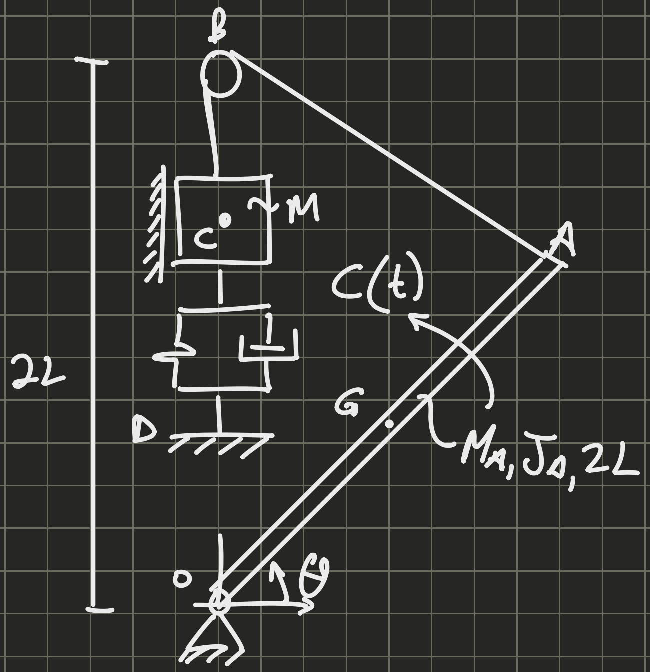
Esercitazione 4 -

Continuation from last time



$$J^*(\theta) = M_A L^2 + J_A + 2M_A L^2 (1+\sin\theta)$$

$$\frac{\partial J^*(\theta)}{\partial\theta} = 2M_A L^2 \cos\theta$$

$$\frac{\partial h_G}{\partial\theta} = L\cos\theta$$

$$\frac{\partial h_C}{\partial\theta} = -\frac{\sqrt{2}\,L\cos\theta}{\sqrt{1-\sin\theta}}$$

$$\frac{\partial \Delta l}{\partial\theta} = \frac{-\sqrt{2}\,L\cos\theta}{\sqrt{1-\sin\theta}}$$

step 6 $\rightarrow$ NL EOM

$$J^*(\theta)\cdot\ddot{\theta} + ML^2\cos\theta\,\dot{\theta}^2 + r^*(\theta)\,\dot{\theta} + M_A gL\cos\theta - M_C g\frac{L\cos\theta}{\sqrt{1-\sin\theta}} +$$

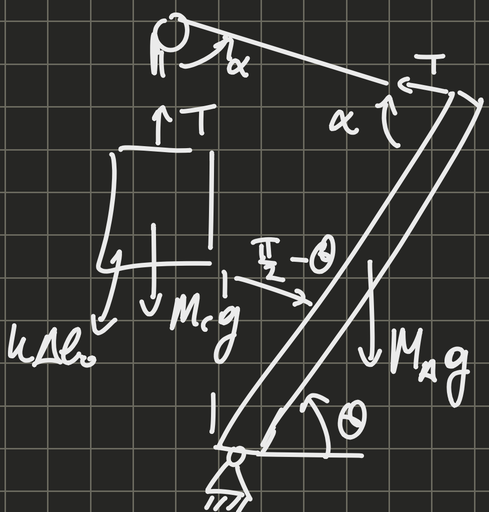
$$+ k\left(2\sqrt{2}L\cdot\sqrt{1-\sin\theta}\right)\cdot\frac{\sqrt{2}\,L\cos\theta}{\sqrt{1-\sin\theta}} = Q_\theta$$

Step 7

$$\left.\frac{\partial V}{\partial \theta}\right|_{\theta_0} = \sum Q_s = 0$$

$$\left.\frac{\partial V}{\partial \theta}\right|_{\theta_0} = M_A L\cos\theta_0 - M_c g L \frac{\cos\theta_0}{\sqrt{1-\sin\theta_0}} + k(2\sqrt{2}L)\sqrt{2}L\cos\theta_0 = 0 \rightarrow \theta_0$$

