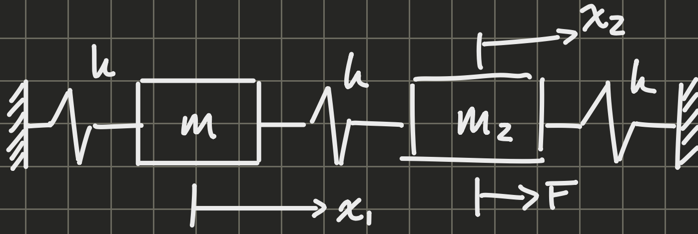
Lezione 11 - Undampened - Forced Motion & Modal Approach.



The system will move according to the frequency, Ω, of the force.

no force on m_1

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ F \end{bmatrix} e^{i\Omega t}$$

$$\underline{x} = \underline{X} e^{i\Omega t} \qquad [M]\ddot{\underline{x}} + [k]\underline{x} = F\cancel{e^{i\Omega t}}$$

$$\left(-\Omega^2 [M] + [k]\right)\underline{x} = \underline{F} \longrightarrow \text{Solution to equation of motion.}$$

We can re-write it as: $\underline{x} = \left(-\Omega^2 [M] + [K]\right)^{-1}\underline{F} = [H(\Omega)]\,\underline{F}$

In it's expaned form this equation is:

Matrix of transfer functions.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} h_{11}(\Omega) & h_{21}(\Omega) \\ h_{21}(\Omega) & h_{22}(\Omega) \end{bmatrix}\begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

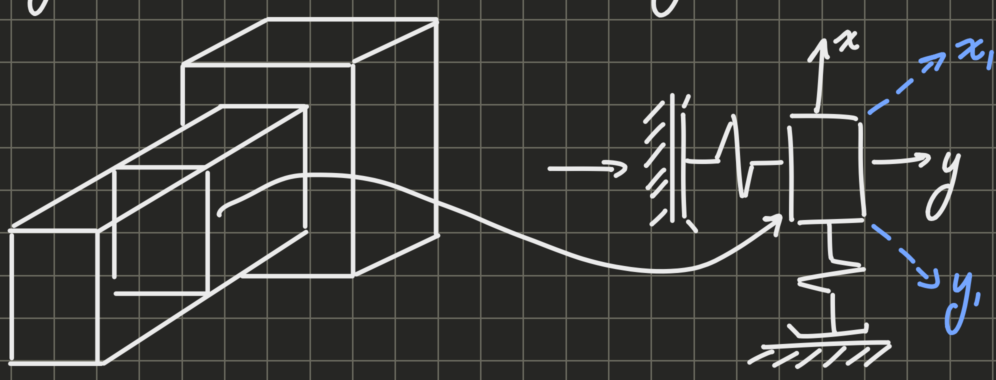
1 dof system is single input, single output.

↳ All forces and output are connected.

Not diagonal, F influences all the dofs.

h_{ij} = Damping factor of force j on the degree of freedom i. (Or the other way around)

This is not the smartest way of solving the system, the smartest way is to change our reference system.



x, y are the principal axes of inertia, while x_1 and y_1 are any two degrees of freedom, we can change the dofs because dofs are the set of coordinates with which we can describe the system without any kinematic relationship between them.

We can do the same with any system, changing the dofs to what we want. Is it possible for a multi-dof system to change the reference system to simply our equation, and make our variable not talk (i.e. diagonal matrices)?

In our example , the stiffness matrix is full, we want to make it diagonal. To convert from our initial reference system to our new system, we use a transformation matrix:

$$\underline{x} = [\phi]\, \underline{q}$$

→ What is this matrix?

This matrix is the modal matrix, which can be built by putting together all the modes of vibration:

$$[\phi] = \left[\underline{X}^{(I)}, \underline{X}^{(II)}, \dots, \underline{X}^{(n)}\right]$$

In the case of the example from last this matrix would be:

$$\underline{X}^{(I)} = \begin{bmatrix}1\\1\end{bmatrix} \quad \underline{X}^{(II)} = \begin{bmatrix}1\\-1\end{bmatrix} \Rightarrow [\phi] = \begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$$

↳ For this specific system.

