$L = \{\omega \in \{a, b, c, p, q, r, \#\}^* : a^i \#^n c^k p^{2x} q^y r^z b^j,$

where $i = j + k$, $y = 3x + z$, n is odd and

$i, j, k, n, x, y, z \geq 0\}$

$$a^i \#^n c^k p^{2x} q^y r^z b^j$$

$$\Rightarrow a^{j+k} \#^n c^k p^{2x} q^{3x+z} r^z b^j$$

$$\Rightarrow a^j a^k \#^n c^k p^{2x} q^{3x} q^z r^z b^j$$

$S \rightarrow aSb \mid T$

$T \rightarrow ABC$

$A \rightarrow aAc \mid X$

$X \rightarrow \#\#X \mid \#$

$B \rightarrow ppBqqq \mid \varepsilon$

$C \rightarrow qCr \mid \varepsilon$

CSE331
Assignment - 02
CFG Solution
Faculty: KKP
(IF you spot any error, please notify me)

3) Design a Context Free Grammar for the Language:

a) L = {w ∈ {a,b,c,p,q,r,#}*: $a^i \#^n c^k p^{2x} q^y r^z b^j$ where i=j+k, y=3x+z, n is odd and i,j,k,n,x,y,z ≥ 0}

b) L = {w ∈ {0,1,2}*: w = $0^i 2^j 1^k$, [where ….conditions….] }

where…

~~i) i = k, i,k ≥ 1 and j ≥ 2~~

ii) i = 3k, j is odd and i,j,k ≥ 0

iii) i is a multiple of two, k is two more than a multiple of 3, j = k+i, and i,j,k ≥ 0

iv) i+j > k and i,j,k ≥ 0

v) i+k is even, j = i+k and j>=1

c) L = {w ∈ {0,1}*: the parity of 0s and 1s is different in w}

d) L = {w ∈ {0,1}*: the number of 0s and 1s are different in w} [Hint: First, try to solve for an equal number of 0s and 1s in w]

e) L = {$1^i 0 2^j 1^k$| i, j, k ≥ 0, 3i ≥ 4k + 2, j is not divisible by three}

f) Recall that for a string w, |w| denotes the length of w. Σ = {0,1}

L1 = {w ∈ Σ^*: w contains exactly two 1s}

L2 = {x#y : x ∈ Σ^*, y ∈ L1, |x| = |y|}

Construct a CFG for L2.

g) Recall that for a string w, |w| denotes the length of w. Σ = {0,1}

L1 = {w ∈ Σ^*: w contains at least three 1s}

L2 = {x#y : x ∈ $(\Sigma\Sigma)^*$, y ∈ L1, |x| = |y|}

Construct a CFG for L2.

CSE331

Assignment - 02

CFG Solution

Faculty: KKP

(IF you spot any error, please notify me)

3) Design a Context Free Grammar for the Language:

a) L = {w ∈ {a,b,c,p,q,r,#}*: $a^i \#^n c^k p^{2x} q^y r^z b^j$ where i=j+k, y=3x+z, n is odd and i,j,k,n,x,y,z ≥ 0}

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Construct a CFG for L2.

$$L = \{ \omega \in \{0,1,2\}^* : w = 0^i 2^j 1^K, \text{ where } i = K,\ i,K \geq 1,\ J \geq 2 \}$$

$0^i 2^j 1^K$

$\Rightarrow 0^i 2^j 1^i$

0221, 00022111, 022221 $\in L$

22, 01, 021, 022 $\notin L$

Solution:

$i,K \geq 1$

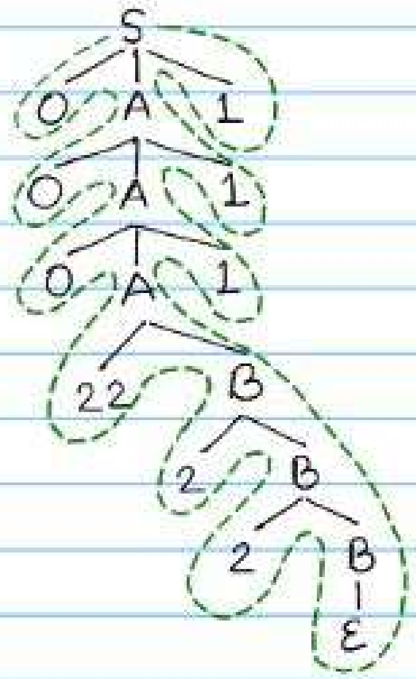
$S \rightarrow 0A1$

$A \rightarrow 0A1 \mid 22B$

$B \rightarrow 2B \mid \varepsilon$

$J \geq 2$

0002222111



Another Solution:

