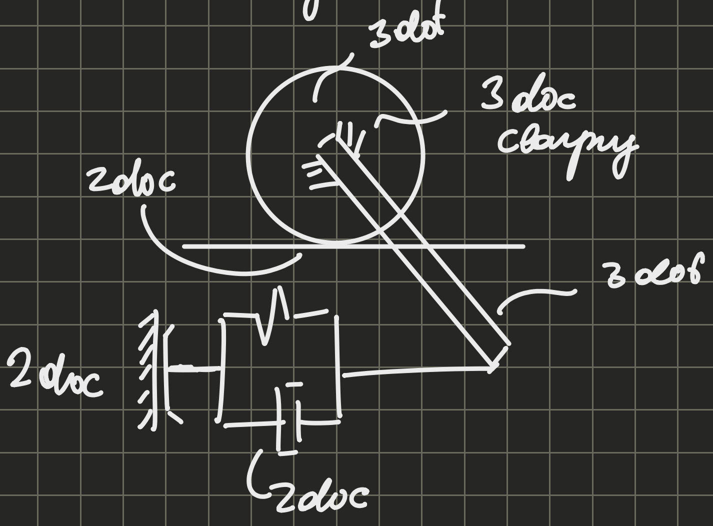
Esercitazione 2

Continuing with the exercise from last time



The presence of the spring eliminates one of the doc by the slider.

$$v_{ax} = \left(2L\cos\theta \cdot \dot{\theta} - R\dot{\theta}\right)\hat{i}$$

$$\int f(x)\,dx = \int f(x(t)) \cdot x'(t)\,dt$$

in our case $\int v_{ax}\,d\theta = \int v(\theta(t)) \cdot \dot{\theta}(t)\,dt$

$$x_A = \left(2L\sin\theta - R\theta\right)$$

static, dynamic

$$\frac{d}{dt}\left(\frac{\partial E_c}{\partial \dot{\theta}} - \frac{\partial E_c}{\partial \theta}\right) + \frac{\partial D}{\partial \dot{\theta}} + \frac{\partial V}{\partial \theta} = Q_S + Q_D$$

$$J^*(\theta)\ddot{\theta} + \left(M_A L R \sin\theta\right)\dot{\theta}^2 + r^*(\theta)\cdot\dot{\theta} + k\left[\Delta l_0 + (2L\sin\theta - R\theta) - (2L\sin\theta_0 - R\theta_0)\right] \cdot (2L\cos\theta - R) +$$

$$+ M_2 g L \sin\theta = (F_S + F_D)\left[-R + 2L\cos\theta\right]$$

This is difficult to calculate, so we can evaluate it at the equilibrium point, making it valid at only that point.

Stationary $\begin{cases} \theta = \theta_0 \\ \dot{\theta} = \end{cases}$

$$\cancel{\frac{d}{dt}\left(\frac{\partial E_c}{\partial \dot{\theta}} - \frac{\partial E_c}{\partial \theta}\right)} + \cancel{\frac{\partial D}{\partial \dot{\theta}}} + \frac{\partial V}{\partial \theta} = Q_s + \cancel{Q_D}$$

$Q_s = \dfrac{\partial V}{\partial \theta}$, equation we have to calculate to find the equilibrium point

$\Delta l_0 = ?$ Such that $\theta = \theta_0$

Method 1

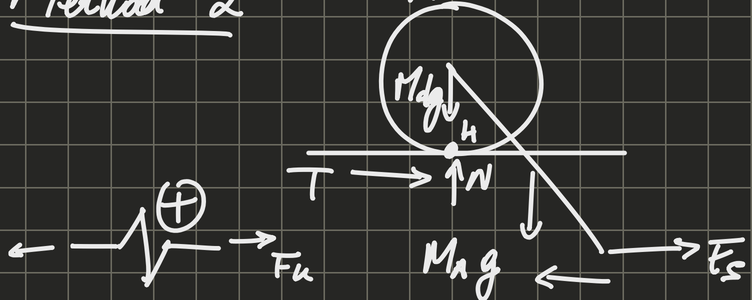
$$\boxed{\left.\frac{\partial V}{\partial \theta}\right|_{\theta_0} = \sum_i Q_s}$$

