## 关系的运算

School of Computer Wuhan University

- 1/145 -

- 1 关系的合成
  - 关系的合成
  - 关系的幂
  - 关系的闭包
  - 传递闭包的求解算法

## 关系上的运算

#### Remark

由于关系就是集合,因此集合上的运算也是关系的运算.

- " $\leq$ " " $\mathbb{1}_A$ " = "<";
- ℝ上有: "≤"∩"≥"= "=";
- "≤"」"≥"= ℝ 上的全域关系:
- a "□"」"□" → 人瑞光系
- 由于关系的对象是n重组,因此还有些一般集合不具有的运算.

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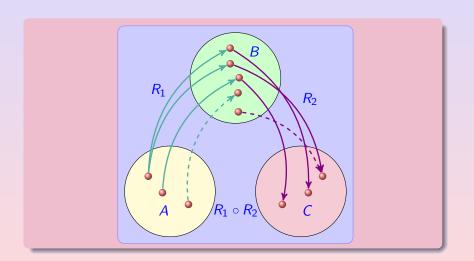
## 关系上的运算

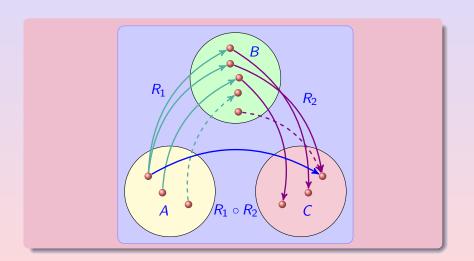
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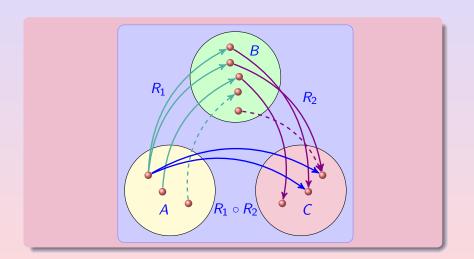
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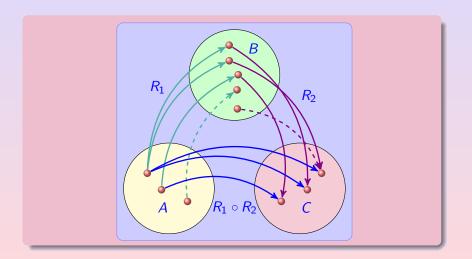
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由于关系的对象是n重组,因此还有些一般集合不具有的运算.









## 合成的定义

### Definition (合成关系, Composite Relation)

设 $\mathcal{R}_1 \subseteq A \times B$ ,  $\mathcal{R}_2 \subseteq B \times C$ ,  $\mathcal{R}_1$ 和 $\mathcal{R}_2$ 的合成记为 $\mathcal{R}_1 \circ \mathcal{R}_2$ ( $\mathcal{R}_1 \mathcal{R}_2$ )定义为:

 $\mathcal{R}_1 \mathcal{R}_2 \triangleq \{ \langle a, c \rangle \mid a \in A, c \in C \land \exists b \in B \land a \mathcal{R}_1 b \land b \mathcal{R}_2 c \}$  是A到C上的关系.

#### Remark

合成的条件:第一个关系的陪域(codomain)和第二个关系的域(domain)是相同的集合.

### Example

R1是兄弟关系; Ro父子关系; R1 Ro是叔侄关系;

 $\mathcal{R} = \{\langle a, b \rangle \mid \text{anbinf直航航线}\}, \mathcal{R}\mathcal{R}$ 是城市之间经过一个城市转机的间接航线(记为 $\mathcal{R}^2$ );

- $\bullet \ (=_4)^2 = =_4;$
- $\emptyset \mathcal{R} = \mathcal{R} \emptyset = \emptyset$ ;
- 合成对应的SQL语句: SELECT R<sub>1</sub> .first, R<sub>2</sub> .second FROM R<sub>3</sub> .JOIN R<sub>3</sub> ON R<sub>4</sub> .second = R<sub>2</sub> .first.

- $R_1$ 是兄弟关系;  $R_2$ 父子关系;  $R_1$   $R_2$ 是叔侄关系;
  - $\mathcal{R} = \{\langle a, b \rangle \mid a \Rightarrow b \in \mathbb{R}$  和b间有直航航线  $\}$ , R R 是城市之间经过一个城市转机的间接航线(记为 $\mathcal{R}^2$ );
  - $(=_4)^2 = =_4;$
- $\bullet \varnothing \mathcal{R} = \mathcal{R} \varnothing = \varnothing;$
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- 。 R1是兄弟关系; R2父子关系; R1 R2是叔侄关系;
- - $(=_4)^2 = =_4;$
  - $\mathcal{R} \subseteq A \times B$ ;  $\mathbb{N}$ ,  $\mathbb{1}_A \mathcal{R} = \mathcal{R} \mathbb{1}_B = \mathcal{R}$ ;
- $\bullet \varnothing \mathcal{R} = \mathcal{R} \varnothing = \varnothing;$
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- $R_1$ 是兄弟关系;  $R_2$ 父子关系;  $R_1$   $R_2$ 是叔侄关系;
- $(=_4)^2 = =_4;$   $\mathcal{R} \subseteq A \times B; \ M, \ \mathbb{1}_A \mathcal{R} = \mathcal{R} \mathbb{1}_B = \mathcal{R};$   $\varnothing \mathcal{R} = \mathcal{R} \varnothing = \varnothing.$
- 合成对应的SQL语句:  $SELECT \mathcal{R}_1$  .first,  $\mathcal{R}_2$  .second FROM  $\mathcal{R}_1$  . $IOIN \mathcal{R}_2$  . $ON \mathcal{R}_1$  second  $= \mathcal{R}_2$  .first