We can also do a static equilibrium to find θ_0



$$\pi = \left(\frac{\pi}{2} - \theta\right) + 2\alpha \rightarrow \alpha = \frac{1}{2}\left(\frac{\pi}{2} + \theta\right)$$

$$\sum_{V(M_c)} = 0 \qquad T = k\Delta l_0 + M_c g$$

$$\sum_{M(OA)} = 0 \qquad -M_A g\cos\theta + Tc$$

Step 7 Linear EOM

$$\tilde{\theta} = \theta - \theta_0$$

$$E_c \simeq \frac{1}{2} J^*(\theta_0)\dot{\tilde{\theta}}^2 \rightarrow \frac{d}{dt}\frac{\partial E_c}{\partial \dot\theta} - \frac{\partial E_c}{\partial \theta} = J^*(\theta_0)\ddot{\tilde{\theta}}$$

$$D = \frac{1}{2} r^*(\theta_0)\dot{\tilde{\theta}}^2 \rightarrow \frac{\partial D}{\partial \dot\theta} = r^*(\theta_0)\dot{\tilde{\theta}}$$

$$V \simeq \cancel{V(\theta_0)} + \cancel{\left.\frac{\partial V}{\partial \theta}\right|_{\theta_0}}^{\,0}\tilde{\theta} + \frac{1}{2}\left.\frac{\partial^2 V}{\partial \theta^2}\right|_{\theta_0}\tilde{\theta}^2 =$$

Negligible

$$\simeq \cancel{V(\theta_0)} + Q_s\tilde{\theta} + (k_J + k_2 + k_3)\tilde{\theta}^2$$

$$- k_J = k\left(\left.\frac{\partial \Delta l}{\partial \theta}\right|_{\theta_0}\right)^2 = k\left(\frac{-\sqrt{2}L\cos\theta}{\sqrt{1-\sin\theta}}\right)^2 \boxed{>0}$$

$$-k_{II} = k \, \Delta l_0 \, \frac{\partial^2 \Delta l}{\partial \theta^2}\Big|_{\theta_0} = -\sqrt{2}\, L\left(\sin\theta_0 (1-\sin\theta_0)^{-1/2} + \cos\theta_0 (1-\sin\theta_0)^{-3/2} \cdot \frac{1}{2} - \cos\theta_0\right)$$

$\boxed{\lessgtr 0} \rightarrow$ Depends on θ_0

$$k_{III} = M_{G} g \frac{\partial h_G^2}{\partial \theta^2} + M_g \underbrace{\frac{\partial h_c^2}{\partial \theta^2}}_{= \frac{\partial^2 \Delta l}{\partial \theta^2}\Big|_{\theta_0}} \quad \boxed{\gtrless 0}$$

↳ EOM

$$J^* \ddot{\tilde{\theta}} + r^* \dot{\tilde{\theta}} + k^* \tilde{\theta} = Q_d$$

Free Response (linear System)

$$\tilde{\theta}(t) = \tilde{\theta}_g(t) + \tilde{\theta}_p(t)$$

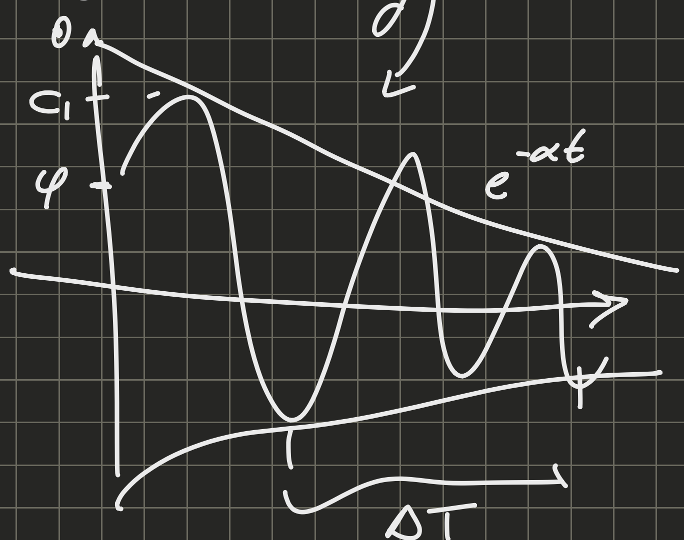
$$\tilde{\theta}_g(t) = \theta_0 e^{\lambda t}$$

