Ans. to the Q. NO-01 (a)

Given,

$f(x) = x \ln(x)$

nodes, $x_0 = 1$

$x_1 = 3$

To obtain a degree 3 interpolating polynomial, we need to use Hermite interpolation.

So, we need to :

1) Calculate Function Values:

↳ find $f(x_0)$ and $f(x_1)$
$= f(1)$ and $f(3)$

2) Caculate their derivative Values:

↳ To find the derivative of $f'(x_0)$ and $f'(x_1)$
$\Rightarrow f'(1)$ and $f'(3)$

3) Constructing the Hermite basis functions using the given nodes;

4) Forming the Hermite polynomial using these basis functions.

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We know,

Hermite interpolation polynomial,

$$H_3(x) = f(x_0)\, h_0(x) + f(x_1) \cdot h_1(x) + f'(x_0) \cdot h_2(x) + f'(x_1) \cdot h_3(x)$$

Ans. to the Q. NO - 01 (b)

Given,

$f(x) = x \ln(x)$

nodes, $x_0 = 1$,

$x_1 = 3$

i) The function values:

$\therefore f(1) = 1 \cdot \ln(1) = 0$

$\therefore f(3) = 3 \cdot \ln(3) = 3 \cdot 2958$

ii) Derivative values:

$$f'(x) = \ln(x) + x * \left(\frac{1}{x}\right)$$

$$= \ln(x) + 1$$

$$f'(1) = \ln(1) + 1 = 1$$

$$f'(3) = \ln(3) + 1 = 2 \cdot 0986$$

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iii) Constructing the Hermite basis functions:

a) for the 1^{st} divided difference:

$$f[x_0, x_0] = f'(x_0) = 1$$

$$f[x_1, x_1] = f'(x_1) = 2.0986$$

$$f[x_0, x_1] = \frac{f(3) - f(1)}{(3-1)}$$

$$= 1.6479$$

b) for the 2nd divided differece:

$$f[x_0, x_0, x_1] = \frac{f[x_0, x_1] - f[x_0, x_0]}{x_1 - x_0}$$

$$= \frac{1.6479 - 1}{3-1}$$

$$= 0.32395$$

$$f[x_0, x_1, x_1] = \frac{f[x_1, x_1] - f[x_0, x_1]}{x_1 - x_0}$$

$$= \frac{2.0986 - 1.6479}{3-1}$$

$$= 0.22535$$

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iii) Constructing the Hermite basis functions:

a) for the 1^{st} divided difference:

$$f[x_0, x_0] = f'(x_0) = 1$$

$$f[x_1, x_1] = f'(x_1) = 2.0986$$

$$f[x_0, x_1] = \frac{f(3) - f(1)}{(3-1)}$$

$$= 1.6479$$

b) for the 2nd divided difference:

$$f[x_0, x_0, x_1] = \frac{f[x_0, x_1] - f[x_0, x_0]}{x_1 - x_0}$$

$$= \frac{1.6479 - 1}{3-1}$$

$$= 0.32395$$

$$f[x_0, x_1, x_1] = \frac{f[x_1, x_1] - f[x_0, x_1]}{x_1 - x_0}$$

$$= \frac{2.0986 - 1.6479}{3-1}$$

$$= 0.22535$$

[P.T.O.]

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(c) for the 3^{rd} divided difference :

$$f[x_0, x_0, x_1, x_1] = \frac{f[x_0, x_1, x_1] - f[x_0, x_0, x_1]}{x_1 - x_0}$$

$$= \frac{0.22535 - 0.32395}{3-1}$$

$$= -0.0493$$

$\therefore$ The bases of the degree 3 polynomial are :

$f[x_0, x_1] = 1.6479$

$f[x_0, x_0, x_1] = 0.22535$

$f[x_0, x_0, x_1, x_1] = -0.0493$

Calculating the Hermite Basis Functions :

$$h_0(x) = (1-2(x-x_0) \cdot l_0'(x_0)) \cdot l_0^2(x)$$

$$h_1(x) = (x-x_0) \cdot l_0^2(x)$$

$$h_2(x) = (1-2 \cdot (x-x_1) \cdot l_1'(x_1)) \cdot l_1^2 \cdot (x)$$

$$h_3(x) = (x-x_1) \cdot l_1^2 \cdot (x)$$

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Where,

the Lagrange basis polynomials are:

$$l_0(x) = \frac{x-3}{1-3} = \frac{x-3}{-2}$$

$$l_1(x) = \frac{x-1}{3-1} = \frac{x-1}{2}$$

Calculating their derivatives:

$$l_0'(x) = -\frac{1}{2} = -0.5$$

$$l_1'(x) = \frac{1}{2} = 0.5$$

calculating their squares:

$$l_0^2(x) = \left(\frac{x-3}{-2}\right)^2 = \frac{(x-3)^2}{4}$$

$$l_1^2(x) = \left(\frac{x-1}{2}\right)^2 = \frac{(x-1)^2}{4}$$

Computing Hermite Basis Functions:

$$h_0(x) = (1-2(x-1)(-0.5)) \cdot \frac{(x-3)^2}{4}$$

$$= 1+(x-1)) \cdot \frac{(x-3)^2}{4}$$

$$h_1(x) = (x-1)\ \frac{(x-3)^2}{4}$$

$$h_2(x) = (1-2(x-3)\cdot(0.5)) \cdot \frac{(x-1)^2}{4} = (1-(x-3)) \cdot \frac{(x-1)^2}{4}$$

$$h_3(x) = (x-3) \cdot \frac{(x-1)^2}{4}$$

(Ans:)

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Ans. to the Q. NO - 01 (C)

