Lezione 13 -

Continuing from last lesson

Exercise

$X \sim N(\mu, \sigma^2)$, energy demand

μ = mean (unknown)

$\sigma^2 = 44\ kWh^2/day^2$

$x_1 = 61{,}9 \quad x_2 = 59{,}7 \quad x_3 = 60{,}8 \quad x_4 = 58{,}6 \quad x_5 = 61{,}3$

a) $H_0 : \mu \geq \underset{}{\overset{=\mu_0}{66}}$ vs. $H_1 : \mu < 66$ $\qquad \mu_0 = 66$, $\sigma_0^2 = 44$

Reject H_0 at significance level 5% $\iff z_0 = \dfrac{\bar{x} - \mu_0}{\sqrt{\dfrac{\sigma_0^2}{n}}} \leq -z_{1-\alpha}$

$\bar{x}_5 = 60{,}46$

$z_0 = -1.87 \leq -z_{0,95} = -1.645$

True $\Rightarrow$ we are rejecting H_0 at 5% significance level.

This decision is strong (any time we reject H_0 it is a strong decision) since the error is a type I probability so it's linked to α, so it's strong since we can control the error of being wrong in our rejection.

b) Type II error probability for $\mu = 60$

Type II error = reject H_1, given that H_1 is true.

$\underbrace{\mu < 66}$

We are asked for $\mu = 60$

$$P_{\mu=60}(\text{accept } H_0) = P_{\mu=60}\left(\frac{\bar{x}_n - \mu_0}{\sqrt{\sigma_0^2/n}} > -z_{1-\alpha}\right)$$

$$P_{\mu=60}\left(\bar{x}_n > \mu_0 - z_{1-\alpha}\sqrt{\frac{\sigma_0^2}{n}}\right)$$

$$\bar{X}_n \underset{\mu=60}{\overset{H_1 \text{ true}}{\sim}} N\left(60, \frac{\sigma_0^2}{n}\right)$$

$$= 1 - P_{\mu=60}\left(\bar{x}_n \leq \mu_0 - z_{1-\alpha}\sqrt{\frac{\sigma_0^2}{n}}\right) =$$

$$= 1 - P_{\mu=\mu_1}\left(\frac{\bar{x}_n - \mu_1}{\sqrt{\sigma_0^2/n}} \leq \frac{\mu_0 - \mu_1 - z_{1-\alpha}\sqrt{\frac{\sigma_0^2}{n}}}{\sqrt{\sigma_0^2/n}}\right)$$

General Type II error probability formula.

$$= 1 - \Phi\left(\frac{\mu_0 - \mu_1}{\sqrt{\sigma_0^2/n}} - z_{1-\alpha}\right) = 1 - \Phi\left(\frac{66-60}{\sqrt{44/5}} - 1.645\right) = 1 - \Phi(.308)$$

$$= 0{,}357$$

c) $H_0: \mu \leq 66 \qquad H_1: \mu > 66$

Reject H_0 at significance level 5%

$$\Longleftrightarrow t_0 = \frac{\bar{x}_n - \mu_0}{\sqrt{\sigma_0^2/n}} \geq z_{1-\alpha}$$

$$= -1.87 \geq 1.645$$

False $\Rightarrow$ We cannot reject H_0, since $\bar{x}_n$ is not within C_0

Not rejecting H_0. This is a weak conclusion, because if we were wrong (H_0 is accepted and H_1 is false), we would not able to control the error if we were wrong, since the error is type II which we cannot control.

Any time we don't reject H_0, the conclusion is weak.

$$\text{Power} = 1 - P(\text{Type II error}) = P_{H_1}(H_1 \text{ is not rejected})$$

<u>Choice of Unilateral</u> - <u>Hypothesis:</u>

If $H_0 \leq \mu_0$ and $H_1: \mu > \mu_0$ or $H_0: \mu > \mu_0$ and $H_1: \mu < \mu_0$

These are symmetric test cases.

The decision of not rejecting H_0 is weak, which is why don't say we accept H_0.

→ If we have to choose, we choose the one where the most important error is type II error.

Example:

↳ we look at an example of NO_2 and NO_x production.

<u>Testing the mean Gaussian population with an UNKNOWN variance</u>

In this case the test statistic is the only thing that changes, to:

$$t_0 = \frac{\bar{X}_n - \mu_0}{\sqrt{\frac{S_n^2}{n}}} \overset{\mu_0}{\sim} t(n-1)$$

Reject H_0 at α significance level. $\iff \dfrac{|\bar{x}_n - \mu_0|}{\sqrt{S_n^2/n}} \geq t_{1-\frac{\alpha}{2}}(n-1)$

$$p = 2 \cdot \left(1 - F_{t(n-1)}\left(\frac{\bar{x}_n - \mu_0}{\sqrt{S_n^2/n}}\right)\right)$$

<u>Tests and CIs for the population mean for a LARGE sample size n</u>

If n is large ($n \geq 50$) whatever the distribution, we can say:

$$\frac{\bar{X}_n - \mu}{\sqrt{S_n^2/n}} \overset{\text{approx}}{\sim} N(0,1)$$

Testing a proportion where n is large:

$X_1, \ldots, X_n \overset{iid}{\sim} Be(p)$, we test $H_0: p = p_0$, $H_1: p \neq p_0$

Reject H_0 a significance level α $\iff |z_0| \geq z_{1-\frac{\alpha}{2}}$ where:

approximately equal to

$$z_0 = \frac{\bar{x}_n - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$$

Exercise

$x_1, \dots, x_n \qquad n = 200 \qquad x_i = \begin{cases} 1 & \text{if stain remover effective} \\ 0 & \text{if stain remover ineffective} \end{cases}$

on i-th stain

$$\sum_{1}^{n} x_i = 174$$

$$X_1, \dots, X_n \overset{iid}{\sim} Be(p)$$

$X_i = \begin{cases} 1 & \text{if } \dots \\ 0 & \text{otherwise} \end{cases}$ $\qquad p$ = probability stain remover effective.

The company claims that $p \geq 0.9$

False statement from the company $\Rightarrow p < 0.9$.

$H_0 : p > 0.9 \qquad H_1 : p \leq 0.9$

$H_0 : p \leq 0.9 \qquad H_1 = p > 0.9$

which do we chose.

This one because we need to give evidence, since in the other case the Type II error which is our error, would not be controllable and so this is the better case to check.

$H_0 : p \geq 0.9$ vs. $H_1 = p < 0.9 \qquad p_0 = 0.9$

Reject H_0 at the significance level $\alpha = 5\%$ $\iff$ $$z_0 = \frac{\bar{x}_n - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} \leq -z_{1-\alpha}$$ with $\bar{x}_n = \frac{174}{200}$

$z_0 = -1.41 \le -1.645 \rightarrow$ Not true, so we cannot reject H_0.

3) With the available data, for which α can we conclude the company claim to be false
i.e. for which α do we reject H_0 and accept H_1.

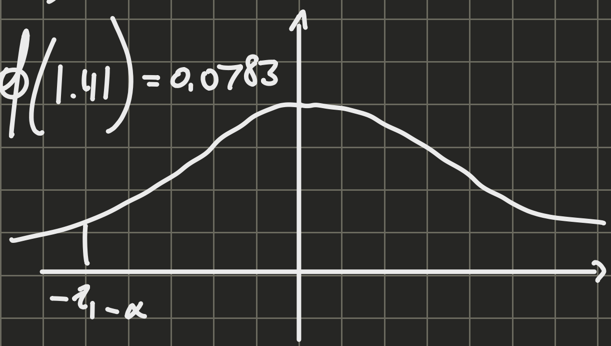
$\Rightarrow$ we need to calculate the p-value.

approximate $\rightarrow$ large population.

Reject H_0 $\Longleftrightarrow$ $z_0 \le -z_{1-\alpha}$

$$\Phi(z_0) \le \Phi(-z_{1-\alpha}) = \alpha$$

$$p\text{-value} = \Phi(z_0) = \Phi(-1.41) = 1 - \Phi(1.41) = 0.0783$$



If $\alpha = 1\%$ or 5% $\Rightarrow$ "accept H_0"
if $\alpha = 10\%$ $\rightarrow$ "reject H_0"

The p-value can be interpreted as the risk of wrongly rejecting H_0.

p-value can be used as an indicator of the strength of the claim against H_0.

Hypothesis testing via CIs → relationship between CIs and hypothesis testing

Let (t_1, t_2) be a two-sided CI for θ of confidence level $1-\alpha$.

We need to check the agreement of data $x = (x_1, \dots, x_n)$ with

$H_0: \theta = \theta_0$ vs. $H_1: \theta \neq \theta_0$

Then:

We reject H_0 if $\theta_0 \notin (t_1, t_2)$

We do not reject H_0 if $\theta_0 \in (t_1, t_2)$

Exercise:

1) Since $\bar{x}$, l, n and $t_{1-\alpha}$ are known, we can find S_n^2

X_i - lengths covered by arrows

$X_i \overset{iid}{\sim} N(\mu, \sigma^2)$

$\bar{x}_n = 173{,}13\,m$

$s_n^2 = 5{,}7403$ since $l = \bar{x}_n - t_{1-\alpha} \frac{S_n^2}{\sqrt{n}} = 172{,}08$

2) Is there empirical evidence that the shot length is > 170 at the 1% confidence level?

$H_0: \mu \leq 170 \qquad H_1: \mu > 170$

↳ so the error is of type I

Reject H_0 at 1% significance level $\iff$ $t_0 = \dfrac{\bar{x}_n - \overset{170}{\mu_0}}{\sqrt{S_n^2/n}} \geq t_{1-\alpha}(n-1) = t_{.99}(15) = 1,753$

opposite of H_0 since we are checking if it's inside rejection region.

Reject H_0 at the 1% significance level $\iff$ $\underset{173,13}{\bar{x}_n} \geq t_{1-\alpha}(n-1)\sqrt{\dfrac{S_n^2}{n}} + \mu_0 = 171.55$

True, so we reject H_0 at the 1% significance level.