### Example

- 。 R1是兄弟关系; R2父子关系; R1 R2是叔侄关系;
- $(=_4)^2 = =_4;$
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### Example

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- $(=_4)^2 = =_4;$
- $\mathcal{R} \subseteq A \times B$ ; M, M<sub>A</sub> $\mathcal{R} = \mathcal{R} M$ <sub>B</sub> =  $\mathcal{R}$ ;
- $\bigcirc \varnothing \mathcal{R} = \mathcal{R} \varnothing = \varnothing;$

合成对应的SQL语句:  $SELECT \mathcal{R}_1$  .first,  $\mathcal{R}_2$  .second FROM  $\mathcal{R}_1$  JOIN  $\mathcal{R}_2$  ON  $\mathcal{R}_1$  .second =  $\mathcal{R}_2$  .first.



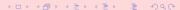
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- $(=_4)^2 = =_4;$
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#### Theorem

设 $\mathcal{R}_1 \subseteq A \times B$ ,  $\mathcal{R}_2, \mathcal{R}_3 \subseteq B \times C$ ,  $\mathcal{R}_4 \subseteq C \times D$ :

- $\mathcal{R}_1(\mathcal{R}_2 \cup \mathcal{R}_3) = \mathcal{R}_1 \mathcal{R}_2 \cup \mathcal{R}_1 \mathcal{R}_3 \ (\circ \mathsf{M} \cup \ \mathsf{hohm});$
- ③  $(R_2 \cup R_3) R_4 = R_2 R_4 \cup R_3 R_4$  (○对∪的分配律);



#### Proof.



#### Proof.

- $\exists b(\langle a,b\rangle \in \mathcal{R}_1 \land \langle b,c\rangle \in \mathcal{R}_2) \land \exists b(\langle a,b\rangle \in \mathcal{R}_1 \land \langle b,c\rangle \in \mathcal{R}_3)$



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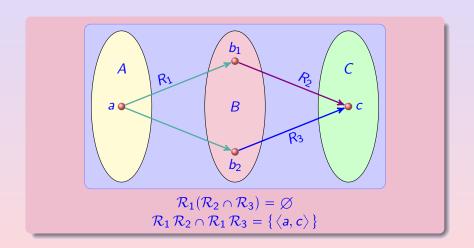


#### Proof.





# ②的反例



### Example

求所有可以间接通航的城市:



a和b可间接通航, iff:

$$\exists a_1, a_2, \ldots, a_{n-1} \ (a \mathcal{R} a_1 \land a_1 \mathcal{R} a_2 \land \ldots \land a_{n-1} \mathcal{R} b)$$

则:  $\langle a, a_1 \rangle \in \mathcal{R}$ ;

$$\langle a,a_2
angle\in\mathcal{R}^2;$$

. . . . .

$$\langle a, a_{n-1} \rangle \in (\mathcal{R}^{n-2}) \mathcal{R} \triangleq \mathcal{R}^{n-1};$$

$$\langle a,b\rangle\in(\mathcal{R}^{n-1})\,\mathcal{R}\triangleq\mathcal{R}^n$$
;

...a和b可间接通航。iff、∃ $n\langle a,b\rangle \in \mathbb{R}^n$ .

### Example

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a和b可间接通航, iff:

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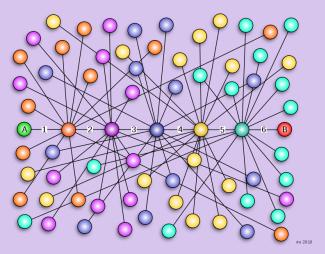
$$\langle a, a_2 \rangle \in \mathcal{R}^2$$
;

$$\langle a, a_{n-1} \rangle \in (\mathcal{R}^{n-2}) \mathcal{R} \triangleq \mathcal{R}^{n-1};$$

$$\langle a,b\rangle\in(\mathcal{R}^{n-1})\,\mathcal{R}\triangleq\mathcal{R}^n$$
;

∴ a n b可间接通航, iff,  $\exists n \langle a, b \rangle \in \mathbb{R}^n$ .

