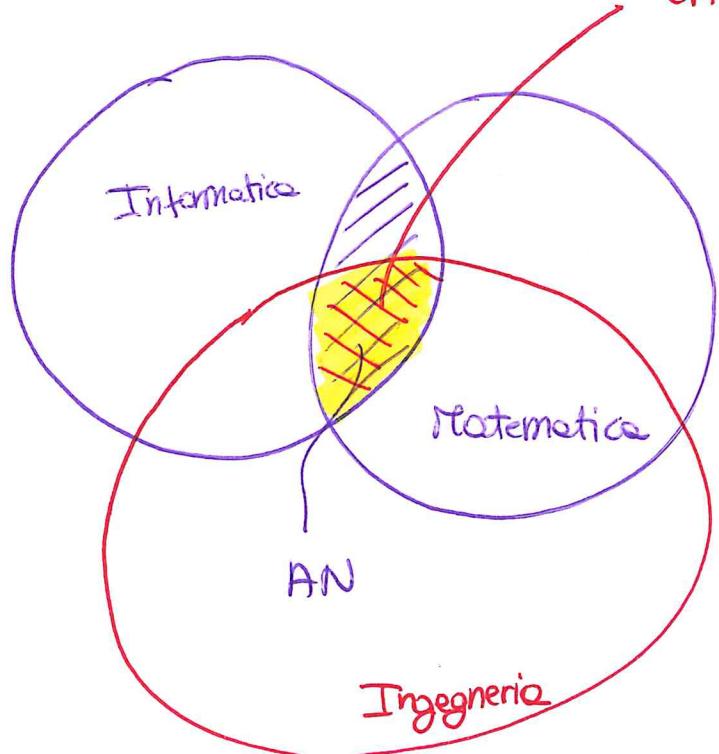


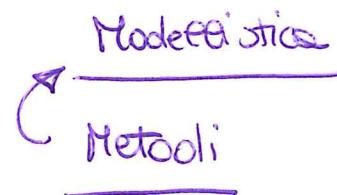
ANALISI NUMERICA (AN)

CALCOLO SCIENTIFICO

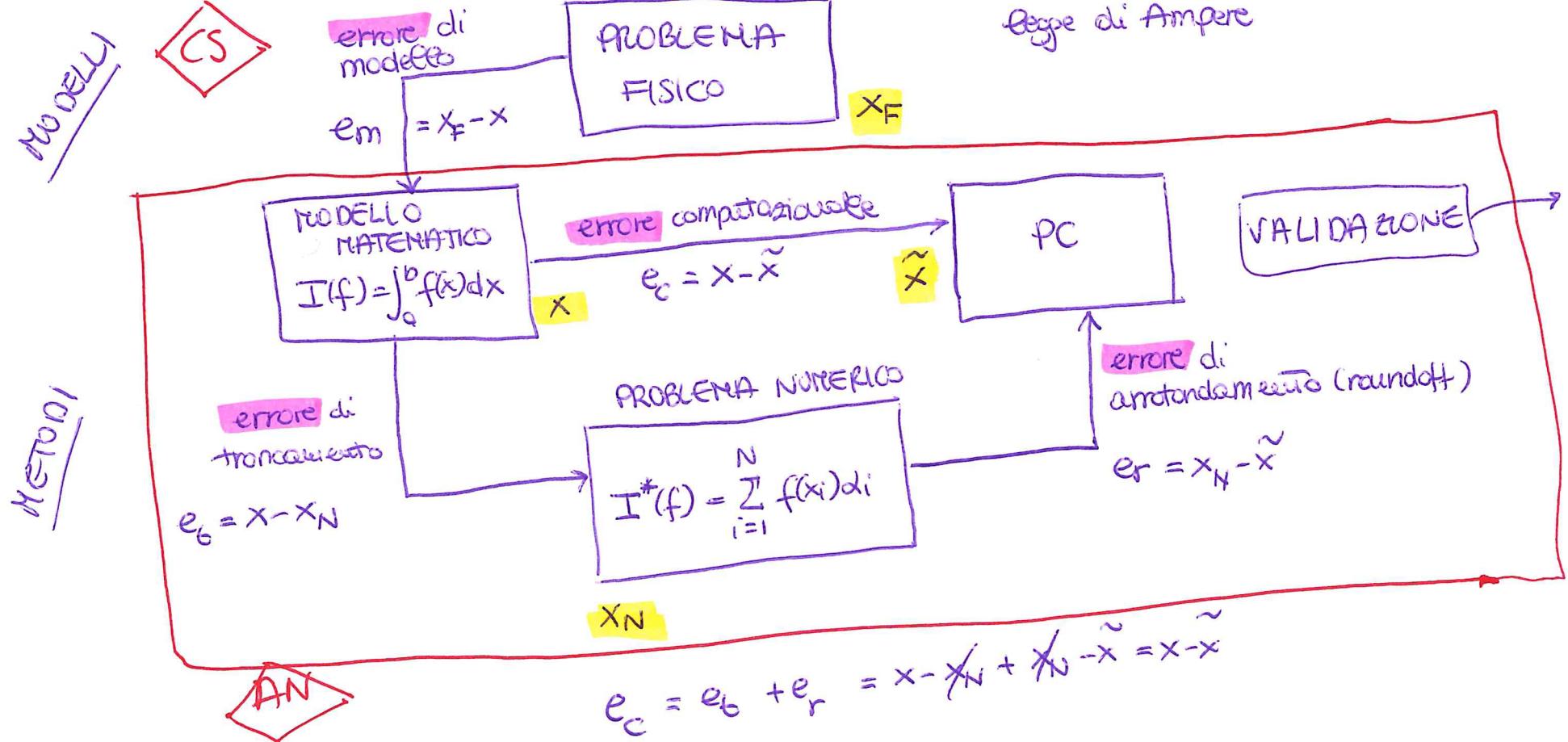
CALCOLO SCIENTIFICO (cs)



CS & AN



mox.polimi.it



errore assoluto

$$|x - \tilde{x}|$$

$$x = 100$$

$$\tilde{x} = 100.1$$

$$|x - \tilde{x}| = 0.1$$

$$\frac{|x - \tilde{x}|}{|x|} = \frac{0.1}{100} = 10^{-3} = 0.1\%$$

floating point

errore relativo

$$\frac{|x - \tilde{x}|}{|x|} \quad x \neq 0$$

$$x = 0.2$$

$$\tilde{x} = 0.1$$

$$|x - \tilde{x}| = 0.1$$

$$\frac{|x - \tilde{x}|}{|x|} = \frac{0.1}{0.2} = 0.5 = 50\%$$

~~STIMA~~

STIMATORI  
DELL'ERRORE

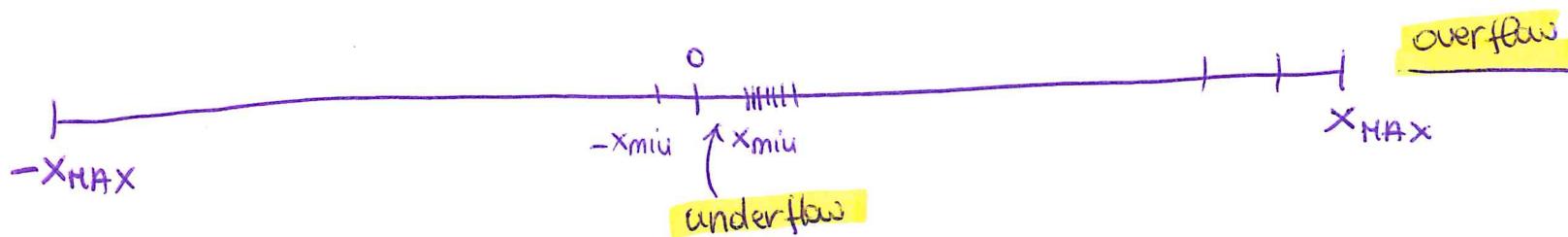
$\mathbb{R}$

arredamento

$$\forall x \in \mathbb{R} \xrightarrow{\downarrow} f(x) \in \mathbb{F}$$

$$x \neq f(x)$$

$$\mathbb{F} \subset \mathbb{R}$$



$$\mathbb{R} \ni x \simeq f(x) = (-1)^s \underbrace{(0.a_1 a_2 \dots a_t)}_{\text{mantissa}} (\beta)^e$$

↑ esponente  
 $\mathbb{F}$        $s \geq 0$  positivi      |  
 $\geq 1$  negativi      base (2)  
 $0 \leq a_i \leq \beta - 1 \quad i=1, \dots, t$

$$L \leq e \leq U$$

$$\boxed{a_1 \neq 0}$$

RAPPRESENTAZIONE  
NORMALIZZATA

$0 \in \mathbb{F}$

$\mathbb{F}(\beta, t, L, U)$

$\mathbb{F}(2, 53, -1021, 1024)$

↓  
unicità di rappresentazione

$$\begin{bmatrix} 0.003 \\ 0.03 \cdot 10^{-1} \\ 0.0003 \cdot 10^{-2} \end{bmatrix}$$

$$\boxed{0.3 \cdot 10^{-2}}$$

$$0.03 \rightarrow 0.3 \cdot 10^{-2}$$

cardinalità di  $\mathbb{F} = \#\mathbb{F}$

$$= 2(\beta-1)\beta^{t-1}(\beta-1+1)$$

$\downarrow$        $\downarrow$        $\downarrow$   
 $20$        $a_1$        $a_2, \dots, a_t$        $e$   
 $20$

# SISTEMI LINEARI

$$Ax = b$$

$$A \in \mathbb{R}^{n \times n}$$

$$b, x \in \mathbb{R}^n$$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n \end{cases}$$

$A$  non singolare  $\Leftrightarrow \exists! x$

CRAMER
+
LAPLACE

$3(n+1)!$

$$x_i = \frac{\det(A_i)}{\det(A)} \quad i = 1, \dots, n \quad (n+1) \text{ determinanti}$$

$$\det(A) = \begin{cases} a_{11} & n=1 \\ \sum_{j=1}^n a_{1j} \Delta_{1j} & n>1 \end{cases}$$

$\downarrow (-1)^{i+j} \det(A_{ij})$

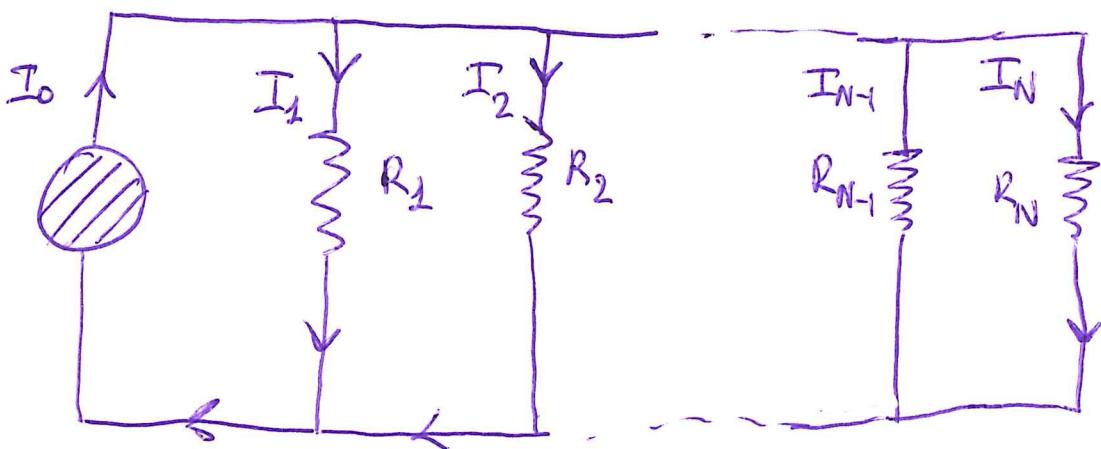
$\mathbb{R}^{(n-1) \times (n-1)}$

$n = 15$  sec

$n = 20$  giorni

$n = 25$  anni

Strassen  $n^{3.8}$



$I_0$  note

$I_1, I_2, \dots, I_N$  ?

$$V = R_k I_k$$

$N$  incognite

$$\vec{I} = [I_1, I_2, \dots, I_N]^T$$

KIRCHHOFF

$$I_0 = I_1 + I_2 + \dots + I_N \quad 1 \text{ equation}$$

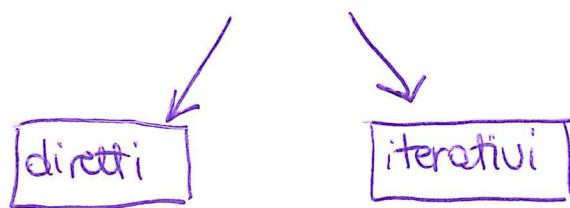
$$R_{k-1} I_{k-1} = R_k I_k \quad k=2, \dots, N \quad (N-1) \text{ equations}$$

$A \vec{I} = \vec{f}$

$N$  equations

$$\begin{bmatrix}
 1 & 1 & \dots & \dots & 1 & 1 \\
 R_1 & -R_2 & 0 & & 0 & \\
 0 & R_2 & -R_3 & 0 & \dots & 0 \\
 \vdots & & & & & \\
 0 & & & & & 0 + R_{N-1} - R_N
 \end{bmatrix} \cdot \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ \vdots \\ I_{N-1} \\ I_N \end{bmatrix} = \begin{bmatrix} I_0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

$\underbrace{A}_{\text{A}}$        $\underbrace{\vec{I}}_{\text{I}}$        $\underbrace{\vec{f}}_{\text{f}}$



## Fattorizzazione LU

$A \in \mathbb{R}^{n \times n}$  non singolare ( $Ax = b$ )

$$A = LU$$

$$L, U \in \mathbb{R}^{n \times n}$$

$$\begin{aligned} l_{ii} &\neq 0 & i = 1, \dots, n \\ u_{ii} &\neq 0 \end{aligned}$$

$$0 \neq \det(A) = \det(LU) = \underbrace{\det(L)}_{\ell_{11}\ell_{22}\dots\ell_{nn}} \underbrace{\det(U)}_{u_{11}u_{22}\dots u_{nn}}$$

$$u_{11}u_{22}\dots u_{nn}$$

$$Ax = b$$

$$A = LU$$

$$\underbrace{LUx = b}_{y} \iff \begin{cases} Ly = b & \textcircled{1} \\ Ux = y & \textcircled{2} \end{cases}$$

"sparsa"

$$a = Cn$$

$$O(n) = a$$

$$A \in \mathbb{R}^{n \times n}$$

$$O(100)$$

$$n^2$$

$$O(n^2)$$

$$O(n) \text{ sparse}$$

$$n = 10^4; 10^2$$

(sparse)

$$Ly = b \quad n=3$$

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$l_{11}y_1 = b_1 \rightarrow y_1 = \frac{b_1}{l_{11}} \neq 0$$

$$l_{21}y_1 + l_{22}y_2 = b_2 \rightarrow y_2 = \frac{1}{l_{22}} \left[ b_2 - l_{21}y_1 \right] \neq 0$$

array

$$l_{31}y_1 + l_{32}y_2 + l_{33}y_3 = b_3 \rightarrow y_3 = \frac{1}{l_{33}} \left[ b_3 - l_{31}y_1 - l_{32}y_2 \right] \neq 0$$

$$L \in \mathbb{R}^{n \times n}$$

$y_1 = \frac{b_1}{l_{11}}$ $y_i = \frac{1}{l_{ii}} \left[ b_i - \sum_{j=1}^{i-1} e_{ij} y_j \right]$ $\neq 0$	$i = 2, 3, \dots, n$ <span style="background-color: yellow;">algoritmo delle sostituzioni in avanti</span>
---------------------------------------------------------------------------------------------------------------------	-----------------------------------------------------------------------------------------------------------------------

$\forall i$  1 divisione

$$\underbrace{\sum_{i=1}^n 1}_n$$

$(i-1)$  sottrazioni

$$+ 2 \sum_{i=1}^n (i-1)$$

$(i-1)$  moltiplicazioni

$$n + 2 \sum_{i=1}^n i - 2 \underbrace{\sum_{i=1}^n 1}_n = -n + 2 \sum_{i=1}^n i$$

$$\underbrace{\sum_{i=1}^n i}_{\frac{n(n+1)}{2}}$$

$$= n^2$$

$$Ux = y$$

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$u_{33}x_3 = y_3 \rightarrow x_3 = \frac{y_3}{u_{33}} \neq 0$$

$$u_{22}x_2 + u_{23}x_3 = y_2 \rightarrow x_2 = \frac{1}{u_{22}} \left[ y_2 - u_{23}x_3 \right]$$

~~x~~  
0

$$u_{11}x_1 + u_{12}x_2 + u_{13}x_3 = y_1 \rightarrow x_1 = \frac{1}{u_{11}} \left[ y_1 - u_{12}x_2 - u_{13}x_3 \right]$$

~~y~~  
0

$$U \in \mathbb{R}^{n \times n}$$

$$x_n = \frac{y_n}{u_{nn}}$$

$$x_i = \frac{1}{u_{ii}} \left[ y_i - \sum_{j=i+1}^n u_{ij}x_j \right]$$

$$i = n-1, \dots, 1$$

algoritmo

delle sostituzioni

all'indietro

$n^2$

pattern di sparsità  
(spy)

$$\begin{bmatrix} * & * & * & \dots & * \\ * & * & \dots & * \\ * & & \ddots & & * \end{bmatrix}$$

$$Ax = b$$

$$A = LU$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} e_{11} & 0 \\ e_{21} & e_{22} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix} = \begin{bmatrix} x & x \\ x & x \end{bmatrix}$$

$$\begin{cases} a_{11} = \boxed{e_{11}} u_{11} \\ a_{12} = \boxed{e_{11}} u_{12} \end{cases} \text{ R1(U)} \quad \begin{array}{l} 4 \text{ eq.} \\ 2 \text{ incognite} \end{array}$$
$$a_{21} = e_{21} u_{11} \quad \text{C1(L)}$$
$$a_{22} = e_{21} u_{12} + \boxed{e_{22}} u_{22} \quad \text{R2(U)}$$

$$e_{11} = e_{22} = 1$$

$$A = \underbrace{\begin{bmatrix} 1 & & & \\ & \ddots & & 0 \\ & & \ddots & \\ & & & 1 \end{bmatrix}}_L \underbrace{\begin{bmatrix} x & \cdots & x \\ \vdots & \ddots & \vdots \\ 0 & \cdots & x \end{bmatrix}}_U$$

$$n \begin{bmatrix} A \\ \vdots \\ A \end{bmatrix} = \left[ \begin{smallmatrix} \Delta & 0 \\ \vdots & \ddots \\ 0 & \Delta \end{smallmatrix} \right] \begin{bmatrix} \Delta \\ \vdots \\ \Delta \end{bmatrix}$$

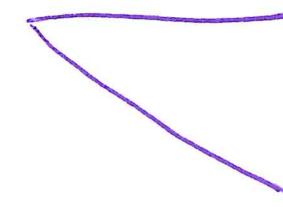
$\frac{n(n+1)}{2} + \frac{(n+1)n}{2}$  incognite  $= n^2 + n$  incognite

$\downarrow$

$$e_{ii} = 1 \quad i=1, \dots, n$$

# Eliminazione Gaussiana ( $Ax = b$ )

$$A = A^{(1)} = \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & b_1^{(1)} \\ a_{21}^{(1)} & a_{22}^{(1)} & a_{23}^{(1)} & b_2^{(1)} \\ a_{31}^{(1)} & a_{32}^{(1)} & a_{33}^{(1)} & b_3^{(1)} \end{bmatrix}$$



$x, y$

$a, b \in \mathbb{R}$

$ax + by$

STEP 1

$$\ell_{21} = \frac{a_{21}^{(1)}}{a_{11}^{(1)}} \quad \ell_{31} = \frac{a_{31}^{(1)}}{a_{11}^{(1)}}$$

$$A^{(1)} \rightarrow A^{(2)} = \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & b_1^{(1)} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & b_2^{(2)} \\ 0 & a_{32}^{(2)} & a_{33}^{(2)} & b_3^{(2)} \end{bmatrix}$$

$$A^{(2)} = \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & b_1^{(1)} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & b_2^{(2)} \\ 0 & 0 & a_{33}^{(3)} & b_3^{(3)} \end{bmatrix} \quad \boxed{b_3^{(3)} = b_3^{(2)} - \ell_{32}^{(2)} b_2^{(2)}}$$

$$R2_{\text{new}} = R2_{\text{old}} - \ell_{21} R1_{\text{old}}$$

$$a_{21}^{(1)} - \ell_{21} a_{11}^{(1)} = a_{21}^{(1)} - \frac{a_{21}^{(1)}}{a_{11}^{(1)}} a_{11}^{(1)} = 0$$

$$a_{22}^{(2)} = a_{22}^{(1)} - \ell_{21} a_{12}^{(1)}$$

$$a_{23}^{(2)} = a_{23}^{(1)} - \ell_{21} a_{13}^{(1)}$$

$$R3_{\text{new}} = R3_{\text{old}} - \ell_{31} R1_{\text{old}}$$

$$a_{31}^{(2)} = a_{31}^{(1)} - \ell_{31} a_{11}^{(1)} = a_{31}^{(1)} - \frac{a_{31}^{(1)}}{a_{11}^{(1)}} a_{11}^{(1)} = 0$$

$$a_{32}^{(2)} = a_{32}^{(1)} - \ell_{31} a_{12}^{(1)}$$

$$a_{33}^{(2)} = a_{33}^{(1)} - \ell_{31} a_{13}^{(1)}$$

$$\ell_{32} = \frac{a_{32}^{(2)}}{a_{22}^{(2)}}$$

X  
O

$$R3_{\text{new}} = R3_{\text{old}} - \ell_{32} R_{2\text{old}}$$

$$a_{32}^{(3)} = a_{32}^{(2)} - \ell_{32} a_{22}^{(2)}$$

$$a_{22}^{(2)} = \cancel{a_{22}^{(2)}} - \frac{\cancel{a_{32}^{(2)}}}{\cancel{a_{22}^{(2)}}} \cancel{a_{22}^{(2)}} = 0$$

$$a_{33}^{(3)} = a_{33}^{(2)} - \ell_{32} a_{23}^{(2)}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix}$$

$$LA^{(n)} = A^{(n)}$$

Esempio:

$$A = A^{(1)} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & -1 \\ -1 & 1 & 5 \end{bmatrix} \longrightarrow A^{(2)} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & -3 \\ 0 & 3 & 6 \end{bmatrix} \longrightarrow A^{(3)} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 4 & -3 \\ 0 & 0 & 15/4 \end{bmatrix} = U$$

$$\ell_{21} = 2$$

$$\ell_{32} = -\frac{3}{4}$$

$$\ell_{31} = -1$$

$$2 - 2 \cdot 1 = 0$$

$$0 - 2 \cdot 2 = -4$$

$$-1 - 2 \cdot 1 = -3$$

$$-1 + (+1) \cdot 1 = 0$$

$$1 + (+2) \cdot 2 = 3$$

$$5 + (+1) \cdot 1 = 6$$

$$3 + \frac{3}{4} \cdot (-4) = 0$$

$$6 + \frac{3}{4} \cdot (-3) = 6 - \frac{9}{4} = \frac{15}{4}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -\frac{3}{4} & 1 \end{bmatrix} \text{ t.c. } \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -\frac{3}{4} & 1 \end{bmatrix}}_L \underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & -3 \\ 0 & 0 & \frac{15}{4} \end{bmatrix}}_U = \underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & -1 \\ -1 & 1 & 5 \end{bmatrix}}_A$$

$$-1 \cdot 2 + \frac{3}{4} \cdot 4 = 1$$

$$-1 \cdot 1 + \frac{3}{4} \cdot 3 + \frac{15}{4} = -1 + \frac{9}{4} + \frac{15}{4} = \frac{20}{4} = 5$$

$\text{eu } [L, U] = \text{eu}(A)$

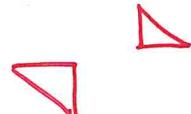
$\text{inv}(A)$

$$\backslash \quad x = A \backslash b \quad \text{sparse}$$

$$\begin{matrix} 1 & (1,1) \\ 1 & (2,2) \\ \vdots & \\ 1 & (10,10) \end{matrix}$$

$$\det(A) = u_{11} \cdots u_{nn}$$

trie  
triu



$$\begin{bmatrix} L & U \end{bmatrix}$$

$$A = LU$$

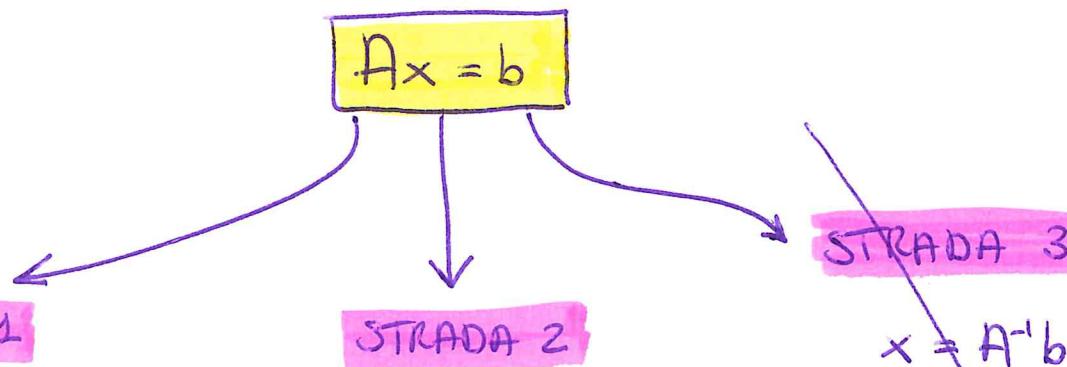
$$\det(A) = \det(LU)$$

$$= \det(L) \det(U)$$

$$= \underbrace{(e_1 \cdots e_n)}_1 (u_1 \cdots u_n)$$

$$\frac{2}{3} n^3 \quad O(n^3)$$

MEG = Metodo di Eliminazione Gaussiana



\* REG

\*  $L\varphi = b$

\*  $Ux = \varphi$

$$\boxed{\frac{2}{3} n^3 + 2n^2}$$

+ MEG

$$\left[ \begin{array}{|c|c} A & |b \end{array} \right]$$

$$\left[ \begin{array}{|c|c} U & |b^* \end{array} \right]$$

$$b^* = [b_1^{(1)}, b_2^{(2)}, b_3^{(3)}]^T$$

\*  $Ux = b^*$

$$\boxed{\left( > \frac{2}{3} n^3 \right) + n^2}$$

$$x = A^{-1}b$$

$$* A^{-1} \frac{8}{3} n^3$$

$$* x = A^{-1}b \quad n^2$$

$$\boxed{\frac{8}{3} n^3 + n^2}$$

$$n = 10$$

$$n^2 = 10^2$$

$$n^3 = 10^3$$

$$Ax = \tilde{b}_1$$

$$Ax = \tilde{b}_2$$

⋮

$$Ax = \tilde{b}_q$$

**I<sup>a</sup> strada**

\* LU di A (1<sup>a</sup> volta)

$$Ax = \tilde{b}_1 \quad \begin{cases} Ly = \tilde{b}_1 \\ Ux = y \end{cases}$$

$$Ax = \tilde{b}_2 \quad \begin{cases} Ly = \tilde{b}_2 \\ Ux = y \end{cases}$$

⋮

$$Ax = \tilde{b}_q \quad \begin{cases} Ly = \tilde{b}_q \\ Ux = y \end{cases}$$

$$\boxed{\frac{2}{3}n^3 + q(2n^2)}$$

$$\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_q \in \mathbb{R}^n$$

$$\Rightarrow [A] \left[ \begin{array}{c|c|c} c_1 & c_2 & c_n \end{array} \right] = \left[ \begin{array}{c|c|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \vdots & \vdots & \vdots \end{array} \right]$$

$$Ac_1 = [1, 0, \dots, 0]^T$$

$$Ac_2 = [0, 1, 0, \dots, 0]^T$$

$$Ac_n = [0, 0, \dots, 0, 1]^T$$

$$\begin{aligned} \frac{2}{3}n^3 + q(n^2) &= \frac{2}{3}n^3 + 2n^3 \\ &= \frac{8}{3}n^3 \end{aligned}$$

~~II<sup>a</sup> strada~~

$$[A | \tilde{b}_1] + Ux = \tilde{b}_1^*$$

$$[A | \tilde{b}_2] + Ux = \tilde{b}_2^*$$

$$\vdots$$

$$[A | \tilde{b}_q] + Ux = \tilde{b}_q^*$$

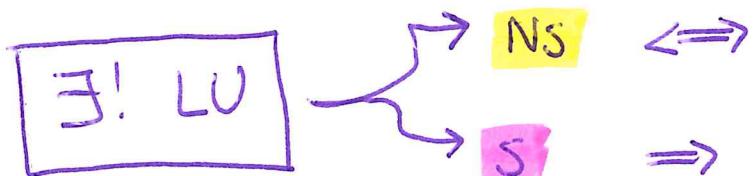
$$q \left( \frac{2}{3}n^3 + n^2 \right)$$

$$A^{(1)} = \left[ \begin{array}{cc|c} 1 & 1 & 3 \\ 2 & 2 & 2 \\ \hline 3 & 6 & 4 \end{array} \right] \longrightarrow A^{(2)} = \left[ \begin{array}{ccc} 1 & 1 & 3 \\ 0 & 0 & -4 \\ 0 & 3 & -5 \end{array} \right] \longrightarrow A^{(3)} = U = \left[ \begin{array}{ccc} 1 & 1 & 3 \\ 0 & 0 & -4 \end{array} \right]$$

$$\begin{array}{l|l|l} l_{21} = 2 & 2 - 2 \cdot 1 & 3 - 3 \cdot 1 \\ l_{31} = 3 & 2 - 2 \cdot 3 & 6 - 3 \cdot 1 \\ & & 4 - 3 \cdot 3 \end{array}$$

$$\begin{array}{l} l_{32} = \frac{3}{0} \neq 1 \\ a_{22}^{(2)} = 0 \end{array}$$

$a_{kk}^{(k)} \neq 0$  pivot



NS Sia  $A \in \mathbb{R}^{n \times n}$ . Allora  $\exists!$  LU di  $A \Leftrightarrow$  le sottomatrici principali di  $A$  denotate con  $A_i$  con  $i = 1, \dots, n-1$  sono tutte singolari.

$$A = \left[ \begin{array}{c|c|c|c} A_1 & & & \\ \hline & A_2 & & \\ \hline & & A_3 & \\ \hline & & & A_{n-1} \end{array} \right]$$

$$A_1 = \left[ \begin{array}{cc} 1 & 2 \\ 1 & 2 \end{array} \right] \rightarrow \left[ \begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array} \right] \left[ \begin{array}{cc} 1 & 2 \\ 0 & 0 \end{array} \right] \text{ singolari!}$$

$$A_2 = \left[ \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right] \not\propto \text{LU}$$

$$A_3 = \left[ \begin{array}{cc} 0 & 1 \\ 0 & 2 \end{array} \right] = \underbrace{\left[ \begin{array}{cc} 1 & 0 \\ \beta & 1 \end{array} \right]}_{L} \underbrace{\left[ \begin{array}{cc} 0 & 1 \\ 0 & 2-\beta \end{array} \right]}_{U} \not\propto \text{LU} \quad \beta \in \mathbb{R}$$

S

1) matrice e dominante diagonale stretta per righe

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^n |a_{ij}| \quad i=1, \dots, n$$

$$\begin{bmatrix} 4 & 0 & -1 \\ 3 & -7 & 2 \\ -2 & 1 & 9 \end{bmatrix} = A$$

2) matrice e dominante diagonale stretta per colonne

$$|a_{jj}| > \sum_{\substack{i=1 \\ i \neq j}}^n |a_{ij}| \quad j=1, \dots, n$$

3) matrice simmetriche e definite positive (sdp)



$$A = A^T$$

$$a_{ij} = a_{ji}$$

$$\begin{bmatrix} * & * & \square \\ * & \square & 0 \\ \square & 0 & 0 \end{bmatrix}$$

$$(A == A^T) \quad (\text{simmetrico})$$

$$x^T A x > 0 \quad \forall x \in \mathbb{R}^n \text{ con } x \neq 0 \quad (\text{definito positivo})$$

$$(1 \times n)(n \times n)(n \times 1)$$

CHECK PRATICO

1) simmetrica

(↳  $\text{eig}(A)^T \in \mathbb{R}^n$ )

2) d.p.

 $\text{eig}(A) > 0$  $(Av = \lambda v)$  $(\lambda, v)$  $\mathbb{R}, \mathbb{R}^n$ tutti autovettori  $> 0$

$$Ax = b$$

LU MEG

1)  $\exists!$  LU di A anche se A è singolare (NS e S)

2) Ax = b A non singolare

3)  $\exists!$  LU di A anche quando NS e S non sono verificate  
se A non singolare

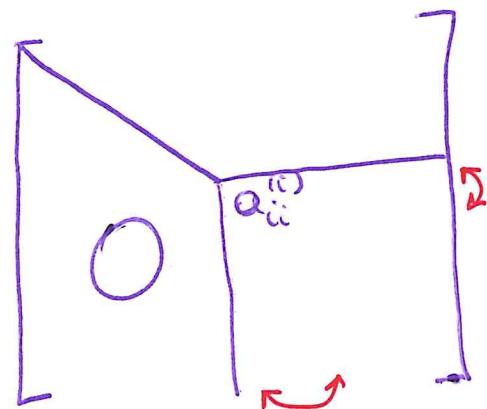
$$A = \tilde{A}^{(1)} = \begin{bmatrix} 1 & 1 & 3 \\ 3 & 6 & 4 \\ 2 & 2 & 2 \end{bmatrix} \rightarrow \tilde{A}^{(2)} = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 3 & -5 \\ 0 & 0 & -4 \end{bmatrix} = U$$

$e_{21} = 3$

$e_{31} = 2$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$e_{32} = 0$



$$\overset{(i)}{a_{ii}} = 0$$

pivoting  
(per righe, per colonne)

- i) quando permutare
- ii) quali righe scambiate ?

$$\overset{\infty}{a_{i+1,i}^{(1)}} \neq 0$$

$$i \leftrightarrow i+1$$

$P$  = matrice di permutazione

$$PA^{(1)} = \tilde{A}^{(1)}$$

ortogonale  $[PP^T = P^TP = I]$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$PA^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 2 & 2 & 2 \\ 3 & 6 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 3 \\ 3 & 6 & 4 \\ 2 & 2 & 2 \end{bmatrix} = \tilde{A}^{(1)}$$

$$A^{(1)}P = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 2 & 2 \\ 3 & 6 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 6 \end{bmatrix}$$

$$I = P$$

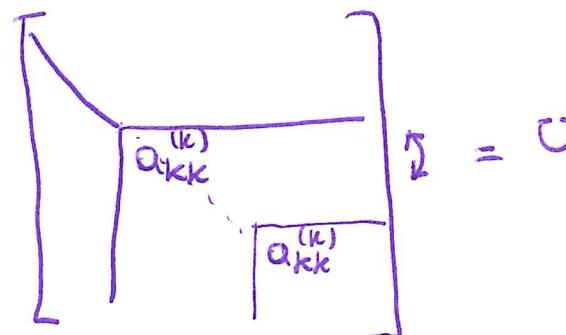
inizio

$$n = 100$$

$$a_{kk}^{(k)} = 0 \quad k = 10, 80$$

$$10 \leftarrow n$$

$$80 \leftarrow 81$$



$$P \equiv I$$

$$P \not\equiv I$$

no pivoting  
pivoting

$$Ax = b$$

$$PAx = Pb$$

$$PA = LU$$

$$\begin{array}{l} L \underbrace{Ux}_{y} = Pb \\ \quad \quad \quad \left\{ \begin{array}{l} Ly = Pb \\ Ux = y \end{array} \right. \end{array}$$

$$a_{kk}^{(k)} = 10^{-15}$$

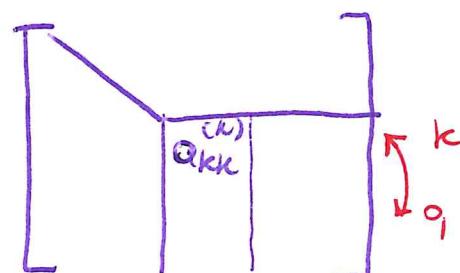
$$e_{k+1,k} = a_{k+1,k}^{(k)} / a_{kk}^{(k)}$$

$$e_{k+2,k} = a_{k+2,k}^{(k)} / a_{kk}^{(k)}$$

:

$$e_{n,k} = a_{n,k}^{(k)} / a_{kk}^{(k)}$$

valori  
metà grandi



$$\max_{i=k+1, \dots, n} |a_{ik}^{(k)}| \quad q$$

$$A = \begin{bmatrix} 1 & 1 + 0.5 \cdot 10^{-15} & 3 \\ 2 & 2 & 20 \\ 3 & 6 & 4 \end{bmatrix}$$

$$A - LU = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$[\tilde{L}, \tilde{U}] = \text{eu}(A);$$

$$[L, U, P] = \text{lu}(A);$$

$$A = \tilde{L} \tilde{U}$$

$$PA = LU$$

$$A = \underbrace{P^{-1}}_{\sim} \underbrace{L}_{\sim} \tilde{U}$$

(sp4)

$$Ax = b$$

$$PAx = Pb$$

$$PA = LU$$

$$P^{-1} L U x = P^{-1} Pb$$

$$\underbrace{P^{-1} L}_{y} \underbrace{Ux}_{\sim} = b$$

$$\begin{cases} P^{-1} Ly = b \\ Ux = y \end{cases}$$

① A sdp

$$A = R^T R$$

$$\begin{array}{c} \diagup \quad \diagdown \\ L \quad U \end{array}$$

Cholesky

$$\frac{n^3}{3}$$

$$r_{ii} \geq 0$$

②

$$A = \begin{bmatrix} & & 0 \\ 0 & & +1 \\ & -1 & \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & & 0 \\ 0 & \ddots & 0 \\ 0 & \cdots & 1 \end{bmatrix}}_L \underbrace{\begin{bmatrix} & & 0 \\ 0 & & 0 \\ 0 & & 0 \end{bmatrix}}_U$$

$$A = \begin{bmatrix} a_1 & c_1 & 0 \\ e_2 & a_2 & c_2 \\ 0 & e_3 & a_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ \beta_2 & 1 & 0 \\ 0 & \beta_3 & 1 \end{bmatrix}}_L \underbrace{\begin{bmatrix} a_1 & 0 & c_1 \\ 0 & a_2 & c_2 \\ 0 & 0 & a_3 \end{bmatrix}}_U$$

$$\begin{array}{c} \cancel{\beta_2}, \cancel{\beta_3} \\ \cancel{a_2}, \cancel{a_3}, \cancel{a_3} \\ \cancel{c_1}, \cancel{c_2} \\ \cancel{c_2} \end{array}$$

$$\begin{bmatrix} R_1 * c_2 \\ R_2 * c_3 \end{bmatrix}$$

$\alpha_1$  $R1 * c_1$ 

$$\alpha_1 = \alpha_1$$

 $\beta_2$  $R2 * c_1$ 

$$\beta_2 \alpha_1 = e_2 \rightarrow \beta_2 = \frac{e_2}{\alpha_1}$$

 $\alpha_2$  $R2 * c_2$ 

$$\beta_2 \alpha_2 + \alpha_2 = \alpha_2 \rightarrow \alpha_2 = \alpha_2 - c_2 \beta_2$$

 $\beta_3$  $R3 * c_2$ 

$$\beta_3 \alpha_2 = e_3 \rightarrow \beta_3 = \frac{e_3}{\alpha_2}$$

 $\alpha_3$  $R3 * c_3$ 

$$\beta_3 \alpha_3 + \alpha_3 = \alpha_3 \rightarrow \alpha_3 = \alpha_3 - c_3 \beta_3$$

$$\begin{array}{l}
 \alpha_1 = \alpha_1 \quad \beta_i = \frac{e_i}{\alpha_{i-1}} \quad i = 2, \dots, n \\
 \alpha_i = \alpha_i - c_{i-1} \beta_i
 \end{array}$$

$3(n-2)$

$$\begin{array}{ll}
 \beta_i & 1 \\
 \alpha_i & 2
 \end{array}$$

$t_i$   
3 operazioni

$$Ax = b \quad LUx = b \quad \left\{ \begin{array}{l} Ly = b \\ Ux = y \end{array} \right. \quad \begin{array}{l} L \\ U \end{array} \text{ bidiagonali}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ \beta_2 & 1 & 0 \\ 0 & \beta_3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$y_1 = b_1$$

$$\beta_2 y_1 + y_2 = b_2 \rightarrow y_2 = b_2 - \beta_2 y_1$$

$$\beta_3 y_2 + y_3 = b_3 \rightarrow y_3 = b_3 - \beta_3 y_2$$

$$\boxed{\begin{aligned} y_1 &= b_1 \\ y_i &= b_i - \beta_i y_{i-1} \\ i &= 2, \dots, n \end{aligned}}$$

$x_i$  2

$2(n-1)$

$$\begin{bmatrix} \alpha_1 & c_1 & 0 \\ 0 & \alpha_2 & c_2 \\ 0 & 0 & \alpha_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\alpha_3 x_3 = y_3 \rightarrow x_3 = \underline{y_3}$$

$$\alpha_2 x_2 + c_2 x_3 = y_2 \quad \xrightarrow{\alpha_3} x_2 = \frac{1}{\alpha_2} [y_2 - c_2 x_3]$$

$$\alpha_1 x_1 + c_1 x_2 = y_1 \rightarrow x_1 = \frac{1}{\alpha_1} [y_1 - c_1 x_2]$$

$$\boxed{\begin{aligned} x_n &= \frac{y_n}{\alpha_n} \\ x_i &= [y_i - c_i x_{i+1}] \frac{1}{\alpha_i} \\ i &= n-1, \dots, 1 \end{aligned}}$$

1  $x_n$   
3  $x_i$

$3(n-1)+1$

$$\underbrace{3(n-1)}_{LU} + \underbrace{2(n-1)}_{Ly=b} + \underbrace{3(n-1)+1}_{Ux=y} = 8(n-1)+1 = \boxed{8n-7}$$

