Lecture: 09

Subspace

Defn: Let, V be a vector space. Then a non-empty subset W of V is called a subspace of V iff —

i) If $\underline{u}, \underline{v} \in W$, then $\underline{u} + \underline{v} \in W$

ii) If K is a scalar & $\underline{u} \in W$, then $K\underline{u} \in W$.

So, W is subspace if it is non-empty, closed under addition & closed under multiplication.

Q: $W \subset \mathbb{R}^n$, $W = \{ (x,y) \mid x \geq 0, y \geq 0 \}$

Is W a subspace of $\mathbb{R}^2$

$\Rightarrow$ No, because, W is not closed under scalar multiplication. as, for e.g.,

$\underline{u} = (1,1) \in W$

but $(-1)\underline{u} = (-1,-1) \notin W$.

Q: Let, M_{nn} be the vector space of $n \times n$ matrices & W be the subset of M_{nn} consisting of all invertible $n \times n$ matrices. Is W a subspace?

$\Rightarrow$ No, for eg. $U = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$, $V = \begin{pmatrix} -1 & 2 \\ -2 & 5 \end{pmatrix} \in W$

since both are invertible (because $|U| \neq 0, |V| \neq 0$)

but, $U+V = \begin{pmatrix} 0 & 4 \\ 0 & 10 \end{pmatrix}$

$$|U+V| = 0$$

So, $U+V$ is not invertible & hence $U+V \notin W$

So, W is not closed under addition.

So, W is not a subspace of M_{nn}.

Q: Let, $F(-\infty, \infty)$ be the vector space of all fu^n s.

Then, the set of all polynomials

$$W = \{ p(x) = a_0 + a_1 x + \cdots + a_n x^n \}$$

is a subspace of $F(-\infty, \infty)$. because sum of two pol^n s is a pol^n and a constant times a pol^n is also a pol^n.

Q: The set of polynomials of degree ~~...~~ n. is not a subspace of $F(-\infty, \infty)$.

for eg., $P_1 = 1 + 2x + 3x^2$
& $P_2 = 5 + 7x - 3x^2$ are both pol^n of degree 2 but $P_1 + P_2 = 6 + 9x$ is not a pol^n of degree 2.

P-200 (Exercise set 4.2)

(1) Is the following subspace of $\mathbb{R}^3$?

(a) all vectors of the form $(a, 1, 1)$.

$\Rightarrow$ let, $W = \{(a,1,1) \in \mathbb{R}^3\}$

$\underline{u} = (a, 1, 1) \in W$

$\underline{v} = (b, 1, 1) \in W$

Then, $\underline{u} + \underline{v} = (a+b, 2, 2) \notin W$

$\therefore$ not subspace.

(c) $W = \{(a, b, c) \in \mathbb{R}^3 \mid b = a + c\}$

clearly, $W \neq \phi$ since $(1, 2, 1) \in W$ $\left(\overset{b}{2} = \overset{a}{1} + \overset{c}{1}\right)$

let, $\underline{u} = (a_1, b_1, c_1) \in W$; $b_1 = a_1 + c_1$

$\underline{v} = (a_2, b_2, c_2) \in W$; $b_2 = a_2 + c_2$

Then, $\underline{u} + \underline{v} = (a_1 + a_2, b_1 + b_2, c_1 + c_2)$

where, $b_1 + b_2 = (a_1 + c_1) + (a_2 + c_2)$

$= (a_1 + a_2) + (c_1 + c_2)$

So, $\underline{u} + \underline{v} \in W$.

Again, let, $k \in \mathbb{R}$, $\underline{u} \in W$.

$k\underline{u} = (ka_1, kb_1, kc_1)$; where, $kb_1 = k(a_1 + c_1)$

$= ka_1 + kc_1$

So, $k\underline{u} \in W$.

W is subspace.

* $W = \{(a, b, c) \in \mathbb{R}^3 \mid b = a + c + 1\}$

clearly, $W \neq \varphi$ since $(1, 3, 1) \in W$ $(\because 3 = 1 + 1 + 1)$

let, $\underline{u} = (a_1, b_1, c_1) \in W$

$\underline{v} = (a_2, b_2, c_2) \in W$; $b_1 = a_1 + c_1 + 1$

$b_2 = a_2 + c_2 + 1$

$\therefore \underline{u} + \underline{v} = (a_1 + a_2, b_1 + b_2, c_1 + c_2)$

where $b_1 + b_2 = (a_1 + c_1 + 1) + (a_2 + c_2 + 1)$

$= a_1 + a_2 + c_1 + c_2 + 2$

So, $\underline{u} + \underline{v} \notin W$

$\therefore$ W is not subspace.