The Hermite polynomial :

$$h_3(x) = f(x_0) + f[x_0, x_0](x-x_0) + f[x_0, x_0, x_1](x-x_0)^2 + f[x_0, x_0, x_1, x_1](x-x_0)^2(x-x_1)$$

Now,

Substituting values : [from 1(a), 1(b)]

$$h_3(x) = 0 \cdot h_0(x) + 1 \cdot h_1(x) + 3.2958 \cdot h_2(x) + 2.0986 \cdot h_3(x)$$

$$= h_1(x) + 3.2958\, h_2(x) + 2.0986\, h_3(x)$$

Now, for h_1, h_2, h_3 :

$$h_3(x) = (x-1) + 0.32395\,(x-1)^2 + (-0.0493)(x-1)^2 \cdot (x-3)$$

$$= (x-1) + 0.32395\,(x-1)^2 - 0.0493\,(x-1)^2(x-3)$$

(Ans:)

Ans. to the Q. NO - 02

Given,

$$f(x) = \frac{1}{1+x^2}$$

x epsilon $[-5,5]$

degree, $n = 4$

We know,

formula for Chebyshev nodes,

$$x_k = \frac{a+b}{2} + \frac{b-a}{2} \cos\left(\frac{(2k+1)\pi}{2(n+1)}\right)$$

here,
$[a,b]$ = lower and upper bounds of the interval

n = degree of the polynomial

k = node index

$a = -5$

$b = 5$

Now,

for 4 degree polynomial,

chebyshev nodes needed $= n+1$
$= 4+1$
$= 5$

Now,

$$x_k = \frac{-5+5}{2} + \frac{5-(-5)}{2} \cos\left(\frac{(2k+1)\pi}{2\cdot(5)}\right)$$

$$= 0 + {}^{10}/_{2} \cos\left(\frac{(2k+1)\pi}{10}\right)$$

$$= 5 \cos\left(\frac{(2k+1)\pi}{10}\right)$$

[P.T.O.]

Again,

for $K=0$,

$$x_0 = 5\cos\left(\frac{(2\cdot(0)+1)\pi}{10}\right)$$

$$\approx 4\cdot 938$$

for $K=1$,

$$x_1 = 5\cos\left(\frac{3\pi}{10}\right)$$

$$= 4\cdot 045$$

for $K=2$,

$$x_2 = 5\cos\left(\frac{(2\cdot(2)+1)\pi}{10}\right)$$

$$= 5\cos\left(\frac{5\pi}{10}\right)$$

$$= 0$$

for $K=3$,

$$x_3 = 5\cos\left(\frac{(2\cdot(3)+1)\pi}{10}\right)$$

$$= -4\cdot 045$$

for $K=4$,

$$x_4 = 5\cos\left(\frac{2\cdot(4)+1}{10}\right)$$

$$= -4\cdot 938$$

[P.T.O.]

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∴ The Chebyshev nodes,

for a 4-degree polynomial in the interval $[-5, 5]$:

$x_0 = 4.938$

$x_1 = 4.045$

$x_2 = 0$

$x_3 = -4.045$

$x_4 = -4.938$

(Ans:)

Ans. to the Q. NO- 03 (a)

Given function,

$$f(x) = x \cdot \ln(x)$$

$x = 1$

$h = 0.1$

We Know,

Forward difference, $f'(x) = \dfrac{f(x+h) - f(x)}{h}$

Now,

$f(1) = 1 \ln(1) = 0$

$f(x+h) = f(1 + 0.1) = f(1.1)$

$= 1.1 \times 0.0953$

$= 0.10483$

$$f'(1) = \frac{f(1.1) - f(1)}{0.1}$$

$$= \frac{0.10483 - 0}{0.1}$$

$= 1.0483$

(Ans:)

[P.T.O.]

Ans. to the Q. No-03(b)

Given function,

$$f(x) = x \ln(x)$$

1st derivative of $f(x)$,

$$f'(x) = \ln(x) + 1$$

We know,

Backward difference, $f'(x) = \dfrac{f(x) - f(x-h)}{h} - \dfrac{h}{2} f''(\xi)$

Truncation error upper bound;

$$\frac{h}{2} \left| f''(\xi) \right| \text{ for } \xi \in \left| x-h, x \right|$$

Now,

2nd derivative of $f(x)$,

$$f''(x) = \frac{d}{dx}\left(1+\ln(x)\right)$$

$$= \frac{1}{x}$$

at $x=1$, $f''(1) = \frac{1}{1} = 1$

3rd derivative, $f'''(x) = \frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$

[P.T.O.]

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At $x = 1$,

$$f'''(1) = -\frac{1}{(1)^2}$$

$$= -1$$

$\therefore$ Backward difference,

$$\text{error} = h/_2 \, f''(\xi)$$

for some $\xi \in (x-h, x)$: Error $\leq \frac{h}{2} f''(\xi)$

here,
$f''(x) = \frac{1}{x}$

worst case $\xi = 1$

$\Rightarrow$ Error $\leq \frac{0 \cdot 1}{2} \times (1)$

$\therefore$ Error $= 0.05$

Again,
We Know,

Central Difference, $f'(x) = \dfrac{f(x+h) - f(x-h)}{2h} - \dfrac{h^2}{6} f'''(\xi)$

Truncation error upper bound $= \dfrac{h^2}{6} \left| f'''(\xi) \right|$

for some $\xi \in$
$[x-h, x+h]$

[P.T.O.]

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Now,

$$\text{error} \Rightarrow \frac{h^2}{6}\left|f'''(\xi)\right|$$

$\because f'''(x) = -\frac{1}{x^2}$

worst case, $\xi = 1$

$$= \frac{(0.1)^2}{6} \times 1$$

$$= 0.00167$$

$\therefore$ Truncation Error upper bounds:

1) For Backward Difference method = 0.05
2) For Central Difference method = 0.00167

(Ans:)

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