diagonalizable.

Example 7. Find the matrix P that diagonalizes the matrix

$A = \begin{bmatrix} 1 & 4 \\ 9 & 1 \end{bmatrix}$ and also determine $P^{-1}AP$.

Solution : The charatcteristic matrix of A is

$$\lambda I - A = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 4 \\ 9 & 1 \end{bmatrix} = \begin{bmatrix} \lambda - 1 & -4 \\ -9 & \lambda - 1 \end{bmatrix}$$

Now the characteristic polynomial of A is

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & -4 \\ -9 & \lambda - 1 \end{vmatrix} = (\lambda - 1)^2 - 36$$

Therefore, the characteristic equation of A is $(\lambda - 1)^2 - 36 = 0$

or, $\lambda^2 - 2\lambda + 1 - 36 = 0$

or, $\lambda^2 - 2\lambda - 35 = 0$,

or, $(\lambda + 5)(\lambda - 7) = 0$

$\therefore \lambda = -5, \lambda = 7$

which are the eigenvalues of the matrix A.

Now by definition $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is an eigenvector of A corresponding to λ if and only if X is a non-trivial solution of $(\lambda I - A)X = 0$, that is, of $\begin{bmatrix} \lambda - 1 & -4 \\ -9 & \lambda - 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (1)

When $\lambda = -5$ equation (1) becomes

$$\begin{bmatrix} -6 & -4 \\ -9 & -6 \end{bmatrix} \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or, $\left.\begin{aligned} -6x_1 - 4x_2 &= 0 \\ -9x_1 - 6x_2 &= 0 \end{aligned}\right\}$ or, $\left.\begin{aligned} 3x_1 + 2x_2 &= 0 \\ 3x_1 + 2x_2 &= 0 \end{aligned}\right\}$

or, $3x_1 + 2x_2 = 0$ (2)

Now it is clear that $x_1 = 2$ and $x_2 = -3$ is a solution of equation (2).

Therefore, $X_1 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = -5$.

When $\lambda = 7$, equation (1) becomes

$$\begin{bmatrix} 6 & -4 \\ -9 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or, $\left.\begin{aligned} 6x_1 - 4x_2 &= 0 \\ -9x_1 + 6x_2 &= 0 \end{aligned}\right\}$

or, $\left.\begin{aligned} 3x_1 - 2x_2 &= 0 \\ 3x_1 - 2x_2 &= 0 \end{aligned}\right\}$

or, $3x_1 - 2x_2 = 0$ (3)

Now it is clear that $x_1 = 2$, $x_2 = 3$ is a solution of the equation given by (3). Therefore, $X_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = 7$.

Suppose that P is the matrix which has the above two eigenvectors as columns.

Then $p = \begin{bmatrix} 2 & 2 \\ -3 & 3 \end{bmatrix}$

One can casily find that the inverse of P is $P^{-1} = \frac{1}{12}\begin{bmatrix} 3 & -2 \\ 3 & 2 \end{bmatrix}$

Now $P^{-1}AP = \frac{1}{12}\begin{bmatrix} 3 & -2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 9 & 1 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ -3 & 3 \end{bmatrix}$

$$= \frac{1}{12}\begin{bmatrix} 3 & -2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} -10 & 14 \\ 15 & 21 \end{bmatrix} = \frac{1}{12}\begin{bmatrix} -60 & 0 \\ 0 & 84 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 0 \\ 0 & 7 \end{bmatrix} = D$$

which is the diagonal matrix of the eigenvalues of the matrix A.

Hecnce $P = \begin{bmatrix} 2 & 2 \\ -3 & 3 \end{bmatrix}$ is the requred matrix that diagonalizes the given matrix $A = \begin{bmatrix} 1 & 4 \\ 9 & 1 \end{bmatrix}$

Example 8. Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$

Also find the matrix P that diagonalizes A and determine $P^{-1}AP$.

