Lecture: 02 01 june 2024

* Using Elementary Row Operations to Solve a system of linear Equations —

Q: Solve the following system using elementary row operations

$$x + y + 2z = 9$$
$$2x + 4y - 3z = 1$$
$$3x + 6y - 5z = 0$$

Solⁿ: The augmented matrix is ⇒

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right)$$

[**NB:** our target is to convert the augmented matrix into $\left(\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right)$ by row operations]

We perform the following row operations —

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 3 & 6 & -5 & 0 \end{array}\right); \quad r_2' = r_2 - 2r_1$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{array}\right) ; \; r_3' = r_3 - 3r_1$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -7/2 & -17/2 \\ 0 & 3 & -11 & -27 \end{array}\right) ; \; r_2' = \frac{1}{2} r_2$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -7/2 & -17/2 \\ 0 & 0 & -\frac{1}{2} & -3/2 \end{array}\right) ; \; r_3' = r_3 - 3r_2$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -7/2 & -17/2 \\ 0 & 0 & 1 & 3 \end{array}\right) ; \; \cancel{r_3 = (-\tfrac{1}{2}) r_3}$$

$$r_3' = (-2) r_3$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right) ; \; r_2' = r_2 + 7/2\, r_3$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right) ; \; r_1' = r_1 - 2r_3$$

or $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right)$; $r_4' = r_4 - r_2$

Hence, we get, $x = 1$

$y = 2$

$z = 3$ (Ans)

Section 1.2 (Gaussian Elimination)

A matrix is ~~called~~ said to be in reduced row echelon form if the following properties hold =>

1. ~~Each row starts~~ The first non-zero element of each row ~~with~~ is 1 (called leading 1)
2. If a row consists of only zero's then it is placed at the last row.
3. In any two successive rows, the leading 1 is the lower row occurs to the right than the upper row.
4. Each column containing a leading 1 has zeros everywhere else in that column.

A matrix which satisfies only the first 3 properties is said to be in row echelon form

Example of reduced row echelon form –

$$\begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Ex: of row echelon form but not reduced row echelon form –

$$\begin{pmatrix} 1 & 4 & -3 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{pmatrix}$$ → Here, the 2nd column contains a leading 1 but it has not 0 everywhere else in that column. So, doesn't satisfy the 4th property.

Different cases of solⁿ → case 1: Unique solⁿ
case 2: No solⁿ (inconsistent)
case 3: many solⁿ

Unique solⁿ:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 \end{array}\right)$$

[Unique solⁿ exists if all the columns have a leading 1]

$x = 3$

$y = 0$

$z = 5$

So, the system has unique solⁿ.

No sol^n.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \rightarrow$$ (এখানে vertical line দেওয়া হয় নাই)

Here, from the last row, we see that —

$$0.x + 0.y + 0.z = 1$$

~~to~~ which is not possible for any values of x, y, z. So, the system has no sol^n or, we say that the system is inconsistent.

Many sol^n.

$$\begin{pmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

[NB: many sol^n occurs when 1 or more rows consists entirely of zeros.]

The corresponding system $\Rightarrow$

$$x + 3z = -1$$
$$y - 4z = 2$$

So, there are 2 eq^{n}s in 3 unknowns. So, we have $(3-2) = 1$ free variable. ~~let, z be~~ We see that z can be treated as free variable. (

[NB: ~~যেখানে~~ যেই column এ leading 1 নাই সেটাই free variable হবে]

so, let, $z = t$; $t \in \mathbb{R}$,

$\therefore$ $y = 2 + 4z$
$= 2 + 4t$

$x = -1 - 3z$
$= -1 - 3t$

Assigning specific values to t, we can get various sol^{n}s. for eg,

$t = 0 \Rightarrow x = -1, y = 2, z = 0$

$t = 1 \Rightarrow x = -4, y = 6, z = 1$

Another eg. of many sol^n.

$$\begin{pmatrix} 1 & -5 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\rightarrow$ There is no leading 1 in 2nd & 3rd column so y & z are free variables.

The corresponding system $\Rightarrow$

$$x - 5y + z = 4$$

$\therefore$ There is 1 eq^n in 3 unknowns, so, $(3-1) = 2$ be free variables. let, $y = s$, $z = t$, then,

$$x = 4 + 5s - t$$