* **Q.** $V = \mathbb{R}^3$, $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x + 2y + 3z = 5 \right\}$

and $F = \mathbb{R}$.

clearly, $W \neq \varphi$ as, $(0, 1, 1) \in W$ since $0 + 2 \cdot 1 + 3 \cdot 1 = 5$

$\Rightarrow$ let, ~~$\underline{u} = (x_1, y_1,$~~

let, $\underline{u} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \in W$

$\underline{v} = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \in W$

where,

$x_1 + 2y_1 + 3z_1 = 5$

$x_2 + 2y_2 + 3z_2 = 5$

Now,

$$\underline{u}+\underline{v}=\begin{pmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \end{pmatrix}=\begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} \text{ (say)}$$

thus $x_3+2y_3+3z_3$

$= (x_1+x_2)+2(y_1+y_2)+3(z_1+z_2)$

$= (x_1+2y_1+3z_1)+(x_2+2y_2+3z_2)$

$= 5+5$

$= 10 \neq 5$

So, $\underline{u}+\underline{v} \notin W$.

W is not subspace.

Q. let, $V=M^{2\times 2}$, field $F=\mathbb{R}$. with usual addition & scalar multiplication. consider a subset containing all the 2×2 symmetric matrices. Is this set a subspace of V?

$\Rightarrow$ ~~let A =~~ $\cancel{\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{11} \end{pmatrix}}$ first, the subset is non-empty as, the 2×2 zero matrix is symmetric ie $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ belongs to this subset.

Now,

let, A & B be two symmetric matrices.

Then, A & B are of the ~~fol~~ form -

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{11} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{12} & b_{11} \end{pmatrix}$$

clearly, $A^T = A$,

$B^T = B$

Now, $$A+B = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{12}+b_{12} & a_1+b_{11} \end{pmatrix}$$

$$(A+B)^T = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{12}+b_{12} & a_{11}+b_{11} \end{pmatrix} = A+B.$$

So, A+B is symmetric.

Again, for any scalar k :-

$$kA = \begin{pmatrix} ka_{11} & ka_{12} \\ ka_{12} & ka_{11} \end{pmatrix}$$

$$(kA)^T = kA.$$

So, subspace.

Ans

Linear Independence

Def^n A vector $\underline{\omega} \in V$ is called a linear combination of the vectors $\underline{v_1}, \underline{v_2}, \ldots \underline{v_n} \in V$ if –

$$\underline{\omega} = k_1 \underline{v_1} + k_2 \underline{v_2} + \cdots + k_n \underline{v_n}$$

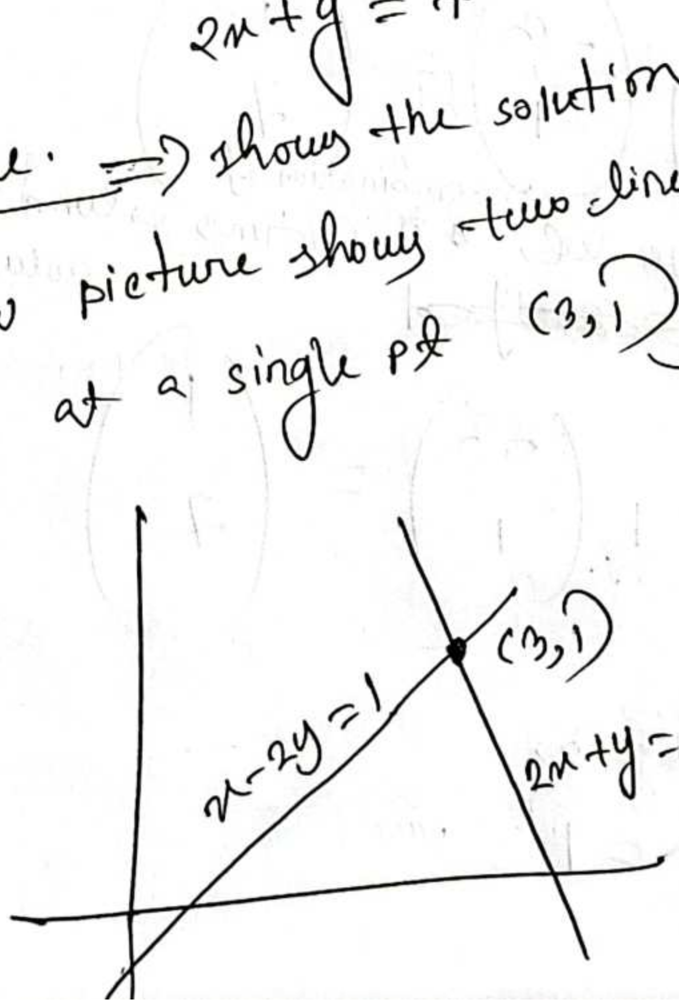
where $k_1, k_2, \ldots k_n$ are scalars, called the co-efficients.

