Lecture: 10

Range/Image of Linear Transformation:-

The range of $T: U \to V$ is the subset of V consisting of all $y \in V$ such that $T(x) = y$ for all $x \in U$. It is denoted by $R(T)$ or $Im(T)$.

Kernel/nullspace of Linear Transformation:

Kernel of $T: U \to V$ is the subset of U consisting all $x \in U$ for which $T(x) = \underline{0}$. It is denoted by $Ker(T)$.

i.e $Ker(T) = \{x \in U \mid T(x) = 0\}$.

Rank of T: The dimension of range of T (i.e. the number of vectors in basis of range of T) is called the rank of T.

Nullity of T: The dimension of kernel of T (i.e. the # of vectors in basis of $Ker(T)$) is called the nullity of T.

NB: ~~rank (T) of~~ rank of T + nullity of T = dimension of U.

domain of Linear Transf

Ex. $T: \mathbb{R}^4 \to \mathbb{R}^3$ be a L.T. defined by
$T(x,y,z,t) = (x-y+z+t,\ x+2z-t,\ x+y+3z-3t)$

find a basis of range of T, basis of kernel / nullspace of T, rank of T & nullity of T.

Sol.ⁿ Given,
$T(x,y,z,t) = (x-y+z+t,\ x+2z-t,\ x+y+3z-3t)$

Basis of range of T:-

We know,
$\{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$ is a basis of $\mathbb{R}^4$.

$\therefore T(1,0,0,0) = (1-0+0+0,\ 1+0-0,\ 1+0+0-0)$
$= (1,1,1)$

$T(0,1,0,0) = (0-1+0+0,\ 0+0-0,\ 0+1+0-0)$
$= (-1,0,1)$

$T(0,0,1,0) = (0-0+1+0,\ 0+2-0,\ 0+0+3-0)$
$= (1,2,3)$

$T(0,0,0,1) = (0-0+0+1,\ 0+0-1,\ 0+0+0-3)$
$= (1,-1,-3)$

So, the corresponding matrix $\Rightarrow$

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 2 & 3 \\ 1 & -1 & -3 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & -2 & -4 \end{pmatrix} \; ; \; \begin{aligned} r_2' &= r_2 + r_4 \\ r_3' &= r_3 - r_4 \\ r_4' &= r_4 - r_4 \end{aligned}$$

$$\sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \; ; \; \begin{aligned} r_3' &= r_3 - r_2 \\ r_4' &= r_4 + 2r_2 \end{aligned}$$

This is in row-echelon form. having two non-zero rows. which will form a Basis of range of T.

$\therefore$ Basis of range of $T = \{(1,1,1), (0,1,2)\}$

$\therefore$ rank of $T = 2$.

[We know, rank T + nullity $T = \dim(\mathbb{R}^4)$

$\Rightarrow 2 +$ nullity of $T = 4$

$\therefore$ nullity of $T = 2$]

Kernel of T :-

We have to find (x, y, z, t) such that -

$$T(x, y, z, t) = (0, 0, 0)$$

$$\Rightarrow (x - y + z + t,\ x + 2z - t,\ x + y + 3z - 3t) = (0, 0, 0)$$

$$\Rightarrow \begin{aligned} x - y + z + t &= 0 \\ x \quad + 2z - t &= 0 \\ x + y + 3z - 3t &= 0 \end{aligned}$$

∴ The ~~eq~~ augmented matrix ⇒

$$\left(\begin{array}{cccc|c} 1 & -1 & 1 & 1 & 0 \\ 1 & 0 & 2 & -1 & 0 \\ 1 & 1 & 3 & -3 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & -1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 2 & 2 & -4 & 0 \end{array}\right) ;\ r_2' = r_2 - r_1 ;\ r_3' = r_3 - r_1$$

$$\sim \left(\begin{array}{cccc|c} 1 & -1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) ;\ r_3' = r_3 - 2r_2$$

$$\therefore \begin{aligned} x - y + z + t &= 0 \\ y + z - 2t &= 0 \end{aligned}$$

∴ 2 eqn's & 4 unknowns so, $(4-2) = 2$ free variables.

