Complex Fourier series

* Using Euler's identity, prove that the complex form of Fourier series can be expressed as.

$$\sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{i n\pi x}{L}\right).$$

Solⁿ. We know, Fourier series is -

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right] \text{ --- (i)}$$

By Euler's identity, we know $\Rightarrow$

$$e^{i\theta} = \cos\theta + i\sin\theta \text{ --- (ii)}$$

$$\therefore e^{-i\theta} = \cos\theta - i\sin\theta \text{ --- (iii)}$$

(ii) + (iii) $\Rightarrow$ $e^{i\theta} + e^{-i\theta} = 2\cos\theta$

$$\therefore \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

Hence, $$\cos\left(\frac{n\pi x}{L}\right) = \frac{e^{i\frac{n\pi x}{L}} + e^{-i\frac{n\pi x}{L}}}{2} \text{ --- (4)}$$

Again, (ii) − (iii) ⇒

$$e^{i\theta} - e^{-i\theta} = 2i \sin\theta$$

$$\Rightarrow \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Hence, $$\sin\left(\frac{n\pi x}{L}\right) = \frac{e^{i\frac{n\pi x}{L}} - e^{-i\frac{n\pi x}{L}}}{2i} \quad \text{--- (5)}$$

Substituting (4) & (5) in (1); we get ⇒

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \frac{e^{\frac{in\pi x}{L}} + e^{-\frac{in\pi x}{L}}}{2} + b_n \frac{e^{i\frac{n\pi x}{L}} - e^{-\frac{in\pi x}{L}}}{2i}\right]$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left\{\frac{a_n}{2}\left(e^{i\frac{n\pi x}{L}} + e^{-\frac{in\pi x}{L}}\right) + \frac{b_n}{2i}\left(e^{\frac{in\pi x}{L}} - e^{-\frac{in\pi x}{L}}\right)\right\}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[\left(\frac{a_n}{2} + \frac{b_n}{2i}\right)e^{\frac{in\pi x}{L}} + \left(\frac{a_n}{2} - \frac{b_n}{2i}\right)e^{-\frac{in\pi x}{L}}\right] \quad \text{--- (6)}$$

Let, $c_0 = \frac{a_0}{2}$, $c_n = \frac{a_n}{2} + \frac{b_n}{2i}$

and $c_{-n} = \frac{a_n}{2} - \frac{b_n}{2i}$

Then, (6) $\Rightarrow$

$$c_0 + \sum_{n=1}^{\infty}\left[c_n e^{\frac{in\pi x}{L}} + c_{-n} e^{-\frac{in\pi x}{L}}\right]$$

$$= c_0 + \sum_{n=1}^{\infty} c_n e^{\frac{in\pi x}{L}} + \sum_{n=1}^{\infty} c_{-n} e^{-i\frac{n\pi x}{L}}$$

$$= c_0 e^{\frac{i\times 0\pi x}{L}} + \sum_{n=1}^{\infty} c_n e^{\frac{in\pi x}{L}} + \sum_{-n=1}^{-n=\infty} c_n e^{\frac{in\pi x}{L}}$$

[replacing n by $-n$ in the last sum]

$$= c_0 e^{\frac{i\cdot 0\cdot\pi x}{L}} + \sum_{n=1}^{\infty} c_n e^{\frac{in\pi x}{L}} + \sum_{n=-\infty}^{-1} c_n e^{\frac{in\pi x}{L}}$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}}$$

$$= \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{in\pi x}{L}\right).$$ [Proved]

Complex Fourier Series

or, Complex Notation / complex form of Fourier Series

* Complex Fourier series of $f(x)$ is ; $-L \le x \le L$ is,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\left(\frac{n\pi x}{L}\right)}$$

$$= c_0 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} c_n e^{i\left(\frac{n\pi x}{L}\right)}$$

where,

$$c_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\, dx$$

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{-i\left(\frac{n\pi x}{L}\right)} dx$$

Ex: 01 Find the complex Fourier series of

$$f(x) = \begin{cases} 0 ; & -\pi \le x \le 0 \\ 1 ; & 0 \le x \le \pi . \end{cases}$$

