Half-range Fourier Sine or Cosine series

In half-range Fourier series =)

~~1. Only sine or cosine~~

(1) Only sine (for odd f_n) or only cosine (for even f_n) terms are present

(2) The limit of integration becomes $(0, L)$ instead of $(-L, L)$ & hence the name is 'Half-range'.

Def^n:

Half range sine series of $f(x)$ with period L is given by where; $0 \le x \le L$,

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

where

$$b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx ;$$

$n = 1, 2, 3, \ldots$

⊞ Half range cosine series of $f(x)$ $: 0 \le x \le L$. with period L is given by ⇒

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

where,

$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx \; ; \; n = 1, 2, \ldots$$

* Complex Notation for Fourier Series :

Fourier series of $f(x)$ can be written in complex form as -

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi x}{L}}$$

where,

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x) e^{-i\frac{n\pi x}{L}}\,dx$$

NB: $e^{i\theta} = \cos\theta + i\sin\theta$, $e^{-i\theta} = \cos\theta - i\sin\theta$.

Half-Range Fourier Series

Ex (01) Express $f(x) = x$ as a half range sine series in the interval $0 < x < 2$.

Sol^n Here, $L = 2$.

We know, half-range sine series $\Rightarrow$

$$x = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$\therefore x = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right) \text{ ———— } (*)$$

where,

$$b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{\cancel{2}}{\cancel{2}}\int_0^2 x \sin\left(\frac{n\pi x}{2}\right) dx$$

$$= \left[-x \frac{\cos\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}} + \frac{\sin\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)^2} \right]_0^2$$

	D	I
		$\sin\left(\frac{n\pi x}{2}\right)$
$+$	x	$-\frac{\cos\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}}$
$-$	1	$-\frac{\sin\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)^2}$
	0	

$$= \left[-2 \cdot \frac{2}{n\pi} \cdot \cos(n\pi) + \frac{4}{n^2\pi^2} \sin(n\pi) + 0 - \frac{4}{n^2\pi^2} \sin(0) \right]$$

$$= -\frac{4}{n\pi}(-1)^n + 0 + 0 - 0 \quad \left[\because \cos(n\pi) = (-1)^n \atop \sin(n\pi) = 0 \right]$$

$$= -\frac{4(-1)^n}{n\pi}$$

$$(*) \Rightarrow \quad x = \sum_{n=1}^{\infty} \frac{-4(-1)^n}{n\pi} \sin\left(\frac{n\pi x}{2}\right)$$

Ans.

Ex (02) find the Half-range fourier series for $f(x) = x^4$ on the interval $0 < x < \pi$.

Sol^n Here, $L = \pi$

Here $f(x) = x^4$ $\therefore$ $f(-x) = (-x)^4 = x^4 = f(x)$

$\Rightarrow$ f is an even function.

So, the corresponding Half-range Fourier series will be a Half-range cosine series. i.e.

$$x^4 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

$$= \sum_{n=1}^{\infty} a_n \cos(nx) \quad \text{--- (A)} \quad [\because L = \pi]$$

where,

$$a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x^4 \cos(nx)\, dx$$

$$= \frac{2}{\pi} \left[\frac{x^4 \sin(nx)}{n} + \frac{4x^3 \cos(nx)}{n^2} - \frac{12x^2 \sin(nx)}{n^3} - \frac{24x \cos(nx)}{n^4} + \frac{24 \sin(nx)}{n^5} \right]_0^{\pi}$$

	D	I
+	x^4	$\cos(nx)$
−	$4x^3$	$\frac{\sin(nx)}{n}$
+	$12x^2$	$-\frac{\cos(nx)}{n^2}$
−	$24x$	$-\frac{\sin(nx)}{n^3}$
+	24	$\frac{\cos(nx)}{n^4}$
	0	$\frac{\sin(nx)}{n^5}$

$$= \frac{2}{\pi}\left[0 + \frac{4\pi^3(-1)^n}{n^2} - 0 - \frac{24\pi(-1)^3}{n^4} + 0 - 0 - 0 + 0 + 0 - 0\right]$$

$$= \frac{2}{\pi}\left[\frac{4\pi^3(-1)^n}{n^2} - \frac{24\pi(-1)^n}{n^4}\right]$$

$$a_n = \frac{8\pi^2(-1)^n}{n^2} - \frac{48(-1)^n}{n^4} = \frac{(8\pi^2 n^2 - 48)(-1)^n}{n^4}$$

