

Numerical Methods

Lab 7 [Solving Linear System]

1. Prerequisites:

- Open the Colab file shared in BUX.
- Create a copy of that shared file.
- Rename the Colab filename using the format Name_ID_Lab Section

2. Lab Tasks:

Task-1: Solving a linear system using an inverse matrix - 1 Marks

We have a linear system

$$\begin{aligned}a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\&\vdots \\a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n\end{aligned}$$

It is convenient to express this system in the matrix form

$$Ax = b$$

where A is an $n \times n$ square matrix with elements a_{ij} , and x, b are $n \times 1$ vectors.

We have to keep in mind that this system will have a unique solution iff A is non-singular, given by $x = A^{-1}b$.

A. You have to **implement** the `get_result_by_inverse_matrix(A, b)`, where A is a $n \times n$ matrix and b is a $n \times 1$ vector.

B. Check if A is a singular matrix or not. If not, find its inverse.

C. Multiply the inverse with the vector b .

Task-2: Gaussian elimination method - 1.5 Marks

Gaussian elimination method uses elementary row operations to transform the system to an upper triangular form $Ux = y$.

Elementary row operations include swapping rows and adding multiples of one row to another. They won't change the solution x but will change the matrix A and the right-hand side b .

The upper triangular matrix, U , is defined as

$$\begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & u_{nn} \end{bmatrix}$$

The algorithm of the Gaussian elimination method:

Algorithm of Gaussian elimination

Let $A^{(1)} = A$ and $b^{(1)} = b$. Then for each k from 1 to $n - 1$, compute a new matrix $A^{(k+1)}$ and right-hand side $b^{(k+1)}$ by the following procedure:

1. Define the row multipliers

$$m_{ij} = \frac{a_{ik}^{(k)}}{a_{kk}^{(k)}}, i = k + 1, \dots, n.$$

2. Use these to remove the unknown x_k from equations $k + 1$ to n , leaving

$$a_{ij}^{(k+1)} = a_{ij}^{(k)} - m_{ik} a_{kj}^{(k)}, b_i^{(k+1)} = b_i^{(k)} - m_{ik} b_k^{(k)}, i, j = k + 1, \dots, n.$$

It is helpful to combine these matrices to form an Augmented matrix (matrix b is the fourth column). We will perform this row operations on the Augmented matrix. It takes care of both A and b matrixes at the same time.

After generating the upper triangular matrix, we have to apply **backward substitution method**. For any $n \times n$ upper triangular system, $Ux = b$, the solution is:

$$x_j = \frac{b_j - \sum_{k=j+1}^n u_{jk} x_k}{u_{jj}}, j = n, n - 1, \dots, 1.$$

Here we assumed that $\det U \neq 0$.

A. You have to **implement** the `get_result_gaussian_elimination(n, A)` method, where n is the number of unknowns and A is the augmented matrix.

Task-3: LU Decomposition - 1.5 Marks

We will transform the $n \times n$ matrix A into a product of two triangular matrices: one lower triangular (L) and the other upper triangular (U).

$$A = LU$$

The algorithm of LU decomposition:

Algorithm of LU decomposition:

1. Initialize L to an identity matrix, I of dimension $n \times n$ and $U = A$.
2. For $i = 1, \dots, n$ do Step 3
3. For $j = i + 1, \dots, n$ do Steps 4-5
4. Set $l_{ji} = u_{ji}/u_{ii}$
5. Perform $U_j = (U_j - l_{ji}U_i)$ (where U_i, U_j represent the i and j rows of the matrix U , respectively)

- We know, The linear system in matrix form is $Ax = b$.
- Using the decomposition, we get $LUx = b$.
- Now, let $Ux = y \implies Ly = b$.
- Since L is lower triangular, we solve $Ly = b$ to obtain y by forward substitution.
- Since U and y are known, we solve $Ux = y$ to obtain x by backward substitution.

A. You have to **implement** the `lu(A)` method.

B. You have to **implement** the `forward_substitution(L, b)` method.

C. You have to **implement** the `back_substitution(U, y)` method.

Note: `lu_solve(A, b)` method has been completed for you.

3. Daily Evaluation - 4 Marks

Students have learned to solve Linear Systems using the Inverse Matrix method, Gaussian Elimination, and LU Decomposition method. They are now required to apply this understanding through a set of implementation exercises, which will be provided separately.