关系的运算

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关系上的运算

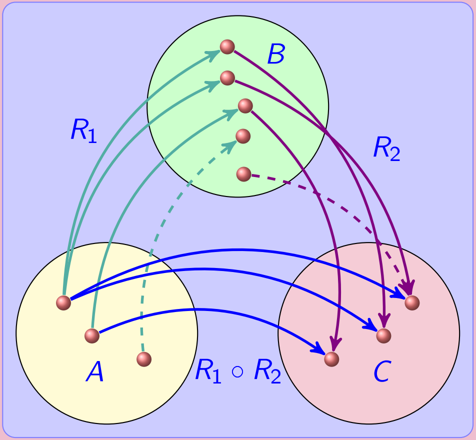
Remark

由于关系就是集合，因此集合上的运算也是关系的运算.

- “$\leqslant$” − “$\mathbb{1}_A$” = “$<$”;
- $\mathbb{R}$上有：“$\leqslant$” $\cap$ “$\geqslant$” = “$=$”;
- “$\leqslant$” $\cup$ “$\geqslant$” = $\mathbb{R}$上的全域关系;
- $\mathscr{P}(A)$上有：“$\subseteq$” $\cap$ “$\supseteq$” = “$=$”;
- “$\subseteq$” $\cup$ “$\supseteq$” $\neq$ 全域关系.

由于关系的对象是n重组，因此还有些一般集合不具有的运算.

关系合成的图示



合成的定义

Definition (合成关系, Composite Relation)

设$\mathcal{R}_1 \subseteq A \times B$, $\mathcal{R}_2 \subseteq B \times C$, $\mathcal{R}_1$和$\mathcal{R}_2$的合成记为$\mathcal{R}_1 \circ \mathcal{R}_2$ $(\mathcal{R}_1\,\mathcal{R}_2)$定义为:

$$\mathcal{R}_1\,\mathcal{R}_2 \triangleq \{\langle a, c\rangle \mid a \in A,\ c \in C \wedge \exists b \in B \wedge a\,\mathcal{R}_1\,b \wedge b\,\mathcal{R}_2\,c\}$$

是A到C上的关系.

Remark

合成的条件: 第一个关系的陪域(codomain)和第二个关系的域(domain)是相同的集合.

Example

Example

- $\mathcal{R}_1$是兄弟关系；$\mathcal{R}_2$父子关系；$\mathcal{R}_1\,\mathcal{R}_2$是叔侄关系；
- $\mathcal{R} = \{\langle a, b\rangle \mid \text{a和b间有直航航线}\}$, $\mathcal{R}\,\mathcal{R}$是城市之间经过一个城市转机的间接航线(记为$\mathcal{R}^2$);
- $(=_4)^2 = =_4$;
- $\mathcal{R} \subseteq A \times B$; 则, $\mathbb{1}_A\,\mathcal{R} = \mathcal{R}\,\mathbb{1}_B = \mathcal{R}$;
- $\varnothing\,\mathcal{R} = \mathcal{R}\,\varnothing = \varnothing$;
- 合成对应的SQL语句：$\mathit{SELECT}\ \ \mathcal{R}_1\,.\mathit{first},\ \mathcal{R}_2\,.\mathit{second}\ \ \mathit{FROM}\ \ \mathcal{R}_1\ \ \mathit{JOIN}\ \ \mathcal{R}_2\ \ \mathit{ON}\ \ \mathcal{R}_1\,.\mathit{second} = \mathcal{R}_2\,.\mathit{first}$.

