Lecture-03 (06-06-24)

* **Gauss-Jordan Elimination:** The procedure/algorithm for reducing a matrix to $\underbrace{\text{reduced row echelon}}$ form is called Gauss-Jordan elimination.

Ex: Solve by Gauss-Jordan elimination –

$$
\begin{aligned}
x_1 + 3x_2 - 2x_3 \qquad\qquad + 2x_5 &= 0\\
2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= -1\\
5x_3 + 10x_4 \qquad + 15x_6 &= 5\\
2x_1 + 6x_2 \qquad + 8x_4 + 4x_5 + 18x_6 &= 6
\end{aligned}
$$

Soln. The augmented matrix $\Rightarrow$

$$
\left[\begin{array}{cccccc|c}
1 & 3 & -2 & 0 & 2 & 0 & 0\\
2 & 6 & -5 & -2 & 4 & -3 & -1\\
0 & 0 & 5 & 10 & 0 & 15 & 5\\
2 & 6 & 0 & 8 & 4 & 18 & 6
\end{array}\right]
$$

$$
\sim \left[\begin{array}{cccccc|c}
1 & 3 & -2 & 0 & 2 & 0 & 0\\
0 & 0 & -1 & -2 & 0 & -3 & -1\\
0 & 0 & 5 & 10 & 0 & 15 & 5\\
0 & 0 & 4 & 8 & 0 & 18 & 6
\end{array}\right] \begin{array}{l} \\ ;\ r_2' = r_2 - 2r_1 \\ ;\ r_4' = r_4 - 2r_1 \\ \\ \end{array}
$$

$$\sim \left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right] \quad r_2' = (-1) \times r_2$$

$$\sim \left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{array}\right] \quad ; r_3' = r_3 - 5r_2, \; r_4' = r_4 - 4r_2$$

$$\sim \left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad r_3 \leftrightarrow r_4$$

$$\sim \left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad ; r_3 = r_3/6$$

this is in row echelon form but not in reduced row echelon

leading 1 যে column এ

এই leading 1 যে column এ আছে সে column এর বাকি element গুলো 0 করতে হবে to make it in reduced row echelon form.

$$\sim \left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad r_2' = r_2 - 3r_3$$

$$\sim \left[\begin{array}{cccccc|c} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad r_1' = r_1 + 2r_2$$

∴ This is now in reduced row echelon form.

The corresponding system ⇒

$$x_1 + 3x_2 + 4x_4 + 2x_5 = 0$$

$$x_3 + 2x_4 = 0$$

$$x_6 = 1/3$$

∴ There are 3 eqⁿs in 6 unknowns. So, $(6-3) = 3$ free variables. We see that there is no leading 1 in 2nd, 4th & 5th columns. So, x_2, x_4 & x_5 are free variables.

Let, $x_2 = r$

$x_4 = s$

$x_5 = t$

$\therefore x_1 = -3x_2 - 4x_4 - 2x_5$

$= -3r - 4s - 2t$

$x_3 = -2x_4$

$= -2s$

$x_6 = 1/3$

$\therefore$ The general soln $\Rightarrow$

$x_1 = -3r - 4s - 2t$

$x_2 = r$

$x_3 = -2s$

$x_4 = s$

$x_5 = t$

$x_6 = 1/3$.

Ans

Homogeneous Linear ~~Systems~~: Equation :-

Linear eq^n $\Rightarrow ax+by=c$

If $c=0$, then it becomes a homogeneous linear eq^n

i.e. $ax+by=0$ is called homogeneous linear eq^n.

Homogeneous Linear Systems:- A system of linear eq^{n}s is said to be homogeneous if the constant terms are all zero, i.e. the system has the form -

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ \vdots \qquad\quad \vdots \qquad\quad & \vdots \quad\; \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

* Every homo. system has at least one sol^n, namely $x_1=0, x_2=0, \cdots x_n=0$. It is called <u>trivial sol^n</u>. If there are other sol^n, they are called <u>non-trivial sol^n</u>.

