关系的运算

School of Computer
Wuhan University

1 关系的合成
- 关系的合成
- 关系的幂
- 关系的闭包
- 传递闭包的求解算法

关系上的运算

Remark

由于关系就是集合，因此集合上的运算也是关系的运算.

- “$\leqslant$” − “$\mathbb{1}_A$” = “$<$”;
- $\mathbb{R}$上有: “$\leqslant$” $\cap$ “$\geqslant$” = “$=$”;
- “$\leqslant$” $\cup$ “$\geqslant$” = $\mathbb{R}$上的全域关系;
- $\mathscr{P}(A)$上有: “$\subseteq$” $\cap$ “$\supseteq$” = “$=$”;
- “$\subseteq$” $\cup$ “$\supseteq$” $\neq$ 全域关系.

由于关系的对象是n重组，因此还有些一般集合不具有的运算.

关系上的运算

Remark

由于关系就是集合，因此集合上的运算也是关系的运算.

- "$\leqslant$" $-$ "$\mathbb{1}_A$" $=$ "$<$";
- $\mathbb{R}$上有："$\leqslant$" $\cap$ "$\geqslant$" $=$ "$=$";
- "$\leqslant$" $\cup$ "$\geqslant$" $=$ $\mathbb{R}$上的全域关系;
- $\mathscr{P}(A)$上有："$\subseteq$" $\cap$ "$\supseteq$" $=$ "$=$";
- "$\subseteq$" $\cup$ "$\supseteq$" $\neq$ 全域关系.

由于关系的对象是n重组，因此还有些一般集合不具有的运算.

关系上的运算

Remark

由于关系就是集合，因此集合上的运算也是关系的运算.

- "$\leqslant$" $-$ "$\mathbb{1}_A$" $=$ "$<$";
- $\mathbb{R}$上有: "$\leqslant$" $\cap$ "$\geqslant$" $=$ "$=$";
- "$\leqslant$" $\cup$ "$\geqslant$" $=$ $\mathbb{R}$上的全域关系;
- $\mathscr{P}(A)$上有: "$\subseteq$" $\cap$ "$\supseteq$" $=$ "$=$";
- "$\subseteq$" $\cup$ "$\supseteq$" $\neq$ 全域关系.

由于关系的对象是n重组，因此还有些一般集合不具有的运算.

关系上的运算

Remark

由于关系就是集合，因此集合上的运算也是关系的运算.

- "$\leqslant$" $-$ "$\mathbb{1}_A$" $=$ "$<$";
- $\mathbb{R}$上有："$\leqslant$" $\cap$ "$\geqslant$" $=$ "$=$";
- "$\leqslant$" $\cup$ "$\geqslant$" $=$ $\mathbb{R}$上的全域关系;
- $\mathscr{P}(A)$上有："$\subseteq$" $\cap$ "$\supseteq$" $=$ "$=$";
- "$\subseteq$" $\cup$ "$\supseteq$" $\neq$ 全域关系.

由于关系的对象是n重组，因此还有些一般集合不具有的运算.

关系上的运算

Remark

由于关系就是集合，因此集合上的运算也是关系的运算.

- "$\leqslant$" $-$ "$\mathbb{1}_A$" $=$ "$<$";
- $\mathbb{R}$上有: "$\leqslant$" $\cap$ "$\geqslant$" $=$ "$=$";
- "$\leqslant$" $\cup$ "$\geqslant$" $=$ $\mathbb{R}$上的全域关系;
- $\mathscr{P}(A)$上有: "$\subseteq$" $\cap$ "$\supseteq$" $=$ "$=$";
- "$\subseteq$" $\cup$ "$\supseteq$" $\neq$ 全域关系.

由于关系的对象是n重组，因此还有些一般集合不具有的运算.

关系上的运算

Remark

由于关系就是集合，因此集合上的运算也是关系的运算.

- “$\leqslant$” − “$\mathbb{1}_A$” = “$<$”;
- $\mathbb{R}$上有: “$\leqslant$” ∩ “$\geqslant$” = “$=$”;
- “$\leqslant$” ∪ “$\geqslant$” = $\mathbb{R}$上的全域关系;
- $\mathscr{P}(A)$上有: “$\subseteq$” ∩ “$\supseteq$” = “$=$”;
- “$\subseteq$” ∪ “$\supseteq$” ≠ 全域关系.

由于关系的对象是n重组，因此还有些一般集合不具有的运算.

关系上的运算

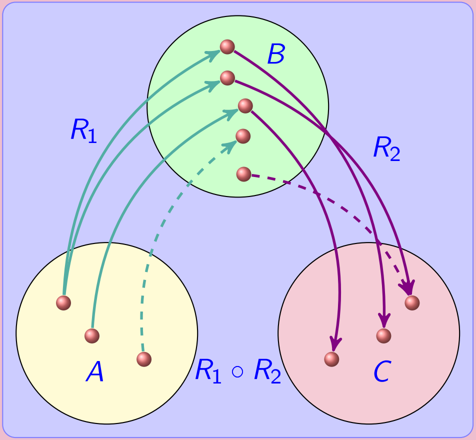
Remark

由于关系就是集合，因此集合上的运算也是关系的运算.

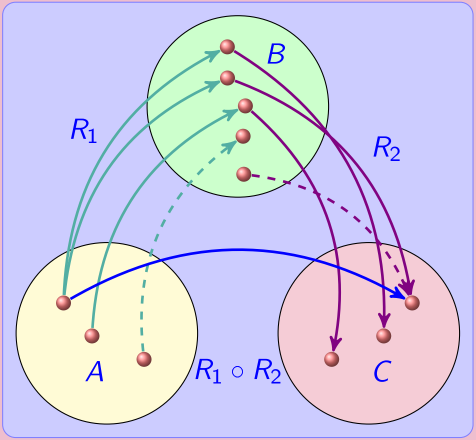
- "$\leqslant$" $-$ "$\mathbb{1}_A$" $=$ "$<$";
- $\mathbb{R}$上有："$\leqslant$" $\cap$ "$\geqslant$" $=$ "$=$";
- "$\leqslant$" $\cup$ "$\geqslant$" $=$ $\mathbb{R}$上的全域关系;
- $\mathscr{P}(A)$上有："$\subseteq$" $\cap$ "$\supseteq$" $=$ "$=$";
- "$\subseteq$" $\cup$ "$\supseteq$" $\neq$ 全域关系.

由于关系的对象是n重组，因此还有些一般集合不具有的运算.

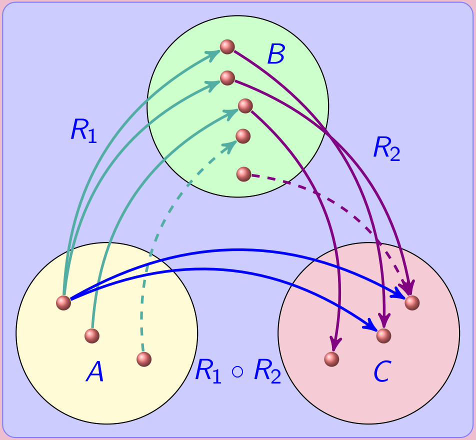
关系合成的图示



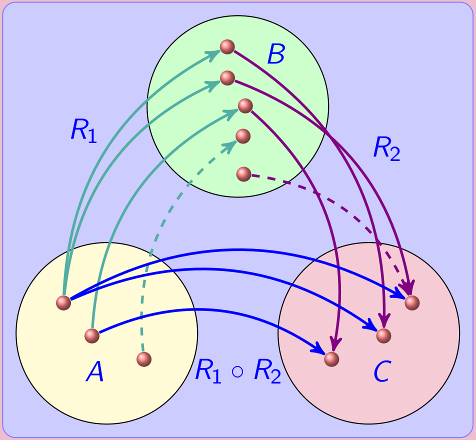
关系合成的图示



关系合成的图示



关系合成的图示



合成的定义

Definition (合成关系, Composite Relation)

设$\mathcal{R}_1 \subseteq A \times B$, $\mathcal{R}_2 \subseteq B \times C$, $\mathcal{R}_1$和$\mathcal{R}_2$的合成记为$\mathcal{R}_1 \circ \mathcal{R}_2$ $(\mathcal{R}_1\,\mathcal{R}_2)$定义为:

$$\mathcal{R}_1\,\mathcal{R}_2 \triangleq \{\langle a, c\rangle \mid a \in A,\ c \in C \wedge \exists b \in B \wedge a\,\mathcal{R}_1\,b \wedge b\,\mathcal{R}_2\,c\}$$

是A到C上的关系.

Remark

合成的条件: 第一个关系的陪域(codomain)和第二个关系的域(domain)是相同的集合.

Example

Example

$\mathcal{R}_1$是兄弟关系；$\mathcal{R}_2$父子关系；$\mathcal{R}_1\,\mathcal{R}_2$是叔侄关系；

$\mathcal{R} = \{\langle a, b\rangle \mid$ a和b间有直航航线$\}$, $\mathcal{R}\,\mathcal{R}$是城市之间经过一个城市转机的间接航线(记为$\mathcal{R}^2$);

- $(=_4)^2 = =_4$;
- $\mathcal{R} \subseteq A \times B$; 则, $\mathbb{1}_A\,\mathcal{R} = \mathcal{R}\,\mathbb{1}_B = \mathcal{R}$;
- $\varnothing\,\mathcal{R} = \mathcal{R}\,\varnothing = \varnothing$;
- 合成对应的SQL语句：*SELECT* $\mathcal{R}_1$*.first*, $\mathcal{R}_2$*.second FROM* $\mathcal{R}_1$ *JOIN* $\mathcal{R}_2$ *ON* $\mathcal{R}_1$*.second* $=$ $\mathcal{R}_2$*.first*.

Example

Example

- $\mathcal{R}_1$是兄弟关系；$\mathcal{R}_2$父子关系；$\mathcal{R}_1\,\mathcal{R}_2$是叔侄关系；
  $\mathcal{R} = \{\langle a, b\rangle \mid$ a和b间有直航航线$\}$, $\mathcal{R}\,\mathcal{R}$是城市之间经过一个城市转机的间接航线(记为$\mathcal{R}^2$);
  $(=_4)^2 = =_4$;
- $\mathcal{R} \subseteq A \times B$; 则, $\mathbb{1}_A \mathcal{R} = \mathcal{R}\,\mathbb{1}_B = \mathcal{R}$;
- $\emptyset\,\mathcal{R} = \mathcal{R}\,\emptyset = \emptyset$;
- 合成对应的SQL语句：*SELECT* $\mathcal{R}_1$*.first*, $\mathcal{R}_2$*.second FROM* $\mathcal{R}_1$ *JOIN* $\mathcal{R}_2$ *ON* $\mathcal{R}_1$*.second* = $\mathcal{R}_2$*.first*.

Example

Example

- $\mathcal{R}_1$是兄弟关系；$\mathcal{R}_2$父子关系；$\mathcal{R}_1\mathcal{R}_2$是叔侄关系；
- $\mathcal{R} = \{\langle a, b\rangle \mid$ a和b间有直航航线$\}$, $\mathcal{R}\mathcal{R}$是城市之间经过一个城市转机的间接航线(记为$\mathcal{R}^2$)；

  $(=_4)^2 = =_4$;

  $\mathcal{R} \subseteq A \times B$; 则, $\mathbb{1}_A\mathcal{R} = \mathcal{R}\mathbb{1}_B = \mathcal{R}$;
- $\emptyset\mathcal{R} = \mathcal{R}\emptyset = \emptyset$;
- 合成对应的SQL语句：*SELECT* $\mathcal{R}_1$*.first*, $\mathcal{R}_2$*.second FROM* $\mathcal{R}_1$ *JOIN* $\mathcal{R}_2$ *ON* $\mathcal{R}_1$*.second* $=$ $\mathcal{R}_2$*.first*.

Example

Example

- $\mathcal{R}_1$是兄弟关系；$\mathcal{R}_2$父子关系；$\mathcal{R}_1\mathcal{R}_2$是叔侄关系；
- $\mathcal{R} = \{\langle a, b\rangle \mid \text{a和b间有直航航线}\}$, $\mathcal{R}\mathcal{R}$是城市之间经过一个城市转机的间接航线(记为$\mathcal{R}^2$)；
- $(=_4)^2 = =_4$;

  $\mathcal{R} \subseteq A \times B$; 则, $\mathbb{1}_A\mathcal{R} = \mathcal{R}\mathbb{1}_B = \mathcal{R}$;

  $\emptyset\mathcal{R} = \mathcal{R}\emptyset = \emptyset$;
- 合成对应的SQL语句：*SELECT* $\mathcal{R}_1$*.first*, $\mathcal{R}_2$*.second FROM* $\mathcal{R}_1$ *JOIN* $\mathcal{R}_2$ *ON* $\mathcal{R}_1$*.second* = $\mathcal{R}_2$*.first*.

Example

Example

- $\mathcal{R}_1$是兄弟关系; $\mathcal{R}_2$父子关系; $\mathcal{R}_1\mathcal{R}_2$是叔侄关系;
- $\mathcal{R} = \{\langle a, b\rangle \mid \text{a和b间有直航航线}\}$, $\mathcal{R}\mathcal{R}$是城市之间经过一个城市转机的间接航线(记为$\mathcal{R}^2$);
- $(=_4)^2 = =_4$;
- $\mathcal{R} \subseteq A \times B$; 则, $\mathbb{1}_A\mathcal{R} = \mathcal{R}\mathbb{1}_B = \mathcal{R}$;
  $\emptyset\mathcal{R} = \mathcal{R}\emptyset = \emptyset$;
  合成对应的SQL语句: $SELECT\ \mathcal{R}_1.first,\ \mathcal{R}_2.second\ FROM\ \mathcal{R}_1\ JOIN\ \mathcal{R}_2\ ON\ \mathcal{R}_1.second = \mathcal{R}_2.first.$

Example

Example

- $\mathcal{R}_1$是兄弟关系；$\mathcal{R}_2$父子关系；$\mathcal{R}_1\,\mathcal{R}_2$是叔侄关系；
- $\mathcal{R} = \{\langle a, b\rangle \mid \text{a和b间有直航航线}\}$, $\mathcal{R}\,\mathcal{R}$是城市之间经过一个城市转机的间接航线(记为$\mathcal{R}^2$)；
- $(=_4)^2 = =_4$;
- $\mathcal{R} \subseteq A \times B$; 则, $\mathbb{1}_A\,\mathcal{R} = \mathcal{R}\,\mathbb{1}_B = \mathcal{R}$;
- $\varnothing\,\mathcal{R} = \mathcal{R}\,\varnothing = \varnothing$;

合成对应的SQL语句：$SELECT\ \mathcal{R}_1.first,\ \mathcal{R}_2.second\ FROM\ \mathcal{R}_1\ JOIN\ \mathcal{R}_2\ ON\ \mathcal{R}_1.second = \mathcal{R}_2.first.$

Example

Example

- $\mathcal{R}_1$是兄弟关系；$\mathcal{R}_2$父子关系；$\mathcal{R}_1\,\mathcal{R}_2$是叔侄关系；
- $\mathcal{R} = \{\langle a, b\rangle \mid \text{a和b间有直航航线}\}$, $\mathcal{R}\,\mathcal{R}$是城市之间经过一个城市转机的间接航线(记为$\mathcal{R}^2$)；
- $(=_4)^2 = =_4$;
- $\mathcal{R} \subseteq A \times B$; 则, $\mathbb{1}_A\,\mathcal{R} = \mathcal{R}\,\mathbb{1}_B = \mathcal{R}$;
- $\emptyset\,\mathcal{R} = \mathcal{R}\,\emptyset = \emptyset$;
- 合成对应的SQL语句：*SELECT* $\mathcal{R}_1$ *.first*, $\mathcal{R}_2$ *.second* *FROM* $\mathcal{R}_1$ *JOIN* $\mathcal{R}_2$ *ON* $\mathcal{R}_1$ *.second* $=$ $\mathcal{R}_2$ *.first*.