## Example 2: Six Degrees of Separation (六度分隔)



见http://en.wikipedia.org/wiki/Six\_degrees\_of\_separation.

### Definition (关系的幂, Power of relation)

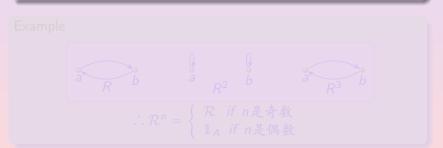
设 $\mathcal{R}$ 是A上的关系,  $n \in \mathbb{N}$ ,  $\mathcal{R}$ 的乘幂递归定义如下:

- $\mathbf{0} \ \mathcal{R}^0 = \mathbb{1}_A;$

### Definition (关系的幂, Power of relation)

设R是A上的关系,  $n \in \mathbb{N}$ , R的乘幂递归定义如下:

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## Definition (关系的幂, Power of relation)

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## Example



$$\therefore \mathcal{R}^n = \begin{cases} \mathcal{R} & \text{if } n \neq 3 \\ \mathbb{1}_A & \text{if } n \neq 3 \end{cases}$$

- $(\mathcal{R}^m)^n = \mathcal{R}^{mn};$

①的证明.

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#### Theorem

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### ①的证明

#### **Theorem**

- $(\mathcal{R}^m)^n = \mathcal{R}^{mn};$

### ①的证明.

- ① n = 0时, $\mathcal{R}^m \mathcal{R}^0 = \mathcal{R}^m \mathbb{1}_A = \mathcal{R}^m = \mathcal{R}^{m+0}$ ;
- ②  $\mathfrak{F}_n = k \mathfrak{H}, \, \mathcal{R}^m \mathcal{R}^k = \mathcal{R}^{m+k};$
- ③ n = k + 1时:

$$\mathcal{R}^{m}\mathcal{R}^{k+1}$$

$$=\mathcal{R}^{m}(\mathcal{R}^{k}\mathcal{R}) \quad \text{(def)}$$

$$=(\mathcal{R}^{m}\mathcal{R}^{k})\mathcal{R} \quad \text{(结合律)}$$

$$=\mathcal{R}^{m+k+1} \quad \text{(归纳假设)}$$

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 $(\mathcal{R}^m)^n = \mathcal{R}^{mn};$ 

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 $=\mathcal{R}^{m}(\mathcal{R}^{k}\mathcal{R})$  (def)
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 $=\mathcal{R}^{m+k+1}$  (def)

① 设|A| = n, 则存在 $i, j \in \{1, 2\}$ 0  $\{1, 2\}$ 0  $\{1, 2\}$ 0  $\{2\}$ 1  $\{2\}$ 2  $\{2\}$ 2  $\{2\}$ 3  $\{2\}$ 4  $\{2\}$ 3  $\{2\}$ 4  $\{2\}$ 5  $\{2\}$ 6  $\{2\}$ 9  $\{$ 

#### Proof.

Corollary

- ① 设|A| = n, 则存在i, j  $0 \le i < j \le 2^{n^2}$ , 使得:  $\mathcal{R}^i = \mathcal{R}^j$ .
- Proof.

- Corollary

#### Theorem

① 设|A| = n, 则存在 $i, j \in \{1, 2\}$ 0 《  $i < j \in \{2\}^n$ ", 使得:  $\mathbb{R}^i = \mathbb{R}^j$ .

#### Proof.

- $\bullet$  而 $\mathcal{R}^0$ ,  $\mathcal{R}^1$ , ...,  $\mathcal{R}^{2''}$  共有 $2^{n'}$  + 1项;
- 根据抽屉原则,  $\exists i, j \ 0 \leq i < j \leq 2^{n^{\epsilon}}$ , 使得:  $\mathcal{R}' = \mathcal{R}'$ .

Corollary

#### Theorem

① 设|A| = n, 则存在 $i, j \in \{0\}$ 0  $\{i < j \le 2^{n^2}\}$ , 使得:  $\mathcal{R}^i = \mathcal{R}^j$ .

#### Proof.

- ⑤ 而 $\mathbb{R}^0$ ,  $\mathbb{R}^1$ , ...,  $\mathbb{R}^{2^{n^2}}$  共有 $2^{n^2}$  + 1项;
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#### Theorem

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Corollary

#### Theorem

① 设|A| = n, 则存在 $i, j \in \{0\}$ 0 《 $i < j \in \{2\}^n$ 》,使得:  $\mathbb{R}^i = \mathbb{R}^j$ .

#### Proof.

- ③ 而 $\mathcal{R}^0, \mathcal{R}^1, \dots, \mathcal{R}^{2^{n^2}}$ 共有 $2^{n^2}$ +1项;
- ④ 根据抽屉原则,  $\exists i, j \in I < j \leq 2^{n^2}$ , 使得:  $\mathcal{R}' = \mathcal{R}^J$ .

