Ans: to the Ques No - 1

(a)

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$|A| = 2$

$tr(A) = 0$

$A_{11} + A_{22} + A_{33} = -1 + (-1) + (-1)$

$= -3$

$\therefore$ Characteristic equation;

$\lambda^3 - 0 \cdot \lambda^2 + (-3)\lambda - 2 = 0$

$\Rightarrow \lambda^3 - 3\lambda - 2 = 0$

$\Rightarrow \lambda = 2, -1$

$\lambda^3 - 3\lambda - 2$

$= \lambda^3 - 2\lambda^2 + 2\lambda^2 - 4\lambda + \lambda - 2$

$= \lambda^2(\lambda - 2) + 2\lambda(\lambda - 2) + 1(\lambda - 2)$

$= (\lambda - 2)(\lambda^2 + 2\lambda + 1)$

$= (\lambda-2)\{\lambda^2+\lambda+\lambda+1\}$

$= (\lambda-2)\{\lambda(\lambda+1)+1(\lambda+1)\}$

$= (\lambda-2)(\lambda+1)(\lambda+1)$

$\therefore \lambda = 2, -1, -1$ (Ans)

(b)

If $\lambda = 2$

$(A-\lambda I)\underline{x} = 0$

$$\Rightarrow \left(\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$= \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Augmented matrix,

$$A = \left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & 0 \\ 0 & \frac{3}{2} & -\frac{3}{2} & 0 \end{array}\right] \quad \begin{array}{l} R_2' = R_2 + \frac{R_1}{2} \\ R_3' = R_3 + \frac{R_1}{2} \end{array}$$

$$= \left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_3' = R_3 + R_2$$

$$= \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} R_1' = \frac{R_1}{-2} \\ R_2' = R_2\left(-\frac{2}{3}\right) \end{array}$$

$\therefore x_1 - \frac{1}{2}x_2 - \frac{1}{2}x_3 = 0$ —— (i)

$x_2 - x_3 = 0$ —— (ii)

Let free variable, $x_3 = t$

$\therefore x_2 = t$

$\therefore x_1 = \frac{t}{2} + \frac{t}{2}$

$= t$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

if $\lambda = -1$,

$$\left(\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} - \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\therefore A = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} R_2' = R_2 - R_1 \\ R_3' = R_3 - R_1 \end{array}$$

$\therefore$ free variable, $x_2 = p$ & $x_3 = q$

$\therefore x_1 = -p - q$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -p-q \\ p \\ q \end{bmatrix}$$

$$= \begin{bmatrix} -p \\ p \\ 0 \end{bmatrix} + \begin{bmatrix} -q \\ 0 \\ q \end{bmatrix}$$

$$= p\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + q\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ Eigen vector, $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

(Ans)

(2)

$$A = \begin{bmatrix} 5 & 1 \\ 0 & 6 \end{bmatrix}$$

characterstic equation; $\lambda^2 - tr(A)\lambda + |A| = 0$

$\Rightarrow \lambda^2 - 11\lambda + 30 = 0$

$\therefore \lambda = 5, 6$

$\therefore$ Eigen vector for $\lambda = 5$,

$$\left(\begin{bmatrix} 5 & 1 \\ 0 & 6 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

Augmented matrix,

$$A = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

$\therefore$ free variable, $x_1 = p$

$\therefore x_2 = 0$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} p \\ 0 \end{bmatrix}$$

$$= p\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

if $\lambda = 6$,

$$\left(\begin{bmatrix} 5 & 1 \\ 0 & 6 \end{bmatrix} - \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\therefore A = \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad r_1' = (-1) r_1$$

$\therefore$ free variable, $x_2 = p$

$\therefore x_1 = p$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} p \\ p \end{bmatrix} = p\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\therefore P_{\text{matrix}} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

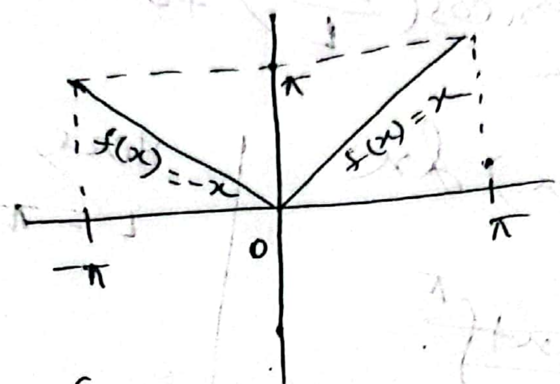
$$\therefore P^{-1}AP = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 5 & 1 \\ 0 & 6 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -5 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 \\ 0 & 6 \end{bmatrix} \text{ (Ans)}$$