合成的运算性质(1/2)

Theorem

设$\mathcal{R}_1 \subseteq A \times B$, $\mathcal{R}_2, \mathcal{R}_3 \subseteq B \times C$, $\mathcal{R}_4 \subseteq C \times D$:

1. $\mathcal{R}_1(\mathcal{R}_2 \cup \mathcal{R}_3) = \mathcal{R}_1 \mathcal{R}_2 \cup \mathcal{R}_1 \mathcal{R}_3$ ($\circ$对$\cup$的分配律);
2. $\mathcal{R}_1(\mathcal{R}_2 \cap \mathcal{R}_3) \boxed{\subseteq} \mathcal{R}_1 \mathcal{R}_2 \cap \mathcal{R}_1 \mathcal{R}_3$;
3. $(\mathcal{R}_2 \cup \mathcal{R}_3) \mathcal{R}_4 = \mathcal{R}_2 \mathcal{R}_4 \cup \mathcal{R}_3 \mathcal{R}_4$ ($\circ$对$\cup$的分配律);
4. $(\mathcal{R}_2 \cap \mathcal{R}_3) \mathcal{R}_4 \boxed{\subseteq} \mathcal{R}_2 \mathcal{R}_4 \cap \mathcal{R}_3 \mathcal{R}_4$;
5. $(\mathcal{R}_1 \mathcal{R}_2) \mathcal{R}_4 = \mathcal{R}_1(\mathcal{R}_2 \mathcal{R}_4)$ (结合律).

合成的运算性质(2/2)

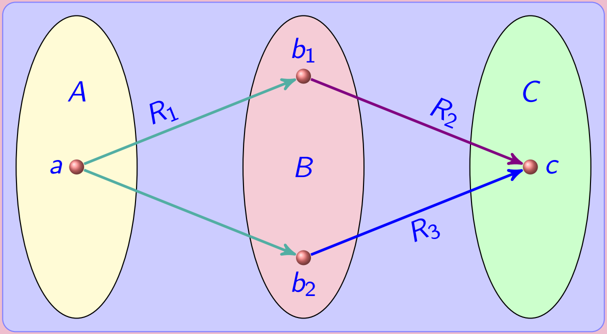
Proof.

②的证明:

1. $\forall \langle a, c\rangle \in \mathcal{R}_1(\mathcal{R}_2 \cap \mathcal{R}_3)$
2. $\Longleftrightarrow \exists b(\langle a, b\rangle \in \mathcal{R}_1 \wedge \langle b, c\rangle \in \mathcal{R}_2 \cap \mathcal{R}_3)$
3. $\Longleftrightarrow \exists b(\langle a, b\rangle \in \mathcal{R}_1 \wedge \langle b, c\rangle \in \mathcal{R}_2 \wedge \langle b, c\rangle \in \mathcal{R}_3)$
4. $\Longleftrightarrow \exists b((\langle a, b\rangle \in \mathcal{R}_1 \wedge \langle b, c\rangle \in \mathcal{R}_2) \wedge (\langle a, b\rangle \in \mathcal{R}_1 \wedge \langle b, c\rangle \in \mathcal{R}_3))$
5. $\Longrightarrow \exists b(\langle a, b\rangle \in \mathcal{R}_1 \wedge \langle b, c\rangle \in \mathcal{R}_2) \wedge \exists b(\langle a, b\rangle \in \mathcal{R}_1 \wedge \langle b, c\rangle \in \mathcal{R}_3)$
6. $\Longleftrightarrow \langle a, c\rangle \in \mathcal{R}_1\mathcal{R}_2 \wedge \langle a, c\rangle \in \mathcal{R}_1\mathcal{R}_3$
7. $\Longleftrightarrow \langle a, c\rangle \in \mathcal{R}_1\mathcal{R}_2 \cap \mathcal{R}_1\mathcal{R}_3$

□

②的反例

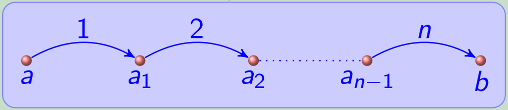


$$\mathcal{R}_1(\mathcal{R}_2 \cap \mathcal{R}_3) = \varnothing$$
$$\mathcal{R}_1\,\mathcal{R}_2 \cap \mathcal{R}_1\,\mathcal{R}_3 = \{\langle a, c\rangle\}$$