Thomas

MEG + pivoting  $\Rightarrow$  LU accurate  
 $LU - PA = 0 \quad ?$   $\Rightarrow Ax = b \quad x \text{ accurate}$

$$A_n x_n = b_n$$

Hiebert

$n \geq 1$

$$b_n : x_n = [1, \dots, 1]^T$$

$\tilde{x}_n$

$$R_n = L_n U_n - P_n A_n \in \mathbb{R}^{n \times n}$$

$$\boxed{E_n = \frac{\|x_n - \tilde{x}_n\|}{\|x_n\|}} \quad (B)$$

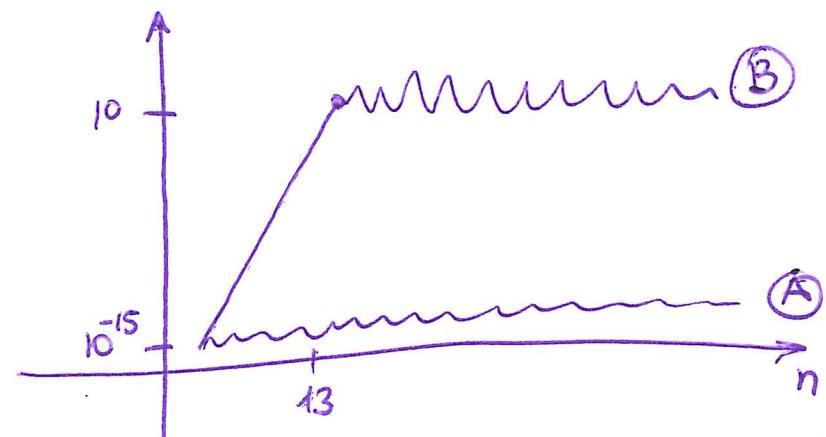
$$n \geq 13 \quad E_n \geq 10$$

$$\boxed{1000 \%}$$

$$a_{ij} = \frac{1}{i+j-1}$$

$$\begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 & \dots \\ 1/2 & 1/3 & 1/4 & \dots \\ 1/3 & 1/4 & \dots \end{bmatrix}$$

5dp



$$Ax = b$$

perturbazione  
sui dati

$$(A + \delta A)(x + \delta x) = b + \delta b$$

$\tilde{x}$

perturbazione  
sui dati

$$A, \delta A \in \mathbb{R}^{n \times n} \quad b, \delta b \in \mathbb{R}^n \quad (\text{noti})$$

$$x, \delta x \in \mathbb{R}^n \quad (\text{incognite})$$

$$\frac{\|\delta A\|}{\|A\|} \quad \begin{array}{|c|c|} \hline & \|\delta b\| \\ \hline & \|b\| \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline & \|\delta x\| \\ \hline & \|x\| \\ \hline \end{array}$$

$k(A)$  numero di condizionamento

$$\delta A = 0$$

$$\frac{\|\delta x\|}{\|x\|} \leq \underbrace{\sqrt{\frac{\lambda_{\max}(A^T A)}{\lambda_{\min}(A^T A)}}}_{k_2(A)} \|\delta b\|$$

$$\begin{matrix} A^T A \\ \lambda_{\max}, \lambda_{\min} \end{matrix}$$

$$k(A) = 1 \quad \text{ideale}$$

$$k(A) = 100$$

$$\frac{\|\delta b\|}{\|b\|} \approx 10^{-5}$$

$$\frac{\|\delta x\|}{\|x\|} \leq 10^{-3}$$

$$k(A) \geq 1$$

$$k(A) = \|A\| \|A^{-1}\|$$

$$k_2(A) = \|A\|_2 \|A^{-1}\|_2$$

$$k_\infty(A) = \|A\|_\infty \|A^{-1}\|_\infty$$

$$k_2(A) = \|A\|_2 \|A^{-1}\|_2$$

$$k_2(A) = \sqrt{\frac{\lambda_{\max}(A^T A)}{\lambda_{\min}(A^T A)}} = \|A\|_2 \|A^{-1}\|_2$$

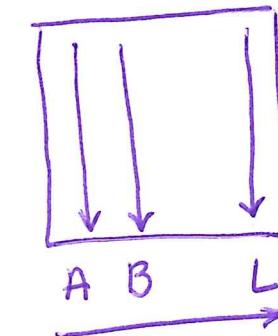
$$\|A^{-1}\|_2 = \frac{1}{\sqrt{\lambda_{\min}(A^T A)}}$$

$$\text{cond}(A) = \|A\|_2$$

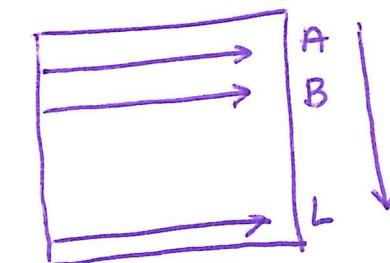
$$\text{condest}(A) = \|A\|_1$$

sparsa,

$$\|A\|_2 = \max_{\text{colonne}} \text{somme} = \max_j \sum_{i=1}^n |a_{ij}|$$



$$\|A\|_\infty = \max_{\text{ligne}} \text{somme} = \max_i \sum_{j=1}^n |a_{ij}|$$



$$\|A\|_2 = \text{norme spectrale} = \sqrt{\lambda_{\max}(A^T A)}$$

se  $A$  è simmetrica

$$\lambda_{\max}(A^T A) = [\lambda_{\max}(A)]^2$$

$$\|A\|_2 = \lambda_{\max}(A)$$

$$\|A^{-1}\|_2 = [\lambda_{\min}(A)]^{-1}$$

$$k_2(A) = \frac{\lambda_{\max}(A)}{\lambda_{\min}(A)}$$

$\delta A = 0$

$$\frac{\|\delta x\|}{\|x\|} \leq k(A) \frac{\|\delta b\|}{\|b\|}$$

\*\*

Hilbert

$k(A_n)$



$k(A_+) > 15.000$

$Ax = b$  ben condizionato  $k(A)$  è piccolo

ma e "  $k(A)$  è grande

$k(A) \geq 1$

$\delta A \neq 0$

Se  $\|\delta A\| \|A^{-1}\| < 1$  allora

$$\frac{\|\delta x\|}{\|x\|} \leq \frac{k(A)}{1 - k(A) \frac{\|\delta A\|}{\|A\|}} \left[ \frac{\|\delta A\|}{\|A\|} + \frac{\|\delta b\|}{\|b\|} \right]$$

Os: • Se  $\|\delta A\| = 0$  trovo \*\*

•  $\|\delta A\| \|A^{-1}\| < 1$

$$\frac{\|\delta A\|}{\|A\|} < \underbrace{\frac{1}{\|A^{-1}\| \|A\|}}_{k(A)}$$

$$\frac{\|\delta A\|}{\|A\|} < \frac{1}{k(A)}$$

$$1 - k(A) \frac{\|\delta A\|}{\|A\|} > 0$$

$$Ax = b$$

$\lim_{k \rightarrow +\infty} x^{(k)} = x$  convergenza

$$\begin{bmatrix} e^{(k)} = x - x^{(k)} \\ \lim_{k \rightarrow +\infty} e^{(k)} = 0 \end{bmatrix}$$

$\cap \mathbb{R}^n$

BB

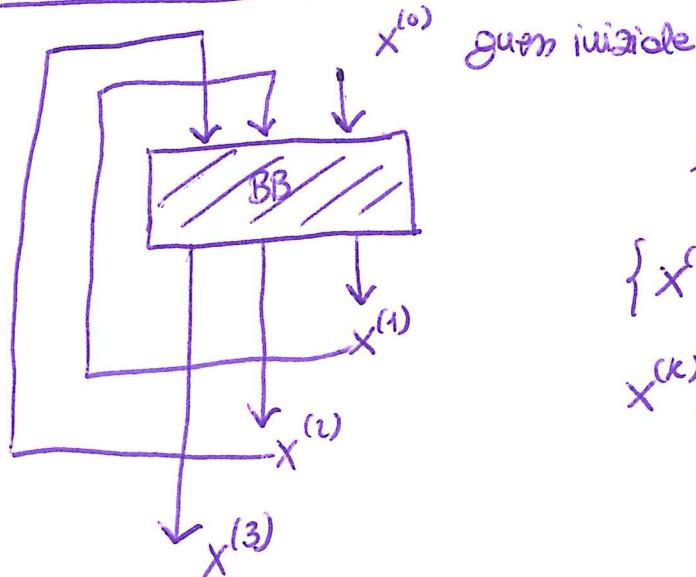
$$x^{(k+1)} = Bx^{(k)} + g$$

$k \geq 0$

$$\begin{array}{l} B \in \mathbb{R}^{n \times n} \\ g \in \mathbb{R}^n \end{array}$$

( $\forall B$   $\forall g$  1 metodo)

### Metodi iterativi



$$x^{(k)} \in \mathbb{R}^n$$

$$\{x^{(k)}\}$$

$$x^{(k)} \approx x$$

### STOPPING CRITERIA

$$\begin{array}{l} 1) k^* \in \mathbb{N} \text{ t.c.} \\ \|x - x^{(k^*)}\| \leq \text{TOL} \\ (10^{-3}) \end{array}$$

controllo accuratezza

2)  $N_{\max}$

$$\|x - x^{(k^*)}\| \leq [S \leq \text{TOL}]$$

↓  
computabile

### consistenza

$$x = Bx + g$$

$$(I - B)x = g$$

$$(I - B)A^{-1}b = g$$

① convergenza

② consistenza

③ stabilità

$$\underbrace{x - x^{(k+1)}}_{e^{(k+1)}} = \cancel{Bx} + \cancel{-Bx^{(k)} - g} = B(\underbrace{x - x^{(k)}}_{e^{(k)}})$$

$$\boxed{e^{(k+1)} = Be^{(k)} \quad \forall k \geq 0}$$

$$e^{(k)} = Be^{(k-1)}$$

$$\|e^{(k)}\| = \|Be^{(k-1)}\| \leq \|B\|_2 \|e^{(k-1)}\| \simeq \underbrace{\rho(B)}_{\text{raggio spettrale di } B} \|e^{(k-1)}\|$$

raggio spettrale di  $B$   $\max(\text{abs}(\text{eig}(B)))$   
se  $B \in \text{sdp}$   $\max(\text{eig}(B))$

$$\boxed{\|e^{(k)}\| \lesssim \rho(B) \|e^{(k-1)}\| \quad \forall k > 0}$$

$$\|e^{(k-1)}\| \lesssim \rho(B) \|e^{(k-2)}\|$$

$$\|e^{(k-2)}\| \lesssim \rho(B) \|e^{(k-3)}\|$$

$$\|e^{(k)}\| \lesssim \rho(B) \|e^{(k-1)}\| \lesssim [\rho(B)]^2 \|e^{(k-2)}\| \lesssim [\rho(B)]^3 \|e^{(k-3)}\| \dots \lesssim [\rho(B)]^k \|e^{(0)}\|$$

$$\|e^{(k)}\| \lesssim [\rho(B)]^k \|e^{(0)}\|$$

$$\lim_{k \rightarrow +\infty} e^{(k)} = 0$$

$$\boxed{\rho(B) < 1} \quad \forall x^{(0)} \in \mathbb{R}^n$$

Teorема: Se lo schema  $x^{(k+1)} = Bx^{(k)} + g$  è consistente allora lo schema è convergente  
 $\forall x^{(0)} \in \mathbb{R}^n \iff g(B) < 1$ . Più  $g(B)$  è piccolo, più la convergenza è veloce.

$B_1, g_1$
$g(B_1) = 0.23$

$B_2, g_2$
$g(B_2) = 0.87$

$B_3, g_3$
$g(B_3) = 1.0003$

$$\|e^{(k)}\| \leq [g(B)]^k \|e^{(0)}\|$$

?  $k_{miu}$  t.c.  $\frac{\|e^{(k_{miu})}\|}{\|e^{(0)}\|} \leq [g(B)]^{k_{miu}} \leq \epsilon = (10^{-s})$



schema iterativo di Richardson

$$Ax = b \quad (Ax - b = 0) \quad \alpha_k \in \mathbb{K}$$

$$\underline{\alpha_k A x = \alpha_k b}$$

$$\alpha_k A = P - (P - \alpha_k A)$$

$$Px - (P - \alpha_k A)x = \alpha_k b$$

$$Px = (P - \alpha_k A)x + \alpha_k b$$

$$\begin{matrix} \downarrow \\ x^{(k+1)} \end{matrix} \qquad \qquad \qquad \begin{matrix} \downarrow \\ x^{(k)} \end{matrix}$$

$$Px^{(k+1)} = (P - \alpha_k A)x^{(k)} + \alpha_k b$$

$$Px^{(k+1)} = \alpha_k \left( \underbrace{b - Ax^{(k)}}_{r^{(k)} \text{ (residuo)}} \right) + Px^{(k)}$$

$$\underline{z^{(k)} = P^{-1}r^{(k)}}$$

$$Pz^{(k)} = r^{(k)}$$

P "semplice"

$\nexists P \in \mathbb{R}^{n \times n}$  precondizionatore, invertibile ( $\exists P^{-1}$ )

$$x^{(k+1)} = Bx^{(k)} + g$$

RICHARDSON

$$x^{(k+1)} = \underbrace{(I - \alpha_k P^{-1}A)}_{Bx^{(k)}} \underbrace{x^{(k)}}_{g^{(k)}} + \underbrace{\alpha_k P^{-1}b}_{\tilde{g}^{(k)}}$$

$$x^{(k+1)} = x^{(k)} + \underbrace{\alpha_k P^{-1}r^{(k)}}_{\underline{z^{(k)}} \text{ (residuo precondizionato)}} \quad k \geq 0$$

$\underbrace{z^{(k)}}_{\text{correzione}}$

stazionario  
 $\alpha_k = \alpha \in \mathbb{R} \quad \forall k$

dinamico  
 $\alpha_k \neq k$

## Metodo di Jacobi

$$\left\{ \begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \rightarrow x_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \rightarrow x_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \rightarrow x_3 \end{array} \right.$$

(A)  $x_j^{(k+1)} = B_J x_j^{(k)} + g_j$

(B)  $x_j^{(k+1)} = x_j^{(k)} + \alpha_{k,j} P_J^{-1} r^{(k)}$

$$x^{(k+1)} = D^{-1}b - D^{-1}(A-D)x^{(k)}$$

(A)  $x^{(k+1)} = (I - D^{-1}A)x^{(k)} + D^{-1}b$

$$x^{(k+1)} = x^{(k)} + D^{-1}b - D^{-1}Ax^{(k)}$$

$$x^{(k+1)} = x^{(k)} + D^{-1}(b - Ax^{(k)})$$

(B)  $x^{(k+1)} = x^{(k)} + D^{-1}r^{(k)}$

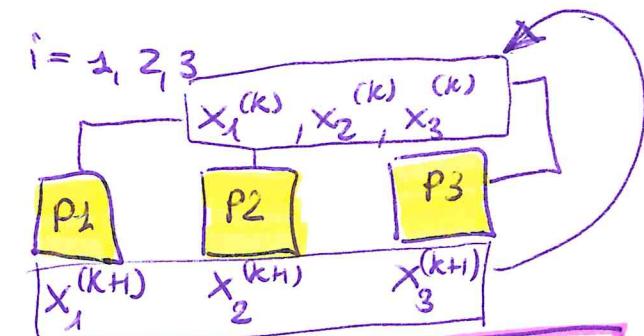
$\alpha_{k,j} = \alpha_j = 1$   
 $P_J = D$

$$x_1^{(k+1)} = \frac{1}{a_{11}} \left[ b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)} \right] \quad P_1$$

$$x_2^{(k+1)} = \frac{1}{a_{22}} \left[ b_2 - a_{21}x_1^{(k)} - a_{23}x_3^{(k)} \right] \quad P_2$$

$$x_3^{(k+1)} = \frac{1}{a_{33}} \left[ b_3 - a_{31}x_1^{(k)} - a_{32}x_2^{(k)} \right] \quad P_3$$

$a_{ii} \neq 0 \quad i=1, 2, 3$



$x_i^{(k+1)} = \frac{1}{a_{ii}} \left[ b_i - \sum_{\substack{j=1 \\ j \neq i}}^n a_{ij} x_j^{(k)} \right]$

$i=1, \dots, n$

$$D = \begin{bmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{bmatrix} \quad x^{(k)} = \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \\ \vdots \\ x_n^{(k)} \end{bmatrix}^T$$

$$x^{(k+1)} = \begin{bmatrix} x_1^{(k+1)} \\ x_2^{(k+1)} \\ \vdots \\ x_n^{(k+1)} \end{bmatrix}^T$$

$$a_{ii} x_i^{(k+1)} = [Dx^{(k+1)}]_i \quad b = [b_1, \dots, b_n]^T$$

$Dx^{(k+1)} = b - (A - D)x^{(k)}$

# Metodo di Gauss-Seidel

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left[ b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^n a_{ij} x_j^{(k)} \right] \quad i=1, \dots, n$$

$k \geq 0$   
 $a_{ii} \neq 0$

(A)  $x^{(k+1)} = B_{GS} x^{(k)} + g_{GS}$

(B)  $x^{(k+1)} = x^{(k)} + \alpha_{k, GS} P_{GS}^{-1} r^{(k)}$

$$x^{(k+1)} = (D-E)^{-1} b + (D-E)^{-1} ((D-E)A)x^{(k)}$$

(A)  $x^{(k+1)} = \underbrace{(I - (D-E)^{-1}A)}_{B_{GS}} x^{(k)} + \underbrace{(D-E)^{-1}b}_{g_{GS}}$

$$x^{(k+1)} = x^{(k)} + (D-E)^{-1} b - (D-E)^{-1} A x^{(k)}$$

$$x^{(k+1)} = x^{(k)} + \underbrace{(D-E)^{-1} (b - A x^{(k)})}_{r^{(k)}}$$

(B)  $x^{(k+1)} = x^{(k)} + (D-E)^{-1} r^{(k)}$

$$A = \begin{bmatrix} & & & & \\ & D & & & \\ & & A_{21} & \ddots & \\ & & & A_{32} & \ddots \\ & & & & \ddots & 0 \\ & & & & & A_{n1} & A_{n2} & \ddots & 0 \end{bmatrix}$$

$$-E = \begin{bmatrix} 0 & \cdots & 0 \\ 0 & \ddots & \\ 0 & & 1 \\ \vdots & & \vdots \\ 0 & & 0 \end{bmatrix} \in \mathbb{R}^{n \times n}$$

$$Dx^{(k+1)} = b + E x^{(k+1)} - (A-D+E)x^{(k)}$$

$$Dx^{(k+1)} = b + E x^{(k+1)} + (D-E-A)x^{(k)}$$

$$(D-E)x^{(k+1)} = b + (D-E-A)x^{(k)}$$

$$\alpha_{k, GS} = \alpha_{GS} = 1$$

$$P_{GS} = D - E$$

# CONVERGENZA JACOBI e GAUSS-SEIDEL

## JACOBI

CNS

consistenza ( $\times$  costruzione)

$$f(B_J) < 1 \quad \text{con} \quad B_J = I - D^{-1}A$$

CS

1)  $A$  dominante diagonale  
 $\times$  righe

2)  $A$  dominante "  $\times$  colonne

CNS

consistenza ( $\times$  costruzione)

$$f(B_G) < 1 \quad B_G = I - (D - E)^{-1}A$$

CS

1)  $A$  dominante diagonale stretta  
 $\times$  righe

2)  $A$  " " "  
 $\times$  colonne

3)  $A$  sdip

Proposizione: Se  $A \in \mathbb{R}^{n \times n}$  sia tridiagonale, non singolare, con  $a_{ii} \neq 0$  per  $i=1, \dots, n$ .

Allora J e GS sono o entrambi convergenti o entrambi divergenti.

Se entrambi convergenti allora

$$g(B_{GS}) = [g(B_J)]^2.$$

$\varepsilon$

$$g(B_J) = \frac{1}{4} \quad \left(\frac{1}{4}\right)^{k_{\text{miu}}} \leq \varepsilon \quad \frac{1}{\varepsilon} \leq 4^{k_{\text{miu}}} \quad \log_4 \frac{1}{\varepsilon} \leq k_{\text{miu}}$$

$$g(B_{GS}) = \frac{1}{4^2} \quad \left(\frac{1}{4^2}\right)^{k_{\text{miu}}} \leq \varepsilon \quad \frac{1}{\varepsilon} \leq 4^{2k_{\text{miu}}} \quad \log_4 \frac{1}{\varepsilon} \leq 2k_{\text{miu}}$$

$$\frac{1}{2} \left[ \log_4 \frac{1}{\varepsilon} \right] \leq k_{\text{miu}}$$

Ex

$$A = \begin{bmatrix} 3 & -1 & 0 & 0 \\ -2 & 5 & -1 & 0 \\ 0 & -1 & 3 & -2 \\ 0 & 0 & -2 & 3 \end{bmatrix}$$

$$Ax = b$$

b t.c.  $x = [x_1, \dots, x_4]^T$

$$g(B_J) < 1$$

$$g(B_{GS}) < 1$$

$$\begin{array}{ll} J & GS \\ x^{(0)} = 0 & \varepsilon = 10^{-12} \end{array}$$

$$\begin{array}{ll} J & 277 \text{ iterazioni} \\ GS & 143 \text{ iterazioni} \end{array}$$

Proposizione: Siano  $A$  e  $P$  sdp. Allora Richardson stazionano convergono  $\forall x^{(0)} \in \mathbb{R}^n$  se e solo se  $0 < \alpha < \frac{2}{\lambda_{\max}}$  dove  $\lambda_{\max} = \max_i \lambda_i$  con  $\lambda_i$  autovalore di  $P^{-1}A$ .

La scelta ottimale per  $\alpha$  è  $\alpha_{\text{opt}} = \frac{2}{\lambda_{\max} + \lambda_{\min}}$  con  $\lambda_{\min}$  autovalore minimo di  $P^{-1}A$ .

Inoltre

$$\frac{\|e^{(k)}\|_A}{\|x - x^{(k)}\|} \leq \underbrace{\left( \frac{k(P^{-1}A) - 2}{k(P^{-1}A) + 1} \right)^k}_{< 1} \|e^{(0)}\|_A \quad \text{dove } \|w\|_A = \sqrt{w^T A w}$$

$\forall w \in \mathbb{R}^n$  (norma in energia)

Dimostrazione:

$$g(B_\alpha) < 1 \quad B_\alpha = I - \alpha P^{-1}A$$

$\lambda_i$  autovalori di  $P^{-1}A$  ( $> 0$ )

$\mu_i$  " di  $B_\alpha$

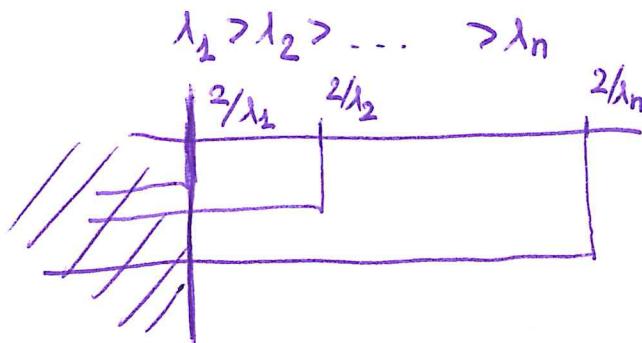
$$\mu_i = 1 - \alpha \lambda_i$$

$$|1 - \alpha \lambda_i| < 1$$

$$\underbrace{-1 < 1 - \alpha \lambda_i < 1}_{-\alpha \lambda_i < 0} \rightarrow \boxed{\alpha > 0}$$

$$\alpha \lambda_i < 2$$

$$\alpha < \frac{2}{\lambda_i} \quad \forall i \rightarrow \boxed{\alpha < \frac{2}{\lambda_{\max}}}$$



$\alpha$  ottimo t.c.  $\begin{cases} \text{velocità di convergenza è max} \\ g(B_\alpha) \text{ è minimo} \end{cases}$

$$B_\alpha = I - \alpha P^{-1}A$$

$$1 - \alpha \lambda_i \quad \lambda_i \quad P^{-1}A$$

$$\lambda_1 > \lambda_2 > \lambda_3$$

$$\frac{1}{\lambda_1} < \frac{1}{\lambda_2} < \frac{1}{\lambda_3}$$

$$P^{-1}A$$

$$|1 - \alpha \lambda_1|$$

$$|1 - \alpha \lambda_2|$$

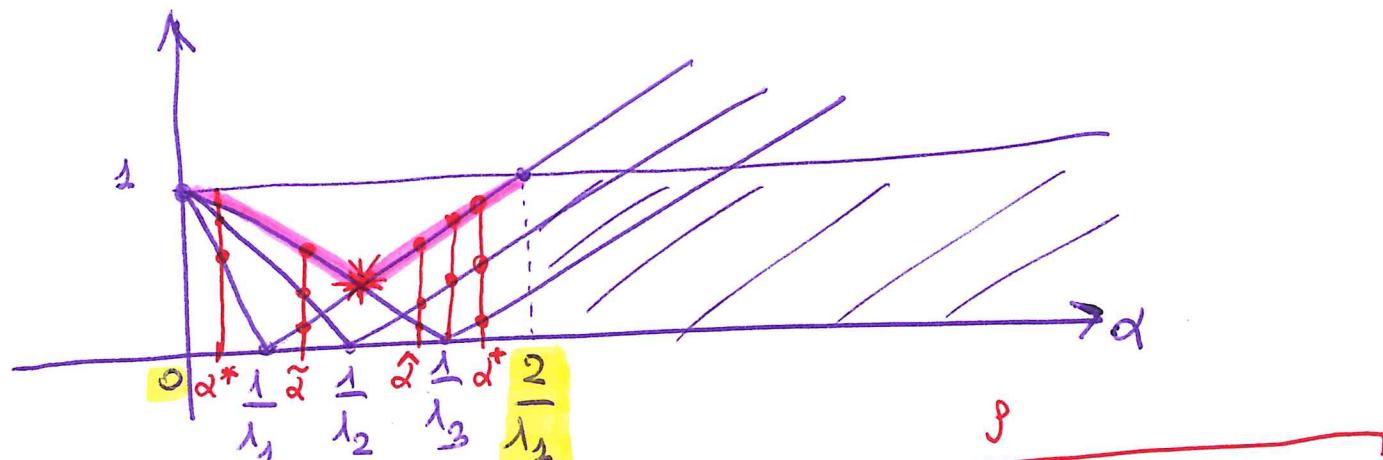
$$|1 - \alpha \lambda_3|$$

$$B_\alpha$$



$$\alpha = 0 \quad (0, 1)$$

$$\alpha = \frac{1}{\lambda_1}$$



$\alpha$  che minimizza il massimo autovalore in modulo

~~$\alpha \lambda_3 - 1$~~

$1 - \alpha \lambda_3$

$\alpha \lambda_1 - 1 = 1 - \alpha \lambda_3$

~~$\alpha \lambda_1 - 1$~~

~~$1 - \alpha \lambda_1$~~

$\alpha(\lambda_1 + \lambda_3) = 2$

$$\alpha_{opt} = \frac{2}{\lambda_1 + \lambda_3} \rightarrow$$

$$\alpha_{opt} = \frac{2}{\lambda_{max} + \lambda_{min}}$$

$$g(B_{\alpha_{opt}}) = 1 - \alpha_{opt} \lambda_{min} = 1 - \frac{2 \lambda_{min}}{\lambda_{max} + \lambda_{min}} =$$

$$\frac{\lambda_{max} - \lambda_{min}}{\lambda_{max} + \lambda_{min}}$$

$P$  non singolare  
 $P$  "facile"  
 $P$   $\underbrace{k(P^{-1}A)}_{\text{matrice } A}$  più piccolo possibile  
 precondizionato

$$Pz^{(k)} = r^{(k)}$$

$$\frac{\frac{10}{k(P^{-1}A) - 1}}{\underbrace{k(P^{-1}A) + 1}_{< 1}} \sim 1$$

$$\frac{\frac{10}{k(P^{-1}A) - 1}}{\underbrace{k(P^{-1}A) + 1}_{< 1}} = \frac{9}{11} < 1$$

Proposizione: siano  $A$  e  $P$  sdip. Allora, Richardson dinamico converge  $\forall x^{(0)} \in \mathbb{R}^n$  se

$$\alpha_k = \alpha_{\text{opt}, k} = \frac{(z^{(k)})^T r^{(k)}}{(z^{(k)})^T A z^{(k)}} \quad k \geq 0 \quad [Pz^{(k)} = r^{(k)}]$$

Inoltre vale

$$\|e^{(k)}\|_A \leq \left( \frac{k(P^{-1}A) - 1}{k(P^{-1}A) + 1} \right)^k \|e^{(0)}\|_A \quad k \geq 0$$

metodo dei gradi ente  
precondizionato

( $P = I$  metodo del gradiente  
 $\alpha_k = \alpha_{\text{opt}, k} = \frac{(r^{(k)})^T r^{(k)}}{(r^{(k)})^T A r^{(k)}}$ )

4 passi

$$x^{(0)} \rightarrow r^{(0)} = b - Ax^{(0)}$$

$k = 0, 1, \dots$

$$\left[ \begin{array}{l} \rightarrow 1) \boxed{p_2^{(k)} = r^{(k)}} \rightarrow z^{(k)} \\ \rightarrow 2) \alpha_k = \frac{(z^{(k)})^T r^{(k)}}{(z^{(k)})^T A z^{(k)}} \rightarrow \alpha_k \\ \rightarrow 3) x^{(k+1)} = x^{(k)} + \alpha_k z^{(k)} \rightarrow x^{(k+1)} \\ \rightarrow 4) r^{(k+1)} = b - Ax^{(k+1)} = b - A(x^{(k)} + \alpha_k z^{(k)}) = \underbrace{b - Ax^{(k)}}_{r^{(k)}} - \alpha_k A z^{(k)} = r^{(k)} - \alpha_k A z^{(k)} \rightarrow r^{(k+1)} \end{array} \right]$$

GP

G (3 passi)

Richardson stazionario  
(3 passi)

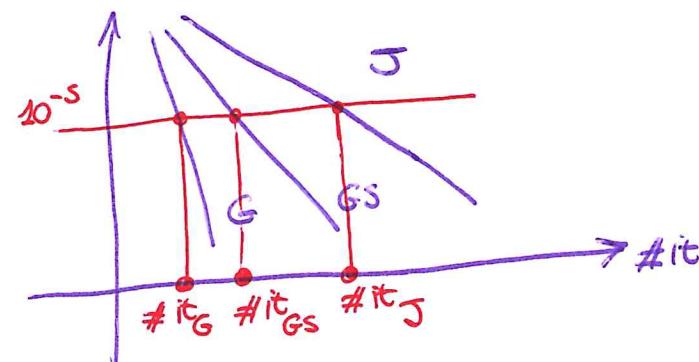
STOPPING CRITERIA

J, GS, G

$$\begin{cases} 2x_1 + x_2 = 1 \\ x_1 + 3x_2 = 0 \end{cases}$$

$$x = \left( \frac{3}{5}, -\frac{1}{5} \right)^T$$

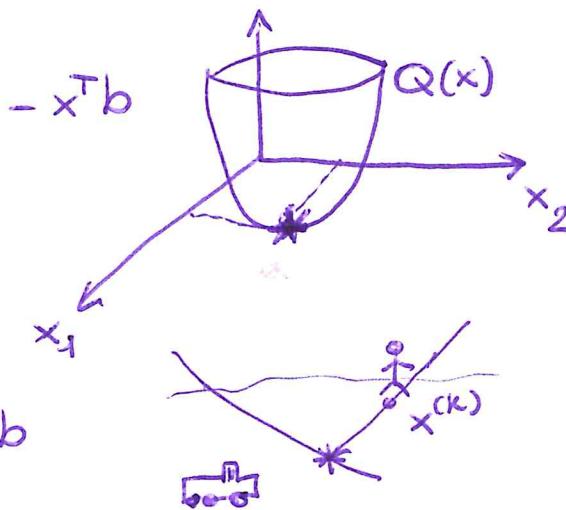
$$A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \text{ sdp}$$



$$\begin{aligned}x^{(k+1)} &= x^{(k)} + \alpha_k r^{(k)} \\ \alpha_k &= \frac{(r^{(k)})^T r^{(k)}}{(r^{(k)})^T A r^{(k)}}\end{aligned}$$

A sdp  
 $Ax = b \iff$  minimize  $Q(x) = \frac{1}{2} x^T A x - x^T b$

$$x = [x_1, x_2]^T$$



$$\nabla Q(x) = Ax - b$$

$$\nabla Q(x) = 0 \quad Ax - b = 0 \quad Ax = b$$

$$x^{(k+1)} = x^{(k)} + \beta_k d^{(k)}$$

steepest descent method

$$d^{(k)} = -\nabla Q(x^{(k)}) = -[Ax^{(k)} - b] = b - Ax^{(k)} = r^{(k)}$$

$$x^{(k+1)} = x^{(k)} + \beta_k r^{(k)} \quad ? \beta_k$$

$$Q(x^{(k)} + \beta_k r^{(k)}) = \tilde{Q}(\beta_k) \quad \frac{d\tilde{Q}}{d\beta_k} = 0$$

$$\frac{1}{2} (x^{(k)} + \beta_k r^{(k)})^T A (x^{(k)} + \beta_k r^{(k)}) - (x^{(k)} + \beta_k r^{(k)})^T b = \tilde{Q}(\beta_k)$$

$$\frac{d\tilde{Q}}{d\beta_k} = r^{(k)^T} A x^{(k)} + \beta_k r^{(k)^T} A r^{(k)} - r^{(k)^T} b = 0 \rightarrow \beta_k = \frac{r^{(k)^T} b - r^{(k)^T} A x^{(k)}}{r^{(k)^T} A r^{(k)}}$$

$$= \frac{r^{(k)^T} r^{(k)}}{r^{(k)^T} A r^{(k)}}$$

## Metodo dei gradienti coniugato (precondizionato)

A sdp

$$x^{(k+1)} = x^{(k)} + \tilde{\alpha}_k p^{(k)}$$

$\dagger$

$$r^{(k)}$$

$$p^{(k+1)} \text{ t.c. } (Ap^{(i)})^T p^{(k+1)} = 0 \quad \forall i=0, \dots, k$$

Algoritmo a 5 passi:  $x^{(0)} \rightarrow r^{(0)} = b - Ax^{(0)}$ ;  $p^{(0)} = r^{(0)}$

$$\{p^{(k)}\}$$

$$p^{(i)}, p^{(j)}$$

A-coniugate ( $A$ -ortogonale)

$$p^{(i)T} p^{(j)} = 0$$

$$(Ap^{(i)})^T p^{(j)} = 0$$

$$[p^{(i)T} A^T p^{(j)} = 0]$$

1)  $\tilde{\alpha}_k = \frac{(p^{(k)T} r^{(k)})}{(p^{(k)T} A p^{(k)})} \rightarrow \tilde{\alpha}_k$

2)  $x^{(k+1)} = x^{(k)} + \tilde{\alpha}_k p^{(k)} \rightarrow x^{(k+1)}$

3)  $r^{(k+1)} = r^{(k)} - \tilde{\alpha}_k A p^{(k)} \rightarrow r^{(k+1)}$

4)  $\beta_k = \frac{(A p^{(k)T} r^{(k+1)})}{(A p^{(k)T} p^{(k)})} \rightarrow \beta_k$

5)  $p^{(k+1)} = r^{(k+1)} - \beta_k p^{(k)} \rightarrow p^{(k+1)}$

$$\|e^{(k)}\|_A \leq [C^*]^k \|e^{(0)}\|_A$$

$$C^* = C^* (\sqrt{k(A)})$$

2<sup>(k+1)</sup>

|| |

Hiebert  
sdp

$n$	$K(H_n)$	$\searrow$	PG	PGC	
4	$O(10^4)$	$O(10^{-13})$	$O(10^{-3})$	985	$O(10^{-2})$
6	$O(10^6)$	$O(10^{-10})$	$O(10^{-3})$	1813	$O(10^{-3})$
8	:	:	:	:	:
10	:	:	:	:	:
12	:	:	:	:	:
14	$O(10^{17})$	$O(10)$	$O(10^{-3})$	1373	$O(10^{-3})$

$$P = D = \begin{bmatrix} a_{11} & a_{12} & 0 \\ 0 & \ddots & a_{nn} \end{bmatrix}; \quad x^{(0)} = 0 \\ TOL = 10^{-6}$$

### CRITERI D'ARRESTO

$N_{\text{max}}$   
accuracy

$$\|x - x^{(k)}\| \leq \boxed{\leq TOL} (10^{-9})$$

stimatore (computabile)

affidabilità

residuo  $r^{(k)}$

incremento  $x^{(k+1)} - x^{(k)}$

Residuo  $r^{(k)} = b - Ax^{(k)} \in \mathbb{R}^n$

?  $k_{\text{miu}}$  t.e.

$$\frac{\|r^{(k_{\text{min}})}\|}{\|b\|} \leq \text{TOL}$$

residuo relativo

$\downarrow$

$$(x^{(0)} = 0 \quad r^{(0)} = b - Ax^{(0)} = b)$$

$$\frac{\|e^{(k_{\text{miu}})}\|}{\|x\|} \leq C$$

$10^2$

$$\frac{\|r^{(k_{\text{miu}})}\|}{\|b\|} \leq \text{TOL}$$

0.5  $10^{-6}$   $10^{-6}$

$$C = k(A)$$

$$(\delta A=0) \quad \frac{\|\delta x\|}{\|x\|} \leq k(A) \frac{\|\delta b\|}{\|b\|}$$

$$Ax = b$$

$$(A + \delta A)(x + \delta x) = b + \delta b$$

$$A(x + \delta x) = b + \delta b$$

$$\tilde{x} = x + \delta x$$

$$\tilde{x} = x^{(k_{\text{miu}})}$$

$$A(\underbrace{x + \delta x}_{\tilde{x}}) = b + \delta b$$

$$\delta b = A\tilde{x} - b = -(b - A\tilde{x}) = -\tilde{r}$$

$$\frac{\|x^{(k_{\text{miu}})} - x\|}{\|x\|} \leq k(A)$$

$$\frac{\|\delta x\|}{\|x\|} \leq k(A) \frac{\|\tilde{r}\|}{\|b\|}$$

$$\frac{\|b - Ax^{(k_{\text{miu}})}\|}{\|b\|} = k(A) \frac{\|r^{(k_{\text{miu}})}\|}{\|b\|}$$

## Incremento

$$x^{(k+1)} - x^{(k)} = \delta^{(k)}$$

?  $x^{(kmin)}$  t.c.  $\|e^{(kmin)}\| \leq C$   $\|\delta^{(kmin)}\| \leq \text{TOL } (10^{-s})$

$$B \text{ sdp} \quad x^{(k+1)} = Bx^{(k)} + g$$

$$e^{(kmin)} = x - x^{(kmin)} = \underbrace{x - x^{(kmin+1)}}_{e^{(kmin+1)}} + \underbrace{x^{(kmin+1)} - x^{(kmin)}}_{\delta^{(kmin)}}$$

$$\|e^{(kmin)}\| = \|x - x^{(kmin+1)} + \delta^{(kmin)}\|$$

$$\leq \underbrace{\|e^{(kmin+1)}\|}_{\leq p(B) \|e^{(kmin)}\|} + \|\delta^{(kmin)}\|$$

$$\leq p(B) \|e^{(kmin)}\| + \|\delta^{(kmin)}\|$$

$$\|e^{(kmin)}\| \leq \frac{C}{1-p(B)} \|\delta^{(kmin)}\|$$

$$[\|e^{(k+1)}\| \leq \|B\|_2 \|e^{(k)}\|]$$

$$\simeq p(B)$$

B sdp  $\|B\|_2 = \sqrt{\lambda_{\max}(B^T B)}$

$$\Rightarrow \|B\|_2 = \sqrt{\lambda_{\max}(B^2)} = \sqrt{[\lambda_{\max}(B)]^2} \equiv p(B)$$

## Approssimazione di autovetori e autovettori

$$A \in \mathbb{C}^{n \times n}$$

$$\exists \lambda \in \mathbb{C}(\mathbb{R}) \text{ e } x \in \mathbb{C}^n(\mathbb{R}^n) \text{ t.c. } Ax = \lambda x$$

$$\begin{matrix} x \\ 0 \end{matrix}$$

$$( \alpha x \quad \forall \alpha \in \mathbb{C})$$

Quoziente di Rayleigh: noto  $x$ , l'autovalore  $\lambda$  associato è

$$\lambda = \frac{x^H A x}{\|x\|^2}$$

$$\downarrow \begin{matrix} x^H & x^T \\ (x_i = a+ib \rightarrow x_i^H = a-ib) \end{matrix}$$

$$\|x\|^2 = x^T x \rightarrow \|x\|^2 = x^H x$$

trasposto completo coniugato

Polinomio caratteristico:  $p_A(\lambda) = \det(A - \lambda I)$

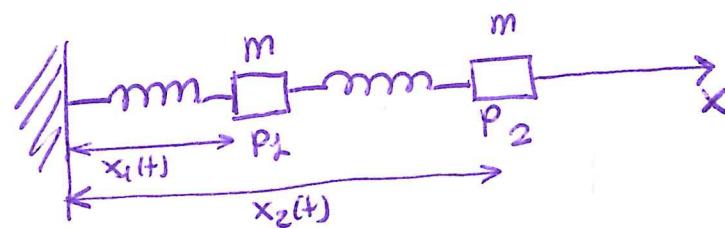
Matrice diagonalizzabile:  $A$  si dice diagonalizzabile se  $\exists U \in \mathbb{C}^{n \times n}$  invertibile e t.c.

$$U^{-1} A U = \text{diag}(\lambda_1, \dots, \lambda_n)$$

$\downarrow$  colonne  $\equiv$  autovettori di  $A$

Spettro:  $\{\lambda_i\}_{i=1}^n \quad g(A) = \max_i |\lambda_i|$

## Sistema di 2 molle



$x_i(t)$  posizione occupata da  $P_i$  all'istante  $t$

Ho 2 corpi puntiformi  $P_1$  e  $P_2$  di massa  $m$ ; 2 molle con stesso coefficiente  $k$  di elasticità;  
liberi di muoversi lungo la direzione  $x$

Per le seconde leggi della dinamica  $m\ddot{x}_1 = k(x_2 - x_1) - kx_1$

$$m\ddot{x}_2 = k(x_1 - x_2)$$

$$m \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} -2k & k \\ k & -k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

siamo interessati ad oscillazioni libere  
Ansatz  $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} e^{i\omega t}$

$$-\omega^2 m \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} e^{i\omega t} = \begin{bmatrix} -2k & k \\ k & -k \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} e^{i\omega t}$$

$$\dot{x} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} i\omega e^{i\omega t} \quad \ddot{x} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} (-\omega^2) e^{i\omega t}$$

$$\begin{bmatrix} 2k & -2k \\ -2k & 2k \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \omega^2 m \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$

matrice di stiffn

$$\underbrace{\begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix}}_A \underbrace{\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}}_V = \underbrace{\omega^2 m}_{\lambda} \underbrace{\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}}_V$$

problema agli autovetori

EX2

$$m\ddot{x}_1 = -kx_1$$

$$\ddot{x}_1 = -\frac{k}{m}x_1$$

$$\begin{cases} \dot{x}_1 = y \\ \dot{y} = -\frac{k}{m}x_1 \end{cases}$$

$$iw A_1 e^{iwt} = A_2 e^{iwt}$$

$$iw A_2 e^{iwt} = -\frac{k}{m} A_1 e^{iwt}$$

$$\underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix}}_A \underbrace{\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}}_v = \underbrace{iw}_{\lambda} \underbrace{\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}}_v$$