Corollary

#### **Theorem**

① 设|A| = n, 则存在 $i, j \in \{0\}$ 0  $\leq i < j \leq 2^{n^2}$ , 使得:  $\mathcal{R}^i = \mathcal{R}^j$ .

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- ③ 而 $\mathcal{R}^0, \mathcal{R}^1, \dots, \mathcal{R}^{2^{n^2}}$ 共有 $2^{n^2}$ +1项;
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#### **Theorem**

① 设|A| = n, 则存在 $i, j \in \{0\}$ 0  $\{i < j \le 2^{n^2}\}$ , 使得:  $\mathcal{R}^i = \mathcal{R}^j$ .

### Proof.

- $2 : |\mathscr{P}(A \times A)| = 2^{n^2};$
- ③ 而 $\mathcal{R}^0, \mathcal{R}^1, \dots, \mathcal{R}^{2^{n^2}}$ 共有 $2^{n^2}$ +1项;
- ④ 根据抽屉原则,  $\exists i, j \ 0 \leq i < j \leq 2^{n^2}$ , 使得:  $\mathcal{R}^i = \mathcal{R}^j$ .

### Corollary

闭包

### Description (闭包, Closure)

········

数学上把包含某个给定的集合,并且具有某个性质的最小集合称 为闭包.

### Example

● 所有的可以间接通航的城市之间的关系, 是直接通航城市的传递闭包;

设 $\mathcal{R} \subseteq A \times B$ , 关系 $\mathcal{R}$ 的逆关系, 记为 $\widetilde{\mathcal{R}}$ (读作tilde), 定义如下:  $\widetilde{\mathcal{R}} = \{\langle y, x \rangle \mid \langle x, y \rangle \in \mathcal{R} \} \subseteq B \times A$ 

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### Example

 $\widetilde{\leqslant} = \geqslant$ ;  $\widetilde{\mathbb{1}}_A = \mathbb{1}_A$ ;  $\widetilde{\subseteq} = \supseteq$ ;

- 关系的逆是关系的对偶概念;如果R具有五性,则R也相应的具有;
- 关系的逆与关系的补是不同的概念:
  - $\overline{\mathcal{R}} = \{\langle x, y \rangle \mid \langle x, y \rangle \notin \mathcal{R}\} \subseteq A \times B$

设 $\mathcal{R} \subseteq A \times B$ , 关系 $\mathcal{R}$ 的逆关系, 记为 $\widetilde{\mathcal{R}}$ (读作tilde), 定义如下:  $\widetilde{\mathcal{R}} = \{\langle y, x \rangle \mid \langle x, y \rangle \in \mathcal{R}\} \subseteq B \times A$ 

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- 关系的逆与关系的补是不同的概念:
  - $\overline{\mathcal{R}} = \{\langle x, y \rangle \mid \langle x, y \rangle \notin \mathcal{R} \} \subseteq A \times B$

设 $\mathcal{R} \subseteq A \times B$ , 关系 $\mathcal{R}$ 的逆关系, 记为 $\widetilde{\mathcal{R}}$ (读作tilde), 定义如下:  $\widetilde{\mathcal{R}} = \{\langle y, x \rangle \mid \langle x, y \rangle \in \mathcal{R}\} \subseteq B \times A$ 

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#### Proof.

- ⇒  $\forall \langle x, y \rangle \in \mathcal{R}, \therefore \langle y, x \rangle \in \mathcal{R}; So \langle x, y \rangle \in \mathcal{R}$ ∴  $\mathcal{R} \subseteq \widetilde{\mathcal{R}}, but \stackrel{\widetilde{\mathcal{R}}}{\mathcal{R}} = \mathcal{R};$   $So, \stackrel{\widetilde{\mathcal{R}}}{\mathcal{R}} \subseteq \stackrel{\widetilde{\mathcal{R}}}{\mathcal{R}} = \mathcal{R} (\cdots \widetilde{\mathcal{R}} d \mathcal{L} d \mathcal{L$
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#### Definition

设 $\mathcal{R} \subseteq A^2$ ,  $\mathcal{R}$ 的自反(对称、传递)闭包 $\mathcal{R}'$ 是满足下述三条件的关系:

- ② R'是自反的(对称的、传递的);
- ③ 设 $\mathbb{R}''$ 是满足上述两条件的关系,则 $\mathbb{R}' \subseteq \mathbb{R}''$ .

分别记R的自反、对称和传递闭包为: r(R), s(R)和t(R).

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$$r(\mathcal{R}) = \mathcal{R} \cup \mathbb{1}_A$$
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# 闭包的构造(1/2)

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### ③的证明.

