Answer to the Q. NO - 01 (a)

let,

$\underline{u} = (1, x_1)$

$\underline{v} = (1, x_2)$

(i) for $\underline{u}, \underline{v} \in V$,

$\underline{u} + \underline{v} = (1, x_1 + x_2) \in V$; which are real number

(ii) $\underline{u} + \underline{v} = (1, x_1 + x_2)$

$= (1, x_2 + x_1)$

$= (1, x_2) + (1, x_1)$

$= \underline{v} + \underline{u}$

(iii) let,

$w = (1, x_3)$

$\underline{u} + (\underline{v} + \underline{w}) = (1, x_1) + \{(1, x_2) + (1, x_3)\}$

$= (1, x_1) + (1, x_2 + x_3)$

$= (1, (x_1 + x_2) + x_3)$

$= (1, (x_1 + x_2) + (1, x_3)$

$= \{(1, x_1) + (1, x_2)\} + (1, x_3)$

$= (\underline{u} + \underline{v}) + \underline{w}$

[P.T.O.]

(iv) let,

$$\underline{0} = (1, 0)$$

$$\underline{u} + \underline{0} = (1, x_1) + (1, 0)$$
$$= (1, (x_1 + 0))$$
$$= (1, x_1)$$
$$= \underline{u}$$

(v) let,

$$-u = (1, -x_1)$$

$$\underline{u} + (-\underline{u}) = (1, x_1) + (1, -x_1)$$
$$= (1, x_1 - x_1)$$
$$= (1, 0)$$
$$= \underline{0}$$

(vi) for all $\underline{u} \in V$ and $K \in F$,

$$K\underline{u} = K(1, x_1)$$
$$= (1, Kx) \in V$$

(vii)
$$ab(\underline{u}) = ab(1, x_1)$$
$$= (1, abx_1)$$
$$= \{1, a(bx_1)\}$$
$$= a(1, bx_1)$$
$$= a\{b(1, x_1)\}$$
$$= a(b\underline{u})$$

(VIII) $a(\underline{u}+\underline{v}) = a(1, x_1+x_2)$

$= (1, ax_1 + ax_2)$

$= (1, ax_1) + (1, ax_2)$

$= a(1, x_1) + a(1, x_2)$

$= a\underline{u} + a\underline{v}$

(IX) $(a+b)\underline{u} = (a+b)(1, x_1)$

$= \{1, (a+b)x_1\}$

$= (1, ax_1 + bx_1)$

$= (1, ax_1) + (1, bx_1)$

$= a(1, x_1) + b(1, x_1)$

$= a\underline{u} + b\underline{u}$

(X) $1\underline{u} = 1(1, x_1)$

$= 1, (1 \cdot x_1)$

$= (1, x_1)$

$= \underline{u}$

$\therefore$ V is a vector space as it here satisfies the given 10 axioms.

(Ans:)

Answer to the Q. NO- 01(b)

According to given all data,

Here,

$$(a, b, c), \text{ where } b = a + c + 1$$

Again,

let, $b = 3$

$\therefore (a, c) = (1, 1)$

$L.H.S. = b = 3$

$\therefore R.H.S. = a + c + 1$

$= 1 + 1 + 1$

$= 3$

Therefore,

$$W \neq \varphi$$

$$\because (1, 3, 1) \in W$$

let,

$$\underline{u} = (a_1, b_1, c_1) \in W$$

$$\therefore b_1 = a_1 + c_1 + 1$$

And $\underline{v} = (a_2, b_2, c_2) \in W$

$$\therefore b_2 = a_2 + c_2 + 1$$

$$\therefore \underline{u} + \underline{v} = (a_1 + a_2, b_1 + b_2, c_1 + c_2)$$

where as, $b_1 + b_2 = (a_1 + a_2) + (c_1 + c_2) + 2$

$$\therefore \underline{u} + \underline{v} \in W$$

$\therefore$ W is not a subspace of $\mathbb{R}$.

(Ans:)

Answer to the Q. NO - 02 (a)

Given,

$V_1 = (2, 2, 2)$

$V_2 = (0, 0, 3)$

$V_3 = (0, 1, 1)$

We know,

A vector spans R^3 if every vector $\underline{v}$ in R^3 can be written as a linear combination of the vectors in the set S.

$$\therefore \quad K_1 \underline{V_1} + K_2 \underline{V_2} + K_3 \underline{V_3} + \cdots K_n \underline{V_n} = \vec{V}$$

where $\underline{V}$ is a vector to be any vector in R^3.

Now,

$$\begin{bmatrix} 2 & 0 & 0 \\ 2 & 0 & 1 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

Next,

$$\begin{vmatrix} 2 & 0 & 0 \\ 2 & 0 & 1 \\ 2 & 3 & 1 \end{vmatrix} = (2) \begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix} - (0) \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} + (0) \begin{vmatrix} 2 & 0 \\ 2 & 3 \end{vmatrix}$$

$$= 2(0 \times 1 - 1 \times 3) + 0 + 0$$

$$= -6$$

$\therefore$ the determinent is nonzero $\therefore$ [P.T.O.]