& $$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^4\,dx = \frac{2}{\pi}\left[\frac{x^5}{5}\right]_0^{\pi} = \frac{2}{\pi}\left[\frac{\pi^5}{5} - 0\right]$$

$$\therefore a_0 = \frac{2\pi^5}{5\pi} = \frac{2\pi^4}{5}$$

$(x) \Rightarrow$

$$x^4 = \frac{1}{\cancel{2}} \frac{\cancel{2}\pi^4}{5} + \sum_{n=1}^{\infty} \frac{(8\pi^2 n^2 - 48)(-1)^n}{n^4} \cos(nx)$$

$$= \frac{\pi^4}{5} + \sum_{n=1}^{\infty} \frac{8(\pi^2 n^2 - 6)}{n^4} (-1)^n \cos(nx).$$

Ans.

Fourier Series

expresses a function as a sum of trigonometric (sine/cosine) fun's (periodic fun's).

Def^n (periodic function)

A fu^n $f(x)$ is said to have a period p if $f(x+p) = f(x), \forall x$ — (i)

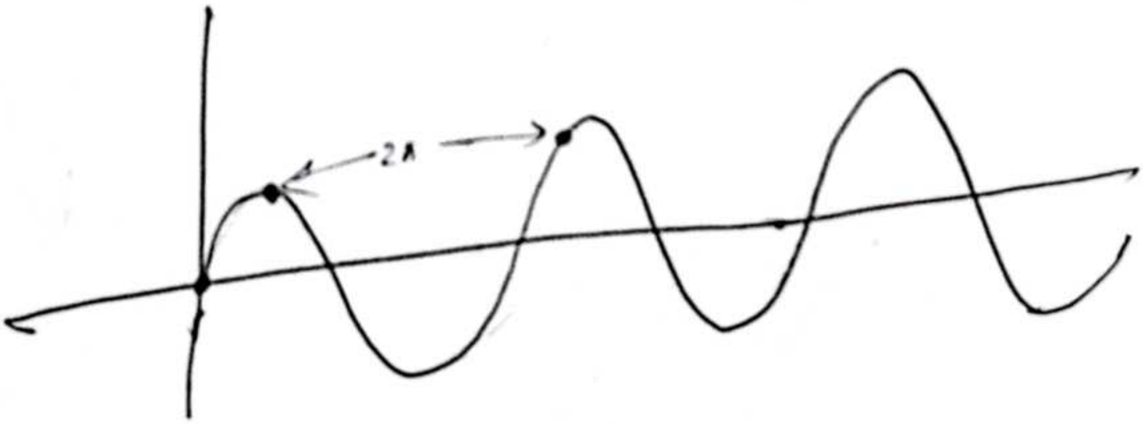
The least value of $p > 0$ satisfying (i) is called the least period of $f(x)$.

Ex.

1) $\sin(x+2\pi) = \sin(x+4\pi) = \sin(x+6\pi) = \sin x$

So, the fu^n $\sin x$ has period $2\pi, 4\pi, 6\pi, \ldots$ etc.

However, 2π is the least period of $\sin x$.



Periodic functions

① ~~sketch~~ $f(x) = \begin{cases} x \ ; & 0 < x < \pi \\ -x \ ; & -\pi < x < 0 \end{cases}$

i) sketch $f(x)$.

ii) Is $f(x)$ even or odd ?

(ii) $f(-x) = \begin{cases} -x \,; & 0 < -x < \pi \\ -(-x) \,; & -\pi < -x < 0 \end{cases}$