we have substitute θ with θ_0

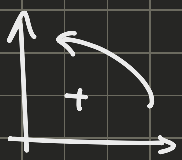
$$k\Big[\Delta l_0 + \underbrace{(2L\sin\theta_0 - R\theta_0) - (2L\sin\theta_0 - R\theta_0)}_{\Delta l_d = 0 \to \text{ static equilibrium}}\Big](2L\cos\theta_0 - R) + M_A g L \sin\theta_0$$

$$= F_s(2L\cos\theta_0 - R)$$

Method 2 → Apply static forces only to the drawing



$F_k = k\,\Delta l_0$ $\qquad\qquad\qquad$ F_k



$\sum M_H = 0$ ← Point H at bottom of dish

$$\Rightarrow -M_A g L \sin\theta_0 + \left(-k\Delta l_0 + F_s\right)\left(2L\cos\theta_0 - R\right) = 0$$

$$\Delta l_0 = \frac{-M_A g\, L \sin\theta_0 + F_s\left(2L\cos\theta_0 - R\right)}{k\left(2L\cos\theta_0 - R\right)}$$

These signs correspond to the convention for the spring.

Check the sign of Δl_0

Linearisation

$\theta = \theta_0 \longrightarrow$ small perturbation around θ_0

$$\theta_0 \rightarrow \tilde{\theta} = \theta - \theta_0$$
$$\dot{\tilde{\theta}} = \dot{\theta}$$
$$\ddot{\tilde{\theta}} = \ddot{\theta}$$

Need to know all of this for the oral exam.

$$E_c \simeq \frac{1}{2} J^*(\theta_0)\,\dot{\tilde{\theta}}^2 \longrightarrow \frac{d}{dt}\frac{\partial E_c}{\partial \dot{\theta}} - \frac{\partial E_c}{\partial \theta} \simeq J^*(\theta_0)\cdot\ddot{\tilde{\theta}}$$

$$D \simeq \frac{1}{2} r^*(\theta_0)\,\dot{\tilde{\theta}}^2 \longrightarrow \frac{\partial D}{\partial \dot{\theta}} \simeq r^*(\theta_0)\cdot\dot{\tilde{\theta}}$$

$$V \simeq V(\theta_0) + \left.\frac{\partial V}{\partial \theta}\right|_{\theta_0}\tilde{\theta} + \frac{1}{2}\left.\frac{\partial V^2}{\partial \theta^2}\right|_{\theta_0}\cdot\tilde{\theta}^2$$

Equivalent Stiffness

$$\frac{\partial V}{\partial \theta} = \cancel{\frac{\partial \overbrace{V(\theta_0)}^{\text{const}}}{\partial \theta}} + \frac{\partial}{\partial \theta}\Bigg(\underbrace{\left.\frac{\partial V}{\partial \theta}\right|_{\theta_0}}_{Q_s}\tilde{\theta}\Bigg) + \frac{\partial}{\partial \theta}\Bigg(\frac{1}{2}\underbrace{\left.\frac{\partial^2 V}{\partial \theta^2}\right|_{\theta_0}}_{k_s + k_{II} + k_{III}}\cdot\tilde{\theta}^2\Bigg) = Q_s + k^*\tilde{\theta}$$

$$k^* = \frac{\partial^2 V}{\partial \theta^2}\Big|_{\theta_0} = k_I + k_{II} + k_{III}$$ → while they are not springs, in this case they do.