$\therefore$ The vectors spans R^3

(Ans:)

Ans. to the Q. NO - 02(b)

Given,

$$V_1 = (2, -1, 3)$$
$$V_2 = (4, 1, 2)$$
$$V_3 = (8, -1, 8)$$

Here,

$$\begin{bmatrix} 2 & 4 & 8 \\ -1 & 1 & -1 \\ 3 & 2 & 8 \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 & 8 \\ -1 & 1 & -1 \\ 3 & 2 & 8 \end{bmatrix} = (2)\begin{vmatrix} 1 & -1 \\ 2 & 8 \end{vmatrix} - (4)\begin{vmatrix} -1 & -1 \\ 3 & 8 \end{vmatrix} + (8)\begin{vmatrix} -1 & 1 \\ 3 & 2 \end{vmatrix}$$

$$= (2)\times(10) - (4)(-5) + (8)(-5)$$

$$= 0$$

$\because$ The determinent $= 0$

$\therefore$ The set does not span R^3

(Ans:)

Answer to the Q. NO- 03 (a)

Given,

$V_1 = (-3, 0, 4)$

$V_2 = (5, -1, 2)$

$V_3 = (1, 1, 3)$

We know,

$S = \{ \underline{V}_1, \underline{V}_2, \underline{V}_3, \dots, \underline{V}_n \}$ is called linearly independent if the only coefficients satisfying $\Rightarrow$

$$K_1V_1 + K_2V_2 + \dots + K_nV_n = 0$$

are $K_1 = K_2 = \dots K_n = 0$

let,

$$K_1V_1 + K_2V_2 + K_3V_3 = 0$$

$$\Rightarrow K_1(-3,0,4) + K_2(5,-1,2) + K_3(1,1,3) = (0,0,0)$$

$$\Rightarrow (-3K_1 + 5K_2 + K_3, -K_2 + K_3, 4K_1 + 2K_2 + 3K_3) = (0,0,0)$$

So, we get,

$$-3K_1 + 5K_2 + K_3 = 0 \quad \text{—— (i)}$$

$$-K_2 + K_3 = 0 \quad \text{—— (ii)}$$

$$4K_1 + 2K_2 + 3K_3 = 0 \quad \text{—— (iii)}$$

from (ii) $\Rightarrow$

$$K_2 = K_3$$

from (i) $\Rightarrow$ $-3K_1 + 5K_2 + K_3 = 0$

$$\Rightarrow -3K_1 + 5K_2 + K_2 = 0 \quad [K_2 = K_3]$$

$\Rightarrow -3K_1 + 6K_2 = 0$

$\Rightarrow K_1 = 2K_2$

from (III) $\Rightarrow$

$4K_1 + 2K_2 + 3K_3 = 0$

$\Rightarrow 8K_2 + 2K_2 + 3K_2 = 0$ $\left[\because K_1 = 2K_2, \; K_3 = K_2\right]$

$\Rightarrow 13K_2 = 0$

$\Rightarrow K_2 = 0$

$\because K_2 = 0$

$\therefore K_1 = 2K_2 = 0$ and

$K_3 = K_2 = 0$

$\therefore (-3, 0, 4), (5, -1, 2), (1, 1, 3)$ are linearly independent.

(Ans:)

Answer to the Q. NO - 03 (b)

Given,

$$V_1 = (-2, 0, 1)$$
$$V_2 = (3, 2, 5)$$
$$V_3 = (6, -1, 1)$$
$$V_4 = (7, 0, -2)$$

We know,

$S = \{ V_1, V_2, V_3, \ldots, V_n \}$ is called linearly independent if the only coefficients satisfying $\Rightarrow$

$$K_1V_1 + K_2V_2 + \ldots + K_nV_n = 0$$

are $K_1 = K_2 = \ldots K_n = 0$

let,

$$K_1 V_1 + K_2 V_2 + K_3 V_3 + K_4 V_4 = 0$$

$$\Rightarrow K_1(-2, 0, 1) + K_2(3, 2, 5) + K_3(6, -1, 1) + K_4(7, 0, -2) = 0$$

$$\Rightarrow (-2K_1 + 3K_2 - 6K_3 + 7K_4,\ 3K_2 - K_3,\ K_1 + 5K_2 + K_3 - 2K_4) = (0, 0, 0)$$

Now,

$$-2K_1 + 3K_2 - 6K_3 + 7K_4 = 0 \quad \text{--- (i)}$$
$$3K_2 - K_3 = 0 \quad \text{--- (ii)}$$
$$K_1 + 5K_2 + K_3 - 2K_4 = 0 \quad \text{--- (iii)}$$

[P.T.O.]

