Example 4. Find the eigenvalues and the corresponding eigevectors of the matrix $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$

Solution : The characteristic matrix of A is

$$\lambda I - A = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} \lambda-2 & -3 \\ -1 & \lambda-4 \end{bmatrix}$$

Now the determinant of $\lambda I - A$ (the characteristic polynomial of A) is $|\lambda I - A| = \begin{vmatrix} \lambda-2 & -3 \\ -1 & \lambda-4 \end{vmatrix} = (\lambda-2)(\lambda-4) - 3.$

Therefore, the characteristic equation of A is

$$(\lambda - 2)(\lambda - 4) - 3 = 0$$

or, $\lambda^2 - 6\lambda + 8 - 3 = 0$

or, $\lambda^2 - 6\lambda + 5 = 0$

or, $(\lambda - 5)(\lambda - 1) = 0 \quad \therefore \lambda = 5. \quad \lambda = 1$

which are the eigenvalues of A.

Now by definition $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is an eigenvector of A corresponding to λ if and only if X is a non-trivial solution of $(\lambda I - A) X = 0$, that is, of

$$\begin{bmatrix} \lambda-2 & -3 \\ -1 & \lambda-4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (1)$$

If $\lambda = 5$. equation no (1) becomes

$$\begin{bmatrix} 3 & -3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or, $\left. \begin{array}{r} 3x_1 - 3x_2 = 0 \\ -x_1 + x_2 = 0 \end{array} \right\} \Rightarrow x_1 - x_2 = 0$

This system is in echelon form and consistent. Since there are more unknowns than equation in echelon form, the system has an infinte number of solutions. Again, the equation begins with x_1 only, the other unknown x_2 is a free variable.

Let us take $x_2 = a$ (a is an arbitrary real number). Therefore, the eigenvectors of A corresponding to the eigenvaluc $\lambda = 5$ are non-zero vectors of the form $X = \begin{bmatrix} a \\ a \end{bmatrix}$

In particular, let a = 1, then $X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = 5$.

If $\lambda = 1$. equation no (1) becomes

$$\begin{bmatrix} -1 & -3 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or, $\left. \begin{array}{r} -x_1 - 3x_2 = 0 \\ -x_1 - 3x_2 = 0 \end{array} \right\} \Rightarrow x_1 + 3x_2 = 0$

This system is in echelon form and consistent, Since there are more unknowns than equation in echelon form, the system has an infinte number of solutions, Again, the equation begins with x_1 only, the other unknown x_2 is a free variable.

Let us take $x_2 = b$ (b is an arbitrary real number). $x_1 = -3b$,

Therefore, the eigen vectors of A corresponding to the eigenvalue $\lambda = 1$ are the non-zero vectors of the form

$$X = \begin{bmatrix} -3b \\ b \end{bmatrix}$$

In particular, let $b = 1$, then $X = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = 1$.

Example 6. Find all eigenvalues and the corresponding eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -2 & 0 \\ 0 & -5 & 2 \end{bmatrix}$$ [D. U. P. 1991]

Solution : The characteristic matrix of A is

$$\lambda I - A = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 2 & -1 \\ 0 & -2 & 0 \\ 0 & -5 & 2 \end{bmatrix} = \begin{bmatrix} \lambda-1 & -2 & 1 \\ 0 & \lambda+2 & 0 \\ 0 & 5 & \lambda-2 \end{bmatrix}$$

The characteristic polynomial of A is

$$\Delta(\lambda) = |\lambda I - A| = \begin{vmatrix} \lambda-1 & -2 & 1 \\ 0 & \lambda+2 & 0 \\ 0 & 5 & \lambda-2 \end{vmatrix}$$

$$= (\lambda - 1)(\lambda + 2)(\lambda - 2).$$

Therefore, the characteristic equation of A is

$$(\lambda - 1)(\lambda + 2)(\lambda - 2) = 0$$

$$\therefore \quad \lambda = 1, \lambda = -2, \lambda = 2$$

which are the eigenvalues of A.