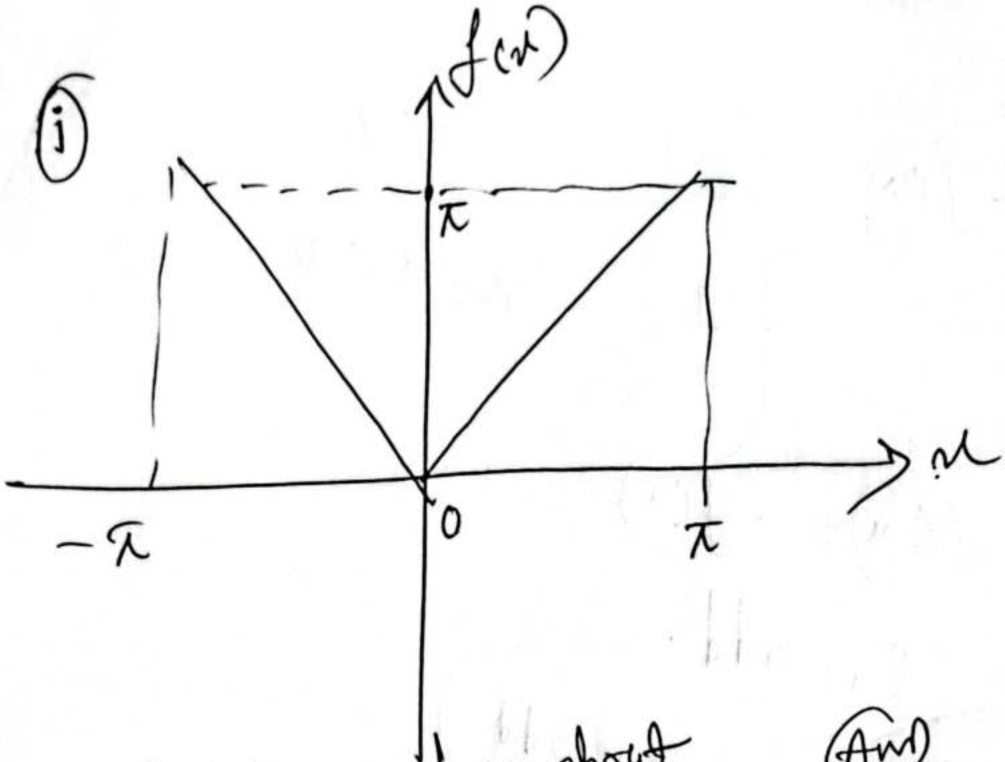
$$= \begin{cases} -x \,; & 0 > x > -\pi \\ x \,; & \pi > x > 0 \end{cases}$$

$$= \begin{cases} -x \,; & -\pi < x < 0 \\ x \,; & 0 < x < \pi \end{cases}$$

$\therefore f$ is even. $= \begin{cases} x \,; & 0 < x < \pi \\ -x \,; & -\pi < x < 0 \end{cases} = f(x)$

CamScanner

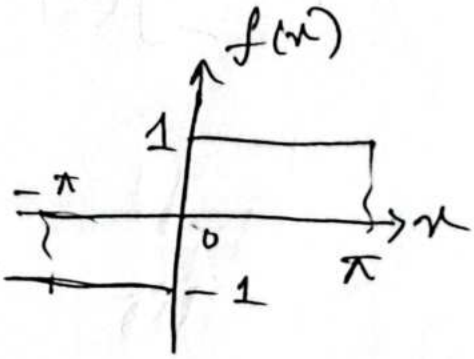
(j)



NB: even f^n is symmetric about y-axis. (Ans)

Ex: (odd f^n)

$$f(x) = \begin{cases} 1 ; & 0 \le x \le \pi \\ -1 ; & -\pi \le x < 0 \end{cases}$$



$$f(-x) = \begin{cases} 1 ; & 0 \le -x \le \pi \\ -1 ; & -\pi \le -x < 0 \end{cases}$$

$$= \begin{cases} 1 ; & 0 \ge x \ge -\pi \\ -1 ; & \pi \ge x > 0 \end{cases}$$

$$= \begin{cases} 1 ; & -\pi \le x \le 0 \\ -1 ; & 0 < x \le \pi \end{cases} = \begin{cases} -1 ; & 0 < x \le \pi \\ 1 ; & -\pi \le x \le 0 \end{cases}$$

Again,

$$-f(x) = \begin{cases} -1; & 0 \le x \le \pi \\ 1; & -\pi \le x \le 0. \end{cases}$$

$\therefore f(-x) = -f(x)$

$\therefore f$ is odd.

Ex: (Neither even nor odd)

$$f(x) = \begin{cases} 2x; & x < 5 \\ 15 - x; & x \ge 5 \end{cases}$$

Also,

$$-f(x) = \begin{cases} -2x; & x < 5 \\ -(15-x); & x \ge 5 \end{cases}$$

$$= \begin{cases} -2x; & x < 5 \\ x - 15; & x \ge 5 \end{cases}$$

Now,

$$f(-x) = \begin{cases} -2x; & -x < 5 \\ 15 + x; & -x \ge 5 \end{cases} = \begin{cases} -2x; & x > -5 \\ 15 + x; & x \le -5. \end{cases}$$

Neither, $\therefore f(-x) \ne f(x)$ nor $f(-x) \ne -f(x)$

$\therefore f$ is neither even nor odd.