final lec: 02

Def^n: (Similar matrix)

If A and B are square matrices, then we say that B is similar to A if there is an invertible matrix P such that $B = P^{-1}AP$; where P is invertible matrix.

& we write $B \sim A$.

N.B: If B is similar to A, then A is also similar to B.

* Since diagonal matrix has simple form, our target is to investigate whether any matrix A is similar to a diagonal matrix.

Def^n: (Diagonalizable)

A square matrix A is called diagonalizable if it is similar to some diagonal matrix. i.e. $\exists$ an invertible matrix P such that $P^{-1}AP$ is diagonal.

In this case, P is said to diagonalize A.

* Why do we want to find a diagonal matrix which is similar to any matrix A ?

$\Rightarrow$ Because the matrix A & it's similar diagonal matrix have same determinant, rank, nullity, trace, characteristic eqn, eigenvalues etc. for diagonal matrix, these are easy to find.

for eg, if $B \sim A$, ie $B = P^{-1}AP$,

then, $|B| = |P^{-1}AP| = |P^{-1}||A||P| = \frac{1}{\cancel{|P|}} \cdot |A| \cdot \cancel{|P|} = |A|$

__Example__ (finding a matrix P that diagonalizes a matrix A).

[P is the matrix formed by taking the eigenvectors of A as columns]

Q. Find a matrix P that diagonalizes

$$A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}.$$ Also, verify.

Soln characteristic eqn $\Rightarrow$

$$\lambda^3 - tr(A)\lambda^2 + (A_{11} + A_{22} + A_{33})\lambda - |A| = 0$$

Here $tr(A) = 0 + 2 + 3 = 5$

$A_{11} + A_{22} + A_{33}$

$$= \begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 0 & -2 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} a & 0 \\ 1 & 2 \end{vmatrix}$$

$$= (6-0) + (0+2) + (0-0)$$

$$= 8$$

$$|A| = 0 - 0 - 2(0-2) = 4$$

$\therefore$ Charac. eqn of $A \Rightarrow$

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

using calculator, we get $\lambda = 1, 2$
repeated eigenvalue!
so, use factor method

$$\Rightarrow \lambda^3 - \lambda^2 - 4\lambda^2 + 4\lambda + 4\lambda - 4 = 0$$

$$\Rightarrow \lambda^2(\lambda-1) - 4\lambda(\lambda-1) + 4(\lambda-1) = 0$$

$$\Rightarrow (\lambda-1)(\lambda^2 - 4\lambda + 4) = 0$$

$$\Rightarrow (\lambda-1)(\lambda-2)^2 = 0$$

$$\therefore \lambda = 1, 2, 2.$$

Now, we find the eigenvectors.

<u>for $\lambda = 1$:</u> $(A - \lambda I)\underline{x} = \underline{0}$

$$\Rightarrow (A - I)\underline{x} = 0$$

$$\Rightarrow \begin{pmatrix} -1 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left[\begin{array}{ccc|c} -1 & 0 & -2 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right) : r_1' = r_1 \times (-1)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) ; \begin{array}{l} r_2' = r_2 - r_1 \\ r_3' = r_3 - r_1 \end{array}$$

$\therefore$

$$x_1 + 2x_3 = 0$$
$$x_2 - x_3 = 0$$

2 eq^n in 3 variables $\therefore (3-2) = 1$ free variable

let, $x_3 = t \in \mathbb{R}$

$\therefore x_2 = t$

$x_1 = -2t$

$$\therefore \underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2t \\ t \\ t \end{pmatrix} = t \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

$\therefore \underline{x} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$ is the eigenvector corresponding to $\lambda = 1$.

<u>for $\lambda = 2$:</u>

$$(A - \lambda I)\underline{x} = \underline{0}$$

$$\Rightarrow (A - 2I)\underline{x} = \underline{0}$$

$$\Rightarrow \begin{pmatrix} -2 & 0 & -2 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\therefore \left(\begin{array}{ccc|c} -2 & 0 & -2 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) ; r_1' = r_1 \times \left(-\tfrac{1}{2}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) ; \begin{array}{l} r_2' = r_2 - r_1 \\ r_3' = r_3 - r_1 \end{array}$$

$x_1 + 0.x_2 + x_3 = 0$.