Now by definition $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is an eigenvector of A

corresponding to the eigenvalue λ if and only if X is a non-trivial solution of $(\lambda I - A) X = 0$.

i.e $= \begin{bmatrix} \lambda-1 & -2 & 1 \\ 0 & \lambda+2 & 0 \\ 0 & 5 & \lambda-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

When $\lambda = 1$, $\begin{bmatrix} 0 & -2 & 1 \\ 0 & 3 & 0 \\ 0 & 5 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Forming the linear system, we have

$\left.\begin{array}{l} -2x_2 + x_3 = 0 \\ 3x_2 \quad\quad = 0 \\ 5x_2 - x_3 = 0 \end{array}\right\}$ Solving we get $x_2 = x_3 = 0$

Hence x_1 is a free variable. Let $x_1 = a$ where a is any real number. Therefore, the eigenvectors of A corresponding to the eigenvalue $\lambda = 1$ are non-zero vectors of the form $X = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}$.

In particular, let $a = 1$, then $X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = 1$.

When $\lambda = -2$, $\begin{bmatrix} -3 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 5 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Forming the linear system, we get

$\left.\begin{array}{l} -3x_1 - 2x_2 + x_3 = 0 \\ \quad\quad 5x_2 - 4x_3 = 0 \end{array}\right\}$

This system is in echelon form and has one free variable which is x_3. Let $x_3 = b$ where b is any real number. Then from

second equation we get $x_2 = \dfrac{4b}{5}$. Putting $x_2 = \dfrac{4b}{5}$ and $x_3 = b$ in the

first equation. we get $x_1 = -\dfrac{b}{5}$.

Therefore, the eigenvectors of A corresponding to the eigenvalue $\lambda = -2$ are the non-zero vectors of the form

$$X = \begin{bmatrix} \dfrac{-b}{5} \\[2mm] \dfrac{4b}{5} \\[2mm] b \end{bmatrix}$$

In particular, let $b = 5$, then $X = \begin{bmatrix} -1 \\ 4 \\ 5 \end{bmatrix}$ is an eigenvector

corresponding to the eigenvalue $\lambda = -2$.

When $\lambda = 2$, $\begin{bmatrix} 1 & -2 & 1 \\ 0 & 4 & 0 \\ 0 & 5 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Forming linear system, we get

$$\left. \begin{array}{l} x_1 - 2x_2 + x_3 = 0 \\ 4x_2 = 0 \\ 5x_2 = 0 \end{array} \right\} \text{ i.e } \left. \begin{array}{l} x_1 - 2x_2 + x_3 = 0 \\ x_2 = 0 \end{array} \right\}$$

Here x_3 is a free variable. Let $x_3 = c$ where c is any real number. Therefore, the eigenvectors of A corresponding to the eigenvalue $\lambda = 2$ are non-zero vectors of the form.

$$X = \begin{bmatrix} -c \\ 0 \\ c \end{bmatrix}$$ In particular, let $c = 1$, then

$$X = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$ is an eigenvector corresponding to the eigenvalue $\lambda = 2$.

2. (a) Find the eigenvalues and the associated eigenvectors of the linear operator given by the matrix :

$$\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$$

[D. U. P. 1984]

Answer : $\lambda = 4, \lambda = -1$.

Eigenvector corresponding to $\lambda = 4$ is $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

Eigenvector corresponding to $\lambda = -1$ is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

3. Find the eigenvalues and eigenvectors for the matrices

(i) $\begin{bmatrix} 2 & -2 & 1 \\ 2 & -8 & -2 \\ 1 & 2 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 0 & 4 \\ 0 & 5 & 4 \\ 4 & 4 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$

Answers : (i) $\lambda = 0, \lambda = 3, \lambda = -7$.

Eigenvector corresponding to $\lambda = 0$ is $\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

"   "   " $\lambda = 3$ is $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

"   "   " $\lambda = -7$ is $\begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}$

(ii) $\lambda = 9$, $\lambda = 3$, $\lambda = -3$

Eigenvector corresponding to $\lambda = 9$ is $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

" $\lambda = 3$ is $\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

" $\lambda = -3$ is $\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

(iii) $\lambda = -1$, $\lambda = -1$, $\lambda = 8$

Linearly independent eigenvectors are

$\begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$