Generalized Stiffness

$$k_I = k\left(\frac{\partial \Delta l_d}{\partial \theta}\Big|_{\theta_0}\right)^2 = k\left(2L\cos\theta_0 - R\right)^2 > 0$$ ← Always → stabilizes the system

↳ Substituted with θ_0 at the end

Souplesse

$$k_{II} = k\,\Delta l_0\,\frac{\partial \Delta l_d^2}{\partial \theta^2}\Big|_{\theta_0} = k\,\Delta l_0\left[\left(\frac{\partial}{\partial\theta}\left(\frac{\partial \Delta l_d}{\partial \theta}\right)\right)\right]_{\theta_0}$$

$$= k\,\Delta l_0\left[\frac{\partial}{\partial\theta}\left(2L\cos\theta - R\right)\right]_{\theta_0}$$

$$= k\,\Delta l_0\left(-2L\sin\theta_0\right)$$

↳ Depends on the θ_0 and Δl_0

Gravitational Stiffness

$$k_{III} = M_A g \cdot \frac{\partial^2 h_c}{\partial \theta^2}\Big|_{\theta_0} - M_A L\cos\theta$$

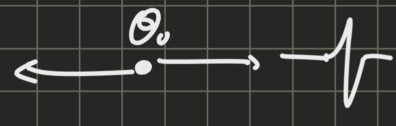
↳ also depends on θ_0

We have to check the sign since they give us the stability of our system.

k_I is always positive

The sum is the stability, if ⊕ the stable, if ⊖ it's unstable

Practical sense

k_I -> linearized contribution of the spring 

} → If preload is present, very small variations don't cause souplesse to be great.

k_{II} 

↳ Contribution of the preload

k_{III} -> contribution of gravity

↳ like what happens when we taught a rope and pull it, (like a guitar string), we can calculate a frequency, which is why we call it a spring, which is dependent on δl_0.

Rod/Spaghetti

Euler instability, at a certain point the system becomes unstable and starts to bend until it breaks.

$$Q = Q(\theta) = (F_S + F_D)\,\frac{\delta s_F}{\delta \theta}\bigg|_{\theta_0}\tilde{\theta}$$

$$= Q_s(\theta) + Q_D(\theta, t)$$

$$= Q_s(\theta_0) + \frac{\partial Q_s}{\partial \theta}\bigg|_{\theta_0}\cdot\tilde{\theta} + Q_d(\theta_0) + \frac{\partial Q_D}{\partial \theta}\bigg|_{\theta_0}\tilde{\theta}$$

For now we neglect, we will not later in the course

$$= Q_s(\theta_0) + Q_d(\theta_0)$$

(LEOM)

-> linearised equation of motion -> form is always the same.

$$\underbrace{J^*(\theta_0)}_{E_c}\ddot{\tilde{\theta}} + \underbrace{r(\theta_0)}_{D}\dot{\tilde{\theta}} + \underbrace{(k_s + k_{II} + k_{III})\tilde{\theta}}_{V} + \underbrace{\cancel{Q_s}} = \cancel{Q_s} + Q_D$$

The static contribution of the forces define the equilibrium but don't impact the vibrations around it.

Free motion of the system → solution of LEOM

$$\tilde{\Theta} = \tilde{\Theta}_p(\Theta) + \tilde{\Theta}_g(\Theta)$$

$\tilde{\Theta}_p$ → partial

$\tilde{\Theta}_g$ → general

$\tilde{\Theta}_g(\Theta)$

$$J^*\ddot{\tilde{\Theta}} + R^*\dot{\tilde{\Theta}} + k^*\tilde{\Theta} = 0$$

$$\tilde{\Theta} = \tilde{\Theta}_0 e^{\lambda t} \longrightarrow (J^*\lambda^2 + r^*\lambda + k^*) = 0 \rightarrow \lambda_{1,2} = -\alpha \pm i\omega_0\sqrt{1-h^2}$$

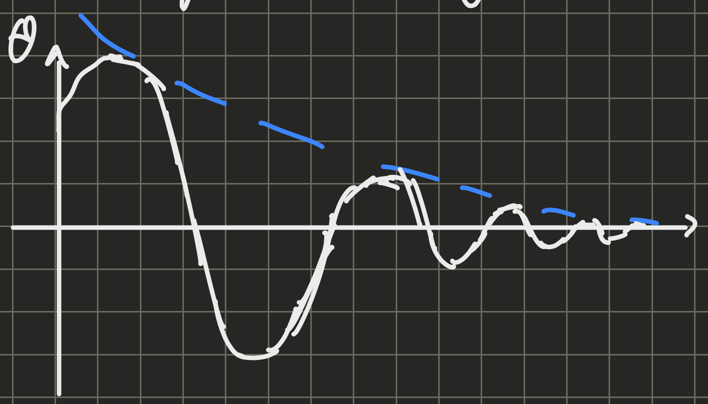
$\alpha = \dfrac{r^*}{2J^*}$ → damping coefficient