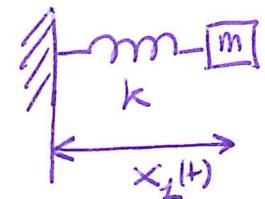
$$\begin{bmatrix} x_1 \\ y \end{bmatrix} = X$$

$$\dot{X} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ y \end{bmatrix}$$

$$x_1 = A_1 e^{iwt}$$

$$y = A_2 e^{iwt}$$

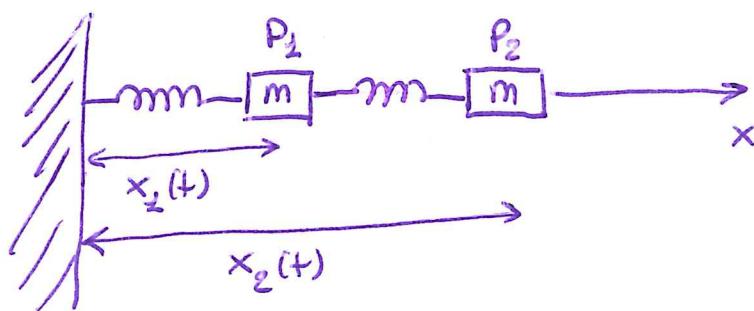
Oscillatore armonico semplice



problema agli autovettori

EX 1

## Sistema di 2 molle



$x_i(t)$        $P_i(t)$

$$\begin{cases} m \ddot{x}_1 = k(x_2 - x_1) - kx_1 \\ m \ddot{x}_2 = k(x_1 - x_2) \end{cases}$$

$$m \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} -2k & k \\ k & -k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$-\omega^2 m \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} e^{i\omega t} = \begin{bmatrix} -2k & k \\ k & -k \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} e^{i\omega t}$$

$$\underbrace{\begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix}}_A \underbrace{\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}}_V = \underbrace{\omega^2 m}_{\lambda} \underbrace{\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}}_V$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} e^{i\omega t}$$

$$\dot{x} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} i\omega e^{i\omega t}, \quad \ddot{x} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} (-\omega^2) e^{i\omega t}$$

TACOMA

METODI  
NUMERICI

$\lambda, x$  specifici

spettro (QR)

Metodo delle potenze ( $\lambda_{\max}, x$ )

$A \in \mathbb{R}^{n \times n}$

$$\textcircled{A1} \quad |\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \dots \geq |\lambda_n|$$

↓ autovettore dominante

$(\lambda_1, x_1)$

↓ ampiezza unitaria

$\textcircled{A2}$  autovettori di  $A$  linearmente indipendenti

$x^{(0)} \in \mathbb{C}^n$  (guess iniziale), sia  $y^{(0)} = \frac{x^{(0)}}{\|x^{(0)}\|}$

$k = 1, 2, \dots$

$$x^{(k)} = Ay^{(k-1)}$$

$$y^{(k)} = \frac{x^{(k)}}{\|x^{(k)}\|} \simeq x_1$$

$$\lambda^{(k)} = (y^{(k)})^H A y^{(k)} \simeq \lambda_1$$

$\textcircled{A} \quad N_{MAX}$   
 $\textcircled{B} \quad |1^{(k)} - \lambda^{(k-1)}|$

$$< \varepsilon \quad |\lambda^{(k)}|$$

$$k=1 \quad [x^{(1)} = Ay^{(0)} \quad y^{(1)} = \frac{Ay^{(0)}}{\|x^{(1)}\|} \quad \lambda^{(1)} = (y^{(1)})^H A y^{(1)}$$

$$k=2 \quad \begin{cases} x^{(2)} = Ay^{(1)} & y^{(2)} = \frac{Ay^{(1)}}{\|x^{(2)}\|} = \frac{A^2 y^{(0)}}{\|x^{(1)}\| \|x^{(2)}\|} = \beta^{(2)} A^2 y^{(0)} \\ \beta^{(2)} = \left( \frac{1}{\|x^{(1)}\| \|x^{(2)}\|} \right) & \lambda^{(2)} = (y^{(2)})^H A y^{(2)} \end{cases}$$

$$k \quad \begin{cases} x^{(k)} = Ay^{(k-1)} & y^{(k)} = A^k y^{(0)} \beta^{(k)} \\ \lambda^{(k)} = (y^{(k)})^H A y^{(k)} & \beta^{(k)} = \left( \prod_{i=1}^k \|x^{(i)}\| \right)^{-1} \end{cases}$$

## Analisi di convergenza

$\{x_i\}$  linearmente indipendenti  $\rightarrow$  base per  $\mathbb{C}^n$

$$x^{(0)} = \sum_{i=1}^n \alpha_i x_i$$

$\alpha_i \in \mathbb{C}$

$$y^{(0)} = \beta^{(0)} \sum_{i=1}^n \alpha_i x_i$$

$$\beta^{(0)} = \frac{1}{\|x^{(0)}\|}$$

$k=1$

$$x^{(1)} = Ax^{(0)} = \beta^{(0)} \sum_{i=1}^n \alpha_i A x_i = \beta^{(0)} \sum_{i=1}^n \alpha_i \lambda_i x_i$$

$$\beta^{(1)} = \frac{1}{\|x^{(0)}\| \|x^{(1)}\|}$$

$$y^{(1)} = \beta^{(1)} \sum_{i=1}^n \alpha_i \lambda_i x_i$$

$k=2$

$$x^{(2)} = Ax^{(1)} = \beta^{(1)} \sum_{i=1}^n \alpha_i \lambda_i A x_i = \beta^{(1)} \sum_{i=1}^n \alpha_i \lambda_i^2 x_i$$

$$y^{(2)} = \beta^{(2)} \sum_{i=1}^n \alpha_i \lambda_i^2 x_i$$

$$\beta^{(2)} = \frac{1}{\|x^{(0)}\| \|x^{(1)}\| \|x^{(2)}\|}$$

generalizzando

$$y^{(k)} = \beta^{(k)} \sum_{i=1}^n \alpha_i \lambda_i^k x_i$$

$$= \beta^{(k)} \left[ \alpha_1 \lambda_1^k x_1 + \sum_{i=2}^n \alpha_i \lambda_i^k x_i \right]$$

$$= \beta^{(k)} \alpha_1 \lambda_1^k \left[ x_1 + \underbrace{\sum_{i=2}^n \frac{\alpha_i}{\alpha_1} \left( \frac{\lambda_i}{\lambda_1} \right)^k}_{< 1} x_i \right]$$

$$\beta^{(k)} = \frac{1}{\|x^{(0)}\| \|x^{(1)}\| \cdots \|x^{(k)}\|}$$

$$\alpha_1 \neq 0$$

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \dots \geq |\lambda_n|$$

$\hookrightarrow 0 \quad k \rightarrow +\infty$

1)  $y^{(k)}$  si allinea a  $x_1$  per  $k \rightarrow +\infty$  se  $\alpha_1 \neq 0$

2) se  $y^{(k)}$  si allinea a  $x_1 \rightarrow \lambda^{(k)} \rightarrow \lambda_1$  per  $k \rightarrow +\infty$

3) velocità di convergenza

$$\|x_1 - y^{(k)}\|$$

$$\left\| x_1 - y^{(k)} - \sum_{i=2}^n \frac{\alpha_i(\lambda_1^k)}{\alpha_1(\lambda_1)} x_i \right\| = \left\| \sum_{i=2}^n \frac{\alpha_i(\lambda_1^k)}{\alpha_1(\lambda_1)} x_i \right\|$$

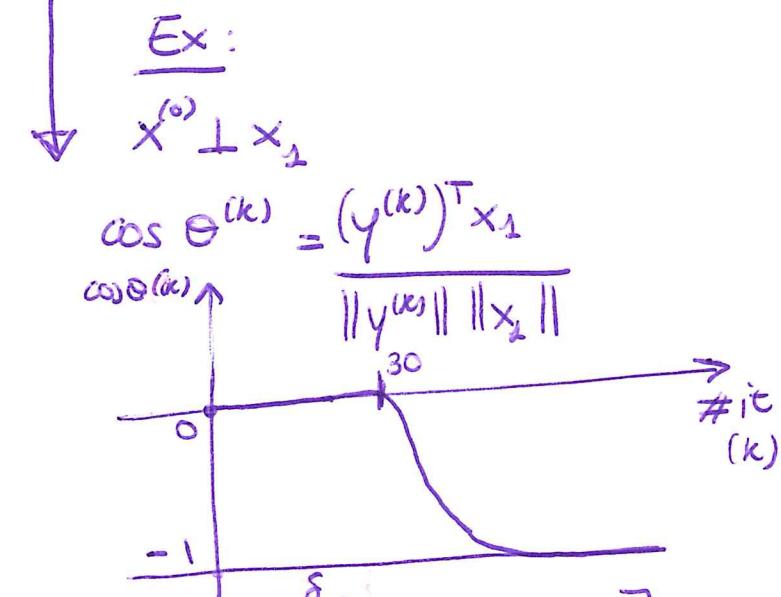
↑ norma unitaria

$$= \left[ \sum_{i=2}^n \left[ \frac{\alpha_i}{\alpha_1} \right]^2 \left( \frac{\lambda_1}{\lambda_2} \right)^{2k} \right]^{1/2}$$

$$\leq \underbrace{\frac{\lambda_2}{\lambda_1}}_1^k \underbrace{\left[ \sum_{i=2}^n \left[ \frac{\alpha_i}{\alpha_1} \right]^2 \right]}_{\text{quantità fissata}}^{1/2}$$

$\hookrightarrow 0$  per  $k \rightarrow +\infty$

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \dots \geq |\lambda_n|$$



16	2	3	13
5	11	10	8
9	7	6	12
4	14	15	1

(separabilità degli autovettori)

Ex:  $\delta = 30$  → 22 iterazioni

$$\lambda_1 = 39.396 \quad \lambda_2 = 17.821$$

$$\begin{array}{l} \delta = -30 \\ E = 10^{-10} \end{array} \quad \lambda_1 = -30.643 \quad \lambda_2 = 29.736$$

→ 408 iterazioni

4) A Hermitiana  $A^H = A$

Se A è Hermitiana,

$$\|x_1 - y^{(k)}\| \leq \left| \frac{1}{\lambda_1} \right|^{2k} \left[ \sum_{i=2}^n \left[ \frac{\alpha_i}{\alpha_1} \right]^2 \right]^{1/2}$$

5) Potenze convergono anche se  $\lambda_1$  è una radice multipla di  $p_A(\lambda)$ .

6) Se  $\exists 2$  autovекторi distinti ma entrambi di modulo massimo, potenze non convergono.

### Generalizzazioni

① Metodo delle potenze inverse: approssima autovettore in modulo minimo (+ corrispondente autovettore)

A non singolare ( $\exists A^{-1}$ )

$$Ax = \lambda x$$

$$A^{-1}Ax = \lambda A^{-1}x$$

$$x = \lambda A^{-1}x$$

$$x^{(0)} \in \mathbb{C}^n, y^{(0)} = \frac{x^{(0)}}{\|x^{(0)}\|}$$

$k = 1, 2, \dots$

$$x^{(k)} = A^{-1}y^{(k-1)}$$

$$y^{(k)} = \frac{\|x^{(k)}\|}{\|x^{(k)}\|} \hat{\sim} x_n$$

$$\mu^{(k)} = (y^{(k)})^H A^{-1} y^{(k)}$$

$$\hat{\sim} \frac{1}{\lambda_n}$$

$$\begin{matrix} A \\ \{ \lambda, x \} \end{matrix}$$

$$\begin{matrix} A^{-1} \\ \left\{ \frac{1}{\lambda}, x \right\} \end{matrix}$$

$$|\lambda_1| > |\lambda_2| \geq \dots \geq |\lambda_n|$$

$$\left| \frac{1}{\lambda_1} \right| < \left| \frac{1}{\lambda_2} \right| \leq \dots \leq \left| \frac{1}{\lambda_n} \right|$$

$$\begin{matrix} (\lambda_n, x_n) \\ A \end{matrix}$$

$$\begin{matrix} \left( \frac{1}{\lambda_n} (\sim \lambda_n), x_n \right) \\ A^{-1} \end{matrix}$$

$$1) Ax^{(k)} = y^{(k-1)}$$

$$2) \mu^{(k)} = (y^{(k)})^H A y^{(k)} \hat{\sim} \lambda_n$$

Potenze inverse con shift: approssimare e' l'autovettore di  $A$  più vicino a  $\mu \in \mathbb{C}$ .

$$A_\mu = A - \mu I$$

$$\lambda(A_\mu) = \lambda(A) - \mu$$

$\lambda_\mu$

SHIFT

$\lambda_{\min}(A_\mu)$  potenze inverse  $A_\mu$

$$\hookrightarrow \lambda_\mu = \lambda_{\min}(A_\mu) + \mu$$

$$\begin{array}{c} A \\ \{\lambda, x\} \end{array} \qquad \begin{array}{c} A_\mu \\ \{\lambda - \mu, x\} \end{array}$$

$$Ax = \lambda x$$

$$Ax - \mu I x = \lambda x - \mu x$$

$$\underbrace{(A - \mu I)}_{A_\mu} x = (\lambda - \mu)x$$

$$\begin{array}{ccccc} x_1 & x_2 & & x_n & \\ |\lambda_1| > |\lambda_2| \geq \dots & > |\lambda_n| & A \\ \lambda_1 - \mu & \lambda_2 - \mu & & \lambda_n - \mu & A_\mu \\ \lambda_{\min}(A_\mu) & \circled{|\lambda_i - \mu|} & x_i & & \end{array}$$

$$x^{(k)} = A_\mu^{-1} y^{(k-1)}$$

$$y^{(k)} = \frac{x^{(k)}}{\|x^{(k)}\|} \simeq x_\mu (x_i)$$

$$\mu^{(k)} = (y^{(k)})^H A_\mu^{-1} y^{(k)} \simeq \frac{1}{\lambda_{\min}(A_\mu)}$$

$$\begin{array}{c}
 \left\{ \lambda_i, \underset{\textcircled{H}}{x_i} \right\} \quad \left\{ \frac{\mu}{\lambda_i - \mu}, x_i \right\} \quad \left\{ \frac{\mu}{\lambda_i - \mu}, \underset{\textcircled{H}}{x_i} \right\} \\
 \left( \begin{array}{l}
 x_i^H A x_i = \lambda_i \\
 x_i^H A^{-1} x_i = \frac{1}{\lambda_i - \mu}
 \end{array} \right) \rightarrow
 \end{array}$$

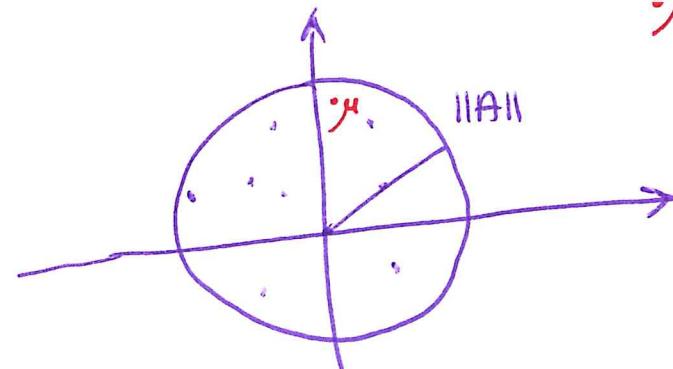
52/bis

## Scelta dello shift

1)

$$\lambda_0 = \sigma_A$$

$$|\lambda|_{\text{lett}} = \|\lambda_0\| = \|A\sigma\| \leq \|A\| \|\sigma\| \rightarrow |\lambda| \leq \|A\|$$



$$\|A\|_2 = \sqrt{\lambda_{\max}(A^T A)}$$

2) cerchi di Gershgorin:

$$C_i^r = \left\{ z \in \mathbb{C} \text{ t.c. } |z - a_{ii}| \leq \sum_{\substack{j=1 \\ j \neq i}}^n |a_{ij}| \right\} \quad i=1, \dots, n$$

$$C_i^c = \left\{ z \in \mathbb{C} \text{ t.c. } |za_{ii}| \leq \sum_{\substack{j=1 \\ j \neq i}}^n |a_{ji}| \right\} \quad i=1, \dots, n$$

Proposizione: gli autovalori di  $A \in \mathbb{C}^{n \times n}$  appartengono alla regione di  $\mathbb{C}$  identificata da

$$\left[ \bigcup_{i=1}^n C_i^r \right] \cap \left[ \bigcup_{i=1}^n C_i^c \right].$$

Inoltre se  $\exists m$  cerchi  $r_{ij}$  (colonne)  $1 \leq m \leq n$  sono disgiunti (sconnessi) dell'unione dei restanti  $n-m$ , allora le loro unioni contiene esattamente  $m$  autovalori.

## Equazioni nonlineari

$f$  non lineare

$\mathbb{R}$

$$P_r = \left\{ a_0 + a_1 x + a_2 x^2 + \dots + a_r x^r \right\} \quad ax + b$$

$\begin{cases} \sin, \cos, \tanh \\ \exp, \log, \\ \sqrt{\cdot} \end{cases}$

$$\exists \alpha \in \mathbb{R} \text{ t.c. } f(\alpha) = 0$$

?  $V$  gas  $T, P$

$$\left[ p + a \left( \frac{N}{V} \right)^2 \right] (V - Nb) = kNT$$

$$\exists V \text{ t.c. } f(V) = 0$$

$$\left[ p + a \left( \frac{N}{V} \right)^2 \right] (V - Nb) - kNT = f(V)$$

Abele

$P_r$  r74

## Metodi di punto fisso

$$1 \quad \cos$$

$$\alpha^* = 0.73908513$$

$$\cos \alpha^* = \alpha^*$$

$\Phi : [a,b] \subset \mathbb{R} \rightarrow \mathbb{R}$ ,  $\Phi \in C^0([a,b])$

$\alpha^*$  è punto fisso di  $\Phi$  se  $\Phi(\alpha^*) = \alpha^*$

$$x^{(k+1)} = \phi(x^{(k)})$$

$$x^{(0)} = 1$$

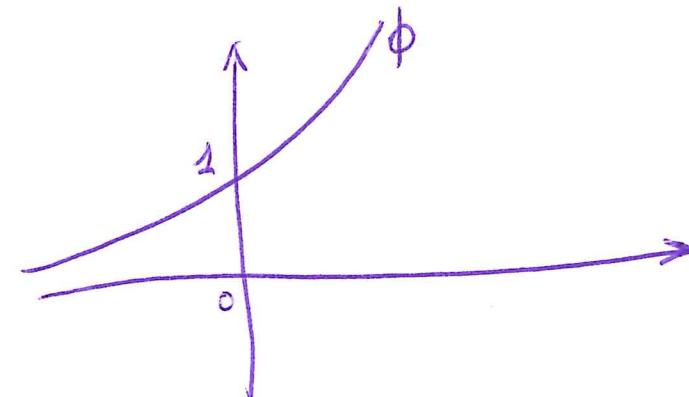
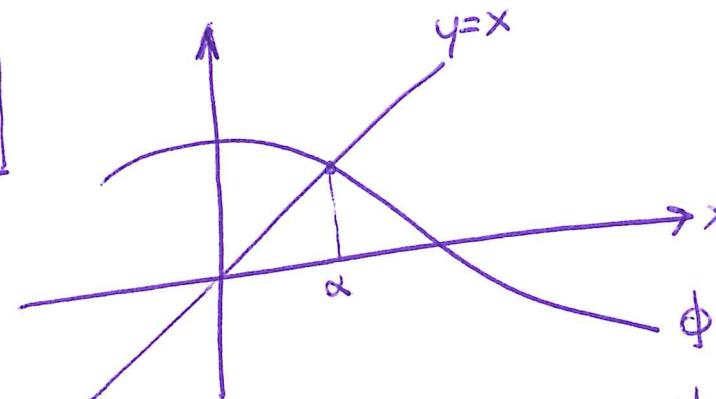
$$x^{(1)} = \cos(x^{(0)})$$

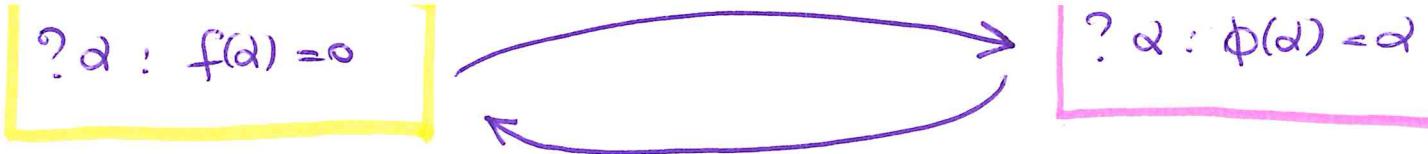
$$x^{(2)} = \cos(x^{(1)})$$

$$x^{(3)} = \cos(x^{(2)}) \rightarrow x^{(k+1)} = \cos(x^{(k)})$$

⋮

$$\phi(x) = \cos(x)$$





$$f(x) = 0 \leftrightarrow x = \phi(x) \quad x^{(k+1)} = \phi(x^{(k)})$$

$$\underbrace{f(x) + x}_{\phi(x)} = x$$

$$\boxed{H_p} \quad f(d) = 0$$

$$\boxed{\text{Tei}} \quad \phi(d) = d$$

$$\phi(d) = \underbrace{f(d)}_{=0} + d = d$$

$$\boxed{f(x) + x = \phi(x)}$$

$$\boxed{H_p} \quad d = \phi(d)$$

$$\text{Tei} \quad f(d) = 0$$

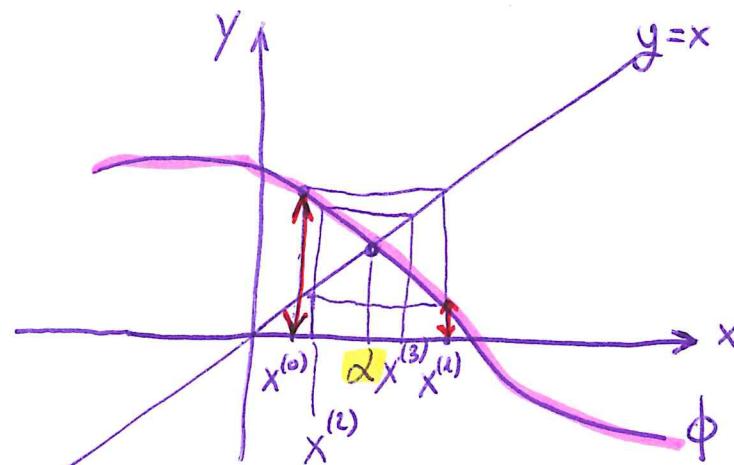
$$\phi(d) = f(d) + d = d$$

$\underbrace{f(d)}_{=0} = 0$

$$\begin{aligned} & \phi(x) \\ & f(x) = 0 \\ & f(x) + 3x = 3x \end{aligned}$$

$\boxed{\frac{f(x) + 3x}{3} = x}$

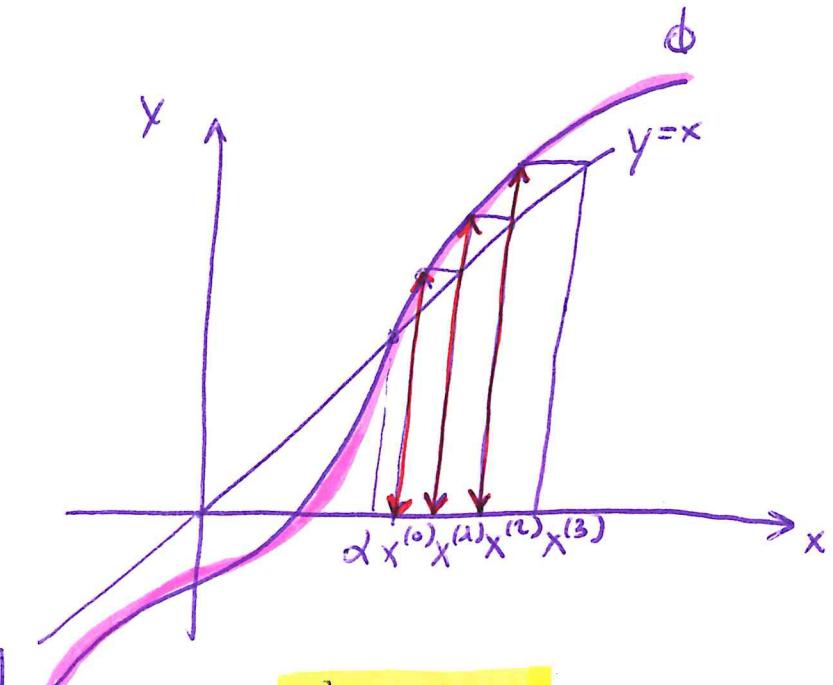
$$x^{(k+1)} = \phi(x^{(k)})$$



$\alpha$  attrattore

$$\begin{aligned} x^{(0)} \\ x^{(1)} &= \phi(x^{(0)}) \\ x^{(2)} &= \phi(x^{(1)}) \\ x^{(3)} &= \phi(x^{(2)}) \\ &\vdots \end{aligned}$$

Convergenza condizionata



$\alpha$  repulsore

$$\phi'$$

Ostrowski :  $\exists \alpha$  punto fisso di  $\phi$  in  $I_\alpha$ ;  $\phi \in C^1(\bar{I}_\alpha)$ . Se  $|\phi'(\alpha)| < 1 \Rightarrow \exists \delta > 0$  t.c.

(locale) se  $x^{(0)}$ :  $|x^{(0)} - \alpha| < \delta$ ,  $\{x^{(k)}\}$  converge a  $\alpha$  e vale le relazioni

$$\lim_{k \rightarrow +\infty} \frac{x^{(k+1)} - \alpha}{x^{(k)} - \alpha} = \phi'(\alpha)$$

$p=1$  (convergenza  
obliqua lineare)  
 $C = \phi'(\alpha)$

$I_\alpha$  = intorno di  $\alpha$

$$\frac{\alpha}{\alpha - \varepsilon} < \frac{\alpha}{\alpha + \varepsilon}$$

$$I_\alpha = (\alpha - \varepsilon, \alpha + \varepsilon); \quad \bar{I}_\alpha = [\alpha - \varepsilon, \alpha + \varepsilon]$$

Ordine di convergenza :  $\{x^{(k)}\}$   $\Leftrightarrow$  ordine di convergenza = p se

$$\frac{|x^{(k+1)} - \alpha|}{|x^{(k)} - \alpha|^p} \leq C \quad \forall k \geq k_0$$

↓  
indipendente da k

Se  $p=1$  allora  $C < 1$

( $p=1$  lineare,  $p=2$  quadratica . . . )

Ex  $p=2$   $|x^{(k+1)} - \alpha| \leq \underbrace{C^*}_{10} \underbrace{|x^{(k)} - \alpha|^2}_{10^{-1}}$

$\underbrace{\phantom{0}}_{10^{-2}}$   
 $\underbrace{\phantom{00}}_{10^{-4}}$

$p=3$   $|x^{(k+1)} - \alpha| \leq \underbrace{\tilde{C}}_{10^{-1}} \underbrace{|x^{(k)} - \alpha|^3}_{10^{-3}}$

$\underbrace{\phantom{0}}_{10^{-9}}$

$p=1$   $|x^{(k+1)} - \alpha| \leq C \underbrace{|x^{(k)} - \alpha|}_{0.5 \cdot 10^{-1}}$

$\downarrow \downarrow$

$$x = \phi_1(x) \quad t.c. \quad |\phi'_1(\alpha_1)| = 0.32$$



$|\phi'(x)| < 1$  convergenza

$|\phi'(x)| > 1$  divergenza

$|\phi'(x)| = 1$

non unicità di  $\phi$

- ↗  $\exists \alpha$
- ↘ convergenza o meno
- ↙ ordine diverso o velocità di convergenza  $\neq$

Ex : 1)  $\phi(x) = \cos(x)$

$$\phi'(x) = -\sin(x)$$

OK

$$|\phi'(\alpha)| = |\sin(\alpha)| < 1$$

→ ordine di convergenza almeno  $\frac{1}{2}$

$$2) \phi(x) = x^2 - 1$$

$$x = \phi(x) \quad x = x^2 - 1 \quad x^2 - x - 1 = 0$$

$$\phi'(x) = 2x$$

$$\alpha_{\pm} = \frac{1 \pm \sqrt{5}}{2}$$

$$|\phi'(\alpha_{\pm})| = |1 \pm \sqrt{5}| > 1$$

KO

$$3) f(x) = \log(x) - \beta x = 0 \quad \beta \in \mathbb{R}$$

$$\phi_1 \quad \underbrace{\log(x) - \beta x + x}_{\Phi_1(x)} = x$$

$$\phi_2 \quad x \log(x) - \beta x^2 = 0$$

$$\phi_3 \equiv \Phi_N$$

$$\times \log x = \beta x$$

$$\beta = -2$$

$$\Phi_2(x) \quad \frac{x \log x}{\beta} = x \quad \beta \neq 0$$

KO

Φ₂ OK

Φₙ OK

53

?  $x^{(0)}$

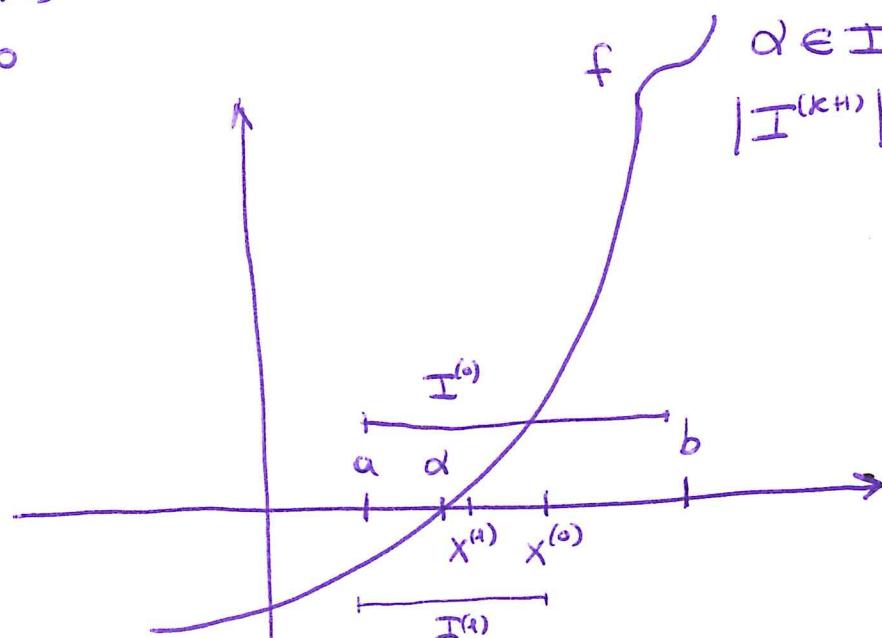
bisezione

$(a, b) \ni \alpha$

$I^{(0)}, I_1^{(1)}, \dots, I^{(k)}$

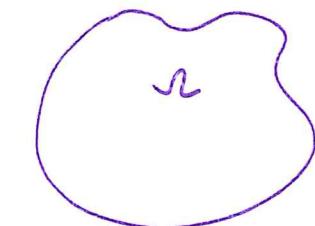
$I^{(0)} \supset I^{(1)} \supset I^{(2)} \supset \dots \supset I^{(k)}$  intervalli incapsulati

tp:  $f \in C^0([a, b])$   
 $f(a)f(b) < 0$



$$|I^{(k+1)}| = \frac{|I^{(k)}|}{2}$$

$f(x) = 0$   
?  $\alpha : f(\alpha) = 0$



$|R|$  = misura di  $R$

$I$

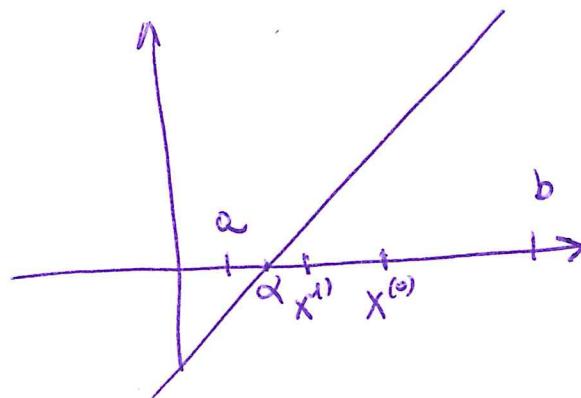
$|I|$  = lunghezza di  $I$

$$I^{(0)} = [a, b]; \quad a^{(0)} = a, \quad b^{(0)} = b; \quad x^{(0)} = \frac{a+b}{2}$$

$$I^{(1)} = [a^{(1)}, b^{(1)}]; \quad a^{(1)} = a^{(0)}, \quad b^{(1)} = x^{(0)}; \quad x^{(1)} = \frac{a^{(0)} + b^{(0)}}{2}$$

$$y = a_0 + a_1 x$$

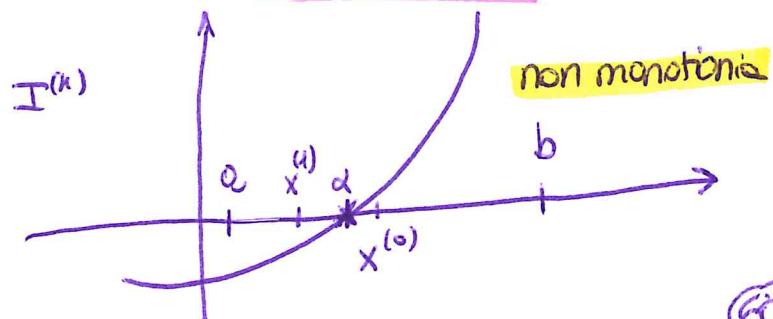
$$\alpha = -\frac{a_0}{a_1}$$



$\{x^{(k)}\}$   
punti medi  $I^{(k)}$

convergenza

incondizionata



non monotonia

bisezione  $(x^{(0)})$

PREDICTOR-CORRECTOR

punto fisso  
(Newton)

$$|e^{(k)}| = |\alpha - x^{(k)}| \leq \frac{1}{2} |I^{(k)}| = \frac{1}{2} \frac{|I^{(0)}|}{2^k} = \frac{b-a}{2^{k+1}} = (b-a) \left(\frac{1}{2}\right)^{k+1} \rightarrow 0$$

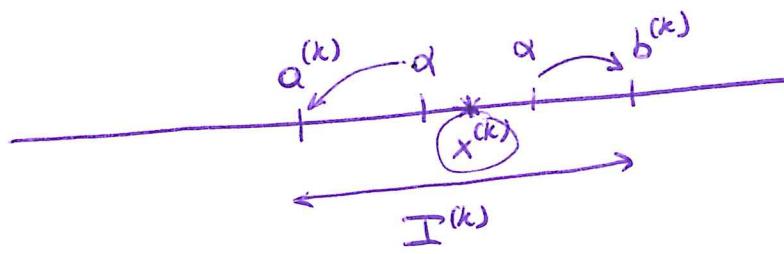
$k \rightarrow +\infty$

$|I^{(0)}|$

$$|I^{(1)}| = \frac{|I^{(0)}|}{2}$$

$$|I^{(2)}| = \frac{|I^{(1)}|}{2} = \frac{|I^{(0)}|}{2^2}$$

$$|I^{(k)}| = \frac{|I^{(0)}|}{2^k}$$



$$|e^{(k)}| \leq \boxed{(b-a) \left(\frac{1}{2}\right)^{k+1} < TOL^{20^{-5}}}$$

?  $k_{miu}$        $|e^{(k_{miu})}| < TOL$

$$\frac{(b-a)}{TOL} < 2^{k+1}$$

$$\log_2 \left( \frac{b-a}{TOL} \right) < k+1$$

$$\Downarrow k > \underbrace{\log_2 \left( \frac{b-a}{TOL} \right) - 1}_{N_{max}}$$

Teorema : Sia  $\alpha$  un punto fisso per  $\phi$ ;  $\Phi \in C^p(\bar{I}_\alpha)$ ,  $p \in \mathbb{N}$ . Se  $\phi^{(i)}(\alpha) = 0$   $1 \leq i \leq p-1$  e  $\phi^{(p)}(\alpha) \neq 0 \Rightarrow \exists \delta$  t.c.  $|x^{(0)} - \alpha| < \delta$   $\{x^{(k)}\}$  converge ad  $\alpha$  e

vale

$$\lim_{K \rightarrow +\infty} \frac{x^{(k+1)} - \alpha}{[x^{(k)} - \alpha]^p} = \underbrace{\frac{\Phi^{(p)}(\alpha)}{p!}}_C$$

ordine  $p$

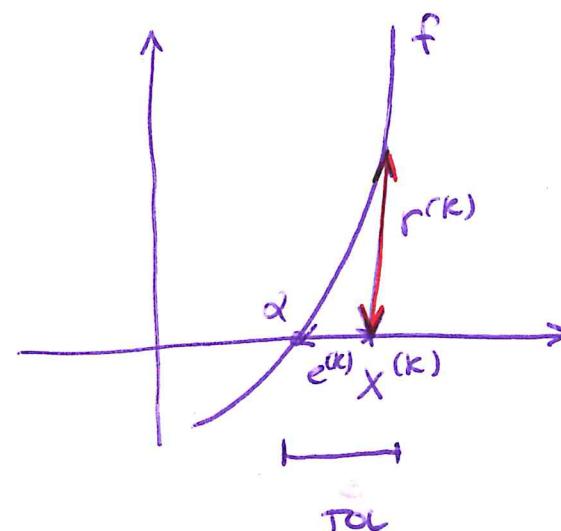
$[N_{MAX}]$

accuratezza  $\rightarrow$  stimatore  $\begin{cases} \text{residuo} \\ \text{incremento} \end{cases}$

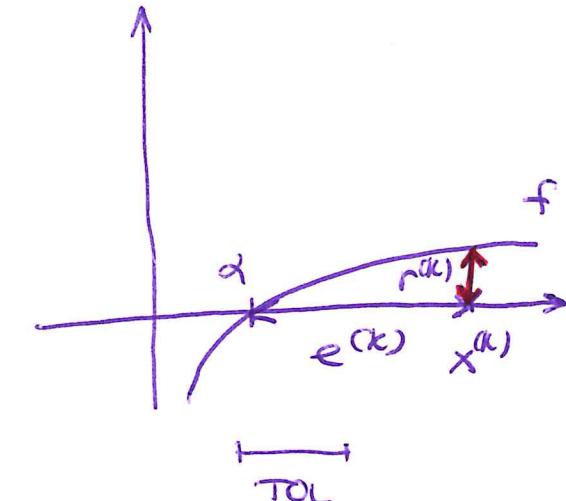
$$S_1 \quad r^{(k)} \quad x^{(k)} \quad \text{affidabilità di } S_1 \rightarrow |f'(\alpha)| \approx 1$$

$\parallel$

$$f(x^{(k)})$$



SOTTOSTRAZIONE



SOMMASTRAZIONE

$S_2$ 

$x^{(k+1)} - x^{(k)}$

affidabilità

$\alpha - x^{(k)}$

 $S_2$ 

$$\alpha - x^{(k)} = \underbrace{\alpha - x^{(k+1)}}_{S_2} + \underbrace{x^{(k+1)} - x^{(k)}}_{S_2}$$

$$\alpha - x^{(k+1)} = \phi(\alpha) - \phi(x^{(k)}) = \phi'(\beta_k) (\alpha - x^{(k)})$$

$$\alpha - x^{(k)} = \underbrace{\phi'(\beta_k) (\alpha - x^{(k)})}_{\leftarrow} + x^{(k+1)} - x^{(k)}$$

$$[1 - \phi'(\beta_k)] (\alpha - x^{(k)}) = x^{(k+1)} - x^{(k)}$$

$$\alpha - x^{(k)} = \underbrace{\frac{1}{1 - \phi'(\beta_k)}}_{\downarrow} (x^{(k+1)} - x^{(k)})$$

$k$  sufficientemente  
grande

$$\boxed{\frac{1}{1 - \phi'(\alpha)}}$$

$$\alpha \leftrightarrow$$

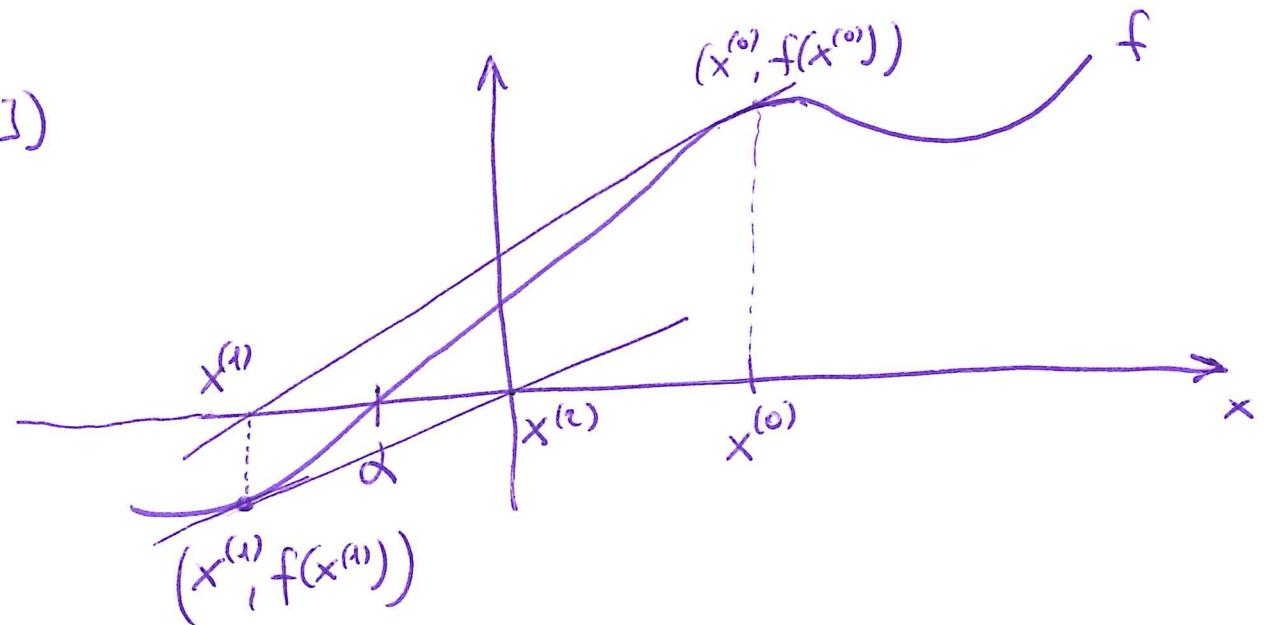
CASO IDEALE  $\phi'(\alpha) = 0$   $p \approx 2$   
APPROXIMABILITÀ  $\phi'(\alpha)$  vicino a zero

