1. Example Applications of Pool Models

2. Reducing Model Complexity
   - The Carbon Cycle example
   - Asking Simpler Questions
   - Answer Questions more simply

Pool Models

An Introduction

Markus Müller, Holger Metzler, Carlos Sierra

August 8, 2019

Max Planck Institute
for Biogeochemistry

- What are pool models?
- Why do we need them?
- What can they be used for?
  - What is needed?
  - What can we learn from them?

Pool Models

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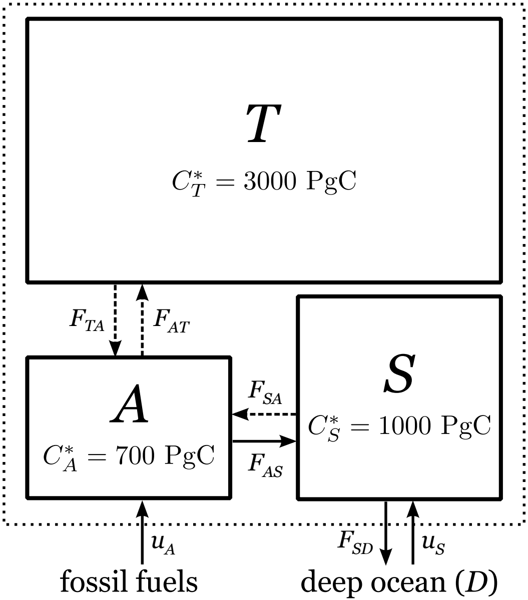
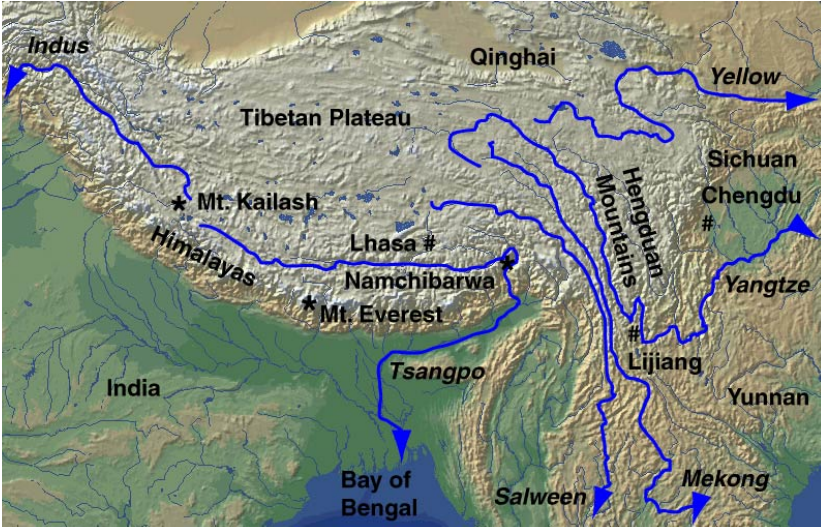
Outline

Outline

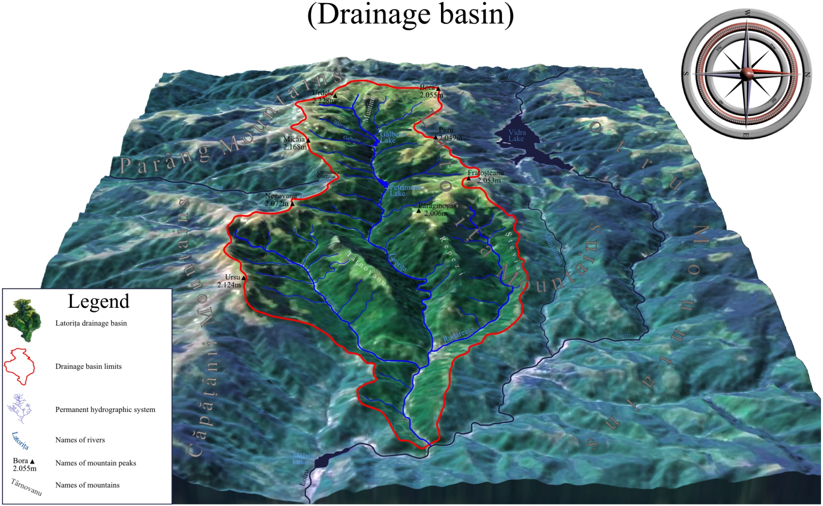

T
$C_T^* = 3000$ PgC
F_{TA}
F_{AT}
A
$C_A^* = 700$ PgC
F_{SA}
F_{AS}
S
$C_S^* = 1000$ PgC
u_A
F_{SD}
u_S
fossil fuels
deep ocean (D)

Hydrology Watersheds

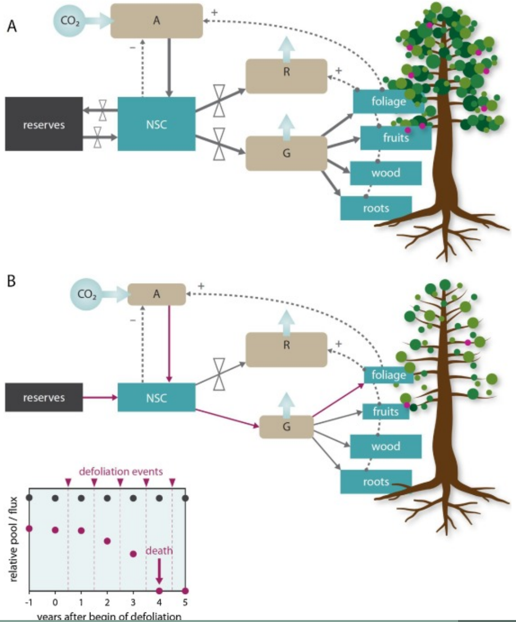


Latorița River, tributary of the Lotru River

(Drainage basin)



Plant Physiology/ Carbon allocation



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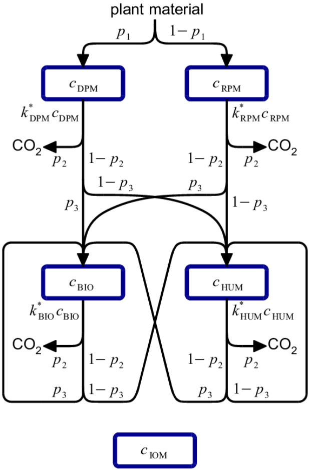
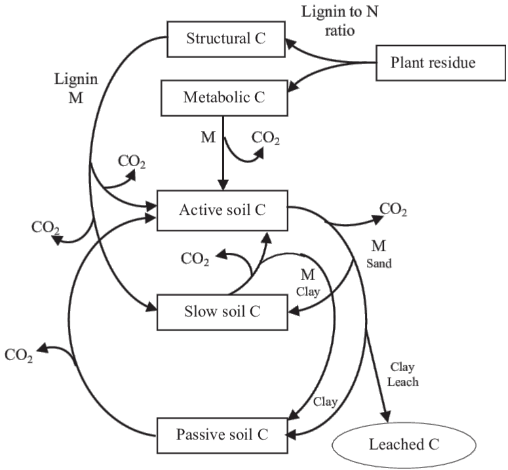
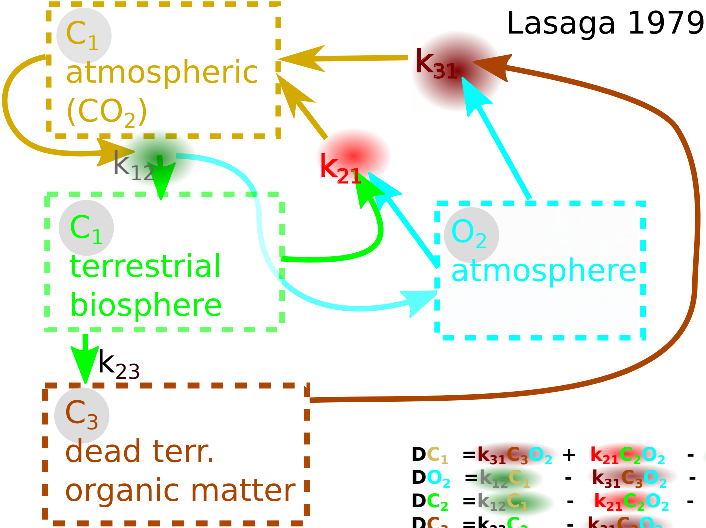
Organic Matter Decomposition / Soil Models



