Def^n (Fourier Series)

Let, $f(x)$ be defined in the interval $(-L, L)$ & is determined outside of this interval by $f(x+2L) = x$ i.e. $f(x)$ has period $2L$.

Then, the fourier series / Fourier expansion corresponding to $f(x)$ is defined as –

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$$

where,

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\, dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx\ ;$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx\ ;$$

$n = 1, 2, \ldots$

NB. for odd fu^n, $a_n = 0$

for even fu^n, $b_n = 0$

Defⁿ: (odd & even function).

* A function $f(x)$ is called odd if $f(-x) = -f(x)$

Ex. • $f = x^5 - 3x^3 + 2x$

$$f(-x) = (-x)^5 - 3(-x)^3 + 2(-x)$$
$$= -x^5 + 3x^3 - 2x$$
$$= -(x^5 - 3x^3 + 2x)$$
$$= -f(x)$$

• $f = x^3 \therefore f(-x) = (-x)^3 = -x^3 = -f(x)$

* A funⁿ $f(x)$ is called even if $f(-x) = f(x)$.

Ex. • $f = \cos x \quad \therefore f(-x) = \cos(-x) = \cos x = f(x)$

• $f = x^2 \quad \therefore f(-x) = (-x)^2 = x^2 = f(x)$.

NB: * The Fourier series corresponding to an odd function can be represented by only 'sine' terms (i.e $a_n = 0$)

* The Fourier series corresponding to an even function can be represented by only 'cosine' terms (plus a constant) (i.e. $b_n = 0$)

Examples

Fourier Series

(01) Find the Fourier series for $f(x) = e^x$ in the interval $-\pi < x < \pi$.

Sol^n Here, $L = \pi$.
Let, the Fourier series for $f(x) = e^x$ be

$$e^x = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{\pi}\right) + b_n \sin\left(\frac{n\pi x}{\pi}\right)\right] \quad [\because L = \pi]$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos(nx) + b_n \sin(nx)\right] \text{ --- } (*)$$

Now,
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\,dx = \frac{1}{\pi}\left[e^x\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[e^{\pi} - e^{-\pi}\right]$$

$$= \frac{2}{\pi}\left[\frac{e^{\pi} - e^{-\pi}}{2}\right]$$

$\left[\because \text{ we know, } \sinh x = \frac{e^x - e^{-x}}{2}\right]$

$$\boxed{a_0 = \frac{2}{\pi}\sinh \pi}$$

Again,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{x}\cos\left(\frac{n\pi x}{\pi}\right)dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{x}\cos(nx)\,dx$$

NB:

$$\int e^{ax}\cos(bx)\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos(bx) + b\sin(bx)\right]$$

$$\cancel{\int e^{ax}\sin(bx)\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos(bx) + b\sin(bx)\right]}$$

$$\int e^{ax}\sin(bx)\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin(bx) - b\cos(bx)\right]$$

$$\therefore a_n = \frac{1}{\pi}\left[\frac{e^{x}}{1+n^2}\left(\cos(nx) + n\sin(nx)\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{\pi}}{1+n^2}\left(\cos(n\pi) + n\sin(n\pi)\right) - \frac{e^{-\pi}}{1+n^2}\left(\cos(n\pi) \cancel{+} - n\sin(n\pi)\right)\right]$$

$$\left[\text{since, } \cos(-x) = \cos x \atop \sin(-x) = -\sin x\right]$$

$$\therefore a_n = \frac{1}{\pi}\left[\frac{e^{\pi}}{1+n^2}\left\{(-1)^n + 0\right\} - \frac{e^{-\pi}}{1+n^2}\left\{(-1)^n - 0\right\}\right]$$

$$\left[\because \cos(n\pi) = (-1)^n \atop \sin(n\pi) = 0\right]$$

$$= \frac{1}{\pi}\left\{\frac{(-1)^n e^{\pi}}{1+n^2} - \frac{(-1)^n e^{-\pi}}{1+n^2}\right\}$$