$S \rightarrow 0S1 \mid 0\underbrace{22}_{J \geq 2}A1$ ($i,K \geq 1$)

$A \rightarrow 2A \mid \varepsilon$

Another Solution:

$S \rightarrow 0S1 \mid 0A1$ ($\rightarrow i,K \geq 1$)

$A \rightarrow 2A \mid 22$ ($\rightarrow J \geq 2$)

$L = \{ \omega \in \{0,1,2\}^* : w = 0^i 2^j 1^K$, where $i = 3K$, J is odd and $i, J, K \geq 0 \}$

$0^i 2^j 1^K$

$\rightarrow 0^{3K} 2^J 1^K$

$K = 0$, 'odd 2s'

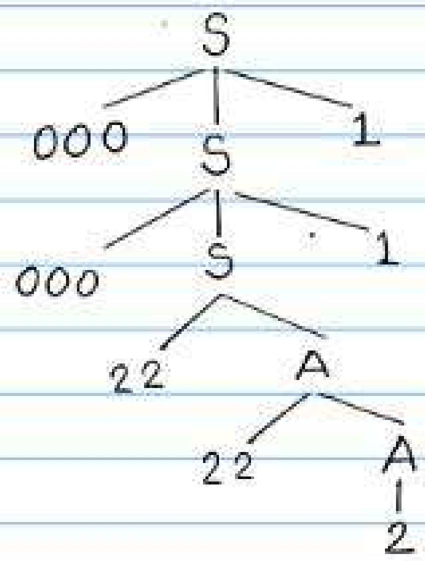
$K = 1$, 000 'odd 2s' 1

$K = 2$, 000000 'odd 2s' 11

Solution:

$S \rightarrow 000S1 \mid A$

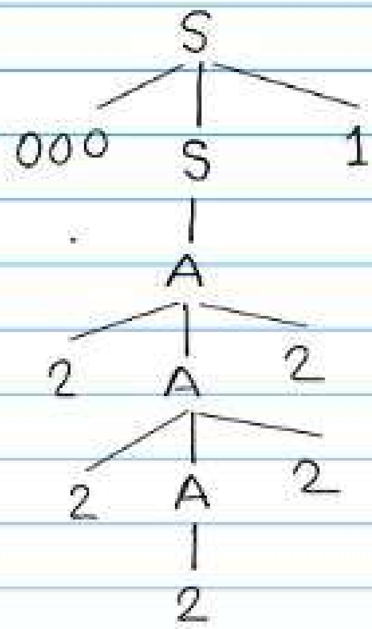
$A \rightarrow 22A \mid 2$



Another Solution:

$S \rightarrow 000S1 \mid A$

$A \rightarrow 2A2 \mid 2$



* It is not mandatory to draw the parse tree in the q/s of CFG. I have drawn for the understanding purpose.

$L = \{ w \in \{0,1,2\}^* : w = 0^i 2^j 1^K,\ i$ is multiple of two, K is two more than multiple of three, $J = K + i$, and $i, J, K \geq 0 \}$

Be careful with maintaining the same order

$$0^i 2^J 1^K$$

$$\Rightarrow 0^{2i} \, 2^{3K+2+2i} \, 1^{3K+2}$$

$$\Rightarrow 0^{2i} \, 2^2 \, 2^{2i} \, 2^{3K} \, 1^2 \, 1^{3K}$$

A B

$S \rightarrow AB$

$A \rightarrow 00A22 \mid 22$

$B \rightarrow 222B111 \mid 11$

some 0s.... some 2s.... some 1s

Another Solution:

$$0^i 2^J 1^K$$

$$\Rightarrow 0^{2i} \, 2^{3K+2+2i} \, 1^{3K+2}$$

$$\Rightarrow \underbrace{0^{2i} \, 2^{2i}} \, \underbrace{2^2} \, \underbrace{2^{3K} \, 1^{3K}} \, \underbrace{1^2}$$