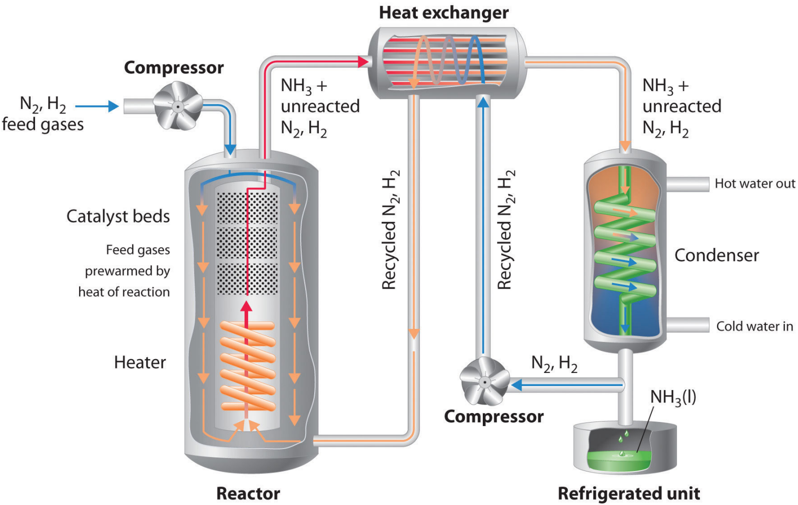


Lasaga 1979

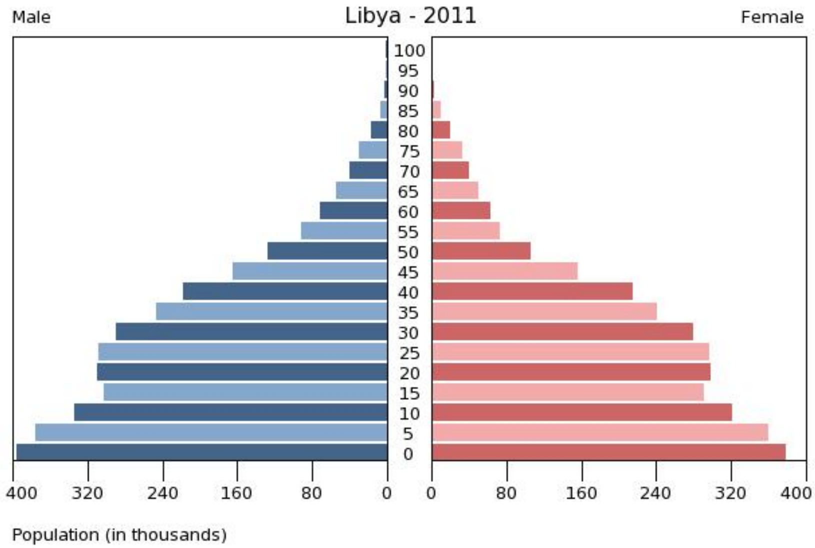


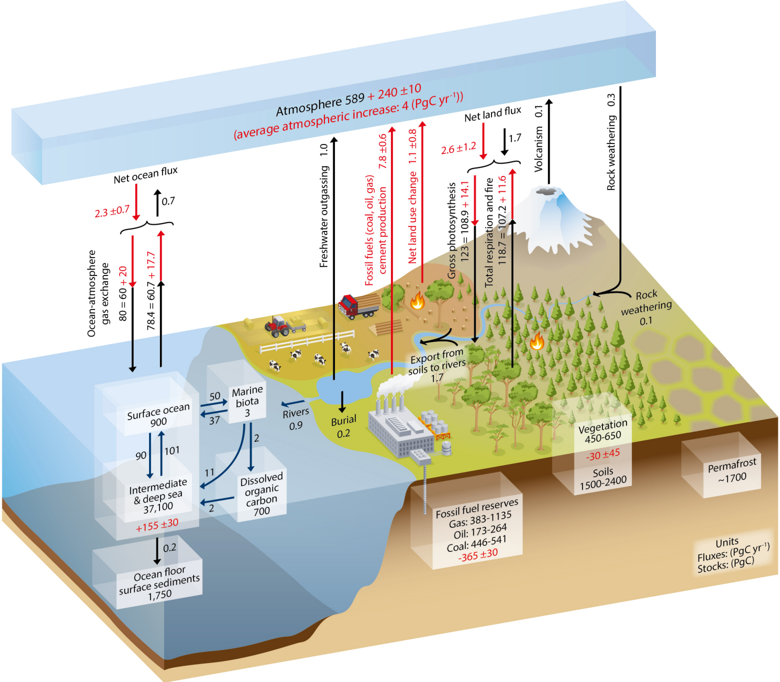
$$
\begin{aligned}
DC_1 &= k_{31}C_3O_2 + k_{21}C_2O_2 - k_{12}C_1\\
DO_2 &= k_{12}C_1 - k_{31}C_3O_2 - k_{21}C_2O_2\\
DC_2 &= k_{12}C_1 - k_{21}C_2O_2 - k_{23}C_2\\
DC_3 &= k_{23}C_2 - k_{31}C_3O_2
\end{aligned}
$$

Chemical Reactors

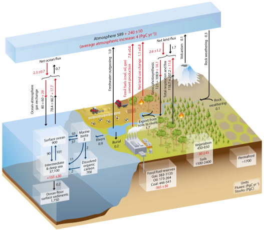


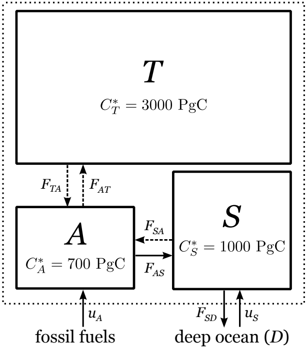
Population Dynamics





Atmosphere 589 + 240 ±10
(average atmospheric increase: 4 (PgC yr⁻¹))
Net ocean flux
2.3 ±0.7
0.7
Ocean-atmosphere gas exchange
80 = 60 + 20
78.4 = 60.7 + 17.7
Freshwater outgassing 1.0
Fossil fuels (coal, oil, gas) cement production 7.8 ±0.6
Net land use change 1.1 ±0.8
Net land flux
2.6 ±1.2
1.7
Gross photosynthesis 123 = 108.9 + 14.1
Total respiration and fire 118.7 = 107.2 + 11.6
Volcanism 0.1
Rock weathering 0.3
Rock weathering 0.1
Export from soils to rivers 1.7
Rivers 0.9
Burial 0.2
Surface ocean 900
50
37
Marine biota 3
90
101
2
11
Intermediate & deep sea 37,100
+155 ±30
Dissolved organic carbon 700
2
0.2
Ocean floor surface sediments 1,750
Vegetation 450-650
-30 ±45
Soils 1500-2400
Permafrost ~1700
Fossil fuel reserves
Gas: 383-1135
Oil: 173-264
Coal: 446-541
-365 ±30
Units
Fluxes: (PgC yr⁻¹)
Stocks: (PgC)