Example 1

Example

- 求所有可以间接通航的城市：



a和b可间接通航, iff:

$\exists a_1, a_2, \ldots, a_{n-1} \left(a \,\mathcal{R}\, a_1 \wedge a_1 \,\mathcal{R}\, a_2 \wedge \ldots \wedge a_{n-1} \,\mathcal{R}\, b \right)$

则: $\langle a, a_1 \rangle \in \mathcal{R}$;

$\langle a, a_2 \rangle \in \mathcal{R}^2$;

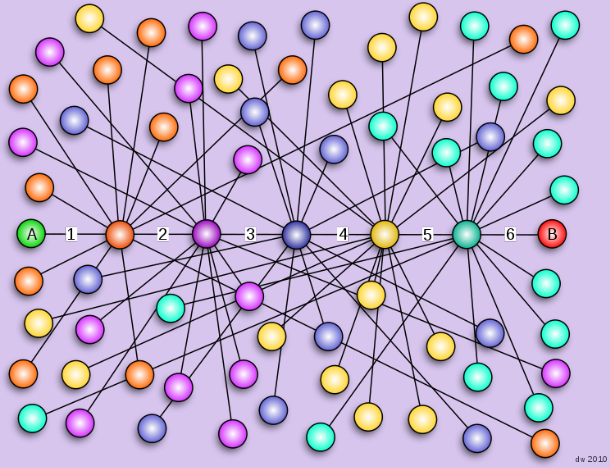
$\ldots\ldots$

$\langle a, a_{n-1} \rangle \in (\mathcal{R}^{n-2}) \,\mathcal{R} \triangleq \mathcal{R}^{n-1}$;

$\langle a, b \rangle \in (\mathcal{R}^{n-1}) \,\mathcal{R} \triangleq \mathcal{R}^{n}$;

$\therefore$ a和b可间接通航, iff, $\exists n \, \langle a, b \rangle \in \mathcal{R}^n$.

Example 2: Six Degrees of Separation (六度分隔)



见http://en.wikipedia.org/wiki/Six_degrees_of_separation.

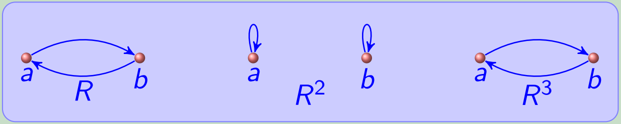
关系的幂

Definition (关系的幂, Power of relation)

设$\mathcal{R}$是A上的关系，$n \in \mathbb{N}$, $\mathcal{R}$的**乘幂**递归定义如下：

1. $\mathcal{R}^0 = \mathbb{1}_A$;
2. $\mathcal{R}^{n+1} = \mathcal{R}^n \mathcal{R}$.

Example



$$\therefore \mathcal{R}^n = \begin{cases} \mathcal{R} & \text{if } n\text{是奇数} \\ \mathbb{1}_A & \text{if } n\text{是偶数} \end{cases}$$

相关性质

Theorem

1. $\mathcal{R}^m\mathcal{R}^n = \mathcal{R}^{m+n}$;
2. $(\mathcal{R}^m)^n = \mathcal{R}^{mn}$;

①的证明.

对n用归纳法:

1. $n = 0$时, $\mathcal{R}^m\mathcal{R}^0 = \mathcal{R}^m\mathbb{1}_A = \mathcal{R}^m = \mathcal{R}^{m+0}$;
2. 设$n = k$时, $\mathcal{R}^m\mathcal{R}^k = \mathcal{R}^{m+k}$;
3. $n = k + 1$时:

$$
\begin{aligned}
& \mathcal{R}^m\mathcal{R}^{k+1} \\
&= \mathcal{R}^m(\mathcal{R}^k\mathcal{R}) && \text{(def)} \\
&= (\mathcal{R}^m\mathcal{R}^k)\mathcal{R} && \text{(结合律)} \\
&= \mathcal{R}^{m+k}\mathcal{R} && \text{(归纳假设)} \\
&= \mathcal{R}^{m+k+1} && \text{(def)}
\end{aligned}
$$

□

相关性质

Theorem

1. 设 $|A| = n$, 则存在 i, j $0 \leqslant i < j \leqslant 2^{n^2}$, 使得: $\mathcal{R}^i = \mathcal{R}^j$.

