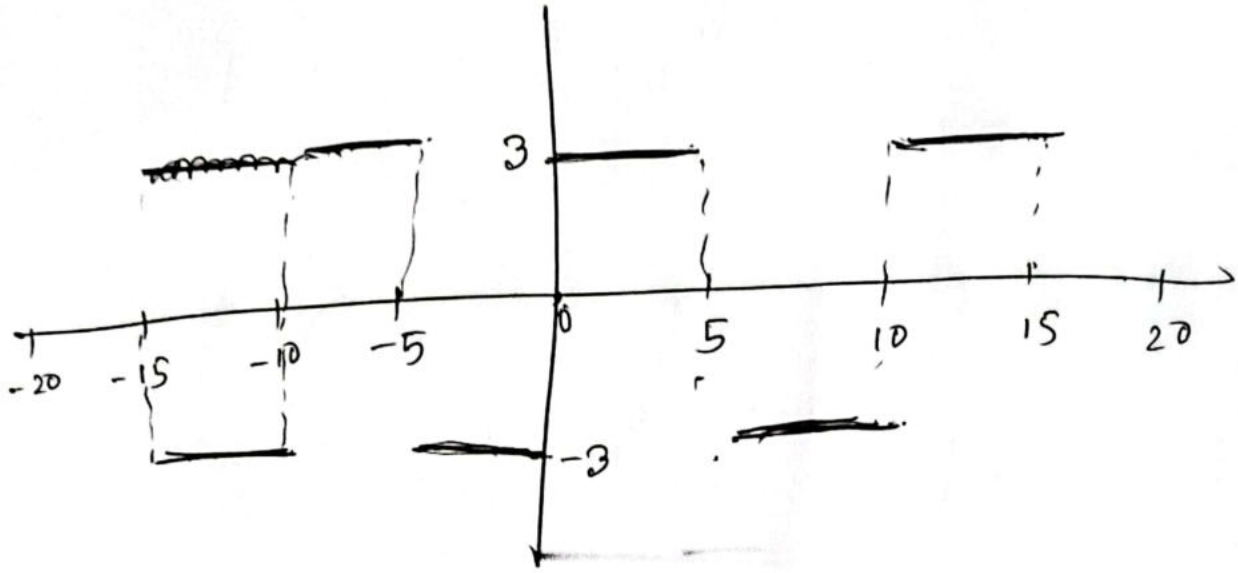
Q. Graph the following fun^n s

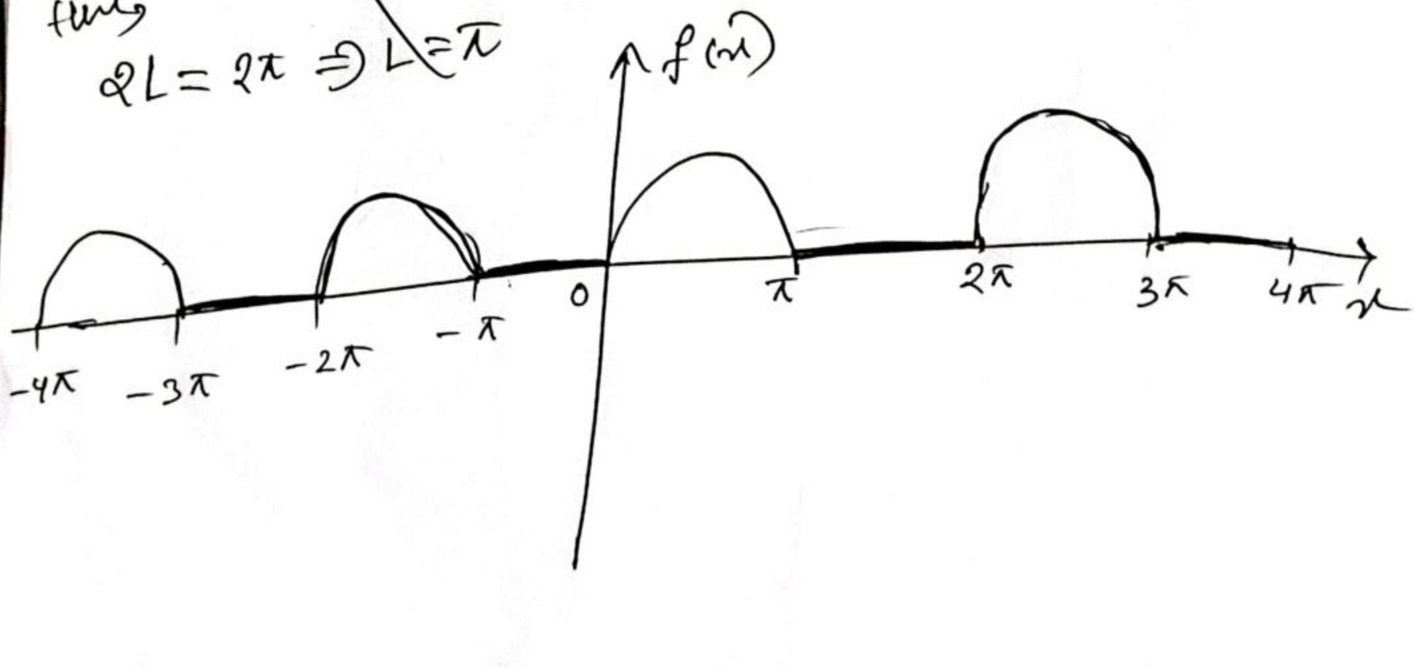
i) $f(x) = \begin{cases} 3; & 0 < x < 5 \\ -3; & -5 < x < 0 \end{cases}$: period $= 10$



ii) $f(x) = \begin{cases} \sin x; & 0 \le x \le \pi \\ 0; & \pi < x < 2\pi \end{cases}$ period $= 2\pi$