合成的运算性质(1/2)

Theorem

设$\mathcal{R}_1 \subseteq A \times B$, $\mathcal{R}_2, \mathcal{R}_3 \subseteq B \times C$, $\mathcal{R}_4 \subseteq C \times D$:

1. $\mathcal{R}_1(\mathcal{R}_2 \cup \mathcal{R}_3) = \mathcal{R}_1\mathcal{R}_2 \cup \mathcal{R}_1\mathcal{R}_3$ (∘对∪的分配律);
2. $\mathcal{R}_1(\mathcal{R}_2 \cap \mathcal{R}_3) \boxed{\subseteq} \mathcal{R}_1\mathcal{R}_2 \cap \mathcal{R}_1\mathcal{R}_3$;
3. $(\mathcal{R}_2 \cup \mathcal{R}_3)\mathcal{R}_4 = \mathcal{R}_2\mathcal{R}_4 \cup \mathcal{R}_3\mathcal{R}_4$ (∘对∪的分配律);
4. $(\mathcal{R}_2 \cap \mathcal{R}_3)\mathcal{R}_4 \boxed{\subseteq} \mathcal{R}_2\mathcal{R}_4 \cap \mathcal{R}_3\mathcal{R}_4$;
5. $(\mathcal{R}_1\mathcal{R}_2)\mathcal{R}_4 = \mathcal{R}_1(\mathcal{R}_2\mathcal{R}_4)$ (结合律).

合成的运算性质(2/2)

Proof.

②的证明:

1. $\forall\langle a,c\rangle\in\mathcal{R}_1(\mathcal{R}_2\cap\mathcal{R}_3)$
2. $\Longleftrightarrow \exists b(\langle a,b\rangle\in\mathcal{R}_1\wedge\langle b,c\rangle\in\mathcal{R}_2\cap\mathcal{R}_3)$
3. $\Longleftrightarrow \exists b(\langle a,b\rangle\in\mathcal{R}_1\wedge\langle b,c\rangle\in\mathcal{R}_2\wedge\langle b,c\rangle\in\mathcal{R}_3)$
4. $\Longleftrightarrow \exists b((\langle a,b\rangle\in\mathcal{R}_1\wedge\langle b,c\rangle\in\mathcal{R}_2)\wedge(\langle a,b\rangle\in\mathcal{R}_1\wedge\langle b,c\rangle\in\mathcal{R}_3))$
5. $\Longrightarrow \exists b(\langle a,b\rangle\in\mathcal{R}_1\wedge\langle b,c\rangle\in\mathcal{R}_2)\wedge\exists b(\langle a,b\rangle\in\mathcal{R}_1\wedge\langle b,c\rangle\in\mathcal{R}_3)$
6. $\Longleftrightarrow \langle a,c\rangle\in\mathcal{R}_1\mathcal{R}_2\wedge\langle a,c\rangle\in\mathcal{R}_1\mathcal{R}_3$
7. $\Longleftrightarrow \langle a,c\rangle\in\mathcal{R}_1\mathcal{R}_2\cap\mathcal{R}_1\mathcal{R}_3$

□

合成的运算性质(2/2)

Proof.

②的证明:

1. $\forall\langle a,c\rangle \in \mathcal{R}_1(\mathcal{R}_2 \cap \mathcal{R}_3)$
2. $\iff \exists b(\langle a,b\rangle \in \mathcal{R}_1 \wedge \langle b,c\rangle \in \mathcal{R}_2 \cap \mathcal{R}_3)$
3. $\iff \exists b(\langle a,b\rangle \in \mathcal{R}_1 \wedge \langle b,c\rangle \in \mathcal{R}_2 \wedge \langle b,c\rangle \in \mathcal{R}_3)$
4. $\iff \exists b((\langle a,b\rangle \in \mathcal{R}_1 \wedge \langle b,c\rangle \in \mathcal{R}_2) \wedge (\langle a,b\rangle \in \mathcal{R}_1 \wedge \langle b,c\rangle \in \mathcal{R}_3))$
5. $\Longrightarrow \exists b(\langle a,b\rangle \in \mathcal{R}_1 \wedge \langle b,c\rangle \in \mathcal{R}_2) \wedge \exists b(\langle a,b\rangle \in \mathcal{R}_1 \wedge \langle b,c\rangle \in \mathcal{R}_3)$
6. $\iff \langle a,c\rangle \in \mathcal{R}_1\mathcal{R}_2 \wedge \langle a,c\rangle \in \mathcal{R}_1\mathcal{R}_3$
7. $\iff \langle a,c\rangle \in \mathcal{R}_1\mathcal{R}_2 \cap \mathcal{R}_1\mathcal{R}_3$

$\square$

合成的运算性质(2/2)

Proof.

②的证明:

1. $\forall\langle a,c\rangle \in \mathcal{R}_1(\mathcal{R}_2 \cap \mathcal{R}_3)$
2. $\iff \exists b(\langle a,b\rangle \in \mathcal{R}_1 \wedge \langle b,c\rangle \in \mathcal{R}_2 \cap \mathcal{R}_3)$
3. $\iff \exists b(\langle a,b\rangle \in \mathcal{R}_1 \wedge \langle b,c\rangle \in \mathcal{R}_2 \wedge \langle b,c\rangle \in \mathcal{R}_3)$
4. $\iff \exists b((\langle a,b\rangle \in \mathcal{R}_1 \wedge \langle b,c\rangle \in \mathcal{R}_2) \wedge (\langle a,b\rangle \in \mathcal{R}_1 \wedge \langle b,c\rangle \in \mathcal{R}_3))$
5. $\Longrightarrow \exists b(\langle a,b\rangle \in \mathcal{R}_1 \wedge \langle b,c\rangle \in \mathcal{R}_2) \wedge \exists b(\langle a,b\rangle \in \mathcal{R}_1 \wedge \langle b,c\rangle \in \mathcal{R}_3)$
6. $\iff \langle a,c\rangle \in \mathcal{R}_1\mathcal{R}_2 \wedge \langle a,c\rangle \in \mathcal{R}_1\mathcal{R}_3$
7. $\iff \langle a,c\rangle \in \mathcal{R}_1\mathcal{R}_2 \cap \mathcal{R}_1\mathcal{R}_3$

□

合成的运算性质(2/2)

Proof.

②的证明:

1. $\forall\langle a,c\rangle\in\mathcal{R}_1(\mathcal{R}_2\cap\mathcal{R}_3)$
2. $\Longleftrightarrow\exists b(\langle a,b\rangle\in\mathcal{R}_1\wedge\langle b,c\rangle\in\mathcal{R}_2\cap\mathcal{R}_3)$
3. $\Longleftrightarrow\exists b(\langle a,b\rangle\in\mathcal{R}_1\wedge\langle b,c\rangle\in\mathcal{R}_2\wedge\langle b,c\rangle\in\mathcal{R}_3)$
4. $\Longleftrightarrow\exists b((\langle a,b\rangle\in\mathcal{R}_1\wedge\langle b,c\rangle\in\mathcal{R}_2)\wedge(\langle a,b\rangle\in\mathcal{R}_1\wedge\langle b,c\rangle\in\mathcal{R}_3))$
5. $\Longrightarrow\exists b(\langle a,b\rangle\in\mathcal{R}_1\wedge\langle b,c\rangle\in\mathcal{R}_2)\wedge\exists b(\langle a,b\rangle\in\mathcal{R}_1\wedge\langle b,c\rangle\in\mathcal{R}_3)$
6. $\Longleftrightarrow\langle a,c\rangle\in\mathcal{R}_1\mathcal{R}_2\wedge\langle a,c\rangle\in\mathcal{R}_1\mathcal{R}_3$
7. $\Longleftrightarrow\langle a,c\rangle\in\mathcal{R}_1\mathcal{R}_2\cap\mathcal{R}_1\mathcal{R}_3$

$\square$

合成的运算性质(2/2)

Proof.

②的证明:

1. $\qquad \forall \langle a, c\rangle \in \mathcal{R}_1(\mathcal{R}_2 \cap \mathcal{R}_3)$
2. $\iff \exists b(\langle a, b\rangle \in \mathcal{R}_1 \wedge \langle b, c\rangle \in \mathcal{R}_2 \cap \mathcal{R}_3)$
3. $\iff \exists b(\langle a, b\rangle \in \mathcal{R}_1 \wedge \langle b, c\rangle \in \mathcal{R}_2 \wedge \langle b, c\rangle \in \mathcal{R}_3)$
4. $\iff \exists b((\langle a, b\rangle \in \mathcal{R}_1 \wedge \langle b, c\rangle \in \mathcal{R}_2) \wedge (\langle a, b\rangle \in \mathcal{R}_1 \wedge \langle b, c\rangle \in \mathcal{R}_3))$
5. $\Longrightarrow \exists b(\langle a, b\rangle \in \mathcal{R}_1 \wedge \langle b, c\rangle \in \mathcal{R}_2) \wedge \exists b(\langle a, b\rangle \in \mathcal{R}_1 \wedge \langle b, c\rangle \in \mathcal{R}_3)$
6. $\iff \langle a, c\rangle \in \mathcal{R}_1\mathcal{R}_2 \wedge \langle a, c\rangle \in \mathcal{R}_1\mathcal{R}_3$
7. $\iff \langle a, c\rangle \in \mathcal{R}_1\mathcal{R}_2 \cap \mathcal{R}_1\mathcal{R}_3$

□

合成的运算性质(2/2)

Proof.

②的证明:

1. $\quad\forall\langle a,c\rangle \in \mathcal{R}_1(\mathcal{R}_2 \cap \mathcal{R}_3)$
2. $\iff \exists b(\langle a,b\rangle \in \mathcal{R}_1 \wedge \langle b,c\rangle \in \mathcal{R}_2 \cap \mathcal{R}_3)$
3. $\iff \exists b(\langle a,b\rangle \in \mathcal{R}_1 \wedge \langle b,c\rangle \in \mathcal{R}_2 \wedge \langle b,c\rangle \in \mathcal{R}_3)$
4. $\iff \exists b((\langle a,b\rangle \in \mathcal{R}_1 \wedge \langle b,c\rangle \in \mathcal{R}_2) \wedge (\langle a,b\rangle \in \mathcal{R}_1 \wedge \langle b,c\rangle \in \mathcal{R}_3))$
5. $\Longrightarrow \exists b(\langle a,b\rangle \in \mathcal{R}_1 \wedge \langle b,c\rangle \in \mathcal{R}_2) \wedge \exists b(\langle a,b\rangle \in \mathcal{R}_1 \wedge \langle b,c\rangle \in \mathcal{R}_3)$
6. $\iff \langle a,c\rangle \in \mathcal{R}_1\mathcal{R}_2 \wedge \langle a,c\rangle \in \mathcal{R}_1\mathcal{R}_3$
7. $\iff \langle a,c\rangle \in \mathcal{R}_1\mathcal{R}_2 \cap \mathcal{R}_1\mathcal{R}_3$

□

合成的运算性质(2/2)

Proof.

②的证明:

1. $\forall\langle a,c\rangle \in \mathcal{R}_1(\mathcal{R}_2 \cap \mathcal{R}_3)$
2. $\iff \exists b(\langle a,b\rangle \in \mathcal{R}_1 \wedge \langle b,c\rangle \in \mathcal{R}_2 \cap \mathcal{R}_3)$
3. $\iff \exists b(\langle a,b\rangle \in \mathcal{R}_1 \wedge \langle b,c\rangle \in \mathcal{R}_2 \wedge \langle b,c\rangle \in \mathcal{R}_3)$
4. $\iff \exists b((\langle a,b\rangle \in \mathcal{R}_1 \wedge \langle b,c\rangle \in \mathcal{R}_2) \wedge (\langle a,b\rangle \in \mathcal{R}_1 \wedge \langle b,c\rangle \in \mathcal{R}_3))$
5. $\Longrightarrow \exists b(\langle a,b\rangle \in \mathcal{R}_1 \wedge \langle b,c\rangle \in \mathcal{R}_2) \wedge \exists b(\langle a,b\rangle \in \mathcal{R}_1 \wedge \langle b,c\rangle \in \mathcal{R}_3)$
6. $\iff \langle a,c\rangle \in \mathcal{R}_1\mathcal{R}_2 \wedge \langle a,c\rangle \in \mathcal{R}_1\mathcal{R}_3$
7. $\iff \langle a,c\rangle \in \mathcal{R}_1\mathcal{R}_2 \cap \mathcal{R}_1\mathcal{R}_3$

□

合成的运算性质(2/2)

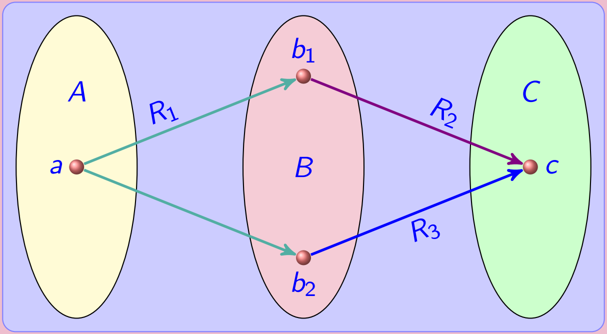
Proof.