(3)

(a)



$$f(x)=\begin{cases} x, & 0<x<\pi \\ -x, & -\pi<x<0 \end{cases}$$

$$\therefore f(-x)=\begin{cases} -x, & 0<-x<\pi \\ x, & -\pi<-x<0 \end{cases}$$

$$=\begin{cases} -x, & 0>x>-\pi \\ x, & \pi>x>0 \end{cases}$$

$\cancel{-x, 0<}$

$$=\begin{cases} x, & \pi>x>0 \\ -x, & 0>x>-\pi \end{cases}$$

$$=\begin{cases} x, & 0<x<\pi \\ -x, & -\pi<x<0 \end{cases}$$

$$=f(x)$$

$\therefore f(-x)=f(x)$

$\therefore f=$ even (Ans)

(b)

As $f(x)$ is even, $b_n = 0$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$$

Here, $L = \pi$

$$\cancel{= \frac{1}{\pi}\left[\int_{-\pi}^{0} x\,dx + \int_{0}^{\pi}\right.}$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -x\,dx + \int_{0}^{\pi} x\,dx\right]$$

$$= \frac{1}{\pi}\left(-\left[\frac{x^2}{2}\right]_{-\pi}^{0} + \left[\frac{x^2}{2}\right]_{0}^{\pi}\right)$$

$$\cancel{= \frac{1}{\pi}\left[-\left(\frac{\pi^2}{2} + \frac{\pi^2}{2}\right.\right.}$$

$$= \frac{1}{\pi}\left(-\left[0 - \frac{(-\pi)^2}{2}\right] + \frac{\pi^2}{2}\right)$$

$$= \frac{1}{\pi}\left(-\frac{-\pi^2}{2} + \frac{\pi^2}{2}\right)$$

$$= \frac{1}{\pi}\left(\frac{\pi^2}{2} + \frac{\pi^2}{2}\right) = \frac{\pi^2}{\pi} = \pi$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos\left(\frac{n\pi x}{\pi}\right)dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -x\,\cancel{dx}\cos(nx)\,dx + \int_{0}^{\pi} x\cos(nx)\,dx\right.$$

$$= \frac{1}{\pi}\left[\left[-x\frac{\sin(nx)}{n} - \frac{\cos(nx)}{n^2}\right]_{-\pi}^{0} + \left[x\frac{\sin(nx)}{n} + \frac{\cos(nx)}{n^2}\right]_{0}^{\pi}\right]$$

	D	I
$+$	x	$\cos(nx)$
$-$	1	$\frac{\sin(nx)}{n}$
	0	$-\frac{\cos(nx)}{n^2}$

$$\cancel{= \frac{1}{\pi}\left[-\left(-\frac{(-\pi)\sin(-n\pi)}{n} - \frac{\cos(-n\pi)}{n^2}\right)\right.}$$

	D	I
$+$	x	$\cos(nx)$
$-$	1	$\frac{\sin(nx)}{n}$
	0	$-\frac{\cos(nx)}{n^2}$

$$= \frac{1}{\pi}\left[\left(\left(0 - \frac{\cos(0)}{n^2}\right) - \left(\frac{\pi\sin(-n\pi)}{n} - \frac{\cos(-n\pi)}{n^2}\right)\right) + \left(\left(\frac{\pi\sin(n\pi)}{n} + \frac{\cos(n\pi)}{n^2}\right) - \frac{\cos(0)}{n^2}\right)\right]$$

$$= \frac{1}{\pi}\left[\left(-\frac{1}{n^2} + 0 + \frac{(-1)^n}{n^2}\right) + \left(0 + \frac{(-1)^n}{n^2} - \frac{1}{n^2}\right)\right]$$

$$= \frac{1}{\pi}\left[-\frac{1}{n^2} + \frac{(-1)^n}{n^2} + \frac{(-1)^n}{n^2} - \frac{1}{n^2}\right]$$

$$= \frac{1}{\pi}\Big[$$

$$= \frac{1}{\pi}\left[-\frac{2}{n^2} + \frac{2(-1)^n}{n^2}\right]$$

$$= \frac{1}{\pi} \cdot \frac{-2+2(-1)^n}{n^2}$$

$$= \frac{-2+2(-1)^n}{n^2\pi}$$

$$\therefore f(x) = \frac{\pi}{2} \sum_{n=1}^{\infty} \left\{\frac{-2+2(-1)^n}{n^2\pi} \times \cos(nx)\right\}$$

(Ans)

(4)