$$= \frac{(-1)^n}{\pi(n^2+1)}\left(e^{\pi} - e^{-\pi}\right)$$

$$= \frac{2(-1)^n}{\pi(n^2+1)} \cdot \frac{e^{\pi} - e^{-\pi}}{2}$$

$$\left[\because \sinh x = \frac{e^{x} - e^{-x}}{2}\right]$$

$$\therefore \boxed{a_n = \frac{2(-1)^n}{\pi(n^2+1)} \cdot \sinh \pi}$$

Again,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{x} \sin\left(\frac{n\pi x}{\pi}\right) dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{x} \sin(nx)\, dx$$

$$= \frac{1}{\pi}\left[\frac{e^{x}}{1+n^2}\left\{\sin(nx) - n\cos(nx)\right\}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{\pi}}{1+n^2}\left\{\sin(n\pi) - n\cos(n\pi)\right\} - \frac{e^{-\pi}}{1+n^2}\left\{-\sin(n\pi) - n\cos(n\pi)\right\}\right]$$

$$= \frac{1}{\pi}\left[\frac{e^{\pi}}{1+n^2}\left\{0 - n(-1)^n\right\} - \frac{e^{-\pi}}{1+n^2}\left\{0 - n(-1)^n\right\}\right]$$

$$= \frac{1}{\pi}\left[-\frac{n(-1)^n e^{\pi}}{1+n^2} + \frac{e^{-\pi}\cdot n(-1)^n}{1+n^2}\right]$$

$$= \frac{1}{\pi}\cdot\frac{n(-1)^n}{1+n^2}\left(e^{-\pi} - e^{\pi}\right)$$

$$b_n = -\frac{n(-1)^n}{\pi(n^2+1)}\left[e^{\pi} - e^{-\pi}\right]$$

$$= \frac{-2n(-1)^n}{\pi(n^2+1)}\left(\frac{e^{\pi} - e^{-\pi}}{2}\right)$$

$$\boxed{b_n = \frac{-2n(-1)^n}{\pi(n^2+1)} \sinh\pi}$$

$\therefore$ Substituting a_0, a_n, b_n in $(*) \Rightarrow$

$$e^{x} = \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{\pi}\sinh\pi + \sum_{n=1}^{\infty}\left[\frac{2(-1)^n \sinh\pi}{\pi(n^2+1)}\cos(nx) - \frac{2n(-1)^n}{\pi(n^2+1)}\sinh\pi \sin(nx)\right]$$

$$e^{x} = \frac{\sinh\pi}{\pi} + \sum_{n=1}^{\infty}\left[\frac{2(-1)^n \sinh\pi}{\pi(n^2+1)}\cos(nx) - \frac{2n(-1)^n \sinh\pi}{\pi(n^2+1)}\sin(nx)\right]$$

Ans

(02) find fourier series of $f(x) = x^2$; $-\pi \leq x \leq \pi$.

<u>Soln</u> Here, $L = \pi$.

F.S =>

$$x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos(nx) + b_n \sin(nx)\right] \longleftarrow (*)$$

[putting $L = \pi$]

Here $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi}$

$$= \frac{1}{3\pi}\left(\pi^3 + \pi^3\right)$$

$$\boxed{a_0 = \frac{2\pi^2}{3}}$$

CamScanner

Now,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos(nx)\,dx$$

	Differentiation	Integration
	D	I
$+$	x^2	$\cos(nx)$
$-$	$2x$	$\frac{\sin(nx)}{n}$
$+$	2	$-\frac{\cos(nx)}{n^2}$
	0	$-\frac{\sin(nx)}{n^3}$

$$= \frac{1}{\pi}\left[\frac{x^2 \sin(nx)}{n} + \frac{2x\cos(nx)}{n^2} - \frac{2\sin(nx)}{n^3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[0 + \frac{2\pi(-1)^n}{n^2} - 0 - 0 + \frac{2\pi(-1)^n}{n^2} + 0\right]$$

$$\left[\because \begin{array}{l} \sin(n\pi) = 0 \\ \cos(n\pi) = (-1)^n \end{array}\right]$$

$$= \frac{1}{\pi}\cdot\frac{4\pi(-1)^n}{n^2}$$

$$= \frac{4(-1)^n}{n^2}.$$

$$\boxed{\therefore a_n = \frac{4(-1)^3}{n^2}}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin(nx)\,dx$$

