Lezione 21 -

Last lecture we introduced gas-dynamics and put it in a system and introduced the Mach number which is a fluid-dynamic property, the Mach number explains how knowledge of an updated boundary condition is transferred to the particles in the system.

Solution to 1D compressible Flow - Isoentropic

System we found:

$$\begin{cases} \rho(x)\, v(x)\, A(x) = \dot{m} \text{ (const)} & \circledast_4 \\ h(x) + \dfrac{v(x)^2}{2} = h_T = \text{const} & Ⓐ_1 \\ \dfrac{P(x)}{\rho(x)^\gamma} = \text{const} \quad (\Rightarrow \Delta S = 0 \rightarrow \text{se last lesson}) & \circledast_3 \end{cases}$$

↳ (Technically already includes equations of state)

Equations of State: (must also hold)

$$+\begin{cases} P/\rho = RT \\ \Delta h = c_p \Delta T \end{cases}$$

Temperature reached after an isoentropic decelleration

$\exists h_T \Rightarrow \exists$ Total Temperal (T_T)

$$Ⓐ_1 \Rightarrow h_T = h + \frac{V^2}{2} \Rightarrow h_T - h = \frac{V^2}{2} \Longrightarrow c_p(T_T - T) = \frac{V^2}{2} \Rightarrow *$$

$$* \Longrightarrow T_T - T = \frac{V^2}{2c_p} \Rightarrow \frac{T_T}{T} = 1 + \frac{V^2}{2c_pT} =$$

$$\left(c_p = \frac{\gamma}{\gamma - 1} R\right) \qquad = 1 + \frac{\gamma - 1}{2} \cdot \frac{V^2}{\gamma R T} = \circledast_2$$

$$a = \sqrt{\left(\frac{\partial P}{\partial \rho}\right)_s} \quad ; \text{ Perfect Gas} \Rightarrow \ln\left(\frac{P}{\rho^\gamma}\right) = \ln(\text{const}) \Rightarrow \frac{dP}{P} = \gamma \frac{d\rho}{\rho}$$

speed of Sound

$$\frac{P}{\rho^\gamma} = \text{const}$$

$$\Rightarrow \left(\frac{\partial P}{\partial \rho}\right)_s = \gamma \underbrace{\frac{P}{\rho}}_{RT} = \gamma RT$$

$$a = \sqrt{\gamma RT}$$

$$\circledast \Rightarrow \frac{T_T}{T} = 1 + \frac{\gamma - 1}{\gamma} M^2 \rightarrow$$ Parametrisation of the Energy Balance as a function of M

$P_T \rightarrow$ pressure due to isoentropic decelleration

$$\circledast_3 \rightarrow \frac{P_2}{P_1} = \left(\frac{T_2}{T_1}\right)^{\frac{\gamma}{\gamma-1}} \rightarrow \frac{P_T}{P} = \left(\frac{T_T}{T}\right)^{\frac{\gamma}{\gamma-1}} = \left(1 + \frac{\gamma-1}{2} M^2\right)^{\frac{\gamma}{\gamma-1}}$$

since already have $\frac{T_T}{T_1}$

we can find P_T

for compressible flows

$$\Rightarrow \frac{\rho_T}{\rho} = \left(1 + \frac{\gamma-1}{\gamma} M^2\right)^{\frac{1}{\gamma-1}}$$

From here we can also get the total density ρ_T

$\circledast_4$

$$\hookrightarrow \dot{m} = \rho v A = \frac{P}{RT} \cdot \underbrace{aM}_{v} \cdot A = \frac{P}{RT} \cdot \sqrt{\gamma RT} \cdot M \cdot A$$

$$= \frac{PA}{\sqrt{RT}} \sqrt{\gamma}\, M$$

h_T, P_T and T_T are all good candidates for boundary condition.

We can write in function of P_T and T_T:

$$\Rightarrow \frac{P_T A}{\sqrt{\gamma T_T}} \underbrace{\left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{1}{2}-\frac{\gamma}{\gamma-1}} \sqrt{\gamma}\, M}_{f(\gamma, M)} \rightarrow \text{Dimensionless}$$

$$f(\gamma, M) = \frac{\dot{m}\sqrt{R T_T}}{P_T A} \rightarrow \text{This is how we introduce a dimensionless mass flow rate}$$