②的证明:

1. $\forall \langle a, c\rangle \in \mathcal{R}_1(\mathcal{R}_2 \cap \mathcal{R}_3)$
2. $\iff \exists b(\langle a, b\rangle \in \mathcal{R}_1 \wedge \langle b, c\rangle \in \mathcal{R}_2 \cap \mathcal{R}_3)$
3. $\iff \exists b(\langle a, b\rangle \in \mathcal{R}_1 \wedge \langle b, c\rangle \in \mathcal{R}_2 \wedge \langle b, c\rangle \in \mathcal{R}_3)$
4. $\iff \exists b((\langle a, b\rangle \in \mathcal{R}_1 \wedge \langle b, c\rangle \in \mathcal{R}_2) \wedge (\langle a, b\rangle \in \mathcal{R}_1 \wedge \langle b, c\rangle \in \mathcal{R}_3))$
5. $\Longrightarrow \exists b(\langle a, b\rangle \in \mathcal{R}_1 \wedge \langle b, c\rangle \in \mathcal{R}_2) \wedge \exists b(\langle a, b\rangle \in \mathcal{R}_1 \wedge \langle b, c\rangle \in \mathcal{R}_3)$
6. $\iff \langle a, c\rangle \in \mathcal{R}_1\mathcal{R}_2 \wedge \langle a, c\rangle \in \mathcal{R}_1\mathcal{R}_3$
7. $\iff \langle a, c\rangle \in \mathcal{R}_1\mathcal{R}_2 \cap \mathcal{R}_1\mathcal{R}_3$

□

②的反例



$$\mathcal{R}_1(\mathcal{R}_2 \cap \mathcal{R}_3) = \varnothing$$
$$\mathcal{R}_1\,\mathcal{R}_2 \cap \mathcal{R}_1\,\mathcal{R}_3 = \{\langle a, c\rangle\}$$

Example 1

Example

求所有可以间接通航的城市:

$$a \xrightarrow{1} a_1 \xrightarrow{2} a_2 \cdots\cdots a_{n-1} \xrightarrow{n} b$$

a和b可间接通航, iff:

$$\exists a_1, a_2, \ldots, a_{n-1}\ (a\,\mathcal{R}\,a_1 \wedge a_1\,\mathcal{R}\,a_2 \wedge \ldots \wedge a_{n\text{-}1}\,\mathcal{R}\,b)$$

则: $\langle a, a_1\rangle \in \mathcal{R}$;

$\langle a, a_2\rangle \in \mathcal{R}^2$;

$\cdots\cdots$

$\langle a, a_{n-1}\rangle \in (\mathcal{R}^{n-2})\,\mathcal{R} \triangleq \mathcal{R}^{n-1}$;

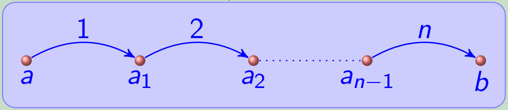
$\langle a, b\rangle \in (\mathcal{R}^{n-1})\,\mathcal{R} \triangleq \mathcal{R}^{n}$;

$\therefore$ a和b可间接通航, iff, $\exists n\ \langle a, b\rangle \in \mathcal{R}^n$.

Example 1

Example

- 求所有可以间接通航的城市:



a和b可间接通航, iff:

$$\exists a_1, a_2, \ldots, a_{n-1}\ (a\,\mathcal{R}\,a_1 \wedge a_1\,\mathcal{R}\,a_2 \wedge \ldots \wedge a_{n-1}\,\mathcal{R}\,b)$$

则: $\langle a, a_1\rangle \in \mathcal{R}$;

$\langle a, a_2\rangle \in \mathcal{R}^2$;

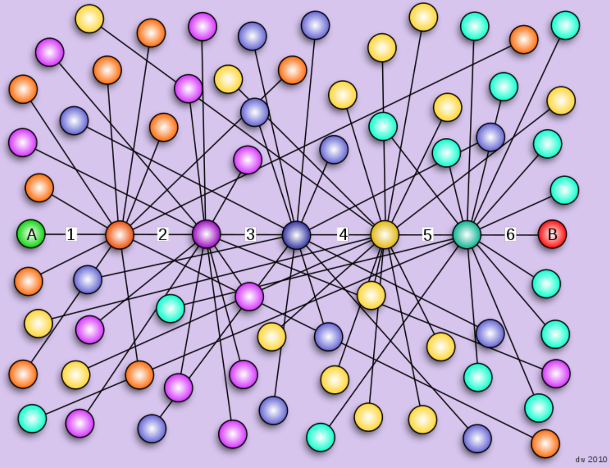
......

$\langle a, a_{n-1}\rangle \in (\mathcal{R}^{n-2})\,\mathcal{R} \triangleq \mathcal{R}^{n-1}$;

$\langle a, b\rangle \in (\mathcal{R}^{n-1})\,\mathcal{R} \triangleq \mathcal{R}^{n}$;

$\therefore$ a和b可间接通航, iff, $\exists n\ \langle a, b\rangle \in \mathcal{R}^n$.

Example 2: Six Degrees of Separation (六度分隔)



见http://en.wikipedia.org/wiki/Six_degrees_of_separation.

关系的幂

Definition (关系的幂, Power of relation)

设$\mathcal{R}$是A上的关系，$n \in \mathbb{N}$, $\mathcal{R}$的乘幂递归定义如下：

1. $\mathcal{R}^0 = \mathbb{1}_A$;
2. $\mathcal{R}^{n+1} = \mathcal{R}^n\mathcal{R}$.

Example

$$\therefore \mathcal{R}^n = \begin{cases} \mathcal{R} & \text{if } n\text{是奇数} \\ \mathbb{1}_A & \text{if } n\text{是偶数} \end{cases}$$

关系的幂

Definition (关系的幂, Power of relation)

设$\mathcal{R}$是A上的关系，$n \in \mathbb{N}$, $\mathcal{R}$的**乘幂**递归定义如下:

1. $\mathcal{R}^0 = \mathbb{1}_A$;
2. $\mathcal{R}^{n+1} = \mathcal{R}^n \mathcal{R}$.

Example

$$\therefore \mathcal{R}^n = \begin{cases} \mathcal{R} & \text{if } n\text{是奇数} \\ \mathbb{1}_A & \text{if } n\text{是偶数} \end{cases}$$

关系的幂

Definition (关系的幂, Power of relation)

设$\mathcal{R}$是A上的关系，$n \in \mathbb{N}$, $\mathcal{R}$的**乘幂**递归定义如下：

1. $\mathcal{R}^0 = \mathbb{1}_A$;
2. $\mathcal{R}^{n+1} = \mathcal{R}^n \mathcal{R}$.

Example

$$\therefore \mathcal{R}^n = \begin{cases} \mathcal{R} & \text{if } n\text{是奇数} \\ \mathbb{1}_A & \text{if } n\text{是偶数} \end{cases}$$

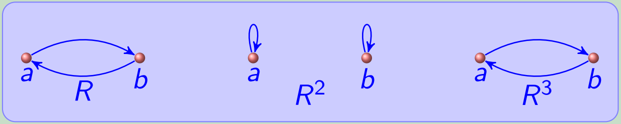
关系的幂

Definition (关系的幂, Power of relation)

设$\mathcal{R}$是A上的关系，$n \in \mathbb{N}$, $\mathcal{R}$的**乘幂**递归定义如下：

1. $\mathcal{R}^0 = \mathbb{1}_A$;
2. $\mathcal{R}^{n+1} = \mathcal{R}^n \mathcal{R}$.

Example



$$\therefore \mathcal{R}^n = \begin{cases} \mathcal{R} & \text{if } n\text{是奇数} \\ \mathbb{1}_A & \text{if } n\text{是偶数} \end{cases}$$

相关性质

Theorem

1. $\mathcal{R}^m\mathcal{R}^n = \mathcal{R}^{m+n}$;
2. $(\mathcal{R}^m)^n = \mathcal{R}^{mn}$;

①的证明.

对n用归纳法:

1. $n = 0$时，$\mathcal{R}^m\mathcal{R}^0 = \mathcal{R}^m I_A = \mathcal{R}^m = \mathcal{R}^{m+0}$;
2. 设$n = k$时，$\mathcal{R}^m\mathcal{R}^k = \mathcal{R}^{m+k}$;
3. $n = k+1$时：

$$
\begin{aligned}
& \mathcal{R}^m\mathcal{R}^{k+1} \\
= & \mathcal{R}^m(\mathcal{R}^k\mathcal{R}) && (\text{def}) \\
= & (\mathcal{R}^m\mathcal{R}^k)\mathcal{R} && (\text{结合律}) \\
= & \mathcal{R}^{m+k}\mathcal{R} && (\text{归纳假设}) \\
= & \mathcal{R}^{m+k+1} && (\text{def})
\end{aligned}
$$

相关性质

Theorem

1. $\mathcal{R}^m\mathcal{R}^n = \mathcal{R}^{m+n}$;
2. $(\mathcal{R}^m)^n = \mathcal{R}^{mn}$;

①的证明.

对n用归纳法:

相关性质

Theorem

1. $\mathcal{R}^m\mathcal{R}^n = \mathcal{R}^{m+n}$;
2. $(\mathcal{R}^m)^n = \mathcal{R}^{mn}$;

①的证明.

对n用归纳法:

相关性质

Theorem

1. $\mathcal{R}^m\mathcal{R}^n = \mathcal{R}^{m+n}$;
2. $(\mathcal{R}^m)^n = \mathcal{R}^{mn}$;

①的证明.

对n用归纳法:

1. $n = 0$时, $\mathcal{R}^m\mathcal{R}^0 = \mathcal{R}^m\mathbb{1}_A = \mathcal{R}^m = \mathcal{R}^{m+0}$;
2. 设$n = k$时, $\mathcal{R}^m\mathcal{R}^k = \mathcal{R}^{m+k}$;
3. $n = k+1$时:

$$\begin{aligned}
&\mathcal{R}^m\mathcal{R}^{k+1} \\
&= \mathcal{R}^m(\mathcal{R}^k\mathcal{R}) && \text{(def)} \\
&= (\mathcal{R}^m\mathcal{R}^k)\mathcal{R} && \text{(结合律)} \\
&= \mathcal{R}^{m+k}\mathcal{R} && \text{(归纳假设)} \\
&= \mathcal{R}^{m+k+1} && \text{(def)}
\end{aligned}$$

□

相关性质

Theorem

1. $\mathcal{R}^m\mathcal{R}^n = \mathcal{R}^{m+n}$;
2. $(\mathcal{R}^m)^n = \mathcal{R}^{mn}$;

①的证明.

对n用归纳法:

1. $n = 0$时, $\mathcal{R}^m\mathcal{R}^0 = \mathcal{R}^m\mathbb{1}_A = \mathcal{R}^m = \mathcal{R}^{m+0}$;
2. 设$n = k$时, $\mathcal{R}^m\mathcal{R}^k = \mathcal{R}^{m+k}$;
3. $n = k + 1$时:

$$
\begin{aligned}
& \mathcal{R}^m\mathcal{R}^{k+1} \\
= {} & \mathcal{R}^m(\mathcal{R}^k\mathcal{R}) && \text{(def)} \\
= {} & (\mathcal{R}^m\mathcal{R}^k)\mathcal{R} && \text{(结合律)} \\
= {} & \mathcal{R}^{m+k}\mathcal{R} && \text{(归纳假设)} \\
= {} & \mathcal{R}^{m+k+1} && \text{(def)}
\end{aligned}
$$

□

相关性质

Theorem

1. $\mathcal{R}^m\mathcal{R}^n = \mathcal{R}^{m+n}$;
2. $(\mathcal{R}^m)^n = \mathcal{R}^{mn}$;

①的证明.

对n用归纳法:

1. $n = 0$时, $\mathcal{R}^m\mathcal{R}^0 = \mathcal{R}^m\mathbb{1}_A = \mathcal{R}^m = \mathcal{R}^{m+0}$;
2. 设$n = k$时, $\mathcal{R}^m\mathcal{R}^k = \mathcal{R}^{m+k}$;
3. $n = k+1$时:

$$\begin{aligned}
&\mathcal{R}^m\mathcal{R}^{k+1} \\
&= \mathcal{R}^m(\mathcal{R}^k\mathcal{R}) && \text{(def)} \\
&= (\mathcal{R}^m\mathcal{R}^k)\mathcal{R} && \text{(结合律)} \\
&= \mathcal{R}^{m+k}\mathcal{R} && \text{(归纳假设)} \\
&= \mathcal{R}^{m+k+1} && \text{(def)}
\end{aligned}$$

□

相关性质

Theorem

1. $\mathcal{R}^m\mathcal{R}^n = \mathcal{R}^{m+n}$;
2. $(\mathcal{R}^m)^n = \mathcal{R}^{mn}$;

①的证明.

对n用归纳法:

1. $n = 0$时, $\mathcal{R}^m\mathcal{R}^0 = \mathcal{R}^m\mathbb{1}_A = \mathcal{R}^m = \mathcal{R}^{m+0}$;
2. 设$n = k$时, $\mathcal{R}^m\mathcal{R}^k = \mathcal{R}^{m+k}$;
3. $n = k+1$时:

$$
\begin{aligned}
& \mathcal{R}^m\mathcal{R}^{k+1} \\
= {} & \mathcal{R}^m(\mathcal{R}^k\mathcal{R}) && \text{(def)} \\
= {} & (\mathcal{R}^m\mathcal{R}^k)\mathcal{R} && \text{(结合律)} \\
= {} & \mathcal{R}^{m+k}\mathcal{R} && \text{(归纳假设)} \\
= {} & \mathcal{R}^{m+k+1} && \text{(def)}
\end{aligned}
$$

□

相关性质

Theorem

1. 设 $|A| = n$，则存在 i, j $0 \leqslant i < j \leqslant 2^{n^2}$，使得：$\mathcal{R}^i = \mathcal{R}^j$.

Proof.

1. $|A| = n \Rightarrow |A \times A| = n^2$.
2. $\Rightarrow |\mathcal{P}(A \times A)| = 2^{n^2}$;
3. 而 $\mathcal{R}^0, \mathcal{R}^1, \ldots, \mathcal{R}^{2^{n^2}}$ 共有 $2^{n^2} + 1$ 项;
4. 根据抽屉原则，$\exists i, j$ $0 \leqslant i < j \leqslant 2^{n^2}$，使得：$\mathcal{R}^i = \mathcal{R}^j$.

□

Corollary

$\forall m \in \mathbb{N}$ $\mathcal{R}^m \in \{\mathcal{R}^0, \mathcal{R}^1, \ldots, \mathcal{R}^{2^{n^2}-1}\}$.

相关性质

Theorem

1. 设$|A| = n$, 则存在i, j $0 \leqslant i < j \leqslant 2^{n^2}$, 使得: $\mathcal{R}^i = \mathcal{R}^j$.

Proof.

1. $|A| = n$, $\therefore |A \times A| = n^2$;
2. $\therefore |\mathcal{P}(A \times A)| = 2^{n^2}$;
3. 而$\mathcal{R}^0, \mathcal{R}^1, \ldots, \mathcal{R}^{2^{n^2}}$共有$2^{n^2}+1$项;
4. 根据抽屉原则, $\exists i, j$ $0 \leqslant i < j \leqslant 2^{n^2}$, 使得: $\mathcal{R}^i = \mathcal{R}^j$.

$\square$

Corollary

$\forall m \in \mathbb{N}$ $\mathcal{R}^m \in \{\mathcal{R}^0, \mathcal{R}^1, \ldots, \mathcal{R}^{2^{n^2}-1}\}$.