$\omega_0 = \sqrt{\dfrac{k^*}{J_0^*}}$ → natural frequency

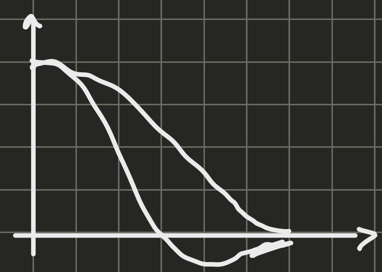
$h = \dfrac{r}{r_{cr}} = \dfrac{r}{2J^*\omega_0}$

$$\begin{cases} h < 1 \\ h = 1 \\ h > 1 \end{cases}$$

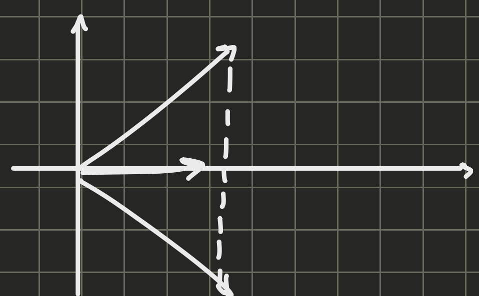
The response of the system is:



No vibrations



$$\Theta_g(t) = x_1 e^{(-\alpha + i\omega_d)t} + x_2 e^{(-\alpha - \omega_d)t}$$



$$= e^{-\alpha t}\left(x_1 e^{i\omega_d t} + x_2 e^{-i\omega_d t}\right)$$

$$= e^{-\alpha t}\left(A\cos(\omega_d t) + B\sin(\omega_d t)\right)$$

$$= e^{-\alpha t}\left(A\cos(\omega_d t + \varphi)\right)$$

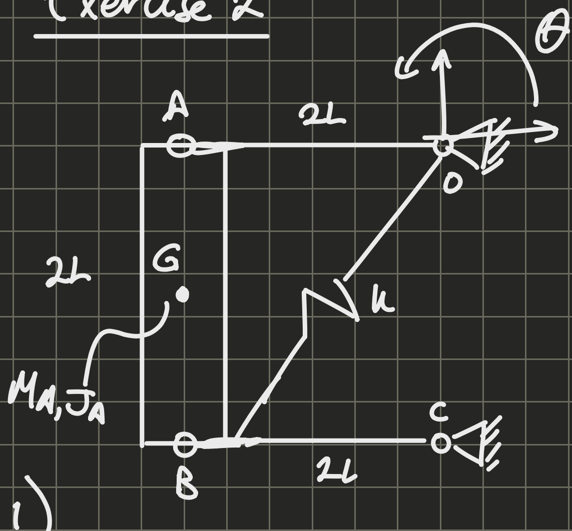
To find A & φ, we need to apply the initial conditions.

$$\begin{cases} x(\theta_i) = x_i \leftarrow \text{initial condition} \\ \dot{x}(\theta_i) = v_i \end{cases}$$

We get that

$$\begin{cases} x_i = A \\ v_i = B\omega_d \end{cases}$$

Exercise 2



1) NL EOM

2) Static Equilibrium

3) L EOM

1)

Step 1) dof

bodies $3 \times 3 = 9$ dof

on body ← hinges $2 \times 2 = 4$ doc

at ground ← hinges $2 \times 2 = 4$ doc

1 dof left

Always check the lengths since they simplify some calculation

Step 2

$$E_C = \frac{1}{2} M_A V_G^2 + \frac{1}{2} J_A \omega_A^2 \rightarrow$$

$\hookrightarrow$ the rod moves but does not rotate.