This matrix converts our system into what will be a more useful system which will have diagonal matrices:

With this conversion, our equation of motion becomes:

$$[M][\phi]\ddot{\underline{q}} + [k][\phi]\underline{q} = \underline{0}$$

An then:

$$\underbrace{[\phi]^T[M][\phi]}_{\begin{bmatrix}2m & 0\\0 & 2m\end{bmatrix}} + \underbrace{[\phi]^T[k][\phi]}_{\begin{bmatrix}2k & 0\\0 & 6k\end{bmatrix}}\underline{q} = \underline{0}$$

Because $[\phi]$ is orthogonal $\Rightarrow [\phi]^T = [\phi]$

The multiplication of $[\phi]^T$ and $[\phi]$ before and after each terms because it to become diagonal and therefore the resulting matrix is what is aligned with our new axis, and the degrees of freedom become independent.

$$\begin{bmatrix} 2m & 0 \\ 0 & 2m \end{bmatrix}\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} 2k & 0 \\ 0 & 6k \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = 0$$

Independent Equations $\rightarrow$ $\begin{cases} 2m\,\ddot{q}_1 + 2k\,q_1 = 0 \\ 2m\,\ddot{q}_1 + 6k\,q_1 = 0 \end{cases} \Rightarrow \begin{matrix} \sqrt{\frac{k}{m}} \\ \sqrt{\frac{3k}{m}} \end{matrix}$

The solutions that we find are the same, and it makes sense since it's the exact same system.

We now need to see if this works generally or if we have just gotten lucky.

To do this we use the Lagrangian:

$$E_c = \frac{1}{2}\,\underline{\dot{x}}^T [M]\,\underline{\dot{x}} = \frac{1}{2}\,\dot{q}^T \underbrace{[\phi]^T [M][\phi]}_{[m]}\,\dot{q}$$

$$V_k = \frac{1}{2}\,\underline{x}^T [k]\,\underline{x} = \frac{1}{2}\,q^T \underbrace{[\phi]^T [K][\phi]}_{[k]}\,q$$

Applying Lagrange we get:

$$[m]\,\ddot{q} + [k]\,q = \underline{0}$$

In general are $[m]$ and $[k]$ diagonal?

$[\phi]$ is built like:

$$[\phi] = \begin{bmatrix} X_1^{(I)} & X_1^{(II)} & \cdots & X_1^{(s)} & X_1^{(r)} & \cdots & X_1^{(n)} \\ X_2^{(I)} & X_2^{(II)} & & \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ X_n^{(I)} & X_n^{(II)} & \cdots & X_n^{(s)} & X_n^{(r)} & \cdots & X_n^{(n)} \end{bmatrix}$$

$\hookrightarrow \mu_n^{I} X_1^{(I)}$ $\qquad$ $\rightarrow n^{th}$ mode

Chosing the modes, $\underline{X}^{(s)}$ and $\underline{X}^{(r)}$, with corresponding associated natural frequency, ω_s and ω_r.

The equation of free motion for our system is:

$$\left(-\omega^2[M]+[k]\right)\underline{X} = 0$$

And so:

$$\begin{cases} -\omega_s^2[M]\underline{X}^{(s)} + [k]\underline{X}^{(s)} = 0 \\ -\omega_r^2[M]\underline{X}^{(r)} + [k]\underline{X}^{(r)} = 0 \end{cases}$$

Multiplying by the transpose of the other mode, we get:

$$\underline{X}^{(r)T}[k]\underline{X}^{(s)} = \omega_s\ \underline{X}^{(r)T}[M]\underline{X}^{(s)}$$

$$\underline{X}^{(s)T}[k]\underline{X}^{(r)} = \omega_s\ \underline{X}^{(s)T}[M]\underline{X}^{(r)}$$

Taking the transpose of the first equation:

$$\Rightarrow \underline{X}^{(s)T}[k]\underline{X}^{(r)} = X^{(s)T}[M]\underline{X}^{(r)}$$

we see that the first part is the same as the second equation, and so will the second parts be the same as a consequence:

$$\Rightarrow \omega_s^2 \underline{X}^{(s)T}[M]\underline{X}^{(r)} = \omega_r^2 \underline{X}^{(s)T}[M]\underline{X}^{(r)}$$

$$\Rightarrow (\omega_s^2 - \omega_r^2)\underline{X}^{(s)T}[M]\underline{X}^{(r)} = \underline{0}$$

