Assignment – 02

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Course Title : Integral Calculus & Differential Equations

Course Code : MAT120

Section : 17

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Answer to the Q. NO - 01 (a)

Given,

$$\int \frac{x^4+6x^3+10x^2+x}{x^2+6x+10}\,dx$$

$$\Rightarrow \quad x^2+6x+10 \overline{\big)\, x^4+6x^3+10x^2+x\,}\big(\, x^2$$
$$\underline{x^4+6x^3+10x^2}$$
$$x$$

Now,

$$\int \frac{x^4+6x^3+10x^2+x}{x^2+6x+10} = \int\left\{\frac{x}{x^2+6x+10}\right\}dx$$

$$\therefore \int \frac{x}{x^2+6x+10} = \int \frac{x}{x^2+6x+9+1}$$

$$= \int \frac{x}{(x+3)^2+1}$$

$$= \int \frac{u-3}{u^2+1}\,du$$

Let,
$u = x+3$
$\Rightarrow du = dx$
$\Rightarrow u-3 = x$

$$= \int (x-3)\cdot\frac{1}{u^2+1}\cdot du \int \frac{u}{u^2+1}\,dx - 3\int\frac{1}{u^2+1}$$

$$= \frac{1}{2}\ln\left|x^2+1\right| - 3\tan^{-1}|x| + c$$

$$= \frac{1}{2}\ln\left|x^2+6x+10\right| - 3\tan^{-1}|x+3| + c$$

$$\therefore \int x^2 + \frac{x}{x^2+6x+10} = \frac{1}{3}x^3 + \frac{1}{2}\ln\left|x^2+6x+10\right| - 3\tan^{-1}(x+3) + c$$

(Ans:)

Answer to the Q. NO - 1(b)

Given,

$$\int \frac{2x^3-4x-8}{x^4-x^3+4x^2-4x}\,dx$$

$$= \int \frac{2x^3-4x-8}{x^3(x-1)+4x(x-1)}\,dx$$

$$= \int \frac{2x^3-4x-8}{(x^3+4x)(x-1)}\,dx$$

$$= \int \frac{2x^3-4x-8}{x(x^2+4)(x-1)}\,dx$$

$$\therefore \frac{2x^3-4x-8}{x(x^2+4)(x-1)} \equiv \frac{A}{x}+\frac{B}{x-1}+\frac{Cx+D}{x^2+4}$$

$$\Rightarrow 2x^3-4x-8 = A(x-1)(x^2+4)+Bx(x^2+4)+(Cx+D)(x-1)x$$

$$= Ax^3-Ax^2+4Ax-4A+Bx^3+4Bx+Cx^3+Dx^2-Cx^2-Dx$$

$$= x^3(A+B+C)+x^2(-A-C+D)+x(4A+4B-D)+(-4A)$$

Here,

$$A+B+C=2$$
$$-A-C+D=0$$
$$4A+4B-D=-4$$
$$-4A=-8$$
$$\Rightarrow A=2$$

Again,
$$B+C=0$$
$$-C+D=2$$
$$4B-D=-12$$

$\therefore$ Getting values for $A, B, C, D \Rightarrow$

$$A=2$$
$$B=-2$$
$$C=2$$
$$D=-4$$

$$\therefore \int \frac{2x^3 - 4x - 8}{x^4 - x^3 + 4x^2 - 4x}\,dx = \int \frac{2}{x}\,dx + \int \frac{(-2)}{x-1}\,dx + \int \frac{2x+4}{x^2+4}\,dx$$

$$= 2\ln|x| - 2\ln|x-1| + \int \frac{2x}{x^2+4}\,dx + \int \frac{4}{x^2+4}\,dx$$

$$= 2\ln|x| - 2\ln|x-1| + \ln|u|$$
$$+ \int \frac{2\times 4\sec^2\theta}{4\sec^2\theta}\cdot d\theta$$

let,
$u = x^2+4$
$\Rightarrow du = 2x\,dx$

$x = \tan\theta$
$\Rightarrow dx = 2\sec^2\theta \cdot d\theta$
$\therefore \theta = \tan^{-1}\left(\frac{x}{2}\right)$

$$= 2\ln|x| - 2\ln|x-1| + \ln|x^2+4| + 2\int 1 \cdot d\theta$$

$$= 2\ln|x| - 2\ln|x-1| + \ln|x^2+4| + 2\tan^{-1}\left(\frac{x}{2}\right) + C$$

(Ans:)