	D	I
+	x^2	$\sin(nx)$
−	$2x$	$-\frac{\cos(nx)}{n}$
+	2	$-\frac{\sin(nx)}{n^2}$
	0	$\frac{\cos(nx)}{n^3}$

$$= \frac{1}{\pi}\left[-\frac{x^2\cos(nx)}{n} + \frac{2x\sin(nx)}{n^2} + \frac{2\cos(nx)}{n^3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[-\cancel{\frac{\pi^2(-1)^n}{n}} + 0 + \cancel{\frac{2(-1)^n}{n^3}} + \cancel{\frac{\pi^2(-1)^n}{n}} - 0 - \cancel{\frac{2(-1)^n}{n^3}}\right]$$

$$= \frac{1}{\pi}\times 0$$

$$= 0 \qquad \boxed{b_n = 0}$$

NB:
- for odd $fu^n \Rightarrow a_n = 0$
- for even $fu^n \Rightarrow b_n = 0$

$\therefore f(x) = x^2$ is even fu^n, so, we can directly say $b_n = 0$ instead of calculating the integral.

(*) ⇒ The required fourier series ⇒

$$x^2 = \frac{1}{\cancel{2}}\frac{\cancel{2}\pi^2}{3} + \sum_{n=1}^{\infty}\left[\frac{4(-1)^n}{n^2}\cos(nx) + 0\right]$$

$$= \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos(nx).$$

Ans.

(13) Find Fourier series of -

$$f(x) = \begin{cases} 0; & -\pi \le x \le 0 \\ 1; & 0 \le x \le \pi. \end{cases}$$

Sol^n Here, $L = \pi$

Let, the F.S be ⇒

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos(nx) + b_n\sin(nx)\right] \quad \text{---} (*)$$

[putting $L = \pi$]

Here

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx\right]$$

$$\Rightarrow a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0 \cdot dx + \int_{0}^{\pi} 1 \cdot dx\right]$$

$$= \frac{1}{\pi}\left[0 + [x]_0^{\pi}\right]$$

$$= \frac{1}{\pi}(\pi - 0)$$

$$\boxed{a_0 = 1}$$

Again,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0} f(x)\cos(nx)\,dx + \int_{0}^{\pi} f(x)\cos(nx)\,dx\right\}$$

$$= \frac{1}{\pi}\left\{0 + \int_{0}^{\pi}\cos(nx)\,dx\right\}$$

CamScanner

$$a_n = \frac{1}{\pi}\int_0^{\pi} \cos(nx)\,dx$$

$$= \frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left(\frac{\sin(n\pi)}{n} - 0\right)$$

$$= \frac{1}{\pi} \times 0 \qquad [\because \sin(n\pi) = 0]$$

$$\boxed{a_n = 0}$$

Again, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$

$$= \frac{1}{\pi}\left\{0 + \int_0^{\pi} \sin(nx)\,dx\right\}$$

CamScanner

$$b_n = \frac{1}{\pi}\left[-\frac{\cos(nx)}{n}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{(-1)^n}{n} + \frac{1}{n}\right]$$

$$= \frac{1}{n\pi}\left[1 - (-1)^n\right]$$

$$b_n = \begin{cases} 0; & \text{when } n \text{ is even} \\ \frac{2}{n\pi}; & \text{when } n \text{ is odd.} \end{cases}$$

i.e $b_1 = \frac{2}{\pi}$, $b_2 = 0$, $b_3 = \frac{2}{3\pi}$, $b_4 = 0$, $b_5 = \frac{2}{5\pi}$, $b_6 = 0$, ...etc.

(*) $\Rightarrow$

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \left[0 + b_n \sin(nx) \right]$$

$$= \frac{1}{2} + b_1 \sin x + b_2 \sin(2x) + b_3 \sin(3x) + b_4 \sin(4x) + b_5 \sin(5x) + \cdots$$

$$= \frac{1}{2} + \frac{2}{\pi} \sin x + 0 + \frac{2}{3\pi} \sin 3x + 0 + \frac{2}{5\pi} \sin(5x) + \cdots$$

$$= \frac{1}{2} + \frac{2}{\pi} \sin x + \frac{2}{3\pi} \sin 3x + \frac{2}{5\pi} \sin(5x) + \cdots$$

$$= \frac{1}{2} + \frac{2}{\pi} \left[\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots \right]$$

Ans

which is same as in 6 (a).