$x_1 + x_3 = 0 \Rightarrow$

1 eq^n in 3 variable $\therefore (3-1) = 2$ free variables.

let, $x_2 = r$, $x_3 = t$

$x_1 = -x_3 = -t$

-1

$$\therefore \underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -t \\ r \\ t \end{pmatrix}$$

$$= \begin{pmatrix} -t \\ 0 \\ t \end{pmatrix} + \begin{pmatrix} 0 \\ r \\ 0 \end{pmatrix}$$

$$= t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + r \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$\therefore \underline{x} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ & $\underline{x} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ are the eigenvectors corresponding ~~to~~ $\lambda = 2$.

So, the matrix P will have these eigenvectors as its column. i.e.

$$P = \begin{pmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

So, P diagonalizes A. i.e. $P^{-1}AP$ ~~is a~~ will be a diagonal matrix.

verify.

~~$P^{-1}AP$~~

$$P^{-1} = \begin{pmatrix} -1 & 0 & -1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix} \longrightarrow$$ use calculator to find P^{-1}.

$$P^{-1}AP = \begin{pmatrix} -1 & 0 & -1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix} \begin{pmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 & -1 \\ 2 & 0 & 4 \\ 2 & 2 & 2 \end{pmatrix} \begin{pmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Ans.

Note.

There is no rule to place the columns in P.

for eg. if we set, $P = \begin{pmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$

then $P^{-1}AP = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

If $P = \begin{pmatrix} -1 & -2 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$, then $P^{-1}AP = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

Ans.

Example. (A matrix that is NOT diagonalizable)

Show that $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{pmatrix}$ is not diagonalizable.

Sol^n $\cancel{\text{chr}}$ $tr(A) = 1+2+2 = 5$

$$A_{11} + A_{22} + A_{33} = 4 + 2 + 2 = 8$$

$$|A| = 4$$

$$\therefore \lambda^3 - tr(A)\lambda^2 + (A_{11} + A_{22} + A_{33})\lambda - |A| = 0$$

$$\Rightarrow \lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

$$\Rightarrow (\lambda - 1)(\lambda - 2)^2 = 0$$

$$\lambda = 1, 2, 2.$$

for $\lambda = 1$: $(A - \lambda I)\underline{x} = 0$

$$\Rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ -3 & 5 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\therefore \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -3 & 5 & 1 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & -\frac{5}{3} & -\frac{1}{3} & 0 \end{array}\right); \quad r_1 \leftrightarrow r_2, \quad r_3' = r_3 \times \left(-\frac{1}{3}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & -\frac{5}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) ; \; r_2 \longleftrightarrow r_3$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -\frac{8}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \; r_2' = r_2 - r_1$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{8} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) ; \; r_2' = r_2 \times \left(-\frac{3}{8}\right)$$

$x_1 + x_2 = 0$

$x_2 + \frac{1}{8} x_3 = 0$

$\therefore$ 2 eq^n in 3 var $\therefore$ $(3-2) = 1$ free var

let, $x_3 = t \in \mathbb{R}$

$x_2 = -\frac{1}{8} t$,

$x_1 = \frac{1}{8} t$

$$\therefore \underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} 1/8 \\ -1/8 \\ 1 \end{pmatrix}$$

$$\therefore \underline{x} = \begin{pmatrix} 1/8 \\ -1/8 \\ 1 \end{pmatrix}.$$

for $\lambda = 2$:

$(A - \lambda I)\underline{x} = 0$

$$\Rightarrow \begin{pmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ -3 & 5 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\therefore \left(\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -3 & 5 & 0 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -3 & 5 & 0 & 0 \end{array}\right) \quad ; r_1' = r_1 \times (-1)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -5/3 & 0 & 0 \end{array}\right) \quad \begin{array}{l} r_2' = r_2 - r_1 \\ r_3' = r_3 \times (-1/3) \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & -5/3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad r_2 \leftrightarrow r_3$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & -5/3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad r_2' = r_2 - r_1$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad r_2' = r_2 \times (-3/5)$$

$x_1 = 0$

$x_2 = 0$

2 eq^n in 3 unknowns. $(3-2)=1$ free variable.