$$\lambda_{1,2} = -\alpha \pm i\omega_0\sqrt{1-h^2}$$

$$\theta_g(t) = X_1 e^{\lambda_1 t} + X_2 e^{\lambda_2 t} = C_1 e^{-\alpha t} \cos(\omega_d t + \varphi)$$

We want to write this exponential in another way

$$S = \ln\left(\frac{x_1(t_1)}{x_2(t_2)}\right) = \ln\left(\frac{C_1 e^{-\alpha T_1}}{C_1 e^{-\alpha(T_1+\Delta T)}}\right) = \ln\left(e^{\alpha \Delta T}\right) = \alpha \Delta T$$



(Useful for oral exam)

Frequency Response Function (FRF)

Non linear response is more complicated, we will see later.

Linearized $\left\{ \begin{array}{l} \tilde{\theta}_p(t) = Re\left[\vec{\theta}_0 \, e^{i\Omega t}\right] \\ C(t) = Re\left[\vec{C}_0 e^{i\Omega t}\right] \end{array} \right.$

↓

Solution will have same equation

$$\left(-J\Omega^2 + i\Omega r^* + k^*\right)\vec{\theta}_0 \cancel{e^{i\Omega t}} = \vec{C}_0 \cancel{e^{i\Omega t}}$$

system response → $\vec{\tilde{\theta}}_0 = \boxed{\dfrac{1}{-J^*\Omega^2 + i\Omega r^* + k^*}}\vec{C}_0$ → FRF

Input → FRF → Output → Response of System.

Hypothesis of Linearisation:

- Number of harmonics of external force and response are the same.
- $\Omega_S = \Omega_F$ — Ω_S: system, of response; Ω_F: force

$$|FRF| = \frac{1}{\sqrt{\left(-J^*\Omega^2 + k^*\right)^2 + \left(\Omega r^*\right)^2}}$$

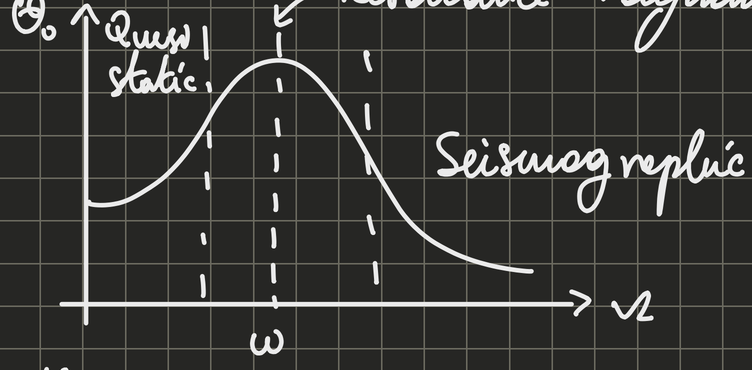
$$\varphi_{FRF} = \operatorname{atan}(0) - \operatorname{atan}\left(\frac{\Omega r^*}{-J^*\Omega^2 + k^*}\right)$$