Proof.

1. $|A| = n$, $\therefore |A \times A| = n^2$;
2. $\therefore |\mathscr{P}(A \times A)| = 2^{n^2}$;
3. 而$\mathcal{R}^0, \mathcal{R}^1, \ldots, \mathcal{R}^{2^{n^2}}$ 共有$2^{n^2} + 1$项;
4. 根据抽屉原则, $\exists i, j$ $0 \leqslant i < j \leqslant 2^{n^2}$, 使得: $\mathcal{R}^i = \mathcal{R}^j$.

□

Corollary

$\forall m \in \mathbb{N}$ $\mathcal{R}^m \in \{\mathcal{R}^0, \mathcal{R}^1, \ldots, \mathcal{R}^{2^{n^2}-1}\}$.

闭包

Description (闭包, Closure)

数学上把包含某个给定的集合，并且具有某个性质的**最小**集合称为*闭包*.

Example

1. 所有的可以间接通航的城市之间的关系，
   是直接通航城市的传递闭包;

关系的逆

Definition (关系的逆)

设$\mathcal{R} \subseteq A \times B$，关系$\mathcal{R}$的逆关系，记为$\widetilde{\mathcal{R}}$(读作tilde)，定义如下：

$$\widetilde{\mathcal{R}} = \{\langle y, x\rangle \mid \langle x, y\rangle \in \mathcal{R}\} \subseteq B \times A$$

Example

- $\widetilde{\leqslant} = \geqslant$; $\widetilde{\mathbb{1}_A} = \mathbb{1}_A$; $\widetilde{\subseteq} = \supseteq$;
- 关系的逆是关系的对偶概念；如果$\mathcal{R}$具有五性，则$\widetilde{\mathcal{R}}$也相应的具有；
- 关系的逆与关系的补是不同的概念：

$$\overline{\mathcal{R}} = \{\langle x, y\rangle \mid \langle x, y\rangle \notin \mathcal{R}\} \subseteq A \times B$$

相关性质

Theorem

$\mathcal{R}$是对称关系，*iff*, $\mathcal{R} = \tilde{\mathcal{R}}$.

Proof.

$\Longrightarrow$ $\forall \langle x, y\rangle \in \mathcal{R}$, $\therefore \langle y, x\rangle \in \mathcal{R}$; *So* $\langle x, y\rangle \in \tilde{\mathcal{R}}$

$\therefore \mathcal{R} \subseteq \tilde{\mathcal{R}}$, *but* $\tilde{\tilde{\mathcal{R}}} = \mathcal{R}$;

So, $\tilde{\mathcal{R}} \subseteq \tilde{\tilde{\mathcal{R}}} = \mathcal{R}$ ($\because \tilde{\mathcal{R}}$也是对称关系), $\therefore \mathcal{R} = \tilde{\mathcal{R}}$;

$\Longleftarrow$ $\forall \langle x, y\rangle \in \mathcal{R}$ $\therefore \langle x, y\rangle \in \tilde{\mathcal{R}}$; *So* $\langle y, x\rangle \in \mathcal{R}$

所以$\mathcal{R}$是对称关系.

□

特性关系的闭包

Definition

设$\mathcal{R} \subseteq A^2$, $\mathcal{R}$的自反(对称、传递)闭包$\mathcal{R}'$是满足下述三条件的关系:

1. $\mathcal{R} \subseteq \mathcal{R}'$;
2. $\mathcal{R}'$是自反的(对称的、传递的);
3. 设$\mathcal{R}''$是满足上述两条件的关系, 则$\mathcal{R}' \subseteq \mathcal{R}''$.