Newton (metodo delle tangenti)

$f, f'$

$f \in C^1([a,b])$

?  $\alpha$   $f(\alpha) = 0$   $f$  non lineare



$$P_k = (x^{(k)}, f(x^{(k)})) \quad x^{(k+1)} = "y=0" \cap \text{tg a f in } P_k$$

$$\begin{cases} y=0 \\ y(x) = f(x^{(k)}) + f'(x^{(k)})(x - x^{(k)}) \end{cases}$$

$$0 = f(x^{(k)}) + f'(x^{(k)})(x^{(k+1)} - x^{(k)})$$

$$x^{(k+1)} = x^{(k)} - \frac{f(x^{(k)})}{f'(x^{(k)})} \neq 0$$

Newton

$$f(x) = 0$$

$$\boxed{\phi_N(x) = x - \frac{f(x)}{f'(x)}}$$

$$x^{(k+1)} = \phi_N(x^{(k)})$$

Newton  $\simeq$  Taylor

$$f \in C^2([a,b]) \quad x^{(k)} \quad x^{(k+1)} \quad \text{II}^{\circ} \text{ ordine}$$

$$O = f(a) \underset{k \text{ suff. grande}}{\simeq} f(x^{(k+1)}) = f(x^{(k)}) + (x^{(k+1)} - x^{(k)}) f'(x^{(k)}) + O((x^{(k+1)} - x^{(k)})^2)$$

$$O = f(x^{(k)}) + (x^{(k+1)} - x^{(k)}) f'(x^{(k)})$$

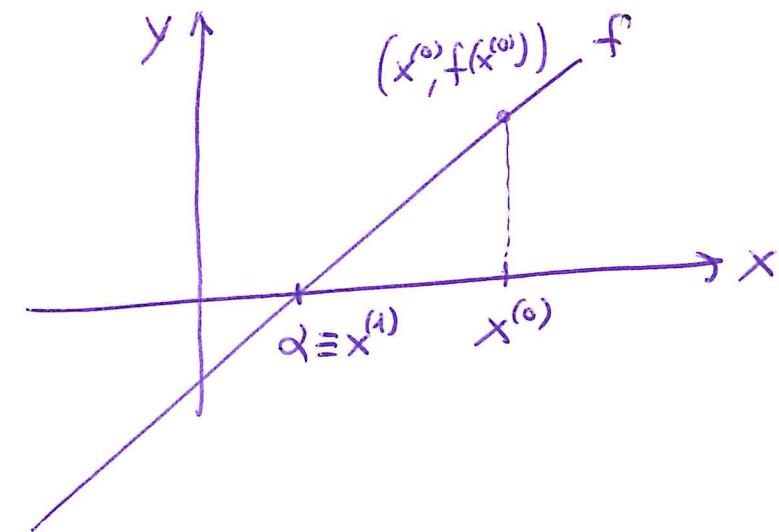
$$f(x) = a_0 + a_1 x$$

$$\alpha = -\frac{a_0}{a_1}$$

$$x^{(1)} = x^{(0)} - \frac{f(x^{(0)})}{f'(x^{(0)})}$$

$$= x^{(0)} - \frac{a_0 + a_1 x^{(0)}}{a_1}$$

$$= \frac{a_1 x^{(0)} - a_0 - a_1 x^{(0)}}{a_1} = \frac{-a_0}{a_1}$$



## Convergenza (Newton)

H1)  $x^{(0)}$  sufficientemente vicino ad  $\alpha$  (BISEZIONE/NEWTON)

H2)  $\alpha$  zero semplice di  $f$  [ $f(\alpha)=0$  e  $f'(\alpha) \neq 0$ ]

H3)  $f \in C^2([a,b])$

$$\Rightarrow \lim_{k \rightarrow \infty} \frac{x^{(k+1)} - \alpha}{[x^{(k)} - \alpha]^2} = \frac{f''(\alpha)}{2f'(\alpha)} \quad p=2$$

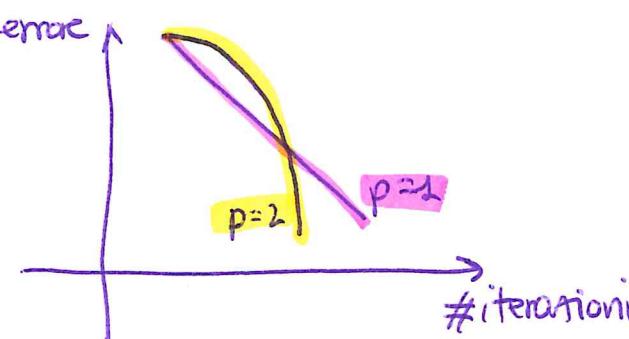
C  
(convergenza quadratica)

Newton modif cotto

$$x^{(k+1)} = x^{(k)} - \frac{f(x^{(k)})}{f'(x^{(k)})} \quad k \geq 0$$

$m$   
moeteguale  
 $\alpha$

$$f(x) = (x-1)\log(x) \quad \alpha=1 \quad m=2$$



H1) + H3) converge quadraticamente ( $p=2$ )

Newton converge

$(x-\alpha)^3$

## Scalare

$$f(x) = 0$$

$$x^{(k+1)} = x^{(k)} - \frac{f(x^{(k)})}{f'(x^{(k)})}$$

$$x^{(k+1)} = x^{(k)} + \delta x^{(k)} \text{ dove } \delta x^{(k)} = -\frac{f(x^{(k)})}{f'(x^{(k)})}$$

$$f'(x^{(k)}) \delta x^{(k)} = -f(x^{(k)})$$

## Vettoriale

$$\begin{cases} f_1(x_1, x_2, \dots, x_n) = 0 \\ f_2(x_1, x_2, \dots, x_n) = 0 \\ \vdots \\ f_n(x_1, x_2, \dots, x_n) = 0 \end{cases}$$

$$\vec{x}^T \in \mathbb{R}^n \ni 0$$

$$\vec{f} = [f_1, f_2, \dots, f_n]^T$$

$$\boxed{\vec{f}(\vec{x}) = 0}$$

$$\vec{x}^{(k)} = (x_1^{(k)}, x_2^{(k)}, \dots, x_n^{(k)})^T \in \mathbb{R}^n$$

$$\vec{x}^{(k+1)} = (x_1^{(k+1)}, x_2^{(k+1)}, \dots, x_n^{(k+1)})^T \in \mathbb{R}^n$$

$$\vec{x}^{(k+1)} = \vec{x}^{(k)} + \vec{\delta x}^{(k)}$$

$$(J_F)_{ij}(\vec{x}) = \frac{\partial f_i}{\partial x_j}(\vec{x}) \quad i, j = 1, \dots, n$$

$$J_F(\vec{x}^{(k)}) \boxed{\vec{\delta x}^{(k)}} = -\vec{f}(\vec{x}^{(k)})$$

$$\underbrace{A}_{A} \quad \underbrace{x}_{x} \quad \underbrace{b}_{b}$$

Es  $n=2$

$$\begin{cases} x_1^2 + x_2^2 - \pi = 0 \\ \tan\left(\frac{\pi}{3}x_1\right) + \sqrt{x_2} + 1 = 0 \end{cases}$$

## Approssimazione di dati e funzioni

$f \in C^0([a,b])$

$(t_i, m_i) \quad i=0, \dots, n$

$f$

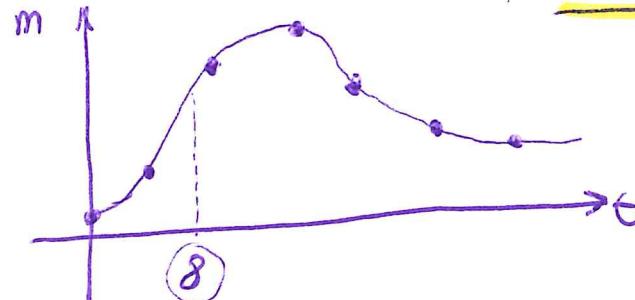
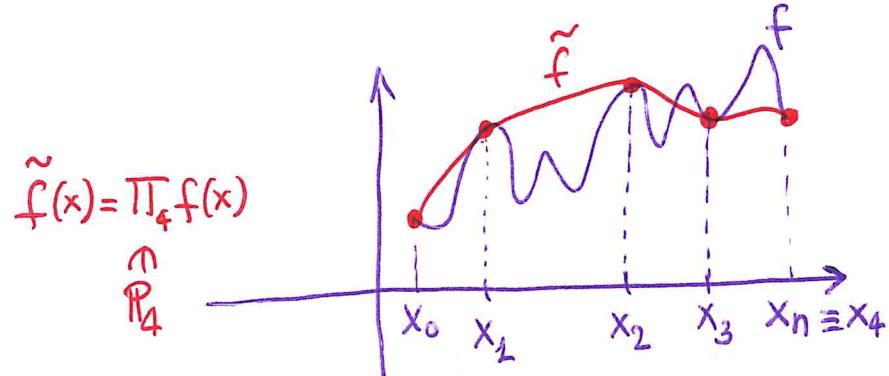
$$I(f) = \int_a^b f(x) dx \simeq \int_a^b \tilde{f}(x) dx$$

Taylor

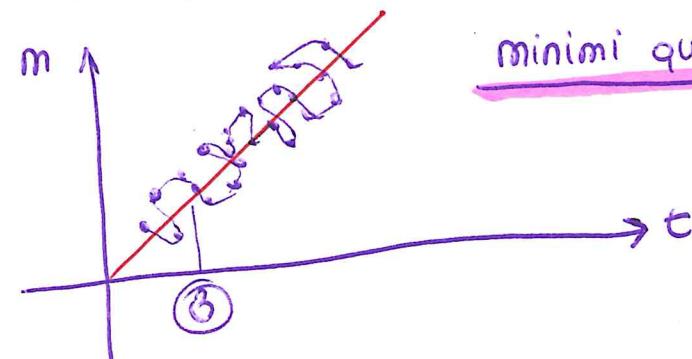
$f(x) \quad x_0$

$\frac{1}{x} \quad x_0 = 1$

$e^x \quad x_0 = 0$



interpolazione



minimi quadrati

Interpolazione

$$\left\{ (x_i, y_i) \right\}_{i=0}^n \quad (n+1)$$

nodi valori

$x_i \neq$

?  $\tilde{f}$  t.c.

$$\tilde{f}(x_i) = y_i \quad i=0, \dots, n$$

condizioni di interpolazione

dati  $y_i = \text{misurazione}$   
 $x_i = t_i$

funzione  $y_i = f(x_i)$

$$\begin{array}{ll}
 \tilde{f} = \begin{cases} \text{polinomio} & f \in P_n = \{ p_n(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n; a_i \in \mathbb{R} \} \\
 \text{trigonometrica} & \hat{f}(x) = \sum_k a_k e^{ikx} \quad (\text{Fourier}) \\
 \text{razionali} & \hat{f}(x) = \frac{p_k(x)}{q_s(x)} = \frac{a_0 + a_1 x + \dots + a_k x^k}{b_0 + b_1 x + \dots + b_s x^s} \quad k, s \in \mathbb{N} \\
 & a_i, b_i \in \mathbb{R} \end{cases}
 \end{array}$$

## Interpolazione polinomiale Lagrangiana

Proposizione:  $\{(x_i, y_i)\}_{i=0}^n$  con  $x_i \neq x_j$ .  $\exists!$  <sup>Th</sup> polinomio di grado  $\leq n$  t.c.  $\pi_n(x_i) = y_i$   
 $\pi_n$  polinomio interpolatore

$$\begin{cases}
 \tilde{f} = \pi_n & \text{dati } y_i = m_i \\
 \tilde{f} = \pi_n f & \text{funzione } y_i = f(x_i) \quad f \in C^0([a, b])
 \end{cases}$$

Dim: unicità

$$\pi_n \in P_n \quad \pi_n(x_i) = y_i \quad i=0, \dots, n$$

$$\pi_n^* \in P_n \quad \pi_n^*(x_i) = y_i \quad i=0, \dots, n$$

$$M_n(x) = \pi_n(x) - \pi_n^*(x) \in P_n$$

$$M_n(x_i) = \pi_n(x_i) - \pi_n^*(x_i) = y_i - y_i = 0 \quad i=0, \dots, n$$

$M_n \in P_n$  ha  $(n+1)$  zeri  $\rightarrow$

$$M_n(x) = 0 \quad \forall x \in \mathbb{R}$$

$$\pi_n(x) - \pi_n^*(x) = 0 \quad \forall x \in \mathbb{R}$$

$$\pi_n(x) = \pi_n^*(x) \quad \boxed{\text{ASSURDO}}$$

3

Th

$$\{(x_i, y_i)\}_{i=0}^2$$

$$x_0 = 0$$

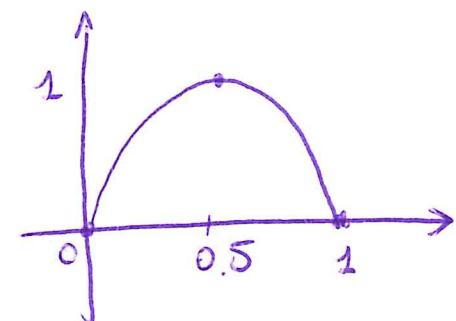
$$x_1 = 0.5$$

$$x_2 = 1$$

$$y_0 = 0$$

$$y_1 = 1$$

$$y_2 = 0$$



$$\varphi_1 \in P_2$$

$$\boxed{\text{I}} \quad \varphi_1(0) = 0$$

$$\boxed{\text{II}} \quad \varphi_1(0.5) = 1$$

$$\boxed{\text{III}} \quad \varphi_1(1) = 0$$

condizioni  
di interpolazione

$$\boxed{\text{I}} \text{ e } \boxed{\text{III}}$$

$$\begin{matrix} a \in \mathbb{R} \\ (x-a) \end{matrix}$$

$$x(x-1)$$

$$(x-0)(x-1) \in P_2$$

$$\boxed{\text{II}}$$

$$\frac{(x-0)(x-1)}{(0.5-0)(0.5-1)} = \frac{x(x-1)}{\frac{1}{2}(-\frac{1}{2})} = -4(x-1)x = -4(x^2-x)$$

$$x_0 \quad x_1 \quad \dots \quad x_{k-1} \quad x_k \quad x_{k+1} \quad \dots \quad x_n$$

$$y_0 = 0 \quad y_1 = 0 \quad \dots \quad y_{k-1} = 0 \quad y_k = 1 \quad y_{k+1} = 0 \quad \dots \quad y_n = 0$$

→ polinomi  
di Lagrange

$$\varphi_k(x) = \sum_{j=0}^n \underbrace{\delta_{jk}}_{\downarrow \text{delta di Kronecker}} : P_n \ni \varphi_k(x) = \frac{(x-x_0)(x-x_1) \dots (x-x_{k-1})(x-x_{k+1}) \dots (x-x_n)}{(x_k-x_0)(x_k-x_1) \dots (x_k-x_{k-1})(x_k-x_{k+1}) \dots (x_k-x_n)} = \prod_{\substack{j=0 \\ j \neq k}}^n \frac{(x-x_j)}{(x_k-x_j)}$$

delta di Kronecker

$$\chi_{\omega}(x) = \begin{cases} 0 & x \notin \omega \\ 1 & x \in \omega \end{cases}$$

$$\begin{array}{c} x_0 \\ y_0 \\ \downarrow \varphi_0, \end{array} \quad \begin{array}{c} x_1 \\ y_1 \\ \downarrow \varphi_1, \end{array} \quad \begin{array}{c} x_2 \\ y_2 \\ \downarrow \varphi_2, \end{array} \quad \varphi_0, \varphi_1, \varphi_2 \in P_2$$

$$\Pi_2 \in P_2 \quad \Pi_2(x_0) = y_0$$

$$\Pi_2(x_1) = y_1$$

$$\Pi_2(x_2) = y_2$$

$$\varphi_0(x_0) = 1 \quad \varphi_0(x_1) = 0 \quad \varphi_0(x_2) = 0$$

$$\varphi_1(x_0) = 0 \quad \varphi_1(x_1) = 1 \quad \varphi_1(x_2) = 0$$

$$\varphi_2(x_0) = 0 \quad \varphi_2(x_1) = 0 \quad \varphi_2(x_2) = 1$$

$$\Pi_2(x) = a\varphi_0(x) + b\varphi_1(x) + c\varphi_2(x)$$

$$? \quad a, b, c \in \mathbb{R}$$

$$\Pi_2 \in P_2$$

$$\Pi_2(x_0) = \underbrace{a\varphi_0(x_0)}_1 + \underbrace{b\varphi_1(x_0)}_0 + \underbrace{c\varphi_2(x_0)}_0 = y_0$$

$$a = y_0$$

$$\Pi_2(x) = y_0 \varphi_0(x) + y_1 \varphi_1(x) + y_2 \varphi_2(x)$$

$$\Pi_2(x_1) = \underbrace{a\varphi_0(x_1)}_0 + \underbrace{b\varphi_1(x_1)}_1 + \underbrace{c\varphi_2(x_1)}_0 = y_1$$

$$b = y_1$$

$$\Pi_2(x_2) = \underbrace{a\varphi_0(x_2)}_0 + \underbrace{b\varphi_1(x_2)}_0 + \underbrace{c\varphi_2(x_2)}_1 = y_2$$

$$c = y_2$$

$$\begin{matrix} x_0 & x_1 & \dots & x_n \\ y_0 & y_1 & \dots & y_n \end{matrix}$$

$$\prod_{k=0}^n (x - x_k) = y_0 \varphi_0(x) + y_1 \varphi_1(x) + \dots + y_n \varphi_n(x)$$

$$P_n(x) = \sum_{k=0}^n y_k \varphi_k(x)$$

$$\varphi_k(x) = \prod_{\substack{j=0 \\ j \neq k}}^n \frac{x - x_j}{x_k - x_j}$$

forma di Lagrange delle polinomi interpolatori

$$[ax+by, a, b \in \mathbb{R}]$$

$$\prod_{k=0}^n (x_i - x_k) = ?$$

$$\prod_{k=0}^n y_k \varphi_k(x_i) = y_i \underbrace{\varphi_i(x_i)}_{=1} = y_i$$

polyfit  
polyval

$c = \text{polyfit}(x, y, n)$   
 $\in \mathbb{R}^{n+1}$        $\{x_i\}$        $\{y_i\}$       grado

$$c(1)x^n + c(2)x^{n-1} + \dots + c(n)x + c(n+1) = \prod_{k=0}^n (x - x_k)$$

$$d = \text{polyval}(c, z)$$

$$\in \mathbb{R} \quad \in \mathbb{R}^q \quad \in \mathbb{R} \quad \in \mathbb{R}^q$$

$$\prod_{k=0}^n (z - z_k) = d$$

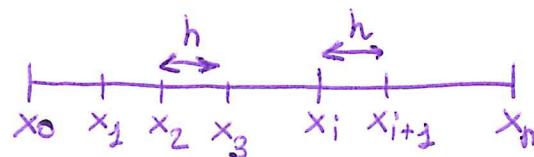
$$q=2 \quad z = [z_1, z_2]^T \quad d = \begin{bmatrix} \prod_{k=0}^n (z_1 - z_k) \\ \prod_{k=0}^n (z_2 - z_k) \end{bmatrix}$$

Errore: Sia  $I = [x_0, x_n]$ ;  $\{f(x_i, y_i)\}_{i=0}^n$   $x_i$  distinti. se  $f \in C^{n+1}(I)$ , allora  $\forall x \in I$ ,  $\exists \alpha = \alpha(x) \in I$  t.c.

$$E_{nf}(x) = f(x) - \underbrace{\tilde{f}}_{\substack{\downarrow \\ \text{errore} \\ \text{di interpolazione}}}(x) = \frac{f^{(n+1)}(\alpha(x))}{(n+1)!} \prod_{i=0}^n (x-x_i)$$

$$E_{nf}(x_i) = 0, i=0, \dots, n$$

$x_i$  uniformi



$$h = \frac{x_n - x_0}{n} = \frac{|I|}{n} \quad \text{passo di discretizzazione}$$

$$x_k = x_{k-1} + h \quad k=1, \dots, n$$

Se  $\{x_i\}$  sono uniformi

$$\left| \prod_{i=0}^n (x-x_i) \right| \leq n! \frac{h^{n+1}}{4} \quad \forall x \in I$$

$$[x_k = x_0 + kh \quad k=1, \dots, n]$$

$$\max_{x \in I} |E_{nf}(x)| \leq \max_{x \in I} \frac{|f^{(n+1)}(x)|}{(n+1)!} \left| \prod_{i=0}^n (x-x_i) \right| \leq \max_{x \in I} \frac{|f^{(n+1)}(x)|}{(n+1)!} \cancel{n!} \frac{h^{n+1}}{4}$$

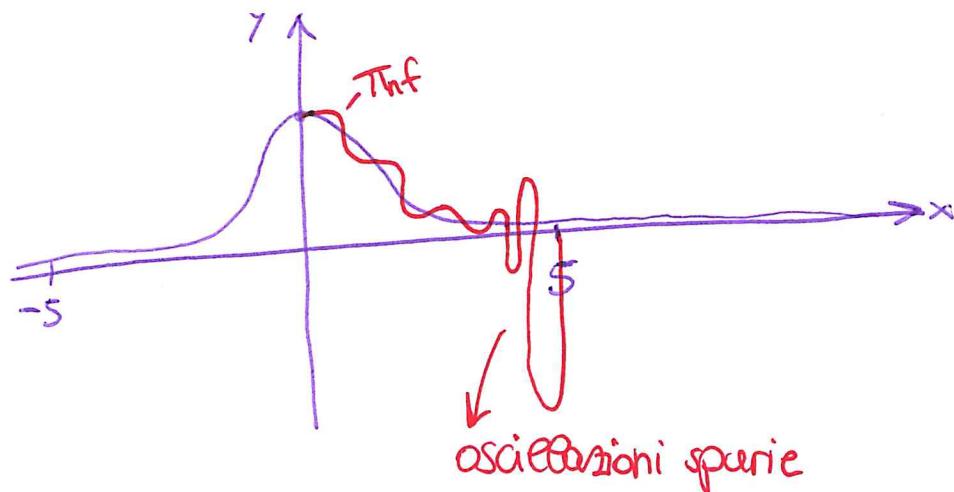
$$\max_{x \in I} |E_{nf}(x)| \leq \boxed{\frac{h^{n+1}}{4(n+1)}} \boxed{\max_{x \in I} |f^{(n+1)}(x)|}$$

$\downarrow n \rightarrow +\infty$

$0$	$0$	$C$	$+\infty$
ok	ok	ok	ok

(73)

$$f(x) = \frac{1}{1+x^2} \quad [-5, 5]$$



	$\boxed{1}$	$\boxed{2}$	$\boxed{2} \cdot \boxed{2}$
$n=3$	$O(10)$	$O(10^0)$	$O(10^1)$
$n=9$	$O(10^{-2})$	$O(10^5)$	$O(10^4)$
$n=15$	$O(10^{-5})$	$O(10^{12})$	$O(10^7)$
$n=21$	$O(10^{-10})$	$O(10^{18})$	$O(10^{10})$
	$\downarrow n \rightarrow +\infty$	$\downarrow n \rightarrow +\infty$	$\downarrow n \rightarrow +\infty$
	0	$+\infty$	$+\infty$

FENOMENO DI  
RUNGE

Soluzioni al fenomeno di Runge

① nodi di Chebyshev ; nodi di Chebyshev - Gauss - Lobatto

$$\mathcal{I} = [a, b]$$

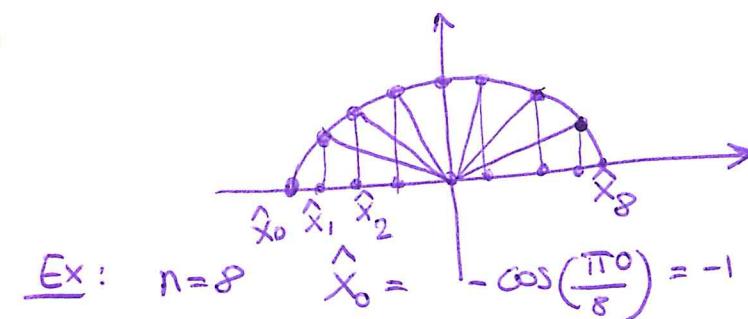
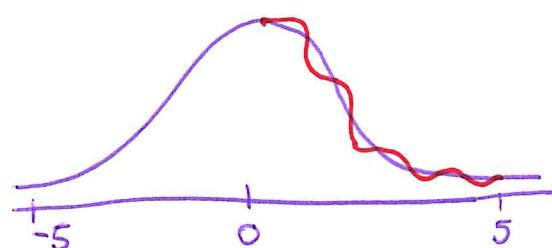
$$x_i = \frac{b+a}{2} + \frac{b-a}{2} \hat{x}_i$$

$[a, b]$

$$\hat{x}_i = -\cos\left(\frac{\pi i}{n}\right) \quad i=0, \dots, n$$

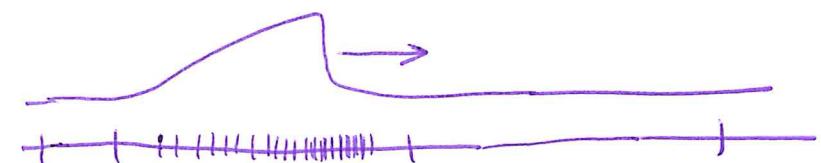
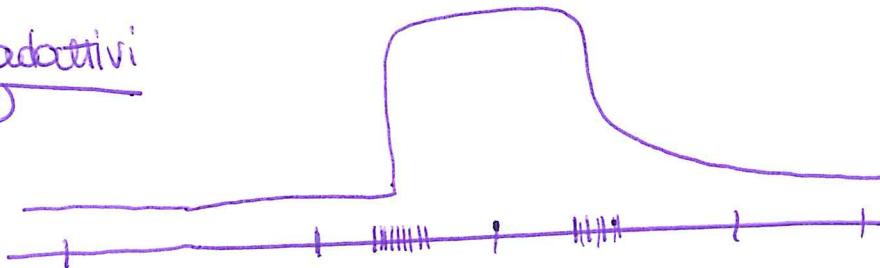
$[1, 1]$

Se  $f \in C^2([a, b]) \Rightarrow T_{nf} \rightarrow f \quad n \rightarrow +\infty$



$$\begin{aligned} \gamma_1 &= -\omega \left( \frac{\pi}{8} \right) \\ \hat{x}_2 &= -\cos \left( \frac{\pi}{4} \right) \\ \vdots \\ \hat{x}_8 &= -\cos(\pi) \end{aligned}$$

② Nodi adattivi (plot)



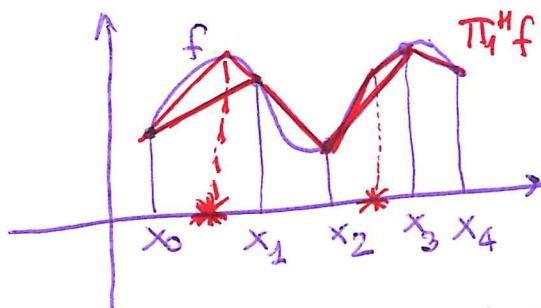
③ Interpolazione Lineare a tratti

$$(\Pi_1^H f) \in C^0(\bar{I}), I = [x_0, x_n]$$

$$\Pi_1^H f \Big|_{I_i} \in P_1(I_i), \quad \Pi_1^H f(x_i) = y_i \quad i = 0, \dots, n$$

$$\Pi_1^H f(x) \Big|_{I_i} = f(x_i) + \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} (x - x_i)$$

$$I_i = (x_i, f(x_i)), (x_{i+1}, f(x_{i+1}))$$

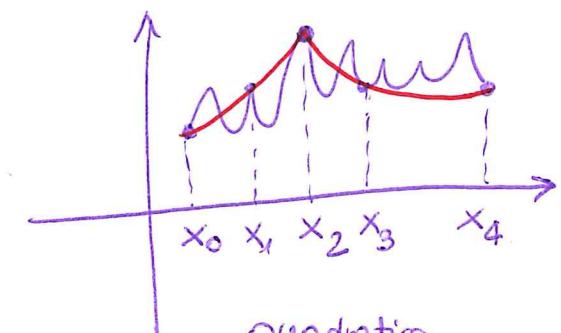


$x_i$  non necessariamente uniformi

$$I_i = [x_i, x_{i+1}] \quad i = 0, \dots, n-1$$

$$h_i = |I_i| = x_{i+1} - x_i$$

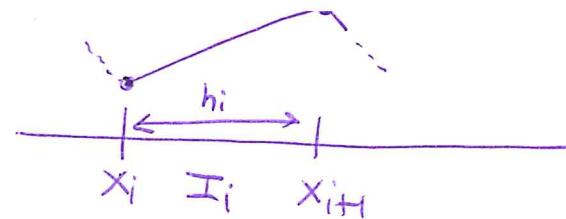
$$H = \max_i h_i$$



quadratici  
a tratti

$$\Pi_2^H f$$

$$I: \max_{x \in I_i} |E_n f(x)|_{I_i} \stackrel{n=1}{\leq} \frac{h_i^2}{8} \max_{x \in I_i} |f''(x)|$$



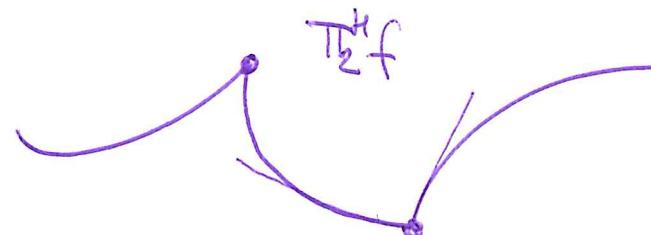
$$I: \max_{x \in I} |E_n f(x)| \leq \frac{H^2}{8} \max_{x \in I} |f''(x)| \quad f \in C^2(\bar{I})$$

$\downarrow n \rightarrow +\infty$  costante

EF (elementi finiti)

$$d = \text{interp1}(x, y, z)$$

$R \downarrow R^9$   
 nodi valori  
 $R^9 \nearrow R^9$



$$n=2 \quad \frac{H^3}{12} \max_{x \in I} |f'''(x)|$$

$f \in C^3(\bar{I})$

④ CG CAD



$C^1$   
 $C^2$

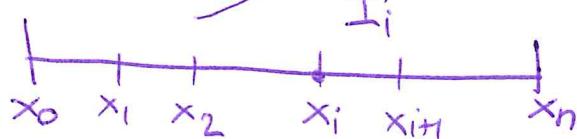
Interpolazione cubica spline

$s_3$  t.c.  $s_3|_{I_i} \in P_3(I_i)$

A  $\Rightarrow s_3(x_i) = f(x_i) \quad i=0, \dots, n$

B  $s_3 \in C^2(\bar{I})$   
(smooth)

$I_i$ : 4 incognite  
4n incognite



A

$n+1$  condizioni

$$\boxed{B} \quad s_3 \in C^2(\bar{\mathbb{I}}) \quad \begin{array}{l} \xrightarrow{s_3 \in C^0(\bar{\mathbb{I}})} \boxed{B1} \\ \xrightarrow{s_3' \in C^0(\bar{\mathbb{I}})} \boxed{B2} \\ \xrightarrow{s_3'' \in C^0(\bar{\mathbb{I}})} \boxed{B3} \end{array}$$

$$\boxed{B1} \quad s_3^-(x_i) = s_3^+(x_i) \quad i=1, \dots, n-1 \quad (n-1) \text{ condizioni}$$

$$\boxed{B2} \quad [s_3'(x_i)]^- = [s_3'(x_i)]^+ \quad i=1, \dots, n-1 \quad (n-1) \text{ condizioni}$$

$$\boxed{B3} \quad [s_3''(x_i)]^- = [s_3''(x_i)]^+ \quad i=1, \dots, n-1 \quad (n-1) \text{ condizioni}$$

$$(n+1) + 3(n-1) = 4n-2 \text{ condizioni}$$

4n incognite

MANCANO 2 condizioni

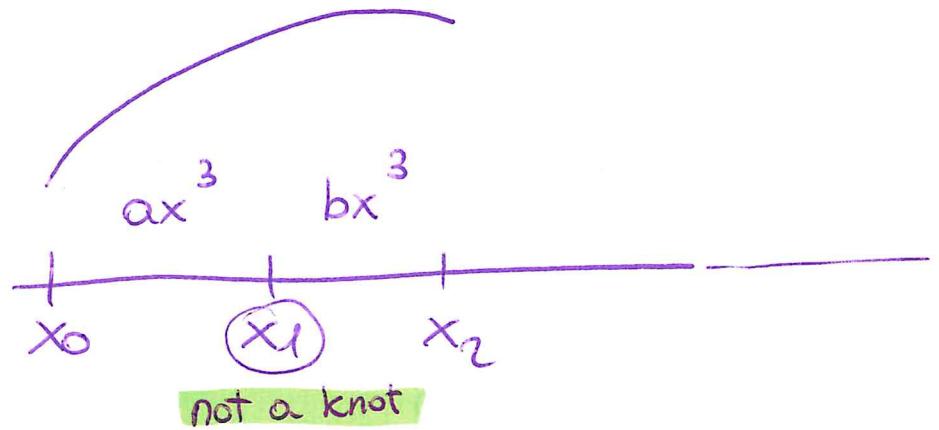
$$1) s_3''(x_0) = s_3''(x_n) = 0 \quad (\text{naturale})$$

$$2) s_3''' \text{ continua in } x_1 \text{ e } x_{n-1} \quad (\text{not-a-knot})$$

co.

$$\left. \begin{array}{l} R \\ R^q \end{array} \right\} \xrightarrow{d} d = \text{spline}(x, y, z)$$

$\in \mathbb{R} \quad \in \mathbb{R}^q$



$$[S_3^{(0)}(x_1)]^- = [S_3^{(0)}(x_1)]^+ \rightarrow a = b$$

$$S_3^{(1)} \quad 3ax^2 \quad 3bx^2$$

$$S_3^{(2)} \quad 6ax \quad 6bx$$

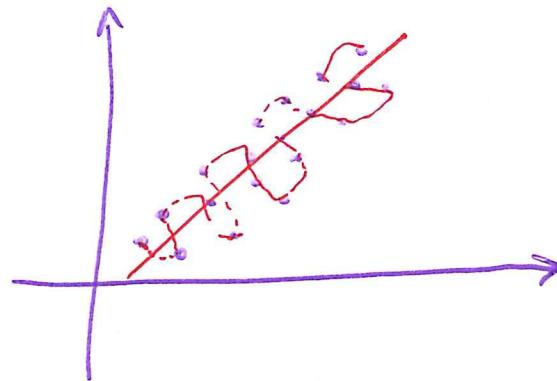
$$S_3^{(3)} \quad 6a \quad 6b$$

$S_3 \simeq f$	$H^4$	$C_0$
$S_3^{(1)} \simeq f'$	$H^3$	$C_1$
$S_3^{(2)} \simeq f''$	$H^2$	$C_2$

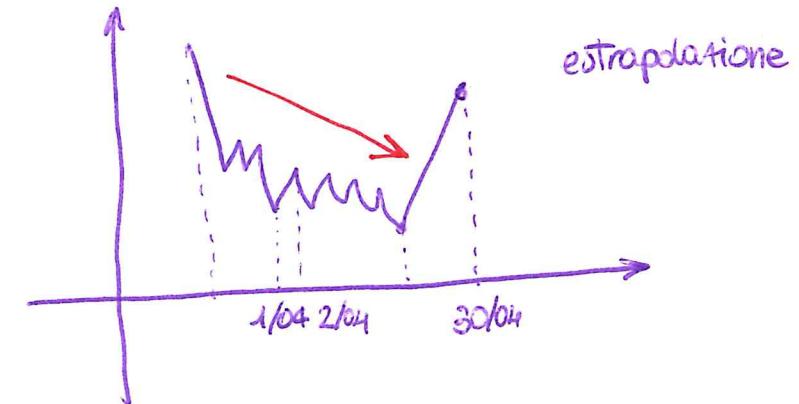
$$\max_{x \in I} |S_3^{(q)}(x) - f^{(q)}(x)| \leq C_q H^{4-q} \max_{x \in I} |f^{(q)}(x)|, \quad q=0,1,2$$

$$f \in C^4(\bar{I})$$

## Minimi quadrati



$$\{(x_i, y_i)\}_{i=0}^n$$



interpolazione

$$\{(x_i, y_i)\}_{i=0}^n \quad x_i \neq \\ f(x_i)$$

?  $\hat{f} \in P_m$        $m \leq n$       t.c.

$$\sum_{i=0}^n [y_i - \hat{f}(x_i)]^2 \leq \sum_{i=0}^n [y_i - p_m(x_i)]^2 \quad p_m \in P_m$$

m scelto da utente

• DATA FITTING

• Se  $\hat{f} \exists$ ,  $\hat{f}$  approssimazione nel senso dei minimi quadrati  
( $m=1$ ,  $\hat{f}$  retta di regressione)

•  $m=n$ ,  $\hat{f} \equiv T_n f$

• rinuncia alle  $(n+1)$  condizioni  $\hat{f}(x_i) = y_i \quad i=0, \dots, n$

$$? \tilde{f}(x) = a_0 + a_1 x + \dots + a_m x^m$$

$$p_m(x) = b_0 + b_1 x + \dots + b_m x^m$$

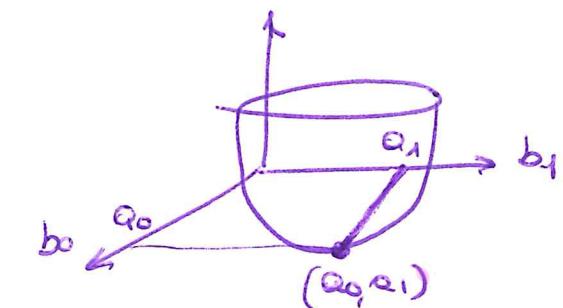
?  $a_0, a_1, \dots, a_m$

$$\Phi(b_0, b_1, \dots, b_m) = \sum_{i=0}^n [y_i - (b_0 + b_1 x_i + b_2 x_i^2 + \dots + b_m x_i^m)]^2 \quad \textcircled{D}$$

$$\Phi(a_0, a_1, \dots, a_m) = \sum_{i=0}^n [y_i - (a_0 + a_1 x_i + a_2 x_i^2 + \dots + a_m x_i^m)]^2 \quad \textcircled{C}$$

$$\boxed{\Phi(a_0, a_1, \dots, a_m) = \min_{b_0, b_1, \dots, b_m} \Phi(b_0, b_1, \dots, b_m)}$$

$$m=1 \quad \tilde{f}(x) = a_0 + a_1 x \quad ? a_0, a_1 \quad p_1(x) = b_0 + b_1 x$$



$$\Phi(a_0, a_1) = \min_{b_0, b_1} \Phi(b_0, b_1) \quad \text{con} \quad \Phi(b_0, b_1) = \sum_{i=0}^n [y_i - b_0 - b_1 x_i]^2$$

$$= \sum_{i=0}^n [y_i^2 + b_0^2 + b_1^2 x_i^2 - 2y_i b_0 - 2y_i b_1 x_i + 2b_0 b_1 x_i]$$

$$\left\{ \begin{array}{l} \frac{\partial \Phi}{\partial b_0} (a_0, a_1) = 0 \quad (1) \\ \frac{\partial \Phi}{\partial b_1} (a_0, a_1) = 0 \quad (2) \end{array} \right. \rightarrow \frac{\partial \Phi}{\partial b_0} (b_0, b_1) = \sum_{i=0}^n [2b_0 - 2y_i + 2b_1 x_i]$$

$$\frac{\partial \Phi}{\partial b_0} (a_0, a_1) = \sum_{i=0}^n [2a_0 - 2y_i + 2a_1 x_i] = 0$$

$$\sum_{i=0}^n [a_0 - y_i + a_1 x_i] = 0 \quad (1)$$

$$\frac{\partial \Phi}{\partial b_1} (b_0, b_1) = \sum_{i=0}^n [2b_1 x_i^2 - 2y_i x_i + 2b_0 x_i]$$

$$\frac{\partial \Phi}{\partial b_0} (a_0, a_1) = \sum_{i=0}^n [2a_1 x_i^2 - 2y_i x_i + 2a_0 x_i] = 0$$

$$\sum_{i=0}^n [a_1 x_i^2 - y_i x_i + a_0 x_i] = 0 \quad (2)$$

$$\left\{ \begin{array}{l} \sum_{i=0}^n [a_0 - y_i + a_1 x_i] = 0 \\ \sum_{i=0}^n [a_1 x_i^2 - y_i x_i + a_0 x_i] = 0 \end{array} \right. \quad (1)$$

$$\left\{ \begin{array}{l} a_0 \begin{bmatrix} n \\ i=0 \end{bmatrix}_{(n+1)} + a_1 \begin{bmatrix} n \\ i=0 \end{bmatrix}_{x_i} = \sum_{i=0}^n y_i \\ a_0 \begin{bmatrix} n \\ i=0 \end{bmatrix}_{B_{21}} + a_1 \begin{bmatrix} n \\ i=0 \end{bmatrix}_{B_{22}} = \sum_{i=0}^n x_i y_i \end{array} \right.$$