Answer to the Q. NO- 01 (c)

Given,
$$\int_{-\infty}^{0} \frac{e^{\frac{1}{x}}}{x^{2}}\, dx$$

$$= \lim_{K \to -\infty} \int_{K}^{0} \frac{e^{1/x}}{x^{2}}\, dx$$

$$= \lim_{K \to -\infty} - \int_{K}^{0} e^{u}.\, du$$

$$= \lim_{K \to -\infty} - \left[e^{u} \right]_{K}^{0}$$

$$= \lim_{K \to -\infty} \left[-e^{1/x} + e^{1/x} \right]_{K}^{0}$$

$= 1$

(Ans:)

Answer to the Q. NO- 01 (d)

Given,

$$\int_0^{\pi/6} \frac{\cos x}{\sqrt{1-2\sin x}} dx$$

let,

$u = 1 - 2\sin x \, dx$

$\Rightarrow du = -2\cos x \, dx$

$\Rightarrow -\frac{du}{2} = \cos x \cdot dx$

$$= -\frac{1}{2}\int_0^{\pi/6} \frac{1}{\sqrt{u}} \cdot du$$

$$= -\frac{1}{2}\int_0^{K} \frac{1}{\sqrt{u}} \cdot du$$

$$= -\left[\sqrt{1-2\sin x}\right]_0^K$$

$$= -\sqrt{1-2\sin\frac{\pi}{6}} + \sqrt{1-2\sin 0}$$

$$= -\sqrt{1-1} + \sqrt{1}$$

$= 1$

(Ans:)

Answer to the Q. NO - 1 (e)

Given,

$$\int_0^1 \frac{1}{x \ln\left(\frac{1}{x}\right)} dx$$

$$= \int_0^1 x^{-1/2} \cdot \ln\left(\frac{1}{x}\right)^{-\frac{1}{2}} \cdot dx$$

$$= -\int_\infty^0 e^{z/2} \cdot z^{-1/2} \cdot e^{-z} \cdot dz$$

$$= \int_0^\infty e^{-z/2} \cdot z^{-1/2} \cdot dz$$

$$= \int_0^\infty e^{-y} \cdot (2y)^{-1/2} \cdot 2 \cdot dy$$

$$= 2 \cdot 2^{-1/2} \int_0^\infty e^{-y} \cdot y^{-1/2} \cdot dy$$

$$= \sqrt{2}\, \Gamma\frac{1}{2}$$

$$= \sqrt{2} \cdot \sqrt{\pi}$$

$$= \sqrt{2\pi}$$

(Ans:)

Let,

$\ln \frac{1}{x} = z$

$\Rightarrow \ln 1 - \ln x = z$

$\Rightarrow \ln x = -z$

$\Rightarrow \log x = -z$

$\Rightarrow e^{-z} = x$

$\Rightarrow -e^{-z} \cdot dz = dx$

Now,

if, $x = 1$, $z = 0$

$x = 0$, $z = \infty$

Again,

$z/2 = y$

$\Rightarrow z = 2y$

$\Rightarrow dz = 2dy$

Now,

if, $z = \infty$, $y = \infty$

$z = 0$, $y = 0$

Answer to the Q. NO- 1(f)

Given,

$$\int_0^\infty e^{-x^2}.\,dx$$

$$= \frac{1}{2}\int_0^\infty e^{-z}.\, z^{-1/2}.\, dz$$

$$= \frac{1}{2}\int_0^\infty z^{1/2-1}.\, e^{-z}.\, dz$$

$$= \frac{1}{2}\cdot \Gamma\frac{1}{2}$$

$$= \sqrt{\pi}/2$$

$$\therefore \int_0^\infty e^{-x^2}.\, dx = \frac{\sqrt{\pi}}{2}$$

(Ans:)

let,

$x^2 = z$

$\Rightarrow dx = \frac{1}{2} z^{-1/2}.\, dz$

Now,

if,

$x = \infty,\ z = \infty$

$x = 0,\ z = 0$

Answer to the Q. NO- 01 (g)