相关性质

Theorem

1. 设$|A| = n$, 则存在i, j $0 \leqslant i < j \leqslant 2^{n^2}$, 使得: $\mathcal{R}^i = \mathcal{R}^j$.

Proof.

1. $|A| = n$, $\therefore |A \times A| = n^2$;
2. $\therefore |\mathscr{P}(A \times A)| = 2^{n^2}$;
3. 而$\mathcal{R}^0, \mathcal{R}^1, \ldots, \mathcal{R}^{2^{n^2}}$ 共有$2^{n^2} + 1$项;
4. 根据抽屉原则, $\exists i, j$ $0 \leqslant i < j \leqslant 2^{n^2}$, 使得: $\mathcal{R}^i = \mathcal{R}^j$.

$\square$

Corollary

$\forall m \in \mathbb{N}$ $\mathcal{R}^m \in \{\mathcal{R}^0, \mathcal{R}^1, \ldots, \mathcal{R}^{2^{n^2}-1}\}$.

相关性质

Theorem

1. 设$|A| = n$, 则存在i, j $0 \leqslant i < j \leqslant 2^{n^2}$, 使得: $\mathcal{R}^i = \mathcal{R}^j$.

Proof.

1. $|A| = n$, $\therefore$ $|A \times A| = n^2$;
2. $\therefore$ $|\mathscr{P}(A \times A)| = 2^{n^2}$;
3. 而$\mathcal{R}^0, \mathcal{R}^1, \ldots, \mathcal{R}^{2^{n^2}}$共有$2^{n^2} + 1$项;
4. 根据抽屉原则, $\exists i, j$ $0 \leqslant i < j \leqslant 2^{n^2}$, 使得: $\mathcal{R}^i = \mathcal{R}^j$.

□

Corollary

$\forall m \in \mathbb{N}$ $\mathcal{R}^m \in \{\mathcal{R}^0, \mathcal{R}^1, \ldots, \mathcal{R}^{2^{n^2}-1}\}$.

相关性质

Theorem

1. 设$|A| = n$, 则存在i, j $0 \leqslant i < j \leqslant 2^{n^2}$, 使得: $\mathcal{R}^i = \mathcal{R}^j$.

Proof.

1. $|A| = n$, $\therefore |A \times A| = n^2$;
2. $\therefore |\mathscr{P}(A \times A)| = 2^{n^2}$;
3. 而$\mathcal{R}^0, \mathcal{R}^1, \ldots, \mathcal{R}^{2^{n^2}}$ 共有$2^{n^2} + 1$项;
4. 根据抽屉原则, $\exists i, j$ $0 \leqslant i < j \leqslant 2^{n^2}$, 使得: $\mathcal{R}^i = \mathcal{R}^j$.

$\square$

Corollary

$\forall m \in \mathbb{N}$ $\mathcal{R}^m \in \{\mathcal{R}^0, \mathcal{R}^1, \ldots, \mathcal{R}^{2^{n^2}-1}\}$.

相关性质

Theorem

1. 设$|A| = n$, 则存在i, j $0 \leqslant i < j \leqslant 2^{n^2}$, 使得: $\mathcal{R}^i = \mathcal{R}^j$.

Proof.

1. $|A| = n$, $\therefore |A \times A| = n^2$;
2. $\therefore |\mathscr{P}(A \times A)| = 2^{n^2}$;
3. 而$\mathcal{R}^0, \mathcal{R}^1, \ldots, \mathcal{R}^{2^{n^2}}$ 共有$2^{n^2} + 1$项;
4. 根据抽屉原则, $\exists i, j$ $0 \leqslant i < j \leqslant 2^{n^2}$, 使得: $\mathcal{R}^i = \mathcal{R}^j$.

□

Corollary

$\forall m \in \mathbb{N}$ $\mathcal{R}^m \in \{\mathcal{R}^0, \mathcal{R}^1, \ldots, \mathcal{R}^{2^{n^2}-1}\}$.

相关性质

Theorem

1. 设$|A| = n$, 则存在i, j $0 \leqslant i < j \leqslant 2^{n^2}$, 使得: $\mathcal{R}^i = \mathcal{R}^j$.

Proof.

1. $|A| = n$, $\therefore$ $|A \times A| = n^2$;
2. $\therefore$ $|\mathscr{P}(A \times A)| = 2^{n^2}$;
3. 而$\mathcal{R}^0, \mathcal{R}^1, \ldots, \mathcal{R}^{2^{n^2}}$ 共有$2^{n^2} + 1$项;
4. 根据抽屉原则, $\exists i, j$ $0 \leqslant i < j \leqslant 2^{n^2}$, 使得: $\mathcal{R}^i = \mathcal{R}^j$.

$\square$

Corollary

$\forall m \in \mathbb{N}$ $\mathcal{R}^m \in \{\mathcal{R}^0, \mathcal{R}^1, \ldots, \mathcal{R}^{2^{n^2}-1}\}$.

相关性质

Theorem

1. 设$|A| = n$, 则存在i, j $0 \leqslant i < j \leqslant 2^{n^2}$, 使得: $\mathcal{R}^i = \mathcal{R}^j$.

Proof.

1. $|A| = n$, $\therefore$ $|A \times A| = n^2$;
2. $\therefore$ $|\mathscr{P}(A \times A)| = 2^{n^2}$;
3. 而$\mathcal{R}^0, \mathcal{R}^1, \ldots, \mathcal{R}^{2^{n^2}}$共有$2^{n^2}+1$项;
4. 根据抽屉原则, $\exists i, j$ $0 \leqslant i < j \leqslant 2^{n^2}$, 使得: $\mathcal{R}^i = \mathcal{R}^j$.

$\square$

Corollary

$\forall m \in \mathbb{N}$ $\mathcal{R}^m \in \{\mathcal{R}^0, \mathcal{R}^1, \ldots, \mathcal{R}^{2^{n^2}-1}\}$.

闭包

Description (闭包, Closure)

数学上把包含某个给定的集合，并且具有某个性质的最小集合称为*闭包*.

Example

1. 所有的可以间接通航的城市之间的关系，
   是直接通航城市的传递闭包;

关系的逆

Definition (关系的逆)

设$\mathcal{R} \subseteq A \times B$，关系$\mathcal{R}$的逆关系，记为$\widetilde{\mathcal{R}}$(读作tilde)，定义如下：

$$\widetilde{\mathcal{R}} = \{\langle y, x\rangle \mid \langle x, y\rangle \in \mathcal{R}\} \subseteq B \times A$$

Example

关系的逆

Definition (关系的逆)

设$\mathcal{R} \subseteq A \times B$，关系$\mathcal{R}$的逆关系，记为$\tilde{\mathcal{R}}$(读作tilde)，定义如下：

$$\tilde{\mathcal{R}} = \{\langle y, x\rangle \mid \langle x, y\rangle \in \mathcal{R}\} \subseteq B \times A$$

Example

- $\widetilde{\leqslant} = \geqslant$; $\widetilde{\mathbb{1}_A} = \mathbb{1}_A$; $\widetilde{\subseteq} = \supseteq$;
- 关系的逆是关系的对偶概念；如果$\mathcal{R}$具有五性，则$\tilde{\mathcal{R}}$也相应的具有；
- 关系的逆与关系的补是不同的概念：
  $$\overline{\mathcal{R}} = \{\langle x, y\rangle \mid \langle x, y\rangle \notin \mathcal{R}\} \subseteq A \times B$$

关系的逆

Definition (关系的逆)

设$\mathcal{R} \subseteq A \times B$，关系$\mathcal{R}$的逆关系，记为$\tilde{\mathcal{R}}$(读作tilde)，定义如下：

$$\tilde{\mathcal{R}} = \{\langle y, x\rangle \mid \langle x, y\rangle \in \mathcal{R}\} \subseteq B \times A$$

Example

- $\tilde{\leqslant} = \geqslant$; $\widetilde{\mathbb{1}_A} = \mathbb{1}_A$; $\tilde{\subseteq} = \supseteq$;

关系的逆是关系的对偶概念；如果$\mathcal{R}$具有五性，则$\tilde{\mathcal{R}}$也相应的具有；

关系的逆与关系的补是不同的概念：

$$\overline{\mathcal{R}} = \{\langle x, y\rangle \mid \langle x, y\rangle \notin \mathcal{R}\} \subseteq A \times B$$

关系的逆

Definition (关系的逆)

设$\mathcal{R} \subseteq A \times B$，关系$\mathcal{R}$的逆关系，记为$\tilde{\mathcal{R}}$(读作tilde)，定义如下：

$$\tilde{\mathcal{R}} = \{\langle y, x\rangle \mid \langle x, y\rangle \in \mathcal{R}\} \subseteq B \times A$$

Example

- $\tilde{\leqslant} = \geqslant$; $\widetilde{\mathbb{1}_A} = \mathbb{1}_A$; $\tilde{\subseteq} = \supseteq$;
- 关系的逆是关系的对偶概念；如果$\mathcal{R}$具有五性，则$\tilde{\mathcal{R}}$也相应的具有；

关系的逆与关系的补是不同的概念：

$$\overline{\mathcal{R}} = \{\langle x, y\rangle \mid \langle x, y\rangle \notin \mathcal{R}\} \subseteq A \times B$$

关系的逆

Definition (关系的逆)

设$\mathcal{R} \subseteq A \times B$，关系$\mathcal{R}$的逆关系，记为$\tilde{\mathcal{R}}$(读作tilde)，定义如下：

$$\tilde{\mathcal{R}} = \{\langle y, x\rangle \mid \langle x, y\rangle \in \mathcal{R}\} \subseteq B \times A$$

Example

- $\tilde{\leqslant} = \geqslant$; $\widetilde{\mathbb{1}_A} = \mathbb{1}_A$; $\tilde{\subseteq} = \supseteq$;
- 关系的逆是关系的对偶概念；如果$\mathcal{R}$具有五性，则$\tilde{\mathcal{R}}$也相应的具有；
- 关系的逆与关系的补是不同的概念：

$$\overline{\mathcal{R}} = \{\langle x, y\rangle \mid \langle x, y\rangle \notin \mathcal{R}\} \subseteq A \times B$$

相关性质

Theorem

$\mathcal{R}$是对称关系，*iff*, $\mathcal{R} = \tilde{\mathcal{R}}$.

相关性质

Theorem

$\mathcal{R}$是对称关系，iff, $\mathcal{R} = \tilde{\mathcal{R}}$.

Proof.

$\Longrightarrow$ $\forall \langle x, y \rangle \in \mathcal{R}$, $\therefore \langle y, x \rangle \in \mathcal{R}$; So $\langle x, y \rangle \in \tilde{\mathcal{R}}$

$\therefore \mathcal{R} \subseteq \tilde{\mathcal{R}}$, but $\tilde{\tilde{\mathcal{R}}} = \mathcal{R}$;

So, $\tilde{\mathcal{R}} \subseteq \tilde{\tilde{\mathcal{R}}} = \mathcal{R}$ ($\because \tilde{\mathcal{R}}$也是对称关系), $\therefore \mathcal{R} = \tilde{\mathcal{R}}$;

$\Longleftarrow$ $\forall \langle x, y \rangle \in \mathcal{R}$ $\therefore \langle x, y \rangle \in \tilde{\mathcal{R}}$; So $\langle y, x \rangle \in \mathcal{R}$

所以$\mathcal{R}$是对称关系.

$\square$

相关性质

Theorem

$\mathcal{R}$是对称关系，*iff*, $\mathcal{R} = \tilde{\mathcal{R}}$.

Proof.

$\Longrightarrow$ $\forall\langle x, y\rangle \in \mathcal{R}$, $\therefore \langle y, x\rangle \in \mathcal{R}$; *So* $\langle x, y\rangle \in \tilde{\mathcal{R}}$

$\therefore \mathcal{R} \subseteq \tilde{\mathcal{R}}$, *but* $\tilde{\tilde{\mathcal{R}}} = \mathcal{R}$;

So, $\tilde{\mathcal{R}} \subseteq \tilde{\tilde{\mathcal{R}}} = \mathcal{R}$ ($\because \tilde{\mathcal{R}}$也是对称关系), $\therefore \mathcal{R} = \tilde{\mathcal{R}}$;

$\Longleftarrow$ $\forall\langle x, y\rangle \in \mathcal{R}$ $\therefore \langle x, y\rangle \in \tilde{\mathcal{R}}$; *So* $\langle y, x\rangle \in \mathcal{R}$

所以$\mathcal{R}$是对称关系.

$\square$

相关性质

Theorem

$\mathcal{R}$是对称关系，*iff*, $\mathcal{R} = \tilde{\mathcal{R}}$.

Proof.

$\Longrightarrow$ $\forall\langle x, y\rangle \in \mathcal{R}$, $\therefore \langle y, x\rangle \in \mathcal{R}$; *So* $\langle x, y\rangle \in \tilde{\mathcal{R}}$

$\therefore \mathcal{R} \subseteq \tilde{\mathcal{R}}$, *but* $\tilde{\tilde{\mathcal{R}}} = \mathcal{R}$;

So, $\tilde{\mathcal{R}} \subseteq \tilde{\tilde{\mathcal{R}}} = \mathcal{R}$ ($\because \tilde{\mathcal{R}}$也是对称关系), $\therefore \mathcal{R} = \tilde{\mathcal{R}}$;

$\Longleftarrow$ $\forall\langle x, y\rangle \in \mathcal{R}$ $\therefore \langle x, y\rangle \in \tilde{\mathcal{R}}$; *So* $\langle y, x\rangle \in \mathcal{R}$

所以$\mathcal{R}$是对称关系. $\square$

特性关系的闭包

Definition

设$\mathcal{R} \subseteq A^2$, $\mathcal{R}$的自反(对称、传递)闭包$\mathcal{R}'$是满足下述三条件的关系：

1. $\mathcal{R} \subseteq \mathcal{R}'$;
2. $\mathcal{R}'$是自反的(对称的、传递的);
3. 设$\mathcal{R}''$是满足上述两条件的关系，则$\mathcal{R}' \subseteq \mathcal{R}''$.

分别记$\mathcal{R}$的自反、对称和传递闭包为：$r(\mathcal{R})$, $s(\mathcal{R})$和$t(\mathcal{R})$.

Propostion

$\mathcal{R}$是自反的(对称的、传递的), iff, $\mathcal{R} = r(\mathcal{R})$ $(s(\mathcal{R}), t(\mathcal{R}))$.

特性关系的闭包

Definition

设$\mathcal{R} \subseteq A^2$, $\mathcal{R}$的自反(对称、传递)闭包$\mathcal{R}'$是满足下述三条件的关系:

1. $\mathcal{R} \subseteq \mathcal{R}'$;
2. $\mathcal{R}'$是自反的(对称的、传递的);
3. 设$\mathcal{R}''$是满足上述两条件的关系, 则$\mathcal{R}' \subseteq \mathcal{R}''$.