The above system in matrix form is given by $\Rightarrow$

$$\begin{bmatrix} -2 & 3 & -6 & 7 \\ 0 & 3 & -1 & 0 \\ 1 & 5 & 1 & -2 \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \\ K_3 \\ K_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{--- (1)}$$

Let,

$$A = \begin{bmatrix} -2 & 3 & -6 & 7 \\ 0 & 3 & -1 & 0 \\ 1 & 5 & 1 & -2 \end{bmatrix}$$

$\because$ A is a 3×4 matrix

$\therefore$ rank $(A) \leq 3$

$\therefore$ System of equation has infinite solution in fact non-zero solution.

That all 'K' are not identically zero.

$\therefore (-2, 0, 1), (3, 2, 5), (6, -1, 1), (7, 0, -2)$ is linearly independent.

(Ans;)

Answer to the Q. NO - 04

Given,

$$A = \begin{pmatrix} 1 & -2 & 2 & 3 & -1 \\ -3 & 6 & -1 & 1 & -7 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

Now,

$$x_1 - 2x_2 + 2x_3 + 3x_4 - x_5 = 0$$

$$-3x_1 + 6x_2 - x_3 + x_4 - 7x_5 = 0$$

$$2x_1 - 4x_2 + 5x_3 + 8x_4 - 4x_5 = 0$$

Here,

$$\left(\begin{array}{ccccc|c} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 5 & 10 & -10 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \end{array}\right) ; \quad \begin{array}{l} R_2' = R_2 + 3R_1 \\ R_3' = R_2 - 2R_1 \end{array}$$

$$\sim \left(\begin{array}{ccccc|c} 1 & 2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 5 & 10 & -10 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccccc|c} 1 & 2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 2 & 0 \end{array}\right) ; \quad R_2' = R_2 - 5R_3$$

$$\sim \left(\begin{array}{ccccc|c} 1 & 2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) ; \quad R_2 \longleftrightarrow R_3$$

$\therefore$ x_5 is a variable (free) here. [P.T.O.]

$\therefore x_5 = t$

Now,

$$x_1 + 2x_2 + 2x_3 + 3x_4 - t = 0 \quad \text{--- (i)}$$

$$x_3 + 2x_4 - 2t = 0 \quad \text{--- (ii)}$$

There, 2 equations & 5 unknowns.

$\therefore (5-2) = 3$ free variables.

$\therefore x_3, x_4, x_5$ are free variables here.

$\because$ There are two non-zero rows in the row echelon form.

$\therefore$ Rank of matrix, $A = 2$

$\therefore$ The nullity of matrix, $A = \dim(\text{Ker}(A))$
$= 3$

(Ans:)

Answer to the Q. NO - 05

Given,

$T: \mathbb{R}^4 \to R^3$ be the linear tramformation defined by

$$T(x_1, x_2, x_3, x_4) = (x_1 - x_2 + x_3 + x_4,\ x_1 + 2x_3 - x_4,\ x_1 + x_2 + 3x_3 - 3x_4)$$

Finding Basis of range of T:

We know,

$\{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$ is the basis of $\mathbb{R}^4$.

$$T(1,0,0,0) = (1-0+0+0,\ 1+2\cdot 0-0,\ 1+0+3\cdot 0-3\cdot 0)$$
$$= (1,1,1)$$

$$T(0,1,0,0) = (0-1+0+0,\ 0+2\cdot 0-0,\ 0+1+3\cdot 0-3\cdot 0)$$
$$= (-1,0,1)$$

$$T(0,0,1,0) = (0-0+1+0,\ 0+2\cdot 1-0,\ 0+0+3\cdot 1-3\cdot 0)$$
$$= (1,2,3)$$

$$T(0,0,0,1) = (0-0+0+1,\ 0+2\cdot 0-1,\ 0+0+3\cdot 0-3\cdot 1)$$
$$= (1,-1,-3)$$

[P.T.O.]

Now, the corresponding matrix will be $\Rightarrow$

$$\sim \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 2 & 3 \\ 1 & -1 & -3 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & -2 & -4 \end{pmatrix} \quad ; \quad \begin{aligned} r_2' &= r_2 + r_1 \\ r_3' &= r_3 - r_1 \\ r_4' &= r_4 - r_1 \end{aligned}$$

$$\sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix} \quad ; \quad \begin{aligned} r_2' &= r_2 - r_1 \\ r_3' &= r_3 - \frac{r_4}{-2} \\ r_4 &= \frac{r_4}{-2} \end{aligned}$$

$$\sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad ; \quad r_2 \leftrightarrow r_4$$

Here, this is in the row echelon form with having two non-zero rows.

It will form a basis of range of T.

$\therefore$ Basis of range of $T = \{(1,1,1), (0,1,2)\}$

$\therefore$ Rank of $T = 2$

(Ans:)