Solution : The characteristic matrix of A is

$$\lambda I - A = \lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix} = \begin{bmatrix} \lambda-4 & -6 & -6 \\ -1 & \lambda-3 & -2 \\ 1 & 4 & \lambda+3 \end{bmatrix}$$

Now the determinant of $\lambda I - A$ (the characteristic polynomial of A) is $\Delta(\lambda) = |\lambda I - A| = \begin{vmatrix} \lambda-4 & -6 & -6 \\ -1 & \lambda-3 & -2 \\ 1 & 4 & \lambda+3 \end{vmatrix}$

$$= (\lambda-4)(\lambda^2 - 9 + 8) + 6(-\lambda - 3 + 2) - 6(-4 - \lambda + 3)$$

$$= (\lambda-4)(\lambda^2 - 1) - 6\lambda - 6 + 6 + 6\lambda = (\lambda-4)(\lambda^2-1)$$

Therefore, the characteristic equation of A is

$(\lambda-4)(\lambda^2-1) = 0 \therefore \lambda = 4, \lambda = -1, \lambda = 1.$

which are the eigenvalues of A.

Now by definition $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is an eigenvector of A

corresponding to the eigenvalue λ if and only if X is a non-trivial solution of $(\lambda I - A)X = 0$

that is, of $\begin{bmatrix} \lambda-4 & -6 & -6 \\ -1 & \lambda-3 & -2 \\ 1 & 4 & \lambda+3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ (1)

If $\lambda = 4$, equation (1) becomes

$$\begin{bmatrix} 0 & -6 & -6 \\ -1 & 1 & -2 \\ 1 & 4 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or, $\left.\begin{aligned} -6x_2 - 6x_3 &= 0 \\ -x_1 + x_2 - 2x_3 &= 0 \\ x_1 + 4x_2 + 7x_3 &= 0 \end{aligned}\right\}$

Reduce the system to echelon form by elementary transformations. Divide 1st equation by -6 and multiply 2nd eqn by -1 and then interchange with the 1st eqn.

Then $\left.\begin{aligned} x_1 - x_2 + 2x_3 &= 0 \\ x_2 + x_3 &= 0 \end{aligned}\right\}$ Subtract first equation from the third

$$x_1 + 4x_2 + 7x_3 = 0$$

Thus $\left.\begin{array}{r} x_1 - x_2 + 2x_3 = 0 \\ x_2 + x_3 = 0 \end{array}\right\} \Rightarrow \left.\begin{array}{r} x_1 - x_2 + 2x_3 = 0 \\ x_2 + x_3 = 0 \end{array}\right\}$

$5x_2 + 5x_3 = 0$

In echelon form there are only two equations in three unknowns. Hence the system has a non-zero solution. Here x_3 is a free variable. Let $x_3 = -1$, then $x_2 = 1$ and $x_1 = 3$. Therefore, $X = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = 4$.

If $\lambda = 1$, equation (1) becomes

$$\begin{bmatrix} -3 & -6 & -6 \\ -1 & -2 & -2 \\ 1 & 4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{or, } \left.\begin{array}{r} -3x_1 - 6x_2 - 6x_3 = 0 \\ -x_1 - 2x_2 - 2x_3 = 0 \\ x_1 + 4x_2 + 4x_3 = 0 \end{array}\right\}$$

Reduce the system to echelon form by elementary transformations. We multiply 2nd eqn by 3 and then subtract from 1st eqn. Also we multiply 2nd eqn by (–1).

Then we have the equivalent system $\left.\begin{array}{r} x_1 + 2x_2 + 2x_3 = 0 \\ x_1 + 4x_2 + 4x_3 = 0 \end{array}\right\}$

We subtract 1st eqn from 2nd eqn. Then we have the equivalent system.

$$\text{or, } \left.\begin{array}{r} x_1 + 2x_2 + 2x_3 = 0 \\ 2x_2 + 2x_3 = 0 \end{array}\right\} \Rightarrow \begin{array}{r} x_1 + 2x_2 + 2x_3 = 0 \\ x_2 + x_3 = 0 \end{array}$$

$$\text{or, } \left.\begin{array}{r} x_1 = 0 \\ x_2 + x_3 = 0 \end{array}\right\} \text{ which is in echelon form.}$$

Since in echelon form there are two equations in three unknowns, the system has non-zero solutions. Here x_3 is a free variable.