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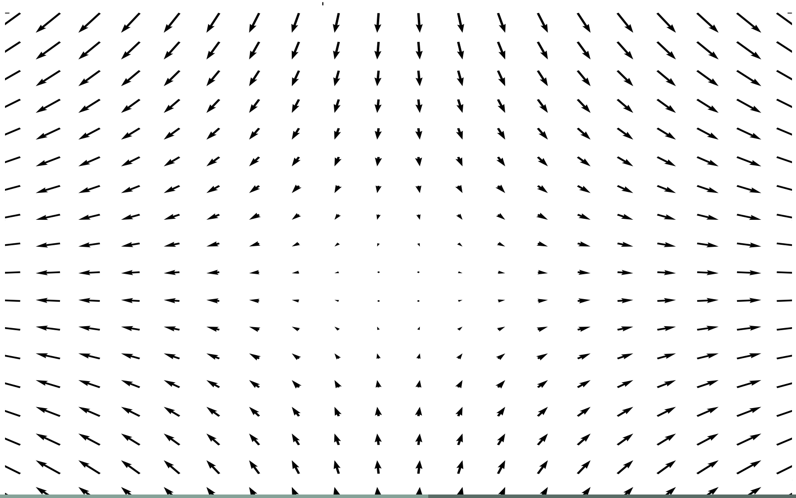
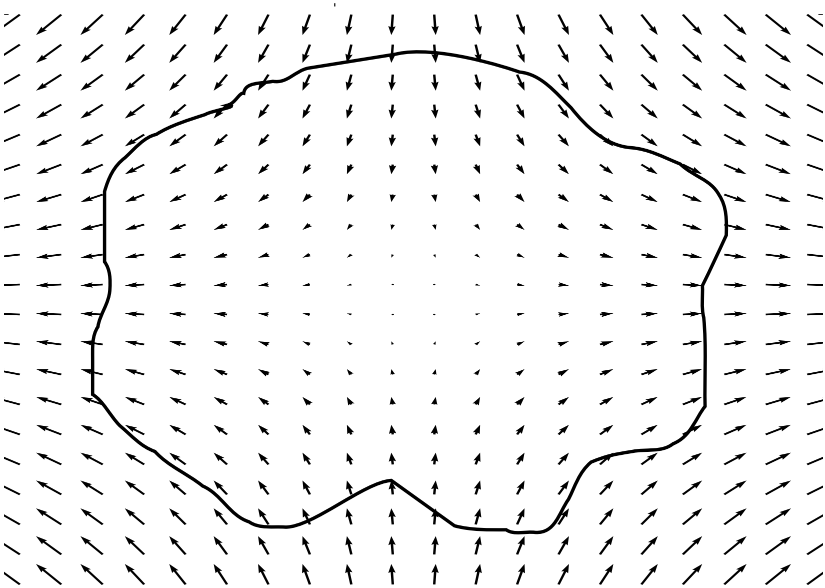
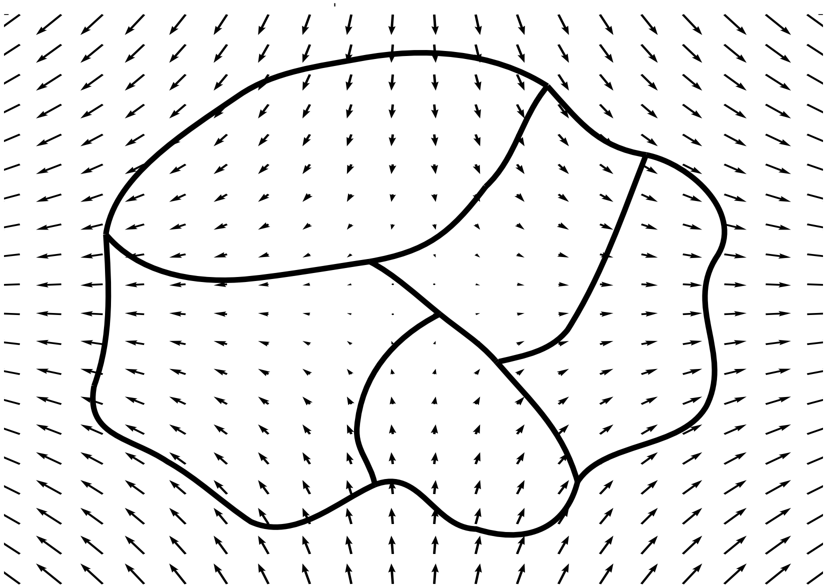
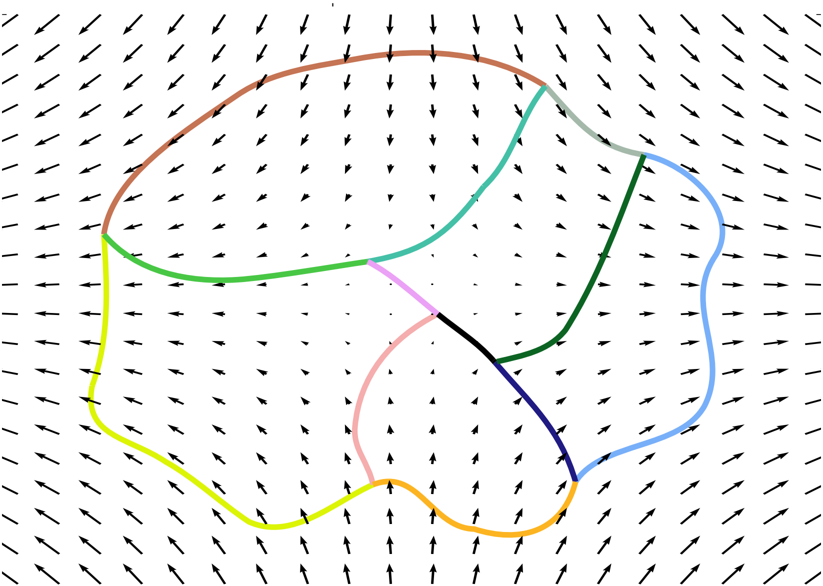
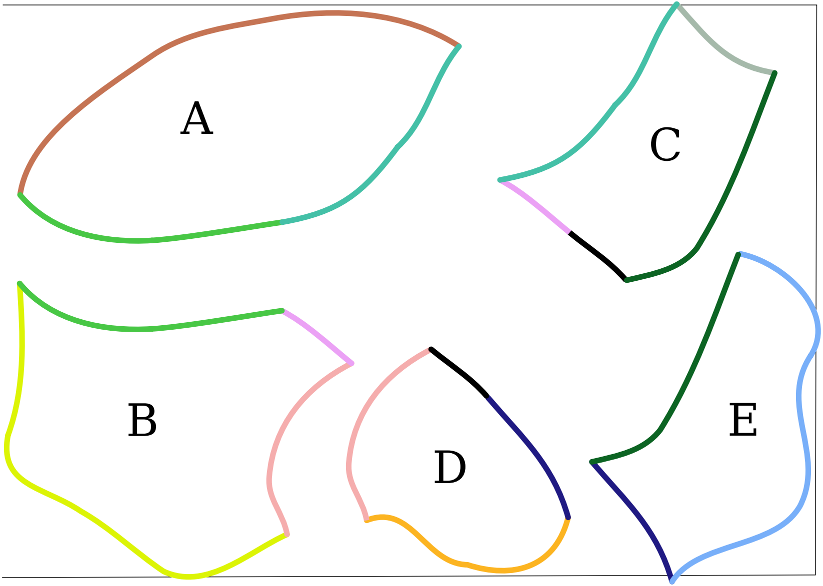
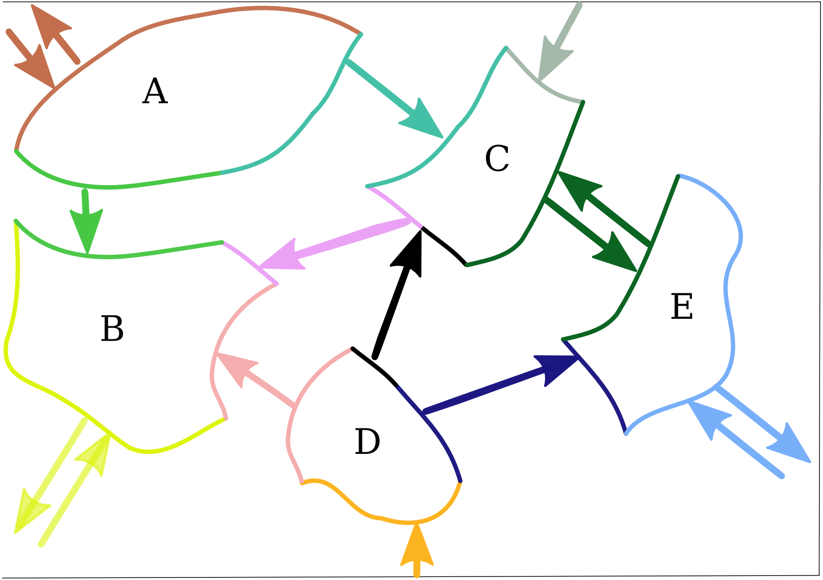