 $\exists m, n \ \langle x, y \rangle \in \mathcal{R}^m, \ \langle y, z \rangle \in \mathcal{R}^n; \ \therefore \langle x, z \rangle \in \mathcal{R}^m \mathcal{R}^n = \mathcal{R}^{m+n} \subseteq \bigcup_{i=1}^m \mathcal{R}^i;$ 

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③ 设传递关系 $\mathbb{R}' \supseteq \mathbb{R}$ ,则要证明:  $\bigcup_{i=1} \mathbb{R}' \subseteq \mathbb{R}'$ ;
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- ②  $\dot{\mathbf{Q}}_n = k$  时结论成立,n = k + 1 时:
  - $\therefore \exists v \langle x, v \rangle \in \mathcal{R}^k \land \langle v, z \rangle \in \mathcal{R}^k$ 
    - $S_{\alpha}/x$   $\sqrt{x}$   $C_{\alpha}/x$   $\sqrt{x}$   $C_{\alpha}/x$   $\sqrt{x}$
    - So  $\langle x, y \rangle \in \mathcal{R}' \land \langle y, z \rangle \in \mathcal{R}';$
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### Example

$$r(<) = <; s(<) = \neq;$$
  
 $s(\leq) =$  全域关系:  $r(\neq) =$  全域关系:

• 设尺是城市之间有直接航线的关系,则城市之间有间接航线的关系等于 $| \mathcal{R}^i |$ 

## **Examples**

### Example

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#### **Theorem**

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#### **Theorem**

$$\mathcal{E}|A|=n,\ \mathcal{R}\subseteq A^2,\ M:\ t(\mathcal{R})=\bigcup_{i=1}^n\mathcal{R}^i;$$

- ① 设 $\langle x_0, x_{n+1} \rangle \in \mathcal{R}^{n+1}$ ;
- $\exists x_1, x_2, \ldots, x_n x_0 \mathcal{R} x_1 \land x_1 \mathcal{R} x_2 \land \ldots x_n \mathcal{R} x_{n+1};$
- ③ 即尺关系图中有从x₀到xn+1长度为n+1的有向路径;
- ⑥ 而X1, X2,...,Xn+1 n+ 1个元素只能在|A| = n个元素中选取;
- ⑥ 所以根据抽屉原则,  $\exists 1 \leq i < j \leq n+1 \times_i = x_j$ ;
- $\bigcirc \therefore x_0 \mathcal{R} x_1 \wedge x_1 \mathcal{R} x_2 \wedge \cdots \wedge x_i \mathcal{R} x_{j+1} \wedge \cdots \wedge x_n \mathcal{R} x_{n+1};$ 
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- ③ 即 $\mathbb{R}$ 关系图中有从 $x_0$ 到 $x_{n+1}$ 长度为n+1的有向路径;
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## Examples

$$\mathcal{R} = \{\langle a,b \rangle, \langle b,c \rangle, \langle c,a \rangle \}$$
的传递闭包 
$$t(R) = R \cup R^2 \cup R^3$$

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### Propostion

$$rt(\mathcal{R}) = tr(\mathcal{R});$$

Proof.

$$rt(\mathcal{R}) = \bigcup_{i=0}^{\infty} \mathcal{R}^i; \quad tr(\mathcal{R}) = \bigcup_{i=1}^{\infty} (\mathbb{1}_A \cup \mathcal{R})^i$$

用归纳法证明下述等式即可:

$$\forall n \in \mathbb{N} \quad (\mathbb{1}_A \cup \mathcal{R})^n = \bigcup_{i \in \mathcal{R}} \mathcal{R}^i;$$

由此:

$$\forall n \ (\mathbb{1}_A \cup \mathcal{R})^n \subseteq \bigcup \mathcal{R}^i \subseteq \bigcup \mathcal{R}^i;$$

所以:

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 $rt(\mathcal{R}) = tr(\mathcal{R}).$ 

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$$\forall n \ (\mathbb{1}_A \cup \mathcal{R})^n \subseteq \bigcup_{i=0}^n \mathcal{R}^i \subseteq \bigcup_{i=0}^\infty \mathcal{R}^i;$$

所以:

$$\bigcup_{i=0}^{\infty} \mathcal{R}^{i} \subseteq \bigcup_{i=0}^{\infty} (\mathbb{1}_{A} \cup \mathcal{R})^{i} \subseteq \bigcup_{i=0}^{\infty} \mathcal{R}^{i};$$

$$rt(\mathcal{R}) = tr(\mathcal{R}).$$

### Propostion

$$rt(\mathcal{R}) = tr(\mathcal{R});$$

Proof.

$$rt(\mathcal{R}) = \bigcup_{i=0}^{\infty} \mathcal{R}^{i}; \quad tr(\mathcal{R}) = \bigcup_{i=1}^{\infty} (\mathbb{1}_{A} \cup \mathcal{R})^{i}$$

用归纳法证明下述等式即可:

$$\forall n \in \mathbb{N} \quad (\mathbb{1}_A \cup \mathcal{R})^n = \bigcup_{i=0}^n \mathcal{R}^i;$$

由此:

$$\forall n \ (\mathbb{1}_A \cup \mathcal{R})^n \subseteq \bigcup_{i=0}^n \mathcal{R}^i \subseteq \bigcup_{i=0}^\infty \mathcal{R}^i;$$