分别记$\mathcal{R}$的自反、对称和传递闭包为: $r(\mathcal{R})$, $s(\mathcal{R})$和$t(\mathcal{R})$.

Propostion

$\mathcal{R}$是自反的(对称的、传递的), iff, $\mathcal{R} = r(\mathcal{R})$ $(s(\mathcal{R}), t(\mathcal{R}))$.

闭包的构造(1/2)

Theorem

$$① \ r(\mathcal{R}) = \mathcal{R} \cup \mathbb{1}_A; \quad ② \ s(\mathcal{R}) = \mathcal{R} \cup \tilde{\mathcal{R}}\ ; \quad ③ \ t(\mathcal{R}) = \bigcup_{i=1}^{\infty} \mathcal{R}^i.$$

③的证明.

1. $\mathcal{R} \subseteq \bigcup_{i=1}^{\infty} \mathcal{R}^i$;
2. $\forall \langle x, y \rangle \in \bigcup_{i=1}^{\infty} \mathcal{R}^i, \ \langle y, z \rangle \in \bigcup_{i=1}^{\infty} \mathcal{R}^i$;

   $\exists m, n \ \ \langle x, y \rangle \in \mathcal{R}^m, \ \langle y, z \rangle \in \mathcal{R}^n; \ \therefore \langle x, z \rangle \in \mathcal{R}^m \mathcal{R}^n = \mathcal{R}^{m+n} \subseteq \bigcup_{i=1}^{\infty} \mathcal{R}^i$;

   所以$\bigcup_{i=1}^{\infty} \mathcal{R}^i$是传递的.

$\square$

闭包的构造(2/2)

③的证明.

3. 设传递关系$\mathcal{R}' \supseteq \mathcal{R}$，则要证明：$\bigcup_{i=1}^{\infty} \mathcal{R}^i \subseteq \mathcal{R}'$;

   用归纳法证明：$\forall n\ \mathcal{R}^n \subseteq \mathcal{R}'$.

   1. $n = 1$时，$\mathcal{R} \subseteq \mathcal{R}'$;
   2. 设$n = k$时结论成立，$n = k + 1$时：
      设$\langle x, z\rangle \in \mathcal{R}^{k+1} = \mathcal{R}^k \mathcal{R}$;
      $\therefore \exists y\ \langle x, y\rangle \in \mathcal{R}^k \wedge \langle y, z\rangle \in \mathcal{R}$;
      $So\ \langle x, y\rangle \in \mathcal{R}' \wedge \langle y, z\rangle \in \mathcal{R}'$;
      $\therefore \langle x, z\rangle \in \mathcal{R}'$($\mathcal{R}'$是传递的).

□

Examples

Example

- $r(<) = \leqslant$; $s(<) =$ "$\neq$";
- $s(\leqslant) =$ 全域关系; $r(\neq) =$ 全域关系;
- 设$\mathcal{R}$是城市之间有直接航线的关系，则城市之间有间接航线的关系等于$\bigcup_{i=1}^{\infty} \mathcal{R}^i$.

有限集合的传递闭包

Theorem

设$|A| = n$, $\mathcal{R} \subseteq A^2$, 则: $t(\mathcal{R}) = \bigcup_{i=1}^{n} \mathcal{R}^i$;

Proof.

有限集合的传递闭包

Theorem

设$|A| = n$, $\mathcal{R} \subseteq A^2$, 则: $t(\mathcal{R}) = \bigcup_{i=1}^{n} \mathcal{R}^i$;

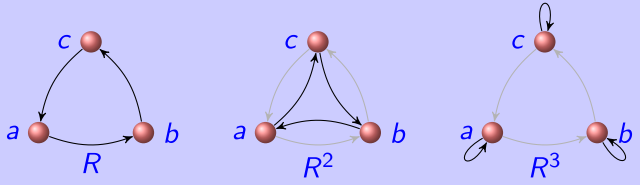
Proof.