Example 7. Find the Fourier series expansion of the function. $f(x) = \begin{cases} 0, & -\pi < x \leq 0 \\ x, & 0 < x \leq \pi \end{cases}$

Solution : By definition of Fourier series we have

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \cos nx) \quad (1)$$

where $a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$, $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx$

and $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx$.

Now $a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} f(x)\, dx + \int_{0}^{\pi} f(x)\, dx\right]$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} o.dx + \int_{0}^{\pi} x\ dx\right]$$

$$= o + \frac{1}{2\pi}\left[\frac{x^2}{2}\right]_{o}^{\pi} = \frac{1}{4\pi}(\pi^2 - o) = \frac{\pi}{4}.$$

Again $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx \ (n \neq o)$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x) \cos nx\, dx + \int_{0}^{\pi} f(x) \cos nx\, dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} o. \cos nx\, dx + \int_{0}^{\pi} x \cos nx\, dx\right]$$

$$= 0 + \frac{1}{\pi n}\left[x \sin nx\right]_{\pi}^{o} - \frac{1}{\pi n} \int_{0}^{\pi} \sin nx\, dx$$

$$= 0 + \frac{1}{\pi n^2}\left[\cos nx\right]_{o}^{\pi}$$

$$= \frac{1}{\pi n^2}[\cos n\pi - \cos o] = \frac{1}{\pi n^2}[(-1)^n - 1]$$

since $[\cos n\pi = (-1)^n]$

Finally, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx.$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x) \sin nx\, dx + \int_{0}^{\pi} f(x) \sin nx dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} o. \sin nx dx + \int_{0}^{\pi} x \sin nx dx\right]$$

$$= o - \frac{1}{\pi n}[x \cos nx]_o^{\pi} + \frac{1}{\pi n}\int_{0}^{\pi} \cos nx dx$$

$$= -\frac{1}{\pi n}(\pi \cos n\pi - o) + \frac{1}{\pi n^2}[\sin nx]_o^{\pi}$$

$$= -\frac{1}{n}\cos n\pi + o \text{ since } \sin n\pi = o, \sin 0 = 0$$

$$= -\frac{1}{n}.(-1)^n = \frac{(-1)^{n+1}}{n}.$$

Now putting the values of a_o, a_n and b_n in (1), we get

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{1}{\pi n^2}\{(-1)^n - 1\} \cos nx + \frac{(-1)^{n+1}}{n} \sin nx\right]$$

$$= \frac{\pi}{4} + \left[\left(-\frac{2}{\pi 1^2}\cos x + o - \frac{2}{\pi 3^2}\cos 3x + o - \frac{2}{\pi 5^2}\cos 5x + o - \dots\right)\right.$$

$$\left. + \left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \dots \dots \dots\right)\right]$$

$$= \frac{\pi}{4} - \frac{2}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots \dots\right)$$

$$+ \left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \dots \dots\right) \quad (2)$$

Example 9. Find the Fourier series expansion of the function $f(x) = |x|$ in the interval $[-\pi, \pi]$.

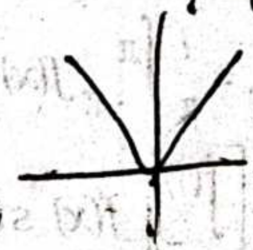
[D. U. S. 1986]

Solution : By definition of the Fourier series, we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad (1)$$

where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx \ (n \neq 0)$

and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx dx$.



symm about y-axis, even fun

Now by definition $|x| = \begin{cases} x, & x > 0 \\ -x, & x < 0 \end{cases}$

Hence the given function $f(x) = |x|$ is an even function and for the even function $b_n = 0$, $(n = 1, 2, 3, \ldots)$ in the Fourier series expansion (1) of $f(x)$ and

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} |x|\, dx = \frac{2}{\pi}\int_0^{\pi} x dx = \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \pi$$

Also $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx dx \ (n \neq 0)$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos nx dx = \frac{2}{\pi}\int_{0}^{\pi} |x| \cos nx dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x \cos nx dx$$

$$= \frac{2}{\pi n}[x \sin nx]_0^{\pi} - \frac{2}{\pi n}\int_{0}^{\pi} \sin nx dx$$

$$= 0 + \frac{2}{\pi n^2}[\cos nx]_0^{\pi}$$

$$= \frac{2}{\pi n^2}(\cos n\pi - \cos 0)$$

$$= \frac{2}{\pi n^2}\{(-1)^n - 1\}$$

$$= \begin{cases} \dfrac{-4}{\pi n^2} & \text{when } n = 1, 3, 5, \ldots \\ 0 & \text{when } n = 2, 4, 6, \ldots \end{cases}$$