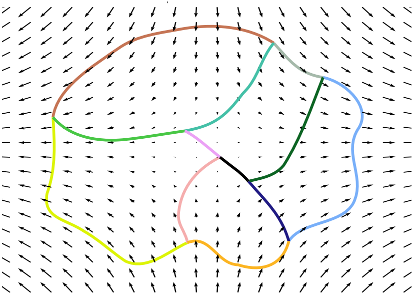


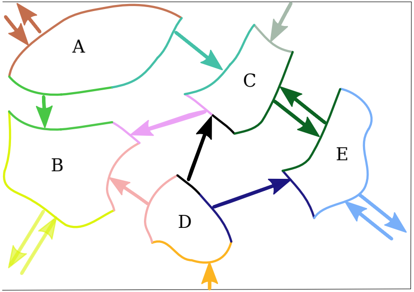



A
B
C
D
E



A
C
B
E
D




A
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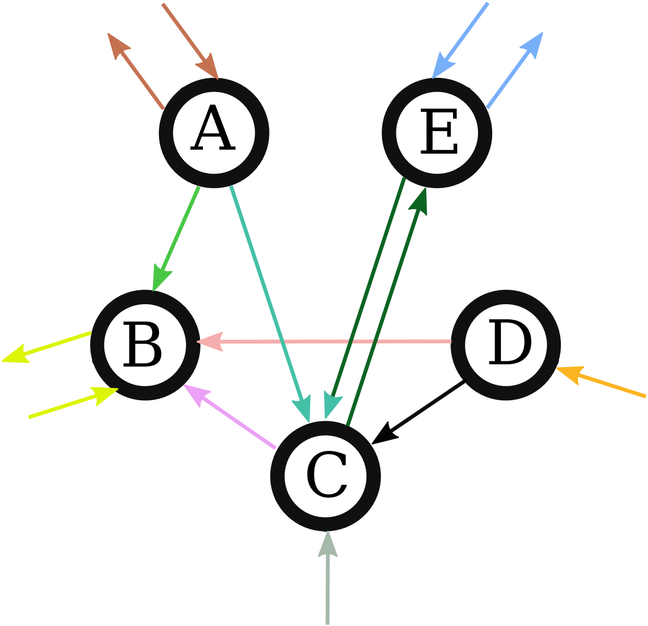
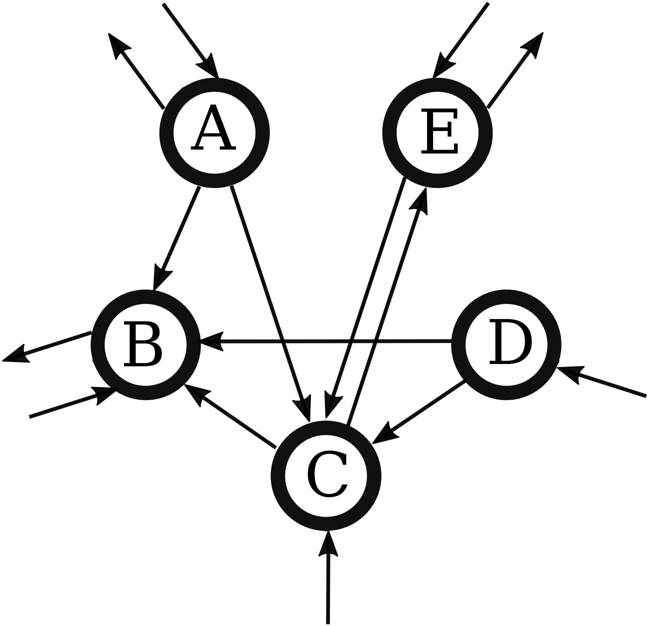
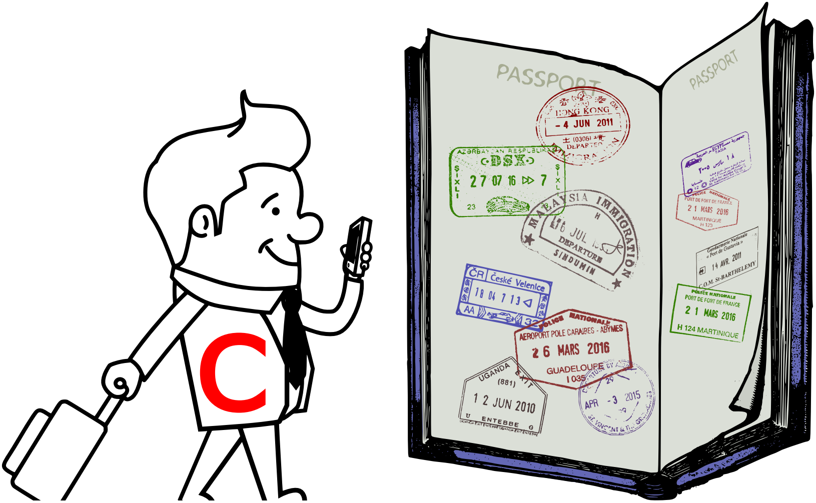
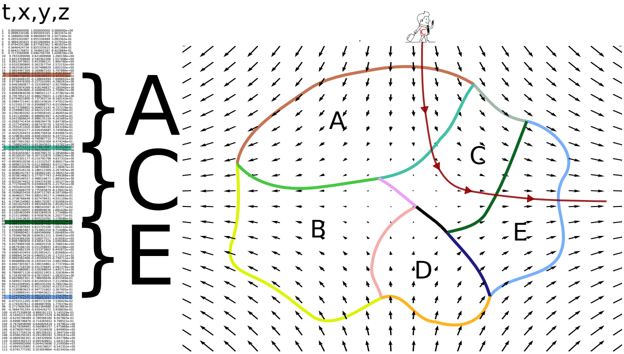
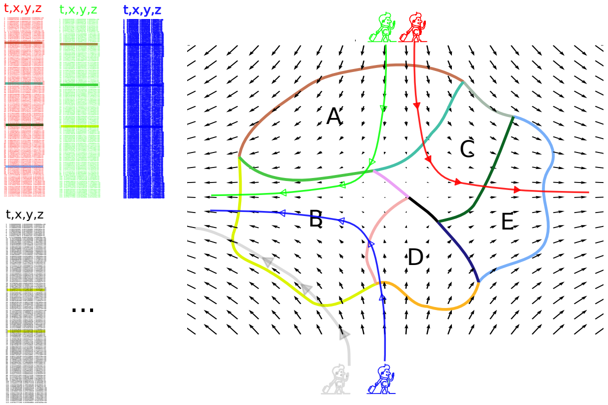

A
E
B
D
C


A
E
B
D
C


C
PASSPORT
PASSPORT
HONG KONG
- 4 JUN 2011
DEPARTED
DSX
27 07 16 7
MALAYSIA IMMIGRATION
DEPARTURE
SINDUMIN
ČR České Velenice
18 04 7 13
POLICE NATIONALE
AEROPORT POLE CARAIBES - ABYMES
26 MARS 2016
GUADELOUPE
UGANDA
(881)
12 JUN 2010
ENTEBBE
APR -3 2015
PORT DE FORT DE FRANCE
21 MARS 2016
MARTINIQUE
Gendarmerie Nationale
« Port de Gustavia »
14 AVR. 2011
C.O.M. St-BARTHELEMY
POLICE NATIONALE
PORT DE FORT DE FRANCE
21 MARS 2016
H 124 MARTINIQUE



	Pools	arrival
1	A	16.45
2	C	42.45
3	E	68.56
4	Ext	94.23
...		