Given function,

$$\int_{-3}^{1} \frac{1}{\omega^2 + 2\omega}\, d\omega$$

Here,

$$\frac{1}{\omega^2+2\omega} = \frac{1}{\omega(\omega+2)}$$

$$\therefore \frac{1}{\omega(\omega+2)} = \frac{A}{\omega} + \frac{B}{\omega+2}$$

$$= A(\omega+2) + B\omega$$

Now,

$$1 = A(0+2) + 0$$

$$\Rightarrow A = 1/2$$

$$1 = 0 + B(-2)$$

$$\Rightarrow B = -1/2$$

$$\therefore \int \frac{1}{\omega^2+2\omega}\, d\omega = \int \frac{1/2}{d\omega} - \int \frac{1/2}{\omega+2}\, d\omega$$

$$= \frac{1}{2} \ln|\omega| - \frac{1}{2} \ln|\omega+2| + c$$

$$= \left[\frac{1}{2} \ln|\omega| - \frac{1}{2} \ln|\omega+2|\right]_{-3}^{1}$$

$$= \left(\frac{1}{2} \ln|1| - \frac{1}{2} \ln|3|\right) - \left(\frac{1}{2} \ln|-3| - \frac{1}{2} \ln|-1|\right)$$

[P.T.O.]

$$= \left\{\frac{1}{2}\cdot 0 - \frac{1}{2}\ln(3)\right\} - \left\{-\frac{1}{2}\ln(3) - \frac{1}{2}\cdot 0\right\}$$

$$= \frac{1}{2}\ln(3) - \frac{1}{2}\ln(3)$$

$$= 0$$

$$\therefore \int_{-3}^{1} \frac{1}{\omega^2 + 2\omega}\, d\omega = 0$$

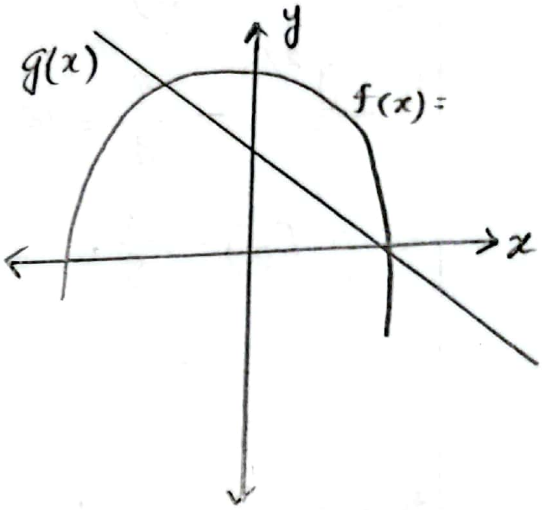
(Ans:)

Answer to the Q. NO - 2

Given,

$f(x) = 9 - \left(\frac{x}{2}\right)^2$

$g(x) = 6 - x$



Now,

$6 - x = 9 - \frac{x^2}{4}$

$\Rightarrow x^2 - 4x - 9 \times 4 + 4 \times 6 = 0$

$\Rightarrow x^2 - 4x - 12 = 0$

$\Rightarrow x = 6, -2$

$\therefore$ the area of the region, $R = \int_{-2}^{6} \left(9 - \frac{x^2}{4} - 6 + x\right) dx$

$= \left[9x - \frac{x^3}{12} - 6x + \frac{x^2}{2}\right]_{-2}^{6}$

$= 18 + \frac{10}{3}$

$= \frac{64}{3} (\text{unit})^2$

(Ans:)

Answer to the Q. NO − 03

$$y = (8x+3)^{3/2}$$

$$\Rightarrow \frac{y^{2/3}}{8} - \frac{3}{8} = x$$

$$\Rightarrow g'(y) = \frac{1}{8} \cdot \frac{2}{3} y^{-1/3}$$

$$= \frac{1}{12\sqrt[3]{y}}$$

$$\Rightarrow (g'(y))^2 = \frac{1}{144\sqrt[3]{y^2}}$$

Given interval,
$11^{3/2} \leq y \leq 27^{3/2}$
with respect to y-axis.