分别记$\mathcal{R}$的自反、对称和传递闭包为: $r(\mathcal{R})$, $s(\mathcal{R})$和$t(\mathcal{R})$.

Propostion

$\mathcal{R}$是自反的(对称的、传递的), iff, $\mathcal{R} = r(\mathcal{R})$ $(s(\mathcal{R}), t(\mathcal{R}))$.

特性关系的闭包

Definition

设$\mathcal{R} \subseteq A^2$, $\mathcal{R}$的自反(对称、传递)闭包$\mathcal{R}'$是满足下述三条件的关系：

1. $\mathcal{R} \subseteq \mathcal{R}'$;
2. $\mathcal{R}'$是自反的(对称的、传递的);
3. 设$\mathcal{R}''$是满足上述两条件的关系，则$\mathcal{R}' \subseteq \mathcal{R}''$.

分别记$\mathcal{R}$的自反、对称和传递闭包为：$r(\mathcal{R})$, $s(\mathcal{R})$和$t(\mathcal{R})$.

Propostion

$\mathcal{R}$是自反的(对称的、传递的), iff, $\mathcal{R} = r(\mathcal{R})$ $(s(\mathcal{R}), t(\mathcal{R}))$.

特性关系的闭包

Definition

设$\mathcal{R} \subseteq A^2$, $\mathcal{R}$的自反(对称、传递)闭包$\mathcal{R}'$是满足下述三条件的关系：

1. $\mathcal{R} \subseteq \mathcal{R}'$;
2. $\mathcal{R}'$是自反的(对称的、传递的);
3. 设$\mathcal{R}''$是满足上述两条件的关系，则$\mathcal{R}' \subseteq \mathcal{R}''$.

分别记$\mathcal{R}$的自反、对称和传递闭包为：$r(\mathcal{R})$, $s(\mathcal{R})$和$t(\mathcal{R})$.

Propostion

$\mathcal{R}$是自反的(对称的、传递的), iff, $\mathcal{R} = r(\mathcal{R})$ $(s(\mathcal{R}), t(\mathcal{R}))$.

特性关系的闭包

Definition

设$\mathcal{R} \subseteq A^2$, $\mathcal{R}$的自反(对称、传递)闭包$\mathcal{R}'$是满足下述三条件的关系：

1. $\mathcal{R} \subseteq \mathcal{R}'$;
2. $\mathcal{R}'$是自反的(对称的、传递的);
3. 设$\mathcal{R}''$是满足上述两条件的关系，则$\mathcal{R}' \subseteq \mathcal{R}''$.

分别记$\mathcal{R}$的自反、对称和传递闭包为：$r(\mathcal{R})$, $s(\mathcal{R})$和$t(\mathcal{R})$.

Propostion

$\mathcal{R}$是自反的(对称的、传递的), iff, $\mathcal{R} = r(\mathcal{R})$ $(s(\mathcal{R}), t(\mathcal{R}))$.

闭包的构造(1/2)

Theorem

$$① \ r(\mathcal{R}) = \mathcal{R} \cup \mathbb{1}_A; \quad ② \ s(\mathcal{R}) = \mathcal{R} \cup \tilde{\mathcal{R}}\ ; \quad ③ \ t(\mathcal{R}) = \bigcup_{i=1}^{\infty} \mathcal{R}^i.$$

③的证明.

闭包的构造(1/2)

Theorem

$$① \ r(\mathcal{R}) = \mathcal{R} \cup \mathbb{1}_A; \quad ② \ s(\mathcal{R}) = \mathcal{R} \cup \tilde{\mathcal{R}}\ ; \quad ③ \ t(\mathcal{R}) = \bigcup_{i=1}^{\infty} \mathcal{R}^i.$$

③的证明.

1. $\mathcal{R} \subseteq \bigcup_{i=1}^{\infty} \mathcal{R}^i$;
2. $\forall \langle x, y \rangle \in \bigcup_{i=1}^{\infty} \mathcal{R}^i, \langle y, z \rangle \in \bigcup_{i=1}^{\infty} \mathcal{R}^i$;

   $\exists m, n \ \langle x, y \rangle \in \mathcal{R}^m, \langle y, z \rangle \in \mathcal{R}^n; \ \therefore \langle x, z \rangle \in \mathcal{R}^m \mathcal{R}^n = \mathcal{R}^{m+n} \subseteq \bigcup_{i=1}^{\infty} \mathcal{R}^i$;

   所以$\bigcup_{i=1}^{\infty} \mathcal{R}^i$是传递的.

$\square$

闭包的构造(1/2)

Theorem

$$① \ r(\mathcal{R}) = \mathcal{R} \cup \mathbb{1}_A; \quad ② \ s(\mathcal{R}) = \mathcal{R} \cup \tilde{\mathcal{R}} \ ; \quad ③ \ t(\mathcal{R}) = \bigcup_{i=1}^{\infty} \mathcal{R}^i.$$

③的证明.

1. $\mathcal{R} \subseteq \bigcup_{i=1}^{\infty} \mathcal{R}^i$;
2. $\forall \langle x, y \rangle \in \bigcup_{i=1}^{\infty} \mathcal{R}^i, \ \langle y, z \rangle \in \bigcup_{i=1}^{\infty} \mathcal{R}^i$;

   $\exists m, n \ \ \langle x, y \rangle \in \mathcal{R}^m, \ \langle y, z \rangle \in \mathcal{R}^n; \ \ \therefore \langle x, z \rangle \in \mathcal{R}^m \mathcal{R}^n = \mathcal{R}^{m+n} \subseteq \bigcup_{i=1}^{\infty} \mathcal{R}^i$;

   所以$\bigcup_{i=1}^{\infty} \mathcal{R}^i$是传递的.

□

闭包的构造(1/2)

Theorem

$$① \ r(\mathcal{R}) = \mathcal{R} \cup \mathbb{1}_A; \quad ② \ s(\mathcal{R}) = \mathcal{R} \cup \tilde{\mathcal{R}}\ ; \quad ③ \ t(\mathcal{R}) = \bigcup_{i=1}^{\infty} \mathcal{R}^i.$$

③的证明.

1. $\mathcal{R} \subseteq \bigcup_{i=1}^{\infty} \mathcal{R}^i$;
2. $\forall \langle x, y\rangle \in \bigcup_{i=1}^{\infty} \mathcal{R}^i,\ \langle y, z\rangle \in \bigcup_{i=1}^{\infty} \mathcal{R}^i$;

   $\exists m, n\ \ \langle x, y\rangle \in \mathcal{R}^m,\ \langle y, z\rangle \in \mathcal{R}^n;\ \ \therefore \langle x, z\rangle \in \mathcal{R}^m\mathcal{R}^n = \mathcal{R}^{m+n} \subseteq \bigcup_{i=1}^{\infty} \mathcal{R}^i$;

   所以$\bigcup_{i=1}^{\infty} \mathcal{R}^i$是传递的.

□

闭包的构造(2/2)

③的证明.

3. 设传递关系$\mathcal{R}' \supseteq \mathcal{R}$，则要证明：$\bigcup\limits_{i=1}^{\infty} \mathcal{R}^i \subseteq \mathcal{R}'$；

   用归纳法证明：$\forall n\ \mathcal{R}^n \subseteq \mathcal{R}'$.

   1. $n = 1$时，$\mathcal{R} \subseteq \mathcal{R}'$；
   2. 设$n = k$时结论成立，$n = k + 1$时：

      设$\langle x, z\rangle \in \mathcal{R}^{k+1} = \mathcal{R}^k \circ \mathcal{R}$,

      $\exists y(\langle x, y\rangle \in \mathcal{R}^k \wedge \langle y, z\rangle \in \mathcal{R})$

      So $\langle x, y\rangle \in \mathcal{R}' \wedge \langle y, z\rangle \in \mathcal{R}'$

      $\langle x, z\rangle \in \mathcal{R}'$ ($\mathcal{R}'$是传递的)

$\square$

闭包的构造(2/2)

③的证明.

3. 设传递关系$\mathcal{R}' \supseteq \mathcal{R}$，则要证明：$\bigcup_{i=1}^{\infty} \mathcal{R}^i \subseteq \mathcal{R}'$;

用归纳法证明：$\forall n\ \mathcal{R}^n \subseteq \mathcal{R}'$.

1. $n=1$时，$\mathcal{R} \subseteq \mathcal{R}'$;
2. 设$n=k$时结论成立，$n=k+1$时：
   设$\langle x,z\rangle \in \mathcal{R}^{k+1} = \mathcal{R}^k \mathcal{R}$;
   $\therefore \exists y\ \langle x,y\rangle \in \mathcal{R}^k \wedge \langle y,z\rangle \in \mathcal{R}$;
   So $\langle x,y\rangle \in \mathcal{R}' \wedge \langle y,z\rangle \in \mathcal{R}'$;
   $\therefore \langle x,z\rangle \in \mathcal{R}'$($\mathcal{R}'$是传递的).

□

闭包的构造(2/2)

③的证明.

3. 设传递关系$\mathcal{R}' \supseteq \mathcal{R}$，则要证明：$\bigcup_{i=1}^{\infty} \mathcal{R}^i \subseteq \mathcal{R}'$;

   用归纳法证明：$\forall n \ \mathcal{R}^n \subseteq \mathcal{R}'$.

   1. $n = 1$时，$\mathcal{R} \subseteq \mathcal{R}'$;
   2. 设$n = k$时结论成立，$n = k + 1$时：
      设$\langle x, z \rangle \in \mathcal{R}^{k+1} = \mathcal{R}^k \mathcal{R}$;
      $\therefore \exists y \ \langle x, y \rangle \in \mathcal{R}^k \wedge \langle y, z \rangle \in \mathcal{R}$;
      So $\langle x, y \rangle \in \mathcal{R}' \wedge \langle y, z \rangle \in \mathcal{R}'$;
      $\therefore \langle x, z \rangle \in \mathcal{R}'$($\mathcal{R}'$是传递的).

$\square$

闭包的构造(2/2)

③的证明.

3. 设传递关系$\mathcal{R}' \supseteq \mathcal{R}$，则要证明：$\bigcup_{i=1}^{\infty} \mathcal{R}^i \subseteq \mathcal{R}'$;

   用归纳法证明：$\forall n\ \mathcal{R}^n \subseteq \mathcal{R}'$.

   1. $n = 1$时，$\mathcal{R} \subseteq \mathcal{R}'$;
   2. 设$n = k$时结论成立，$n = k + 1$时：
      设$\langle x, z\rangle \in \mathcal{R}^{k+1} = \mathcal{R}^k \mathcal{R}$;
      $\therefore \exists y\ \langle x, y\rangle \in \mathcal{R}^k \wedge \langle y, z\rangle \in \mathcal{R}$;
      $So\ \langle x, y\rangle \in \mathcal{R}' \wedge \langle y, z\rangle \in \mathcal{R}'$;
      $\therefore \langle x, z\rangle \in \mathcal{R}'$($\mathcal{R}'$是传递的).

□

Examples

Example

$r(<) = \leqslant$; $s(<) =$ "$\neq$";

$s(\leqslant) =$ 全域关系; $r(\neq) =$ 全域关系;

- 设$\mathcal{R}$是城市之间有直接航线的关系，则城市之间有间接航线的关系等于$\bigcup_{i=1}^{\infty} \mathcal{R}^i$.

Examples

Example

- $r(<) = \leqslant$; $s(<) =$ “$\neq$”;

  $s(\leqslant) =$ 全域关系; $r(\neq) =$ 全域关系;

  设$\mathcal{R}$是城市之间有直接航线的关系，则城市之间有间接航线的关系等于$\bigcup_{i=1}^{\infty}\mathcal{R}^i$.

Examples

Example

- $r(<) = \leqslant$; $s(<) =$ "$\neq$";
- $s(\leqslant) =$ 全域关系; $r(\neq) =$ 全域关系;

设$\mathcal{R}$是城市之间有直接航线的关系，则城市之间有间接航线的关系等于$\bigcup_{i=1}^{\infty} \mathcal{R}^i$.

Examples

Example

- $r(<) = \leqslant$; $s(<) =$ "$\neq$";
- $s(\leqslant) =$ 全域关系; $r(\neq) =$ 全域关系;
- 设$\mathcal{R}$是城市之间有直接航线的关系，则城市之间有间接航线的关系等于$\bigcup_{i=1}^{\infty} \mathcal{R}^i$.

有限集合的传递闭包

Theorem

设$|A| = n$, $\mathcal{R} \subseteq A^2$, 则: $t(\mathcal{R}) = \bigcup_{i=1}^{n} \mathcal{R}^i$;

有限集合的传递闭包

Theorem

设$|A| = n$, $\mathcal{R} \subseteq A^2$, 则: $t(\mathcal{R}) = \bigcup_{i=1}^{n} \mathcal{R}^i$;

Proof.

1. 设$\langle x_0, x_{n+1}\rangle \in \mathcal{R}^{n+1}$;
2. $\exists\, x_1, x_2, \ldots, x_n$ $x_0 \mathcal{R} x_1 \wedge x_1 \mathcal{R} x_2 \wedge \ldots x_n \mathcal{R} x_{n+1}$;
3. 即$\mathcal{R}$关系图中有从x_0到x_{n+1}长度为$n+1$的有向路径;
4. 而$x_1, x_2, \ldots, x_{n+1}$ $n+1$个元素只能在$|A| = n$个元素中选取;
5. 所以根据抽屉原则, $\exists 1 \leqslant i < j \leqslant n+1$ $x_i = x_j$;
6. $\therefore \underbrace{x_0 \mathcal{R} x_1 \wedge x_1 \mathcal{R} x_2 \wedge \cdots \wedge x_i \mathcal{R} x_{j+1} \wedge \cdots \wedge x_n \mathcal{R} x_{n+1}}_{n+1-(j-i)\text{个}}$;
7. $\therefore \langle x_0, x_{n+1}\rangle \in \mathcal{R}^{n+1-(j-i)} \subseteq \bigcup_{i=1}^{n} \mathcal{R}^i$.

$\square$

有限集合的传递闭包

Theorem

设$|A| = n$, $\mathcal{R} \subseteq A^2$, 则: $t(\mathcal{R}) = \bigcup_{i=1}^{n} \mathcal{R}^i$;

Proof.