$\hat{f} \in P_m$  ?  $a_0, a_1, \dots, a_m$

$$\left\{ \begin{array}{l} a_0(n+1) + a_1 \sum x_i + a_2 \sum x_i^2 + \dots + a_m \sum x_i^m = \sum y_i \\ a_0 \sum x_i + a_1 \sum x_i^2 + a_2 \sum x_i^3 + \dots + a_m \sum x_i^{m+1} = \sum x_i y_i \\ \vdots \\ a_0 \sum x_i^m + a_1 \sum x_i^{m+1} + a_2 \sum x_i^{m+2} + \dots + a_m \sum x_i^{2m} = \sum x_i^m y_i \end{array} \right.$$

simmetrico

equazioni normali

$c = \text{polyfit}(x, y, m)$   
 $d = \text{polyval}(c, z)$

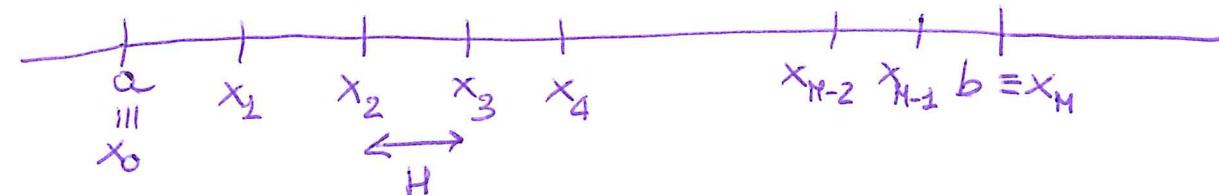
## Approssimazione di integrali definiti

$$I(f) = \int_a^b f(x) dx \quad f \in C^0([a,b])$$

I  
↓

$$\tilde{I}(f) = \int_a^b \tilde{f}(x) dx \quad \text{formula di quadratura}$$

Pq



$$H = \frac{b-a}{N} \quad ; \quad I_k = [x_{k-1}, x_k] ; \quad x_k = x_{k-1} + H \quad \text{partizione uniforme}$$

$$x_k = x_0 + kH$$

$$I(f) = \int_a^b f(x) dx = \sum_{k=1}^N \int_{I_k} f(x) dx \underset{\approx}{=} \sum_{k=1}^N \int_{I_k} \tilde{f}(x) dx = \tilde{I}(f)$$

$$\tilde{f}(x) \in P_q(I_k)$$

q = 0 rettangolo

q = 1 trapezio

q = 2 Simpson



$$\bar{x}_k = \frac{x_{k-1} + x_k}{2}; I_k = [x_{k-1}, x_k]$$

$$\tilde{I}_{PH}^c(f) = \sum_{k=1}^n H f(\bar{x}_k) = H \sum_{k=1}^n f(\bar{x}_k)$$

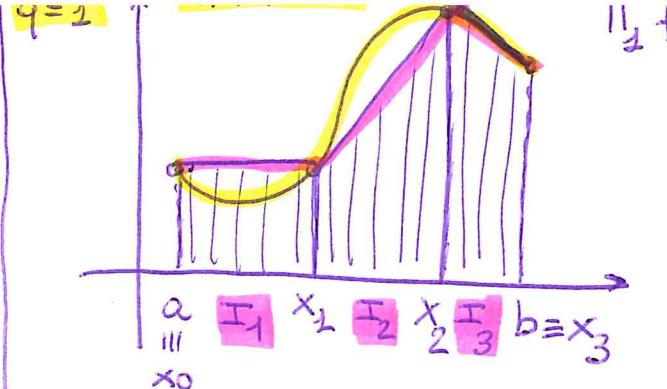
$$\tilde{I}_{PH}(f) = (b-a) f\left(\frac{a+b}{2}\right)$$

$$I(f) - \tilde{I}_{PH}(f) = \frac{1}{24} (b-a)^3 f''(\beta) \quad f \in C^2([a,b])$$

$\alpha, \beta \in [a,b]$

$$I(f) - \hat{I}_{PH}^c(f) = \frac{1}{24} (b-a) H^2 f''(\alpha) \quad f \in C^2([a,b])$$

oda: 2  
gde: 1



$$\tilde{I}_T^c(f) = \frac{H}{2} \sum_{k=1}^n [f(x_{k-1}) + f(x_k)]$$

$$= \frac{H}{2} [f(a) + f(b)] + H \sum_{k=1}^{n-1} f(x_k)$$

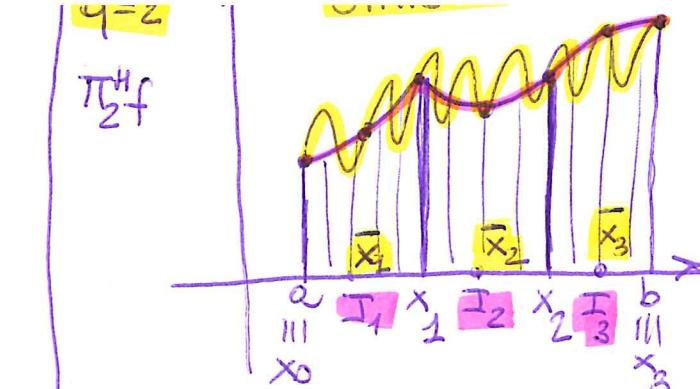
$$\tilde{I}_T(f) = \frac{(b-a)}{2} [f(a) + f(b)]$$

$$I(f) - \tilde{I}_T^c(f) = -\frac{1}{12} (b-a)^3 f''(\sigma)$$

$\sigma \in [\alpha, \beta]$   
 $f \in C^2([\alpha, \beta])$

$$I(f) - \tilde{I}_T(f) = -\frac{1}{12} (b-a)^2 f''(\delta)$$

oda: 2  
gde: 1



$$\tilde{I}_S^c(f) = \frac{H}{6} \sum_{k=1}^n [f(x_{k-1}) + 4f(\bar{x}_k) + f(x_k)]$$

$$= \frac{H}{6} [f(a) + f(b)] + \frac{H}{3} \sum_{k=1}^{n-1} f(x_k) + \frac{2}{3} H \sum_{k=1}^n f(\bar{x}_k)$$

$$\tilde{I}_S(f) = \frac{(b-a)}{6} [f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)]$$

$$I(f) - \tilde{I}_S^c(f) = -\frac{1}{2880} (b-a)^5 f^{(4)}(\nu)$$

$\nu \in [\alpha, \beta]$   
 $f \in C^4([\alpha, \beta])$

$$I(f) - \tilde{I}_S(f) = -\frac{1}{2880} (b-a) H^4 f^{(4)}(\delta)$$

oda: 4  
gde: 3

Errore punto medio

$$f(\bar{x})(b-a)$$

$$\underbrace{I(f) - \tilde{I}_{PM}(f)}_{E_{PM}(f)} = \int_a^b f(x) dx - \overline{\int_a^b f(\bar{x}) dx} = \int_a^b [f(x) - f(\bar{x})] dx = \textcircled{*}$$

$$\bar{x} = \frac{a+b}{2}$$

Taylor  $f \in C^2([a,b])$   $f(x) = f(\bar{x}) + f'(\bar{x})(x-\bar{x}) + \frac{f''(\alpha(x))}{2} (x-\bar{x})^2$

$$\textcircled{*} = \int_a^b \cancel{f'(\bar{x})(x-\bar{x})} dx + \frac{1}{2} \int_a^b f''(\alpha(x))(x-\bar{x})^2 dx = \frac{1}{2} f''(\beta) \int_a^b (x-\bar{x})^2 dx = \frac{1}{2} f''(\beta) \cdot \frac{1}{12} (b-a)^3$$

$$\int_a^b (x-\bar{x}) dx = \frac{(x-\bar{x})^2}{2} \Big|_a^b = \frac{1}{2} \left[ \left( b - \frac{a+b}{2} \right)^2 - \left( a - \frac{a+b}{2} \right)^2 \right] = \frac{1}{2} \left[ \left( \frac{b-a}{2} \right)^2 - \left( \frac{a-b}{2} \right)^2 \right] = 0$$

$$\int_a^b (x-\bar{x})^2 dx = \frac{(x-\bar{x})^3}{3} \Big|_a^b = \frac{1}{3} \left[ \left( \frac{b-a}{2} \right)^3 - \left( \frac{a-b}{2} \right)^3 \right] = \frac{1}{3} \cdot \frac{1}{8} \left[ 2(b-a)^3 \right] = \frac{1}{12} (b-a)^3$$

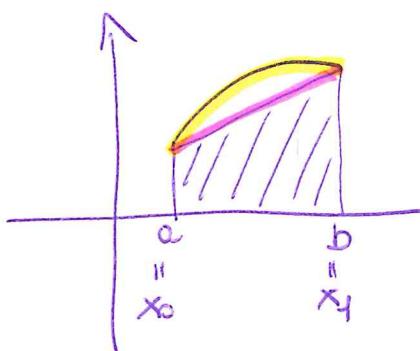
$$(b-a)^2 = (a-b)^2$$

$$(b-a)^3 = -(a-b)^3$$

$$I(f) - \tilde{I}_{PM}(f) = \frac{1}{24} (b-a)^3 f''(\beta)$$

$$\begin{aligned}
 I(f) - \tilde{I}_{PH}^c(f) &= \sum_{k=1}^2 \left[ \int_{I_k} f(x) dx - I_{PH}(f|_{I_k}) \right] = \frac{H^3}{24} \sum_{k=1}^2 f''(\beta_k) = \frac{H^3}{24} \underbrace{f''(x)}_{\substack{\in \\ [a,b]}} \sum_{k=1}^2 \frac{1}{H} \\
 &\quad \frac{1}{24} H^3 f''(\beta_k) \\
 &= \frac{H^3}{24} \sum_{k=1}^2 f''(\alpha) = \frac{1}{24} \frac{(b-a)H^2}{H} f''(\alpha) \quad f \in C^2([a,b])
 \end{aligned}$$

### Errore trapezio



$$E_n f(x) = \frac{f^{(n+1)}(\gamma(x))}{(n+1)!} \prod_{i=0}^n (x-x_i)$$

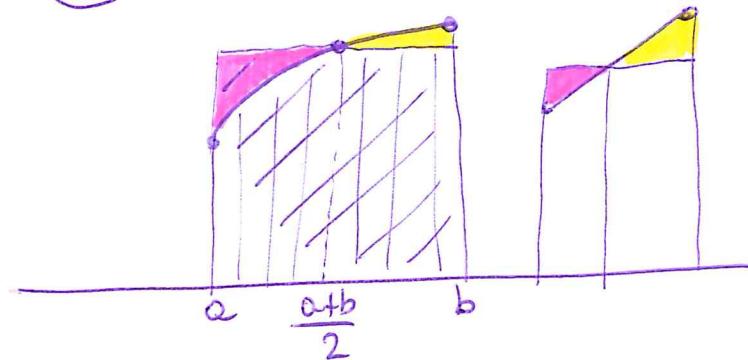
$n=1$

$$\begin{aligned}
 I(f) - \tilde{I}_T(f) &= \int_a^b E_1 f(x) dx = \int_a^b \frac{f''(\gamma(x))}{2} (x-a)(x-b) dx \\
 &= \frac{1}{2} f''(\alpha) \int_a^b (x-a)(x-b) dx \\
 &= \frac{1}{2} f''(\alpha) \left(-\frac{1}{6}\right) (b-a)^3 = -\frac{1}{12} (b-a)^3 f''(\alpha) \\
 \int_a^b (x-a)(x-b) dx &= -\int_a^b \frac{(x-a)^2}{2} dx + \left. \frac{(x-a)^2}{2} (x-b) \right|_a^b = -\frac{1}{2} \int_a^b (x-a)^2 dx \\
 \text{x parti} &= -\frac{1}{2} \frac{(x-a)^3}{3} \Big|_a^b = -\frac{1}{6} (b-a)^3
 \end{aligned}$$

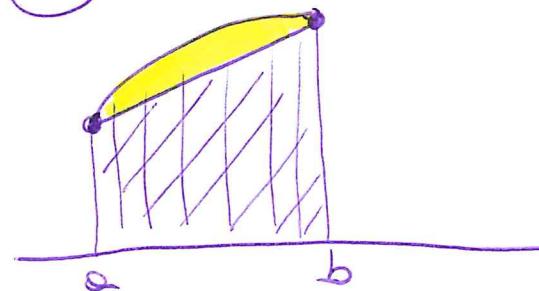
$$\max_{x \in [a,b]} |I(f) - \tilde{I}(f)| \leq C (b-a)^{\frac{p}{p}} \max_{x \in [a,b]} |f^{(p)}(x)|$$

$$\max_{x \in [a,b]} |I(f) - \tilde{I}^c(f)| \leq C (b-a)^{\frac{H^m}{p}} \max_{x \in [a,b]} |f^{(m)}(x)| \leq \text{TOL } (10^{-5}) \rightarrow H$$

(PM)



(T)



grado di esattezza : il max grado dei polinomi integrati esattamente dalle formule di quadratura

ordine di accuratezza : velocità con cui l'errore va a zero  
(formule composite)

$$P_0 \quad 1 \quad \int_a^b z dx \stackrel{?}{=} \tilde{I}(z)$$

$$P_1 \quad x \quad \int_a^b x dx \stackrel{?}{=} \tilde{I}(x)$$

$$P_2 \quad x^2 \quad \vdots$$

$$P_3 \quad x^3$$

$$I(f) \neq \tilde{I}(f)$$

$$f \in P_n \quad I(f) = \tilde{I}(f)$$

gde = 1  $\leftarrow$

$f \in P_0$	? $I(f) = \hat{I}(f)$	✓
$f \in P_1$	? $I(f) = \hat{I}(f)$	✓
$f \in P_2$	? $I(f) = \hat{I}(f)$	✗
$f \in P_3$	? $I(f) = \hat{I}(f)$	✓

Formule di quadratura interpolatorie

$\left\{ \begin{array}{l} \text{perotto menu } (\omega=2) ; n_2 = \left(\frac{b-a}{2}\right)^{-2} \\ \text{trapezi } (\omega=1) ; n_1 = a; n_2 = b; \alpha_1 = \alpha_2 = \frac{(b-a)}{2} \\ \text{Simpson } (\omega=3) ; n_1 = a; n_2 = \frac{a+b}{2}; n_3 = b; \alpha_1 = \alpha_3 = \frac{b-a}{6} \end{array} \right.$

Newton-Cotes  
⊗

Gaussiane

$$I(f) \approx \tilde{I}(f) = \sum_{j=1}^J f(n_j) \alpha_j$$

↑  
nodi di quadratura  
↖ pesi di quadratura

$\boxed{gde = 0}$        $\underbrace{I(1)}_{\sim} = \tilde{I}(1) = \boxed{\sum_{j=1}^J \alpha_j}$

$\alpha_j \in \mathbb{R}$   
 $n_j \in [a,b]$

$\int_a^b 1 dx = \boxed{(b-a)}$

### Formule di Gaussiane

$J=2$        $n_1, n_2, \alpha_1, \alpha_2$  (4 incognite) t.c.  $gde$  sia massimo

condizione 1  $gde = 0$

condizione 2  $gde = 1$

condizione 3  $gde = 2$

condizione 4  $gde = 3$

$$\left\{ \begin{array}{l} (b-a) = \alpha_1 + \alpha_2 \\ \int_a^b x dx = \frac{b^2}{2} - \frac{a^2}{2} = \tilde{I}(x) = \sum_{j=1}^2 n_j \alpha_j = n_1 \alpha_1 + n_2 \alpha_2 \\ \int_a^b x^2 dx = \frac{b^3}{3} - \frac{a^3}{3} = \tilde{I}(x^2) = n_1^2 \alpha_1 + n_2^2 \alpha_2 \\ \int_a^b x^3 dx = \frac{b^4}{4} - \frac{a^4}{4} = \tilde{I}(x^3) = n_1^3 \alpha_1 + n_2^3 \alpha_2 \end{array} \right.$$

$$\int_a^b f(x) dx \approx \int_a^b \hat{f}(x) dx \quad \hat{f} \in P_3$$

$$\tilde{I}_G(f) = \frac{b-a}{2} [f(\bar{x}_0) + f(\bar{x}_2)]$$

$$\bar{x}_1 = \bar{x}_0 = a + \left(2 - \frac{1}{\sqrt{3}}\right) \left(\frac{b-a}{2}\right); \quad \bar{x}_2 = a + \left(1 + \frac{1}{\sqrt{3}}\right) \left(\frac{b-a}{2}\right)$$

$\approx \frac{1}{5}$

$$\alpha_1 = \alpha_2 = \frac{b-a}{2}$$

$$\tilde{I}_G^c(f) = \frac{H}{2} \sum_{k=1}^N [f(\bar{x}_0^k) + f(\bar{x}_2^k)]$$

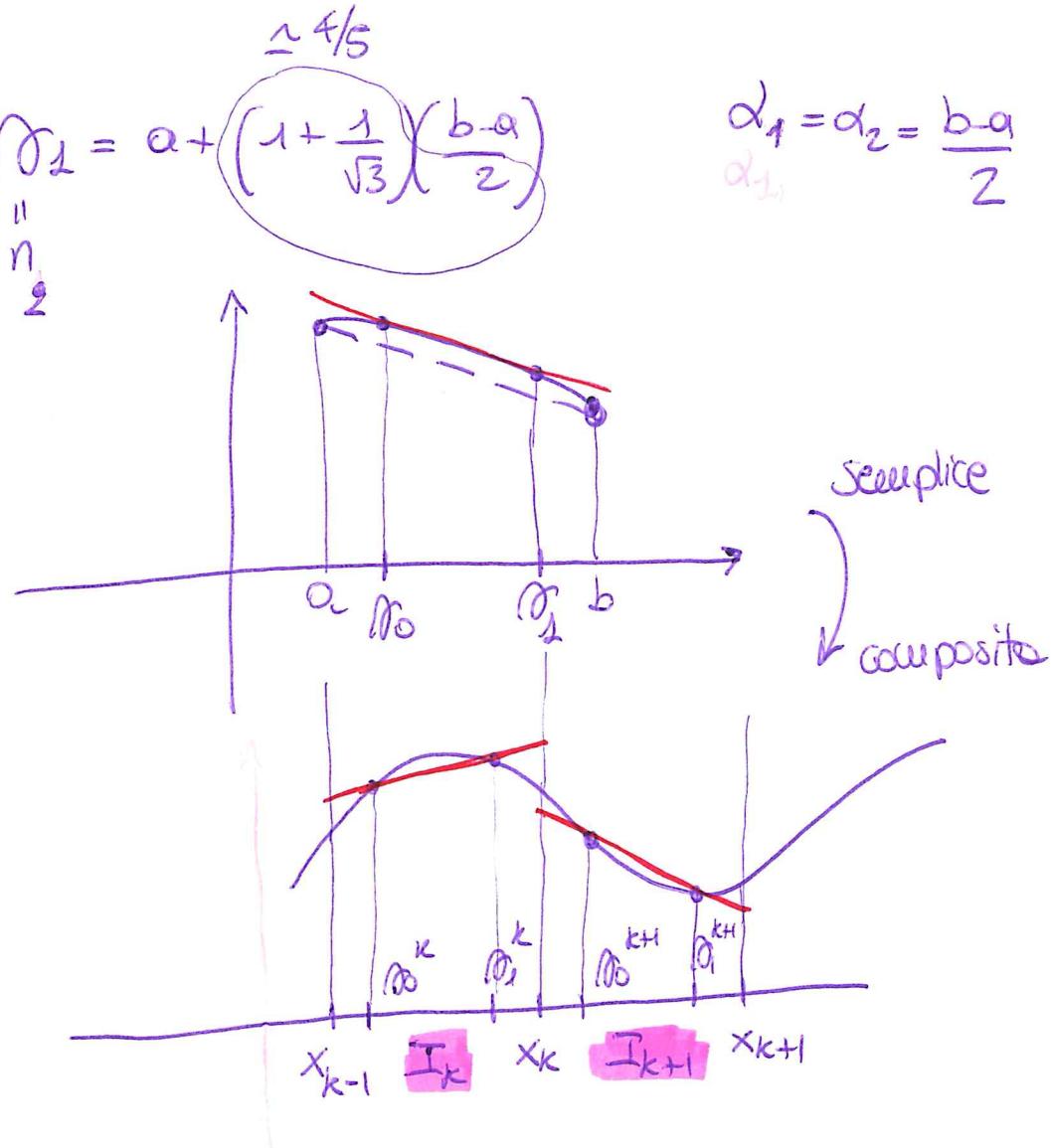
$$I(f) - \tilde{I}_G(f) = \frac{1}{4320} (b-a)^5 f^{(4)}(\xi)$$

$\xi, \xi \in [a, b]$   
 $f \in C^4([a, b])$

$$I(f) - \tilde{I}_G^c(f) = \frac{1}{4320} (b-a)^5 f^{(4)}(\xi)$$

ode: 4

gde: 3



## Approssimazione di derivate

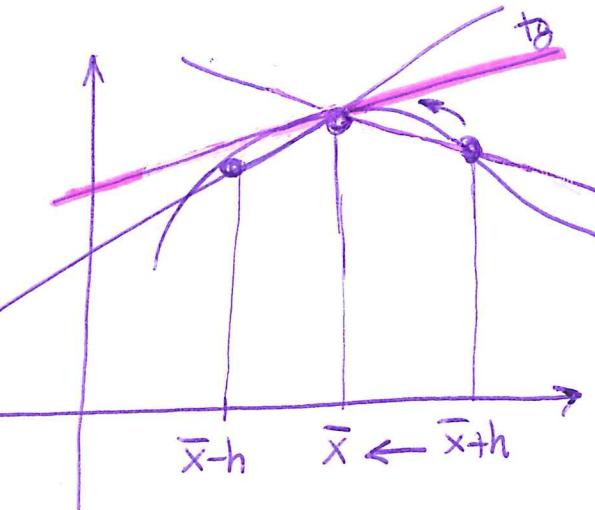
$$f'(\bar{x})$$

$$f: [a,b] \subset \mathbb{R} \rightarrow \mathbb{R}$$

$$f \in C^2([a,b])$$

$$\bar{x} \in [a,b]$$

lim  $\frac{f(\bar{x}+h) - f(\bar{x})}{h} = f'(\bar{x}) \left[ \lim_{h \rightarrow 0} \frac{f(\bar{x}) - f(\bar{x}-h)}{h} \right]$



$$f'(\bar{x}) \approx \delta_+ f(\bar{x}) = \frac{f(\bar{x}+h) - f(\bar{x})}{h} \quad (h \text{ piccolo})$$

differenza finita in avanti

$$f'(\bar{x}) - \delta_+ f(\bar{x})$$

Taylor

$$f \in C^2([a,b])$$

$$f(\bar{x}+h) = f(\bar{x}) + h f'(\bar{x}) + \frac{h^2}{2} f''(\alpha)$$

$$\alpha \in [\bar{x}, \bar{x}+h]$$

$$h f'(\bar{x}) - \frac{f(\bar{x}+h) - f(\bar{x})}{h} + \frac{f(\bar{x})}{h} = -\frac{h^2}{2} f''(\alpha)$$

$$f'(\bar{x}) - \underbrace{\frac{f(\bar{x}+h) - f(\bar{x})}{h}}_{\delta_+ f(\bar{x})} = -\frac{h}{2} f''(\alpha)$$

ACCURATA  
AL PRIM'ORDINE  
RISPETTO AD h

$$f'(\bar{x}) \approx \delta_- f(\bar{x}) = \frac{f(\bar{x}) - f(\bar{x}-h)}{h} \quad (h \text{ piccolo})$$

↓

DIFFERENZA FINITA ALL'INDIETRO

$$f'(\bar{x}) - \delta_- f(\bar{x})$$

Taylor

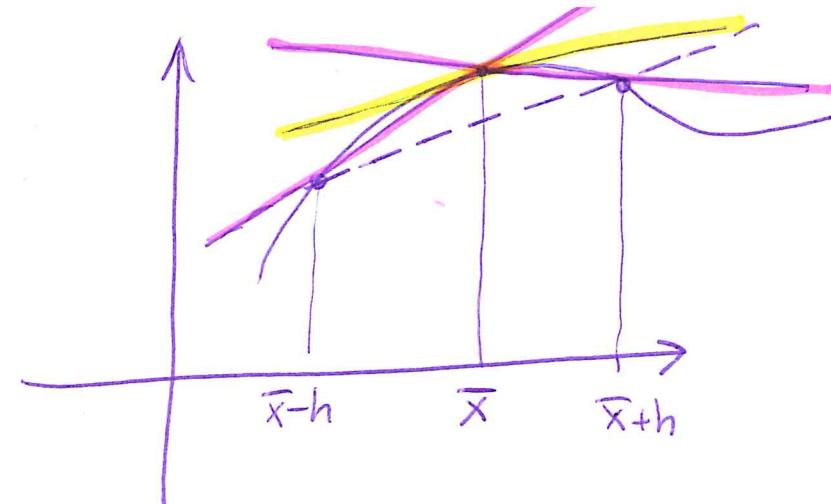
$f \in C^2([a, b])$

$$f(\bar{x}-h) = f(\bar{x}) - h f'(\bar{x}) + \frac{h^2}{2} f''(\delta) \quad \delta \in [\bar{x}-h, \bar{x}]$$

$\curvearrowleft \curvearrowleft$

$$h f'(\bar{x}) + \frac{f(\bar{x}-h) - f(\bar{x})}{h} = \frac{h^2}{2} f''(\delta)$$

$$f'(\bar{x}) - \underbrace{\frac{f(\bar{x}) - f(\bar{x}-h)}{h}}_{\delta_- f(\bar{x})} = \frac{h}{2} f''(\delta) \quad \text{Acurato al 2° ordine rispetto ad } h$$



Esercizio

- 1) RETTA che interpolà  $f$  nei punti  $(\bar{x}-h, f(\bar{x}-h)), (\bar{x}, f(\bar{x})) \Rightarrow \delta f(\bar{x})$
- 2) DERIVATA

$$f'(\bar{x}) = \lim_{h \rightarrow 0} \frac{f(\bar{x}+h) - f(\bar{x}-h)}{2h}$$

$$f'(\bar{x}) \simeq \delta f(\bar{x}) = \frac{f(\bar{x}+h) - f(\bar{x}-h)}{2h}$$

DIFFERENZA  
FINITA CENTRATA

$$f'(\bar{x}) - \delta f(\bar{x})$$

$$f \in C^3([a,b])$$

$$\textcircled{-} \quad f(\bar{x}+h) = f(\bar{x}) + h f'(\bar{x}) + \frac{h^2}{2} f''(\bar{x}) + \frac{h^3}{6} f'''(\sigma)$$

$$f(\bar{x}-h) = f(\bar{x}) - h f'(\bar{x}) + \frac{h^2}{2} f''(\bar{x}) - \frac{h^3}{6} f'''(y)$$

$$\sigma \in [\bar{x}, \bar{x}+h]$$

$$y \in [\bar{x}-h, \bar{x}]$$

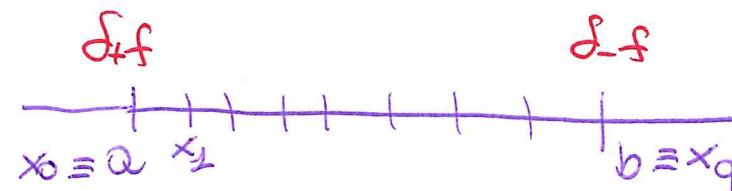
$$f(\bar{x}+h) - f(\bar{x}-h) = 2h f'(\bar{x}) + \frac{h^3}{6} \{ f'''(\sigma) + f'''(y) \}$$

$$2h f'(\bar{x}) - \frac{f(\bar{x}+h) + f(\bar{x}-h)}{2h} = -\frac{h^2}{6 \cdot 2} \{ f'''(\sigma) + f'''(y) \}$$

$$f'(\bar{x}) - \underbrace{\frac{f(\bar{x}+h) + f(\bar{x}-h)}{2h}}_{\delta f(\bar{x})} = -\frac{h^2}{12} \{ f'''(\sigma) + f'''(y) \}$$

ACCURATA  
AL SECONDO ORDINE  
RISPETTO AD  $h$

$$f'(\bar{x}) \left\{ \begin{array}{l} \delta_+ f(\bar{x}) \text{ (I)} \\ \delta_- f(\bar{x}) \text{ (II)} \\ \delta f(\bar{x}) \text{ (III)} \end{array} \right.$$



# EQUAZIONI DIFFERENZIALI ORDINARIE (EDO)

① Legge di Newton

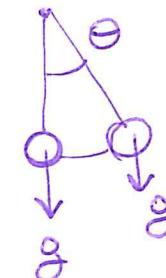
$$m \frac{d^2x}{dt^2}(t) = f(t)$$

②  $f = f(x(t))$

$$m \frac{d^2x}{dt^2}(t) = -kx(t) \quad k > 0$$

③ pendolo semplice

$$\underbrace{\ell \frac{d^2\theta}{dt^2}(t)} + \underbrace{\alpha \frac{d\theta}{dt}(t)} + g \sin\theta(t) = 0$$



PROBLEMI DI CAUCHY :

$$F(t, y(t), y'(t), \dots, y^{(p)}(t)) \stackrel{*}{=} 0 \quad t \in I = [t_0, T]$$

$$y^{(p)}(t) \stackrel{*}{=} f(t, y(t), y'(t), \dots, y^{(p-1)}(t)) \quad t \in I$$

$y : I \rightarrow \mathbb{R}$   
derivabile p volte  
soddisfa \*

] SOLUZIONE ODE

## ODE ordine 1 ( $p=1$ )

scalar  $y'(t) = f(t, y(t)) \quad \forall t \in I$

$y: I \rightarrow \mathbb{R}$   $\hookrightarrow$  nota  $f: I \times \mathbb{R} \rightarrow \mathbb{R}$

vettoriale  $\vec{y}'(t) = \vec{f}(t, \vec{y}(t)) \quad \forall t \in I$

$\vec{y}(t) = (y_1(t), y_2(t), \dots, y_n(t))^T$

$\vec{f}: I \times \mathbb{R}^n \rightarrow \mathbb{R}^n$

$\vec{f}(t, \vec{y}(t)) = (f_1(t, \vec{y}(t)), f_2(t, \vec{y}(t)), \dots, f_n(t, \vec{y}(t)))^T$

$f_i: I \times \mathbb{R}^n \rightarrow \mathbb{R}$

### CONDIZIONI INIZIALI

ODE  $p=1$  scalare  
 $y(t_0) = y_0 \rightarrow$  noto

ODE  $p=1$  vettoriale  
 $\vec{y}(t_0) = \vec{y}_0 \in \mathbb{R}^n \rightarrow$  noto

ODE  $p=n$   
 $y(t_0) = y_0$   
 $y'(t_0) = y_{0,1} \rightarrow$  noti  
 $\vdots$   
 $y^{(n-1)}(t_0) = y_{0,n-1}$

## ODE ordine n ( $p=n$ )



sistema di  $n$  odes ordine 1

$\boxed{y^{(n)}(t) = f(t, y(t), y'(t), \dots, y^{(n-1)}(t)) \quad t \in I}$

$y_1(t) = y(t)$

$y_2(t) = y'(t) = y_1'(t)$

$y_3(t) = y''(t) = y_2'(t)$

;

$y_n(t) = y^{(n-1)}(t) = y_{n-1}'(t)$

$\begin{cases} y_1'(t) = y_2(t) \\ y_2'(t) = y_3(t) \\ \vdots \end{cases}$

$y_{n-1}'(t) = y_n(t)$

$y_n'(t) = f(t, y_1(t), y_2(t), \dots, y_n(t))$  (50)

## Problema di Cauchy

$$? \quad y : I \subset \mathbb{R} \rightarrow \mathbb{R} \quad \text{t.c.} \quad \begin{cases} y'(t) = f(t, y(t)) & t \in I \\ y(t_0) = y_0 \end{cases}$$

dati                  incognita  
 $I$                    $y = y(t)$   
 $y_0$   
 $f : I \times \mathbb{R} \rightarrow \mathbb{R}$

Osserviamo: se  $f$  è continua rispetto alle prime variabili, allora

$$\int_{t_0}^t \frac{dy}{dz}(z) dz = \int_{t_0}^t f(z, y(z)) dz \quad t \in I$$

se  $y \in C^1(\bar{I})$

$$y(t) = \underbrace{y(t_0)}_{y_0} + \int_{t_0}^t f(z, y(z)) dz$$

Esempio: problema modello

$$\begin{cases} y'(t) = \lambda y(t) & t \in I = [t_0, T], \lambda \in \mathbb{R}^- \\ y(t_0) = y_0 \end{cases}$$

$$y(t) = y_0 e^{\lambda(t-t_0)}$$

problema di Cauchy

→ lineare :  $f$  è lineare rispetto a entrambi gli argomenti

$$\begin{cases} y'(t) = 3y(t) - 3t & t \geq 0 \\ y(0) = 1 \end{cases}$$

Risolvibile

→ non lineare

$$\begin{cases} y'(t) = \sqrt[3]{y(t)} & t \geq 0 \\ y(0) = 0 \end{cases}$$

con soltuzione locale

$$\begin{cases} y'(t) = 1 + [y(t)]^2 & t \geq 0 \\ y(0) = 0 \end{cases}$$
$$y(t) = \tan(t) \quad 0 < t < \frac{\pi}{2}$$

Esistenza e unicità

BUONA POSIZIONE

(3)

dipendenza continua dai dati (stabilità)

Osservazione : se  $f$  è continua rispetto ad entrambi gli argomenti, allora  $\exists$  (almeno) una funzione  $y$  di classe  $C^1$  (in un intorno di  $t_0$ ) soluzione del problema di Cauchy MA non è necessariamente unica

$$\begin{cases} y'(t) = \frac{3}{2}y^{1/3}(t) & t \geq 0 \\ y(0) = 0 \end{cases}$$
$$y(t) = 0 ; y(t) = t^{3/2} \text{ etc...}$$

LIPSCHITZIANITÀ uniforme :  $f : I \times \mathbb{R} \rightarrow \mathbb{R}$  è lipschitziana rispetto ad  $y$  in  $I$ , uniformemente rispetto a  $t$ , se  $\exists$  costante  $L \geq 0$  t.c.

$$|f(t, y_1) - f(t, y_2)| \leq L |y_1 - y_2| \quad \forall y_1, y_2 \in \mathbb{R}, \forall t \in I$$

Teorema ( $\exists!$  globale): Se  $f : I \times \mathbb{R} \rightarrow \mathbb{R}$  è
 

- i) continua rispetto ad entrambi gli argomenti
- ii) Lipschitziana in  $I$  rispetto ad  $y$ , uniformemente rispetto a  $t$ ,

allora  $\exists! y : I \rightarrow \mathbb{R}$  soluzione
 
$$\begin{cases} y'(t) = f(t, y(t)) & t \in I \\ y(0) = y_0 \end{cases}$$
 ed è  $y \in C^1(\bar{I})$ .

Oss: una regolarità di tipo  $C^1$  è maggiore della Lipschitz-continuità. Overo, se  $f : I \times \mathbb{R} \rightarrow \mathbb{R}$  è continua di classe  $C^1$  rispetto ad  $y$ , allora è anche Lipschitz-continua nel II° argomento

$$|f(t, y_1) - f(t, y_2)| \leq \max_{\substack{t \in I \\ y \in \mathbb{R}}} \left| \frac{\partial f}{\partial y}(t, y) \right| |y_1 - y_2|$$

$L > 0$

Esempi

1)  $\begin{cases} y'(t) = 3y(t) - 3t & t > 0 \\ y(0) = 1 \end{cases}$

$$f(t, y(t)) = 3(t - 3y(t)) - 3t$$

$$|f(t, y_1) - f(t, y_2)| = |3y_1 - 3t - 3y_2 + 3t| \leq 3 |y_1 - y_2| \quad \forall y_1, y_2 \in \mathbb{R}$$

$$\hookrightarrow \exists! \left[ y(t) = \left(1 - \frac{1}{3}\right)e^{3t} + t + \frac{1}{3} \right] \quad L$$

2)  $f(t, y(t)) = |y(t)| \notin C^1(\bar{I})$

$$|y_1 - y_2| \leq \frac{1}{L} |y_1 - y_2| \quad \forall y_1, y_2 \in \mathbb{R}$$

$L$

STABILITÀ SECONDO LYAPUNOV :

$$\textcircled{+} \quad \begin{cases} y'(t) = f(t, y(t)) & t \in I \\ y(t_0) = y_0 \end{cases}$$

$$\begin{cases} z'(t) = f(t, z(t)) + \delta(t) & t \in I \\ z(t_0) = y_0 + \delta_0 \end{cases}$$

$\delta: I \rightarrow \mathbb{R}$  perturbazione su  $f$   
 $\delta_0 \in \mathbb{R}$  " " sue date iniziali

$\textcircled{+}$  si dice stabile (secondo Lyapunov) sull'intervallo  $I$  se,  $\forall$  perturbazione  $(\delta_0, \delta(t))$  t.c.  $|\delta_0| < \varepsilon$  e  $|\delta(t)| \leq \varepsilon \quad \forall t \in I$ , con  $\varepsilon > 0$ , e tali da garantire l'esistenza di soluzione  $z = z(t)$ ,  $\exists$  una costante  $C > 0$  t.c.

 $|y(t) - z(t)| \leq C\varepsilon \quad \forall t \in I.$ 

[asintoticamente stabile se per  $\lim_{t \rightarrow +\infty} |\delta(t)| = 0$  si ottiene]

$$\lim_{t \rightarrow +\infty} |y(t) - z(t)| = 0$$

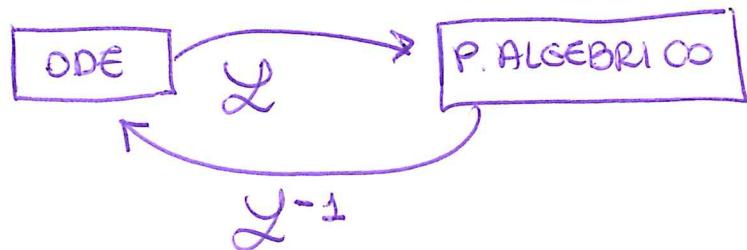
In particolare,  $\forall t \in I$  con  $t > t_0$

$$|y(t) - z(t)| \leq |\delta_0| + L \int_{t_0}^t |y(\tau) - z(\tau)| d\tau + \int_{t_0}^t |\delta(\tau)| d\tau$$

Gronwall 

$$|y(t) - z(t)| \leq \underbrace{\varepsilon(1 + \int_{t_0}^t e^{L(t-\tau)} d\tau)}_C \quad \forall t \in I$$

## La Trasformata di Laplace



Definizione: Data  $f = f(t)$  con  $t \geq 0$ , la trasformata di Laplace di  $f$  è

$$F(s) = \mathcal{L}[f(t)](s) = \int_0^{+\infty} f(t) e^{-st} dt \quad s \in \mathbb{R}$$

Viceversa, dato  $F = F(s)$ , trovare  $f = f(t)$  t.c.  $\mathcal{L}[f(t)](s) = F(s)$  nell'ambito del calcolo dell'inverse trasformata di Laplace

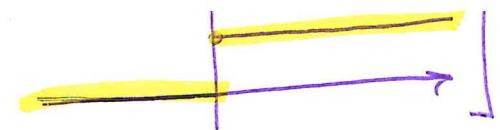
$$\mathcal{L}^{-1}[F(s)](t) = f(t) \quad \text{se e solo se } \mathcal{L}[f(t)](s) = F(s)$$

Osservazione:  $F(s)$  dipende da  $f(t)$  solo per  $t \geq 0$ .  $\Rightarrow f(t) = 0$  per  $t < 0$

Osservazione:  $F(s)$  esiste se A)  $\exists$  primitive su un generico intervallo limitato;  
B) garantire la convergenza dell'integrale improprio.

Examp 1)  $f(t) = 1$

$f(t) = H(t) = \text{Heaviside function}$



$$F(s) = \mathcal{L}[1](s) = \int_0^{+\infty} 1 \cdot e^{-st} dt = -\frac{1}{s} e^{-st} \Big|_0^{+\infty} = \lim_{b \rightarrow +\infty} \left[ -\frac{1}{s} e^{-sb} \right]_0^b = \lim_{b \rightarrow +\infty} \left[ -\frac{1}{s} e^{-sb} + \frac{1}{s} \right] = \frac{1}{s}$$

2)  $f(t) = e^{at}$

$$\begin{aligned} F(s) = \mathcal{L}[e^{at}](s) &= \int_0^{+\infty} e^{at} e^{-st} dt = \int_0^{+\infty} e^{(a-s)t} dt = \lim_{b \rightarrow +\infty} \int_0^b e^{(a-s)t} dt \\ &= \lim_{b \rightarrow +\infty} \left[ \frac{e^{(a-s)t}}{(a-s)} \right]_0^b = \frac{1}{a-s} = \frac{1}{s-a} \end{aligned}$$

$s > 0$

$a-s < 0$        $a < s$

3)  $f(t) = e^{t^2}$      $\exists \mathcal{L}[f(t)](s)$

$$\begin{aligned} F(s) = \mathcal{L}[e^{t^2}](s) &= \lim_{b \rightarrow +\infty} \int_0^b e^{t^2} e^{-st} dt = \lim_{b \rightarrow +\infty} \int_0^b e^{-st+t^2} dt \\ \int_0^b e^{-st+t^2} dt &\geq \int_{|s|}^b e^{-st+t^2} dt \geq \int_{|s|}^b e^0 dt = 1 \cdot (b - |s|) \xrightarrow[b \rightarrow +\infty]{} +\infty \end{aligned}$$

$b > |s| ; \quad t \geq |s| \Rightarrow t^2 \geq |s|t \Rightarrow t^2 - |s|t \geq 0$

$t^2 - st \geq t^2 - |s|t \geq 0$

## Proprietà

1) LINEARITÀ: siano  $f(t)$  e  $g(t)$  due funzioni che ammettono le trasformate di Laplace  $F(s)$  e  $G(s)$ .

Allora

$$\mathcal{L}[f(t) + g(t)](s) = F(s) + G(s)$$

$$\mathcal{L}[c f(t)](s) = c F(s) \quad \forall c \in \mathbb{R}$$

Es t.c.  $F(s)$  e  $G(s)$  siano definite. Risulta lineare anche l'antitrasformata, ovvero

$$\mathcal{L}^{-1}[F(s) + G(s)](t) = f(t) + g(t)$$

$$\mathcal{L}^{-1}[c F(s)](t) = c f(t) \quad \forall c \in \mathbb{R}.$$

Ex: 2)a  $f(t) = \sin(3t) + 2$

$$\begin{aligned} F(s) &= \mathcal{L}[f(t)](s) = \mathcal{L}[\sin(3t)](s) + \mathcal{L}[2](s) \\ &= \frac{3}{s^2+9} + 2 \mathcal{L}[1](s) = \frac{3}{s^2+9} + \frac{2}{s} \end{aligned}$$

$$2)b \quad F(s) = \frac{3}{s^2+16} - \frac{4}{(s-5)(s-12)} = \frac{3}{4} \frac{4}{s^2+16} - \frac{12-5}{(s-12)(s-5)}$$

$$f(t) = \frac{3}{4} \mathcal{L}^{-1}\left[\frac{4}{s^2+16}\right](t) - \mathcal{L}^{-1}\left[\frac{12-5}{(s-12)(s-5)}\right](t)$$

$$= \frac{3}{4} \sin(4t) - (e^{12t} - e^{5t}).$$

## 2) FORMULE DI SHIFT (O DI TRASLAZIONE):

$$\mathcal{L}[1](s) = \frac{1}{s} ; \quad \mathcal{L}[e^{at}](s) = \frac{1}{s-a}$$

- i) moltiplicare  $f(t) = 1$  per  $e^{at}$  genera uno shift a dx di  $a$  della trasformata  $F(s)$ ;
- ii) sostituire nella trasformata  $s$  con  $s-a$  corrisponde a moltiplicare per  $e^{at}$  la funzione  $f(t)$ .