1. 设$\langle x_0, x_{n+1}\rangle \in \mathcal{R}^{n+1}$;
2. $\exists\, x_1, x_2, \ldots, x_n \;\; x_0 \,\mathcal{R}\, x_1 \wedge x_1 \,\mathcal{R}\, x_2 \wedge \ldots x_n \,\mathcal{R}\, x_{n+1}$;
3. 即$\mathcal{R}$关系图中有从x_0到x_{n+1}长度为n+1的有向路径;
4. 而$x_1, x_2, \ldots, x_{n+1}$ n+1个元素只能在 $|A| = n$个元素中选取;
5. 所以根据抽屉原则, $\exists\, 1 \leqslant i < j \leqslant n+1 \;\; x_i = x_j$;
6. $\therefore \underbrace{x_0 \,\mathcal{R}\, x_1 \wedge x_1 \,\mathcal{R}\, x_2 \wedge \cdots \wedge x_i \,\mathcal{R}\, x_{j+1} \wedge \cdots \wedge x_n \,\mathcal{R}\, x_{n+1}}_{n+1-(j-i)\text{个}}$;
7. $\therefore \langle x_0, x_{n+1}\rangle \in \mathcal{R}^{n+1-(j-i)} \subseteq \bigcup_{i=1}^{n} \mathcal{R}^i$.

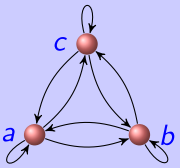
□

Examples

$\mathcal{R} = \{\langle a, b\rangle, \langle b.c\rangle, \langle c, a\rangle\}$的传递闭包



$$t(R) = R \cup R^2 \cup R^3$$



闭包之间的关系(1/3)

Propostion

设$\mathcal{R}$是自反关系，则，$t(\mathcal{R})$和$s(\mathcal{R})$也是自反关系；

$t(\mathcal{R})$是自反关系的证明.

1. $\mathcal{R}$是自反的，而 $I_A \subseteq \mathcal{R}$

闭包之间的关系(1/3)

Propostion

设$\mathcal{R}$是自反关系，则，$t(\mathcal{R})$和$s(\mathcal{R})$也是自反关系；

$t(\mathcal{R})$是自反关系的证明.

1. $\mathcal{R}$是自反的，iff, $\mathbb{1}_A \subseteq \mathcal{R}$
2. 而$t(\mathcal{R}) = \bigcup_{i=1}^{\infty} \mathcal{R}^i \supseteq \mathbb{1}_A$;
3. 所以，$\mathcal{R}$是自反的.

$\square$

闭包之间的关系(2/3)

Propostion

$$rt(\mathcal{R}) = tr(\mathcal{R});$$

Proof.

$$rt(\mathcal{R}) = \bigcup_{i=0}^{\infty} \mathcal{R}^i; \quad tr(\mathcal{R}) = \bigcup_{i=1}^{\infty} (\mathbb{1}_A \cup \mathcal{R})^i$$

用归纳法证明下述等式即可:

$$\forall n \in \mathbb{N} \quad (\mathbb{1}_A \cup \mathcal{R})^n = \bigcup_{i=0}^{n} \mathcal{R}^i;$$

由此:

$$\forall n \ (\mathbb{1}_A \cup \mathcal{R})^n \subseteq \bigcup_{i=0}^{n} \mathcal{R}^i \subseteq \bigcup_{i=0}^{\infty} \mathcal{R}^i;$$

所以:

$$\bigcup_{i=0}^{\infty} \mathcal{R}^i \subseteq \bigcup_{i=1}^{\infty} (\mathbb{1}_A \cup \mathcal{R})^i \subseteq \bigcup_{i=0}^{\infty} \mathcal{R}^i;$$

即:

$$rt(\mathcal{R}) = tr(\mathcal{R}).$$

□

闭包之间的关系(3/3)