$$\therefore \text{exact length, } L = \int \sqrt{1 + \frac{1}{144\sqrt[3]{y^2}}}\, dy$$

$$= \int \sqrt{1 + \frac{1}{144 y^{2/3}}}\, dy$$

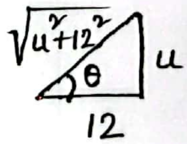
$$= \frac{1}{12} \int \sqrt{144 + (y^{1/3})^2}\, dy$$

$$= \frac{1}{12} \int \sqrt{(12)^2 + (u)^2}\, (-3) u^{-4}\, du$$

$$= \frac{1}{12} \times (-3) \int \frac{\sqrt{(12)^2 + (u)^2}}{u^4} . du$$

$$= \frac{1}{12} \times (-3) \int \frac{\sqrt{(12)^2 + (12\tan\theta)^2}}{12^4 \tan^4\theta}\, d\theta$$

let,
$u = y^{-1/3} \mid y = u^{-3}$
$\Rightarrow -3u^{-4} du = dy$



let,
$u = 12 \tan\theta$
$\Rightarrow du = 12 \sec^2\theta .\, d\theta$
$\Rightarrow \sin\theta = \frac{u}{\sqrt{u^2+144}}$

[P.T.O.]

$$= -3 \times \frac{1}{12} \times 12 \int \frac{\sec\theta \cdot \sec^2\theta}{\tan^4\theta} \cdot \frac{12}{12^4}\, d\theta$$

$$= -3 \times \frac{1}{(12)^3} \int \frac{1}{\cos^3\theta} \cdot \frac{\cos^4\theta}{\sin^4\theta} \cdot d\theta$$

let,
$\sin\theta = v$
$\Rightarrow \cos\theta \cdot d\theta = dv$

$$= -3 \times \frac{1}{(12)^3} \cdot \int \frac{\cos\theta}{\sin^4\theta} \cdot d\theta$$

$$= -3 \times \frac{1}{(12)^3} \times -\frac{1}{3v^3} + c$$

$$= \frac{1}{(12)^3 (v^3)} + c$$

$$= \frac{1}{(12)^3} \cdot \frac{\left(\sqrt{u^2+144}\right)^3}{u^3} + c$$

$$= \frac{1}{(12)^3} \cdot \frac{\left(\sqrt{(y^{-1/3})^2 + 144}\right)^3}{(y^{-1/3})^3} + c$$

$$\therefore \text{Length, } L = \left[\frac{y\left(\sqrt{y^{-2/3} + 144}\right)^3}{(12)^3} \right]_{11^{3/2}}^{27^{3/2}}$$

$$= 103.8334 \text{ unit}$$

(Ans:)

[P.T.O.]

Answer to the Q. NO - 04

Given,

$$24xy = y^4 + 48$$

$$\Rightarrow x = \frac{1}{24} y^3 + \frac{2}{y}$$

let,

$$g(y) = \frac{1}{24} y^3 - \frac{2}{y}$$

$$g'(y) = \frac{1}{8} y^2 - \frac{2}{y^2}$$

Given interval, $2 \leq y \leq 4$ with respect to y-axis.

$\therefore$ The exact length of the curve,

$$L = \int_2^4 \sqrt{1 + (g'(y))^2}\, dy$$

$$= \int_2^4 \sqrt{1 + \frac{1}{64} y^4 - \frac{1}{2} + \frac{4}{y^4}}\, dy$$

$$= \int_2^4 \sqrt{\left(\frac{1}{8} y^2 + \frac{2}{y^2}\right)^2}\, dy$$

$$= \int_2^4 \left(\frac{1}{8} y^2 + \frac{2}{y^2}\right) dy$$

$$= \left[\frac{1}{24} y^3 - \frac{2}{y}\right]_2^4$$

$$= \frac{1}{12} \times (4)^3 - \frac{2}{4} - \frac{1}{24} \times (2)^3 + \frac{2}{2}$$

$$= \frac{17}{6}$$

$$= 2.8334 \text{ units.}$$

(Ans:)