Soln Here, $L = \pi$

We know,

$$f(x) = c_0 + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} c_n e^{i\left(\frac{n\pi x}{L}\right)} \quad \text{---} (i)$$

Here

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{2\pi} \left[\int_{-\pi}^{0} f(x)\, dx + \int_{0}^{\pi} f(x)\, dx \right]$$

$$= \frac{1}{2\pi} \left[0 + \int_{0}^{\pi} dx \right]$$

$$= \frac{1}{2\pi} \left[x \right]_0^{\pi}$$

$$= \frac{1}{2\pi} \times \left(\pi - 0 \right)$$

$$= \frac{1}{2}$$

$$\boxed{c_0 = \frac{1}{2}}$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-i(nx)}\, dx \quad \left[L=\pi\right]$$

$$= \frac{1}{2\pi}\left[0 + \int_{0}^{\pi} e^{-inx}\, dx\right]$$

$$= \frac{1}{2\pi}\left[\frac{e^{-inx}}{-in}\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{e^{-in\pi}}{-in} + \frac{1}{in}\right]$$

$$= \frac{1}{2\pi i n}\left(1 - e^{-in\pi}\right)$$

(1) $\Rightarrow$ The required complex series $\Rightarrow$

$$f(x) = \frac{1}{2} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{2\pi i n}\left(1 - e^{-in\pi}\right) e^{inx}.$$

Ans

Fourier Transformation

* $F(u) = \int_{-\infty}^{\infty} f(x)\, e^{-iux}\, dx$ is called the Fourier transform of $f(x)$.

Ex: (01) Find the Fourier transform of

$$f(x) = \begin{cases} 1; & |x| < a \\ 0; & |x| > a \end{cases}$$

Sol^n: $f(x)$ can be written as $\Rightarrow$

$$f(x) = \begin{cases} 1; & -a < x < a \\ 0; & \text{otherwise} \end{cases}$$

We Know,

Fourier transform of $f(x) \Rightarrow$

$$F(u) = \int_{-\infty}^{\infty} f(x)\, e^{-iux}\, dx$$

$$F(u) = \int_{-\infty}^{-a} f(x)\, e^{-iux}\, dx + \int_{-a}^{a} f(x)\, e^{-iux}\, dx + \int_{a}^{\infty} f(x)\, e^{-iux}\, dx$$

$$F(u) = 0 + \int_{-a}^{a} e^{-iux}\, dx + 0 \quad \text{——— (1)}$$

$$= \left[\frac{e^{-iux}}{-iu} \right]_{-a}^{a}$$

$$= \left[\frac{e^{-iua}}{-iu} + \frac{e^{iua}}{iu} \right]$$

$$= \frac{1}{iu} \left[e^{iua} - e^{-iua} \right]$$

$$= \frac{2}{u} \, \frac{e^{iua} - e^{-iua}}{2i}$$

$$= \frac{2}{u} \sin(au) \qquad \left[\because \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}, \quad \underline{NB}:\ \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \right]$$

$\therefore F(u) = \frac{2}{u}\sin(au)$; $u \neq 0$

~~if u=0~~ ~~Ans~~

If $u = 0$, (i) $\Rightarrow$

$$F(0) = \int_{-a}^{a} 1 . dx = [x]_{-a}^{a} = a + a = 2a.$$

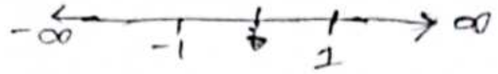
So, $\boxed{F(u) = \frac{2}{u}\sin(au); \; u \neq 0 \;\; \& \;\; F(0) = 2a.}$ Ans.