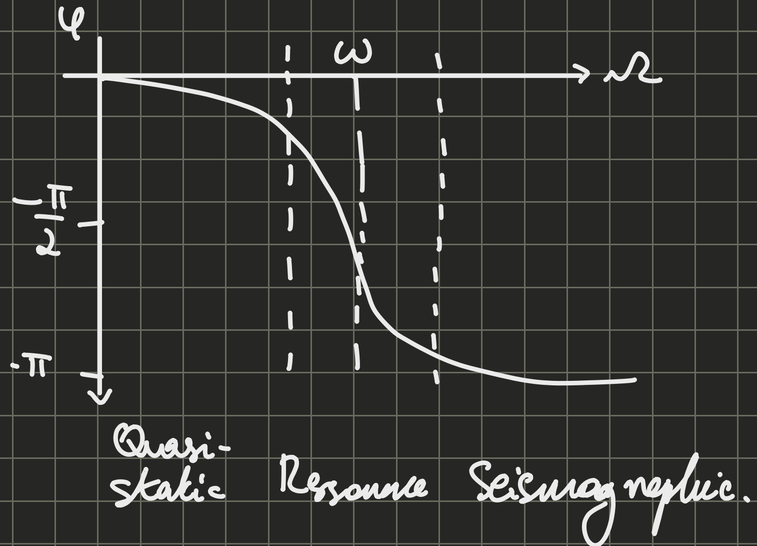
$$\tilde{\theta}_p(t) = Re\left[\left(\vec{C}_0 \cdot \left|\frac{\vec{\theta}_0}{\vec{C}_0}\right| \cdot e^{i\varphi}\right) e^{i\Omega t}\right]$$

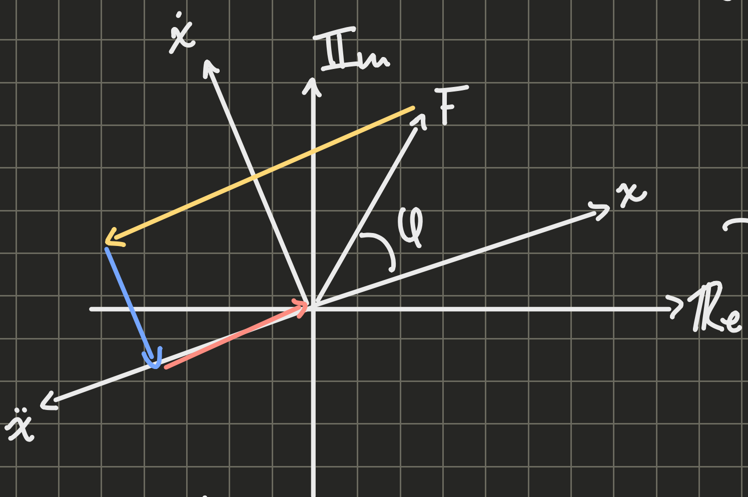
- $\tilde{\theta}_p(t)$ → Particular solution
- $\vec{C}_0$ → Input (phase + amplitude)
- $\left|\frac{\vec{\theta}_0}{\vec{C}_0}\right|$ → Amplification
- $e^{i\varphi}$ → Delay
- $e^{i\Omega t}$ → Rotation

Resonance → dynamic amplification

↳ The energy doesn't change, what changes is the source of the forces in the system.

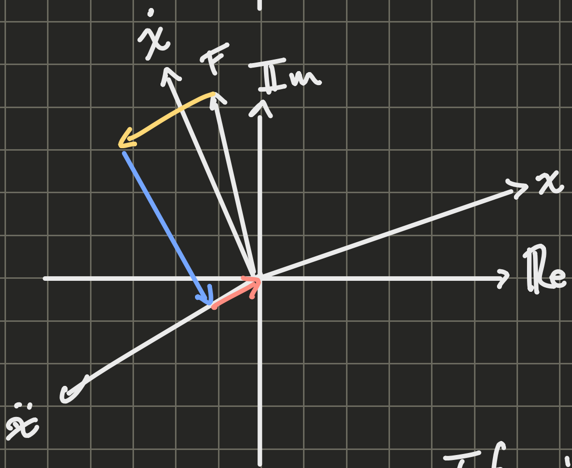






$$\underbrace{m\ddot{x}}_{F_m} + \underbrace{r\dot{x}}_{F_v} + \underbrace{kx}_{F_{el}} = F$$

⇝ In quasi-static the main force responding to F is Fel.