Proof(continued).

$$\forall n \in \mathbb{N} \quad (\mathbb{1}_A \cup \mathcal{R})^n = \bigcup_{i=0}^{n} \mathcal{R}^i;$$

1. $n = 0$时上述等式成立;
2. 设$n = k$时上述等式成立，则$n = k + 1$时:

$$
\begin{aligned}
& (\mathbb{1}_A \cup \mathcal{R})^{k+1} \\
= & (\mathbb{1}_A \cup \mathcal{R})^k (\mathbb{1}_A \cup \mathcal{R}) && \text{(by 乘幂的定义)} \\
= & \left(\bigcup_{i=0}^{k} \mathcal{R}^i\right) (\mathbb{1}_A \cup \mathcal{R}) && \text{(by 归纳假设)} \\
= & \left(\left(\bigcup_{i=0}^{k} \mathcal{R}^i\right) \mathcal{R}\right) \bigcup \left(\left(\bigcup_{i=0}^{k} \mathcal{R}^i\right) \mathbb{1}_A\right) && \text{(by 合成对并的分配率)} \\
= & \left(\bigcup_{i=1}^{k+1} \mathcal{R}^i\right) \bigcup \left(\bigcup_{i=0}^{k} \mathcal{R}^i\right) && \text{(by 合成对并的分配率)} \\
= & \bigcup_{i=0}^{k+1} \mathcal{R}^i
\end{aligned}
$$

关系的合成与关系矩阵乘积

Theorem

设$\mathcal{R} \subseteq A \times B$, $\mathcal{S} \subseteq B \times C$; $|A| = m$, $|B| = n$和$|C| = p$, 则:

$$M_{\mathcal{R}\,\mathcal{S}} = M_{\mathcal{R}} \cdot M_{\mathcal{S}};$$

其中: $M_{\mathcal{R}} = (a_{ij})_{m\times n}$; $M_{\mathcal{S}} = (b_{ij})_{n\times p}$

$$M_{\mathcal{R}} \cdot M_{\mathcal{S}} = (c_{ij})_{m\times p}; \quad c_{ij} \triangleq \bigvee_{k=1}^{n} a_{ik} \wedge b_{kj};$$

Proof.

设$A = \{x_1, x_2, \ldots, x_m\}$, $B = \{y_1, y_2, \ldots, y_n\}$, $C = \{z_1, z_2, \ldots, z_p\}$

1. $c_{ij} = 1$
2. $\iff \exists k \; a_{ik} = 1 \wedge b_{kj} = 1$
3. $\iff \exists k \; x_i \,\mathcal{R}\, y_k \wedge y_k \,\mathcal{S}\, z_j$
4. $\iff x_i \,\mathcal{R}\,\mathcal{S}\, z_j$

□

Example

Example

设: $M_{\mathcal{R}} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$; $M_{\mathcal{S}} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$;

则:

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$M_{\mathcal{R}\,\mathcal{S}} = M_{\mathcal{R}} \cdot M_{\mathcal{S}} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

传递闭包的求解算法

Description

$$M_{t(\mathcal{R})} = \sum_{i=1}^{n} M_{\mathcal{R}}^{i}$$

其中：$M_{\mathcal{R}}$是n阶方阵；

- 计算$M \cdot M$的每个元素$c_{ij} = \bigvee_{k=1}^{n} a_{ik} \wedge b_{kj}$ . . . . $\mathrm{O}(n)$;
- 计算$M \cdot M$ . . . . $\mathrm{O}(n^3)$;
- 计算$\sum_{i=1}^{n} M_{\mathcal{R}}^{i}$ . . . . $\mathrm{O}(n^4)$.

*Warshall*算法可降算法的复杂度为：$\mathrm{O}(n^3)$.