Pools arrival		
1	A	16.45
2	C	42.45
3	E	68.56
4	Ext	94.23

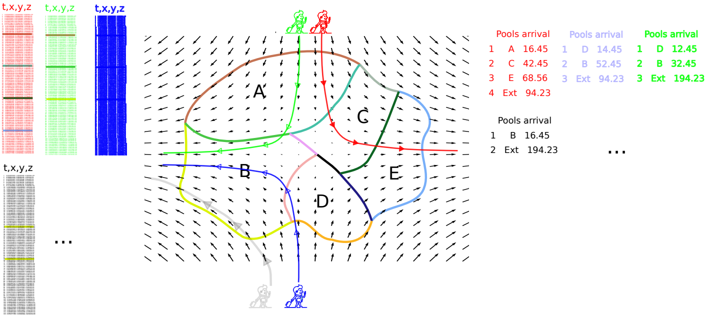
Pools arrival		
1	D	14.45
2	B	52.45
3	Ext	94.23

Pools arrival		
1	D	12.45
2	B	32.45
3	Ext	194.23

Pools arrival		
1	B	16.45
2	Ext	194.23

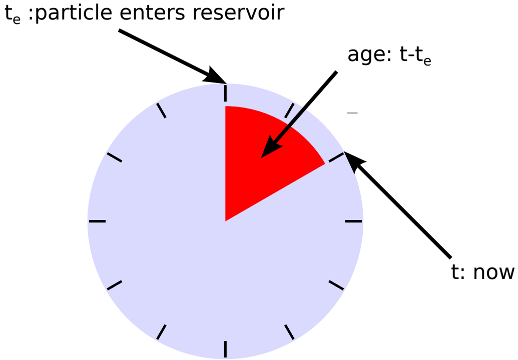
...

Possible Descriptive Statistics

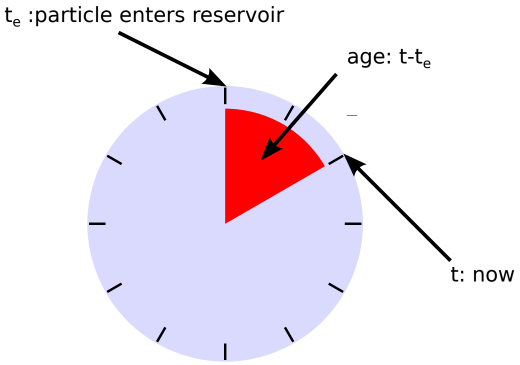


- number / mass of particles in pool A, B, ...
- average time spent in a pool A ,B...
- average time spent in the whole system
- average time of particles spent between pool C and E under the assumption of having entered by pool D (weird but possible...)
- deathrate of pool A.

Age of a Particle

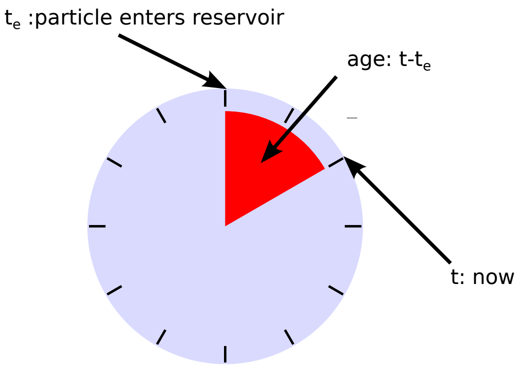


Age of a Particle



- The "age " is always defined in *context* of the reservoir

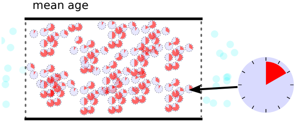
Age of a Particle



- The “age ” can not be negative!

Mean Age

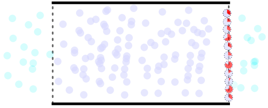
- Which set of particles to use for the average?
  proposition: *all* particles that are in the reservoir at the given time.
  $\rightarrow$ usually depends on input rates as well as the dynamics of the system.



$$\bar{a}(t) = \frac{a_1 + a_2 + \cdots + a_N}{N}$$

With $N = N(t)$ the number of all particles in the reservoir at time t.

Mean Transit Time



$$\bar{t}_r(t) = \frac{a_1 + a_2 + \cdots + a_{n_o}}{n_o}$$

With $n_o = n_o(t)$ the number of particles just leaving at time t

- Can be time dependent as well

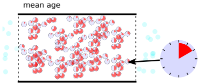
Mean Transit Time

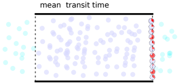
$$\bar{t}_r(t) = \frac{a_1 + a_2 + \cdots + a_{n_o}}{n_o}$$

With $n_o = n_o(t)$ the number of particles just leaving at time t

- Can be time dependent as well
- Includes only the subset of particles that are just leaving at the given time. (Can only be computed when there is an output stream)

Differences between mean age and mean transit time





- Includes all particles that are in the reservoir at the given time.
- Directly coupled to input rates

- Includes only the subset of particles that are just leaving at the given time.
- Indirectly coupled to inputs

Iteration over all particles

1. To compute the mean transit time we have to identify the particles just leaving.
2. Ask every leaving particle when it entered and compute its age.
3. Iterate over all particles and compute the average of their ages.

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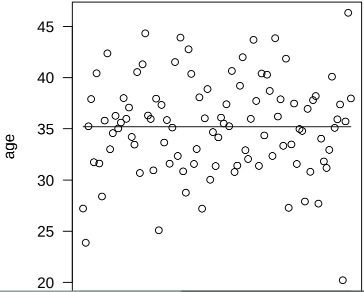
$$\bar{t}_r(t) = \frac{a_1 + a_2 + \cdots + a_{n_o}}{n_o}$$

With $n_o = n_o(t)$ the number of particles just leaving at time t



Iteration over all ages

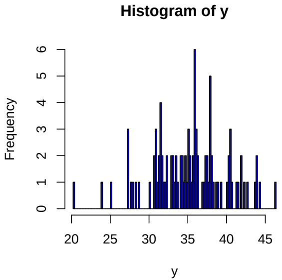
1. as above
2. as above + make a histogram of all ages
3. iterate over all ages and compute their weighted average

Iteration over all ages

1. as above
2. as above + make a histogram of all ages
3. iterate over all ages and compute their weighted average

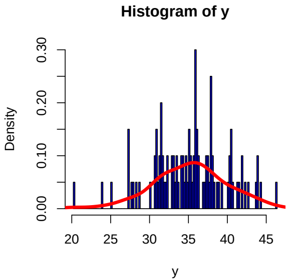
$$\bar{t}_r(t) = \frac{a_1 n_{a_1} + a_2 n_{a_2} + \cdots + a_n n_{a_n}}{n_o}$$

With $n_o = n_o(t) = n_{a_1} + n_{a_2} + \cdots + n_{a_n}$



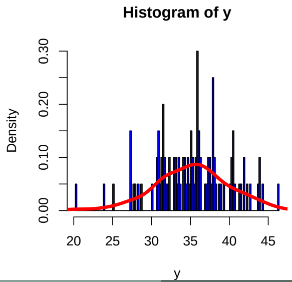
Integration over a density

$$
\begin{aligned}
\bar{t}_r(t) &= \lim_{n\to\infty} \frac{a_1 n_{a_1} + a_2 n_{a_2} + \cdots + a_n n_{a_n}}{n_o} \\
&= \lim_{n\to\infty} \sum_{minage}^{maxage} a \frac{n(a)}{n_o} da \\
&= \int_{minage}^{maxage} a\psi(a) da
\end{aligned}
$$

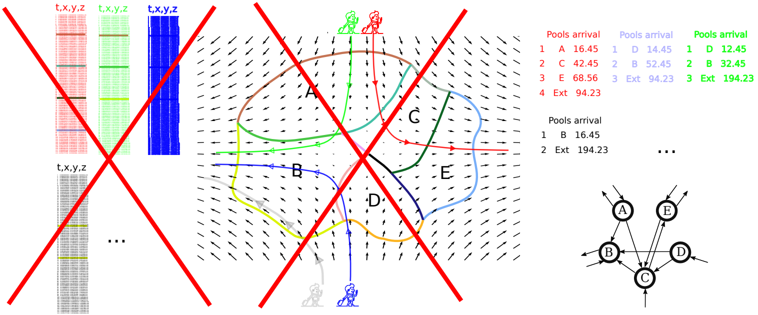


Same procedure for age density

$$\begin{aligned}
\bar{a}(t) &= \lim_{n\to\infty} \frac{a_1 n_{a_1} + a_2 n_{a_2} + \cdots + a_n n_{a_n}}{n_p} \\
&= \lim_{n\to\infty} \sum_{minage}^{maxage} a \frac{n(a)}{n_p} da \\
&= \int_{minage}^{maxage} a\phi(a) da
\end{aligned}$$



Possible Predictive Statistics ?

