Answer to the Q. NO - 01

According to the given equation,

let, $z = 1 + \sqrt{3}i$

Now, polar form' expression:

$$z = r.e^{i\theta}$$
$$= 2e^{i\frac{\pi}{3}}$$

here, $r = \sqrt{(1)^2 + (\sqrt{3})^2}$
$= 2$

$\theta = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right)$
$= \frac{\pi}{3}$

$$\text{L.H.S.} = \left(1+\sqrt{3}i\right)^{-10}$$
$$= \left(2e^{i\pi/3}\right)^{-10}$$
$$= 2^{-10} \cdot e^{-i \cdot 10 \cdot \pi/3}$$
$$= 2^{-10}\left(\cos\left(\frac{10\pi}{3}\right) - i\sin\left(\frac{10\pi}{3}\right)\right)$$
$$= 2^{-10}\left(-\frac{1}{2} + i \cdot \frac{\sqrt{3}}{2}\right)$$
$$= 2^{-10} \cdot \frac{1}{2} \cdot \left(-1+\sqrt{3}i\right)$$
$$= 2^{-11}\left(-1+\sqrt{3}i\right)$$
$$= \text{R.H.S.}$$

$\therefore (1+\sqrt{3}i)^{-10} = 2^{-11}(-1+\sqrt{3}i)$ $\boxed{\text{Showed}}$

Answer to the Q. NO - 02

We know,

$$z = x + iy$$

$$\therefore \bar{z} = x - iy,$$

$$\therefore z^3 = (x+iy)^3$$

$$= x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3$$

$$= x^3 + 3x^2(iy) - 3xy^2 - iy^3$$

$$= (x^3 - 3xy^2) + i\,(3x^2y - y^3)$$

Here,

The Real Part, $Re(z^3) = (x^3 - 3xy^2)$

The Imaginary Part, $Im(z^3) = (3x^2y - y^3)$

Again here,

$$|Re(2 + \bar{z} + z^3)|$$

$$= |2 + x + x^3 - 3x^2|$$

Answer to the Q. NO-02

Since,

$|z| \leq 1,$

$\therefore |x| \leq 1$ and

$|y| \leq 1.$

We know,

$|z| = |\bar{z}|$

$\therefore |\bar{z}| \leq 1.$

Now, for the expression,

$$|z^3| = |z \cdot z \cdot z|$$

$$= |z| \cdot |z| \cdot |z| \quad [\because \text{rule of modulus}]$$

$$= |z|^3$$

$$\therefore |z^3| \leq 1^3$$

$$\Rightarrow |z^3| \leq 1.$$

Again,

$$|2| + |\bar{z}| + |z^3| \leq 2+1+1$$

$$\Rightarrow |2+\bar{z}+z^3| \leq 2+1+1$$

$\because$ The modulus of this entire complex number's expression is whether less or equal to the value of left $\rightarrow (2+1+1)$

$= 4$.

Therefore, The Real part of the complex number supposed to be less or equal comparing to the value of left side.

[When the value of 'imaginary part' = 0] the real part can be equal to the value of right side of the equation of complex number]

$\therefore \left| Re\left(2+\bar{z}+z^{3}\right)\right| \leq 4$

(Showed)

Answer to the Q. NO - 03

Given identity,

$$1+z+z^2+ ---- +z^n = \frac{1-z^{n+1}}{1-z}$$

Lagrange's trigonometric identity:

$$1+\cos\theta+\cos 2\theta + \dots\dots + \cos n\theta = \frac{1}{2} + \frac{\sin(2n+1)\frac{\theta}{2}}{2\sin\frac{\theta}{2}};$$

We know, $0<\theta<2\pi$

The formula for the sum of a geometric series:

$$S_n = a\,\frac{1-r^{n+1}}{1-r}$$

here
$a = 1^{st}$ term $= 1$
$r =$ common ratio
$= z$

$$\therefore S_n = 1 \cdot \frac{1-z^{n+1}}{1-z}$$
$$= \frac{1-z^{n+1}}{1-z}$$

Now, for deriving Lagrange's identity:-

expression using Euler's forma for cosine terms;

$$\cos k\theta = \frac{e^{ik\theta} + e^{-ik\theta}}{2}$$

Now,

by rewriting that,

$$1+\cos\theta+\cos 2\theta+\cdots+\cos n\theta = 1+\frac{1}{2}\Big(e^{i\theta}+e^{-i\theta}+e^{2i\theta}+e^{-2i\theta}+\cdots+e^{in\theta}+e^{-in\theta}\Big).$$