$$\hat{=}\ \dot{m}_{AD} \rightarrow \text{Dimensionless Mass Flow Rate}$$

$\longrightarrow$ <u>Area-Velocity Relation</u>

$$\begin{cases} \frac{d\rho}{\rho} + \frac{dv}{v} + \frac{dA}{A} = 0 \Rightarrow \frac{dA}{A} = \frac{dv}{v} - \frac{d\rho}{\rho} \quad (*)_6 \\ dh + d\left(\frac{v^2}{2}\right) = 0 \Rightarrow \cancel{Tds}^{\,0,\text{ isentropic}} + v\,dP + \frac{2}{2}v\,dv = 0 \Rightarrow v\,dP = -v\,dv \\ ds = 0 \Rightarrow dP/P = \gamma\frac{d\rho}{\rho} \Rightarrow dP = \gamma \underbrace{\frac{P}{\rho}}_{RT} d\rho = a^2 d\rho \end{cases}$$

$$+ \begin{cases} P/\rho = RT \\ \Delta h = C_p \Delta T \end{cases}$$

$$\Rightarrow \frac{1}{\rho} a^2 d\rho = -v\,dv$$

$$\Rightarrow \frac{d\rho}{\rho} = -\frac{v^2 dv}{a^2 v} = -M^2\frac{dv}{v}$$

$$\Rightarrow \frac{d\rho}{\rho} = -M^2 \frac{dv}{v} \quad (*)_5$$

If $M<1$, $\frac{d\rho}{\rho} < \frac{dv}{v}$, if $M>1$, $\frac{d\rho}{\rho} > \frac{dv}{v}$

$$(*)_6 + (*)_5 \Rightarrow \frac{dA}{A} = \frac{-dv}{v} + M^2\frac{dv}{v} = (M^2-1)\frac{dv}{v}$$

$$\frac{dA}{A} = (M^2-1)\frac{dV}{V}$$ → two different behaviour depending on M.

Since $v = \frac{1}{\rho} \Rightarrow \frac{d\rho}{\rho} = -\frac{dv}{v}$

we can therefore write

$$\frac{dv}{v} = M^2 \frac{dV}{V}$$ → v → specific volume
V → velocity.

Cases:

1) $M<1$ (Subsonic)

$$\frac{dv}{v} < \frac{dV}{V} \longrightarrow M^2-1<0 \Rightarrow \frac{dA}{A} = -|M^2-1|\frac{dV}{V}$$

⇒ Rate of change is opposite

$\frac{dV}{V} > 0 \Longleftrightarrow \frac{dA}{A}$ — converging nozzles
— diverging diffusers

2) $M>1$ (Supersonic)

$$\frac{dv}{v} > \frac{dV}{V} \Longrightarrow \frac{dA}{A} = |M^2-1|\frac{dV}{V}$$

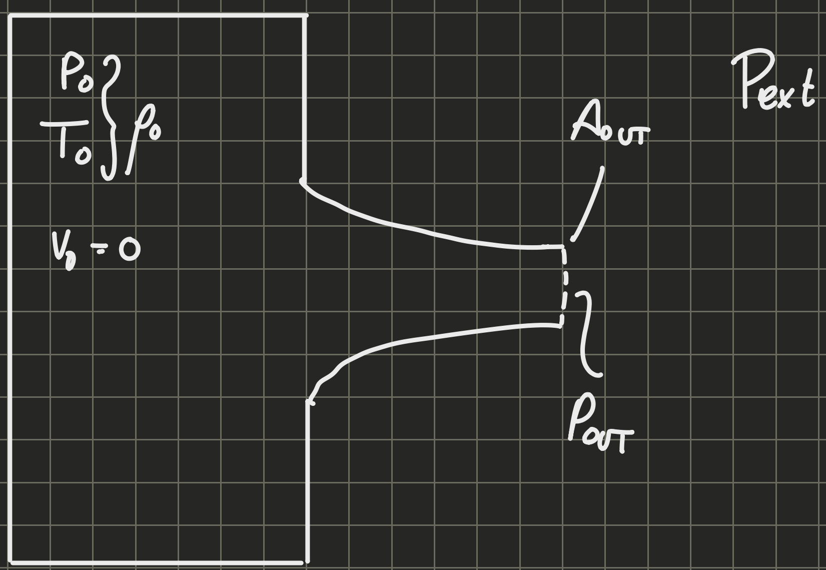
⇒ Diverging Nozzles and Converging Diffusers

3) $M=1$ (Sonic)

$$\frac{dA}{A} = 0 \implies \text{Sonic Throat}$$

Application

Efflux from Reservoir, 1D, compressible, isoentropic, purely converging nozzle.