If we choose $r \neq s$: $(\omega_r^2 - \omega_s^2) \neq 0$

$$\Rightarrow \underline{X}^{(s)T}[M]\underline{X}^{(r)} = 0 \Rightarrow$$ the natural modes of vibration are orthogonal

So in general:

If $r=s$:

$$\Rightarrow (\omega_r^2 - \omega_s^2) = 0$$

$\Rightarrow X^{(s)T}[M]X^{(r)}$ is not forced to be 0 to be able to still satisfy the equation $\Rightarrow$ diagonal terms will be $\neq 0$.

If $r \neq s$:

$$\Rightarrow (\omega_r^2 - \omega_s^2) \neq 0$$

$\Rightarrow \underline{X}^{(s)T}[M]\underline{X}^{(r)}$ will have to be 0 to satisfy the equation, so the non-diagonal terms will be 0.

So in general:

$$[\phi]^T[m][\phi] = \begin{bmatrix} \ddots_m & \\ & \ddots \end{bmatrix} \rightarrow \text{Diagonal}$$

Since $[\phi] = [\underline{X}^{(1)}, \dots, \underline{X}^{(n)}]$ $\quad [\phi]^T = \begin{bmatrix} \underline{X}^{(1)T} \\ \vdots \\ X^{(n)T} \end{bmatrix}$

Therefore:

$$[\phi]^T[M][\phi] = \begin{bmatrix} \underline{X}^{(1)T}[M]\underline{X}^{(1)} & \cdots & \overset{=0}{\underline{X}^{(1)T}[M]\underline{X}^{(n)}} \\ & \ddots & \\ 0 & & \underline{X}^{(n)T}[M]\underline{X}^{(n)} \end{bmatrix}$$

The terms where $r \neq s$ will be null, and the diagonal terms will be $\neq 0$.

The general case works because the modes of vibration are diagonal, and having diagonal modes of vibration means we can apply the transformation to our system and recover a system in which our matrices are diagonal.

We can also write this as:

$$[\phi]^T[M][\phi] = \mathrm{diag}\left(\underline{X}^{(i)T}[M]\underline{X}^{(i)}\right)$$

Let's now consider the dampened system:

$$[M]\ddot{\underline{x}} + [R]\dot{\underline{x}} + [K]\underline{x} = \underline{F}$$

We want to get the natural frequencies and modes of vibration:

we switch to the undamped system:

$$[M]\ddot{\underline{x}} + [K]\underline{x} = \underline{F}$$ $\rightarrow$ from here we get the natural frequency and modes.

If we solve, we get:

$$\underline{x} = \left[H(\Omega)\right] \underline{F}$$

But if we want to have something more managable, we say:

$$\underline{x} = [\phi] \underline{q}$$

$$\Rightarrow [\phi]^T [M][\phi] \ddot{\underline{q}} + [\phi]^T [R][\phi] \dot{\underline{q}} + [\phi]^T [k][\phi] \underline{q} = [\phi]^T \underline{F}$$

We know the mass and stiffness matrices will be diagonal, but since we never considered the damping we don't know if it also becomes diagonal.

With the damping, all the frequencies and modes become complex.

$\rightarrow$ Not good since every real system is dampened

In most cases we add damping in parallel to our spring, so we can say that the damping matrix is a linear combination of the mass and stiffness matrices.

$$\Rightarrow [R] = \alpha [M] + \beta [k]$$

not integers, the convert units.

$$\Rightarrow \underbrace{[\phi]^T [R][\phi]}_{\text{Diagonal}} = \alpha \underbrace{[\phi]^T [M][\phi]}_{\text{Diagonal}} + \beta \underbrace{[\phi]^T [k][\phi]}_{\text{Diagonal}}$$

Doing this means:

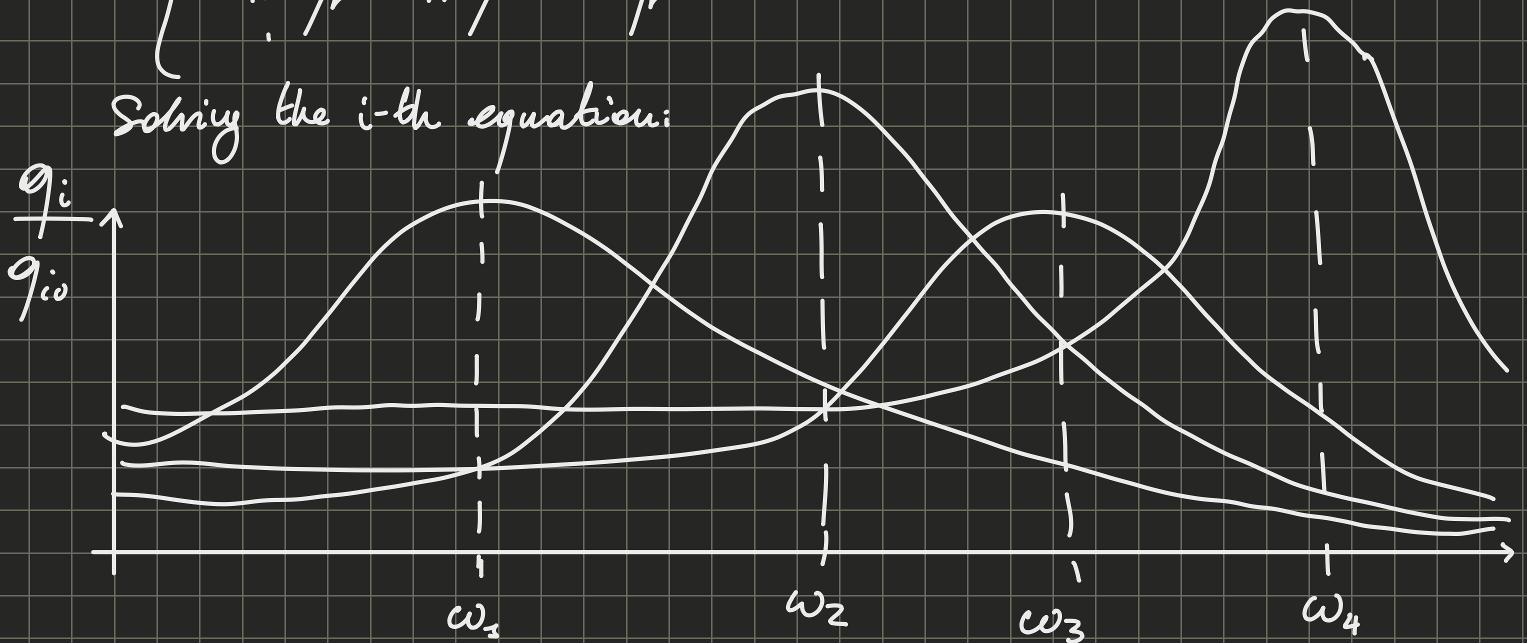
$$\begin{bmatrix} \ddots & & \\ & m & \\ & & \ddots \end{bmatrix} \ddot{q} + \begin{bmatrix} \ddots & & \\ & r & \\ & & \ddots \end{bmatrix} \dot{q} + \begin{bmatrix} \ddots & & \\ & k & \\ & & \ddots \end{bmatrix} q = \underline{Q}$$

we get our equations:

$$\begin{cases} m_{11} \ddot{q}_1 + r_{11} \dot{q} + k_{11} q = Q_1 \\ \vdots \\ m_{rr} \ddot{q}_r + r_{rr} \dot{q} + k_{rr} q_r = Q_r \\ \vdots \end{cases}$$

By using $[\Phi]$, we went from having to solve n equation together to n equations alone.

Solving the i-th equation:



If a force we apply have frequency ω_2, we only have to solve the second equation since all others will be ignored. This is the power of this approach, it allows us to limit the number of equations we solve to the only ones that are activated by our forces.

The diagonalisation is a consequence of this need to reduce equations, it's not the main objective of it.