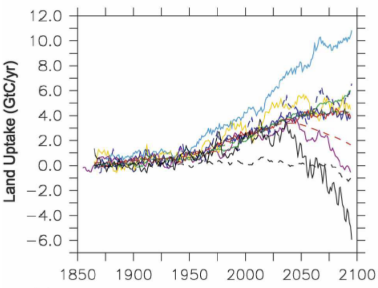


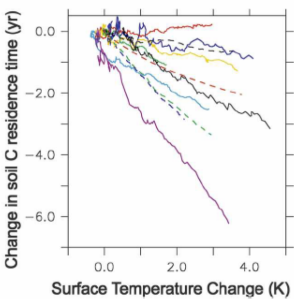
- Could we make a rule to predict the number of particles exiting from Pool E at time using only the particle passports? (Assuming that the exit from E is not recorded.)
- Could we make a rule to predict the age distribution of particles exiting from Pool E at time using only the particle passports? (Assuming again that the exit from E is not recorded.)

Intermediate Summary

1. Pool descriptions condense complex information to a time series of pool changes.(A series of stamps in the passport)
2. There are many possible statistics on sets of these time series, usually related to numbers of particles and times. (e.g. the number of particles in a pool,
3. Pool Models predict = model some of these exclusively with respect to the information obtainable from (all) passports.
4. The most common even disregard most of the information in the passports.

Diverse model predictions





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Model comparison

Key quantities

Model comparison

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- **transit time**
  - **the time that particles need to travel through the system**
  - **exit time - entry time**

Model comparison

Key quantities

- **transit time**
  - **the time that particles need to travel through the system**
  - **exit time - entry time**
- **system age**
  - **for particles in the system**
  - **current time - entry time**

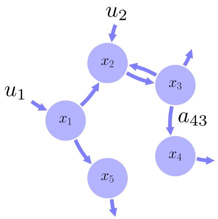
Model comparison

Key quantities

- **transit time**
  - **the time that particles need to travel through the system**
  - **exit time - entry time**
- **system age**
  - **for particles in the system**
  - **current time - entry time**
- **compartment age**
  - **system age of particles in a compartment**

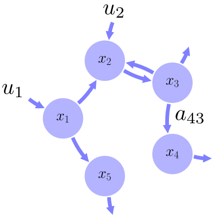
Linear autonomous compartmental models

$$\frac{d}{dt}\,\mathbf{x}(t) = \mathrm{A}\,\mathbf{x}(t) + \mathbf{u}$$



Linear autonomous compartmental models

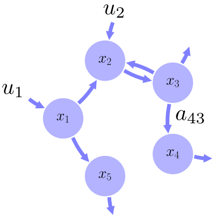
$$\frac{d}{dt}\mathbf{x}(t) = \mathrm{A}\,\mathbf{x}(t) + \mathbf{u}$$



$\mathbf{x}(t)$ vector of compartment content (e.g. C) at time t

Linear autonomous compartmental models

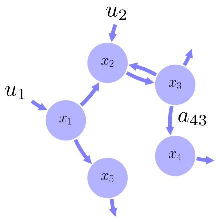
$$\frac{d}{dt}\,\mathbf{x}(t) = \mathrm{A}\,\mathbf{x}(t) + \mathbf{u}$$



$\mathbf{x}(t)$	vector of compartment content (e.g. C) at time t
$\mathbf{u}$	constant input vector

Linear autonomous compartmental models

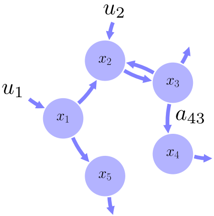
$$\frac{d}{dt}\,\mathbf{x}(t) = \mathrm{A}\,\mathbf{x}(t) + \mathbf{u}$$



$\mathbf{x}(t)$	vector of compartment content (e.g. C) at time t
$\mathbf{u}$	constant input vector
$\mathrm{A} = (a_{ij})$	compartmental matrix

Linear autonomous compartmental models

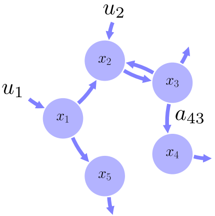
$$\frac{d}{dt}\,\mathbf{x}(t) = \mathrm{A}\,\mathbf{x}(t) + \mathbf{u}$$



$\mathbf{x}(t)$	vector of compartment content (e.g. C) at time t
$\mathbf{u}$	constant input vector
$\mathrm{A} = (a_{ij})$	compartmental matrix
$a_{ij}\ (i \neq j)$	fractional transfer coefficients,
	rate of flow from compartment j to compartment i

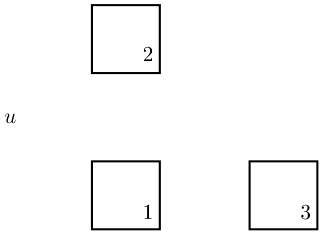
Linear autonomous compartmental models

$$\frac{d}{dt}\mathbf{x}(t) = \mathrm{A}\,\mathbf{x}(t) + \mathbf{u}$$



$\mathbf{x}(t)$	vector of compartment content (e.g. C) at time t
$\mathbf{u}$	constant input vector
$\mathrm{A} = (a_{ij})$	compartmental matrix
$a_{ij}\ (i \neq j)$	fractional transfer coefficients,
	rate of flow from compartment j to compartment i
$-a_{ii} > 0$	rate of flow out of compartment i

A particle travels $\frac{d}{dt}\,\mathbf{x}(t) = \mathrm{A}\,\mathbf{x}(t) + \mathbf{u}$



$$\begin{array}{cc} & \mathrm{A} \qquad\qquad\qquad\qquad \mathbf{u} \\ & \begin{pmatrix} -a_{11} & a_{12} & 0 \\ a_{21} & -a_{22} & 0 \\ a_{31} & 0 & -a_{33} \end{pmatrix} \quad \begin{pmatrix} u_1 \\ u_2 \\ 0 \end{pmatrix} \\ -\sum & \quad 0 \qquad > 0 \qquad > 0 \qquad\quad \|\mathbf{u}\| \end{array}$$

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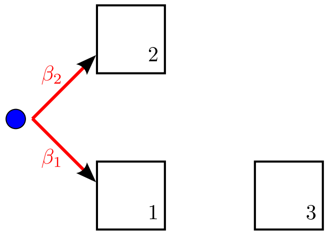
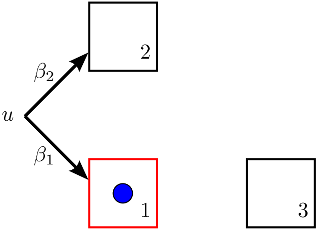


$$\begin{array}{ccc} & \mathrm{A} & \mathbf{u} \\ & \begin{pmatrix} -a_{11} & a_{12} & 0 \\ a_{21} & -a_{22} & 0 \\ a_{31} & 0 & -a_{33} \end{pmatrix} & \begin{pmatrix} u_1 \\ u_2 \\ 0 \end{pmatrix} \\ -\sum & \begin{matrix} 0 & \quad > 0 & \quad > 0 \end{matrix} & \|\mathbf{u}\| \end{array}$$

$$\beta_1 = \frac{u_1}{u_1+u_2} \qquad \beta_2 = \frac{u_2}{u_1+u_2}$$

$T_0 = 0$

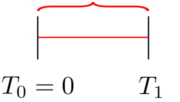
A particle travels $\frac{d}{dt}\mathbf{x}(t) = \mathrm{A}\,\mathbf{x}(t) + \mathbf{u}$



