final, lec: 01

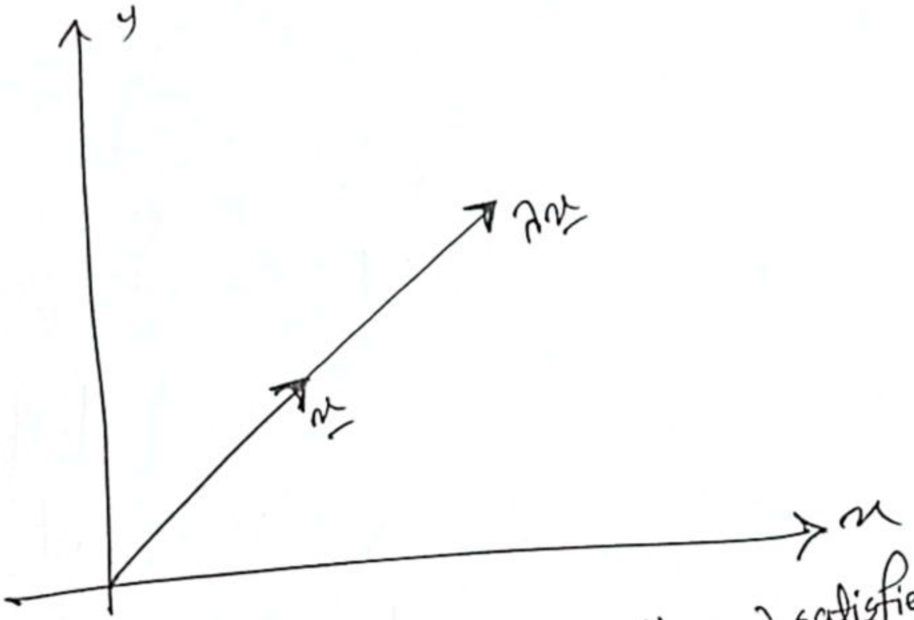
Eigenvalue & Eigenvectors

31-08-24

Def^n: If A is an $n \times n$ matrix, then a non-zero vector $\underline{x} \in \mathbb{R}^n$ is called an eigenvector of A if

$$A\underline{x} = \lambda \underline{x} \text{ ; for some scalar } \lambda.$$

The scalar λ is called an eigenvalue of A and $\underline{x}$ is called eigenvector corresponding to λ.



* If λ is an eigenvalue of A, then λ satisfies the following eq^n ⇒ ~~followi~~

$$|A - \lambda I| = 0 \text{ ——— } (*) \text{ ; } I = \text{identity matrix}$$

$$\text{or, } \det(A - \lambda I) = 0$$

$(*)$ is called the characteristic equation of A.

NB: * Eigenvalues & eigenvectors are only for square matrices

* Eigenvalues may be zero but eigenvectors are always non-zero.

Finding eigenvalues of A

(2 ways —
1) using characteristic eq^n
2) short-cut way)

For 2x2 matrix:

Q: Find eigenvalues of $A = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$.

Using characteristic eq^n: characteristic eq^n =>

$$|A - \lambda I| = 0$$

$$\Rightarrow \left| \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 0 \\ 8 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (3-\lambda)(-1-\lambda) - 0 = 0$$

$$\Rightarrow -3 - 3\lambda + \lambda + \lambda^2 = 0$$

$$\Rightarrow \lambda^2 - 2\lambda - 3 = 0$$

$$\therefore \lambda = 3, -1 \quad \text{Ans}$$

using short-cut way:

* If A is a 2×2 matrix, then the characteristic eq^n of A is $\Rightarrow$

$$\lambda^2 - tr(A)\lambda + |A| = 0$$

where, $tr(A)$ = trace of A (i.e. sum of diagonal elements).

$|A|$ = determinant of A.

Given, $A = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$

Here, $tr(A) = 3 - 1 = 2$

$$|A| = (3)(-1) - (8)(0) = -3$$

$\therefore$ characteristic $eq^n \Rightarrow$

$$\lambda^2 - tr(A)\lambda + |A| = 0$$

$$\Rightarrow \lambda^2 - 2\lambda - 3 = 0$$

$\therefore \lambda = 3, -1$. (Ans).

for 3×3 matrix: (★★★)

Q: find eigenvalues of $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{pmatrix}$

using characteristic eqn:

The charac. eqn ⇒

$$|A - \lambda I| = 0$$

$$\Rightarrow \left| \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 4 & -17 & 8-\lambda \end{vmatrix} = 0$$

$$\Rightarrow -\lambda \left\{ -\lambda(8-\lambda) + 17 \right\} - 1 \left\{ 0 - 4 \right\} + 0 = 0$$

$$\Rightarrow -\lambda \left(-8\lambda + \lambda^2 + 17 \right) + 4 = 0$$

$$\Rightarrow 8\lambda^2 - \lambda^3 - 17\lambda + 4 = 0$$

$$\Rightarrow \lambda^3 - 8\lambda^2 + 17\lambda - 4 = 0$$

use calculator → $\therefore \lambda = 0.27,\ 4,\ 3.73$.

(Ans)

using short-cut way

* If A is a 3×3 matrix, then the charac. eqn of A is $\Rightarrow$

$$\lambda^3 - tr(A)\lambda^2 + (A_{11}+A_{22}+A_{33})\lambda - |A| = 0$$

where $tr(A)$ = trace of A (i.e. sum of diagonal elements

$|A|$ = determinant of A

$A_{11}+A_{22}+A_{33}$ = sum of diagonal minors

Ex. $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

then, $A_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$, $A_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$, $A_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

*

Given, $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{pmatrix}$

$tr(A) = 0+0+8 = 8$

$A_{11} = \begin{vmatrix} 0 & 1 \\ -17 & 8 \end{vmatrix} = 0+17 = 17$

$A_{22} = \begin{vmatrix} 0 & 0 \\ 4 & 8 \end{vmatrix} = 0-0 = 0$

$A_{33} = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0-0 = 0$

$\therefore A_{11}+A_{22}+A_{33} = 17+0+0 = 17$

$$|A| = 0 - 1(0 - 4) + 0 = 4$$

$\therefore$ charac. eqⁿ $\Rightarrow$

$$\lambda^3 - 8\lambda^2 + 17\lambda - 4 = 0$$

$$\therefore \lambda = 0.27,\ 4,\ 3.73.$$

(Ans).

* Eigenvalues of a upper/Lower triangular matrix are just the diagonal elements.

Ex. Eigenvalues of the upper triangular matrix $A = \begin{pmatrix} 6 & 7 & 0 \\ 0 & -8 & 4 \\ 0 & 0 & 3 \end{pmatrix}$

are $\lambda = 6, -8, 3$

Similarly, eigenvalues of lower triangular matrix $A = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ -1 & 2/3 & 0 \\ 5 & -8 & -\frac{1}{4} \end{pmatrix}$

are $\lambda = \frac{1}{2}, \frac{2}{3}, -\frac{1}{4}$.

(Ans).