所以:

$$\bigcup_{i=0}^{\infty} \mathcal{R}^{i} \subseteq \bigcup_{i=0}^{\infty} (\mathbb{1}_{A} \cup \mathcal{R})^{i} \subseteq \bigcup_{i=0}^{\infty} \mathcal{R}^{i};$$

$$rt(\mathcal{R}) = tr(\mathcal{R}).$$

$$\forall n \in \mathbb{N} \quad (\mathbb{1}_A \cup \mathcal{R})^n = \bigcup_{i=0}^n \mathcal{R}^i;$$

- n = 0时上述等式成立;
- ② 设n = k时上述等式成立,则n = k + 1时:

(c) hfwang

## 闭包之间的关系(3/3)

$$\forall n \in \mathbb{N} \quad (\mathbb{1}_A \cup \mathcal{R})^n = \bigcup_{i=0}^n \mathcal{R}^i;$$

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$$\begin{aligned} & (\mathbb{1}_A \cup \mathcal{R})^{k+1} \\ &= (\mathbb{1}_A \cup \mathcal{R})^k (\mathbb{1}_A \cup \mathcal{R}) \\ &= \left(\bigcup_{i=0}^k \mathcal{R}^i\right) (\mathbb{1}_A \cup \mathcal{R}) \\ &= \left(\left(\bigcup_{i=0}^k \mathcal{R}^i\right) \mathcal{R}\right) \left(\bigcup_{i=0}^k \left(\bigcup_{i=0}^k \mathcal{R}\right) \mathcal{R}\right) \left(\bigcup_{i=0}^k \left(\bigcup_{i=0}^k \mathcal{R}\right) \mathcal{R}\right) \left(\bigcup_{i=0}^k \left(\bigcup_{i=0}^k \mathcal{R}\right) \mathcal{R}\right) \left(\bigcup_{i=0}^k \mathcal{R}\right) \left(\bigcup_{i=0}^k \left(\bigcup_{i=0}^k \mathcal{R}\right) \mathcal{R}\right) \left(\bigcup_{i=0}^k \left(\bigcup_{i=0}^k \mathcal{R}\right) \mathcal{R}\right) \left(\bigcup_{i=0}^k \left(\bigcup_{i=0}^k \mathcal{R}\right) \mathcal{R}\right) \left(\bigcup_{i=0}^k \mathcal{R}\right) \left(\bigcup_{i=0}^k$$

$$\int_{i=0}^{\infty} \left( \left( \bigcup_{i=0}^{k} \mathcal{R}^{i} \right) \mathbf{1}_{A} \right) dt$$

# 闭包之间的关系(3/3)

$$\forall n \in \mathbb{N} \quad (\mathbb{1}_A \cup \mathcal{R})^n = \bigcup_{i=0}^n \mathcal{R}^i;$$

- ① n = 0时上述等式成立;
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$$\begin{array}{l} & \textbf{(1}_A \cup \mathcal{R})^{k+1} \\ = (\mathbb{1}_A \cup \mathcal{R})^k (\mathbb{1}_A \cup \mathcal{R}) & \text{(by 乘幂的定义)} \\ = \begin{pmatrix} \binom{k}{i=0} & \mathcal{R}^i \end{pmatrix} (\mathbb{1}_A \cup \mathcal{R}) & \text{(by 归纳假设)} \\ = \begin{pmatrix} \binom{k}{i=0} & \mathcal{R}^i \end{pmatrix} \mathcal{R} \end{pmatrix} \bigcup \begin{pmatrix} \binom{k}{i=0} & \mathcal{R}^i \end{pmatrix} \mathbb{1}_A \end{pmatrix} & \text{(by 合成对并的分配率)} \\ = \begin{pmatrix} \binom{k+1}{i=0} & \mathcal{R}^i \end{pmatrix} \bigcup \begin{pmatrix} \binom{k}{i=0} & \mathcal{R}^i \end{pmatrix} & \text{(by 合成对并的分配率)} \\ = \begin{pmatrix} \binom{k+1}{i=0} & \mathcal{R}^i \end{pmatrix} \bigcup \begin{pmatrix} \binom{k}{i=0} & \mathcal{R}^i \end{pmatrix} & \text{(by 合成对并的分配率)} \\ \end{pmatrix}$$

# 闭包之间的关系(3/3)

Proof(continued).

$$\forall n \in \mathbb{N} \quad (\mathbb{1}_A \cup \mathcal{R})^n = \bigcup_{i=0}^n \mathcal{R}^i;$$

- □ n = 0时上述等式成立;
- ② 设n = k时上述等式成立,则n = k + 1时:

$$(\mathbb{1}_A \cup \mathcal{R})^{k+1}$$
  
=  $(\mathbb{1}_A \cup \mathcal{R})^k (\mathbb{1}_A \cup \mathcal{R})$ 

(by 乘幂的定义)

$$= \left(\bigcup_{i=0}^{k} \mathcal{R}^{i}\right) \mathcal{R} \cup \left(\left(\bigcup_{i=0}^{k} \mathcal{R}^{i}\right) \mathbb{1}_{A}\right)$$

