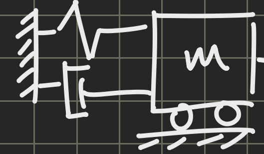
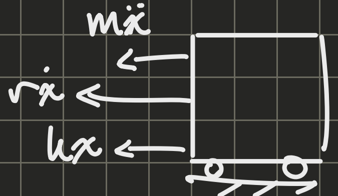
Lezione 4 -

We will look at what to do with the linearized equation (we linearize around the equilibrium position)
↳ Add to last notes

We are taking the system:

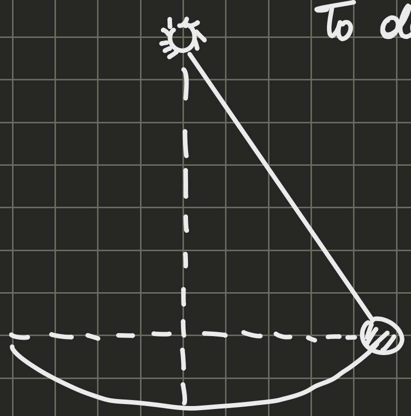


Dynamic equilibrium



Free motion
Equation of motion without force.

$$\boxed{m\ddot{x} + r\dot{x} + kx = 0}$$



To define the motion we need to define the initial condition.

$$\begin{cases} \theta(0) = \theta_0 \\ \dot{\theta}(0) = \omega_0 \end{cases}$$

since we need to solve the differential.

The equation we know: $J\ddot{\theta} + k_T\theta = 0$

With the initial condition

$$\begin{cases} \theta(0) = \theta_0 \\ \dot{\theta}(0) = 0 \end{cases}$$

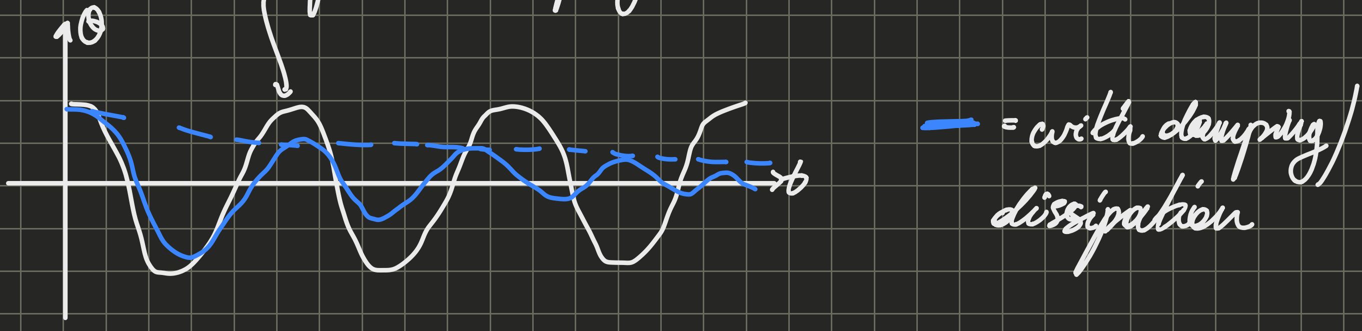
→ we put work to deviate the system from it equilibrium.

This work becomes stored in the potential energy of the system which in this case is nigh., while the kinetic energy is 0.

At the lowest point, all the potential energy is that of kinetic energy, and then back.

If there is no dissipation we have perpetual motion.

Without dissipation / damping.



Every cycle the mechanical energy reduces.

