Cross Entropy under Linear Constraints

Nicholas Barnfield   Michael P. Friedlander   Tim Hoheisel

2024-04-03

Problem statement

Define the cross-entropy and log-exponential functions by

$$
\begin{aligned}
\operatorname{cent}\,(p \mid q) &= \sum_{i=1}^{n} p_i \log(p_i / q_i), \\
\operatorname{logexp}\,(p \mid q) &= \log \sum_{i=1}^{n} q_i \exp(p_i),
\end{aligned}
$$

which form a conjugate pair. We consider the pair of dual optimization problems

$$
\begin{aligned}
&\min_{p,\, y}\left\{\operatorname{cent}\,(p \mid q) + \lambda/2 \,\| y \|^2 \mid Ax + \lambda y = b\right\} \\
&\min_{y}\left\{\operatorname{logexp}\,(A^T y \mid q) + \lambda/2 \,\| y \|^2 - \langle b, y \rangle\right\},
\end{aligned}
$$

where A is an m-by-n matrix, b is an m-vector, and λ is a nonnegative regularization parameter.

Implicit approach

Optimality conditions

A pair (p, y) is primal-dual optimal if and only if it satisfies the optimality conditions

$$
\begin{aligned}
b &= Ax + \lambda y, \\
x_j &= \frac{1}{\sigma(x)} q_j \exp\bigl(\langle a_j, y \rangle\bigr), \quad j \in [n], \\
\sigma(x) &= \sum_{j=1}^{n} q_j \exp\bigl(\langle a_j, y \rangle\bigr),
\end{aligned}
$$