$S \rightarrow ABCD$

$A \rightarrow 00A22 \mid \varepsilon$

$B \rightarrow 22$

$C \rightarrow 222C111 \mid \varepsilon$

$D \rightarrow 11$

some 0s.... some 2s.... some 1s

Be careful with maintaining the same order

Another Solution:

$$0^{i}\, 2^{j}\, 1^{K}$$

$$\Rightarrow 0^{2i}\, 2^{3K+2+2i}\, 1^{3K+2}$$

$$\Rightarrow 0^{2i}\, 2^{3K}\, 2^{2}\, 2^{2i}\, 1^{3K}\, 1^{2}$$

$$\Rightarrow 0^{2i}\, 2^{2i}\, 2^{3K}\, \underbrace{2^{2}}\, \underbrace{1^{2}}\, 1^{3K}$$

$S \rightarrow AB$

$A \rightarrow 00A22 \mid \varepsilon$

$B \rightarrow 222B111 \mid 2211$

$$L = \{ \omega \in \{0,1,2\}^* : w = 0^i 2^j 1^k, \text{ where } i+j > k \text{ and } i,j,k \geq 0 \}$$

let's first solve for $i+j=k$

$$0^i 2^j 1^k$$

$$\Rightarrow 0^i 2^j 1^{i+j}$$

$$\Rightarrow 0^i 2^j 1^j 1^i$$

Now since $i+j>k$,

$0^i 2^j 1^j 1^i$ or $0^i 2^j 1^j 1^i$

there could be more 0s and equal 2s & 1s means, in $i+j>k$, $j=k$ and $i>k$

$$\underbrace{00}_{\text{more 0s}}0022211111$$

there could be more 2s and equal 0s & 1s means, in $i+j>k$, $i=k$ and $j>k$

$$0022222111$$

you can also think in this way

$$00\underbrace{00}_{\text{more 0s}}22211111$$

or, there could be more 0s and 2s than 1s both, means in $i+j>k$ $i>k$ and $j>k$

Too summarize,

equal 0s & 1s

$0^i\,\_\_\,2^j\;1^j\;1^i$

equal 2s & 1s

$000 \cdots 2222$

↳ either more 0s
↳ either more 2s
↳ or both

equal 0s & 1s

$P \to 0P1 \mid X$

$X \to ABQ \mid CDQ$

where $i>k,\ j=k$ $i+j>k$

$A \to 0A \mid 0$ → more 0s

$B \to 2B \mid \varepsilon$ → there could be more 2s (handling the both case) → $i>k$ & $j>k$

where $j>k,\ i=k$ $i+j>k$

$C \to 0C \mid \varepsilon$

$D \to 2D \mid 2$ → more 2s

$Q \to 2Q1 \mid \varepsilon$ → equal 2s & 1s

this also handle the both case, can be skipped, since we already have handled the case previously.

$L = \{w \in \{0,1\}^*$: parity of number of 0s and 1s is different$\}$

Case 1: even 0s and odd 1s

Case 2: odd 0s and even 1s

So, the problem can be boiled down into $L = \{$ length of w is odd $\}$

$$S \rightarrow 00S \mid 01S \mid 10S \mid 11S \mid 0 \mid 1$$

This can also be written as

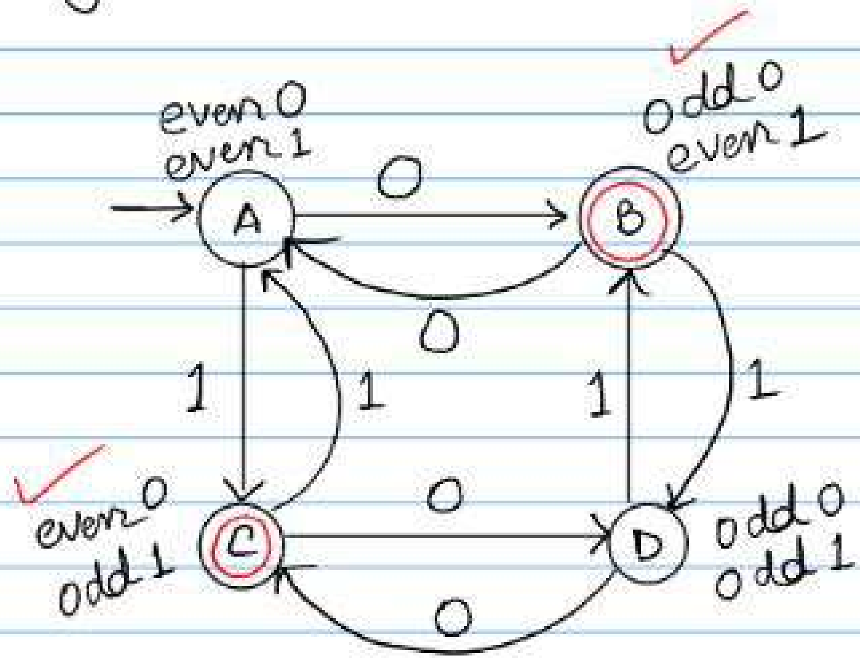
$$S \rightarrow XXS \mid X$$
$$X \rightarrow 0 \mid 1$$