(by 合成对并的分配率)

$$=\bigcup \mathcal{R}^i$$

Chfware

## 闭包之间的关系(3/3)

$$\forall n \in \mathbb{N} \quad (\mathbb{1}_A \cup \mathcal{R})^n = \bigcup_{i=0}^n \mathcal{R}^i;$$

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$$\begin{array}{l} \left(\mathbb{1}_{A} \cup \mathcal{R}\right)^{k+1} \\ = \left(\mathbb{1}_{A} \cup \mathcal{R}\right)^{k} (\mathbb{1}_{A} \cup \mathcal{R}) \\ = \left(\bigcup_{i=0}^{k} \mathcal{R}^{i}\right) (\mathbb{1}_{A} \cup \mathcal{R}) \\ = \left(\left(\bigcup_{i=0}^{k} \mathcal{R}^{i}\right) \mathcal{R}\right) \bigcup \left(\left(\bigcup_{i=0}^{k} \mathcal{R}^{i}\right) \mathbb{1}_{A}\right) \\ = \left(\bigcup_{i=1}^{k+1} \mathcal{R}^{i}\right) \bigcup \left(\bigcup_{i=0}^{k} \mathcal{R}^{i}\right) \mathbb{1}_{A} \end{array} \right) \text{ (by 合成对并的分配率)} \\ = \left(\bigcup_{i=1}^{k+1} \mathcal{R}^{i}\right) \bigcup \left(\bigcup_{i=0}^{k} \mathcal{R}^{i}\right) \\ = \left(\bigcup_{i=1}^{k+1} \mathcal{R}^{i}\right) \bigcup \left(\bigcup_{i=0}^{k} \mathcal{R}^{i}\right) \\ = \left(\bigcup_{i=0}^{k+1} \mathcal{R}^{i}\right) \bigcup \left(\bigcup_{i=0}^{k+1} \mathcal{R}^{i}\right) \\ = \left(\bigcup_{i=0}^{k+1} \mathcal{R}^{i}\right) \bigcup \left(\bigcup_{i=0}^{k+$$

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### Theorem

设
$$\mathcal{R} \subseteq A \times B$$
,  $\mathcal{S} \subseteq B \times C$ ;  $|A| = m$ ,  $|B| = n$ 和 $|C| = p$ , 则:  $M_{\mathcal{R} \mathcal{S}} = M_{\mathcal{R}} \cdot M_{\mathcal{S}}$ ; 其中:  $M_{\mathcal{R}} = (a_{ij})_{m \times n}$ ;  $M_{\mathcal{S}} = (b_{ij})_{n \times p}$   $M_{\mathcal{R}} \cdot M_{\mathcal{S}} = (c_{ij})_{m \times p}$ ;  $c_{ij} \triangleq \bigvee_{i} a_{ik} \wedge b_{kj}$ ;

设
$$A = \{x_1, x_2, \dots, x_m\}, B = \{y_1, y_2, \dots, y_n\}, C = \{z_1, z_2, \dots, z_p\}$$

#### **Theorem**

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- $c_{ij}=1$

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- $2 \iff \exists k \ a_{ik} = 1 \land b_{ki} = 1$

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$$c_{ij} = 1$$

设: 
$$M_{\mathcal{R}} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
;  $M_{\mathcal{S}} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ ;

则:

$$\mathbf{M}_{\mathcal{R}\mathcal{S}} = \mathbf{M}_{\mathcal{R}} \cdot \mathbf{M}_{\mathcal{S}} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$



### Description

$$M_{t(\mathcal{R})} = \sum_{i=1}^{n} M_{\mathcal{R}}^{i}$$

其中: MR是n阶方阵;

• 计算
$$M \cdot M$$
的每个元素 $c_{ij} = \bigvee_{k=1}^{n} a_{ik} \wedge b_{kj} \dots O(n)$ 

• 计算
$$\sum M_{\mathcal{R}}^{i}$$
 ...... $O(n^4)$ 

# 传递闭包的求解算法

### Description

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#### Definition

设 $A = \{a_1, a_2, \dots, a_n\}, \mathcal{R} \subseteq A^2, M$ 是 $\mathcal{R}$ 的关系矩阵; n阶方阵 $W_k$  递归定义如下:

- $\mathbf{0} \ W_0 = M;$
- ②  $W_k = (w_{ij}^k)_{n \times n}$ , 其中:  $w_{ij}^k = 1$ , iff,  $A_i = 1$ , iff,  $A_i = 1$ ,  $A_i$

Propostion

 $W_n = M_{t(\mathcal{R})}$ 

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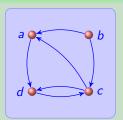
- **1**  $W_0 = M$ ;
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### Propostion

$$W_n = M_{t(\mathcal{R})}$$

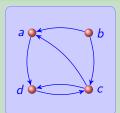


#### Example



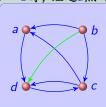
$$W_0 = \begin{pmatrix} a & b & c & d \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