1. 设$\langle x_0, x_{n+1} \rangle \in \mathcal{R}^{n+1}$;
2. $\exists\, x_1, x_2, \ldots, x_n \; x_0 \,\mathcal{R}\, x_1 \wedge x_1 \,\mathcal{R}\, x_2 \wedge \ldots x_n \,\mathcal{R}\, x_{n+1}$;
3. 即$\mathcal{R}$关系图中有从x_0到x_{n+1}长度为$n+1$的有向路径;
4. 而$x_1, x_2, \ldots, x_{n+1}$ $n+1$个元素只能在 $|A| = n$个元素中选取;
5. 所以根据抽屉原则, $\exists\, 1 \leqslant i < j \leqslant n+1 \;\; x_i = x_j$;
6. $\therefore \underbrace{x_0 \,\mathcal{R}\, x_1 \wedge x_1 \,\mathcal{R}\, x_2 \wedge \cdots \wedge x_i \,\mathcal{R}\, x_{j+1} \wedge \cdots \wedge x_n \,\mathcal{R}\, x_{n+1}}_{n+1-(j-i)\text{个}}$;
7. $\therefore \langle x_0, x_{n+1} \rangle \in \mathcal{R}^{n+1-(j-i)} \subseteq \bigcup_{i=1}^{n} \mathcal{R}^i$.

□

有限集合的传递闭包

Theorem

设$|A| = n$, $\mathcal{R} \subseteq A^2$, 则: $t(\mathcal{R}) = \bigcup_{i=1}^{n} \mathcal{R}^i$;

Proof.

1. 设$\langle x_0, x_{n+1}\rangle \in \mathcal{R}^{n+1}$;
2. $\exists\, x_1, x_2, \ldots, x_n\ x_0\,\mathcal{R}\,x_1 \wedge x_1\,\mathcal{R}\,x_2 \wedge \ldots x_n\,\mathcal{R}\,x_{n+1}$;
3. 即$\mathcal{R}$关系图中有从x_0到x_{n+1}长度为$n+1$的有向路径;
4. 而$x_1, x_2, \ldots, x_{n+1}$ $n+1$个元素只能在 $|A| = n$个元素中选取;
5. 所以根据抽屉原则, $\exists\, 1 \leqslant i < j \leqslant n+1$ $x_i = x_j$;
6. $\therefore \underbrace{x_0\,\mathcal{R}\,x_1 \wedge x_1\,\mathcal{R}\,x_2 \wedge \cdots \wedge x_i\,\mathcal{R}\,x_{j+1} \wedge \cdots \wedge x_n\,\mathcal{R}\,x_{n+1}}_{n+1-(j-i)\text{个}}$;
7. $\therefore \langle x_0, x_{n+1}\rangle \in \mathcal{R}^{n+1-(j-i)} \subseteq \bigcup_{i=1}^{n} \mathcal{R}^i$.

□

有限集合的传递闭包

Theorem

设$|A| = n$, $\mathcal{R} \subseteq A^2$, 则: $t(\mathcal{R}) = \bigcup_{i=1}^{n} \mathcal{R}^i$;

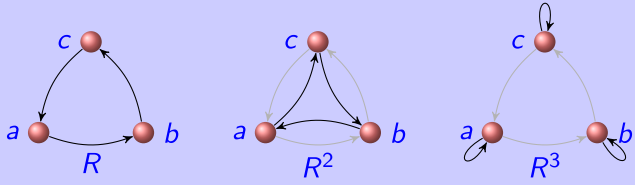
Proof.

1. 设$\langle x_0, x_{n+1} \rangle \in \mathcal{R}^{n+1}$;
2. $\exists\, x_1, x_2, \ldots, x_n\ x_0 \,\mathcal{R}\, x_1 \wedge x_1 \,\mathcal{R}\, x_2 \wedge \ldots x_n \,\mathcal{R}\, x_{n+1}$;
3. 即$\mathcal{R}$关系图中有从x_0到x_{n+1}长度为$n+1$的有向路径;
4. 而$x_1, x_2, \ldots, x_{n+1}$ $n+1$个元素只能在 $|A| = n$个元素中选取;
5. 所以根据抽屉原则, $\exists\, 1 \leqslant i < j \leqslant n+1$ $x_i = x_j$;
6. $\therefore \underbrace{x_0 \,\mathcal{R}\, x_1 \wedge x_1 \,\mathcal{R}\, x_2 \wedge \cdots \wedge x_i \,\mathcal{R}\, x_{j+1} \wedge \cdots \wedge x_n \,\mathcal{R}\, x_{n+1}}_{n+1-(j-i)\text{个}}$;
7. $\therefore \langle x_0, x_{n+1} \rangle \in \mathcal{R}^{n+1-(j-i)} \subseteq \bigcup_{i=1}^{n} \mathcal{R}^i$.

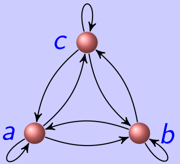
□

Examples

$\mathcal{R} = \{\langle a, b\rangle, \langle b.c\rangle, \langle c, a\rangle\}$的传递闭包



$t(R) = R \cup R^2 \cup R^3$



闭包之间的关系(1/3)

Propostion

设$\mathcal{R}$是自反关系，则，$t(\mathcal{R})$和$s(\mathcal{R})$也是自反关系；

$t(\mathcal{R})$是自反关系的证明.

1. $\mathcal{R}$是自反的，iff, $\mathbb{I}_A \subseteq \mathcal{R}$
2. 而$t(\mathcal{R}) = \bigcup_{i=1}^{\infty} \mathcal{R}^i \supseteq \mathbb{I}_A$;
3. 所以，$\mathcal{R}$是自反的.

□

闭包之间的关系(1/3)

Propostion

设$\mathcal{R}$是自反关系，则，$t(\mathcal{R})$和$s(\mathcal{R})$也是自反关系；

$t(\mathcal{R})$是自反关系的证明.

1. $\mathcal{R}$是自反的，iff, $\mathbb{1}_A \subseteq \mathcal{R}$
2. 而$t(\mathcal{R}) = \bigcup_{i=1}^{\infty} \mathcal{R}^i \supseteq \mathbb{1}_A$;
3. 所以，$\mathcal{R}$是自反的.

□

闭包之间的关系(1/3)

Propostion

设$\mathcal{R}$是自反关系，则，$t(\mathcal{R})$和$s(\mathcal{R})$也是自反关系；

$t(\mathcal{R})$是自反关系的证明.

1. $\mathcal{R}$是自反的，iff, $\mathbb{1}_A \subseteq \mathcal{R}$
2. 而$t(\mathcal{R}) = \bigcup_{i=1}^{\infty} \mathcal{R}^i \supseteq \mathbb{1}_A$;
3. 所以，$\mathcal{R}$是自反的.

□

闭包之间的关系(1/3)

Propostion

设$\mathcal{R}$是自反关系，则，$t(\mathcal{R})$和$s(\mathcal{R})$也是自反关系；

$t(\mathcal{R})$是自反关系的证明.

1. $\mathcal{R}$是自反的，iff, $\mathbb{1}_A \subseteq \mathcal{R}$
2. 而$t(\mathcal{R}) = \bigcup_{i=1}^{\infty} \mathcal{R}^i \supseteq \mathbb{1}_A$;
3. 所以，$\mathcal{R}$是自反的.

$\square$

闭包之间的关系(1/3)

Propostion

设$\mathcal{R}$是自反关系，则，$t(\mathcal{R})$和$s(\mathcal{R})$也是自反关系；

$t(\mathcal{R})$是自反关系的证明.

1. $\mathcal{R}$是自反的，iff, $\mathbb{1}_A \subseteq \mathcal{R}$
2. 而$t(\mathcal{R}) = \bigcup_{i=1}^{\infty} \mathcal{R}^i \supseteq \mathbb{1}_A$;
3. 所以，$\mathcal{R}$是自反的.

$\square$

闭包之间的关系(2/3)

Propostion

$$rt(\mathcal{R}) = tr(\mathcal{R});$$

Proof.

$$rt(\mathcal{R}) = \bigcup_{i=0}^{\infty} \mathcal{R}^i; \quad tr(\mathcal{R}) = \bigcup_{i=1}^{\infty} (\mathbb{1}_A \cup \mathcal{R})^i$$

用归纳法证明下述等式即可：

$$\forall n \in \mathbb{N} \quad (\mathbb{1}_A \cup \mathcal{R})^n = \bigcup_{i=0}^{n} \mathcal{R}^i;$$

由此：

$$\forall n \ \ (\mathbb{1}_A \cup \mathcal{R})^n \subseteq \bigcup_{i=0}^{n} \mathcal{R}^i \subseteq \bigcup_{i=0}^{\infty} \mathcal{R}^i;$$

所以：

$$\bigcup_{i=0}^{\infty} \mathcal{R}^i \subseteq \bigcup_{i=1}^{\infty} (\mathbb{1}_A \cup \mathcal{R})^i \subseteq \bigcup_{i=0}^{\infty} \mathcal{R}^i;$$

即：

$$rt(\mathcal{R}) = tr(\mathcal{R}).$$

□

闭包之间的关系(2/3)

Propostion

$$rt(\mathcal{R}) = tr(\mathcal{R});$$

Proof.

$$rt(\mathcal{R}) = \bigcup_{i=0}^{\infty} \mathcal{R}^i; \quad tr(\mathcal{R}) = \bigcup_{i=1}^{\infty} (\mathbb{1}_A \cup \mathcal{R})^i$$

用归纳法证明下述等式即可:

$$\forall n \in \mathbb{N} \quad (\mathbb{1}_A \cup \mathcal{R})^n = \bigcup_{i=0}^{n} \mathcal{R}^i;$$

由此:

$$\forall n \ \ (\mathbb{1}_A \cup \mathcal{R})^n \subseteq \bigcup_{i=0}^{n} \mathcal{R}^i \subseteq \bigcup_{i=0}^{\infty} \mathcal{R}^i;$$

所以:

$$\bigcup_{i=0}^{\infty} \mathcal{R}^i \subseteq \bigcup_{i=1}^{\infty} (\mathbb{1}_A \cup \mathcal{R})^i \subseteq \bigcup_{i=0}^{\infty} \mathcal{R}^i;$$

即:

$$rt(\mathcal{R}) = tr(\mathcal{R}).$$

□

闭包之间的关系(2/3)

Propostion

$$rt(\mathcal{R}) = tr(\mathcal{R});$$

Proof.

$$rt(\mathcal{R}) = \bigcup_{i=0}^{\infty} \mathcal{R}^i; \quad tr(\mathcal{R}) = \bigcup_{i=1}^{\infty} (\mathbb{1}_A \cup \mathcal{R})^i$$

用归纳法证明下述等式即可:

$$\forall n \in \mathbb{N} \quad (\mathbb{1}_A \cup \mathcal{R})^n = \bigcup_{i=0}^{n} \mathcal{R}^i;$$

由此:

$$\forall n \quad (\mathbb{1}_A \cup \mathcal{R})^n \subseteq \bigcup_{i=0}^{n} \mathcal{R}^i \subseteq \bigcup_{i=0}^{\infty} \mathcal{R}^i;$$

所以:

$$\bigcup_{i=0}^{\infty} \mathcal{R}^i \subseteq \bigcup_{i=1}^{\infty} (\mathbb{1}_A \cup \mathcal{R})^i \subseteq \bigcup_{i=0}^{\infty} \mathcal{R}^i;$$

即:

$$rt(\mathcal{R}) = tr(\mathcal{R}).$$

□

闭包之间的关系(2/3)

Propostion

$$rt(\mathcal{R}) = tr(\mathcal{R});$$

Proof.

$$rt(\mathcal{R}) = \bigcup_{i=0}^{\infty} \mathcal{R}^i; \quad tr(\mathcal{R}) = \bigcup_{i=1}^{\infty} (\mathbb{1}_A \cup \mathcal{R})^i$$

用归纳法证明下述等式即可:

$$\forall n \in \mathbb{N} \quad (\mathbb{1}_A \cup \mathcal{R})^n = \bigcup_{i=0}^{n} \mathcal{R}^i;$$

由此:

$$\forall n \quad (\mathbb{1}_A \cup \mathcal{R})^n \subseteq \bigcup_{i=0}^{n} \mathcal{R}^i \subseteq \bigcup_{i=0}^{\infty} \mathcal{R}^i;$$

所以:

$$\bigcup_{i=0}^{\infty} \mathcal{R}^i \subseteq \bigcup_{i=1}^{\infty} (\mathbb{1}_A \cup \mathcal{R})^i \subseteq \bigcup_{i=0}^{\infty} \mathcal{R}^i;$$

即:

$$rt(\mathcal{R}) = tr(\mathcal{R}).$$

□

闭包之间的关系(2/3)

Propostion

$$rt(\mathcal{R}) = tr(\mathcal{R});$$

Proof.

$$rt(\mathcal{R}) = \bigcup_{i=0}^{\infty} \mathcal{R}^i; \quad tr(\mathcal{R}) = \bigcup_{i=1}^{\infty} (\mathbb{1}_A \cup \mathcal{R})^i$$

用归纳法证明下述等式即可:

$$\forall n \in \mathbb{N} \quad (\mathbb{1}_A \cup \mathcal{R})^n = \bigcup_{i=0}^{n} \mathcal{R}^i;$$

由此:

$$\forall n \ \ (\mathbb{1}_A \cup \mathcal{R})^n \subseteq \bigcup_{i=0}^{n} \mathcal{R}^i \subseteq \bigcup_{i=0}^{\infty} \mathcal{R}^i;$$

所以:

$$\bigcup_{i=0}^{\infty} \mathcal{R}^i \subseteq \bigcup_{i=1}^{\infty} (\mathbb{1}_A \cup \mathcal{R})^i \subseteq \bigcup_{i=0}^{\infty} \mathcal{R}^i;$$

即:

$$rt(\mathcal{R}) = tr(\mathcal{R}).$$

□

闭包之间的关系(2/3)

Propostion

$$rt(\mathcal{R}) = tr(\mathcal{R});$$

Proof.

$$rt(\mathcal{R}) = \bigcup_{i=0}^{\infty} \mathcal{R}^i; \quad tr(\mathcal{R}) = \bigcup_{i=1}^{\infty} (\mathbb{1}_A \cup \mathcal{R})^i$$

用归纳法证明下述等式即可:

$$\forall n \in \mathbb{N} \quad (\mathbb{1}_A \cup \mathcal{R})^n = \bigcup_{i=0}^{n} \mathcal{R}^i;$$

由此:

$$\forall n \ \ (\mathbb{1}_A \cup \mathcal{R})^n \subseteq \bigcup_{i=0}^{n} \mathcal{R}^i \subseteq \bigcup_{i=0}^{\infty} \mathcal{R}^i;$$

所以:

$$\bigcup_{i=0}^{\infty} \mathcal{R}^i \subseteq \bigcup_{i=1}^{\infty} (\mathbb{1}_A \cup \mathcal{R})^i \subseteq \bigcup_{i=0}^{\infty} \mathcal{R}^i;$$

即:

$$rt(\mathcal{R}) = tr(\mathcal{R}).$$

□

闭包之间的关系(3/3)

Proof(continued).

$$\forall n \in \mathbb{N} \quad (\mathbb{1}_A \cup \mathcal{R})^n = \bigcup_{i=0}^{n} \mathcal{R}^i;$$