Ex: 02 find Fourier transform of

$$f(x) = \begin{cases} 1-x^2 & ; |x| < 1 \\ 0 & ; |x| > 1 \end{cases}$$

NB: $|x| < a \Rightarrow -a < x < a$.
$|x| > a \Rightarrow x > a$ or $x < -a$

Soln: We know,
$$F(u) = \int_{-\infty}^{\infty} f(x)\, e^{-iux}\, dx$$



$$F(u) = \int_{-\infty}^{-1} f(x)\, e^{-iux}\, dx + \int_{-1}^{1} f(x)\, e^{-iux}\, dx + \int_{1}^{\infty} f(x)\, e^{-iux}\, dx$$

$$F(u) = 0 + \int_{-1}^{1} (1-x^2)\, e^{-iux}\, dx + 0 \quad \text{——} \quad (i)$$

$$= \int_{-1}^{1} e^{-iux}\, dx - \int_{-1}^{1} x^2 e^{-iux}\, dx$$

$$= \left[\frac{e^{-iux}}{-iu}\right]_{-1}^{1} - \int_{-1}^{1} x^2 e^{-iux}\, dx$$

$$= \left[\frac{e^{-iu}}{-iu} + \frac{e^{iu}}{iu}\right] - \int_{-1}^{1} x^2 e^{-iux}\, dx$$

$$= \frac{1}{iu}\left[e^{iu} - e^{-iu}\right] - \int_{-1}^{1} x^2 e^{-iux}\, dx$$

$$= \frac{\cancel{\theta}}{\cancel{\theta}} \frac{2}{u} \cdot \left(\frac{e^{iu} - e^{-iu}}{2i} \right) - \int_{-1}^{1} x^2 e^{-iux} dx$$

$$= \frac{2}{u} \sin(u) - \int_{-1}^{1} x^2 e^{-iux} dx \quad \left[\because \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \right]$$

$$= \frac{2}{u} \sin(u) - \left[-\frac{x^2 e^{-iux}}{iu} - \frac{2x e^{-iux}}{i^2 u^2} - \frac{2 e^{-iux}}{i^3 u^3} \right]_{-1}^{1}$$

D	I
$+$ x^2	e^{-iux}
$-$ $2x$	$\frac{e^{-iux}}{-iu}$
$+$ 2	$\frac{e^{-iux}}{i^2u^2}$
0	$\frac{e^{-iux}}{-i^3u^3}$

$$= \frac{2}{u} \sin(u) - \left[-\frac{x^2 e^{-iux}}{iu} + \frac{2x e^{-iux}}{u^2} + \frac{2 e^{-iux}}{iu^3} \right]_{-1}^{1}$$

$$\left[\because i^2 = -1, \quad i^3 = -i \right]$$

$$= \frac{2}{u} \sin(u) - \left[-\frac{e^{-iux}}{iu} + \frac{2e^{-iu}}{u^2} + \frac{2e^{-iu}}{iu^3} + \frac{e^{iu}}{iu} + \frac{2e^{iu}}{u^2} - \frac{2e^{iu}}{iu^3} \right]$$

$$= \frac{2}{u}\sin(u) - \left[\frac{1}{iu}\left(e^{iu} - e^{-iu}\right) + \frac{2}{u^2}\left(e^{iu} + e^{-iu}\right) - \frac{2}{iu^3}\left(e^{iu} - e^{-iu}\right)\right]$$

using $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

& $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$, we get,

$$F(u) = \cancel{\frac{2}{u}\sin(u)} - \left[\cancel{\frac{2}{u}\sin(u)} + \frac{4}{u^2}\cos(u) - \frac{4}{u^3}\sin(u)\right]$$

$$= \frac{4}{u^3}\sin(u) - \frac{4}{u^2}\cos(u) \; ; u \neq 0 .$$

$$F(u) = \frac{4}{u^3}\left[\sin(u) - u\cos(u)\right] \; ; u \neq 0$$

If $u=0$, then (i) $\Rightarrow$

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$$F(0) = \int_{-1}^{1} (1-x^2)\, dx$$

$$= \left[x - \frac{x^3}{3} \right]_{-1}^{1}$$

$$= \left[1 - \frac{1}{3} + 1 - \frac{1}{3} \right]$$

$$= 2 - \frac{2}{3}$$

$$= \frac{4}{3}.$$

So, The required Fourier transform is ⇒

$$F(u) = \frac{4}{u^3} \left[\sin(u) - u \cos(u) \right] ; \quad u \neq 0$$

& $F(0) = \frac{4}{3}$.

Ans