Here, z & t are free variables.

$\therefore$ let $z = s$

$t = r$

$\therefore$ $y = -z + 2t = -s + 2r$

$x = y - z - t = -s + 2r - s - r$

$= -2s + r$

$$\therefore \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} -2s + r \\ -s + 2r \\ s \\ r \end{pmatrix}$$

$$= \begin{pmatrix} -2s \\ -s \\ s \\ 0 \end{pmatrix} + \begin{pmatrix} r \\ 2r \\ 0 \\ r \end{pmatrix}$$

$$= s \begin{pmatrix} -2 \\ -1 \\ 1 \\ 0 \end{pmatrix} + r \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}$$

$$\text{Basis} = \left\{ (-2,-1,1,0), (1,2,0,1) \right\}$$

$\therefore$ Nullity of $T = \dim(\mathrm{Ker}(T))$

$= 2.$

(Ans)

Rank of a matrix

* find the rank of $A = \begin{pmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{pmatrix}$

Sol^n

$$\begin{pmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{pmatrix} ; r_2 \leftrightarrow r_4$$

$$\sim \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{pmatrix} ; \begin{array}{l} r_3' = r_3 - 3r_4 \\ r_4' = r_4 - r_4 \end{array}$$

$$\sim \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} ; \begin{array}{l} r_3' = r_3 - r_2 \\ r_4' = r_4 - r_2 \end{array}$$

This is in row echelon form having two non-zero rows.

So, rank of $A = 2$.

Ans

Basis & dimension of a vector space

Def^n $\{v_1, v_2, --- v_n\}$ is called a basis of a vector space V iff —

i) $\{v_1, v_2, --- v_n\}$ is linearly independent

ii) $\{v_1, v_2, --- v_n\}$ spans V.

The number of vectors in a basis of a vector space is called the dimension of that vector space.

Ex. Do the vectors $v_1 = (1,2,0)$, $v_2 = (0,5,7)$ & $v_3 = (-1,1,3)$ form a basis for $\mathbb{R}^3$?

Soln. First, we check v_1, v_2, v_3 are L.I or not.

for any scalars k_1, k_2, k_3, if,

$$k_1 v_1 + k_2 v_2 + k_3 v_3 = 0$$

$$\Rightarrow (k_1, 2k_1, 0) + (0, 5k_2, 7k_2) + (-k_3, k_3, 3k_3) = (0,0,0)$$

$$\Rightarrow (k_1 - k_3,\ 2k_1 + 5k_2 + k_3,\ 7k_2 + 3k_3) = (0,0,0)$$

$$\therefore \begin{aligned} k_1 \quad - k_3 &= 0 \\ 2k_1 + 5k_2 + k_3 &= 0 \\ 7k_2 + 3k_3 &= 0 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 2 & 5 & 1 & 0 \\ 0 & 7 & 3 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 5 & 3 & 0 \\ 0 & 7 & 3 & 0 \end{array}\right) ; \quad r_2' = r_2 - 2r_1$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & \frac{3}{5} & 0 \\ 0 & 1 & \frac{3}{7} & 0 \end{array}\right) ; \quad r_2' = \frac{r_2}{5}, \quad r_3' = \frac{r_3}{7}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & \frac{3}{5} & 0 \\ 0 & 0 & -\frac{6}{35} & 0 \end{array}\right) ; r_3' = r_3 - r_2$$

$$\cancel{\frac{3}{7} - \frac{3}{5}} = \cancel{\frac{15-21}{35}} = -\frac{6}{35}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & \frac{3}{5} & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) ; r_3' = \left(-\frac{35}{6}\right) r_3$$

$\therefore$ $k_1 - k_3 = 0$

$k_2 - \frac{3}{5} k_3 = 0$

$k_3 = 0$

→ by backward substitution,

$k_3 = 0$

$k_2 = 0$

$k_1 = 0$

So, v_1, v_2, v_3 are linearly Independent.