1. $n = 0$时上述等式成立;
2. 设$n = k$时上述等式成立，则$n = k + 1$时:

闭包之间的关系(3/3)

Proof(continued).

$$\forall n \in \mathbb{N} \quad (\mathbb{1}_A \cup \mathcal{R})^n = \bigcup_{i=0}^{n} \mathcal{R}^i;$$

1. $n = 0$时上述等式成立;
2. 设$n = k$时上述等式成立，则$n = k + 1$时:

闭包之间的关系(3/3)

Proof(continued).

$$\forall n \in \mathbb{N} \quad (\mathbb{1}_A \cup \mathcal{R})^n = \bigcup_{i=0}^{n} \mathcal{R}^i;$$

1. $n = 0$时上述等式成立;
2. 设$n = k$时上述等式成立，则$n = k + 1$时:

$$\begin{aligned}
& (\mathbb{1}_A \cup \mathcal{R})^{k+1} \\
= {} & (\mathbb{1}_A \cup \mathcal{R})^k (\mathbb{1}_A \cup \mathcal{R}) && \text{(by 乘幂的定义)} \\
= {} & \left(\bigcup_{i=0}^{k} \mathcal{R}^i\right) (\mathbb{1}_A \cup \mathcal{R}) && \text{(by 归纳假设)} \\
= {} & \left(\left(\bigcup_{i=0}^{k} \mathcal{R}^i\right) \mathcal{R}\right) \bigcup \left(\left(\bigcup_{i=0}^{k} \mathcal{R}^i\right) \mathbb{1}_A\right) && \text{(by 合成对并的分配率)} \\
= {} & \left(\bigcup_{i=1}^{k+1} \mathcal{R}^i\right) \bigcup \left(\bigcup_{i=0}^{k} \mathcal{R}^i\right) && \text{(by 合成对并的分配率)} \\
= {} & \bigcup_{i=0}^{k+1} \mathcal{R}^i
\end{aligned}$$

闭包之间的关系(3/3)

Proof(continued).

$$\forall n \in \mathbb{N} \quad (\mathbb{1}_A \cup \mathcal{R})^n = \bigcup_{i=0}^{n} \mathcal{R}^i;$$

1. $n = 0$时上述等式成立;
2. 设$n = k$时上述等式成立，则$n = k + 1$时:

$$\begin{aligned}
&(\mathbb{1}_A \cup \mathcal{R})^{k+1} \\
&= (\mathbb{1}_A \cup \mathcal{R})^k(\mathbb{1}_A \cup \mathcal{R}) && \text{(by 乘幂的定义)} \\
&= \left(\bigcup_{i=0}^{k} \mathcal{R}^i\right)(\mathbb{1}_A \cup \mathcal{R}) && \text{(by 归纳假设)} \\
&= \left(\left(\bigcup_{i=0}^{k} \mathcal{R}^i\right)\mathcal{R}\right) \bigcup \left(\left(\bigcup_{i=0}^{k} \mathcal{R}^i\right)\mathbb{1}_A\right) && \text{(by 合成对并的分配率)} \\
&= \left(\bigcup_{i=1}^{k+1} \mathcal{R}^i\right) \bigcup \left(\bigcup_{i=0}^{k} \mathcal{R}^i\right) && \text{(by 合成对并的分配率)} \\
&= \bigcup_{i=0}^{k+1} \mathcal{R}^i
\end{aligned}$$

闭包之间的关系(3/3)

Proof(continued).

$$\forall n \in \mathbb{N} \quad (\mathbb{1}_A \cup \mathcal{R})^n = \bigcup_{i=0}^{n} \mathcal{R}^i;$$

1. $n = 0$时上述等式成立;
2. 设$n = k$时上述等式成立，则$n = k + 1$时:

$$\begin{aligned}
& (\mathbb{1}_A \cup \mathcal{R})^{k+1} \\
= & (\mathbb{1}_A \cup \mathcal{R})^k (\mathbb{1}_A \cup \mathcal{R}) && \text{(by 乘幂的定义)} \\
= & \left(\bigcup_{i=0}^{k} \mathcal{R}^i\right) (\mathbb{1}_A \cup \mathcal{R}) && \text{(by 归纳假设)} \\
= & \left(\left(\bigcup_{i=0}^{k} \mathcal{R}^i\right) \mathcal{R}\right) \bigcup \left(\left(\bigcup_{i=0}^{k} \mathcal{R}^i\right) \mathbb{1}_A\right) && \text{(by 合成对并的分配率)} \\
= & \left(\bigcup_{i=1}^{k+1} \mathcal{R}^i\right) \bigcup \left(\bigcup_{i=0}^{k} \mathcal{R}^i\right) && \text{(by 合成对并的分配率)} \\
= & \bigcup_{i=0}^{k+1} \mathcal{R}^i
\end{aligned}$$

闭包之间的关系(3/3)

Proof(continued).

$$\forall n \in \mathbb{N} \quad (\mathbb{1}_A \cup \mathcal{R})^n = \bigcup_{i=0}^{n} \mathcal{R}^i;$$

1. $n = 0$时上述等式成立;
2. 设$n = k$时上述等式成立，则$n = k + 1$时:

$$
\begin{aligned}
& (\mathbb{1}_A \cup \mathcal{R})^{k+1} & \\
= & (\mathbb{1}_A \cup \mathcal{R})^k (\mathbb{1}_A \cup \mathcal{R}) & \text{(by 乘幂的定义)} \\
= & \left(\bigcup_{i=0}^{k} \mathcal{R}^i\right) (\mathbb{1}_A \cup \mathcal{R}) & \text{(by 归纳假设)} \\
= & \left(\left(\bigcup_{i=0}^{k} \mathcal{R}^i\right) \mathcal{R}\right) \bigcup \left(\left(\bigcup_{i=0}^{k} \mathcal{R}^i\right) \mathbb{1}_A\right) & \text{(by 合成对并的分配率)} \\
= & \left(\bigcup_{i=1}^{k+1} \mathcal{R}^i\right) \bigcup \left(\bigcup_{i=0}^{k} \mathcal{R}^i\right) & \text{(by 合成对并的分配率)} \\
= & \bigcup_{i=0}^{k+1} \mathcal{R}^i &
\end{aligned}
$$

闭包之间的关系(3/3)

Proof(continued).

$$\forall n \in \mathbb{N} \quad (\mathbb{1}_A \cup \mathcal{R})^n = \bigcup_{i=0}^{n} \mathcal{R}^i;$$

1. $n = 0$时上述等式成立;
2. 设$n = k$时上述等式成立，则$n = k + 1$时:

$$\begin{aligned}
& (\mathbb{1}_A \cup \mathcal{R})^{k+1} \\
= {} & (\mathbb{1}_A \cup \mathcal{R})^k (\mathbb{1}_A \cup \mathcal{R}) && \text{(by 乘幂的定义)} \\
= {} & \left(\bigcup_{i=0}^{k} \mathcal{R}^i\right) (\mathbb{1}_A \cup \mathcal{R}) && \text{(by 归纳假设)} \\
= {} & \left(\left(\bigcup_{i=0}^{k} \mathcal{R}^i\right) \mathcal{R}\right) \bigcup \left(\left(\bigcup_{i=0}^{k} \mathcal{R}^i\right) \mathbb{1}_A\right) && \text{(by 合成对并的分配率)} \\
= {} & \left(\bigcup_{i=1}^{k+1} \mathcal{R}^i\right) \bigcup \left(\bigcup_{i=0}^{k} \mathcal{R}^i\right) && \text{(by 合成对并的分配率)} \\
= {} & \bigcup_{i=0}^{k+1} \mathcal{R}^i
\end{aligned}$$

闭包之间的关系(3/3)

Proof(continued).

$$\forall n \in \mathbb{N} \quad (\mathbb{1}_A \cup \mathcal{R})^n = \bigcup_{i=0}^{n} \mathcal{R}^i;$$

1. $n = 0$时上述等式成立;
2. 设$n = k$时上述等式成立，则$n = k+1$时:

$$
\begin{aligned}
& (\mathbb{1}_A \cup \mathcal{R})^{k+1} & \\
= & (\mathbb{1}_A \cup \mathcal{R})^k (\mathbb{1}_A \cup \mathcal{R}) & \text{(by 乘幂的定义)} \\
= & \left(\bigcup_{i=0}^{k} \mathcal{R}^i\right) (\mathbb{1}_A \cup \mathcal{R}) & \text{(by 归纳假设)} \\
= & \left(\left(\bigcup_{i=0}^{k} \mathcal{R}^i\right) \mathcal{R}\right) \bigcup \left(\left(\bigcup_{i=0}^{k} \mathcal{R}^i\right) \mathbb{1}_A\right) & \text{(by 合成对并的分配率)} \\
= & \left(\bigcup_{i=1}^{k+1} \mathcal{R}^i\right) \bigcup \left(\bigcup_{i=0}^{k} \mathcal{R}^i\right) & \text{(by 合成对并的分配率)} \\
= & \bigcup_{i=0}^{k+1} \mathcal{R}^i &
\end{aligned}
$$

闭包之间的关系(3/3)

Proof(continued).

$$\forall n \in \mathbb{N} \quad (\mathbb{1}_A \cup \mathcal{R})^n = \bigcup_{i=0}^{n} \mathcal{R}^i;$$

1. $n = 0$时上述等式成立;
2. 设$n = k$时上述等式成立，则$n = k + 1$时:

$$
\begin{aligned}
& (\mathbb{1}_A \cup \mathcal{R})^{k+1} \\
= & (\mathbb{1}_A \cup \mathcal{R})^k (\mathbb{1}_A \cup \mathcal{R}) && \text{(by 乘幂的定义)} \\
= & \left(\bigcup_{i=0}^{k} \mathcal{R}^i\right) (\mathbb{1}_A \cup \mathcal{R}) && \text{(by 归纳假设)} \\
= & \left(\left(\bigcup_{i=0}^{k} \mathcal{R}^i\right) \mathcal{R}\right) \bigcup \left(\left(\bigcup_{i=0}^{k} \mathcal{R}^i\right) \mathbb{1}_A\right) && \text{(by 合成对并的分配率)} \\
= & \left(\bigcup_{i=1}^{k+1} \mathcal{R}^i\right) \bigcup \left(\bigcup_{i=0}^{k} \mathcal{R}^i\right) && \text{(by 合成对并的分配率)} \\
= & \bigcup_{i=0}^{k+1} \mathcal{R}^i
\end{aligned}
$$

关系的合成与关系矩阵乘积

Theorem

设$\mathcal{R} \subseteq A \times B$, $\mathcal{S} \subseteq B \times C$; $|A| = m$, $|B| = n$和$|C| = p$, 则:

$$M_{\mathcal{R}\,\mathcal{S}} = M_{\mathcal{R}} \cdot M_{\mathcal{S}};$$

其中: $M_{\mathcal{R}} = (a_{ij})_{m\times n}$; $\quad M_{\mathcal{S}} = (b_{ij})_{n\times p}$

$$M_{\mathcal{R}} \cdot M_{\mathcal{S}} = (c_{ij})_{m\times p}; \quad c_{ij} \triangleq \bigvee_{k=1}^{n} a_{ik} \wedge b_{kj};$$

Proof.

设$A = \{x_1, x_2, \ldots, x_m\}$, $B = \{y_1, y_2, \ldots, y_n\}$, $C = \{z_1, z_2, \ldots, z_p\}$

关系的合成与关系矩阵乘积

Theorem

设$\mathcal{R} \subseteq A \times B$, $\mathcal{S} \subseteq B \times C$; $|A| = m$, $|B| = n$和$|C| = p$, 则:

$$M_{\mathcal{R}\,\mathcal{S}} = M_{\mathcal{R}} \cdot M_{\mathcal{S}};$$

其中: $M_{\mathcal{R}} = (a_{ij})_{m\times n}$; $M_{\mathcal{S}} = (b_{ij})_{n\times p}$

$$M_{\mathcal{R}} \cdot M_{\mathcal{S}} = (c_{ij})_{m\times p}; \quad c_{ij} \triangleq \bigvee_{k=1}^{n} a_{ik} \wedge b_{kj};$$

Proof.

设$A = \{x_1, x_2, \ldots, x_m\}$, $B = \{y_1, y_2, \ldots, y_n\}$, $C = \{z_1, z_2, \ldots, z_p\}$

1. $c_{ij} = 1$
2. $\iff \exists k \quad a_{ik} = 1 \wedge b_{kj} = 1$
3. $\iff \exists k \quad x_i \, \mathcal{R} \, y_k \wedge y_k \, \mathcal{S} \, z_j$
4. $\iff x_i \, \mathcal{R} \, \mathcal{S} \, z_j$

$\square$

关系的合成与关系矩阵乘积

Theorem

设$\mathcal{R} \subseteq A \times B$, $\mathcal{S} \subseteq B \times C$; $|A| = m$, $|B| = n$和$|C| = p$, 则:

$$M_{\mathcal{R}\,\mathcal{S}} = M_{\mathcal{R}} \cdot M_{\mathcal{S}};$$

其中: $M_{\mathcal{R}} = (a_{ij})_{m\times n}$; $\quad M_{\mathcal{S}} = (b_{ij})_{n\times p}$

$$M_{\mathcal{R}} \cdot M_{\mathcal{S}} = (c_{ij})_{m\times p}; \quad c_{ij} \triangleq \bigvee_{k=1}^{n} a_{ik} \wedge b_{kj};$$

Proof.

设$A = \{x_1, x_2, \ldots, x_m\}$, $B = \{y_1, y_2, \ldots, y_n\}$, $C = \{z_1, z_2, \ldots, z_p\}$

1. $c_{ij} = 1$
2. $\Longleftrightarrow \exists k \quad a_{ik} = 1 \wedge b_{kj} = 1$
3. $\Longleftrightarrow \exists k \quad x_i\, \mathcal{R}\, y_k \wedge y_k\, \mathcal{S}\, z_j$
4. $\Longleftrightarrow x_i\, \mathcal{R}\,\mathcal{S}\, z_j$

□

关系的合成与关系矩阵乘积

Theorem

设$\mathcal{R} \subseteq A \times B$, $\mathcal{S} \subseteq B \times C$; $|A| = m$, $|B| = n$和$|C| = p$, 则:

$$M_{\mathcal{R}\,\mathcal{S}} = M_{\mathcal{R}} \cdot M_{\mathcal{S}};$$

其中: $M_{\mathcal{R}} = (a_{ij})_{m \times n}$; $\quad M_{\mathcal{S}} = (b_{ij})_{n \times p}$

$$M_{\mathcal{R}} \cdot M_{\mathcal{S}} = (c_{ij})_{m \times p}; \quad c_{ij} \triangleq \bigvee_{k=1}^{n} a_{ik} \wedge b_{kj};$$

Proof.