Answer to the Q. NO-05

Given,

$y = 2x^2 + 10,$

$y = 4x + 16$

$x = -2$

$x = 5$

Let,

$f(x) = 2x^2 + 10$

$g(x) = 4x + 16$

$\therefore$ The bounded area of the region,

$$A = \int_{-2}^{5} \left[f(x) - g(x) \right] dx$$

$$= \int_{-2}^{5} \left(2x^2 + 10 - 4x - 16 \right) dx$$

$$= \left[\frac{2x^3}{3} + 10x - \frac{4x^2}{2} - 16x \right]_{-2}^{5}$$

$$= \left[\frac{2}{3} x^3 - \frac{4x^2}{2} - 6x \right]_{-2}^{5}$$

$$= \left\{ \left(\frac{2 \cdot (5)^3}{3} - 2 \cdot (5)^2 - 6 \cdot 5 \right) - \left(\frac{2 \cdot (-2)^3}{3} - 2 \cdot (-2)^2 - 6 \cdot (-2) \right) \right\}$$

$$= \frac{14}{3}$$

$$= 4.67 \text{ units}^2$$

(Ans:)

Answer to the Q. NO- 06.

Given curve,

$$y = \frac{1}{4}\sqrt{6x+2}$$

over the interval $\frac{\sqrt{2}}{2} \leq y \leq \frac{\sqrt{5}}{2}$ with respect to x-axis.

Now,

$$y = \frac{1}{4}\sqrt{6x+2}$$

$$\Rightarrow (4y)^2 = 6x+2$$

$$\Rightarrow x = \frac{16y^2-2}{6}$$

$$\Rightarrow x = \frac{8}{3}y^2 - \frac{1}{3}$$

if the value,

$$y = \frac{\sqrt{2}}{2},\ x = \frac{8}{3}\cdot\left(\frac{\sqrt{2}}{2}\right)^2 - \frac{1}{3}$$

$$= 1$$

$$y = \frac{\sqrt{5}}{2},\ x = \frac{8}{3}\cdot\left(\frac{\sqrt{5}}{2}\right)^2 - \frac{1}{3}$$

$$= 3$$

Now,

$$g(x) = \frac{1}{4}\sqrt{6x+2}$$

$$\Rightarrow g(x) = \frac{\sqrt{2}}{4}\sqrt{3x+1}$$

$$\Rightarrow g(x) = \frac{1}{2\sqrt{2}}\sqrt{3x+1}$$

$$\therefore g'(x) = \frac{1}{2\sqrt{2}}\cdot\frac{1}{2}\left(-\frac{1}{\sqrt{3x+1}}\right)\cdot 3$$

$$= -\frac{3}{4\sqrt{2}(\sqrt{3x+1})}$$

$$\therefore (g'(x))^2 = \frac{9}{32(3x+1)}$$

∴ The surface area,

$$S = \int_1^3 2\pi g(x) \sqrt{1+(g'(x))^2} \cdot dx$$

$$= \int_1^3 2\pi \cdot \frac{1}{4}\sqrt{6x+2} \cdot \sqrt{1+\frac{9}{32(3x+1)}} \cdot dx$$

$$= \int_1^3 2\pi \cdot \frac{1}{4} \cdot \sqrt{(6x+2) \cdot \frac{32(3x+1)+9}{16(6x+2)}} \cdot dx$$

let,

$u = 96x + 41$

$\Rightarrow du = 96\, dx$

$\Rightarrow dx = \frac{du}{96}$

$x=3,\ u = 329$

$x=1,\ u = 137$

$$= \int_1^3 2\pi \cdot \frac{1}{4}\sqrt{\frac{96x+32+9}{16}}\, dx$$

$$= \int_1^3 \frac{\pi}{2} \cdot \frac{\sqrt{96x+41}}{4}\, dx$$

$$= \frac{\pi}{2} \cdot \frac{1}{4} \int_{137}^{329} \sqrt{u} \cdot \frac{du}{96}$$

$$= \frac{\pi}{2} \cdot \frac{1}{4} \cdot \frac{1}{96} \left[\frac{2}{3} \cdot u^{3/2}\right]_{137}^{329}$$

$$= \frac{\pi}{2} \cdot \frac{1}{4} \cdot \frac{1}{96} \left[\frac{2}{3} (329)^{3/2} - (137)^{3/2}\right]$$

$= 3 \cdot 79 \cdot \pi$ units

$= 3 \cdot 79 \times 3 \cdot 1416$ units

$= 11 \cdot 9067$ units (Ans:)