Now we check whether v_1, v_2, v_3 spans $\mathbb{R}^3$ or not.

Let, $(a,b,c) \in \mathbb{R}^3$ be arbitrary.

$(a,b,c) = k_1 v_1 + k_2 v_2 + k_3 v_3$

$\Rightarrow (a,b,c) = (k_1 - k_3,\ 2k_1 + 5k_2 + k_3,\ 7k_2 + 3k_3)$

$$k_1 - k_3 = a$$
$$2k_1 + 5k_2 + k_3 = b$$
$$7k_2 + 3k_3 = c$$

NB: there are 2 ways to proceed —

i) using determinant of co-efficient matrix (as it is applicable only if the co-eff matrix is a square matrix)

ii) using Gaussian elimination.

Way 1. Here the co-efficient matrix is

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 5 & 1 \\ 0 & 7 & 3 \end{pmatrix}$$

$$\therefore |A| = 1(15-7) - 1(14-0)$$
$$= 8 - 14$$
$$= -6 \neq 0$$

So, the system is consistent

Hence, v_1, v_2, v_3 spans $\mathbb{R}^3$. $\square$

Way 2. The corresponding augmented matrix $\Rightarrow$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & a \\ 2 & 5 & 1 & b \\ 0 & 7 & 3 & c \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & a \\ 0 & 5 & 3 & b-2a \\ 0 & 7 & 3 & c \end{array}\right) \quad r_2' = r_2 - 2r_1$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & a \\ 0 & 1 & 3/5 & \frac{b-2a}{5} \\ 0 & 1 & 3/7 & c/7 \end{array}\right) \quad ; r_2' = \frac{r_2}{5}, \quad r_3' = \frac{r_3}{7}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & a \\ 0 & 1 & 3/5 & \frac{b-2a}{5} \\ 0 & 0 & -6/35 & \frac{c}{7} - \frac{b-2a}{5} \end{array}\right) \quad ; r_3' = r_3 - r_2$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & a \\ 0 & 1 & 3/5 & \frac{b-2a}{5} \\ 0 & 0 & 1 & -\frac{5c}{6} + \frac{7(b-2a)}{6} \end{array}\right)$$

$$\therefore \quad k_3 = \frac{1}{6}\left(-14a + 7b - 5c\right)$$

$$k_2 = \frac{b-2a}{5} - \frac{\cancel{3}}{5} \times \frac{1}{\cancel{6}_2}\left(-14a + 7b - 5c\right)$$

$$= \frac{2b - 4a + 14a - 7b + 5c}{10} = \frac{10a - 5b + 5c}{10}$$

$$= \frac{1}{2}\left(2a - b + c\right)$$

$$k_1 = a + k_3 = a + \frac{-14a + 7b - 5c}{6} = \frac{-8a + 7b - 5c}{6}$$

$$k_1 = \frac{1}{6}\left(-8a + 7b - 5c\right)$$

So, v_1, v_2, v_3 spans $\mathbb{R}^3$. D.

[1] Let, U be a subspace of $\mathbb{R}^3$ spanned by vectors $(1,2,1)$, $(0,-1,0)$ & $(2,0,2)$. Find a basis & dimension of U.

Sol^n We form a matrix whose rows are the given vectors -

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & -2 & 0 \end{pmatrix} \; ; \; r_3' = r_3 - 2r_1$$

$$\sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \; ; \; \begin{array}{l} r_2' = (-1)r_2 \\ r_3' = r_3 \times \left(-\frac{1}{2}\right) \end{array}$$

$$\sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \; ; \; r_3' = r_3 - r_2$$

This is in row-echelon form having two non-zero rows, which will form a basis of U.

$\therefore$ Basis of $U = \{(1,2,1), (0,1,0)\}$

dimension $= 2$ (Ans)

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