The sum of the exponentials:-

$$e^{i\theta}+e^{i2\theta}+\cdots+e^{in\theta} = \sum_{k=1}^{n} e^{ik\theta}$$

using the geometric series:

$$\sum_{k=0}^{n} e^{ik\theta} = \frac{1-e^{i(n+1)\theta}}{1-e^{i\theta}}$$

$$\Rightarrow \sum_{k=1}^{n} e^{ik\theta} = \frac{e^{i\theta}-e^{i(n+1)\theta}}{1-e^{i\theta}}$$

substituting $\Rightarrow$ to the sum of cosine:-

$$1+\frac{1}{2}\left(\frac{e^{i\theta}-e^{i(n+1)\theta}}{1-e^{i\theta}}+\frac{e^{-i\theta}-e^{-i(n+1)\theta}}{1-e^{-i\theta}}\right)$$

combining the fractions:

$$1+\frac{1}{2}\left(\frac{e^{i\theta}-e^{i(n+1)\theta}}{1-e^{i\theta}}+\frac{e^{-i\theta}-e^{-i(n+1)\theta}}{1-e^{-i\theta}}\right)$$

$$= 1 + \frac{1}{2}\left(\frac{\left(e^{i\theta} - e^{i(n+1)\theta}\right) - e^{-i\theta} - e^{-i(n+1)\theta}}{\left(1 - e^{i\theta}\right)\left(1 - e^{-i\theta}\right)}\right)$$

$$= 1 + \frac{1}{2} \cdot \frac{2i\sin\left(\frac{(n+1)\theta}{2}\right) \cdot e^{i\frac{n\theta}{2}}}{2\sin\left(\frac{\theta}{2}\right)}$$

$\because e^{i\theta} - e^{-i\theta} = 2i\sin\theta$

from the above simplification,

$$\Rightarrow 1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin(2n+1)\frac{\theta}{2}}{2\sin\frac{\theta}{2}}.$$

Answer to the Q. NO.- 04

Given,

$$(-4+4i)^{1/5}$$

Polar form expression:

$$r = \sqrt{(-4)^2+4^2}$$
$$= \sqrt{32}$$
$$= 4\sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{4}{-4}\right)$$
$$= \tan^{-1}(-1)$$
$$= -\frac{1}{4}\pi$$
$$\therefore \pi - \frac{1}{4}\pi = \frac{3}{4}\pi$$

$$-4+4i = 4\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

applying De Moiver's Theorem:

$$(4\sqrt{2})^{1/5}\left(\cos\frac{\frac{3\pi}{4}+2K\pi}{5}\right) + i\sin\frac{\frac{3\pi}{4}+2K\pi}{5}\Bigg)$$

here,

$K = 0, 1, 2, 3, 4, 5,$

$$(4\sqrt{2})^{1/5} = (2)^{3/5}$$

[P.T.O.]

for $K = 0$,

$$\theta_0 = 2^{3/5}\left(\cos\frac{3\pi}{20} + i\sin\frac{3\pi}{20}\right)$$

for $K = 1$,

$$\theta_1 = 2^{3/5}\left(\cos\frac{11\pi}{20} + i\sin\frac{11\pi}{20}\right)$$

for $K = 2$,

$$\theta_2 = 2^{3/5}\left(\cos\frac{19\pi}{20} + i\sin\frac{19\pi}{20}\right)$$

for $K = 3$

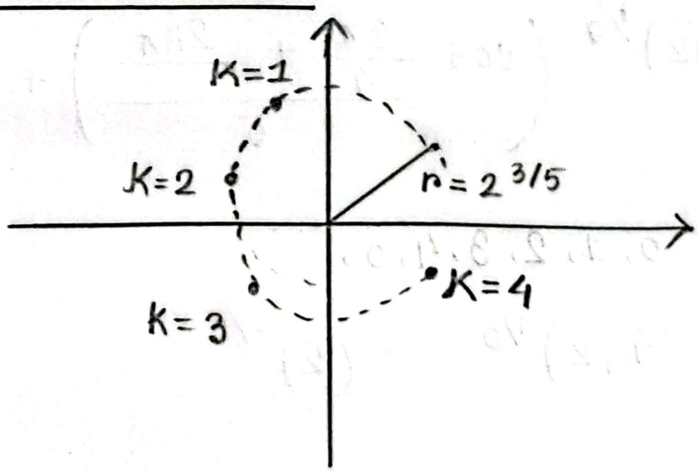
$$\theta_3 = \frac{\frac{3\pi}{4} + 6\pi}{5} = \frac{27\pi}{20}$$

$$\therefore \theta_3 = 2^{3/5}\left(\cos\frac{27\pi}{20} + i\sin\frac{27\pi}{20}\right)$$

for $K_4 = 2^{3/5}\left(\cos\frac{35\pi}{20} + i\sin\frac{35\pi}{20}\right)$

$$= 2^{3/5}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$$

Graphical Representation:



Answere to the Q. NO - 05

Given,

$$z_1 = 4-3i$$

$$\therefore \overline{z_1} = 4+3i$$

$$z_2 = -1+2i$$

$$\therefore \overline{z_2} = -1-2i$$

$$\therefore 2\overline{z_1} = 2(4+3i) = 8+6i$$

and $-3\overline{z_2} = -3(-1-2i) = 3+6i$

Now,

$$|2\overline{z_1} - 3\overline{z_2} - 2|$$

$$= |(8+6i+3+6i)-2|$$

$$= |11+12i-2|$$

$$= |9+12i|$$

$$= \sqrt{(9)^2+(12)^2}$$

$$= \sqrt{81+144}$$

$$= \sqrt{225}$$

$$= 15$$

$$\therefore |2\overline{z_1} - 3\overline{z_2} - 2| = 15$$

(Ans:)