finding Eigenvectors

we will get an eigenvector corresponding to each eigenvalue.

Q. find ~~eigenvector~~ eigenvalue & the corresponding eigenvectors of

$$A = \begin{pmatrix} -1 & 3 \\ 2 & 0 \end{pmatrix}.$$

Soln. Eigen-value: charac. eqn ⇒

$$\lambda^2 - (-1+0)\lambda + (0-6) = 0$$

$$\Rightarrow \lambda^2 + \lambda - 6 = 0$$

$$\lambda = 2, -3. \quad \text{Ans}$$

Eigenvectors:

for $\lambda = 2$: Let, $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be the eigenvector corresponding to the eigen value $\lambda = 2$. Then,

$$A\underline{x} = \lambda \underline{x}$$

$$A\underline{x} - \lambda \underline{x} = 0$$

$$\Rightarrow (A - \lambda I)\underline{x} = 0$$

$$\Rightarrow (A - 2I)\underline{x} = 0$$

$$\Rightarrow (A - $$

$$\Rightarrow \left(\begin{pmatrix} -1 & 3 \\ 2 & 0 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right) \underline{x} = \underline{0}$$

$$\Rightarrow \begin{pmatrix} -3 & 3 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\therefore$ The augmented matrix $\Rightarrow$

$$\left(\begin{array}{cc|c} -3 & 3 & 0 \\ 2 & -2 & 0 \end{array} \right)$$

$$\sim \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & -1 & 0 \end{array} \right) ; \quad \begin{array}{l} R_1' = R_1 \times \left(-\frac{1}{3}\right) \\ R_2' = R_2 \times \left(\frac{1}{2}\right) \end{array}$$

$$\sim \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array} \right) ; \quad R_2' = R_2 - R_1$$

$\therefore$ The corresponding system $\Rightarrow$

$$x_1 - x_2 = 0$$

there is 1 eq^n in 2 unknowns, so $(2-1) = 1$ free variable. Let $x_2 = t$; $t \in \mathbb{R}$

then, $x_1 = t$.

$$\therefore \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} t \\ t \end{pmatrix} = t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\therefore$ Eigenvector corresponding to $\lambda = 2$ is $\underline{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

<u>for $\lambda = -3$:</u>

$$(A - \lambda I)\underline{x} = 0$$

$$\Rightarrow (A + 3I)\underline{x} = 0 \quad [\because \lambda = -3]$$

$$\Rightarrow \left(\begin{pmatrix} -1 & 3 \\ 2 & 0 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \right) \underline{x} = 0$$

$$\Rightarrow \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\therefore$ Augmented matrix $\Rightarrow$

$$\left(\begin{array}{cc|c} 2 & 3 & 0 \\ 2 & 3 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|c} 2 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right) : r_2' = r_2 - r_1$$

$\therefore \quad 2x_1 + 3x_2 = 0$

Here, $(2-1) = 1$ free variable

$\underline{1}$ eq^n in 2 unknowns. So,

let, $x_2 = r : r \in \mathbb{R}$

$\therefore 2x_1 = -3x_2 = -3r$

$\therefore x_1 = -\frac{3}{2}r$

$$\therefore \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\frac{3}{2}r \\ r \end{pmatrix} = r \begin{pmatrix} -3/2 \\ 1 \end{pmatrix}$$

So, eigenvector corresponding to $\lambda = -3$ is $\underline{x} = \begin{pmatrix} -3/2 \\ 1 \end{pmatrix}$.

Ans

3×3 matrix (non-repeated eigenvalue)

Q: Find Eigenvalue & the corresponding eigenvectors of $A = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}$

Soln.