设$A = \{ x_1, x_2, \ldots, x_m \}$, $B = \{ y_1, y_2, \ldots, y_n \}$, $C = \{ z_1, z_2, \ldots, z_p \}$

1. $\quad c_{ij} = 1$
2. $\iff \exists k \quad a_{ik} = 1 \wedge b_{kj} = 1$
3. $\iff \exists k \quad x_i \,\mathcal{R}\, y_k \wedge y_k \,\mathcal{S}\, z_j$
4. $\iff x_i \,\mathcal{R}\,\mathcal{S}\, z_j$

□

关系的合成与关系矩阵乘积

Theorem

设$\mathcal{R} \subseteq A \times B$, $\mathcal{S} \subseteq B \times C$; $|A| = m$, $|B| = n$和$|C| = p$, 则:

$$M_{\mathcal{R}\,\mathcal{S}} = M_{\mathcal{R}} \cdot M_{\mathcal{S}};$$

其中: $M_{\mathcal{R}} = (a_{ij})_{m\times n}$; $\quad M_{\mathcal{S}} = (b_{ij})_{n\times p}$

$$M_{\mathcal{R}} \cdot M_{\mathcal{S}} = (c_{ij})_{m\times p}; \quad c_{ij} \triangleq \bigvee_{k=1}^{n} a_{ik} \wedge b_{kj};$$

Proof.

设$A = \{x_1, x_2, \ldots, x_m\}$, $B = \{y_1, y_2, \ldots, y_n\}$, $C = \{z_1, z_2, \ldots, z_p\}$

1. $c_{ij} = 1$
2. $\iff \exists k \;\; a_{ik} = 1 \wedge b_{kj} = 1$
3. $\iff \exists k \;\; x_i \,\mathcal{R}\, y_k \wedge y_k \,\mathcal{S}\, z_j$
4. $\iff x_i \,\mathcal{R}\,\mathcal{S}\, z_j$

$\square$

关系的合成与关系矩阵乘积

Theorem

设$\mathcal{R} \subseteq A \times B$, $\mathcal{S} \subseteq B \times C$; $|A| = m$, $|B| = n$和$|C| = p$, 则:

$$M_{\mathcal{R}\,\mathcal{S}} = M_{\mathcal{R}} \cdot M_{\mathcal{S}};$$

其中: $M_{\mathcal{R}} = (a_{ij})_{m\times n}$; $M_{\mathcal{S}} = (b_{ij})_{n\times p}$

$$M_{\mathcal{R}} \cdot M_{\mathcal{S}} = (c_{ij})_{m\times p}; \quad c_{ij} \triangleq \bigvee_{k=1}^{n} a_{ik} \wedge b_{kj};$$

Proof.

设$A = \{x_1, x_2, \ldots, x_m\}$, $B = \{y_1, y_2, \ldots, y_n\}$, $C = \{z_1, z_2, \ldots, z_p\}$

1. $c_{ij} = 1$
2. $\iff \exists k \ \ a_{ik} = 1 \wedge b_{kj} = 1$
3. $\iff \exists k \ \ x_i \,\mathcal{R}\, y_k \wedge y_k \,\mathcal{S}\, z_j$
4. $\iff x_i \,\mathcal{R}\,\mathcal{S}\, z_j$

$\square$

Example

Example

设: $M_{\mathcal{R}} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$; $M_{\mathcal{S}} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$;

则:

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$M_{\mathcal{R}\,\mathcal{S}} = M_{\mathcal{R}} \cdot M_{\mathcal{S}} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

传递闭包的求解算法

Description

$$M_{t(\mathcal{R})} = \sum_{i=1}^{n} M_{\mathcal{R}}^{i}$$

其中：$M_{\mathcal{R}}$是n阶方阵；

- 计算$M \cdot M$的每个元素$c_{ij} = \bigvee_{k=1}^{n} a_{ik} \wedge b_{kj}$ .... $O(n)$;
- 计算$M \cdot M$ .... $O(n^3)$;
- 计算$\sum_{i=1}^{n} M_{\mathcal{R}}^{i}$ .... $O(n^4)$.

Warshall算法可降算法的复杂度为：$O(n^3)$.

传递闭包的求解算法

Description

$$M_{t(\mathcal{R})} = \sum_{i=1}^{n} M_{\mathcal{R}}^{i}$$

其中：$M_{\mathcal{R}}$是n阶方阵；

- 计算$M \cdot M$的每个元素$c_{ij} = \bigvee_{k=1}^{n} a_{ik} \wedge b_{kj}$ ……………… $O(n)$;
- 计算$M \cdot M$ ……………… $O(n^3)$;
- 计算$\sum_{i=1}^{n} M_{\mathcal{R}}^{i}$ ……………… $O(n^4)$.

Warshall算法可降算法的复杂度为：$O(n^3)$.

传递闭包的求解算法

Description

$$M_{t(\mathcal{R})} = \sum_{i=1}^{n} M_{\mathcal{R}}^{i}$$

其中：$M_{\mathcal{R}}$是n阶方阵；

- 计算$M \cdot M$的每个元素$c_{ij} = \bigvee_{k=1}^{n} a_{ik} \wedge b_{kj}$ .... $\mathrm{O}(n)$;
- 计算$M \cdot M$ .... $\mathrm{O}(n^3)$;
- 计算$\sum_{i=1}^{n} M_{\mathcal{R}}^{i}$ .... $\mathrm{O}(n^4)$.

Warshall算法可降算法的复杂度为：$\mathrm{O}(n^3)$.

传递闭包的求解算法

Description

$$M_{t(\mathcal{R})} = \sum_{i=1}^{n} M_{\mathcal{R}}^{i}$$

其中：$M_{\mathcal{R}}$是n阶方阵；

- 计算$M \cdot M$的每个元素$c_{ij} = \bigvee_{k=1}^{n} a_{ik} \wedge b_{kj}$ .... $\mathrm{O}(n)$;
- 计算$M \cdot M$ .... $\mathrm{O}(n^3)$;
- 计算$\sum_{i=1}^{n} M_{\mathcal{R}}^{i}$ .... $\mathrm{O}(n^4)$.

Warshall算法可降算法的复杂度为：$\mathrm{O}(n^3)$.

Warshall算法

Definition

设$A = \{a_1, a_2, \ldots, a_n\}$, $\mathcal{R} \subseteq A^2$, M是$\mathcal{R}$的关系矩阵；n阶方阵W_k递归定义如下：

1. $W_0 = M$;
2. $W_k = (w_{ij}^k)_{n\times n}$, 其中：$w_{ij}^k = 1$, iff, 从$a_i$到$a_j$ 有一条仅经过$a_1, a_2, \ldots, a_k$的有向路径.

Propostion

$$W_n = M_{t(\mathcal{R})}$$

Warshall算法

Definition

设$A = \{a_1, a_2, \ldots, a_n\}$, $\mathcal{R} \subseteq A^2$, M是$\mathcal{R}$的关系矩阵; n阶方阵W_k递归定义如下:

1. $W_0 = M$;
2. $W_k = (w_{ij}^k)_{n\times n}$, 其中: $w_{ij}^k = 1$, iff, 从a_i到a_j 有一条仅经过$a_1, a_2, \ldots, a_k$的有向路径.

Propostion

$$W_n = M_{t(\mathcal{R})}$$

Warshall算法

Definition

设$A = \{ a_1, a_2, \ldots, a_n \}$, $\mathcal{R} \subseteq A^2$, M是$\mathcal{R}$的关系矩阵; n阶方阵W_k递归定义如下:

1. $W_0 = M$;
2. $W_k = (w_{ij}^k)_{n\times n}$, 其中: $w_{ij}^k = 1$, iff, 从a_i到a_j 有一条仅经过$a_1, a_2, \ldots, a_k$的有向路径.

Propostion

$$W_n = M_{t(\mathcal{R})}$$

Warshall算法

Definition

设$A = \{a_1, a_2, \ldots, a_n\}$, $\mathcal{R} \subseteq A^2$, M是$\mathcal{R}$的关系矩阵; n阶方阵W_k递归定义如下:

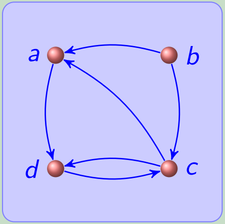
1. $W_0 = M$;
2. $W_k = (w_{ij}^k)_{n\times n}$, 其中: $w_{ij}^k = 1$, iff, 从a_i到a_j 有一条仅经过$a_1, a_2, \ldots, a_k$的有向路径.

Propostion

$$W_n = M_{t(\mathcal{R})}$$

Example

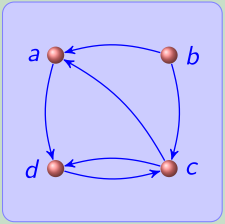
Example



$$W_0 = \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array} \begin{array}{c} a \\ b \\ c \\ d \end{array}$$

Example

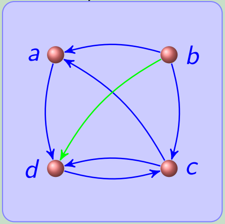
Example



$$W_0 = \begin{array}{c} \\ \\ \\ \\ \\ \end{array}\begin{array}{cccc} a & b & c & d \\ \end{array}$$

$$W_0 = \begin{array}{cc} \begin{array}{cccc} a & b & c & d \end{array} & \\ \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} & \begin{array}{c} a \\ b \\ c \\ d \end{array} \end{array}$$

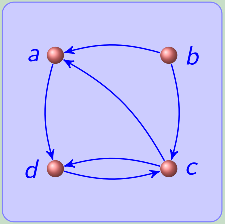
$k = 1$时，经过a点的路径：



$$W_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

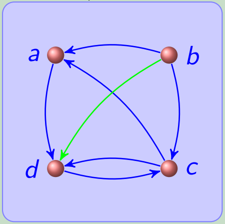
Example

Example



$$W_0 = \begin{array}{cc} \begin{array}{cccc} a & b & c & d \end{array} & \\ \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} & \begin{array}{c} a \\ b \\ c \\ d \end{array} \end{array}$$

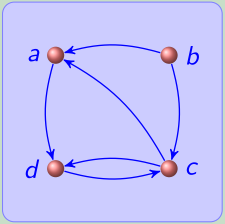
$k=2$时，经过$a,\ b$点的路径：b没有引入的边，所以没有经过b的路径



$$W_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

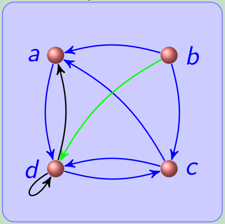
Example

Example



$$W_0 = \begin{array}{cc} & \begin{array}{cccc} a & b & c & d \end{array} \\ & \left(\begin{array}{cccc} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right) \begin{array}{c} a \\ b \\ c \\ d \end{array} \end{array}$$

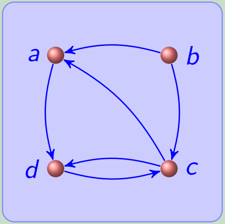
$k=3$时，经过a，b，c点的路径：



$$W_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

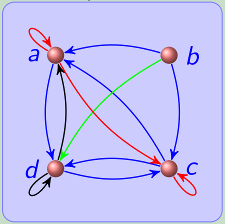
Example

Example



$$W_0 = \begin{array}{c@{}l} \begin{array}{cccc} a & b & c & d \end{array} & \\ \left(\begin{array}{cccc} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right) & \begin{array}{l} a \\ b \\ c \\ d \end{array} \end{array}$$

$k=4$时，经过a，b，c，d点的路径：



$$W_4 = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

W_k的计算

Description (W_k和W_{k+1}的关系)

$w_{ij}^{k+1} = 1$, *iff*, 下述两条件之一成立:

1. $w_{ij}^{k} = 1$, 即从a_i到a_j有一条仅经过$a_1, a_2, \ldots, a_k$的有向路径;
2. 有一条仅经过$a_1, a_2, \ldots, a_{k+1}$, 并且仅经过$a_{k+1}$一次的路径: 如:

$$a_i, x_1, x_2, \ldots, x_p, a_{k+1}, y_1, y_2, \ldots, y_q, a_j$$

其中: $x_1, x_2, \ldots, x_p$和$y_1, y_2, \ldots, y_q$都在$\{a_1, a_2, \ldots, a_k\}$中;

$\therefore$ $$w_{i(k+1)}^{k} = 1 \wedge w_{(k+1)j}^{k} = 1;$$

故:

$$\boxed{w_{ij}^{k+1} = w_{ij}^{k} \vee (w_{i(k+1)}^{k} \wedge w_{(k+1)j}^{k})}$$

W_k的计算

Description (W_k和W_{k+1}的关系)

$w_{ij}^{k+1} = 1$, iff, 下述两条件之一成立：

1. $w_{ij}^{k} = 1$，即从a_i到a_j有一条仅经过$a_1, a_2, \ldots, a_k$的有向路径；
2. 有一条仅经过$a_1, a_2, \ldots, a_{k+1}$，并且仅经过$a_{k+1}$一次的路径：如：

$$a_i, x_1, x_2, \ldots, x_p, a_{k+1}, y_1, y_2, \ldots, y_q, a_j$$

其中：$x_1, x_2, \ldots, x_p$和$y_1, y_2, \ldots, y_q$都在$\{a_1, a_2, \ldots, a_k\}$中；

$\therefore \quad w_{i(k+1)}^{k} = 1 \wedge w_{(k+1)j}^{k} = 1;$

故：

$$\boxed{w_{ij}^{k+1} = w_{ij}^{k} \vee (w_{i(k+1)}^{k} \wedge w_{(k+1)j}^{k})}$$

W_k的计算

Description (W_k和W_{k+1}的关系)

$w_{ij}^{k+1} = 1$, iff, 下述两条件之一成立:

1. $w_{ij}^{k} = 1$, 即从a_i到a_j有一条仅经过$a_1, a_2, \ldots, a_k$的有向路径;
2. 有一条仅经过$a_1, a_2, \ldots, a_{k+1}$, 并且仅经过$a_{k+1}$一次的路径: 如:

$$a_i, x_1, x_2, \ldots, x_p, a_{k+1}, y_1, y_2, \ldots, y_q, a_j$$

其中: $x_1, x_2, \ldots, x_p$和$y_1, y_2, \ldots, y_q$都在$\{a_1, a_2, \ldots, a_k\}$中;

$\therefore \quad w_{i(k+1)}^{k} = 1 \wedge w_{(k+1)j}^{k} = 1;$

故:

$$\boxed{w_{ij}^{k+1} = w_{ij}^{k} \vee (w_{i(k+1)}^{k} \wedge w_{(k+1)j}^{k})}$$

Warshall算法

Warshall算法.

```
procedure warshall(Matrix M_R)
{
  W := M_R;
  for k := 1 to n do {
    for i := 1 to n do {
      for j := 1 to n do {
        w_ij := w_ij ∨ (w_ik ∧ w_kj);
      }
    }
  }
}
```

□

本章小节

1. 关系的合成
   - 关系的合成
   - 关系的幂
   - 关系的闭包
   - 传递闭包的求解算法