#### Example



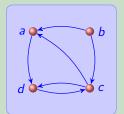
$$W_0 = \begin{pmatrix} a & b & c & d \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}_d^a$$

k = 1时,经过a点的路径:



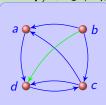
$$W_1 = \left(\begin{array}{cccc} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

#### Example



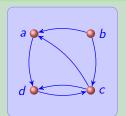
$$W_0 = \begin{pmatrix} a & b & c & d \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

k=2时,经过a,b点的路径:b没有引入的边,所以没有经过b的路径



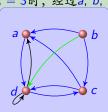
$$W_2 = \left(\begin{array}{cccc} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

#### Example

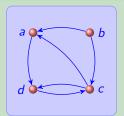


$$W_0 = \begin{pmatrix} a & b & c & d \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}_{d}^{a}$$

k = 3时, 经过a, b, c点的路径:

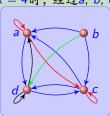


$$W_3 = \left(\begin{array}{cccc} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{array}\right)$$



$$W_0 = \begin{pmatrix} a & b & c & d \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

k = 4时, 经过a, b, c, d点的路径:



$$W_4 = \left(\begin{array}{cccc} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{array}\right)$$

## Description ( $W_k$ 和 $W_{k+1}$ 的关系)

$$w_{ii}^{k+1} = 1$$
, iff, 下述两条件之一成立:

- w<sup>k</sup><sub>ii</sub> = 1, 即从a<sub>i</sub>到a<sub>j</sub>有一条仅经过a<sub>1</sub>, a<sub>2</sub>,..., a<sub>k</sub>的有向路径;
- ② 有一条仅经过a<sub>1</sub>, a<sub>2</sub>,..., a<sub>k+1</sub>, 并且仅经过a<sub>k+1</sub>一次的路径: 如:

 $a_i, x_1, x_2, \dots, x_p, a_{k+1}, y_1, y_2, \dots, y_q, a_j$ 其中:  $x_1, x_2, \dots, x_p$ 和 $y_1, y_2, \dots, y_q$ 都在 $\{a_1, a_2, \dots, a_k\}$ 中;  $w_{(k+1)}^k = 1 \land w_{(k+1)i}^k = 1;$ 

故:

$$w_{ij}^{k+1} = w_{ij}^k \vee (w_{i(k+1)}^k \wedge w_{(k+1)j}^k)$$

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- $w_{ij}^k = 1$ ,  $p_{ij} M_{ai} = 1$ ,  $p_{ij} M_{$
- ② 有一条仅经过 $a_1, a_2, ..., a_{k+1}$ , 并且仅经过 $a_{k+1}$ 一次的路径: 如:

 $a_i, x_1, x_2, \dots, x_p, a_{k+1}, y_1, y_2, \dots, y_q, a_j$ 其中:  $x_1, x_2, \dots, x_p$ 和 $y_1, y_2, \dots, y_q$ 都在 $\{a_1, a_2, \dots, a_k\}$ 中;  $\vdots$   $w_{(k+1)}^k = 1 \land w_{(k+1)}^k = 1;$ 

故:

$$w_{ij}^{k+1} = w_{ij}^k \vee (w_{i(k+1)}^k \wedge w_{(k+1)j}^k)$$

# Wk的计算

### Description ( $W_k$ 和 $W_{k+1}$ 的关系)

$$w_{ii}^{k+1} = 1$$
, iff, 下述两条件之一成立:

- $w_{ij}^k = 1$ ,  $p_{ij} M_{ai} = 1$ ,  $p_{ij} M_{$
- ② 有一条仅经过a<sub>1</sub>, a<sub>2</sub>,..., a<sub>k+1</sub>, 并且仅经过a<sub>k+1</sub>一次的路径: 如:

$$a_i, x_1, x_2, \dots, x_p, a_{k+1}, y_1, y_2, \dots, y_q, a_j$$
  
其中:  $x_1, x_2, \dots, x_p$ 和 $y_1, y_2, \dots, y_q$ 都在 $\{a_1, a_2, \dots, a_k\}$ 中;  
 $w_{i(k+1)}^k = 1 \land w_{(k+1)j}^k = 1;$ 

故:

$$w_{ij}^{k+1} = w_{ij}^k \vee (w_{i(k+1)}^k \wedge w_{(k+1)j}^k)$$

```
Warshall算法.
procedure warshall (Matrix M_R)
  W := M_{\mathcal{R}};
  for k := 1 to n do {
     for i := 1 to n do {
       for j := 1 to n do {
          w_{ii} := w_{ii} \vee (w_{ik} \wedge w_{ki});
```

- ① 关系的合成
  - 关系的合成
  - 关系的幂
  - 关系的闭包
  - 传递闭包的求解算法