Warshall算法

Definition

设$A = \{ a_1, a_2, \ldots, a_n \}$, $\mathcal{R} \subseteq A^2$, M是$\mathcal{R}$的关系矩阵；n阶方阵W_k递归定义如下：

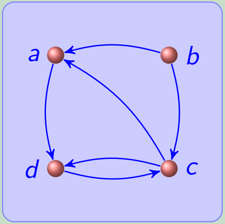
1. $W_0 = M$;
2. $W_k = (w_{ij}^k)_{n\times n}$, 其中：$w_{ij}^k = 1$, iff, 从$a_i$到$a_j$ 有一条仅经过$a_1, a_2, \ldots, a_k$的有向路径.

Propostion

$$W_n = M_{t(\mathcal{R})}$$

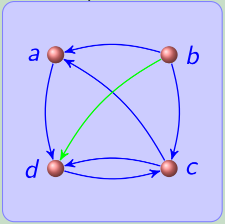
Example

Example



$$W_0 = \begin{array}{c} \\ \\ \\ \\ \\ \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}\begin{array}{c} \\ a \\ b \\ c \\ d \end{array}$$

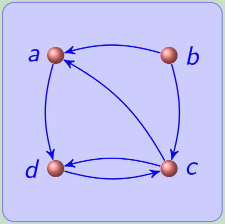
$k = 1$时，经过a点的路径：



$$W_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

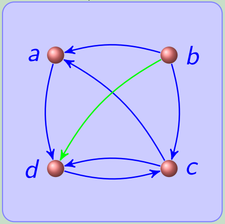
Example

Example



$$W_0 = \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array} \begin{array}{c} a \\ b \\ c \\ d \end{array}$$

$k = 2$时，经过a，b点的路径：b没有引入的边，所以没有经过b的路径



$$W_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Example

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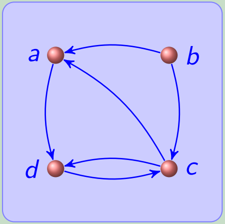
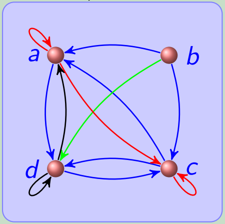


$$W_0 = \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array} \begin{array}{c} a \\ b \\ c \\ d \end{array}$$

$k = 4$时，经过a, b, c, d点的路径：



$$W_4 = \begin{pmatrix} \color{red}{1} & 0 & \color{red}{1} & 1 \\ 1 & 0 & 1 & \color{green}{1} \\ 1 & 0 & \color{red}{1} & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

W_k的计算

Description (W_k和W_{k+1}的关系)

$w_{ij}^{k+1}=1$, iff, 下述两条件之一成立:

1. $w_{ij}^{k}=1$, 即从a_i到a_j有一条仅经过$a_1,a_2,\ldots,a_k$的有向路径;
2. 有一条仅经过$a_1,a_2,\ldots,a_{k+1}$, 并且仅经过a_{k+1}一次的路径: 如:

$$a_i,x_1,x_2,\ldots,x_p,a_{k+1},y_1,y_2,\ldots,y_q,a_j$$

其中: $x_1,x_2,\ldots,x_p$和$y_1,y_2,\ldots,y_q$都在$\{a_1,a_2,\ldots,a_k\}$中;

$\therefore \quad w_{i(k+1)}^{k}=1\wedge w_{(k+1)j}^{k}=1;$

故:

$$\boxed{w_{ij}^{k+1}=w_{ij}^{k}\vee(w_{i(k+1)}^{k}\wedge w_{(k+1)j}^{k})}$$

Warshall算法

Warshall算法.

```
procedure warshall(Matrix M_R)
{
  W := M_R;
  for k := 1 to n do {
    for i := 1 to n do {
      for j := 1 to n do {
        w_ij := w_ij ∨ (w_ik ∧ w_kj);
      }
    }
  }
}
```

□

本章小节

1. 关系的合成
   - 关系的合成
   - 关系的幂
   - 关系的闭包
   - 传递闭包的求解算法