### PRIMO TEOREMA DI SHIFT

$$\begin{cases} \mathcal{L}[e^{at}f(t)](s) = F(s-a) & [\text{i)}] \\ \mathcal{L}^{-1}[F(s-a)](t) = e^{at}f(t) & [\text{ii)}] \end{cases}$$

Ex: 2)a  $\mathcal{L}[\cos(\alpha t)](s) = \frac{s}{s^2 + \alpha^2} = F(s)$

$$\hookrightarrow \mathcal{L}[e^{\beta t} \cos(\alpha t)](s) = F(s-\beta) = \frac{s-\beta}{(s-\beta)^2 + \alpha^2}$$

2)b Calcolare  $\mathcal{L}^{-1}\left[\frac{4}{s^2 + 4s + 20}\right](t).$

$$\frac{4}{s^2 + 4s + 20} = \frac{4}{(s+2)^2 + 16} = F(s+2) \quad F(s) = \frac{4}{s^2 + 16}$$

$$\mathcal{L}^{-1}[F(s)](t) = \sin(4t) \Rightarrow \mathcal{L}^{-1}\left[\frac{4}{s^2 + 4s + 20}\right](t) = \mathcal{L}^{-1}[F(s+2)](t) = e^{-2t} \sin(4t)$$

### 3) TRASFORMATA DEL PRODOTTO DI CONVOLUZIONE

prodotto di convoluzione di due funzioni  $f(t)$  e  $g(t)$  definite per  $t \geq 0$  è data da

$$(f * g)(t) = \int_0^t f(t-u)g(u) du \quad (= \int_0^t f(u)g(t-u) du)$$

$\forall t \geq 0$  tale per cui l'integrale sia ben definito.

[VALE LA P. COMMUTATIVA  $f * g = g * f$ ]

Siano  $f(t)$  e  $g(t)$  due funzioni con trasformate di Laplace  $F(s)$  e  $G(s)$ ; allora,

$$\mathcal{L}[(f * g)(t)](s) = \mathcal{L}\left[\int_0^t f(t-u)g(u) du\right](s) = F(s)G(s)$$

Inoltre

$$\mathcal{L}^{-1}[F(s)G(s)](t) = (f * g)(t) = \int_0^t f(t-u)g(u) du.$$

Ex: ?  $\mathcal{L}^{-1}\left[\frac{1}{s^2(s+1)^2}\right](t)$  ;  $\mathcal{L}^{-1}\left[\frac{1}{s^2}\right](t) = t \leftarrow g$

$$\mathcal{L}^{-1}\left[\frac{1}{(s+1)^2}\right](t) = te^{-t} \leftarrow f$$

$$\begin{aligned} \mathcal{L}^{-1}\left[\frac{1}{s^2(s+1)^2}\right](t) &= \int_0^t (t-u)e^{-(t-u)} u du = t e^{-t} \int_0^t e^u u du - e^{-t} \int_0^t u^2 e^u du \\ &= \text{integrazione per parti} \\ &= (t+2)e^{-t} + (t-2) \end{aligned}$$

#### 4) TRASFORMATA DELLA DERIVATA :

$$\begin{aligned}\mathcal{L}[y'(t)](s) &= \int_0^{+\infty} y'(t) e^{-st} dt = y(t) e^{-st} \Big|_0^{+\infty} - \int_0^{+\infty} y(t)(-s) e^{-st} dt \\ &= -y(0) + s \underbrace{\int_0^t y(t) e^{-st} dt}_{\mathcal{L}[y(t)](s)}\end{aligned}$$

$$\boxed{\mathcal{L}[y'(t)](s) = s \mathcal{L}[y(t)](s) - y(0)}$$

$$\underbrace{\mathcal{L}[y''(t)](s)}_{(y'(t))'} = s \mathcal{L}[y'(t)](s) - y'(0) = s [s \mathcal{L}[y(t)](s) - y(0)] - y'(0)$$

$$\boxed{\mathcal{L}[y''(t)](s) = s^2 \mathcal{L}[y(t)](s) - sy(0) - y'(0)}$$

PROBLEMA 1:

$$\begin{cases} y'(t) = 4y(t) + 1 & t \geq 0 \\ y(0) = 1 \end{cases}$$

$$y'(t) - 4y(t) = 1$$

TRASFORMATA

$$\mathcal{L}[y'(t) - 4y(t)](s) = \mathcal{L}[y'(t)](s) - 4\mathcal{L}[y(t)](s) = s\mathcal{L}[y(t)](s) - y(0) - 4\mathcal{L}[y(t)](s)$$

$$= \mathcal{L}[1](s) = \frac{1}{s}$$

$$s\mathcal{L}[y(t)](s) - 4\mathcal{L}[y(t)](s) - 1 = \frac{1}{s}$$

$$\mathcal{L}[y(t)](s) = Y(s)$$

$$sY(s) - 4Y(s) - 1 = \frac{1}{s}$$

PROBLEMA ALGEBRICO

$$Y(s) = \frac{1}{s-4} \left( 1 + \frac{1}{s} \right) = \frac{1}{s-4} + \frac{1}{s(s-4)}$$

$$\mathcal{L}^{-1}\left[\frac{1}{s-4}\right] = e^{4t}$$

$$\mathcal{L}^{-1}\left[\frac{1}{s(s-4)}\right] = \mathcal{L}^{-1}\left[-\frac{1}{4} \frac{-4}{(s-0)(s-4)}\right] = -\frac{1}{4} (e^{4t} - e^{4t}) = \frac{1}{4} (e^{4t} - 1)$$

$$\begin{aligned} \Rightarrow y(t) &= \mathcal{L}^{-1}[Y(s)](t) = \mathcal{L}^{-1}\left[\frac{1}{s-4} + \frac{1}{s(s-4)}\right](t) = \mathcal{L}^{-1}\left[\frac{1}{s-4}\right](t) + \mathcal{L}^{-1}\left[\frac{1}{s(s-4)}\right](t) \\ &= e^{4t} + \frac{1}{4} e^{4t} - \frac{1}{4} = \frac{5}{4} e^{4t} - \frac{1}{4} = \frac{1}{4} (5e^{4t} - 1) \quad t \geq 0 \end{aligned}$$

PROBLEMA 2 :

$$\begin{cases} y''(t) + 4y'(t) + 3y(t) = e^t & t \geq (q, t_f) \quad t \geq 0 \\ y(0) = 0 \\ y'(0) = 2 \end{cases}$$

$$\begin{aligned} \mathcal{L}[y''(t) + 4y'(t) + 3y(t)](s) &= \mathcal{L}[y''(t)](s) + 4\mathcal{L}[y'(t)](s) + 3\mathcal{L}[y(t)](s) \\ &= s^2 \mathcal{L}[y(t)](s) - \underbrace{sy(0)}_{=0} - \underbrace{y'(0)}_{=2} + 4s \mathcal{L}[y(t)](s) - \underbrace{4y(0)}_{=0} + 3\mathcal{L}[y(t)](s) \\ &= \mathcal{L}[e^t](s) = \frac{1}{s-1} \end{aligned}$$

$$\mathcal{L}[y(t)](s) = Y(s)$$

$$s^2 Y(s) - 2 + 4s Y(s) + 3 Y(s) = \frac{1}{s-1} \quad \boxed{\text{PROBLEMA ALGEBRAICO}}$$

$$Y(s) = \frac{\frac{1}{s-1} + 2}{s^2 + 4s + 3} = \frac{2s+1}{(s-1)(s^2+4s+3)} \quad (s+1)(s+3)$$

$$Y(s) = \frac{1}{8} \frac{1}{s-1} + \frac{3}{4} \frac{1}{s+1} - \frac{7}{8} \frac{1}{s+3}$$

$$y(t) = \mathcal{L}^{-1}[Y(s)](t) = \frac{1}{8} \mathcal{L}^{-1}\left[\frac{1}{s-1}\right](t) + \frac{3}{4} \mathcal{L}^{-1}\left[\frac{1}{s+1}\right](t) - \frac{7}{8} \mathcal{L}^{-1}\left[\frac{1}{s+3}\right](t)$$

$$y(t) = \frac{1}{8} e^t + \frac{3}{4} e^{-t} - \frac{7}{8} e^{-3t}$$

## ODE: metodi numerici

$$y = y(t)$$

↓  
esplicita      ↓  
impatita

$$\begin{cases} y'(t) = f(t, y(t)) \\ y(0) = y_0 \end{cases}$$

↓  
soluzione  
esplicita /  
impatita      \$y\_0, f\$ note

$$y'(t) = e^{-t^2}$$

$$y(t) = \sum_{k=0}^{+\infty} \dots$$

$y = y(t)$  incognita

$$\frac{1}{2} \tan(t^2 + y^2) + \cotan \frac{y}{t} = c$$

tempo finale  
 $t \in (t_0, T) = I$   
↓  
tempo iniziale

$$\{u_0 = y_0, u_1, u_2, \dots, u_{N_h}\}$$

$$\rightarrow u_n \approx y_n = y(t_n) \quad n=0, \dots, N_h$$

Approssimazione discreta di  $y = y(t)$

$y = y(t)$  approssimata da

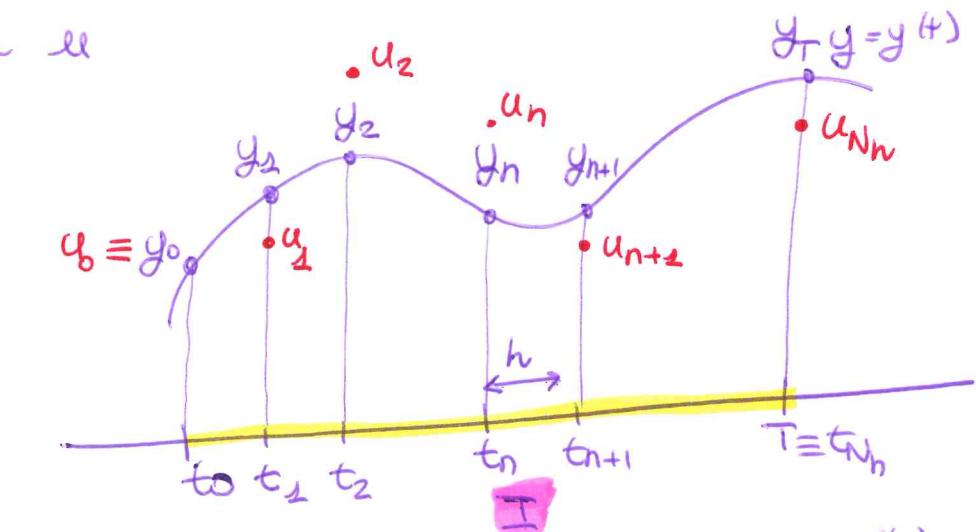
$N_h$  sottointervalli uniformi

$$t_0 < t_1 < t_2 \dots < t_{N_h} = T$$

$$h = \text{passo di discretizzazione} = \frac{T - t_0}{N_h}$$

$t_n \quad 0 \leq n \leq N_h$  istanti

$$\begin{aligned} t_n &= t_{n-1} + h & n = 1, \dots, N_h \\ [t_n &= t_0 + nh & n = 0, \dots, N_h] \end{aligned}$$



valore esatto  $y(t_n) = y_n$   
 $\forall n = 0, \dots, N_h \quad u_n \approx y_n$

$$\begin{cases} y'(t) = f(t, y(t)) & t \in I \\ y(t_0) = y_0 \end{cases}$$

$$t = t_n \quad y'(t_n) = f(t_n, y(t_n)) \quad (\text{collocazione})$$

$$\frac{y(t_{n+1}) - y(t_n)}{h} \simeq f(t_n, y(t_n)) \quad (\text{approssimazione della derivata})$$

$$\frac{y_{n+1} - y_n}{h} \simeq f(t_n, y_n) \quad (\text{cambio notazione})$$

$$\frac{u_{n+1} - u_n}{h} = f(t_n, u_n) \quad (\text{introduce l'approssimazione})$$

$$\begin{array}{|l} \hline u_{n+1} = u_n + h f(t_n, u_n) \quad 0 \leq n \leq N_h-1 \\ \hline u_0 = y_0 \end{array}$$

ex:

$$[u_{n+1} = F(u_n, u_{n-1}, u_{n-2})]$$

a 3 passi

one-step

multi-step]

$$u_0 = y_0$$

$$\downarrow u_1 \simeq y_1$$

$$\downarrow u_2 \simeq y_2$$

:

$$\downarrow u_{N_h-1} \simeq y_{N_h-1}$$

$$\downarrow u_{N_h} \simeq y_{N_h}$$

schema di Euler in avanti  
(" " " " esposto)

metodo ad un passo  
(metodo one-step)

metodo esplicito

esplicito  $u_{n+1} = F(u_n)$  one-step

$$u_{n+1} = F(u_n, u_{n-1}, u_{n-2}, \dots, u_{n-p}) \text{ multi-step}$$

implicito  $u_{n+1} = F(u_{n+1}, u_n)$  one-step

$$u_{n+1} = F(u_{n+2}, u_n, u_{n-1}, u_{n-2}, \dots, u_{n-p}) \text{ multi-step}$$

$$y'(t_{n+1}) = f(t_{n+1}, y(t_{n+1})) \quad (\text{collocazione})$$

$$\frac{y(t_{n+1}) - y(t_n)}{h} \simeq f(t_{n+1}, y(t_{n+1})) \quad (\text{approssimazione derivata})$$

$$\frac{y_{n+1} - y_n}{h} \simeq f(t_{n+1}, y_{n+1}) \quad (\text{cambio notazione})$$

$$\frac{u_{n+1} - u_n}{h} = f(t_{n+1}, u_{n+1}) \quad (\text{approssimo})$$

$$\hookrightarrow \boxed{\begin{array}{l} u_{n+1} = u_n + h f(t_{n+1}, u_{n+1}) \\ u_0 = y_0 \end{array} \quad 0 \leq n \leq N_h - 1}$$

metodo one-step  
metodo implicito

schema di Eulero all'indietro  
(" " " " implicito)

Esempio

$$\begin{cases} y'(t) = c y(t) \left( t - \frac{y(t)}{B} \right) \\ y(0) = y_0 \end{cases} \quad B, c \in \mathbb{R}$$

EE

$$\begin{cases} u_{n+1} = u_n + h f(t_n, u_n) = u_n + h C u_n \left( t_n - \frac{u_n}{B} \right) \\ u_0 = y_0 \end{cases} \quad 0 \leq n \leq N_h - 1$$

EI

$$\begin{cases} u_{n+1} = u_n + h f(t_{n+1}, u_{n+1}) = u_n + h C u_{n+1} \left( t_{n+1} - \frac{u_{n+1}}{B} \right) \\ u_0 = y_0 \end{cases} \quad 0 \leq n \leq N_h - 1$$

$$u_{n+1} \rightarrow x \quad x = u_n + h C x \left( t_{n+1} - \frac{x}{B} \right)$$

equazione non lineare

Schema di Crank-Nicolson

EE  $u_{n+1} = u_n + h f(t_n, u_n)$

EI  $u_{n+1} = u_n + h f(t_{n+1}, u_{n+1})$

$$2u_{n+1} = 2u_n + h [f(t_n, u_n) + f(t_{n+1}, u_{n+1})]$$

$$\begin{cases} u_{n+1} = u_n + \frac{h}{2} [f(t_n, u_n) + f(t_{n+1}, u_{n+1})] \\ u_0 = y_0 \end{cases} \quad 0 \leq n \leq N_h - 1$$

metodo one-step  
metodo implicito

## Analisi metodi one-step

→ convergenza  
 → consistenza  
 → stabilità

### Convergenza

$$t_n = 0, \dots, N_h$$

$$|u_n - y_n| \leq C(h) \quad \text{con } C(h) \text{ infinitesimo rispetto ad } h$$

$e_n = \text{errore associato}$   
 all'istante  $t_n$

$$C(h) = C^* h^p \xrightarrow[h \rightarrow 0]{} 0$$

$$[e_0 = 0]$$

$$e_n = u_n - y_n = (u_n - u_n^*) + (u_n^* - y_n)$$

I                    II

↓ accumulo  
 errore              ↓ 1 passo

$p$  = ordine di convergenza

### Convergenza di EE

$$u_n^* = y_{n-1} + h f(t_{n-1}, y_{n-1})$$

↑ noto              ↓  $y'(t_{n-1})$

$$\begin{aligned} \text{II} \quad u_n^* - y_n &= y_{n-1} - y_n + h f(t_{n-1}, y_{n-1}) \\ &= y_{n-1} - y_n + h y'(t_{n-1}) = -\frac{h^2}{2} y''(\alpha_n) \xrightarrow{h \rightarrow 0} 0 \end{aligned}$$

Se  $y \in C^2(\bar{\mathbb{I}})$

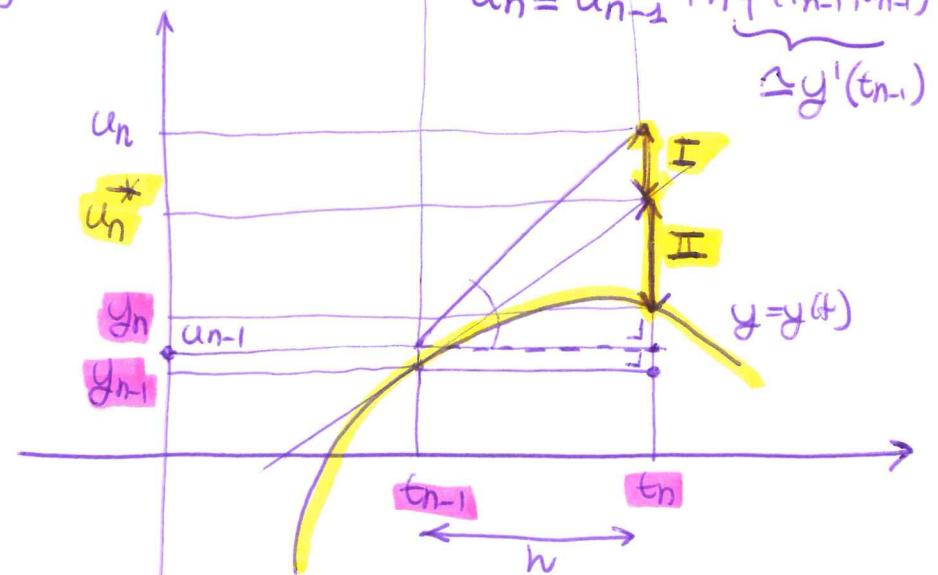
$$\mathbb{I} = (t_0, T)$$

$$\bar{\mathbb{I}} = [t_0, T]$$

Taylor

$$y_n = y_{n-1} + h y'(t_{n-1}) + \frac{h^2}{2} y''(\alpha_n)$$

$$\alpha_n \in (t_{n-1}, t_n)$$



$$\tilde{e}_n(h) = \frac{u_n^* - y_n}{h} \quad \text{errore di truncamento locale} ; \quad \tilde{e}(h) = \max_{n=0,..,N_h} |\tilde{e}_n(h)| \quad \text{errore di truncamento globale}$$

EE

$$\tilde{e}_n(h) = -\frac{h}{2} \underbrace{y''(d_n)}_{f'(x_n, y(x_n))} \Rightarrow \tilde{e}(h) = \frac{h}{2} M \quad M = \max_{t \in I} |f'(t, y(t))|$$

$\hookrightarrow$  EE è consistente;  $q=1$  (ordine di consistenza 1).

Consistenza: uno schema ad un passo  $\square$  è consistente se  $\lim_{h \rightarrow 0} \tilde{e}(h) = 0$ . Inoltre se  $\tilde{e}(h) = O(h^q)$ , lo schema ad un passo ha ordine di consistenza =  $q$ .

## STABILITÀ

ZERO STABILITÀ (intervalli limitati)  $h \rightarrow 0$ ; ( $N_h \rightarrow +\infty$ )

ASSOLUTA STABILITÀ (" " illimitati)

A-STABILITÀ (E)

$$\text{EE} \quad \begin{cases} u_{n+1} = u_n + h f(t_n, u_n) \\ u_0 = y_0 \end{cases}$$

$$I = [t_0, T]$$

$$\text{EE perturbato} \quad \begin{cases} z_{n+1} = z_n + h [f(t_n, z_n) + f_{n+1}] \\ z_0 = y_0 + f_0 \end{cases}$$

$f_0, f_{n+1} \in \mathbb{R}$  perturbazioni sui dati

EE è zero stabile se  $\exists h_0 > 0$  e se  $\exists C > 0$  t.c.  $\forall h \leq h_0$  e  $\forall \varepsilon > 0$  con  $|f_n| \leq \varepsilon$  per  $0 \leq n \leq N_h$

si ha

$$|u_n - z_n| \leq C\varepsilon \quad 0 \leq n \leq N_h.$$

1)  $\exists! z \in \mathbb{R} = z(t)$

2)  $C = C(\exp[(T-t_0)L]) \Rightarrow I$  piccoli

Oss: per i metodi one-step consistenti, la zero-stabilità è una conseguenza di  
EE, EI, CN

$\begin{cases} f \text{ continua rispetto ad} \\ \text{entrambi gli argomenti} \\ f. \text{ Lipschitz continua} \\ \text{rispetto al II argomento} \end{cases}$

consistenza, stabilità, convergenza

Teorema di equivalenza di Lax-Richtmyer

Ogni metodo consistente è convergente se e solo se è zero-stabile.

## ASSOLUTA STABILITÀ

$$I = [t_0, T]$$

$$I = [t_0, +\infty)$$

$h$  fissato

$$N_h \rightarrow +\infty$$

$$t_0 \ t_1 \ t_2$$



problema modello

$$\begin{cases} y'(t) = \lambda y(t) & t \in (0, +\infty), \lambda \in \mathbb{R}^- \\ y(0) = 1 & f(t, y(t)) \\ y_0 & y(t) = e^{\lambda t} \xrightarrow[t \rightarrow +\infty]{} 0 \end{cases}$$

**EE.**

$$u_{n+1} = u_n + h f(t_n, u_n)$$

$$= u_n + h \lambda u_n = (1+h\lambda) u_n$$

$$= (1+h\lambda)^2 u_{n-1}$$

$$= (1+h\lambda)^3 u_{n-2}$$

⋮

$$= (1+h\lambda)^{n+1} u_0$$

$$u_0 = 1$$

condizione di  
assoluta  
stabilità

$$-1 < \underbrace{1+h\lambda}_{\text{vero}} < 1$$

$$h|\lambda| = -h\lambda < 2$$

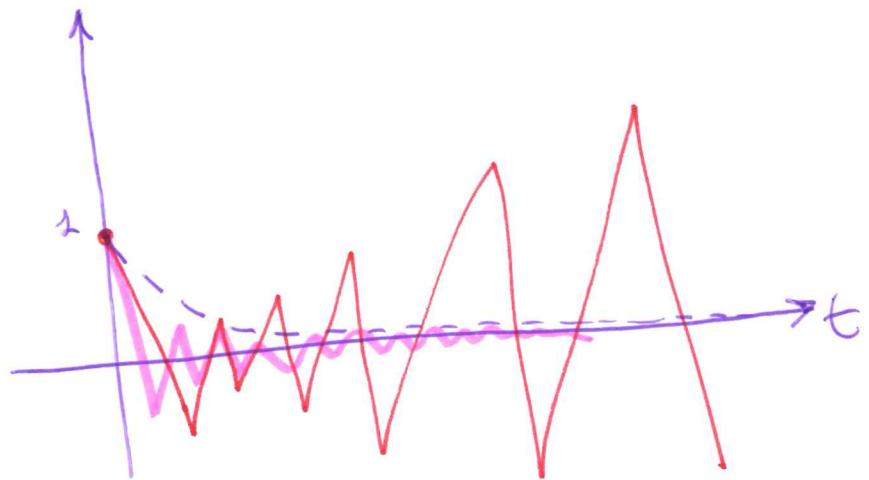
$\hookrightarrow n < \frac{2}{|\lambda|}$

condizione di  
stabilità

$$u_{n+1} = (1+h\lambda)^{n+1}$$

?  $|u_{n+1}| \rightarrow 0$  per  $n \rightarrow +\infty$

(10)



$$\lambda = -1 \quad y(t) = e^{-t}$$

$$|h| < 2$$

$$h_1 = \frac{30}{14} (> 2); \quad h_2 = \frac{30}{16} (< 2); \quad h_3 = \frac{1}{2}$$

— — —

EI

$$u_{n+1} = u_n + h f(t_{n+1}, u_{n+1}) = u_n + h \lambda u_{n+1}$$

$$(1-h\lambda)u_{n+1} = u_n$$

$$\begin{aligned} u_{n+1} &= \left(\frac{1}{1-h\lambda}\right) u_n \\ &= \left(\frac{1}{1-h\lambda}\right)^2 u_{n-1} \\ &= \left(\frac{1}{1-h\lambda}\right)^3 u_{n-2} \end{aligned}$$

$$u_{n+1} = \left(\frac{1}{1-h\lambda}\right)^{n+1}$$

$$\begin{aligned} &= \vdots \\ &= \left(\frac{1}{1-h\lambda}\right)^{n+1} u_0 \\ &\quad || \\ &\quad y_0 = 1 \end{aligned}$$

incondizionalmente  
assolutamente  
stabile

$$? |u_{n+1}| \rightarrow 0, n \rightarrow \infty$$

$$\frac{1}{1-h\lambda} < 1 \quad (\text{f} h)$$

CN

$$u_{n+1} = u_n + \frac{h}{2} [f(t_n, u_n) + f(t_{n+1}, u_{n+1})]$$

$$= u_n + \frac{h}{2} [\lambda u_n + \lambda u_{n+1}]$$

$$\left(1 - \frac{h}{2}\lambda\right)u_{n+1} = \left(1 + \frac{h}{2}\lambda\right)u_n$$

$$u_{n+1} = \left[ \frac{1 + \frac{h}{2}\lambda}{1 - \frac{h}{2}\lambda} \right] u_n = \left[ \frac{1 + \frac{h}{2}\lambda}{1 - \frac{h}{2}\lambda} \right]^2 u_{n-1} = \dots = \left[ \frac{1 + \frac{h}{2}\lambda}{1 - \frac{h}{2}\lambda} \right]^{n+1} u_0$$

$\parallel$   
 $y_0 = 1$

$$u_{n+1} = \left[ \frac{1 + \frac{h}{2}\lambda}{1 - \frac{h}{2}\lambda} \right]^{n+1}$$

$$? |u_{n+1}| \rightarrow 0 \quad n \rightarrow +\infty$$

if  $h$

inconditionnellemente  
absolumente stable

(HZ)

## Metodi predictor-corrector

$$\text{CN} \quad \begin{cases} u_{n+1} = u_n + \frac{h}{2} [f(t_n, u_n) + f(t_{n+1}, u_{n+1})] & n \geq 0 \\ u_0 = y_0 \end{cases}$$

$$u_{n+1}^{(0)} \rightarrow \boxed{u_{n+1}^{(k+1)} = u_n + \frac{h}{2} [f(t_n, u_n) + f(t_{n+1}, u_{n+1}^{(k)})]} \quad k \geq 0$$

$\Phi(u_{n+1}^{(k)})$

$$u_{n+1}^{(k)} \approx u_{n+1}; \quad \{u_{n+1}^{(k)}\}$$

$$\begin{aligned} \alpha &= \phi(\alpha) \\ x^{(k+1)} &= \phi(x^{(k)}) \\ \alpha &\rightarrow u_{n+1} \\ x^{(k+1)} &\rightarrow u_{n+1} \end{aligned}$$

Sé  $u_{n+1}^{(0)}$  é sufficientemente accurata  $\Rightarrow u_{n+1}^{(0)} \approx u_{n+1}$ .

Lo schema predictor/corrector  
ha l'ordine  
di convergenza del  
corrector

Se lo schema Implicito ha ordine p,  
allora  $u_{n+1}^{(0)}$  è ottenuto da uno schema  
esplicito di ordine  $\geq p-1$ .

condizionalmente  
assolutamente  
stabile

Implicito	CN	$p=2$ (corrector)
Esplicito	EE	$p-1=1$ (predictor)

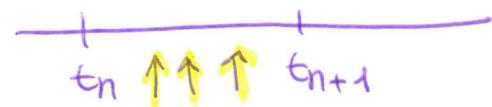
Euler  
migliorato  
(HEUN)  
Ordine 2

$$\begin{cases} u_{n+1}^{(0)} = u_n + h f(t_n, u_n) & \text{PREDIZIONE (EE)} \\ u_{n+1}^{(1)} = u_n + \frac{h}{2} [f(t_n, u_n) + f(t_{n+1}, u_{n+1}^{(0)})] & \text{CORREZIONE (CN)} \\ \hookrightarrow u_{n+1} \end{cases}$$

## METODI DI ALTO ORDINE

"one-step"  
Runge - kutta

multi-step  
(Adams, BDF)



### Metodo di Runge-kutta (RK) a s stadi

$$\begin{cases} u_{n+1} = u_n + h \sum_{i=1}^s b_i k_i & n \geq 0 \\ k_i = f(t_n + c_i h, u_n + h \sum_{j=1}^s a_{ij} k_j) & i = 1, \dots, s \\ u_0 = y_0 \end{cases}$$

### array di Butcher

$$\begin{array}{c|ccccc} \vec{c} & A \\ \hline & \vec{b} \end{array}$$

$$[a_{ij}] = A \in \mathbb{R}^{s \times s}$$

$$\vec{c} = [c_1, c_2, \dots, c_s]^T \in \mathbb{R}^s$$

$$\vec{b} = [b_1, b_2, \dots, b_s]^T \in \mathbb{R}^s$$

$$\begin{array}{c|cccccc} c_1 & a_{11} & a_{12} & \cdots & a_{1s} & \rightarrow k_1 \\ c_2 & a_{21} & a_{22} & \cdots & a_{2s} & \rightarrow k_2 \\ \vdots & \vdots & & & & & \\ c_s & a_{s1} & a_{s2} & \cdots & a_{ss} & \rightarrow k_s \\ \hline & b_1 & b_2 & \cdots & b_s \end{array}$$

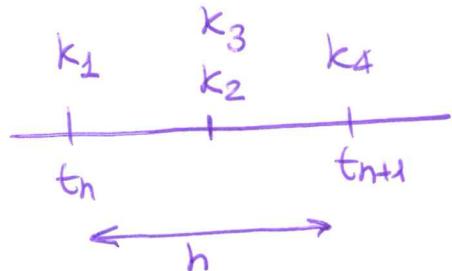
se le sole eurate  
sono solo  
per  $j \leq i$ ,  $a_{ij} \neq 0$

allora lo schema RK è  
esplicito.

Viceversa è implicito.  
(sistema di eq.  
nonlineare)

Rk4

$$\begin{cases} u_{n+1} = u_n + \frac{h}{6} [k_1 + 2k_2 + 2k_3 + k_4] & n \geq 0 \\ u_0 = y_0 \end{cases}$$



$$k_1 = f(t_n, u_n)$$

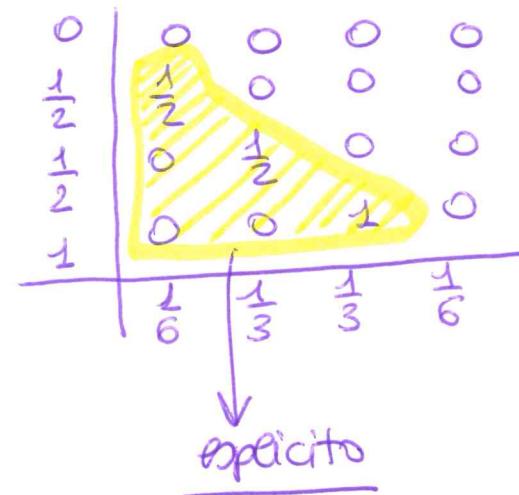
$$k_2 = f\left(t_n + \frac{h}{2}, u_n + \frac{h}{2} k_1\right)$$

$$k_3 = f\left(t_n + \frac{h}{2}, u_n + \frac{h}{2} k_2\right)$$

$$k_4 = f(t_{n+1}, u_n + h k_3)$$

ordine di convergenza 4

$s=4$



EE = Rk1

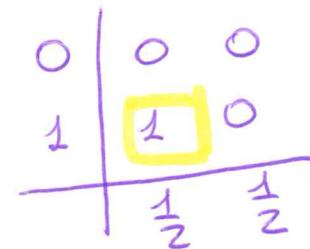
$$u_{n+1} = u_n + \underbrace{h f(t_n, u_n)}_{k_1}$$

$s=1$

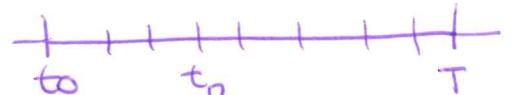
$$\begin{array}{c|cc} 0 & 0 \\ \hline & 1 \end{array}$$

HEUN = Rk2

$$\begin{cases} u_{n+1}^{(1)} = u_n + \frac{h}{2} [\underbrace{f(t_n, u_n)}_{k_1} + \underbrace{f(t_{n+1}, u_{n+1}^*)}_{k_2}] \\ u_{n+1}^* = u_n + h \underbrace{f(t_n, u_n)}_{k_1} \end{cases}$$



Matlab   
 {ode23  
 {ode45

$$y_i : [t_0, T] \longrightarrow \mathbb{R} \quad i = 1, \dots, m \quad y_1, y_2, \dots, y_m$$


$$\begin{cases} \vec{y}'(t) = f_1(t, y_1(+), y_2(+), \dots, y_m(+)) \\ y_1'(+) = f_2(t, y_1(+), y_2(+), \dots, y_m(+)) \\ \vdots \\ y_m'(+) = f_m(t, \underbrace{y_1(+), y_2(+), \dots, y_m(+)}_{\vec{q}(+)}) \end{cases}$$

$$\begin{cases} y_1(t_0) = y_{0,1} \\ y_2(t_0) = y_{0,2} \\ \vdots \\ y_m(t_0) = y_{0,m} \end{cases}$$

$$\vec{y}(+) = \begin{bmatrix} y_1(+) \\ y_2(+) \\ \vdots \\ y_m(+) \end{bmatrix}; \quad \vec{F}(+, \vec{q}(+)) = \begin{bmatrix} f_1(t, \vec{q}(+)) \\ f_2(+, \vec{q}(+)) \\ \vdots \\ f_m(+, \vec{q}(+)) \end{bmatrix}; \quad \vec{y}_0 = \begin{bmatrix} y_{0,1} \\ y_{0,2} \\ \vdots \\ y_{0,m} \end{bmatrix} \in \mathbb{R}^m$$

$$\begin{cases} \vec{y}'(+) = \vec{F}(t, \vec{y}(+)) \\ \vec{y}(t_0) = \vec{y}_0 \end{cases}$$

EE

$$\begin{aligned} u_{n+1} &= u_n + h f(t_n, u_n) \\ u_0 &= y_0 \\ \vec{u}_{n+1} &= \vec{u}_n + h \vec{F}(t_n, \vec{u}_n) \\ \vec{u}_0 &= \vec{y}_0 \end{aligned}$$

$$u_{n+1,i} \approx y_i(t_{n+1})$$

$$u_{n+1} \approx y(t_{n+1})$$

$i = 1, \dots, m$

$$\text{EE } \left\{ \begin{array}{l} u_{n+1,1} = u_{n,1} + h f_2(t_n, u_{n,1}, u_{n,2}, \dots, u_{n,m}) \\ u_{n+1,2} = u_{n,2} + h f_2(t_n, u_{n,1}, u_{n,2}, \dots, u_{n,m}) \\ \vdots \\ u_{n+1,m} = u_{n,m} + h f_m(t_n, u_{n,1}, u_{n,2}, \dots, u_{n,m}) \end{array} \right. ;$$

$\vec{u}_{n+1} \in \mathbb{R}^m$        $\vec{u}_n \in \mathbb{R}^m$        $\vec{F}(t_n, \vec{u}_n)$

$$\left\{ \begin{array}{l} u_{0,1} = y_{0,1} \\ u_{0,2} = y_{0,2} \\ \vdots \\ u_{0,m} = y_{0,m} \end{array} \right. ;$$

$\vec{u}_0$        $\vec{y}_0$

$$\left\{ \begin{array}{l} \vec{u}_{n+1} = \vec{u}_n + h [\theta \vec{F}(t_{n+1}, \vec{u}_{n+1}) + (1-\theta) \vec{F}(t_n, \vec{u}_n)] \\ \vec{u}_0 = \vec{y}_0 \end{array} \right. \quad \theta - \text{metodo}$$

$$0 \leq \theta \leq 1$$

$$\theta = 0 \quad \text{EE}$$

$$\theta = 1 \quad \text{EI}$$

$$\theta = \frac{1}{2} \quad \text{CN}$$

$$aF + bG \quad a, b \in \mathbb{R}$$

c.e.  
convex

$$\begin{cases} a+b=1 \\ \frac{1}{2} \quad \frac{1}{2} \\ \theta, 1-\theta \end{cases}$$

$$\begin{cases} y^{(m)}(t) = f(t, y(t), y'(t), \dots, y^{(m-1)}(t)) & t \in [t_0, T] \\ y(t_0) = y_0, \quad y'(t_0) = y_1, \quad \dots, \quad y^{(m-1)}(t_0) = y_{m-1} \end{cases}$$

CAMBIO DI VARIABILE

$$w_1(t) = y(t), \quad w_2(t) = y'(t), \dots, \quad w_m(t) = y^{(m-1)}(t)$$

$$\begin{cases} w_1'(t) = w_2(t) \\ w_2'(t) = w_3(t) \\ \vdots \\ w_m'(t) = f(t, w_1(t), w_2(t), \dots, w_m(t)) \end{cases} + \begin{cases} w_1(t_0) = y_0 \\ w_2(t_0) = y_1 \\ \vdots \\ w_m(t_0) = y_{m-1} \end{cases}$$

↑  
θ - metodo

Equazioni alle derivate parziali (EDP)  
 [Partial Differential Equations (PDEs)]  
 ODE

ODE

$$y = y(t)$$

PDE

$$u = u(x, t)$$

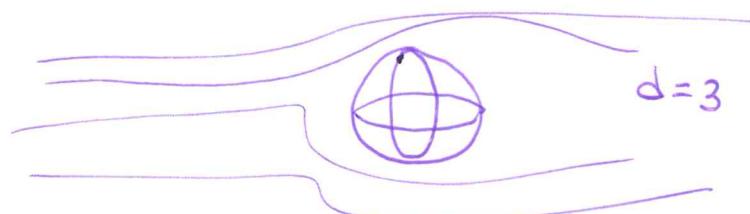
$$0 = F(x, t, u, \underbrace{\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_d}}_{\mathbb{R}^d}) \quad \dots$$

$x = [x_1, \dots, x_d]^T$

$\downarrow$   
 $d=1$

$$\frac{\partial^{p_1 + \dots + p_d + p_t} u}{\partial x_1^{p_1} \dots \partial x_d^{p_d} \partial t^{p_t}}, g \Big)$$

dati  
del  
problema



$$\text{ordine } p = p_1 + \dots + p_d + p_t$$

PDE  $\rightarrow$  lineare  $u_t + \dots + u_{xx} = f$

$\rightarrow$  non lineare  $u_t u_{xx} + \dots = f$

PDE  $\rightarrow$   $x$  stationarie  
 $\rightarrow x, t$  non stationarie

classificazione :

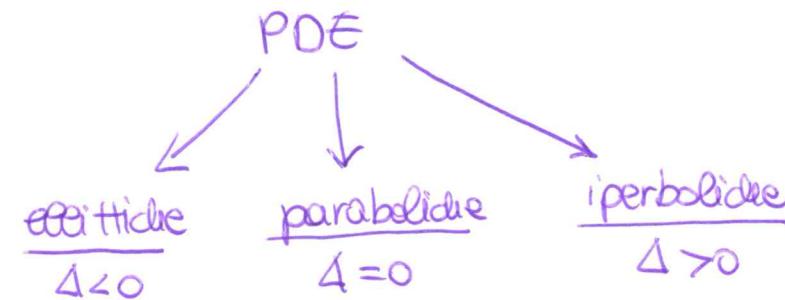
$$Lu = g$$

$$Lu = A \frac{\partial^2 u}{\partial x_1^2} + B \frac{\partial^2 u}{\partial x_1 \partial x_2} + C \frac{\partial^2 u}{\partial x_2^2} + D \frac{\partial u}{\partial x_1} + E \frac{\partial u}{\partial x_2} + Fu$$

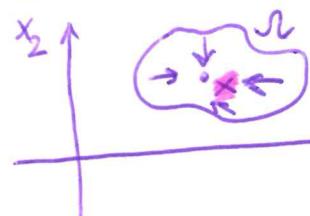
$$\begin{aligned}x_1 &= x \\x_2 &= y, t\end{aligned}$$

$$A, B, C, D, E, F \in \mathbb{R}$$

$$\Delta = B^2 - 4AC$$



Esempi : equazione di Laplace



$$\left\{ \begin{array}{l} -\Delta u = 0 \\ u = g \end{array} \right. \quad \begin{array}{l} \mathcal{N} \subset \mathbb{R}^d \\ \text{su } \partial \mathcal{N} \text{ (CB)} \end{array}$$

$$\left\{ \begin{array}{l} -u'' = f \\ u = g \end{array} \right. \quad \begin{array}{l} d=1 \\ \mathcal{N} = I = [a,b] \\ \text{in } x=a,b \end{array}$$

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$x_1 = x \quad x_2 = y$$

$$\Delta u = \operatorname{div}(\nabla u)$$

$$\begin{array}{ll} A = -1 = C & B = D = E = F = 0 \\ \Delta = -4 < 0 & \text{ellittico} \end{array}$$

$u : \mathbb{R} \rightarrow \mathbb{R}$

$\nabla u \in \mathbb{R}^d$

$$\nabla u = \begin{bmatrix} \frac{\partial u}{\partial x_1} \\ \vdots \\ \frac{\partial u}{\partial x_d} \end{bmatrix}$$

$d=2$

$$\nabla u = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix}$$

$\nabla$

scarsi  $\rightarrow$  vettori

div

$$\nabla \cdot \begin{bmatrix} v_1 \\ \vdots \\ v_d \end{bmatrix} = \frac{\partial v_1}{\partial x_1} + \dots + \frac{\partial v_d}{\partial x_d}$$