Def^n Row & Column Picture. $A x A$

consider the system ⇒

$$x - 2y = 1$$
$$2x + y = 7.$$

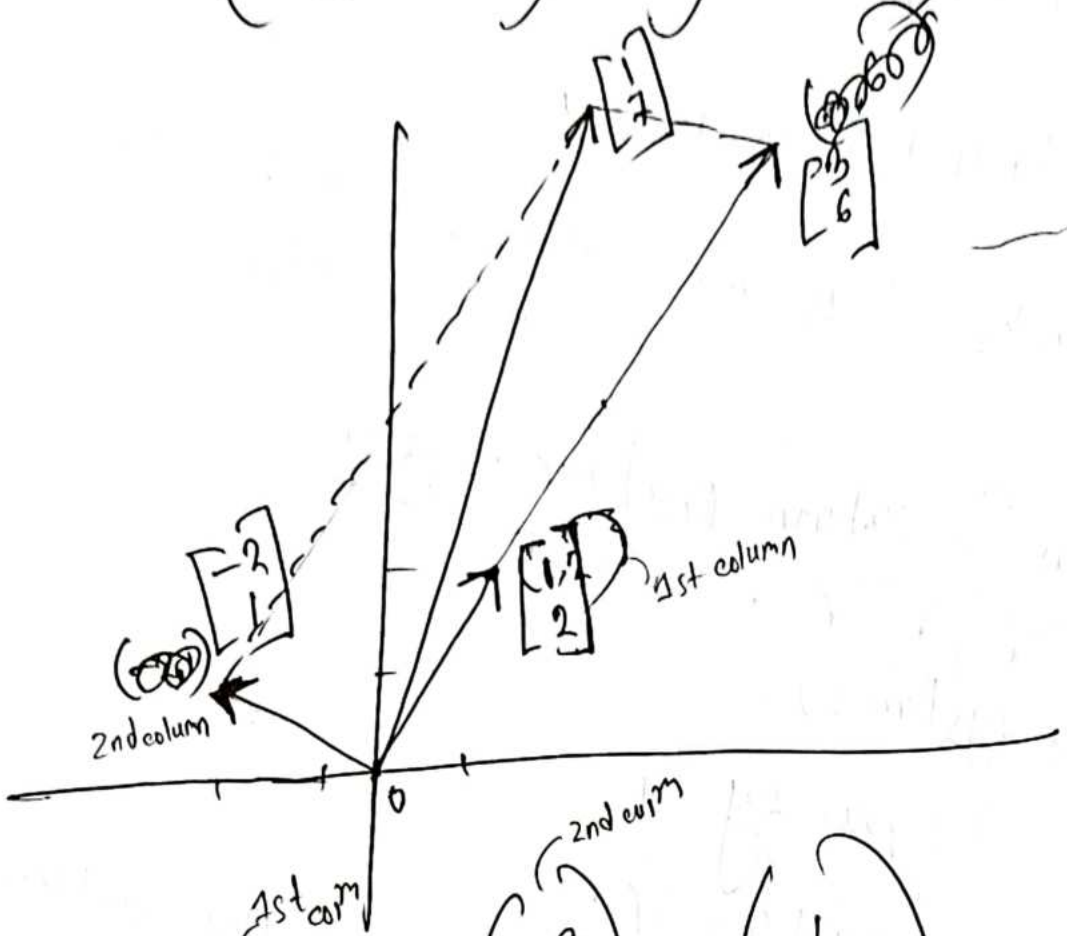
Row picture. ⇒ shows the solution of the system (in this e.g.) ie row picture shows two lines meeting at a single pt $(3,1)$ sol^n



