Lecture: 03

Application of System of Linear Eq^{n}s

1. Balancing chemical Equations

Q. Balance the chemical eq^n using system of linear eq^{n}s.

$$CH_4 + O_2 \longrightarrow CO_2 + H_2O$$

Sol^n: Let,

$$x_1 (CH_4) + x_2 (O_2) \longrightarrow x_3 (CO_2) + x_4 (H_2O)$$

where, x_1, x_2, x_3, x_4 are the integers.

So, To balance the eq^n, the number of atoms on both left & right sides must be equal. i.e.

	left side		right side
# of carbon	x_1	$=$	x_3
# of Hydrogen	$4x_1$	$=$	$2x_4$
# of oxygen	$2x_2$	$=$	$2x_3 + x_4$.

So, the corresponding $\underbrace{\text{homogeneous}}_{RHS=0}$ linear system $\Rightarrow$

$$\begin{aligned} x_1 \quad - x_3 &= 0 \\ 4x_1 \quad - 2x_4 &= 0 \\ 2x_2 - 2x_3 - x_4 &= 0 \end{aligned}$$

$\therefore$ The augmented matrix $\Rightarrow$

$$\left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 4 & 0 & 0 & -2 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 4 & -2 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{array}\right); \quad r_2' = r_2 - 4r_1$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -1 & -\frac{1}{2} & 0 \end{array}\right); \quad r_3' = \frac{r_3}{4}; \quad r_4' = \frac{r_4}{2}$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 \end{array}\right); \quad r_2 \longleftrightarrow r_3$$

This is in row echelon form. The corresponding system ⇒

$$x_1 \quad - x_3 = 0$$
$$x_2 - x_3 - \frac{1}{2}x_4 = 0$$
$$x_3 - \frac{1}{2}x_4 = 0$$

There are 3 eq^{n}s in 4 unknowns. So, $(4-3)=1$ free variable.

Here, x_4 is the free variable.

let, $x_4 = t$

$\therefore$ $x_3 = \frac{t}{2}$

$x_2 = x_3 + \frac{x_4}{2} = \frac{t}{2} + \frac{t}{2} = t$

$x_1 = x_3 = t/2$; $t \in \mathbb{R}$.

The smallest value of t for which x_1, x_2, x_3, x_4 are integers is $t = 2$.

So, $t=2 \Rightarrow x_1 = 1,\ x_2 = 2,\ x_3 = 1,\ x_4 = 2$

$\therefore$ The balanced eq^n ⇒

$$CH_4 + 2O_2 \longrightarrow CO_2 + 2H_2O$$

Ans

Q: Balance ⟹

$HCl + Na_3PO_4 \longrightarrow H_3PO_4 + NaCl.$

Sol^n let, $x_1 (HCl) + x_2 (Na_3PO_4) \longrightarrow x_3 (H_3PO_4) + x_4 (NaCl)$

where x_i; $(i = \overline{1,4})$ are the integers.

$$\begin{array}{lccc} & \text{Left} & & \text{Right} \\ H: & x_1 & = & 3x_3 \\ Cl: & x_1 & = & x_4 \\ Na: & 3x_2 & = & x_4 \\ P: & x_2 & = & x_3 \\ O: & 4x_2 & = & 4x_3 \end{array}$$

$\therefore$

$$\begin{aligned} x_1 \quad\quad - 3x_3 &= 0 \\ x_1 \quad\quad\quad\quad - x_4 &= 0 \\ 3x_2 \quad\quad - x_4 &= 0 \\ x_2 - x_3 \quad\quad &= 0 \\ 4x_2 - 4x_3 \quad\quad &= 0 \end{aligned}$$

Augmented matrix ⟹

$$\left(\begin{array}{cccc|c} 1 & 0 & -3 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 3 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 4 & -4 & 0 & 0 \end{array}\right)$$

[transform the system into reduced row echelon form by row operations] $\to$ do by yourself.

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The reduced row echelon form $\Rightarrow$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$\therefore$ The corresponding system $\Rightarrow$

$$x_1 \quad - x_4 = 0$$
$$x_2 - \frac{1}{3}x_4 = 0$$
$$x_3 - \frac{1}{3}x_4 = 0$$

3 eq^{n}s in 4 unknowns. So, $(4-3) = 1$ free variable.