Let $x_3 = -1$, then $x_2 = 1$.

Therefore, $X = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = 1$.

If $\lambda = -1$, equation (1) becomes

$$\begin{bmatrix} -5 & -6 & -6 \\ -1 & -4 & -2 \\ 1 & 4 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{or, } \left.\begin{aligned} -5x_1 - 6x_2 - 6x_3 &= 0 \\ -x_1 - 4x_2 - 2x_3 &= 0 \\ x_1 + 4x_2 + 2x_3 &= 0 \end{aligned}\right\}$$

Reduce the system to echelon form by elementary transformations. we add 3rd eqn with 2nd eqn. Also we multiply 1st equation by -1.

Then we have the equivalent system $\left.\begin{aligned} 5x_1 + 6x_2 + 6x_3 &= 0 \\ x_1 + 4x_2 + 2x_3 &= 0 \end{aligned}\right\}$

or, $\left.\begin{aligned} 5x_1 + 6x_2 + 6x_3 &= 0 \\ -14x_2 - 4x_3 &= 0 \end{aligned}\right\}$ or, $\left.\begin{aligned} 5x_1 + 6x_2 + 6x_3 &= 0 \\ 7x_2 + 2x_3 &= 0 \end{aligned}\right\}$

Since in echelon form there are two equations in three unknowns, the system has a non-zero solution. Here x_3 is a free variable,

Let $x_3 = -7$, then $x_2 = 2$ and $x_1 = 6$.

Therefore, $X = \begin{bmatrix} 6 \\ 2 \\ -7 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = -1$.

Let us take $P = \begin{bmatrix} 3 & 0 & 6 \\ 1 & 1 & 2 \\ -1 & -1 & -7 \end{bmatrix}$

Now one can easily find that $P^{-1} = -\frac{1}{15}\begin{bmatrix} -5 & -6 & -6 \\ 5 & -15 & 0 \\ 0 & 3 & 3 \end{bmatrix}$

$$P^{-1}AP = -\frac{1}{15}\begin{bmatrix} -5 & -6 & -6 \\ 5 & -15 & 0 \\ 0 & 3 & 3 \end{bmatrix}\begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}\begin{bmatrix} 3 & 0 & 6 \\ 1 & 1 & 2 \\ -1 & -1 & -7 \end{bmatrix}$$

$$= -\frac{1}{15}\begin{bmatrix} -20 & -24 & -24 \\ 5 & -15 & 0 \\ 0 & -3 & -3 \end{bmatrix}\begin{bmatrix} 3 & 0 & 6 \\ 1 & 1 & 2 \\ -1 & -1 & -7 \end{bmatrix}$$

$$= -\frac{1}{15}\begin{bmatrix} -60 & 0 & 0 \\ 0 & -15 & 0 \\ 0 & 0 & 15 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = D$$

which is the diagonal matrix of the eigenvalues of the matrix A. Hence P is the required matrix that diagonalizes the given matrix A.

11. (a) Find a matrix P that diagonalizes the matrix

$$A = \begin{bmatrix} -14 & 12 \\ -20 & 17 \end{bmatrix}$$ and find $P^{-1}AP$.

Answer : $P = \begin{bmatrix} \frac{4}{5} & \frac{3}{4} \\ 1 & 1 \end{bmatrix}$, $P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

(b) Find a matrix P that diagonalizes the matrix

$$A = \begin{bmatrix} -1 & -2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$ and find $P^{-1}AP$.

Answer : $P = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix}$, $P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.