Eigenvalue:

$$\lambda^3 - tr(A)\lambda^2 + (A_{11}+A_{22}+A_{33})\lambda - |A| = 0$$

$$\Rightarrow \lambda^3 - (1+2-1)\lambda^2 + \left((-2+1)+(-1-8)+(2+3)\right)\lambda - \left\{1(-2+1)+1(-3+2)+4(3-4)\right\} = 0$$

$$\Rightarrow \lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0$$

$$\lambda = -2, 3, 1.$$

Ans

Eigenvectors:

for $\lambda = -2$:

$$(A - \lambda I)\underline{x} = \underline{0}$$

$$\Rightarrow (A + 2I)\underline{x} = \underline{0}$$

$$\Rightarrow \left(\begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix} + \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \right) \underline{x} = \underline{0}$$

$$\Rightarrow \begin{pmatrix} 3 & -1 & 4 \\ 3 & 4 & -1 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 & -1 & 4 & | & 0 \\ 3 & 4 & -1 & | & 0 \\ 2 & 1 & 1 & | & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -1/3 & 4/3 & | & 0 \\ 1 & 4/3 & -1/3 & | & 0 \\ 1 & \frac{1}{2} & \frac{1}{2} & | & 0 \end{pmatrix} \quad \begin{array}{l} r_1' = r_1 \times \left(\frac{1}{3}\right) \\ r_2' = r_2 \times \left(\frac{1}{3}\right) \\ r_3' = r_3 \times \left(\frac{1}{2}\right) \end{array}$$

$$\sim \begin{pmatrix} 1 & -1/3 & 4/3 & | & 0 \\ 0 & 5/3 & -5/3 & | & 0 \\ 0 & 5/6 & -5/6 & | & 0 \end{pmatrix} \quad \begin{array}{l} r_2' = r_2 - r_1 \\ r_3' = r_3 - r_1 \end{array}$$

$$\sim \begin{pmatrix} 1 & -1/3 & 4/3 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & 1 & -1 & | & 0 \end{pmatrix} \quad \begin{array}{l} r_2' = r_2 \times \left(\frac{3}{5}\right) \\ r_3' = r_3 \times \left(\frac{6}{5}\right) \end{array}$$

$$\sim \begin{pmatrix} 1 & -1/3 & 4/3 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix} \quad r_3' = r_3 - r_2$$

$$x_1 - \frac{1}{3}x_2 + \frac{4}{3}x_3 = 0$$

$$x_2 - x_3 = 0$$

2 eq^n in 3 unknowns $\therefore (3-2) = 1$ free variables

Let, $x_3 = t$.

$\therefore x_2 = t$

$x_1 = \frac{1}{3}t - \frac{4}{3}t = -t$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -t \\ t \\ t \end{pmatrix} = t\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$\therefore$ Eigenvector corresponding to $\lambda = -2$

is $\underline{x} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$.

For $\lambda = 3$:

$$(A - \lambda I)\underline{x} = 0$$

$$\Rightarrow (A - 3I)\underline{x} = \underline{0}$$

$$\Rightarrow \begin{pmatrix} -2 & -1 & 4 \\ 3 & -1 & -1 \\ 2 & 1 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

[Solve the system by yourself]

$\underline{x} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ is the eigenvector corresponding to $\lambda = 3$.

for $\lambda = 1$:

$$(A - \lambda I)\,\underline{x} = 0$$

$$\Rightarrow (A - I)\,\underline{x} = 0$$

$$\Rightarrow \begin{pmatrix} 0 & -1 & 4 \\ 3 & 1 & -1 \\ 2 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 0 & -1 & 4 & 0 \\ 3 & 1 & -1 & 0 \\ 2 & 1 & -2 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 3 & 1 & -1 & 0 \\ 0 & -1 & 4 & 0 \\ 2 & 1 & -2 & 0 \end{array}\right) \quad ; \; r_1 \longleftrightarrow r_2$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1/3 & -1/3 & 0 \\ 0 & -1 & 4 & 0 \\ 1 & \frac{1}{2} & -1 & 0 \end{array}\right) \quad \begin{array}{l} r_1' = r_1 \times \left(\frac{1}{3}\right) \\ r_3' = r_3 \times \left(\frac{1}{2}\right) \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1/3 & -1/3 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 1/6 & -2/3 & 0 \end{array}\right) \quad \begin{array}{l} r_2' = r_2 \times (-1) \\ r_3' = r_3 - r_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & \frac{1}{3} & -\frac{1}{3} & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 1 & -4 & 0 \end{array}\right) \; \cancel{R_3' = R_3 - \frac{1}{6}R_2} \quad R_3' = R_3 \times 6$$

$$\sim \left(\begin{array}{ccc|c} 1 & \frac{1}{3} & -\frac{1}{3} & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad R_3' = R_3 - R_2$$