Here, x_4 is free variable.

So, let, $x_4 = t$; $t \in \mathbb{R}$.

$\therefore x_3 = t/3$, $x_2 = t/3$, $x_1 = t$.

The smallest value of t for which all x_i's are +ve integers is $t = 3$.

$t = 3 \Rightarrow x_1 = 3, \quad x_2 = 1, \quad x_3 = 1, \quad x_4 = 3$

$\therefore$ The balanced eqⁿ $\Rightarrow$

$$3HCl + Na_3PO_4 \longrightarrow H_3PO_4 + 3NaCl.$$

(Ans)

2. Polynomial Interpolation

** NB: If n points are given then we'll get a polynomial of degree $(n-1)$ or less whose graph passes through those n points.

Q: find a polⁿ whose graph passes through the pts $\Rightarrow$ $(1,3)$, $(2,-2)$, $(3,-5)$, $(4,0)$.

Solⁿ: Since there are 4 pts, we'll ~~assume~~ use a polynomial of degree $(4-1) = 3$.

Let, the required polⁿ be $\Rightarrow$

$$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 \quad \text{——— } (*)$$

Since the polⁿ (i) passes through $(1,3)$, i.e.

when $x = 1$, $p(x) = 3$.

So, (1) $\Rightarrow a_0 + a_1 + a_2 + a_3 = 3$

Similarly,

$$a_0 + 2a_1 + 2^2 a_2 + 2^3 a_3 = -2$$

$$a_0 + 3a_1 + 3^2 a_2 + 3^3 a_3 = -5$$

$$a_0 + 4a_1 + 4^2 a_2 + 4^3 a_3 = 0$$

So, the ~~system~~ Augmented matrix:

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 8 & -2 \\ 1 & 3 & 9 & 27 & -5 \\ 1 & 4 & 16 & 64 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 7 & -5 \\ 0 & 2 & 8 & 26 & -8 \\ 0 & 3 & 15 & 63 & -3 \end{array}\right) ; \quad \begin{array}{l} r_2' = r_2 - r_1 \\ r_3' = r_3 - r_1 \\ r_4' = r_4 - r_1 \end{array}$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 7 & -5 \\ 0 & 1 & 4 & 13 & -4 \\ 0 & 1 & 5 & 21 & -1 \end{array}\right) , \quad \begin{array}{l} r_3' = \frac{r_3}{2} \\ r_4' = \frac{r_4}{3} \end{array}$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 7 & -5 \\ 0 & 0 & 1 & 6 & 1 \\ 0 & 0 & 2 & 14 & 4 \end{array}\right) ; \quad \begin{array}{l} r_3' = r_3 - r_2 \\ r_4' = r_4 - r_2 \end{array}$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 7 & -5 \\ 0 & 0 & 1 & 6 & 1 \\ 0 & 0 & 0 & 2 & 2 \end{array}\right) \quad ; r_4' = r_4 - 2r_3$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 7 & -5 \\ 0 & 0 & 1 & 6 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right) \quad ; r_4' = \frac{r_4}{2}$$

; This is in row echelon form.

The corresponding system $\Rightarrow$

$$a_0 + a_1 + a_2 + a_3 = 3 \quad \text{—— (1)}$$
$$a_1 + 3a_2 + 7a_3 = -5 \quad \text{—— (2)}$$
$$a_2 + 6a_3 = 1 \quad \text{—— (3)}$$
$$a_3 = 1 \quad \text{—— (4)}$$

By backward substitution $\Rightarrow$

$(4) \Rightarrow a_3 = 1$

$(3) \Rightarrow a_2 = 1 - 6a_3 = 1 - 6 = -5$

$(2) \Rightarrow a_1 = -5 - 3(-5) - 7(1) = 3$

$(1) \Rightarrow a_0 = \cancel{3} - \cancel{3} + 5 - 1 = 4$

$(*) \Rightarrow \quad p(x) = 4 + 3x - 5x^2 + x^3.$

Ans

Practice prob^m $\Rightarrow$

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Exercise 9-12, 13, 15, T1.

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