Now substituting the values of a_o, a_n, and b_n in (1) we get

$$f(x) = \frac{\pi}{2} + \left[\left(-\frac{4}{\pi.1^2} + 0 - \frac{4}{\pi.3^2}\cos 3x + 0 - \frac{4}{\pi.5^2}\cos 5x + \ldots\right) + 0\right]$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\ldots\right] \quad (2)$$

Example 13. Find the series of sines and cosines of multiples of x which represents $f(x)$ in the interval $-\pi<x<\pi$ where

$$f(x)=\begin{cases} 0 \text{ where } -\pi<x<0 \\ \dfrac{\pi x}{4} \text{ where } 0<x<\pi \end{cases}$$

[D.U.P. 1969]

Solution : By definition of the Fourier series,

we have $f(x) = \frac{a_o}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ (1)

where $a_o = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx$ $(n \neq 0)$

and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx$.

Now $a_o = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)dx + \int_{0}^{\pi} f(x)dx\right]$

$= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} 0.dx + \int_{0}^{\pi} \frac{\pi x}{4}dx\right]$

$= 0 + \frac{1}{\pi}\int_{0}^{\pi} x\,dx = \frac{1}{4}\left[\frac{x^2}{2}\right]_0^{\pi} = \frac{\pi^2}{8}.$

and $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx$

$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nxdx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nxdx$

$= \frac{1}{\pi}\int_{-\pi}^{0} 0\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} \frac{\pi x}{4}\cos nxdx$

$= 0 + \frac{1}{4}\int_{0}^{\pi} x\cos nxdx$

$= \frac{1}{4n}[x\sin nx]_{0}^{\pi} - \frac{1}{4n}\int_{0}^{\pi} \sin nx\,dx$

$= 0 + \frac{1}{4n^2}[\cos nx]_0^{\pi}$

$= \frac{1}{4n^2}[\cos n\pi - \cos 0] = \frac{1}{4n^2}[(-1)^n - 1].$

Finally, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} 0.\sin nx dx + \frac{1}{\pi}\int_{0}^{\pi} \frac{\pi x}{4}\sin nx dx$$

$$= 0 + \frac{1}{4}\int_{0}^{\pi} x \sin nx\, dx$$

$$= -\frac{1}{4n}[x\cos nx]_{0}^{\pi} + \frac{1}{4n}\int_{0}^{\pi}\cos nx dx$$

$$= -\frac{1}{4n}[\pi\cos n\pi - 0] + \frac{1}{4n^2}[\sin nx]_{0}^{\pi}$$

$$= -\frac{\pi}{4n}.(-1)^n + 0 = -\frac{\pi}{4n}.(-1)^n. \quad \therefore [\cos n\pi = (-1)^n]$$

Thus substituting the values of a_0, a_n and b_n in (1) we get

$$f(x) = \frac{\pi^2}{16} + \sum_{n=1}^{\infty}\left[\frac{1}{4n}\{(-1)^n - 1\}\cos nx + \left\{-\frac{\pi}{4n}(-1)^n\right\}\sin nx\right]$$

$$= \frac{\pi^2}{16} + \left[-\frac{2}{4.1^2}\cos x + 0 - \frac{2}{4.3^2}\cos 3x + 0 - \frac{2}{4.5^2}\cos 5x + \dots\right]$$

$$- \frac{\pi}{4}\left[-\frac{\sin x}{1} + \frac{1}{2}\sin 2x - \frac{1}{3}\sin 3x + \frac{1}{4}\sin 4x - \dots\right]$$

$$= \frac{\pi^2}{16} - \frac{1}{2}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\right]$$

$$+ \frac{\pi}{4}\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \dots\dots\right] \quad (2)$$

11. Find a Fourier series for the function $f(x) = x - x^2$ from $x = -\pi$ to $x = \pi$

Answer : $f(x) = -\frac{\pi^2}{3} + 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \dots\right]$

$+ 2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{} - \frac{\sin 4x}{4} + \dots\right]$

CamScanner