqrt{35}} z_{1-\alpha} = 8.2093$$

We can derived $z_{1-\alpha}$, then α:

$$z_{1-\alpha} = 2.325 \Rightarrow 1-\alpha = 0.9901 \Rightarrow \alpha = 0.0099$$

$\Rightarrow$ 0.99% significance level $\rightarrow$ probability of wrongly rejecting H_0.

c) Compute Type 2 error probability when $\mu = \begin{cases} 8.2093 \\ 8.215 \end{cases}$

Type II error is the error of wrongly accepting H_0
(we cannot control

$$\mathbb{P}_{H_1}(H_0 \text{ accepted}) = P_{H_1}\{X \in CR\} = \mathbb{P}_{H_1}\left\{\bar{X}_n < \mu_0 + \frac{\sigma}{\sqrt{n}} z_{1-\alpha}\right\}?$$

Under H_1,

$$\bar{X}_n \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

$$\Rightarrow P_{H_1}\left\{ \underbrace{\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}}}_{\sim N(0,1)} < \frac{\mu_0 - \mu}{\sigma/\sqrt{n}} + z_{1-\alpha} \right\} = \phi\left(\frac{\mu_0 - \mu}{\sigma/\sqrt{n}} + z_{1-\alpha} \right)$$

Substitute

$$= \begin{cases} \phi(0) = 0{,}5 , & \mu = 8.2093 \\ \phi(-1.43) = 0.07636 , & \mu = 8.215 \end{cases}$$

Numbers.

d) $\bar{x}_{25} = 8.206$

$CR = \{ x : \bar{x}_{25} > 8.2093 \}$

Since $8.206 < 8.2093 \Rightarrow$ we donot reject H_0

Exercise 3

2.3, 1.7, 3.2, 2.1, 2.3, 2.0, 2.2, 1.2

a) Unbiased estimator for mean and variance

$$\bar{X}_8 = \frac{1}{8} \sum_i X_i \rightarrow \bar{x}_8 = 2.125$$

$$S_8^2 = \frac{1}{7} \sum_{i=1}^{8} (x - \bar{x})^2 \rightarrow s_8^2 = 0.325$$

X: concentration certain toxic substance

$X \sim N(\mu, \sigma^2)$

$\mu > 2.7$ there are health risks

b) At level 5%, determine if air is dangerous or not.

Error is greater mistake

$H_0 : \mu > 2.7$ $\quad H_1 : \mu < 2.7$

Unilateral N case, with unknown variance.

$$T_0 = \frac{\bar{X}_n - \mu_0}{S_n/\sqrt{n}} \qquad (t\text{-test})$$

$T_0 < -t_{1-\alpha}(n-1) \rightarrow$ If true, reject H_0.

$\alpha = 0{,}05 \qquad n = 8 \qquad \bar{x}_n = 2.125$

$\mu = 2.7 \qquad s_n = 0.57$

$\Rightarrow T_0 = -2.853 < -1.8946 \Rightarrow$ we rejeb H_0 at level 5%

c) p-value? Would you feel safe to breath the air?

$\alpha^* = T_0 = -t_{1-\alpha^*} \Rightarrow 2.853 = t_{1-\alpha^*}(n-1)$

$T_0 = -2.853$

$t_{0.9875}(7) < 2.853 < t_{0.99}(7)$

$0.975 < 1 - \alpha^* < 0.99$

$0.01 < \alpha^* < 0.025 \Rightarrow$ p-value $(1\%, 2.5\%)$

We have moderate evidence to reject H_0.