Let, $x_3 = t$

$\therefore \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ t \end{pmatrix} = t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

$\therefore \underline{u} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Since, A is a 3×3 matrix & there are only two eigenvectors, so, A is not diagonalizable.

[Showed]

$\square$.

* ~~Eigen~~

Eigenvalues & Eigenvectors of Matrix Powers :-

Th^m: If λ is an eigenvalue of A & k is a +ve integer, and $\underline{x}$ is the eigenvector of A corresponding to λ, then,

i) λ^k is an eigenvalue of A^k, and

ii) $\underline{x}$ is the eigenvector of A^k corresponding to λ^k.

* Eigenvalues & Eigenvectors of Matrix Powers :-

consider $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{pmatrix}$.

- Eigenvalues & eigenvectors are -

$\lambda = 1 : \underline{x} = \begin{pmatrix} 1/8 \\ -1/8 \\ 1 \end{pmatrix}$

$\lambda = 2 : \underline{x} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Q. find eigenvalues & eigenvectors of A^7.

Eigenvalues of A ~~are~~ will be $\lambda = 1^7 = 1$

& $\lambda = 2^7 = 128$

& the eigenvectors of A^7 corresponding to $\lambda = 1$ & $\lambda = 128$ will be same as that of A. i.e.

$\lambda = 1 : \underline{x} = \begin{pmatrix} 1/8 \\ -1/8 \\ 1 \end{pmatrix}$

$\lambda = 128 : \underline{x} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

* Computing powers of a Matrix:-

We know, if A is similar to a diagonal matrix, say D, then, $\exists$ ~~infers~~ an invertible matrix P such that,

$$P^{-1}AP = D \quad \text{---(1)}$$

Now, $$(P^{-1}AP)^2 = P^{-1}APP^{-1}AP$$
$$= P^{-1}AIAP \quad [\because PP^{-1} = I]$$
$$= P^{-1}AAP \quad [AI = A]$$

$$\therefore (P^{-1}AP)^2 = P^{-1}A^2P.$$

$$\Rightarrow D^2 = P^{-1}A^2P \quad [\text{by (1)}]$$

$$\Rightarrow P^{-1}A^2P = D^2$$

$$\Rightarrow PP^{-1}A^2PP^{-1} = PD^2P^{-1} \quad \left[\begin{array}{c}\text{Taking } P \text{ on left \&} \\ P^{-1} \text{ on right}\end{array}\right]$$

$$\Rightarrow A^2 = PD^2P^{-1} \quad [\because PP^{-1} = I]$$

In general,

$$\boxed{A^K = PD^KP^{-1};}$$

Ex: $A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$. find A^{13}.

Soln. eigenvalues $\Rightarrow \lambda = 1, 2, 2$

eigenvectors $\Rightarrow$

$\lambda = 1: \ \underset{\sim}{x} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$

$\lambda = 2: \ \underset{\sim}{x} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \ \underset{\sim}{x} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

So, $P = \begin{pmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$, $P^{-1} = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{pmatrix}$

& $D = P^{-1}AP$

$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

We know,

$A^K = P D^K P^{-1}$

$\therefore A^{13} = P D^{13} P^{-1} = \begin{pmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2^{13} & 0 & 0 \\ 0 & 2^{13} & 0 \\ 0 & 0 & 1^{13} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{pmatrix}$

$= \begin{pmatrix} -8190 & 0 & -16382 \\ 8191 & 8191 & 8191 \\ 8191 & 0 & 16383 \end{pmatrix}$ Ans

Practice Problem:

Howard Anton (Exercise Set 5.2, P-311)

5, 6, 7, 8, 17, 18.