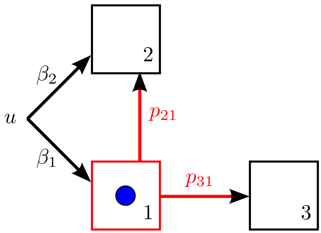
$$
\begin{array}{cc}
\mathrm{A} & \mathbf{u} \\
\begin{pmatrix} -a_{11} & a_{12} & 0 \\ a_{21} & -a_{22} & 0 \\ a_{31} & 0 & -a_{33} \end{pmatrix} & \begin{pmatrix} u_1 \\ u_2 \\ 0 \end{pmatrix}
\end{array}
$$

$$-\sum \quad 0 \quad > 0 \quad > 0 \qquad \|\mathbf{u}\|$$

$T_1 \sim \mathrm{Exp}(a_{11})$



 $\frac{d}{dt}\mathbf{x}(t) = \mathrm{A}\,\mathbf{x}(t) + \mathbf{u}$

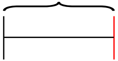


$$\begin{array}{cc} \mathrm{A} & \mathbf{u} \\ \begin{pmatrix} -a_{11} & a_{12} & 0 \\ \color{red}{a_{21}} & -a_{22} & 0 \\ \color{red}{a_{31}} & 0 & -a_{33} \end{pmatrix} & \begin{pmatrix} u_1 \\ u_2 \\ 0 \end{pmatrix} \end{array}$$

$$-\sum \quad \color{red}{0} \quad > 0 \quad > 0 \qquad \|\mathbf{u}\|$$

$$\color{red}{p_{21} = \frac{a_{21}}{a_{11}} \qquad p_{31} = \frac{a_{31}}{a_{11}}}$$

$T_1 \sim \mathrm{Exp}(a_{11})$



$T_0 = 0$ $\quad$ $\color{red}{T_1}$

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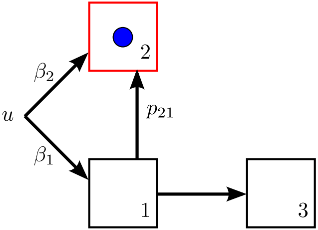
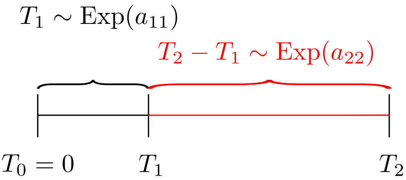


$$
\begin{array}{cc}
\mathrm{A} & \mathbf{u} \\
\begin{pmatrix} -a_{11} & a_{12} & 0 \\ a_{21} & \color{red}{-a_{22}} & 0 \\ a_{31} & 0 & -a_{33} \end{pmatrix} & \begin{pmatrix} u_1 \\ u_2 \\ 0 \end{pmatrix}
\end{array}
$$

$$
-\sum \quad 0 \quad > 0 \quad > 0 \qquad \|\mathbf{u}\|
$$



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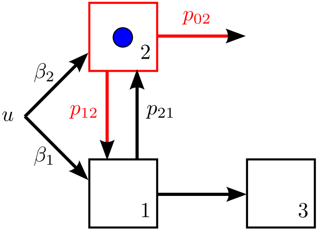
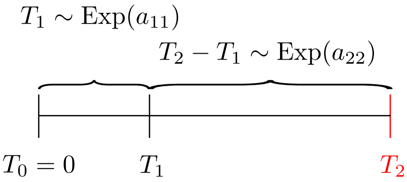
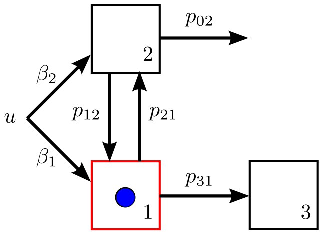


$$\begin{array}{ccc} & \mathrm{A} & \mathbf{u} \\ & \begin{pmatrix} -a_{11} & a_{12} & 0 \\ a_{21} & -a_{22} & 0 \\ a_{31} & 0 & -a_{33} \end{pmatrix} & \begin{pmatrix} u_1 \\ u_2 \\ 0 \end{pmatrix} \\ -\sum & 0 \qquad > 0 \qquad > 0 & \|\mathbf{u}\| \end{array}$$

$$p_{12} = \frac{a_{12}}{a_{22}} \qquad p_{02} = 1 - \frac{a_{12}}{a_{22}}$$



A particle travels $\frac{d}{dt}\mathbf{x}(t) = \mathrm{A}\,\mathbf{x}(t) + \mathbf{u}$

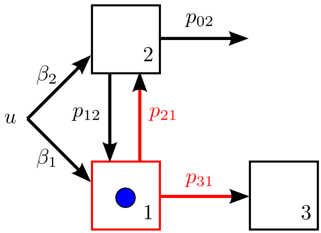


$$\begin{array}{cc}
 & \mathrm{A} \qquad\qquad\qquad\qquad\qquad \mathbf{u} \\
 & \begin{pmatrix} \color{red}{-a_{11}} & a_{12} & 0 \\ a_{21} & -a_{22} & 0 \\ a_{31} & 0 & -a_{33} \end{pmatrix} \quad \begin{pmatrix} u_1 \\ u_2 \\ 0 \end{pmatrix} \\
-\sum & \quad 0 \qquad > 0 \qquad > 0 \qquad\qquad \|\mathbf{u}\|
\end{array}$$

$T_1 \sim \mathrm{Exp}(a_{11})$ $\color{red}{T_3 - T_2 \sim \mathrm{Exp}(a_{11})}$

$T_2 - T_1 \sim \mathrm{Exp}(a_{22})$

A particle travels $\frac{d}{dt}\,\mathbf{x}(t) = \mathrm{A}\,\mathbf{x}(t) + \mathbf{u}$



$$
\begin{array}{cc}
\mathrm{A} & \mathbf{u} \\
\begin{pmatrix} -a_{11} & a_{12} & 0 \\ a_{21} & -a_{22} & 0 \\ a_{31} & 0 & -a_{33} \end{pmatrix} & \begin{pmatrix} u_1 \\ u_2 \\ 0 \end{pmatrix}
\end{array}
$$

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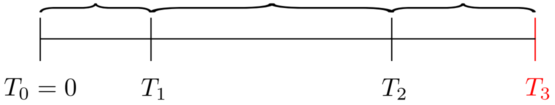
$$p_{21} = \frac{a_{21}}{a_{11}} \qquad p_{30} = \frac{a_{31}}{a_{11}}$$

$T_1 \sim \mathrm{Exp}(a_{11})$

$T_2 - T_1 \sim \mathrm{Exp}(a_{22})$

$T_3 - T_2 \sim \mathrm{Exp}(a_{11})$



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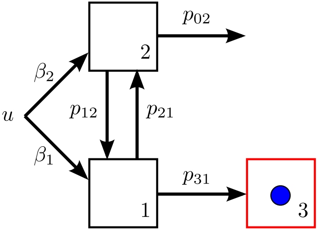
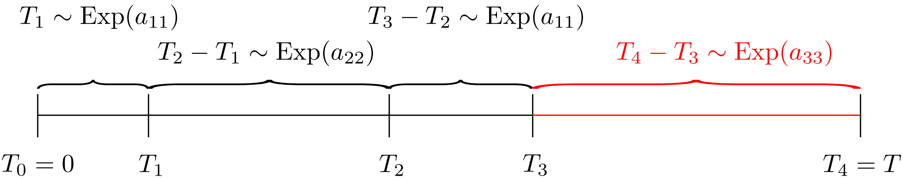


$$\begin{array}{ccc} & \mathrm{A} & \mathbf{u} \\ & \begin{pmatrix} -a_{11} & a_{12} & 0 \\ a_{21} & -a_{22} & 0 \\ a_{31} & 0 & {\color{red}-a_{33}} \end{pmatrix} & \begin{pmatrix} u_1 \\ u_2 \\ 0 \end{pmatrix} \\ -\sum & \quad 0 \qquad > 0 \qquad > 0 & \|\mathbf{u}\| \end{array}$$



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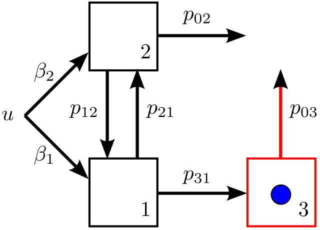
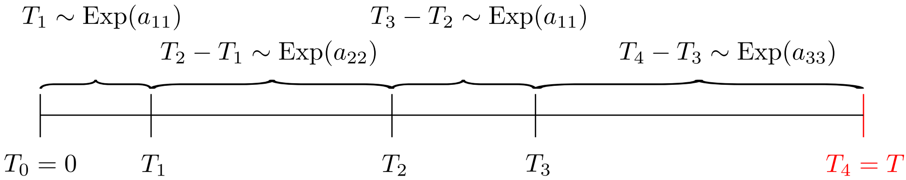


$$
\begin{array}{c|ccc|c}
 & & \mathrm{A} & & \mathbf{u} \\
 & \begin{pmatrix} -a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} & \!\!\!\begin{matrix} a_{12} \\ -a_{22} \\ 0 \end{matrix} & \!\!\!\begin{pmatrix} \color{red}{0} \\ \color{red}{0} \\ -a_{33} \end{pmatrix} & \begin{pmatrix} u_1 \\ u_2 \\ 0 \end{pmatrix} \\
-\sum & 0 & > 0 & \color{red}{> 0} & \|\mathbf{u}\|
\end{array}
$$