$d=2$

$d=2$

$$\nabla \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2}$$

$\nabla$ : vettori  $\rightarrow$  scalari

$$\nabla \cdot (\nabla u) = \nabla \cdot \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix} = \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial y} \right) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \Delta u$$

$\Delta$  scalari  $\rightarrow$  scalari

② equazione del colore ( $d=1$ )

$$\begin{cases} \frac{\partial u}{\partial t} - \mu \frac{\partial^2 u}{\partial x^2} = 0 & (0,L) \times (0,T) \\ + \text{CB} \quad (\text{condizione al bordo}) \\ + \text{CI} \quad (\text{condizione iniziale}) \end{cases}$$

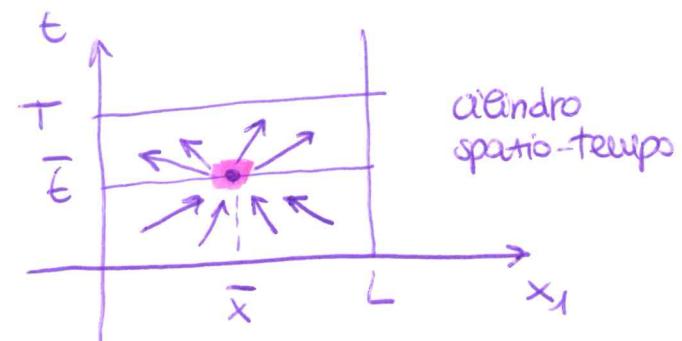
$$d=2,3 \begin{cases} \frac{\partial u}{\partial t} - \mu \Delta u = 0 \\ + \text{CB} \\ + \text{CI} \end{cases}$$

$$x_1 = x \quad x_2 = t$$

$$A = -\mu \quad E = 1$$

$$B = C = D = F = 0$$

$\Delta = 0 \rightarrow$  parabolico



③ equazione delle onde ( $d=1$ )

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 & (0,L) \times (0,T) \\ + \text{CB} \\ + 2 \text{CI} \end{cases}$$

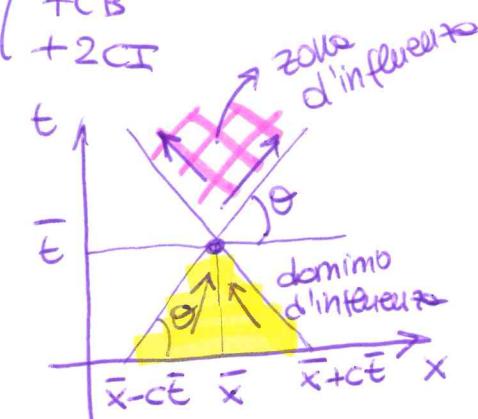
$$x_1 = x$$

$$x_2 = t$$

$$A = -c^2; C = 1; B = D = E = F = 0$$

$\Delta = 4c^2 > 0 \rightarrow$  iperbolico

$$c = \frac{1}{\tan \theta}$$



## Equazione di Poisson (Laplace)

$$\begin{cases} -\mu \Delta u = f \\ +CB \end{cases}$$

$$\begin{pmatrix} 2D/3D & \begin{cases} -\mu \Delta u = 0 \\ +CB \end{cases} \end{pmatrix}$$

$u = u(x) \quad x \in \Omega \subset \mathbb{R}^d \quad d \geq 1, \mu \text{ costante}$

$$\begin{cases} -\mu u'' = f \\ +CB \end{cases}$$

$$\begin{pmatrix} 1D & \begin{cases} -\mu u'' = 0 \\ +CB \end{cases} \end{pmatrix}$$

$$\left[ \begin{array}{lll} -\mu \Delta u & \Delta u = \nabla \cdot (\nabla u) & -\nabla \cdot [\mu(x) \nabla u] \\ \downarrow \text{costante} & & \end{array} \right]$$

Glossario : 1) problemi ai limiti / ai valori al bordo

2)  $\Omega \subset \mathbb{R}^d$  dominio limitato  
(insieme aperto e connesso)

1D intervallo  $\xrightarrow[a]{\hspace{1cm}} b$

3) bordo del dominio  $\partial\Omega$

1D  $x=a, x=b$

4) condizione al bordo : configurazione all'equilibrio al bordo  
CB

Problema stationario  
(configurazione di equilibrio)

EX 1 : termico

$u$  temperatura

$f$  sorgente calore

EX 2 : meccanico

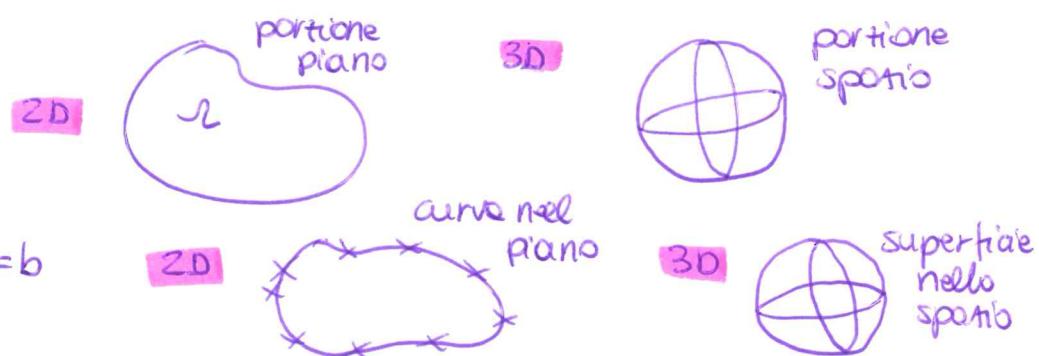
$u$  spostamento

$f$  carico

EX 3 : fluidodinamica

$u$  concentrazione

$f$  sorgente



## CONDIZIONI DI DIRICHLET ( $u$ )

$$u(x) = g(x) \quad x \in \partial\Omega$$

0       $\neq 0$

caso omogeneo      caso non omogeneo

EX 2

[ fissiamo le posizioni del bordo della membrana ]

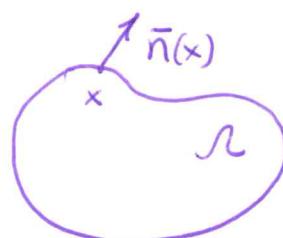
## CONDIZIONE DI NEUMANN (derivata di $u$ )

1D

$$\mu u'(x) = h(x) \quad x=a, x=b$$

0       $\neq 0$

omogeneo      non omogeneo



$$\nabla u \cdot n = \left( \frac{\partial u}{\partial n} \right) \text{ derivata normale}$$

$$\mu \nabla u(x) \cdot n(x) = h(x) \quad x \in \partial\Omega$$

0       $\neq 0$

omogeneo      non omogeneo



[ attacco elastico ]

## CONDIZIONE DI ROBIN ( $u$ e derivata di $u$ )

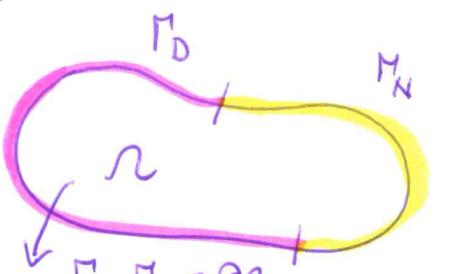
$$\alpha(x)u(x) + \mu(x)\nabla u(x) \cdot n(x) = q(x) \quad x \in \partial\Omega$$

$$\alpha(x) \in \mathbb{R}^+$$

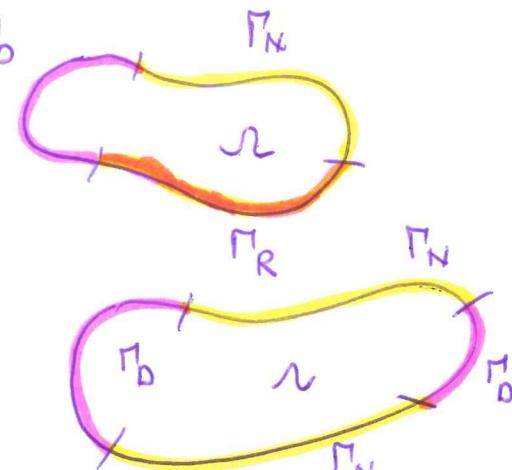
## CONDIZIONI MISTE (combinazione precedenti)

D/N

$$\begin{cases} u(x) = g(x) & x \in \Gamma_D \\ \mu(x)\nabla u(x) \cdot n(x) = h(x) & x \in \Gamma_N \end{cases}$$



$$\Gamma_D \cap \Gamma_N = \emptyset \text{ e } \bar{\Gamma}_D \cup \bar{\Gamma}_N = \partial\Omega$$



## Analisi dell'equazione di Poisson

A Esistenza e unicità: sia  $\Omega \subset \mathbb{R}^d$  un dominio limitato e con frontiera regolare. Allora esiste al più una funzione di classe  $C^2(\Omega) \cap C^1(\bar{\Omega})$  soluzione del problema di Poisson in  $\Omega$  e tale che su  $\partial\Omega$  vengono assegnate CB

$\left\{ \begin{array}{l} \text{Dirichlet} \\ \text{Robin} \\ \text{miste D/N} \end{array} \right.$	per dati sufficientemente regolari.
-----------------------------------------------------------------------------------------------------	-------------------------------------

Nel caso di Neumann invece due soluzioni differiscono per una costante ( $\bar{c}$  garantita solo  $\exists$ )



$$\left\{ \begin{array}{l} -\mu \Delta u = f \quad x \in \Omega \\ \mu \nabla u \cdot n = h \quad x \in \partial\Omega \end{array} \right.$$

$u+c$  con  $c$  costante

$$-\mu \Delta(u+c) = -\mu \Delta u - \underbrace{\mu \Delta c}_{=0} = -\mu \Delta u = f$$

$$\mu \nabla(u+c) \cdot n = \underbrace{\mu \nabla u \cdot n}_h + \underbrace{\mu \nabla c \cdot n}_{=0} = h$$

$$-\mu \Delta(u+c) + \rho(u+c) =$$

$$\cancel{-\mu \Delta u} \cancel{x=0} - \mu \Delta c + \rho u + \rho c = f + \rho c$$

Osservazione: condizione di compatibilità

FORMULA DI GAUSS  
(teorema della divergenza)

$$\int_{\Omega} f(x) dx = - \int_{\Omega} \mu \Delta u(x) dx = \int_{\Omega} \mu \nabla \cdot (\nabla u)(x) dx =$$

$$= - \int_{\partial\Omega} \mu \nabla u(s) \cdot n(s) ds = - \int_{\partial\Omega} h(s) ds$$

$$\int_{\Omega} f(x) dx = - \int_{\partial\Omega} h(s) ds$$

RICEDI

- 1) ulteriore caratterizzazione fisica
- 2) aggiungere un termine di ordine zero ( $\rho u$ )

Ex:

$$\begin{cases} -\mu \Delta u(x) = f & \text{in } \Omega = (0,L) \times (0,H) \\ \mu \frac{\partial u}{\partial y}(x,0) = q_1 & x \in (0,L) \quad (\Gamma_1) \\ -\mu \frac{\partial u}{\partial y}(x,H) = q_2 & x \in (0,L) \quad (\Gamma_2) \\ \mu \frac{\partial u}{\partial x}(0,y) = q_3 & y \in (0,H) \quad (\Gamma_3) \\ -\mu \frac{\partial u}{\partial x}(L,y) = q_4 & y \in (0,H) \quad (\Gamma_4) \end{cases}$$

$$f, q_i \in \mathbb{R} \quad i=1, \dots, 4$$

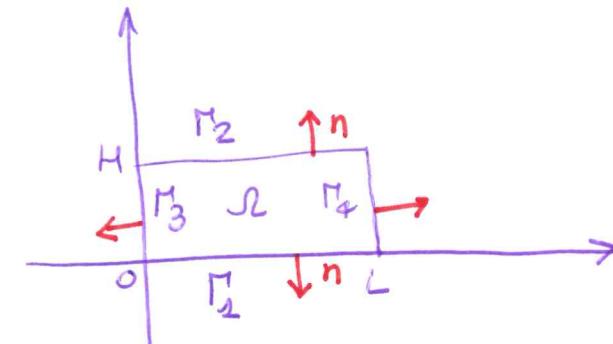
Condizione di compatibilità:

$$\int_{\Omega} f(x) dx = - \int_{\partial\Omega} f(HL) = - \sum_{i=1}^4 \int_{\Gamma_i} q_i(s) ds$$

$$= - \sum_{i=1}^4 q_i |\Gamma_i|$$

$$= - [(q_1 + q_2)L + (q_3 + q_4)H]$$

$$f = - \frac{[(q_1 + q_2)L + (q_3 + q_4)H]}{HL} = - \frac{[q_1 + q_2]}{H} - \frac{[q_3 + q_4]}{L}$$



$$-\mu \frac{\partial u}{\partial n} = -\mu \nabla u \cdot n = Q$$

$$\Gamma_1 \quad n = (0,1)^T$$

$$\Gamma_2 \quad n = (0,-1)^T$$

$$\Gamma_3 \quad n = (-1,0)^T$$

$$\Gamma_4 \quad n = (1,0)^T$$

## Laplace

Definizione: una funzione  $u$  è armonica in un dominio  $\Omega \subset \mathbb{R}^d$  se  $u \in C^2(\Omega)$  e  $\Delta u = 0$  in  $\Omega$ .

Osservazione: la soluzione di  $\Delta u = 0$  è una funzione armonica.

**B** Teorema delle medie: sia  $u$  una funzione armonica in  $\Omega \subset \mathbb{R}^d$ . Allora  $\forall B_R(x) = \{y \in \mathbb{R}^d : \|y-x\| < R\}$  tale che  $\bar{B}_R(x) \subset \Omega$  valgono le formule

$$u(x) = \frac{1}{|B_R(x)|} \int_{B_R(x)} u(y) dy \quad ; \quad u(x) = \frac{1}{|\partial B_R(x)|} \int_{\partial B_R(x)} u(s) ds$$

con  $|\omega|$  misura di un generico insieme  $\omega \subset \mathbb{R}^d$ .

$$\left[ \begin{array}{ll} \text{2D} & |B_R(x)| = \pi R^2 \\ & |\partial B_R(x)| = 2\pi R \\ \text{3D} & |B_R(x)| = \frac{4}{3}\pi R^3 \\ & |\partial B_R(x)| = 4\pi R^2 \end{array} \right]$$

Osservazione: 1) la soluzione di  $\Delta u = 0$  soddisfa la proprietà delle medie

2) Teorema: sia  $u \in C^0(\Omega) \setminus \{x \in \Omega \text{ e } \forall B_R(x) \subset \Omega \text{ vale una delle formule delle medie, allora se } u \text{ è armonica in } \Omega \text{ e } u \in C^\infty(\Omega)\}$

C

### Principio del massimo:

Teorema: se  $u \in C^0(\bar{\Omega})$  soddisfa la proprietà della media in  $\Omega \subset \mathbb{R}^d$  e per  $x \in \Omega$  è un punto interno ad  $\Omega$  ed è un punto estremo di  $u$  (ovvero è un massimo o un minimo di  $u$ ) globale, allora  $u$  è costante.

↳ massimi e minimi sono confinati al bordo

$$\max_{x \in \bar{\Omega}} u(x) = \max_{x \in \partial\Omega} u(x) \quad \text{e} \quad \min_{x \in \bar{\Omega}} u(x) = \min_{x \in \partial\Omega} u(x)$$

$$\left[ \min_{x \in \partial\Omega} u(x) \leq u(x) \leq \max_{x \in \partial\Omega} u(x) \right]$$

- Osservazioni: 1) una funzione armonica soddisfa il principio del massimo.  
 2) le soluzioni di  $\Delta u = 0$  soddisfano " " " " " .

3) Teorema: data una funzione continua  $v \in C^0(\bar{\Omega})$  con  $\Omega \subset \mathbb{R}^d$ . Se  $\Delta v(x) > 0 \quad \forall x \in \Omega$  allora

$$v(x) \leq \max_{x \in \partial\Omega} v(x);$$

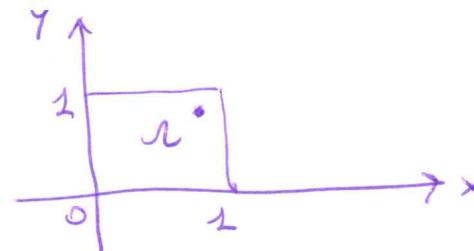
Viceversa se  $\Delta v(x) < 0 \quad \forall x \in \Omega$  allora

$$\boxed{-\Delta u = f} \quad \begin{array}{l} \text{se } f(x) < 0 \quad \forall x \in \Omega \\ \Rightarrow \Delta u > 0 \quad \forall x \in \Omega \Rightarrow u \text{ raggiunge il max sul bordo,} \\ \text{se } f(x) > 0 \quad \forall x \in \Omega \\ \Rightarrow \Delta u < 0 \quad \forall x \in \Omega \Rightarrow u \text{ raggiunge il min sul bordo.} \end{array}$$

Esempio -  $\Delta u = f$  in  $\Omega = (0,1)^2$

$$f = 1 + |x-y|$$

$$(0,1) \times (0,1)$$



CB Dirichlet non omogenee:  $u(x,y) = g(x,y) = x + 2y$

$$f(x) > 0 \quad \forall x \in \Omega \Rightarrow \Delta u(x) < 0 \quad \forall x \in \Omega \Rightarrow \min_{\substack{x \in \partial \Omega}} u(x) \leq u(x) \quad \forall x \in \Omega$$

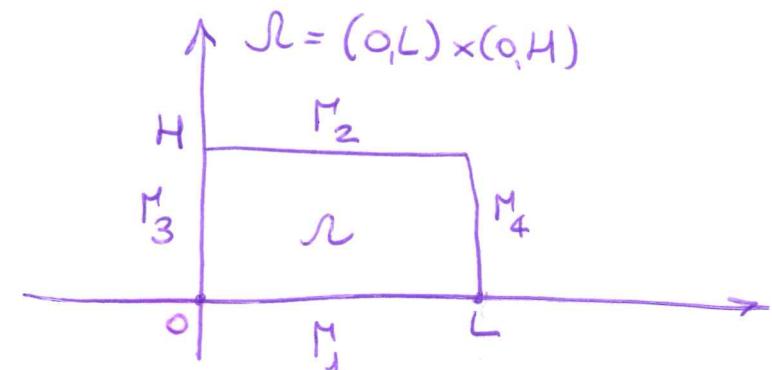
$$0 = \min_{\substack{x \in \partial \Omega}} g(x) = \min_{\substack{x \in \partial \Omega}} u(x) \leq u(x) \quad \forall x \in \Omega$$

Separazione delle variabili in coordinate cartesiane

$$\begin{cases} -\Delta u(x,y) = 0 & x \in \Omega \\ u(x,0) = g_1(x) & x \in (0,L) \quad (\Gamma_1) \\ u(x,H) = 0 & x \in (0,L) \quad (\Gamma_2) \\ u(0,y) = u(L,y) = 0 & y \in (0,H) \quad (\Gamma_3, \Gamma_4) \end{cases}$$

$$\partial \Omega = \bigcup_{i=1}^4 \Gamma_i$$

$$g_1 : (0,L) \rightarrow \mathbb{R}$$



$$\begin{aligned} \Gamma_1 &= (0,L) \times \{0\} \\ &= \{(x,y) \in \mathbb{R}^2 : x \in (0,L), y = 0\} \end{aligned}$$

$$\Gamma_2 = (0,L) \times \{H\};$$

$$\Gamma_3 = \{0\} \times (0,H); \quad \Gamma_4 = \{L\} \times (0,H) \quad (129)$$

$$? \cup : \mathcal{N} \rightarrow \mathbb{R} \quad \text{t.c.} \quad \cup(x,y) = X(x) Y(y)$$

$$X: (0,L) \rightarrow \mathbb{R} ; \quad Y: (0,H) \rightarrow \mathbb{R}$$

PASSO 2

$$-\Delta \cup = 0 \text{ in } \mathcal{N}$$

$$(1) \quad -\Delta \cup(x,y) = -\frac{\partial^2 \cup}{\partial x^2}(x,y) - \frac{\partial^2 \cup}{\partial y^2}(x,y) = 0 \quad x \in (0,L) \\ y \in (0,H)$$

$$\frac{\partial \cup}{\partial x}(x,y) = X'(x) Y(y)$$

$$\frac{\partial \cup}{\partial y}(x,y) = X(x) Y'(y)$$

$$\frac{\partial^2 \cup}{\partial x^2}(x,y) = X''(x) Y(y)$$

$$\frac{\partial^2 \cup}{\partial y^2}(x,y) = X(x) Y''(y)$$

$$+ X''(x) Y(y) + X(x) Y''(y) = 0 \quad x \in (0,L) \quad y \in (0,H)$$

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = k \quad \text{costante} \in \mathbb{R}$$

2 problemi  
ai valori  
di bordo

$$\begin{cases} X''(x) - k X(x) = 0 & x \in (0,L) \\ Y''(y) + k Y(y) = 0 & y \in (0,H) \end{cases} \quad (2) \quad (3)$$

PASSO 2

$$U(0,y) = U(L,y) = 0 \quad y \in (0,H)$$



$$X(0)Y(y) = X(L)Y(y) = 0 \quad y \in (0,H)$$

$$\hookrightarrow X(0) = X(L) = 0 \quad \text{CB per (2)}$$

$$(2) \Leftrightarrow \begin{cases} +X''(x) - kX(x) = 0 & x \in (0,L) \\ X(0) = 0 \\ X(L) = 0 \end{cases} \text{ CB}$$

CASO 2

$$k = \lambda^2 \quad \text{con } \lambda \in \mathbb{R} \quad \text{e A } \lambda \neq 0$$

$$X(x) = Ae^{\lambda x} + Be^{-\lambda x} \quad A, B \in \mathbb{R}$$

$$\begin{cases} X(0) = A + B = 0 \\ X(L) = Ae^{\lambda L} + Be^{-\lambda L} = 0 \end{cases} \Rightarrow A = B = 0$$



CASO 2

$$k = 0 \Rightarrow X(x) = A + Bx, \quad A, B \in \mathbb{R}$$

$$X(x) = 0 \quad \forall x \in [0,L]$$

$$\begin{cases} X(0) = A = 0 \\ X(L) = BL = 0 \end{cases} \Rightarrow A = B = 0 \Rightarrow X(x) = 0 \quad \forall x \in [0,L]$$

CASO 3

$$k = -\lambda^2 \quad \lambda \in \mathbb{R} \text{ ma } \lambda \neq 0$$

$$X(x) = A \cos(\lambda x) + B \sin(\lambda x) \quad A, B \in \mathbb{R}$$

$$\begin{cases} X(0) = A \cos(\lambda \cdot 0) + B \sin(\lambda \cdot 0) = A = 0 \\ X(L) = A \cos(\lambda L) + B \sin(\lambda L) = 0 \end{cases} \Rightarrow \begin{cases} A = 0 \\ B \sin(\lambda L) = 0 \end{cases}$$

$$\begin{cases} A = 0 \\ B = 0 \end{cases}$$

$$\begin{aligned} X(x) &= 0 \\ x \in [0, L] \end{aligned}$$

$$\begin{cases} A = 0 \\ \lambda L = n\pi \quad n = 1, 2, \dots \end{cases}$$

$$\lambda_n = \frac{n\pi}{L}$$

$$X_n(x) = B_n \sin(\lambda_n x) = B_n \sin\left(\frac{n\pi}{L} x\right) \quad n = 1, 2, \dots$$

$$\{B_n\}_{n=1}^{\infty}$$

PASSO 3

$$k \rightarrow k_n = -\lambda_n^2 = -\left(\frac{n\pi}{L}\right)^2$$

$$Y''(y) - \left(\frac{n\pi}{L}\right)^2 Y(y) = 0 \quad y \in (0, L) \quad n = 1, 2, \dots$$

$$Y_n(y) = D_{1,n} e^{\left(\frac{n\pi}{L}\right)y} + D_{2,n} e^{-\left(\frac{n\pi}{L}\right)y} \quad n = 1, 2, \dots$$

$$\{D_{1,n}\}_{n=1}^{\infty} \quad \{D_{2,n}\}_{n=1}^{\infty}$$

## Soluzioni fondamentali

$$U_n(x,y) = X_n(x) Y_n(y) \quad n=1, 2, \dots$$

$$\begin{cases} -\Delta u = 0 \\ \text{CB } M_3, M_4 \end{cases}$$

$$U(x,y) = \sum_{n=1}^{+\infty} U_n(x,y) = \sum_{n=1}^{+\infty} X_n(x) Y_n(y)$$

$$M_2 \quad U(x,H) = 0 \quad x \in (0,L)$$

$$U_n(x,H) = X_n(x) Y_n(H) = 0 \quad x \in (0,L), \quad n=1, 2, \dots$$

$$\Rightarrow Y_n(H) = 0$$

$$Y_n(H) = D_{2,n} e^{(\frac{n\pi}{L})H} + D_{2,n} e^{-\left(\frac{n\pi}{L}\right)H} = 0 \quad n=1, 2, \dots$$

$$\Rightarrow D_{2,n} = - \frac{D_{2,n} e^{\left(\frac{n\pi}{L}\right)H}}{e^{-\left(\frac{n\pi}{L}\right)H}} = - D_{4,n} e^{2\left(\frac{n\pi}{L}\right)H} \quad n=1, 2, \dots$$

$$Y_n(y) = D_{2,n} \left[ e^{\left(\frac{n\pi}{L}\right)y} - e^{2\left(\frac{n\pi}{L}\right)y} e^{-\left(\frac{n\pi}{L}\right)y} \right] \quad n=1, 2, \dots$$

$$\sinh(z) = \frac{e^z - e^{-z}}{2} = \sinh(-z) = \frac{e^{-z} - e^z}{2}$$

$$Q = \frac{n\pi}{L} \quad \left[ e^{Qy} \quad e^{2QH} \quad -e^{-Qy} \right] \cdot \frac{-e^{-QH}}{e^{-QH}} = \frac{e^{Q(H-y)} - e^{Q(H-y)}}{2e^{-QH}} \cdot 2$$

$$Y_n(y) = D_{2,n}^* \sinh \left[ \frac{n\pi}{L}(H-y) \right] \quad n=1, \dots \\ D_{2,n}(-2)e^{QH}$$

$$= 2 \frac{e^{\underline{Q(H-y)}} - e^{-\underline{Q(H-y)}}}{ze^{-QH}}$$

$$= -2 \frac{e^{Q(H-y)} - e^{-Q(H-y)}}{ze^{-QH}}$$

$$\sinh \left[ \frac{n\pi}{L}(H-y) \right]$$

$$U_n(x,y) = X_n(x) Y_n(y)$$

$$= B_n \sin \left( \frac{n\pi}{L} x \right) D_{2,n}^* \sinh \left[ \frac{n\pi}{L}(H-y) \right]$$

$$U_n(x,y) = C_n \sin \left( \frac{n\pi}{L} x \right) \sinh \left[ \frac{n\pi}{L}(H-y) \right] \quad n=1, 2, \dots$$

$$B_n D_{2,n}^*$$

$$\hookrightarrow U(x,y) = \sum_{n=1}^{+\infty} C_n \sin \left( \frac{n\pi}{L} x \right) \sinh \left( \frac{n\pi}{L}(H-y) \right)$$

$$\begin{cases} -\Delta u = 0 \\ \text{CB } \Pi_2, M_3, M_4 \end{cases}$$

PASSO 4

$$U(x, 0) = g_2(x) \quad x \in (0, L)$$

$$\sum_{n=1}^{+\infty} C_n \underbrace{\sin\left(\frac{n\pi}{L}x\right)}_{\text{.}} \sinh\left(\frac{n\pi}{L}H\right) = g_2(x) \quad x \in (0, L)$$

↓  
sufficientemente regolare  
 $(L^2(0, L))$

$$g_2(x) = \sum_{n=1}^{+\infty} A_{2,n} \underbrace{\sin\left(\frac{n\pi}{L}x\right)}_{\uparrow \text{noti}}$$

$$C_n \sinh\left(\frac{n\pi}{L}H\right) = A_{2,n}$$

$$C_n = \frac{A_{2,n}}{\sinh\left(\frac{n\pi}{L}H\right)}$$

$$A_{2,n} = \frac{2}{L} \int_0^L g_2(x) \sin\left(\frac{n\pi}{L}x\right) dx$$

$$U(x, y) = \sum_{n=1}^{+\infty} \frac{A_{2,n}}{\sinh\left(\frac{n\pi}{L}H\right)} \sin\left(\frac{n\pi}{L}x\right) \sinh\left(\frac{n\pi}{L}(H-y)\right)$$

$$\left\{ \sin\left(\frac{n\pi}{L}x\right) \right\}_{n=1}^{+\infty}$$

$$\int_0^L \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{L}x\right) dx = \frac{L}{2} \delta_{mn}$$

$$\sum_{n=1}^{+\infty} A_{1,n} \underbrace{\int_0^L \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{L}x\right) dx}_{\frac{L}{2} \delta_{mn}} = \int_0^L g_1(x) \sin\left(\frac{m\pi}{L}x\right) dx$$

$$A_{2,m} \frac{L}{2} = \int_0^L g_1(x) \sin\left(\frac{m\pi}{L}x\right) dx$$

$$A_{4,m} = \frac{2}{L} \int_0^L g_1(x) \sin\left(\frac{m\pi}{L}x\right) dx$$

## Necessità di avere una nuova formulazione

formulazione forte

$$-\mu \Delta u = f \text{ in } \Omega$$

(differenziale)

$$C^k(\Omega)$$

formulazione debole

(integrale)

spazi di Sobolev

$$-\Delta u = f + \text{DIRICHLET OMOCENO}$$

$$\underline{u \in C^2(\bar{\Omega})}$$

$$u \in C^2(\bar{\Omega}) \cap C^0(\bar{\Omega})$$

$$\boxed{-\Delta u = 1}$$

$$-\Delta u = 1 \quad (0,1)^2 = \Omega$$

$$-\Delta u = 0$$

$$f = 1 \quad f \in C^0(\bar{\Omega})$$

$$-\Delta u(0,0) = -\frac{\partial^2 u}{\partial x^2}(0,0) - \frac{\partial^2 u}{\partial y^2}(0,0) = 0$$

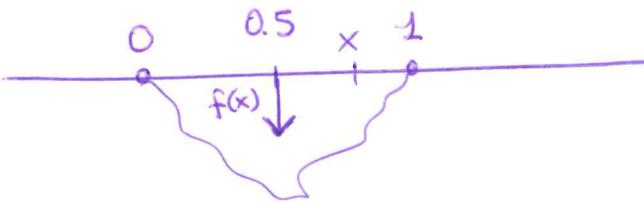
$$u = 0 \quad \partial\Omega$$

$$\boxed{\text{se } f \in C^0(\bar{\Omega}) \Rightarrow u \in C^2(\Omega) \cap C^0(\bar{\Omega})}$$

1D

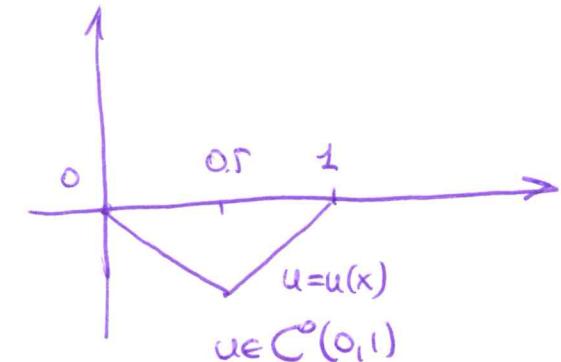
$$u = u(x)$$

$$\mathcal{N} = (0, 1)$$



$$\begin{cases} -u''(x) = f(x) & x \in \mathcal{N} = (0, 1) \\ u(0) = u(1) = 0 \end{cases}$$

$$F(x) = \int_0^x f(y) dy \quad (0, x)$$

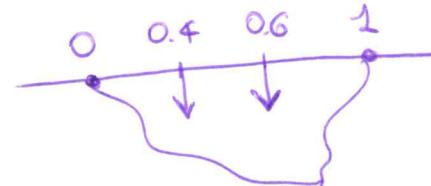
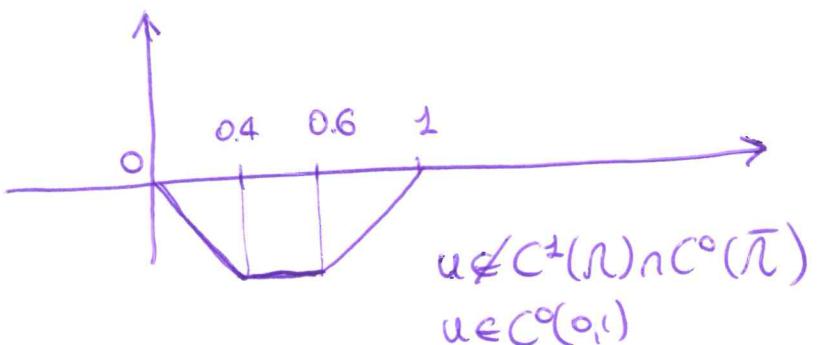
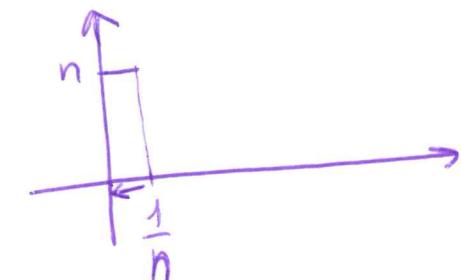


$f$  è di Dirac centrata in 0

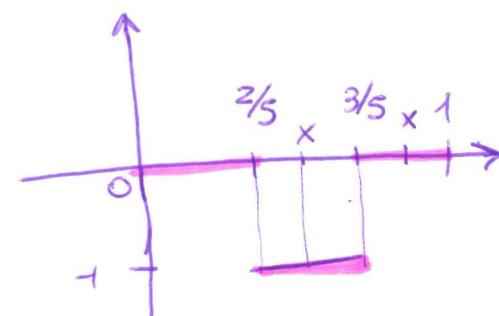
$$u \notin C^4(\mathbb{R}) \cap C^0(\bar{\mathbb{R}})$$

$$f_n(x) = \lim_{n \rightarrow \infty} g_n(x)$$

$$g_n(x) = \begin{cases} 0 & x < 0 \text{ e } x > \frac{1}{n} \\ n & 0 \leq x \leq \frac{1}{n} \end{cases}$$



$$f(x) = \begin{cases} 0 & x \in (0, \frac{2}{5}) \cup [\frac{3}{5}, 1) \\ -1 & x \in [\frac{2}{5}, \frac{3}{5}] \end{cases}$$



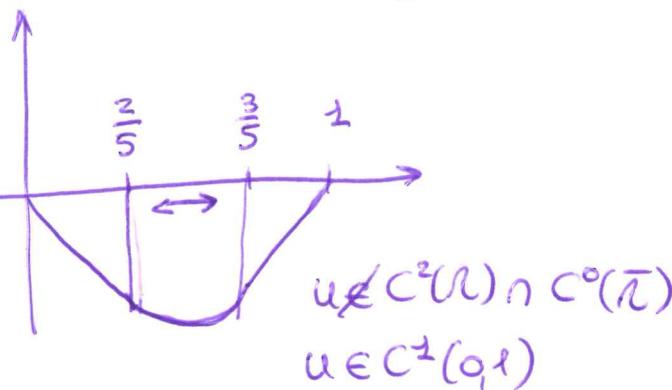
$$u'(x) = \int_0^x u''(y) dy + C = - \underbrace{\int_0^x f(y) dy}_{F(x)} + C = -F(x) + C$$

$$F(x) = \int_0^x f(y) dy = \begin{cases} 0 & x \in (0, \frac{2}{5}) = I_1 \\ -x + \frac{2}{5} & x \in [\frac{2}{5}, \frac{3}{5}] = I_2 \\ -\frac{1}{5} & x \in [\frac{3}{5}, 1) = I_3 \end{cases} \Rightarrow u'(x) = \begin{cases} C & x \in I_1 \\ x - \frac{2}{5} + C & x \in I_2 \\ \frac{1}{5} + C & x \in I_3 \end{cases}$$

$$u(x) = \int_0^x u'(y) dy + C^*$$

$$u(x) = \begin{cases} Cx + D & x \in I_1 \\ \frac{x^2}{2} - \frac{2}{5}x + Cx + E & x \in I_2 \\ \frac{1}{5}x + Cx + F & x \in I_3 \end{cases}$$

$$u(x) = \begin{cases} -\frac{x}{10} & x \in I_1 \\ \frac{x^2}{2} - \frac{x}{2} + \frac{2}{25} & x \in I_2 \\ \frac{x}{10} - \frac{1}{10} & x \in I_3 \end{cases}$$



$$\begin{cases} u(0) = 0 \\ u(1) = 0 \\ u(\frac{2}{5})^- = u(\frac{2}{5})^+ \\ u(\frac{3}{5})^- = u(\frac{3}{5})^+ \end{cases}$$

$\leftarrow C, D, E, F$

$$\begin{cases} -u''(x) = f(x) & x \in (0,1) \\ u(0) = u(1) = 0 \end{cases}$$

$\sigma$  funzione test ( $u$  trial,  $\sigma$  test)  $u, \sigma \in V$

$$\underbrace{-\int_0^1 u''(x)\sigma(x) dx}_{\text{}} = \int_0^1 f(x)\sigma(x) dx \quad \forall x \in V$$

$$\begin{aligned} -\int_0^1 u''(x)\sigma(x) dx &= \int_0^1 u'(x)\sigma'(x) dx - u'(x)\sigma(x) \Big|_0^1 = \int_0^1 u'(x)\sigma'(x) dx \\ &\quad \underbrace{u'(1)\sigma(1)}_{=0} - \underbrace{u'(0)\sigma(0)}_{=0} \end{aligned}$$

**PROPOSTA 1:**  $V = C_0^1(0,1) = \{ \sigma \in C^2(0,1) \text{ t.c. } \sigma(0) = \sigma(1) = 0 \}$

$$? u \in V : \int_0^1 u'(x)\sigma'(x) dx = \int_0^1 f(x)\sigma(x) dx \quad \forall \sigma \in V$$

CB esenziali (D)

CB naturali (N, R)

Def 1: se due funzioni  $f, g : \mathbb{R} \rightarrow \mathbb{R}$  differiscono su un insieme di misura nulla, allora  $f = g$  quasi ovunque in  $\mathbb{R}$  e  $\int_{\mathbb{R}} f d\lambda = \int_{\mathbb{R}} g d\lambda$

Def 2: spazio  $L^p(\mathbb{R}) = \{f : \mathbb{R} \rightarrow \mathbb{R} \text{ t.c. } \int_{\mathbb{R}} |f|^p d\lambda < +\infty\}$

$\uparrow$   
integrali di Lebesgue

LM

$$? u \in V : a(u, v) = F(v) \quad \forall v \in V$$

1)  $V$  Hilbert

2)  $a(\cdot, \cdot)$  bilineare  
continua \*

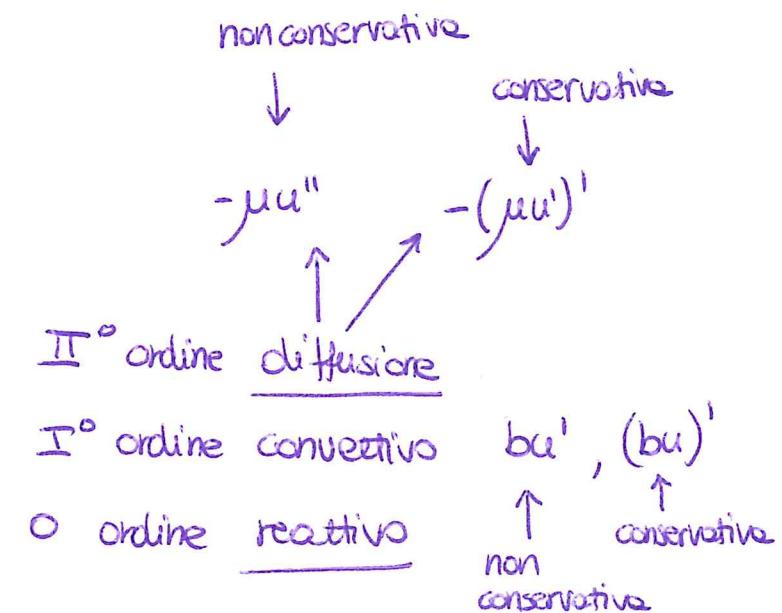
coerziva \*

3)  $F(\cdot)$  lineare

limitata \*

$$\begin{cases} -\mu_0 u''(x) + \sigma_0 u(x) = 0 & x \in (0, L) \\ u'(0) = q_1 \\ -u'(L) = q_2 \end{cases}$$

$\mu_0, \sigma_0$  costanti positive ( $\in \mathbb{R}^+$ ),  $q_1, q_2 \in \mathbb{R}$



$$\int_0^L \mu u'' \sigma = \int_0^L u'(\mu_0)' - \cancel{\mu u' \sigma} \Big|_0^L$$

$\mu$  costante

$$\hookrightarrow = \int_0^L \mu u' \sigma' - \mu u' \sigma \Big|_0^L$$

$$-\int_0^L (\mu u')' \sigma = \int_0^L \mu u' \sigma' - \mu u' \sigma \Big|_0^L$$

$\mu$  non costante

$$-\int_0^L \mu_0 u'' \sigma \, dx + \int_0^L \sigma_0 u \sigma = 0$$

$$\underbrace{\int_0^L \mu_0 u' \sigma' \, dx - \mu_0 u' \sigma \Big|_0^L}_{-q_2} + \int_0^L \sigma_0 u \sigma = 0$$

$$-q_2 - \underbrace{\mu_0 u'(L) \sigma(L)}_{q_2} + \underbrace{\mu_0 u'(0) \sigma(0)}_{q_1}$$

$$\underbrace{\int_0^L \mu_0 u' \sigma' + \int_0^L \sigma_0 u \sigma}_{a(u,v)} = \underbrace{-q_1 \sigma(0) - q_2 \sigma(L)}_{F(v)}$$

$$? u \in V : a(u,v) = F(v) \quad \forall v \in V$$

$H^k(\mathcal{N}) \subset C^0(\bar{\mathcal{N}})$	$k > \frac{d}{2}$	$d=1, k=1$ ok
$d=2,3 \quad H^2 \subset C^0$		$H^1 \subset C^0$