Column Picture

In matrix form,

$$\begin{pmatrix}1 & -2\\ 2 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}1\\ 7\end{pmatrix}$$



$$x\overset{\text{1st colm}}{\begin{pmatrix}1\\ 2\end{pmatrix}}+y\overset{\text{2nd colm}}{\begin{pmatrix}-2\\ 1\end{pmatrix}}=\begin{pmatrix}1\\ 7\end{pmatrix}$$

~~We see that,~~ we see, combination of 3 times column 1 & 1 times column 2 giving the vector $\begin{pmatrix}1\\ 7\end{pmatrix}$

i.e.

$$3\begin{pmatrix}1\\ 2\end{pmatrix}+1\begin{pmatrix}-2\\ 1\end{pmatrix}=\begin{pmatrix}1\\ 7\end{pmatrix}$$

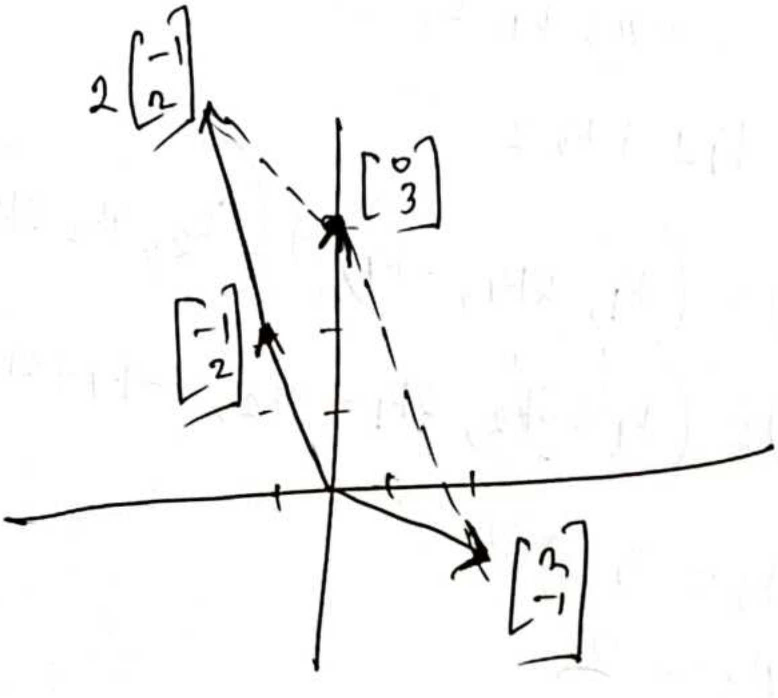
So, $x=3$,
$y=1$.

Ex: Draw column picture for

$$2x - y = 0$$
$$-x + 2y = 3$$

Sol^n Here,

$$x\begin{pmatrix}2\\-1\end{pmatrix} + y\begin{pmatrix}-1\\2\end{pmatrix} = \begin{pmatrix}0\\3\end{pmatrix}.$$



~~So, $x = 2$, $y = 1$~~

We see that a combination of $\frac{1}{2}$ times column 1 & 2 ~~1~~ times column 2 gives the vector $\begin{pmatrix}0\\3\end{pmatrix}$.

So, $x = 1$, $y = 2$.

Ex. $\underline{u} = (1, 2, -1)$, $\underline{v} = (6, 4, 2) \in \mathbb{R}^3$

show — i) $\underline{\omega} = (9, 2, 7)$ is a linear combination of $\underline{u}$ & $\underline{v}$

ii) $\underline{\omega}' = (4, -1, 8)$ is not a linear combination of $\underline{u}$ & $\underline{v}$.

Sol^n i). If $\underline{\omega}$ is a L.C of $\underline{u}$ & $\underline{v}$, then, there must exist scalars k_1, k_2 such that

$$\omega = k_1 \underline{u} + k_2 \underline{v}$$

$$\Rightarrow (9, 2, 7) = (k_1, 2k_1, -k_1) + (6k_2, 4k_2, 2k_2)$$

$$\Rightarrow (9, 2, 7) = (k_1 + 6k_2, 2k_1 + 4k_2, -k_1 + 2k_2)$$

$$\begin{aligned} k_1 + 6k_2 &= 9 \\ 2k_1 + 4k_2 &= 2 \\ -k_1 + 2k_2 &= 7. \end{aligned}$$