Another Solution:

$$S \rightarrow 0S0 \mid 0S1 \mid 1S0 \mid 1S1 \mid 0 \mid 1$$

Another solution:

if you couldn't figure out the previous idea, then no worry. You may also recall the following DFA we had done in the class:



So another solution can be

$A \rightarrow 0B \mid 1C$

$B \rightarrow 0A \mid 1D \mid \varepsilon$

$C \rightarrow 0D \mid 1A \mid \varepsilon$

$D \rightarrow 0C \mid 1B$

$L1 = \{ \omega \in \{0,1\}^* :$ the number of 0s and 1s are different in ω.

Before solving L1, first we try to solve

$L2 = \{ w \in \{0,1\}^* :$ w contains equal numbers of 0s and 1s$\}$

0s & 1s are parsed in pairs, so both the count of 0s & 1s will be same

$$S \rightarrow 0S1 \mid 1S0 \mid \varepsilon$$

However, this solution is partially correct. For example, 0110 can't be parsed.

If we take a string, $w \in L2$, and if it has equal numbers of 0s and 1s, then it means, in w, there are one or more substring in w, having equal 0s & 1s.

$\boxed{001011}\boxed{10}$

↓

We can devide the string into two substrings, having equal 0s and 1s.

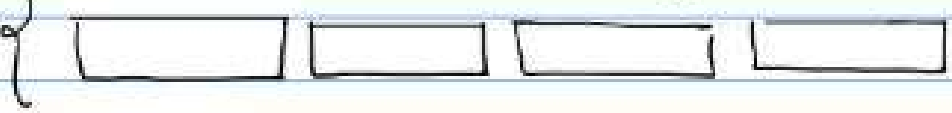
Now, recall the solution for valid parenthesis. and let's fix the grammar.

$$S \rightarrow 0S1 \mid 1S0 \mid SS \mid \varepsilon$$

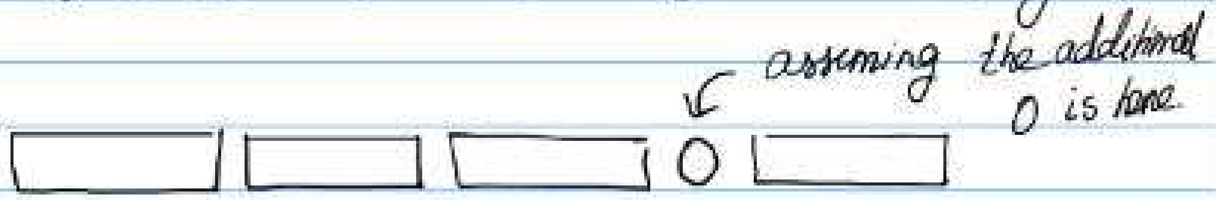
☐ Draw parse tree for 11010100001 1

Now, let's come back to our original question.
let's consider a string having equal 0s & 1s

each block having equal 0s & 1s



Now, if there is more 0s then having at least one additional 0 will be enough.



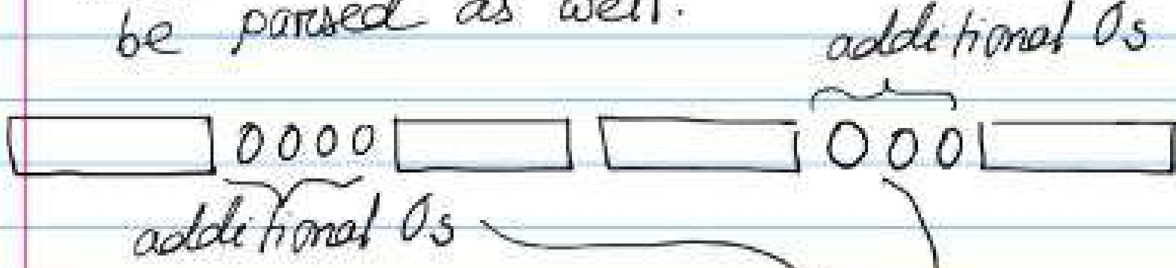
So, we can write

$$T \rightarrow S0S$$

where S produce equal 0s and 1s

$$S \rightarrow 0S1 \mid 1S1 \mid SS \mid \epsilon$$

However, there could be more than one additional 0 than 1s. So, those 0s should be parsed as well.