→ At resonance the main counter balance to the energy generated by F is counter balanced by the damping.

If r is low, we need high $\dot{x}$ to counterbalance $\dot{x}$.

external force.

For now we have $\underbrace{\vec{C}_0}_{\text{input}} \to \text{FRF} \to \underbrace{\theta_0}_{\text{output}}$

What if we want to find find something that isn't the independent variable θ_0 ? e.g. h_G

We find for θ_0 and then go to h_G. Since we are using a linear system we need to find the relation between the two.

Non linear relation: $h_G = L\sin\theta$

Linearization

$$h_G \simeq h_G(\theta_0) + \frac{\partial h_G}{\partial \theta}\Big|_{\theta_0} \tilde{\theta}$$

$$= L\sin\theta_0 + L\cos\theta_0\,\tilde{\theta}$$

$$\tilde{\theta} = \theta - \theta_0$$

$$\boxed{\hat{h}_G} = h_G(\theta) - h_G(\theta_0) = L\cos\theta_0\,\tilde{\theta}$$

↳ New independent variable

$$\hat{h}_G = \alpha \cdot \tilde{\theta} \underset{\text{FRF}}{=} L\cos\theta_0 \cdot \left(\frac{1}{J^* \Omega^2 + r^*\Omega i + k^*}\right)\vec{C}_0$$

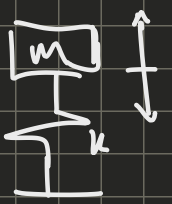
$$C_0 \to \text{FRF } \theta \to \alpha \to \hat{h}_G$$

↳ coefficient to augment the response and allow us to change variable

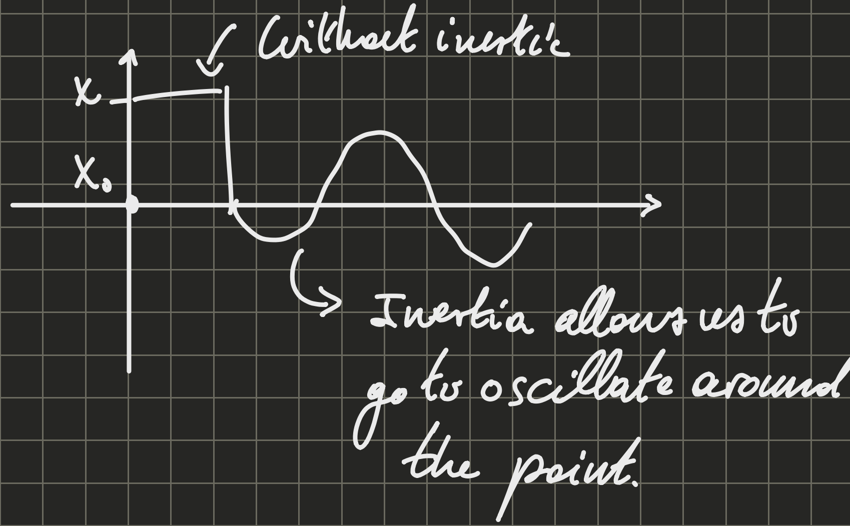
α can be positive or negative.

We only need a coefficient since we are using a linear system, so there will be no phase delay.

① What do we need to make an oscillator?
↳ Spring + Mass



$M\ddot{x} + r\dot{x} + kx = F$



② Static Force
↳ Does it change the equilibrium position?

Yes, because it is the derivative of the potential energy

Does it change the dynamics, yes but not in an intuitive way.

It doesn't change the dynamic force.
Does it change the dynamics? Yes