$$\frac{P_{TOUT}}{P_{OUT}} = \left(1 + \frac{\gamma-1}{2} M_{OUT}^2\right)^{\frac{\gamma}{\gamma-1}} \longrightarrow \frac{P_0}{P_{EXT}} = \left(1 + \frac{\gamma-1}{2} M_{OUT}^2\right)^{\frac{\gamma}{\gamma-1}}$$

$$\frac{T_{TOUT}}{T_{OUT}} = \left(1 + \frac{\gamma-1}{2} M_{OUT}^2\right)$$

$$\left.\begin{aligned} &\implies M_{OUT} = \sqrt{\frac{2}{\gamma-1}\left(\left(\frac{P_0}{P_{EXT}}\right)^{\frac{\gamma-1}{\gamma}} - 1\right)} \\ &T_{OUT} = \frac{T_0}{\left(1 + \frac{\gamma-1}{2} M_{OUT}^2\right)} \end{aligned}\right\} a_{OUT}$$

If we are able to refere to conditions that we know (boundary conditions) then we are able to solve everything.

$$T_{TOUT} \overset{\text{since adiabatic}}{=} \underbrace{T_{T0} = T_0}_{\text{since } M_0 = 0}$$

$$P_{TOUT} = P_{T0} = P_0 \; (M_0 = 0)$$

$\hookrightarrow$ We assume an isoenergetic transformation ($\ell < 0 \to \Delta P = 0$)

$P_{OUT} = P_{EXT}$ (for the moment!, not always true, valid only for $M_{OUT} \le 1$)

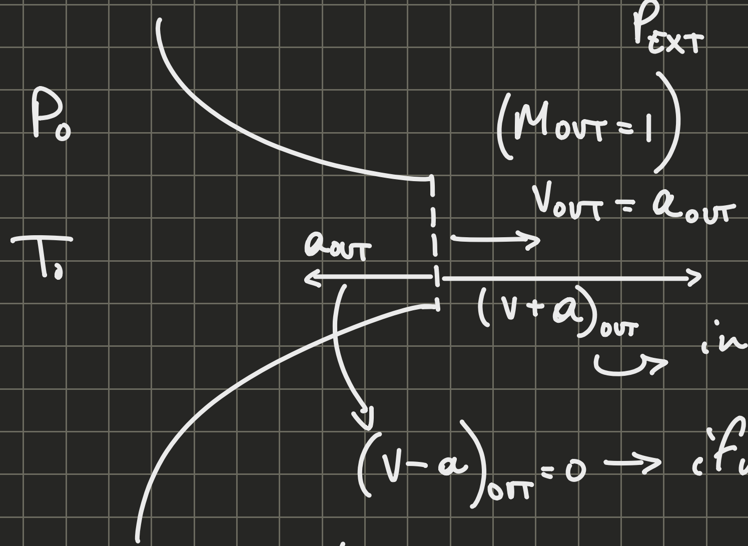
$M_{OUT}, a_{OUT} \to V_{OUT}, T_{OUT}, P_{OUT} \to \rho_{OUT}$ $\quad \rho_{OUT}, V_{OUT}, A_{OUT} \to \dot{m}$

If $M_{OUT} = 1 \Longrightarrow \dfrac{P_{TOUT}}{P_{OUT}} = \left(1+\dfrac{\gamma-1}{2}\right)^{\frac{\gamma}{\gamma-1}} = \left(\dfrac{\gamma+1}{2}\right)^{\frac{\gamma}{\gamma-1}} = \beta_{CRIT}$

↳ Critical Pressure Ratio

$$\frac{T_{TOUT}}{T} = \underbrace{\left(\frac{\gamma+1}{2}\right)}_{\text{Critical Temperature Ratio}}$$

If $P_{ext} < P_0/\beta_{CRIT} \Rightarrow P_{OUT} \neq P_{EXT}$



in normal flow conditions if we change P_0

$(v-a)_{OUT} = 0 \to$ if we reduce P_{EXT} to $\dfrac{P_0}{\beta_{crit}}$, while keeping P_0 const

there is no way for the speed of sound and perturbation to go upstream and renew the boundary condition inside the nozzle. This is called <u>choked flow condition</u>