$$T \rightarrow S0S$$

$$S \rightarrow 0S1 \mid 1S0 \mid SS \mid 0S \mid \epsilon$$

However, we have handled only one case - more 0s than 1s. There could be more 1s than 0s also.

So, the final solution :

$$S \rightarrow A \mid B$$

$$A \rightarrow X0X$$

$$X \rightarrow 0X1 \mid 1X0 \mid XX \mid 0X \mid \varepsilon$$

$$B \rightarrow Y1Y$$

$$Y \rightarrow 0Y1 \mid 1Y0 \mid YY \mid 1Y \mid \varepsilon$$

Recall that for a string w, |w| denotes the length of w. $\Sigma = \{0,1\}$

$L1 = \{w \in \Sigma^*: w \text{ contains exactly two 1s}\}$

$L2 = \{x\#y : x \in \Sigma^*, y \in L1, |x| = |y|\}$

Construct a CFG for L2.

Before solving this problem, let's try to solve a few similar problems.

$$L = \{ W_1 \# W_2 \mid W_1, W_2 \in \{0,1\}^* \text{ and } |W_1| = |W_2| \}$$

1001#0010

$$S \rightarrow 0S0 \mid 0S1 \mid 1S0 \mid 1S1 \mid \#$$

We can also write it as

$$S \rightarrow XSX \mid \#$$
$$X \rightarrow 0 \mid 1$$

Now let's say,

$$L1 = \{ W_1 \# W_2 \mid W_1 \in \{0,1\}^*, W_2 \in L2 \text{ and } |W_1| = |W_2| \}$$

$$L2 = \{ W \in \{0,1\}^* : W \text{ is even} \}$$

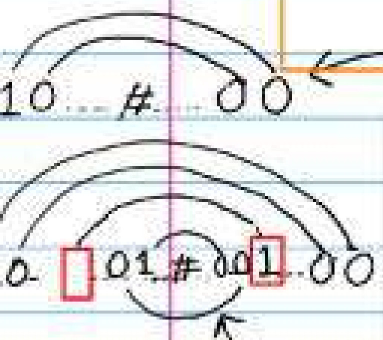
$$S \rightarrow XXSXX \mid \#$$

$$X \rightarrow 0 \mid 1$$

If you understood the previous two solutions. then we try to solve our original question.

$$L1 = \{ w \in \{0,1\}^* : w \text{ contains exactly two 1s} \}$$

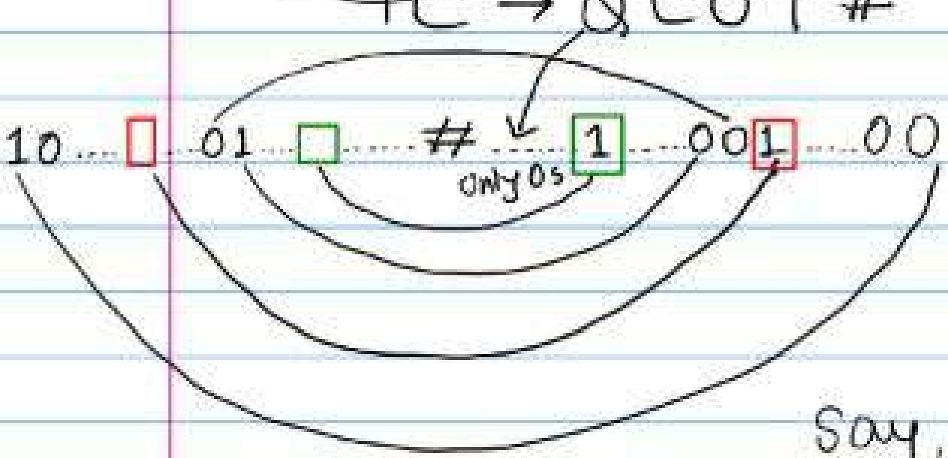
$$L2 = \{ x\#y,\ x \in \{0,1\}^*,\ y \in L1 \text{ and } |x| = |y| \}$$



$A \to QA0 \mid \boxed{QB1}$ — we have found the first 1

$B \to QB0 \mid \boxed{QC1}$ — found the second 1

$C \to QC0 \mid \#$



Here, we can say. if we are at production rule A, then have seen no 1 in y, if we are at the production rule B. then we have found exactly one 1 in y. Next. if we are at rule C, then we have seen exactly two 1s in the y.