$\therefore x_1 + \frac{1}{3}x_2 - \frac{1}{3}x_3 = 0$

$x_2 - 4x_3 = 0$

Let, $x_3 = t$

$\therefore x_2 = 4t$

$x_1 = -\frac{1}{3} \times 4t + \frac{1}{3}t$

$= -t$

$$\therefore \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$$

$\therefore$ Eigenvector corresponding to $\lambda = 1$ is

$$\underline{x} = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} \quad \text{(Ans)}.$$

3×3 matrix (Repeated Eigenvalue)

Q: Find Eigenvalues & Eigenvectors of

$$A = \begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$$

Sol^n^:

Eigenvalue:

$$\lambda^3 - tr(A)\lambda^2 + (A_{11} + A_{22} + A_{33})\lambda - |A| = 0$$

$$\Rightarrow \lambda^3 + \lambda^2 - 21\lambda - 45 = 0$$

$$\cancel{\lambda = 5, -3, -3}$$

using calculator, you will get $\lambda = 5, -3$

but you must get 3 roots since it is a polynomial eq^n^ of degree 3. So, either 5 or -3 must occur two times. To find out this, you have to factor out!

$$\Rightarrow \lambda^3 - 5\lambda^2 + 6\lambda^2 - 30\lambda + 9\lambda - 45 = 0$$

$$\Rightarrow \lambda^2(\lambda - 5) + 6\lambda(\lambda - 5) + 9(\lambda - 5) = 0$$

$$\Rightarrow (\lambda - 5)(\lambda^2 + 6\lambda + 9) = 0$$

$$\Rightarrow (\lambda - 5)(\lambda + 3)^2 = 0 \qquad \therefore \lambda = 5, -3, -3.$$

(Ans)

for $\lambda = $

for $\lambda = 5$:

$$(A - \lambda I)\underline{x} = 0$$

$$\Rightarrow (A - 5I)\underline{x} = 0$$

$$\Rightarrow \begin{pmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

[Solve by yourself] $\therefore \underline{x} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

Eigenvector corresponding to $\lambda = 5$ is $\underline{x} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

for $\lambda = -3$:

$$(A - \lambda I)\underline{x} = 0$$

$$\Rightarrow (A + 3I)\underline{x} = 0$$

$$\Rightarrow \begin{pmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ 1 & 2 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Augmented matrix $\Rightarrow$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 2 & 4 & -6 & 0 \\ 1 & 2 & -3 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 1 & 2 & -3 & 0 \\ 1 & 2 & -3 & 0 \end{array}\right) ; R_2' = \frac{R_2}{2}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) ; \begin{array}{l} R_2' = R_2 - R_1 \\ R_3' = R_3 - R_1 \end{array}$$

$\therefore x_1 + 2x_2 - 3x_3 = 0$

Here 1 eq^n in 3 unknowns.

So, $(3-1) = 2$ free variables :

Let, $x_3 = r$, $x_2 = t$

$\therefore x_1 = -2t + 3r$

$$\therefore \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2t + 3r \\ t \\ r \end{pmatrix}$$

$$= \begin{pmatrix} -2t \\ t \\ 0 \end{pmatrix} + \begin{pmatrix} 3r \\ 0 \\ r \end{pmatrix}$$

$$= t \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + r \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$$

$\therefore$ So, the eigenvectors corresponding to

$\lambda = -3$ are, $\underline{x} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$ and $\underline{x} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$.

Ans

Practice Problems

NB. Eigenvalues of ~~a~~ any matrix ~~is also~~ are same / fixed But Eigen~~values~~ vectors corresponding to each eigenvalue may be different / not unique. So, you may get different answer from the ~~sort~~ given answer.

Q. Find Eigenvalue & Eigenvectors

i) $A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$

Ans. $\lambda = -2 : \underline{x} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ (or any scalar multiplication of it. for eg. $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.)

$\lambda = 3 : \underline{x} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$

$\lambda = 6 : \underline{x} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$.

ii) $A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$

Soln

$\lambda = 8 \; : \; \underset{\sim}{x} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

$\lambda = 2, 2 \; : \; \underset{\sim}{x} =$ find yourself (you'll get 2 eigenvectors for $\lambda = 2$)

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