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

Here,

$L = \pi$

$$b_n = \frac{2}{\pi}\int_0^{\pi} x^3 \sin\left(\frac{n\pi x}{\pi}\right) dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^3 \sin(nx)\, dx$$

$$=\frac{2}{\pi}\left[-x^3\frac{\cos(nx)}{n}+3x^2\frac{\sin(nx)}{n^2}+6x\frac{\cos(nx)}{n^3}-6\frac{\sin(nx)}{n^4}\right]_0^\pi$$

	D	I
$+$	x^3	$\sin(nx)$
$-$	$3x^2$	$-\frac{\cos(nx)}{n}$
$+$	$6x$	$-\frac{\sin(nx)}{n^2}$
$-$	6	$\frac{\cos(nx)}{n^3}$
	0	$\frac{\sin(nx)}{n^4}$

$$=\frac{2}{\pi}\left(-\frac{\pi^3\cos(n\pi)}{n}+\frac{3\pi^2\sin(n\pi)}{n^2}+\frac{6\pi\cos(n\pi)}{n^3}-\frac{6\sin(n\pi)}{n^4}\right)$$

$$=\frac{2}{\pi}\left(\frac{-\pi^3(-1)^n}{n}+0+\frac{6\pi(-1)^n}{n^3}-0\right)$$

$$=\frac{-2\pi^3(-1)^n}{\pi n}+\frac{12\pi(-1)^n}{\pi n^3}$$

$$\therefore f(x)=\sum_{n=1}^{\infty}\left\{\left(\frac{12\pi(-1)^n}{n^3\pi}-\frac{2\pi^3(-1)^n}{n\pi}\right)\sin(nx)\right\}$$

(Ans)

(5)

(a)

$$f(x) = \int_0^\infty e^{-2x} \cos(nx)$$

$$= \left[\frac{e^{-2x}}{4+n^2} \left(-2\cos(nx) + n\sin(nx)\right) \right]_0^\infty$$

$$= 0 - \frac{1}{4+n^2}(-2+0)$$

$$= \frac{2}{4+n^2} \quad \text{(Ans)}$$

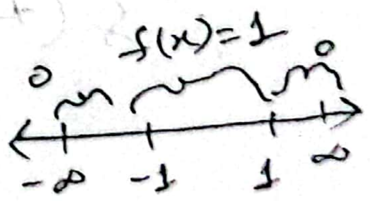
(b)

The fourier transform of $f(x)$

$$F(\alpha) = \int_{-\infty}^{\infty} f(x) e^{-i\alpha x} dx$$

$$= \int_{\infty}^{-1} 0 e^{-i\alpha x} dx + \int_{-1}^{1} 1 e^{-i\alpha x} dx + \int_{1}^{\infty} 0 e^{-i\alpha x} dx$$

$|x| \le 1 \Rightarrow -1 \le x \le 1$



$$= \int_{-1}^{1} e^{-i\alpha x}\, dx \quad \text{---- (1)}$$

$$\int e^{ax} dx = \frac{e^{ax}}{a} + c$$

$$= \left[\frac{e^{-i\alpha x}}{-i\alpha} \right]_{-1}^{1}$$

$$= \frac{e^{-i\alpha}}{-i\alpha} - \frac{e^{i\alpha}}{-i\alpha}$$

$$= \frac{e^{-i\alpha}}{-i\alpha} + \frac{e^{i\alpha}}{i\alpha}$$

$$= \frac{-e^{-i\alpha} + e^{i\alpha}}{i\alpha}$$

$$\left[e^{i\theta} = \cos\theta + i\sin\theta \quad e^{-i\theta} = \cos\theta - i\sin\theta \right]$$

$$= \frac{-(\cos\alpha - i\sin\alpha) + \cos\alpha + i\sin\alpha}{i\alpha}$$

$$= \frac{-\cancel{\cos\alpha} + i\sin\alpha + \cancel{\cos\alpha} + i\sin\alpha}{i\alpha}$$

$$= \frac{2i\sin\alpha}{i\alpha}$$

$$= \frac{2\sin\alpha}{\alpha} \; ; \; \alpha \neq 0$$

(~~ans~~).

From (1),

$$F(0) = \int_{-1}^{1} e^{-i \cdot 0 \cdot x}\, dx$$

$$= \int_{-1}^{1} e^{0}\, dx$$

$$= \int_{-1}^{1} 1\, dx$$

$$= [x]_{-1}^{1}$$

$$= 1 + 1$$

$$= 2$$

Ans: $F(\alpha) = \dfrac{2 \sin\alpha}{\alpha}$; $\alpha \neq 0$

$F(0) = 2$