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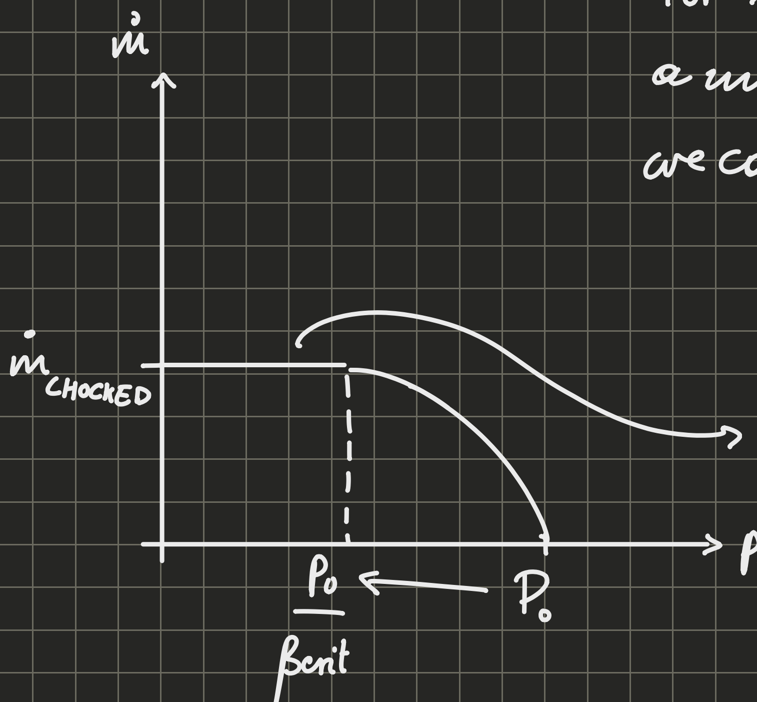
P_{OUT} will be the pressure which the nozzle can reach, to still allow it

$$P_{OUT} = \frac{P_0}{\left(\frac{\gamma+1}{2}\right)^{\gamma/\gamma-1}} \qquad T_{OUT} = \frac{T_0}{\left(\frac{\gamma+1}{2}\right)}$$

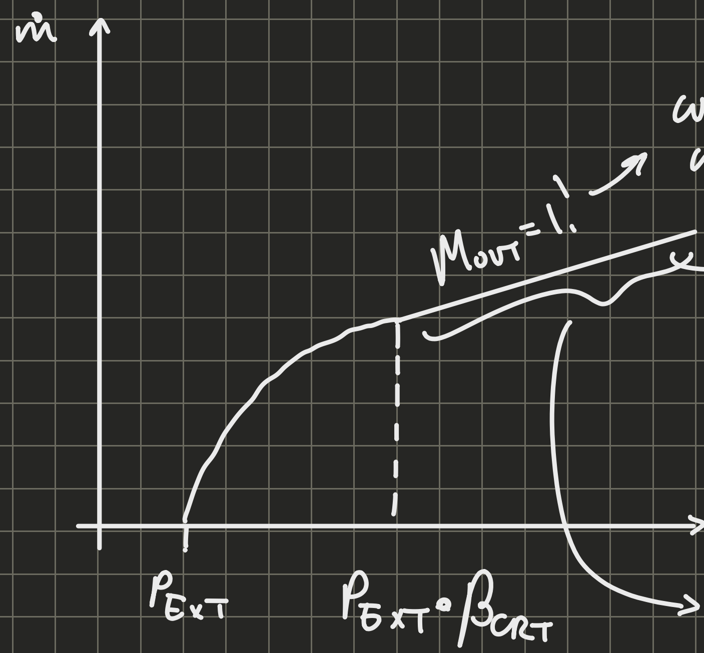
to perturb upstream to update the boundary conditions.

For this case, there is a maximum flow rate, we can extract.



the flow cannot increase since the perturbation wave to increase the flow at some point travels at the same speed as the flow and therefore never reaches the reservoir to be able to update the boundary conditions.

In general though P_{EXT} is fixed, and P_0 variable, so we can change $\dot{m}$ through that:



we cannot exceed 1, because we do not have a diverging duct

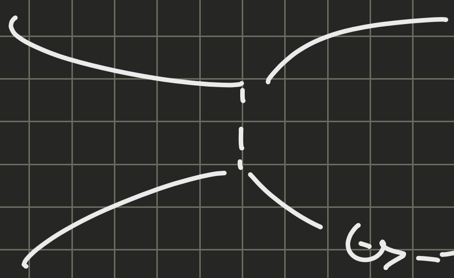
Not blocked and continuously increases, because the a goes with V.

$\dot{m} = P_T \cdot K = P_0 \cdot K$ (K: constant)

$$\dot{m} = \frac{P_T A}{\sqrt{R T_T}} \cdot f(\gamma, M)$$

Difference in plots is asked once a year.

If $P_0 > P_{EXT} \cdot \beta_{crit}$, $P_{OUT} > P_{EXT}$, so to return to P_{EXT} to flow will act as if it were inside a diverging nozzle and so it expands



↳ This was then used in rockets to help guide the fluid.