$\nabla \rightarrow H^1(\mathcal{N})$  full N, full R  
 $\nabla \rightarrow H^1_0(\mathcal{N})$  full D  
 $\nabla \rightarrow H^1_{\Gamma_D}(\mathcal{N}), \Gamma_D \subset \partial \mathcal{N}$  misto D/N  
 D/R

Dirichlet enneiali  $\rightarrow$  spazio

Neumann, Robin naturale  $\rightarrow$  formulazione

miste D/N  
D/R  $\rightarrow$  spazio + formulazione

$V = H^1(0, L)$

$\mathcal{N} = (0, L)$

1)  $H^1(\Omega)$  è di Hilbert

2)  $a(\cdot, \cdot)$  bilineare

3)  $a(\cdot, \cdot)$  continue

?  $M > 0$  t.c.  $|a(u, v)| \leq M \|u\|_V \|v\|_V$

$$\|u\|_V = \|u\|_{H^1(\Omega)}$$

$$\|u\|_{H^1(\Omega)}^2 = \|u\|_{L^2(\Omega)}^2 + \underbrace{\|u'\|_{L^2(\Omega)}^2}_{\|u\|_{H^1(\Omega)}^2}$$

$$\begin{aligned} \|u\|_{L^2(\Omega)} &\leq \|u\|_{H^1(\Omega)} \\ \|u\|_{H^1(\Omega)} &= \|u'\|_{L^2(\Omega)} \leq \|u\|_{H^1(\Omega)} \end{aligned}$$

$\downarrow$   
seminorma  $H^1$

$$|a(u, v)| = \left| \int_0^L \mu_0 u' v' + \int_0^L \sigma_0 u v \right| \leq \mu_0 \left| \int_0^L u' v' \right| + \sigma_0 \left| \int_0^L u v \right|$$

Cauchy-Schwarz

$$\left| \int_\Omega f g \right| \leq \|f\|_{L^2} \|g\|_{L^2} \quad \forall f, g \in L^2(\Omega)$$

$$\stackrel{CS}{\leq} \mu_0 \|u'\|_{L^2} \|v'\|_{L^2} + \sigma_0 \|u\|_{L^2} \|v\|_{L^2}$$

$$\stackrel{*}{\leq} \mu_0 \|u\|_{H^1} \|v\|_{H^1} + \sigma_0 \|u\|_{H^1} \|v\|_{H^1}$$

$$= (\mu_0 + \sigma_0) \|u\|_V \|v\|_V$$

4)  $a(\cdot, \cdot)$  coercive

?  $\alpha > 0$  t.c.  $a(v, v) \geq \alpha \|v\|_V^2$

$$\begin{aligned}
 a(v, v) &= \underbrace{\int_0^L \mu_0 [\dot{v}']^2}_{\text{brace}} + \underbrace{\int_0^L \sigma_0 [v]^2}_{\text{brace}} \\
 &\geq \min (\mu_0, \sigma_0) \int_0^L [\dot{v}']^2 + \min (\mu_0, \sigma_0) \int_0^L [v]^2 \\
 &= \min (\mu_0, \sigma_0) \left[ \underbrace{\int_0^L [\dot{v}']^2}_{\|\dot{v}'\|_{L^2}^2} + \underbrace{\int_0^L [v]^2}_{\|v\|_{L^2}^2} \right] = \min (\mu_0, \sigma_0) \|v\|_{H^1}^2 \\
 a(v, v) &\geq \underbrace{\min (\mu_0, \sigma_0)}_{\alpha} \|v\|_V^2
 \end{aligned}$$

Poincaré: se  $v \in H_0^1(\Omega)$  o  $v \in H_{\Gamma_D}^1(\Omega) = \{v \in H^1(\Omega) \text{ t.c. } v|_{\Gamma_D} = 0\}$  con  $\Gamma_D \subset \partial\Omega$ , allora  $\|v\|_{L^2} \leq C_p \|v\|_{H^1}$

$\{v \in H^1(\Omega) \text{ t.c. } v|_{\partial\Omega} = 0\}$

$$\|\psi\|_{H^1}^2 \stackrel{\text{def}}{=} \|\psi\|_{L^2}^2 + \|\psi'\|_{L^2}^2 \stackrel{P}{\leq} C_p^2 \|\psi'\|_{L^2}^2 + \|\psi'\|_{L^2}^2 = (C_p^2 + 1) \|\psi'\|_{L^2}^2$$

$$\|\psi\|_{H^1}^2 \leq (C_p^2 + 1) \|\psi'\|_{L^2}^2 = (C_p^2 + 1) |\psi|_{H^1}^2$$

(\*)  $|\psi|_{H^1}^2 \geq \frac{1}{C_p^2 + 1} \|\psi\|_{H^1}^2$

Ex:

$$\tilde{a}(\psi, \psi) = \mu_0 \underbrace{\int_0^L [\psi']^2}_{|\psi|_{H^1}^2} \geq \underbrace{\frac{\mu_0}{C_p^2 + 1}}_{\tilde{c}} \|\psi\|_V^2$$

5)  $F(\cdot)$  lineare

6)  $F(\cdot)$  limitato  $\exists C > 0$  t.c.  $|F(\psi)| \leq C \|\psi\|_V$

$$\begin{aligned} |F(\psi)| &= |q_1 \psi(0) + q_2 \psi(L)| \leq |q_1 \psi(0)| + |q_2 \psi(L)| \\ &= |q_1| |\psi(0)| + |q_2| |\psi(L)| \leq C_T [ |q_1| + |q_2| ] \|\psi\|_{H^1} \end{aligned}$$

Stima di traccia:  $\Omega \subset \mathbb{R}^d$  aperto limitato con frontiera regolare.  $\exists$  funzione lineare e continua  $\gamma: H^k(\Omega) \rightarrow L^2(\partial\Omega)$ ,  $k \geq 1$ ,  $\gamma\psi = \psi|_{\partial\Omega} \quad \forall \psi \in H^k \cap C^0$ .

$$\|\gamma\psi\|_{L^2(\partial\Omega)} \leq C_T \|\psi\|_{H^k}$$

$d=1 \quad |\psi(a)| = |\psi(b)|$   
 $\leq C_T \|\psi\|_{H^2}$

Diseguaglianza di Hölder :  $f \in L^p(\Omega)$ ,  $g \in L^q(\Omega)$  t.c.  $\frac{1}{p} + \frac{1}{q} = 1$   $\begin{cases} p=q=2 \\ p=1, q=\infty \end{cases}$

$$\|fg\|_{L^2} \leq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}$$

Ex:

$$\left| \int_0^L u(x) u'(x) v'(x) \right| \stackrel{\begin{array}{l} H \quad q=1 \quad p=\infty \\ \text{f} \quad g \end{array}}{\leq} \|u\|_{L^\infty(\Omega)} \|u' v'\|_{L^2(\Omega)} \stackrel{\text{CS}}{\leq} \|u\|_\infty \|u'\|_{L^2} \|v'\|_{L^2} \stackrel{*}{\leq} \|u\|_\infty \|u'\|_V \|v'\|_V$$

a continuità

⊕  
CS

Hölder

a coercività

Poincaré

⊗⊗

F limitatezza

⊕  
CS

traccia

Corollario (LM): sotto le hp del Lemma di LM, si ha che  $\|u\|_V \leq \frac{1}{\alpha} \|F\|_{V^*}$

costante di coercività

$[V]$   $F$  funzionale  $F: V \rightarrow \mathbb{R}$  lineare e limitato

$$\|F\|_{V^*} = \sup_{\substack{\sigma \in V \\ \sigma \neq 0}} \frac{|F(\sigma)|}{\|\sigma\|_V} \Rightarrow \frac{|F(\sigma)|}{\|\sigma\|_V} \leq \|F\|_{V^*}$$

$$|F(\sigma)| \stackrel{*}{\leq} \|F\|_{V^*} \|\sigma\|_V \quad \forall \sigma \in V$$

Dim:

$$a(u, \sigma) = F(\sigma)$$

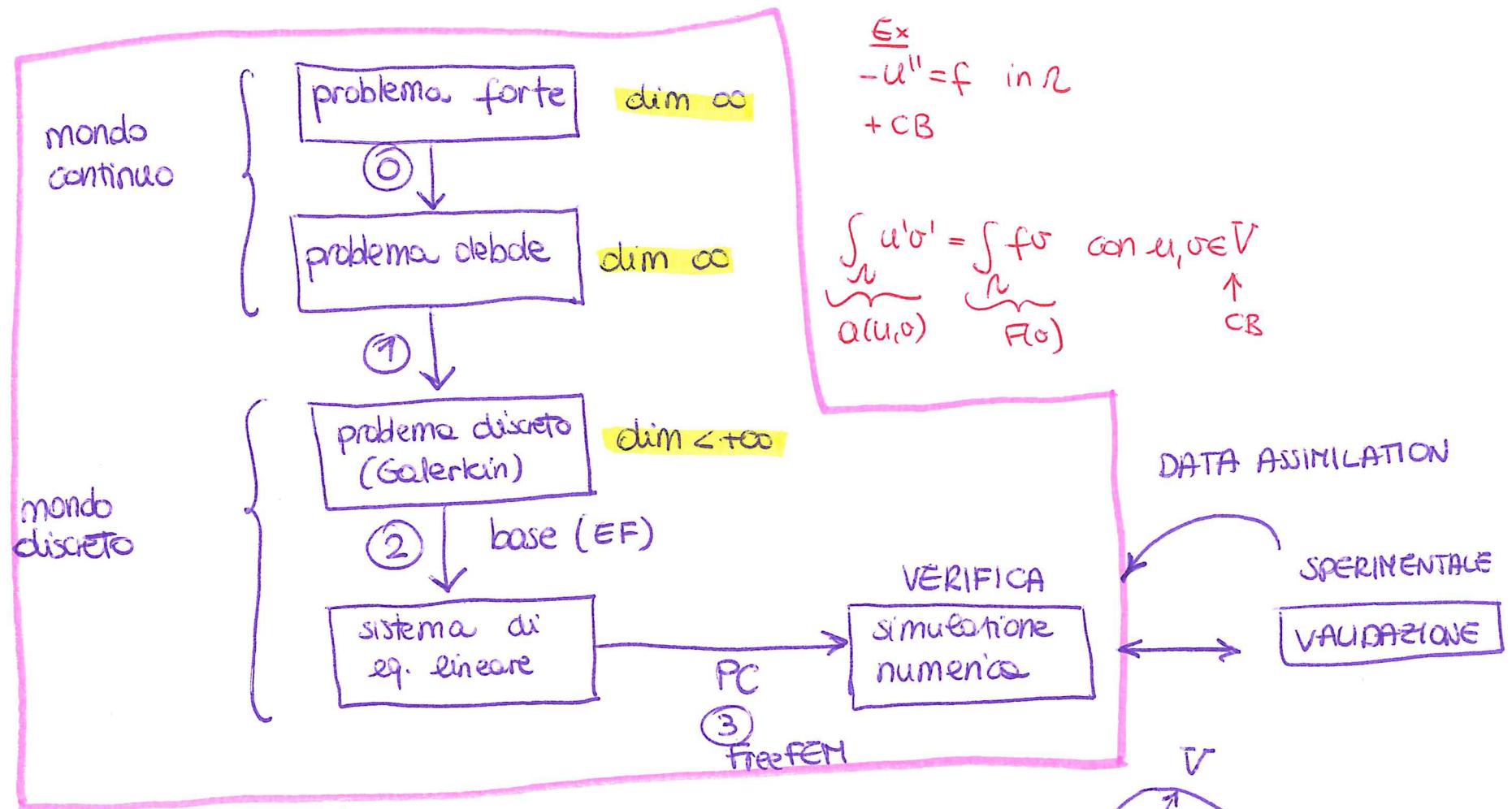
$$\sigma = u$$

$$\alpha \|u\|_V^2 \leq a(u, u) = F(u) \leq |F(u)| \leq \|F\|_{V^*} \|u\|_V$$

$\uparrow$   
coercività

$$\|u\|_V \leq \frac{1}{\alpha} \|F\|_{V^*}$$

■



$$? u \in V : a(u, v) = F(v) \quad \forall v \in V$$

$$\{V_h\} \quad V_h \subset V \quad \begin{cases} h \text{ parametro di discretizzazione} \\ \dim V_h = N_h < +\infty \end{cases}$$

① formulazione di Galerkin:  $? u_h \in V_h \subset V : a(u_h, v_h) = F(v_h) \quad \forall v_h \in V_h$

soluzione discreta

$\uparrow f_{\text{test}} \text{ discreto}$

$$\textcircled{2} \quad \text{base } V_h \quad \left\{ \varphi_j \right\}_{j=1}^{N_h} \quad u_h \leftarrow \varphi_i \quad i = 1, \dots, N_h$$

$$a(u_h, \varphi_i) = F(\varphi_i) \quad i = 1, \dots, N_h$$

$$V_h \ni u_h(x) = \sum_{j=1}^{N_h} u_j \varphi_j(x)$$

incognite

$$a\left(\sum_{j=1}^{N_h} u_j \varphi_j, \varphi_i\right) = F(\varphi_i) \quad i = 1, \dots, N_h$$

$$\sum_{j=1}^{N_h} u_j a(\varphi_j, \varphi_i) = F(\varphi_i) \quad i = 1, \dots, N_h$$

i-esima  
equazione

$$\vec{u} = [u_1, u_2, \dots, u_{N_h}]^T \in \mathbb{R}^{N_h}$$

$$\boxed{\vec{A}\vec{u} = \vec{f}}$$

$$\vec{f} = [f_1, f_2, \dots, f_{N_h}]^T, \quad f_i = F(\varphi_i) \quad i = 1, \dots, N_h$$

$$A = [a_{ij}] \in \mathbb{R}^{N_h \times N_h}$$

$$a_{ij} = a(\varphi_j, \varphi_i) \quad i, j = 1, \dots, N_h$$

matrice di rigidezza

$$\underline{\text{Ex}} \quad -u'' = f \text{ in } \Omega \quad a_{ij} = \int_{\Omega} \varphi_j' \varphi_i'$$

$$f_i = \int_{\Omega} f \varphi_i$$

$$a(u, v) = a(v, u) \text{ simmetrica}$$

$$\underline{\text{Ex}} \quad a(u, v) = \int_{\Omega} u' v' \text{ e simmetrica}$$

$$a(u, v) = \int_{\Omega} u' v \text{ non e simmetrica}$$

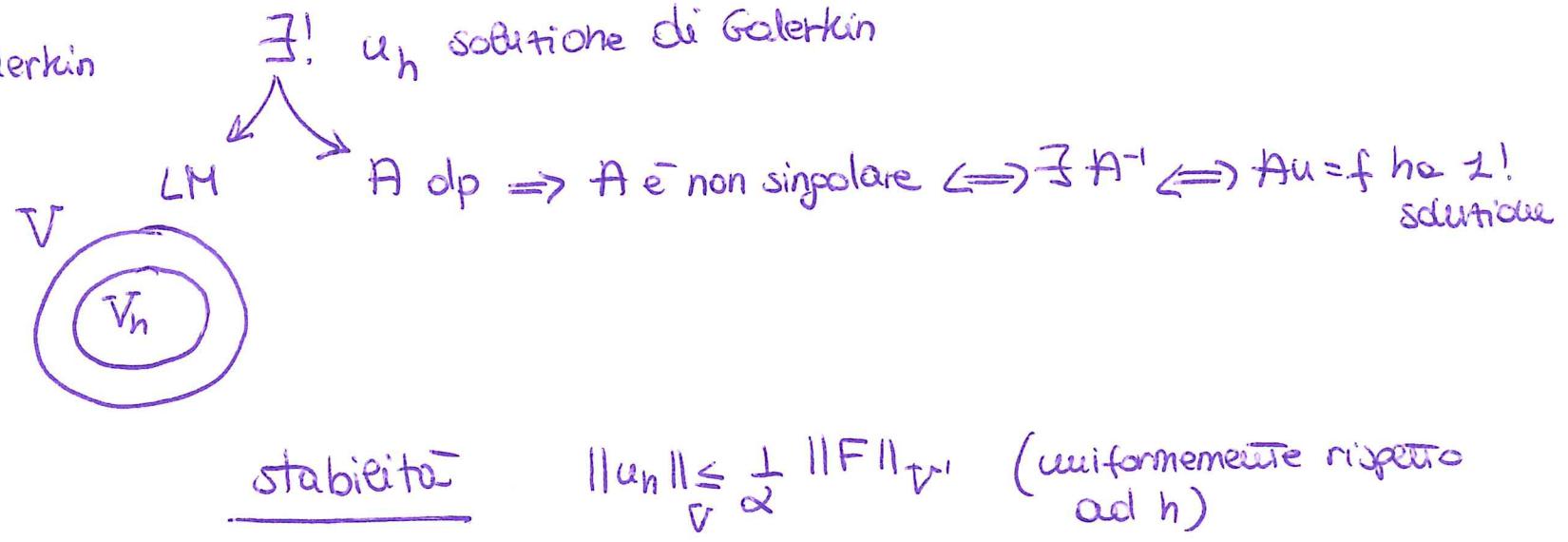
$$A\vec{u} = \vec{f}$$

### problem-dependent

- ① simmetria (Cholesky, PG, PCG)  
 (se  $a(\cdot, \cdot)$  è simmetrico  $\Leftrightarrow A$  simmetrico)
- ② se  $a(\cdot, \cdot)$  è coercivo  $\Rightarrow$   
 $A$  dp.

### ANALISI DI GALERKIN

- $\exists!$
- consistenza
- stabilità
- ortogonalità di Galerkin
- convergenza



$$a(u_h, \sigma_h) = F(\sigma_h) = \int_V f \sigma_h \quad \forall u_h \in V_h$$

$$a(w_h, \sigma_h) = G(\sigma_h) = \int_V (f + p) \sigma_h \quad \forall u_h \in V_h$$

↑  
perturbazione sul dato

$$w_h = u_h + z$$

↑  
soluzione  
perturbata

perturbazione sulla  
soluzione

$$a(u_h - w_h, \sigma_h) = F(\sigma_h) - G(\sigma_h) \quad \forall u_h \in V_h$$

$$\sigma_h = u_h - w_h$$

$$\alpha \|u_h - w_h\|_V^2 \leq a(u_h - w_h, u_h - w_h) = F(u_h - w_h) - G(u_h - w_h) = (F - G)(u_h - w_h)$$

$$\leq |(F - G)(u_h - w_h)| \leq \|F - G\|_{V^{-1}} \|u_h - w_h\|_V$$

$$\|u_h - w_h\|_V \leq \frac{1}{2} \|F - G\|_{V^{-1}}$$

Ortogonalità di Galerkin:  $u_h$  soluzione del problema di Galerkin è t.c.

$$a(\underbrace{u - u_h}_{\text{errore di discretizzazione}}, v_h) = 0 \quad \forall v_h \in V_h$$

errore di  
discretizzazione

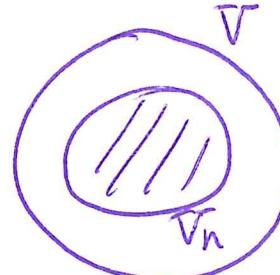
Dim:

$$a(u, v) = F(v) \quad \forall v \in V$$

$$v = v_h$$

consistenza

$$\boxed{a(u, v_h) = F(v_h) \quad \forall v_h \in V_h}$$



$$a(u_h, v_h) = F(v_h) \quad \forall v_h \in V_h$$

$$a(u, v_h) - a(u_h, v_h) = F(v_h) - F(v_h) = 0 \quad \forall v_h \in V_h$$

$$a(u - u_h, v_h) = 0 \quad \forall v_h \in V_h$$

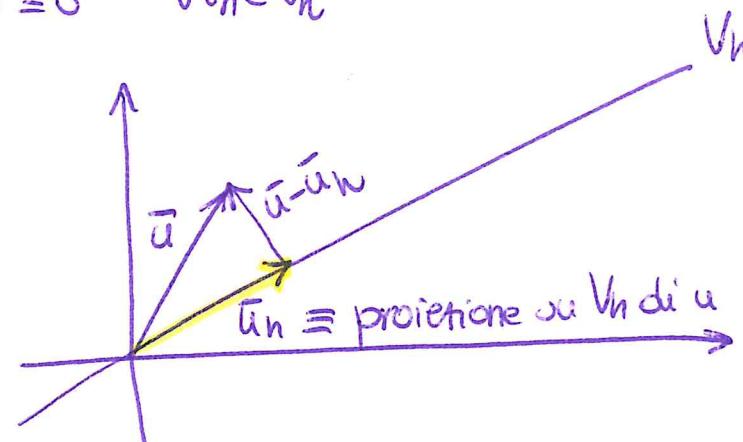
Galerkin è un metodo  
di proiezione ortogonale

Se  $a(\cdot, \cdot)$  è simmetrica,  
allora  $a(\cdot, \cdot)$  definisce un  
prodotto scalare

$$a(u, v) \xrightarrow{\text{norma in energia}} \|u\|_A = \sqrt{a(u_{th}, u_{th})} \quad \forall u_{th} \in V_h$$

$$w_h(x) = \sum_{j=1}^{N_h} w_j \varphi_j(x) \quad \bar{w}_h = [w_1, w_2, \dots, w_{N_h}]^T \quad (\bar{u} \bar{w}_h)^T A (\bar{u} \bar{w}_h)$$

$$\|w\|_A = \sqrt{w^T A w}$$



$$a(u-u_h, u-u_h) = a(u-u_h, u-\sigma_h) + \underbrace{a(u-u_h, \sigma_h - u_h)}_{\stackrel{\sigma_h \in V_h}{=0}} \\ + a(u-u_h, \sigma_h) \quad \forall \sigma_h \in V_h$$

$$2 \|u-u_h\|_V^2 \leq a(u-u_h, u-u_h) = a(u-u_h, u-\sigma_h) \leq |a(u-u_h, u-\sigma_h)| \leq M \|u-u_h\|_V \|u-\sigma_h\|_V$$

↑  
costante di continuità

$$\|u-u_h\|_V \leq \frac{M}{2} \|u-\sigma_h\|_V \quad \forall \sigma_h \in V_h$$

$$\inf_{w_h \in V_h} \|u-w_h\|_V \leq \boxed{\|u-u_h\|_V \leq \frac{M}{2} \inf_{w_h \in V_h} \|u-w_h\|_V}$$

Lemma di Céa  
errore di miglior approssimazione

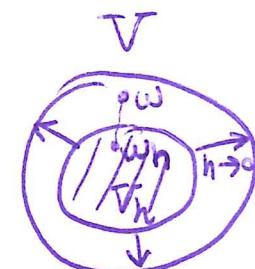
Hp

$$\lim_{h \rightarrow 0} \inf_{w_h \in V_h} \|u-w_h\|_V = 0 \quad \forall w \in V$$

densità di  $V_h$  in  $V$

convergenza

$$\|u-u_h\|_V \rightarrow 0 \quad \text{per } h \rightarrow 0$$



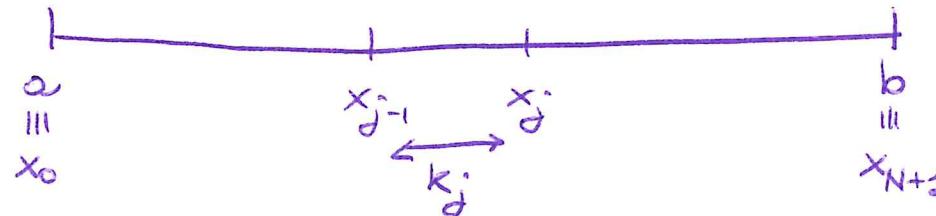
$$V_h = EF$$

$$\{V_h\} \quad V_h \subset V = H_0^1(\Omega)$$

$$\begin{cases} -u''(x) = f(x) & x \in (a,b) = \mathcal{I} \\ u(a) = u(b) = 0 \end{cases}$$

1D

$$T_h = \{k_j\}$$



(N+1) sottointervalli

$$k_j = (x_{j-1}, x_j)$$

$\{x_j\}$  vertici della partizione

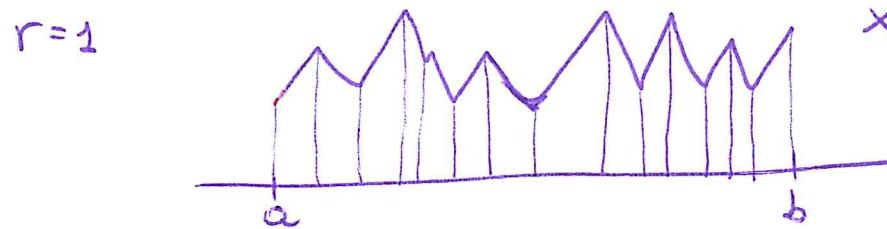
$$h_j = x_j - x_{j-1}$$

$$h = \max_j h_j$$

$$X_h^r = \left\{ v_h \in C^0([a, b]) \text{ t.c. } v_h|_{k_j} \in P_r, \forall k_j \in T_h \right\} \text{ EF space}$$

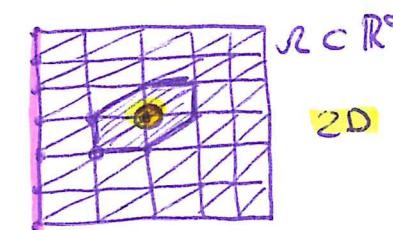
(piecewise polynomials)

$$r=1, 2, \dots$$

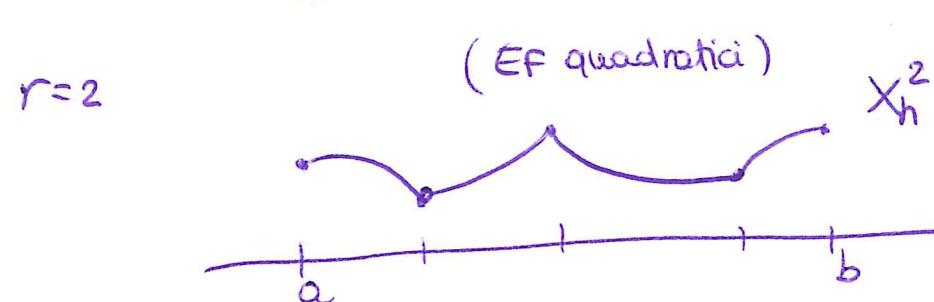


(EF lineari)

$$\boxed{\Pi_H^r f}$$



2D



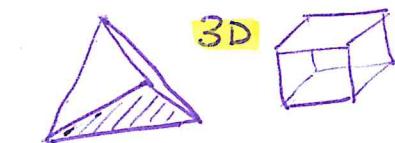
$$X_h^2$$

?

$$1) X_h^r \subset H^2(\Omega) \text{ ST}$$

$$2) CB ? \quad \boxed{V_h = X_h^r \cap V} \text{ ST}$$

$$\rightarrow V_h = \{v_h \in X_h^r \text{ t.c. } v_h(a) = v_h(b) = 0\}$$

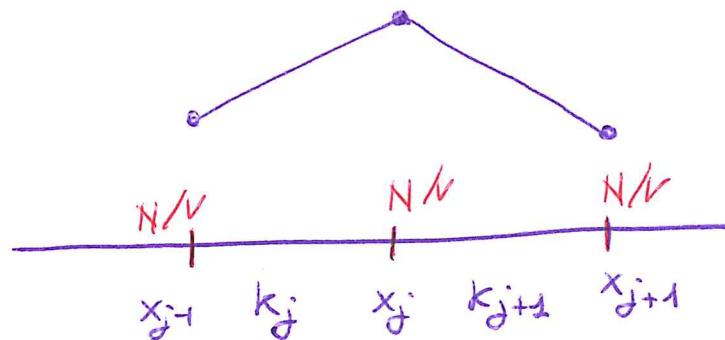


Base per  $V_h$  : 1) funzioni facili da integrare

$\{\varphi_i\}$  2) supporto di  $\varphi_i$  sia piccolo ( $\Rightarrow$  sparsità di  $A$ )

3) base Lagrangiana  $\varphi_i(x_j) = \delta_{ij}$   $\begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

Spazio  $X_h^1$  ( $\Pi_h^1 f$ )



2 informazioni  $\forall k_j$ :

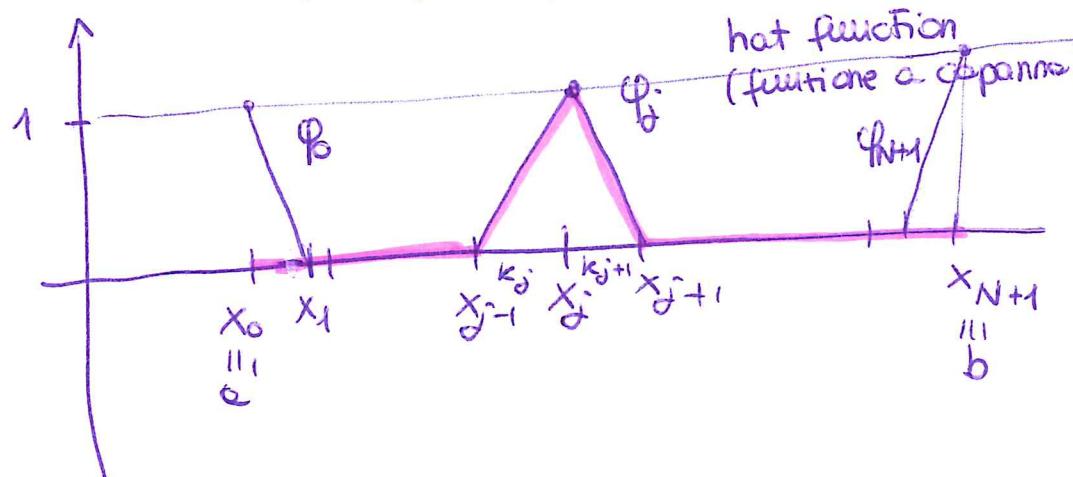
$r=1 \rightarrow \text{vertici}$  (geometrica) "x" }  
 nodi  
 gradi di libertà "y" }  
 (degrees of freedom : dofs)

$\{\text{nodi}\} \supseteq \{\text{vertici}\}$

(nodo, gde)

2 funzione di base  $\forall$  nodo

$\varphi_0, \varphi_1, \dots, \varphi_{N+1}$   $(N+2)$  f. base



$$\left\{ \begin{array}{l} \varphi_j \in X_h^1 \\ \varphi_j(x_i) = \delta_{ij} \\ \text{supp}(\varphi_j) \text{ piccolo} \end{array} \right. \quad i, j = 0, \dots, N+1$$

$$\text{supp}(\varphi_j) = k_j \cup k_{j+1} = [x_{j-1}, x_{j+1}]$$

$$\text{supp}(\varphi_0) = [x_0, x_1] = k_1$$

$$\text{supp}(\varphi_{N+1}) = [x_N, x_{N+1}] = k_{N+1}$$

$$1) \quad x_h^1 \quad V_h \subset H_0^1(\Omega)$$

$$\{\varphi_j\} \quad j=1, \dots, N$$

$$\not\ni \varphi_0, \varphi_{N+1}$$

$$[\text{Ex} \quad V = H_{H_0^1}^1(\Omega) \quad \Gamma_0 = \{b\} \quad \varphi_0, \varphi_1, \dots, \varphi_N \quad \not\ni \varphi_{N+1}]$$

$$V = H_{H_0^1}^1(\Omega) = \Gamma_0 = \{a\} \quad \varphi_1, \varphi_2, \dots, \varphi_N, \varphi_{N+1} \quad \not\ni \varphi_0]$$

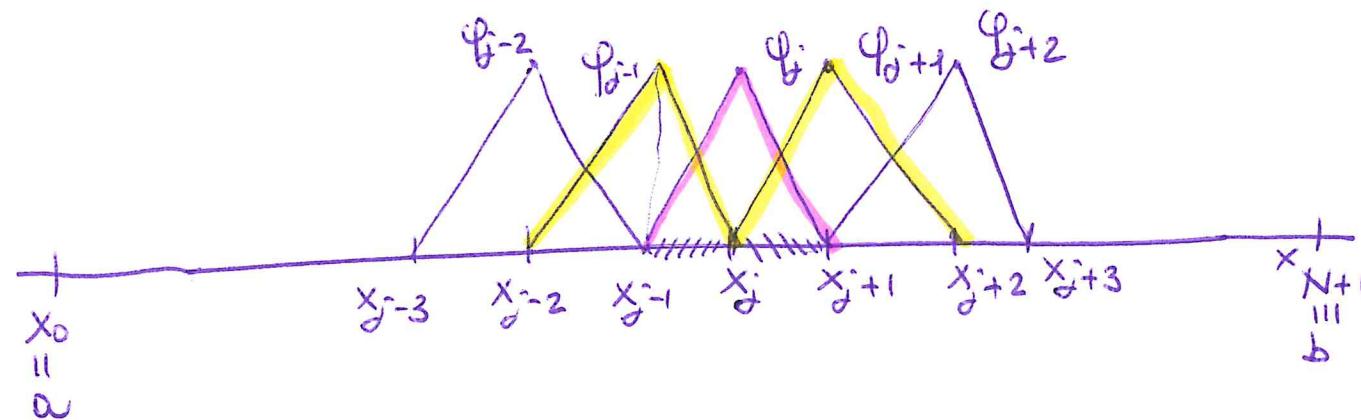
$$2) \quad \varphi_j(x) = \begin{cases} \frac{x - x_{j-1}}{x_j - x_{j-1}} & x \in k_j \\ \frac{x_{j+1} - x}{x_{j+1} - x_j} & x \in k_{j+1} \\ 0 & \text{o. etimewi} \end{cases}$$

$$3) \quad v_h \in V_h \quad \sigma_h(x) = \sum_j \tilde{\sigma}_j \varphi_j(x) \quad \{\tilde{\sigma}_j\} \subset \mathbb{R}$$

$$\sigma_h(x_i) = \sum_j \tilde{\sigma}_j \underbrace{\varphi_j(x_i)}_{\delta_{ij}} = \tilde{\sigma}_i$$

$$\boxed{\tilde{\sigma}_i = \sigma_h(x_i)}$$

$$4) \quad a_{ij} = a(\varphi_j, \varphi_i) = \int_a^b \varphi_j^T \varphi_i^T dx \quad i, j = 1, \dots, N$$



$a_{ij} \neq 0 \quad i = j-1, j, j+1 \rightarrow \underline{\text{tridiagonale}} \quad (1D)$

partizione uniforme

$$a_{j-1,j} = \int_a^b \varphi_j^T \varphi_{j-1}^T dx = \int_{x_{j-1}}^{x_j} \varphi_j^T \varphi_{j-1}^T dx = \int_{x_{j-1}}^{x_j} -\frac{1}{h^2} dx = -\frac{1}{h^2} K = -\frac{1}{h}$$

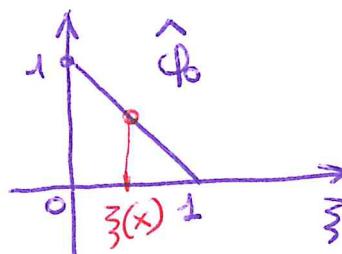
$$a_{j+1,j} = \int_a^b \varphi_j^T \varphi_{j+1}^T dx = \int_{x_j}^{x_{j+1}} \varphi_j^T \varphi_{j+1}^T dx = \int_x^{x_{j+1}} \left(-\frac{1}{h}\right)\left(\frac{1}{h}\right) dx = -\frac{1}{h^2} K = -\frac{1}{h}$$

$$a_{jj} = \int_a^b [\varphi_j^T]^2 dx = \int_{x_{j-1}}^{x_j} \underbrace{[\varphi_j^T]^2}_{\frac{1}{h^2}} dx + \int_{x_j}^{x_{j+1}} \underbrace{[\varphi_j^T]^2}_{\frac{1}{h^2}} dx = \frac{1}{h^2} k + \frac{1}{h^2} K = \frac{2}{h}$$

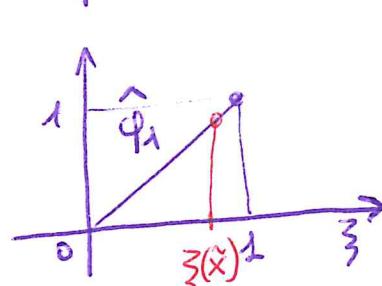
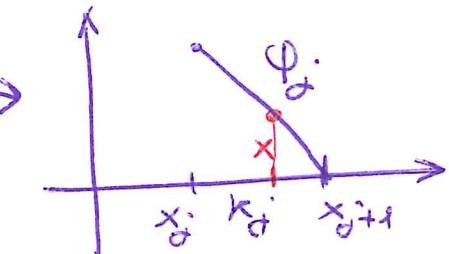
$$\boxed{A = \frac{1}{h} \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & & -1 & \\ 0 & \cdots & 0 & -1 & 2 \end{bmatrix}}$$

1D Poisson  
+ CB Dirichlet  
omogenee +  
partitione uniforme  
EF lineari

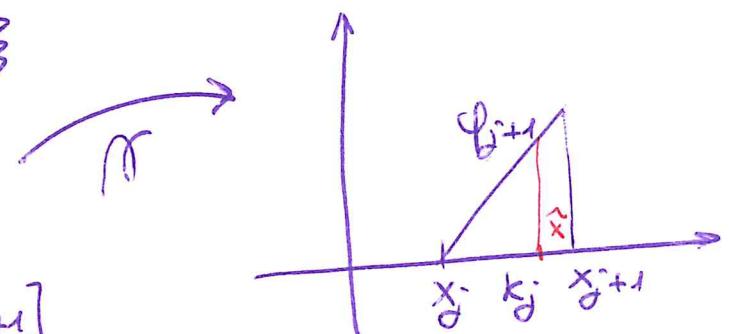
5) RIFERIMENTO  $\xrightarrow{\pi}$  FISICO  
 $[0,1]$   $k_j$



$$\hat{\phi}_0(\xi) = 1 - \xi$$



$$\hat{\phi}_1(\xi) = \xi$$



$$\pi : [0,1] \rightarrow [x_j, x_{j+1}]$$

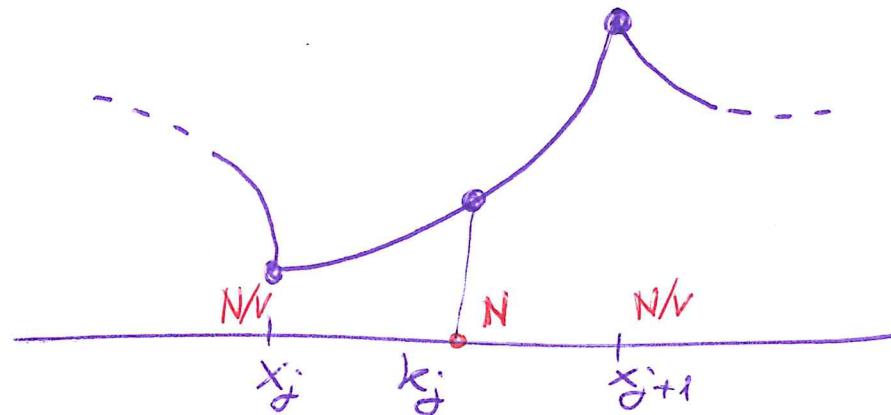
$$x = \pi(\xi) = x_j + \xi(x_{j+1} - x_j)$$

$$\pi(0) = x_j ; \quad \pi(1) = x_{j+1} ; \quad \pi\left(\frac{1}{2}\right) = \frac{x_j + x_{j+1}}{2}$$

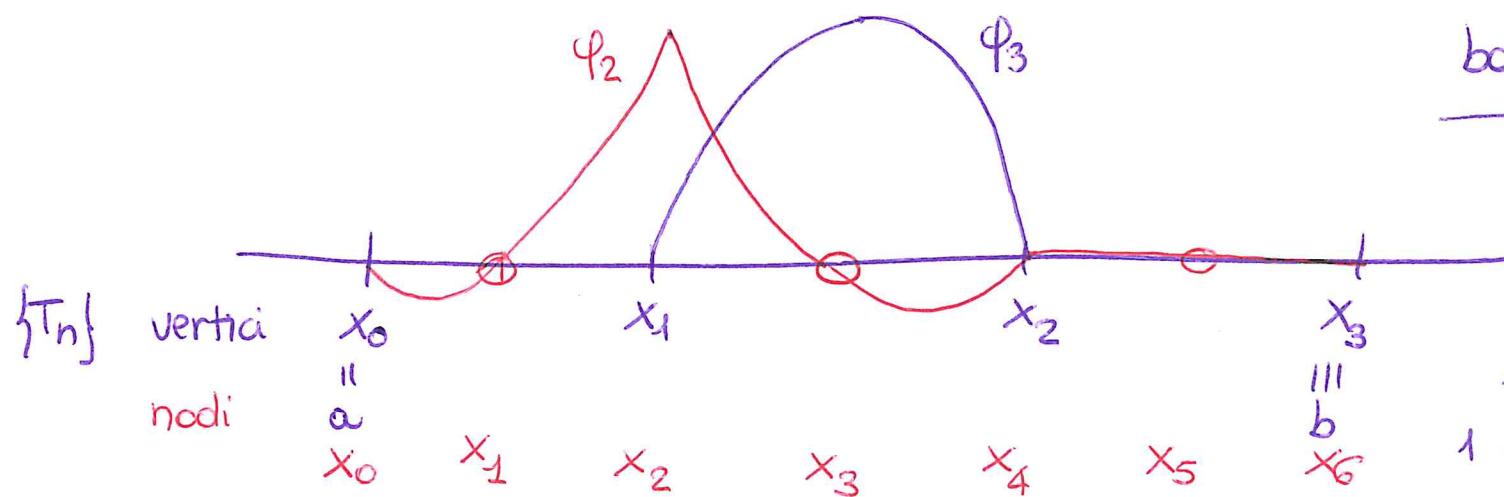
$$\begin{aligned} \phi_{j+1}(x) &= \hat{\phi}_1(\xi(x)) \\ &= \hat{\phi}_1\left(\frac{x - x_j}{x_{j+1} - x_j}\right) \end{aligned}$$

$$\begin{aligned} \phi_{j+1}(x) &= \hat{\phi}_1(\xi(x)) \\ &= \hat{\phi}_1\left(\frac{x - x_j}{x_{j+1} - x_j}\right) \end{aligned}$$

## Spazio $X_h$ EF quadratici



triintervallo 3dof (3 nodi), 2 vertici  
nodi > vertici



base Lagrangiana

$$\begin{aligned}\hat{\varphi}_0(\xi) &= (1-\xi)(1-2\xi) \\ \hat{\varphi}_1(\xi) &= 4\xi(1-\xi) \\ \hat{\varphi}_2(\xi) &= \xi(2\xi-1)\end{aligned}$$