Equation without damping or force: or free motion

$$m\ddot{x} + kx = 0 \quad \rightarrow \quad (\lambda^2 m + k)\, x_0\, e^{\lambda t} = 0$$

True for every t

The tentative solution is:

$$x(t) = x_0 e^{\lambda t}$$

$$\dot{x}(t) = \lambda x_0 e^{\lambda t}$$

$$\ddot{x}(t) = \lambda^2 x_0 e^{\lambda t}$$

The first solution $x_0 = 0 \Rightarrow$ static solution

The other $\lambda^2 m + k = 0 \Rightarrow \lambda^2 = -\frac{k}{m}$

$$\boxed{\Rightarrow \lambda_{1,2} = \pm i\sqrt{\frac{k}{m}} = \pm i\omega_0}$$

pulsazione

$\frac{rad}{s}$

$\omega_0 \rightarrow$ circular frequency, it's a mechanical property since it's linked to the system and in no way to the initial condition.

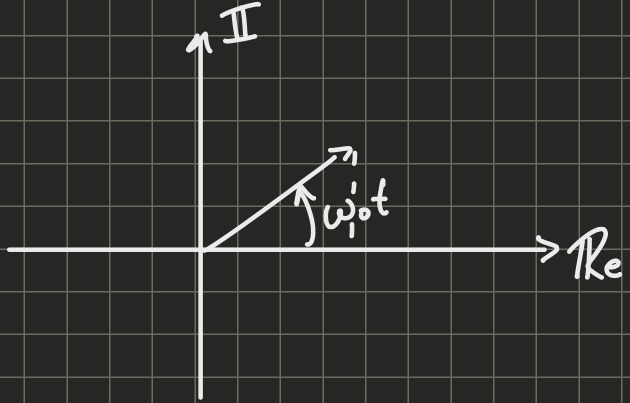
High mass $\Rightarrow$ low frequency. Higher stiffness are higher frequency.

Substituting λ

$$x(t) = x_1 e^{i\omega t} + x_2 e^{-i\omega_0 t}$$

To find x_1 and x_2 we need to use the initial condition.

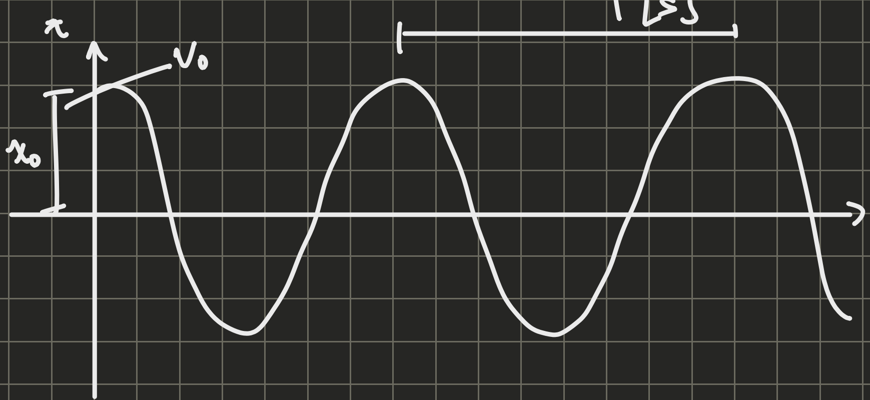
$$x(t) = \underbrace{A}_{\text{Amplitude}}\cos(\omega_0 t + \underbrace{\varphi}_{\text{phase}})$$



We can see the vibration as a rotation on the gauss plane

$$T = \frac{2\pi}{\omega_0}$$

So we get the time history as:



period

$T = \frac{1}{f}$ → natural frequency

$$f = [Hz] = \left[\frac{1}{s}\right]$$

We can change A but T is always the same if T is the same

System with damping

$$\begin{cases} x(0) = x_0 \\ \dot{x}(0) = v_0 \end{cases}$$

$$(\lambda^2 m + \lambda r + k)\, x_0\, \cancel{e^{\lambda t}} = 0 \longrightarrow x_0 = 0 \Rightarrow \text{Static}$$

other solution $\lambda^2 m + \lambda r + k = 0$

$$\hookrightarrow \lambda^2 + \lambda\frac{r}{m} + \frac{k}{m} = 0$$

ω_0^2

$$\lambda_{1,2} = -\frac{r}{2m} \pm \sqrt{\left(\frac{r}{2m}\right)^2 - \omega_0^2}$$

Different solutions:

- $\Delta = 0$ → where $\frac{r}{2m} = \omega_0 \Rightarrow r = 2m\omega_0$

↳ we once again get a mechanical solution to the system.