$L = \{ w \in \{0,1\}^* : 0^i 1^K$ where $i, K \geq 0$ and $3i \geq 4K+2 \}$

This means, if we have i amount of 0s and K amount of 1s, then (3* total 0s) should be greater than or equal to (4* total 1s + 2)

So, let's first figure out, what is the minimum amount of 0s we need to have for $K = 0, 1, 2 \ldots$ satisfying the condition.

if $K=0$, $i \geq 1$

$K=1$, $i \geq 2$

$K=2$, $i \geq 4$

$K=3$, $i \geq 5$

$K=4$, $i \geq 6$

$K=5$, $i \geq 8$

$K=6$, $i \geq 9$

$$3i \geq 4K+2$$

$$i \geq \left\lceil \frac{4K+2}{3} \right\rceil$$

Now, can you find any pattern? Think in respect of K%3.
[since, 3i]

K%3

$K = 0 \Rightarrow K \% 3 = 0, \; i \geq 1$
$K = 1 \Rightarrow K \% 3 = 1, \; i \geq 2$
$K = 2 \Rightarrow K \% 3 = 2, \; i \geq 4$
$K = 3 \Rightarrow K \% 3 = 0, \; i \geq 5$
$K = 4 \Rightarrow K \% 3 = 1, \; i \geq 6$
$K = 5 \Rightarrow K \% 3 = 2, \; i \geq 8$
$K = 6 \Rightarrow K \% 3 = 0, \; i \geq 9$
$K = 7 \Rightarrow K \% 3 = 1, \; i \geq 10$
$K = 8 \Rightarrow K \% 3 = 2, \; i \geq 12$

if we do $K \% 3$, then we have three patterns.
The minimum length of strings for each pattern

$$0^i 1^K$$

0	001	000011
$K=0, K \% 3 = 0$	$K=1, K \% 3 = 1$	$K=2, K \% 3 = 2$

Now, see what amount of 0s and 1s you need to go the next pattern in same $K \% 3$

$0\boxed{0000111}$	$00\boxed{0000111}1$	$0000\boxed{0001111}1$
$K=3, K \% 3 = 0$	$K=4, K \% 3 = 1$	$K=5, K \% 3 = 2$

So, in each pattern we see, we can jump to the next string by adding 0000111

$$L = \{ w \in \{0,1\}^* : 0^i 1^k, \text{ where } i, k \geq 0 \}$$

So, now, if the condition was $\boxed{3i = 4k+2}$ then,

$$S \rightarrow 0A \mid 00A1 \mid 0000A11$$
$$A \rightarrow 0000A111 \mid \varepsilon$$

Now, since, the condition is $\boxed{3i \geq 4k+2}$, hence, we will have some additional 0s as well. So,

$k \% 3 = 0$ $k \% 3 = 1$ $k \% 3 = 2$

$$S \rightarrow 0S \mid 0A \mid 00A1 \mid 0000A11$$
$$A \rightarrow 0000A111 \mid \varepsilon$$

Another Approach: Based on the increment of 0s $\rightarrow i = 0, 1, 2, 4, 5, 6, 8$

$$\underbrace{0,1}_{1}\ \underbrace{1,2}_{1}\ \underbrace{2,4}_{2}\ \underbrace{4,5}_{1}\ \underbrace{5,6}_{1}\ \underbrace{6,8}_{2}$$

$k = 0, 1, 2, 3, 4, 5, 6$

if $3i = 4k+1$

$$S \rightarrow 0A \mid \varepsilon$$
$$A \rightarrow 0B1 \mid \varepsilon$$
$$B \rightarrow 00C1 \mid \varepsilon$$
$$C \rightarrow 0A1 \mid \varepsilon$$

Now, for $3i \geq 4k+1$

$$S \rightarrow 0S \mid 0A \mid \varepsilon$$
$$A \rightarrow 0B1 \mid \varepsilon$$
$$B \rightarrow 00C1 \mid \varepsilon$$
$$C \rightarrow 0A1 \mid \varepsilon$$