Charles Theorem

$D = 0$

$$V = \frac{1}{2} k \Delta l^2 + M_A g\, h_G$$

$$\delta \mathcal{L}^* = 0 \rightarrow Q = 0$$

Step 3)

$$\vec{\omega}_A = 0$$

$$\vec{V}_A = \cancel{V_0} + \vec{\omega}_1 \times (A - O) = \dot{\theta}\hat{k} \times (2L\cos\theta\hat{i} + 2L\sin\theta\hat{j})$$

$$= 2L\dot{\theta}\cos\hat{j} - 2L\dot{\theta}\sin\theta\hat{i}$$

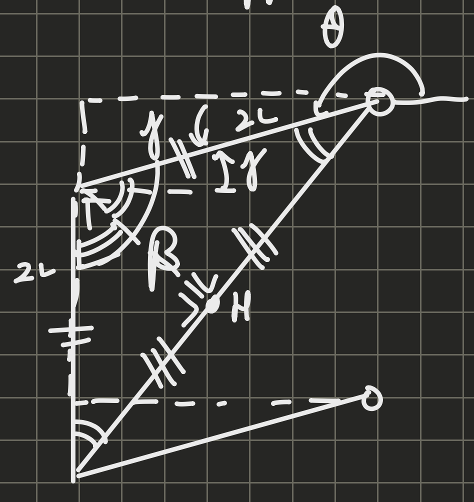
$$\cdot\ \vec{V}_G = \vec{V}_A + \vec{\omega}_A \times \cancel{(G-A)}$$

$$h_G = (G - O) = (A - O) + (G - A) = (2L\cos\theta\hat{i} + 2L\sin\theta\hat{j}) + (-L\hat{j})$$

$$= 2L\cos\theta\,\hat{i} + (2L\sin\theta - L)\hat{j}$$

$\Delta l = ?$

1st Approach



$$\gamma = \theta - \pi$$

$$\beta = \frac{\pi}{2} + \gamma = \frac{\pi}{2} + (\theta - \pi) = \theta - \frac{\pi}{2}$$

$$(BH) = AB\sin\left(\frac{\beta}{2}\right) = 2L\sin\left(\frac{\theta}{2} - \frac{\pi}{4}\right)$$

$$BO = 2\,BH$$

$$BO = 4L\sin\left(\frac{\theta}{2} - \frac{\pi}{4}\right)$$

↳ This is geometrical limited, since BO cannot be negative. So we have to limit θ so the sin does not become negative

ergo $\frac{\pi}{2} \leq \theta \leq \frac{5\pi}{2}$

→ This is easier to use, so we will use this

2nd approach

$$(B-O) = (B-A) + (A-O)$$

$$= -2L\hat{j} + (2L\cos\theta\,\hat{i} + 2L\sin\theta\,\hat{j}) = 2L[\cos\theta\,\hat{i} + (\sin\theta - 1)\hat{j}]$$

$$\overline{BO} = \|B-O\| = 2L\sqrt{\underbrace{\cos^2\theta + \sin^2\theta}_{1} + 1 - 2\sin\theta} =$$

$$= 2L\sqrt{2}\sqrt{1-\sin\theta}$$

Step 4

$$E_c = \frac{1}{2} M_A v_G^2 = \frac{1}{2} M_A(4L^2\cos^2\theta + 4L^2\sin^2\theta)\cdot\dot{\theta}^2 = 2M_A L^2\dot{\theta}^2$$

$$= \frac{1}{2}J^*\dot{\theta} \rightarrow \text{linear}$$

$$V_g = M_A g h_G = M_A g(2L\sin\theta - L)$$

$$V_k = \frac{1}{2}k\Delta l^2 = \frac{1}{2}k\left[\Delta l_0 + \left(4L\sin\left(\frac{\theta}{2} - \frac{\pi}{4}\right) - \underbrace{2\sqrt{2}L}_{l_m(\theta_0 = \pi) = 2\sqrt{2}}\right)\right]^2$$

Step 5

$$\left(\frac{d}{dt}\frac{\partial E_c}{\partial\dot{\theta}} - \frac{\partial E_c}{\partial\theta}\right) + \frac{\partial D}{\partial\dot{\theta}} + \frac{\partial V}{\partial\theta} - Q = 0$$