$$\left(\begin{array}{cc|c} 1 & 6 & 9 \\ 2 & 4 & 2 \\ -1 & 2 & 7 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|c} 1 & 6 & 9 \\ 0 & -8 & -16 \\ 0 & 8 & 16 \end{array}\right) ; \quad \begin{aligned} r_2' &= r_2 - 2r_1 \\ r_3' &= r_3 + r_1 \end{aligned}$$

$$\sim \left(\begin{array}{cc|c} 1 & 6 & 9 \\ 0 & -8 & -16 \\ 0 & 0 & 0 \end{array}\right) ; r_3' = r_3 + r_2$$

$$\sim \left(\begin{array}{cc|c} 1 & 6 & 9 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right) ; r_2' = r_2 \times \left(-\tfrac{1}{8}\right)$$

$$k_1 + 6k_2 = 9 \quad \text{--- (1)}$$

$$\therefore \; k_2 = 2$$

$$(1) \Rightarrow k_1 = 9 - 6k_2 = 9 - (6 \times 2) = 9 - 12 = -3$$

$$\therefore \; \underline{w} = -3\underline{u} + 2\underline{v}.$$ Ans

(ii) w, $w' = k_1\underline{u} + k_2\underline{v}$; k_1, k_2 are scalars.

$$\Rightarrow (4, -1, 8) = (k_1 + 6k_2, \; 2k_1 + 4k_2, \; -k_1 + 2k_2)$$

$$\therefore \; \begin{aligned} k_1 + 6k_2 &= 4 \\ 2k_1 + 4k_2 &= -1 \\ -k_1 + 2k_2 &= 8 \end{aligned}$$

$$\left(\begin{array}{cc|c} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 8 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 6 & 4 \\ 0 & -8 & -9 \\ 0 & 8 & 12 \end{array}\right) ; \begin{aligned} r_2' &= r_2 - 2r_1 \\ r_3' &= r_3 + r_1 \end{aligned}$$

$$\sim \left(\begin{array}{cc|c} 1 & 6 & 4 \\ 0 & -8 & -9 \\ 0 & 0 & 3 \end{array}\right) ; \; r_3' = r_3 + r_2$$

The last row $\Rightarrow 0 = 3$ (not possible)

So, the system is inconsistent.

Span.

A set of vectors $S = \{v_1, v_2, \ldots v_n\}$ is said to span a vector space V if each vector in V can be written as a linear combination of the vectors in S.

Ex. Determine if $v_1 = (1, 1, 2)$, $v_2 = (1, 0, 1)$, $v_3 = (2, 1, 3)$ span $\mathbb{R}^3$.

$\Rightarrow$ Let $\underline{b} = (b_1, b_2, b_3) \in \mathbb{R}^3$ be arbitrary.

Now, $\underline{b} = k_1 v_1 + k_2 v_2 + k_3 v_3$; k_i are scalars.

$\Rightarrow (b_1, b_2, b_3) = (k_1 + k_2 + 2k_3,\; k_1 + k_3,\; 2k_1 + k_2 + 3k_3)$

$$\left.\begin{aligned} k_1 + k_2 + 2k_3 &= b_1 \\ k_1 + \phantom{k_2 +{}} k_3 &= b_2 \\ 2k_1 + k_2 + 3k_3 &= b_3 \end{aligned}\right\} \longleftarrow (*)$$

b can be written as a linear combination of v_1, v_2, v_3 iff the system (A) is consistent.

i.e. iff the co-efficient matrix

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{pmatrix}$$

has non-zero determinant.

We see that, $|A| = 1(0-1) - 1(3-2) + 2(1-0)$

$= -1 - 1 + 2$

$= 0.$

So, the system (A) is not consistent.

So, v_1, v_2, v_3 do not span $\mathbb{R}^3$. Ans

Def^n (Linearly independent):

A non-empty set $S = \{v_1, v_2, \dots v_n\}$ is called linearly independent iff the only co-efficients satisfying $k_1 v_1 + \dots + k_n v_n = 0$ are $k_1 = 0, k_2 = 0, \dots k_n = 0$.

[illegible]

Ex. Determine whether $v_1 = (1,-2,3)$, $v_2 = (5,6,-1)$, $v_3 = (3,2,1)$ are linearly independent in $\mathbb{R}^3$.