$\Rightarrow r_{crit} = 2m\omega_0$

$\frac{r}{r_{crit}} = h$ → to define something dimentioless

$$\lambda_{1,2} = -\frac{r\omega_0}{2m\omega_0} \pm \sqrt{\left(\frac{r\omega_0}{2m\omega_0}\right)^2 - \omega_0^2}$$

$$= -h\omega_0 \pm \sqrt{h^2\omega_0^2 - \omega_0^2}$$

$$= \omega_0\left(-h \pm \sqrt{h^2 - 1}\right.$$

$\Rightarrow \lambda$ can related to relationship between the real damping and the critical damping.

$\Rightarrow \Delta > 0 \qquad h > 1 \Rightarrow r > r_{crit}$

$\Delta = 0 \qquad h = 1 \Rightarrow r = r_{crit}$

$\Delta < 0 \qquad h < 1 \Rightarrow r < r_{crit}$

Let's start solving

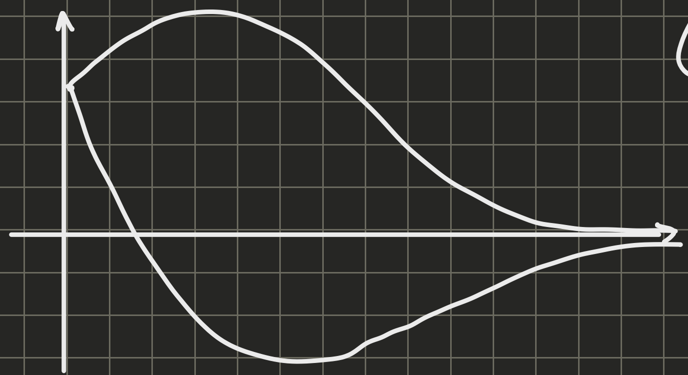
$\Delta > 0 \Rightarrow h > 1$, there is so much damping that it's more than r_{crit}

$\Rightarrow \lambda_{1,2} \in \mathbb{R}^-$

Our solution is therefore:

$$x(t) = x_1 e^{\lambda_1 t} + x_2 e^{\lambda_2 t}$$

for $t \to \infty$, since $\lambda_{1,2} < 0 \to x(t) \to 0$



The system will tend to 0, without oscillation, and will cross the 0 line at most once.

The system will directly go to the equilibrium position. We don't care for these since they are not vibrations

$\Delta = 0$, $h = 1$, $r = r_{crit}$

$\Rightarrow \lambda_1 = \lambda_2 = -h\omega_0$

$x(t) = x_1 t e^{\lambda t} + x_2 e^{\lambda t}$ → among all the solutions it's the fastest to go to equilibrium position.

$\Delta < 0$, $h < 1$, $r < r_{crit}$

$\lambda_{1,2} = -\alpha \pm i\omega$

$\neq \omega_0$ mathematically, $= \omega_0$ engineering wise.