Here

$2L = 2\pi \Rightarrow L = \pi$



Imp. formula

$$* \int \sin(ax)\sin(bx)\,dx = \frac{\sin(a-b)x}{2(a-b)} - \frac{\sin(a+b)x}{2(a+b)}$$

$$* \int \cos(ax)\cos(bx)\,dx = \frac{\sin(a-b)x}{2(a-b)} + \frac{\sin(a+b)x}{2(a+b)}$$

$$* \int \sin(ax)\cos(bx)\,dx = -\frac{\cos(a+b)x}{2(a+b)} - \frac{\cos(a-b)x}{2(a-b)}$$

Q. find the Fourier sine series for

$f(x) = \cos x; \quad 0 \le x \le \pi.$

Sol^n $L = \pi$

$$\cos x = \sum_{n=1}^{\infty} b_n \sin(nx) \quad \text{——} (1)$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} \underset{= f(x)}{\cos x}\, \sin(nx)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \sin(nx)\cos x \, dx \qquad \left[\begin{array}{l}\text{use formula 3} \\ \text{with } a = n \\ \qquad\ \ b = 1\end{array}\right]$$

$$= \frac{2}{\pi}\left[-\frac{\cos(n+1)x}{2(n+1)} - \frac{\cos(n-1)x}{2(n-1)}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{-\cos(n+1)\pi}{2(n+1)} - \frac{\cos(n-1)\pi}{2(n-1)} + \frac{1}{2(n+1)} + \frac{1}{2(n-1)}\right]$$

$$b_n = -\frac{\cos(n+1)\pi}{\pi(n+1)} - \frac{\cos(n-1)\pi}{\pi(n-1)} + \frac{1}{\pi(n+1)} + \frac{1}{\pi(n-1)}.$$

$$\text{——}(*).$$

$\therefore \cos x = \sum_{n=1}^{\infty} b_n \sin(nx)$ when b_n is given by (*).

complex notation

Fourier Transformations

~~Fourier Transform:~~

Fourier Integral:

Fourier Integral of $f(x)$ is =>

$$f(x) = \int_0^\infty \left\{ A(u) \cos(ux) + B(u) \sin(ux) \right\} du$$

where

$$A(u) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \cos(ux)\, dx$$

$$B(u) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \sin(ux)\, dx$$

* Alternative form / complex form of Fourier integral is $\Rightarrow$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(t)\, e^{-iut}\, dt\right] e^{iux}\, du \quad \text{—} (*)$$

If $\quad F(u) = \int_{-\infty}^{\infty} f(x)\, e^{-iux}\, dx$

then $(*) \Rightarrow$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)\, e^{iux}\, du$$

The function $F(u)$ is called the Fourier transform of $f(x)$ & is written as-

$$F(u) = \mathcal{F}\{f(x)\}$$

* Fourier sine transform:-

The Fourier sine transform of a fun $F(x)$; $0<x<\infty$ is denoted by $f_s(n)$ & is defined as -

$$f_s(n) = \int_0^\infty F(x) \sin(nx)\, dx$$

* Fourier cosine transform.

The Fourier cosine transform of a fun $F(x)$; $0<x<\infty$ is denoted by $f_c(n)$ & is defined as $\Rightarrow$

$$f_c(n) = \int_0^\infty F(x) \cos(nx)\, dx$$

Examples

(1) Find Fourier sine transform of e^{-x}; $x \geq 0$

Soln We know, Fourier sine transform of $F(x) = e^{-x}$ is $\Rightarrow$

$$f_s(n) = \int_0^\infty F(x) \sin(nx)\, dx$$

$$= \int_0^\infty e^{-x} \sin(nx)\, dx$$

$$= \left[\frac{e^{-x}}{1+n^2}\left(-\sin(nx) - n\cos(nx)\right)\right]_0^\infty$$

[$\because$ using the formula $\Rightarrow$

$$\int e^{ax}\sin(bx)\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin(bx) - b\cos(bx)\right]$$]

$$\Rightarrow f_s(n) = \left[0 - \frac{1}{1+n^2}(-0-n)\right] \quad [\because e^{-\infty}=0]$$

$$\boxed{f_s(n) = \frac{n}{1+n^2}} \text{ (Ans).}$$

(02) find Fourier cosine transform of e^{-x}; $x \geq 0$.

Sol^n let, $F(x) = e^{-x}$; $x \geq 0$

$\left[\int e^{ax}\cos(bx)\,dx = \frac{e^{ax}}{a^2+b^2}\left(a\cos(bx)+b\sin(bx)\right)\right]$

We know,

$$f_c(n) = \int_0^\infty F(x)\cos(nx)\,dx$$

$$= \int_0^\infty e^{-x}\cos(nx)\,dx$$

$$= \left[\frac{e^{-x}}{1+n^2}\left(-\cos(nx) + n\sin(nx)\right)\right]_0^\infty$$

$$= \left[0 - \frac{1}{1+n^2}\left(-1 + n \times 0\right)\right]$$

$$= \frac{1}{1+n^2}$$

$$\therefore \boxed{f_c(n) = \frac{1}{1+n^2}}$$ is the required Fourier cosine transform.

Ans