<u>NB</u>: Homogeneous system has either trivial sol^n or many solution.

Gaussian Elimination Method Vs Gauss-jordan Elimination-method

Gaussian Elimination Method ↓ The augmented matrix is transformed into row echelon (R.E) form

Gauss-jordan Elimination method ↓ The augmented matrix is transformed into reduced row echelon (R.R.E) form.

Ex: Use Gauss-jordan elimination to solve the homogeneous linear system -

$$x_1 + 3x_2 - 2x_3 + 2x_5 = 0$$
$$2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 = 0$$
$$5x_3 + 10x_4 + 15x_6 = 0$$
$$2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 = 0.$$

Soln: The corresponding augmented matrix $\Rightarrow$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & 0 \\ 0 & 0 & 5 & 10 & 0 & 15 & 0 \\ 2 & 6 & 0 & 8 & 4 & 18 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & 0 \\ 0 & 0 & 5 & 10 & 0 & 15 & 0 \\ 0 & 0 & 4 & 8 & 0 & 18 & 0 \end{array}\right] \quad \begin{array}{l} ; r_2' = r_2 - 2r_4 \\ ; r_4' = r_4 - 2r_1 \end{array}$$

$$\sim \left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 2 & 0 & \frac{18}{4} & 0 \end{array}\right] \quad \begin{array}{l} ; r_2' = (-1) r_2 \\ ; r_3' = \frac{1}{5} \times r_3 \\ ; r_4' = \frac{1}{4} \times r_4 \end{array}$$

$$\sim \left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{3}{2} & 0 \end{array}\right] \quad \begin{array}{l} ; r_3' = r_3 - r_2 \\ ; r_4' = r_4 - r_2 \end{array}$$

$$\sim \left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad ; r_3 \longleftrightarrow r_4$$

$$\sim \left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] ; r_3' = \frac{2}{3} \times r_3 \sim$$

$$\sim \left[\begin{array}{cccccc|c} 1 & 3 & 0 & 4 & 2 & 6 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] ; r_1' = r_1 + 2r_2$$

$$\sim \left[\begin{array}{cccccc|c} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} r_4' = r_4 - 6r_3 \\ r_2' = r_2 - 3r_3 \end{array}$$

This is in reduced row echelon form.

The corresponding system $\Rightarrow$

$$x_1 + 3x_2 + 4x_4 + 2x_5 = 0 \quad \text{——} (1)$$

$$x_3 + 2x_4 = 0 \quad \text{——} (2)$$

$$x_6 = 0 \quad \text{——} (3)$$

There are 3 equations in 6 unknowns.

So, $(6-3) = 3$ free variables.

here, x_2, x_4, and x_5 are free variables

because there is no leading 1 in column 2, 4, and 5.

$\therefore$ let, $x_2 = r$, $x_4 = s$, $x_5 = t$.

$(1) \Rightarrow x_1 = -3x_2 - 4x_4 - 2x_5 = -3r - 4s - 2t$

$(2) \Rightarrow x_3 = -2x_4 = -2s$

$(3) \Rightarrow x_6 = 0$

So, the required sol^n $\Rightarrow$

$$
\left.
\begin{aligned}
x_1 &= -3r - 4s - 2t \\
x_2 &= r \\
x_3 &= -2s \\
x_4 &= s \\
x_5 &= t \\
x_6 &= 0
\end{aligned}
\right\} ; \quad r, s, t \in \mathbb{R}.
$$

(Ans)

NB: we get the trivial sol^n when $r = s = t = 0$

i.e. then we have $\Rightarrow$

$x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ x_4 = 0,\ x_5 = 0,\ x_6 = 0$

Practice Prob^m $\Rightarrow$ Exercise set 1.2 এর

$\boxed{1, 3, 5, 7, 15, 23,}$ — do yourself

imp. for basic

25, 27

** V.V.S

(Answers are given in the back side of the book)