Soln. Let, $k_1 v_1 + k_2 v_2 + k_3 v_3 = 0$

$$\Rightarrow (k_1, -2k_1, 3k_1) + (5k_2, 6k_2, -k_2) + (3k_3, 2k_3, k_3) = (0,0,0)$$

$$\therefore \left.\begin{aligned} k_1 + 5k_2 + 3k_3 &= 0 \\ -2k_1 + 6k_2 + 2k_3 &= 0 \\ 3k_1 - k_2 + k_3 &= 0 \end{aligned}\right\} \text{---} (*)$$

$$\left(\begin{array}{ccc|c} 1 & 5 & 3 & 0 \\ -2 & 6 & 2 & 0 \\ 3 & -1 & 1 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 5 & 3 & 0 \\ 0 & 16 & 8 & 0 \\ 0 & -16 & -8 & 0 \end{array}\right) ; \; r_2' = r_2 + 2r_1, \; r_3' = r_3 - 3r_1$$

$$\sim \left(\begin{array}{ccc|c} 1 & 5 & 3 & 0 \\ 0 & 16 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) ; \; r_3' = r_3 + r_2$$

$$\sim \left(\begin{array}{ccc|c} 1 & 5 & 3 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) ; \; r_2' = r_2 \times \frac{1}{16}.$$

$\therefore$ $k_1 + 5k_2 + 3k_3 = 0$

$k_2 + \frac{1}{2}k_3 = 0$

$k_3 =$ free variable

let, $k_3 = t$; $t \in \mathbb{R}$

$\therefore$ $k_2 = -t/2$

$k_1 = -t/2$

This shows that the system (**) has non-trivial soln. So, v_1, v_2, v_3 are NOT linearly independent.

Ex: P-205, Ex: 3.

~~Sho~~ Check, $v_1 = (1, 2, 2, -1)$, $v_2 = (4, 9, 9, -4)$,

$v_3 = (5, 8, 9, -5)$ are L.I or not

Soln. [do yourself]

Linear Transformation

Def^n: A mapping $T: V \longrightarrow W$ from a vector space V to W is called a linear transformation if the following two properties hold for all $\underline{u}, \underline{v} \in V$ & for all scalars k -

i) $T(\underline{u}+\underline{v}) = T(\underline{u}) + T(\underline{v})$

ii) $T(k\underline{u}) = k\,T(\underline{u})$.

If $V = W$, then, T is called a linear operator.

Ex $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$T(x_1, x_2, x_3) = (x_1, x_2, 0)$

Is T a linear transformation?

1) let, $\underline{u} = (u_1, u_2, u_3)$, $\underline{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$

$\underline{u}+\underline{v} = (u_1+v_1, u_2+v_2, u_3+v_3)$

$\therefore T(\underline{u}+\underline{v}) = T(u_1+v_1, u_2+v_2, u_3+v_3)$

$= (u_1+v_1, u_2+v_2, 0)$

$= (u_1, u_2, 0) + (v_1, v_2, 0) = T(\underline{u}) + T(\underline{v})$

Again, $k\underline{u} = (ku_1, ku_2, ku_3)$

$$T(k\underline{u}) = T(ku_1, ku_2, ku_3)$$
$$= (ku_1, ku_2, 0)$$
$$= k(u_1, u_2, 0)$$
$$= k\,T(\underline{u}).$$

$\therefore$ T is a. L.T. Ans

Ex. $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$T(x_1, x_2) = (x_1^2, x_1 + x_2)$$

Sol^n. $\underline{u} = (u_1, u_2),\ \underline{v} = (v_1, v_2) \in \mathbb{R}^2$

$$T(\underline{u} + \underline{v}) = T(u_1 + v_1, u_2 + v_2)$$
$$= \left((u_1 + v_1)^2, u_1 + v_1 + u_2 + v_2\right)$$
$$= \left(u_1^2 + 2u_1v_1 + v_1^2, u_1 + u_2 + v_1 + v_2\right)$$

$$T(\underline{u}) + T(\underline{v}) = T(u_1, u_2) + T(v_1, v_2)$$
$$= (u_1^2, u_1 + u_2) + (v_1^2, v_1 + v_2)$$
$$= (u_1^2 + v_1^2, u_1 + u_2 + v_1 + v_2)$$

$\therefore T(\underline{u} + \underline{v}) \neq T(\underline{u}) + T(\underline{v})$. $\therefore$ T is not a L.T.

Ans