$\omega_0 \sqrt{h^2-1}$

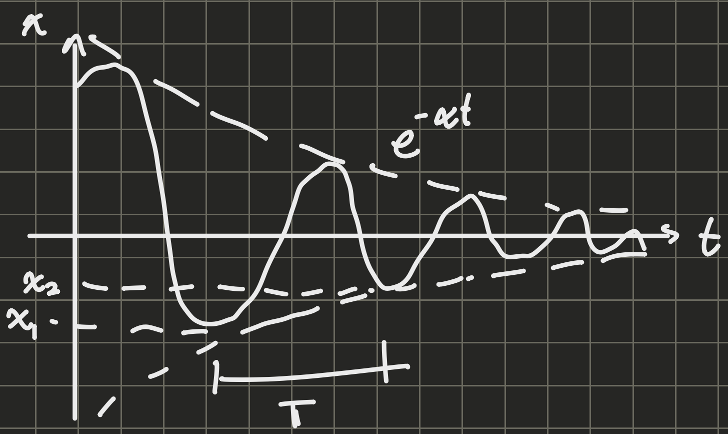
↳ since $h<1$, we can confuse ω and ω_0

$$x(t) = x_1 e^{(-\alpha+i\omega_0)t} + x_2 e^{(-\alpha-i\omega_0)t}$$

Manipulating it we get

$$x(t) = e^{-\alpha t}\underbrace{\left(x_1 e^{i\omega t} + x_2 e^{-i\omega t}\right)}_{\text{Solution for the undamped system}}$$

↳ This is causes the damping, modulating $x(t)$



It takes a number of vibrations to return to equilibrium.

Dampening is difficult to define, since it's the englobation of many factors.

Bike dampers work by making porous plate pass through oil.

The damping is usually ever constant since it itself is dependent on $\dot{x}$, and also many other factors.

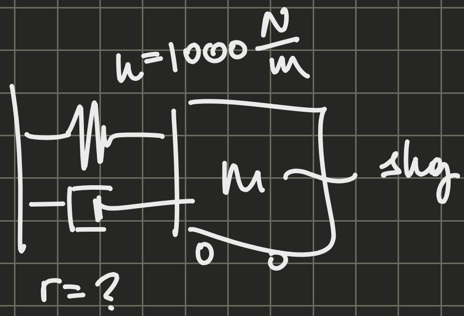
In many cases $h \ll 1$

Every mechanical system has its own natural frequency which is a propriety of the mechanical system.

Mechanical systems also have a root, the real damping on the mechanical system.

To get the free motion we need to define $x(0)$ and $\dot{x}(0)$

Damping is difficult to define so its also hard to find.

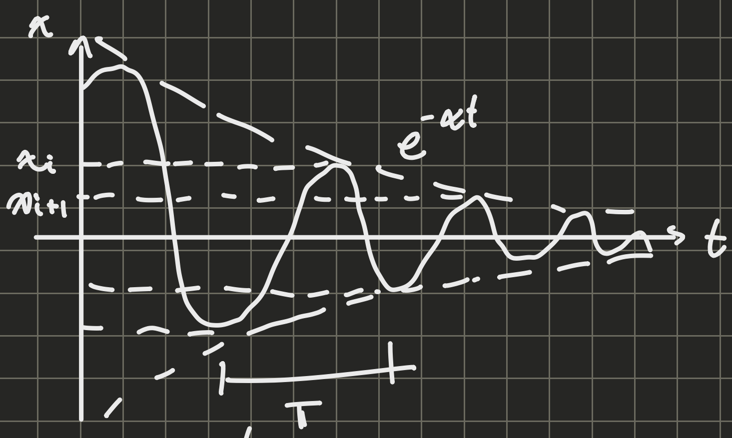


How do we get r ? Assuming $h < 1$

$\alpha = h\omega_0$

$$x(t) = |X| \, e^{\overbrace{-h\omega_0 t}^{\alpha = h\omega_0}} \cos(\omega t + \varphi)$$

If we know T, and x_1 and x_2



$$\delta = \frac{x_t}{x_{t+T}} = \ln\left(\frac{x_i}{x_{i+1}}\right) = \ln\left(\frac{\cancel{|X|} e^{-h\omega_0 t} \cancel{\cos(\omega t + \varphi)}}{\cancel{|X|} e^{-h\omega_0 (t+T)} \cancel{\cos(\omega(t+T) + \varphi)}}\right)$$

$$= \ln\left(e^{h\omega_0 T}\right)$$

$$\delta = h\omega_0 T \Rightarrow h = \frac{\delta}{\omega_0 T} = \frac{\delta}{2\pi}$$

In reality we do an average over more peaks so to have less measuring error.