$\color{red}{p_{03} = 1}$



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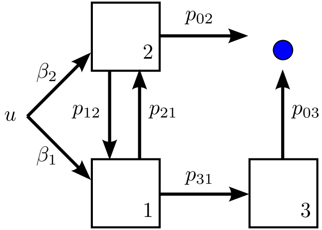
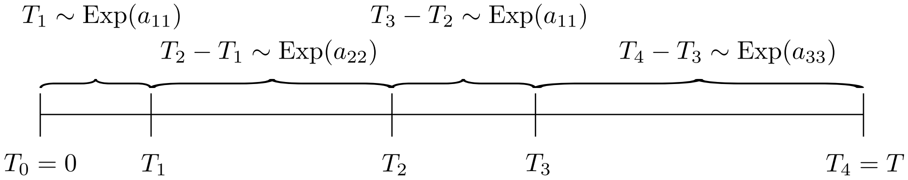


$$
\begin{array}{rcc}
 & \mathrm{A} & \mathbf{u} \\
 & \begin{pmatrix} -a_{11} & a_{12} & 0 \\ a_{21} & -a_{22} & 0 \\ a_{31} & 0 & -a_{33} \end{pmatrix} & \begin{pmatrix} u_1 \\ u_2 \\ 0 \end{pmatrix} \\
-\sum & \begin{matrix} 0 & \quad > 0 & \quad > 0 \end{matrix} & \|\mathbf{u}\|
\end{array}
$$



Transit time distribution

- transit time computation:

$$T = \underbrace{(T_1 - T_0)}_{\sim \mathrm{Exp}(a_{11})} + \underbrace{(T_2 - T_1)}_{\sim \mathrm{Exp}(a_{22})} + \underbrace{(T_3 - T_2)}_{\sim \mathrm{Exp}(a_{11})} + \underbrace{(T_4 - T_3)}_{\sim \mathrm{Exp}(a_{33})}$$

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- mixture of exponential distributions

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- mixture of exponential distributions
- length of mixture is stochastic

Transit time distribution

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$$T = \underbrace{(T_1 - T_0)}_{\sim \mathrm{Exp}(a_{11})} + \underbrace{(T_2 - T_1)}_{\sim \mathrm{Exp}(a_{22})} + \underbrace{(T_3 - T_2)}_{\sim \mathrm{Exp}(a_{11})} + \underbrace{(T_4 - T_3)}_{\sim \mathrm{Exp}(a_{33})}$$

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$\rightarrow$ T follows phase-type distribution with parameters β, A

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$\beta = \frac{\mathbf{u}}{\|\mathbf{u}\|}$

Transit time distribution

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- mixture of exponential distributions
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$\rightarrow$ T follows phase-type distribution with parameters β, A

$$\beta = \frac{\mathbf{u}}{\|\mathbf{u}\|}$$

$\rightarrow$ $T \sim \mathrm{PH}(\beta, \mathrm{A})$

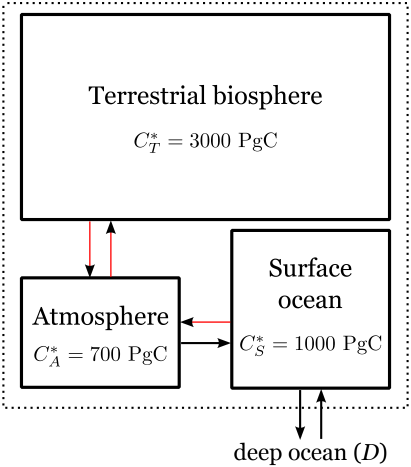
General, simple, explicit formulas

- phase-type distribution is well known:
  - probability density
  - cumulative distribution function
  - quantiles
  - mean and higher order moments
    - $\mathbb{E}[T] = \|\mathrm{A}^{-1}\,\beta\| = \frac{\|\mathbf{x}^*\|}{\|\mathbf{u}\|}$ (mean transit time)

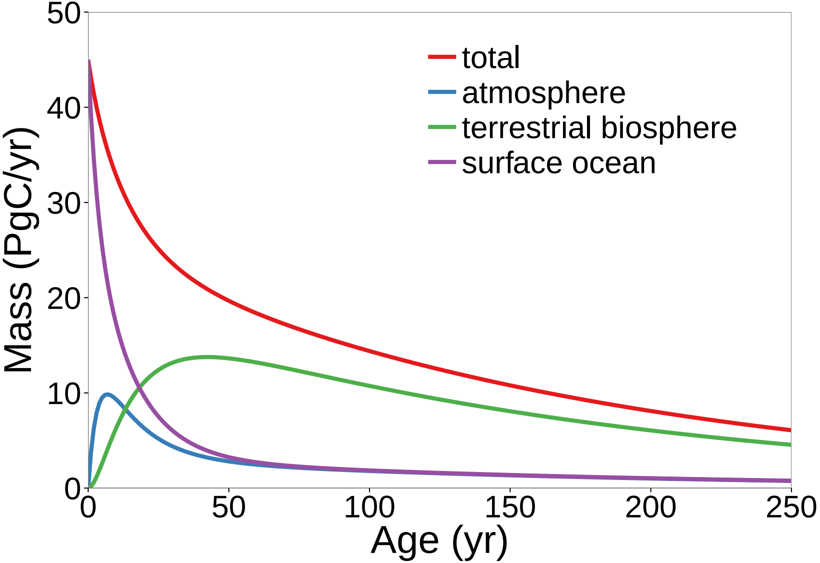
- system age is also phase-type distributed
  - parameters $\eta := \frac{\mathbf{x}^*}{\|\mathbf{x}^*\|}, \mathrm{A}$

- probability density of compartmental age
  - $\mathbf{f_a}(y) = (\mathrm{X}^*)^{-1}\, e^{y\,\mathrm{A}}\,\mathbf{u}$

Nonlinear model in steady state [?] with three compartments



Equilibrium age densities



Linear nonautonomous systems

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- **general solution**

$$\mathbf{x}(t) = \Phi(t,0)\,\mathbf{x}^0 + \int_0^t \Phi(t,s)\,\mathbf{u}(s)\,ds$$

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- **special case of no time dependence**

$$\mathbf{x}(t) = e^{t\,\mathrm{A}}\,\mathbf{x}^0 + \int_0^t e^{(t-s)\,\mathrm{A}}\,\mathbf{u}\,ds$$

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$\longrightarrow$ compartmental age probability density

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$$\Rightarrow \underbrace{\Phi(t,t-a)\,\mathbf{u}(t-a)}_{\text{mass with age } a \text{ at time } t}$$

The state transition operator Φ

- solution to the matrix ordinary differential equation

$$\frac{d}{dt}\,\Phi(t,s) = \mathrm{A}(t)\,\Phi(t,s), \quad \text{initial condition } \Phi(\mathrm{s},\mathrm{s}) = \mathrm{I}$$

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- time-independent system

$$\Phi(t,s) = e^{(t-s)\,\mathrm{A}} \quad \text{(matrix exponential)}$$

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$$\Phi(t,s) = e^{-\lambda\,(t-s)} \quad \text{(exponential function)}$$

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$\rightarrow$ describes the time evolution of mass in the system

$\rightarrow$ we can always obtain it (at least numerically)

Splitting the system

- we split the system into two parts

$$\text{mass in the system at time } t \text{ with age } a = \left\{ \begin{array}{l} \\ \\ \end{array} \right.$$

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$$\begin{array}{l}\text{mass in the system}\\ \text{at time } t \text{ with age} =\\ a\end{array} \begin{cases} \qquad\qquad , & a < t \\ \qquad\qquad , & a \geq t \end{cases}$$

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$$\begin{array}{l}\text{mass in the system}\\ \text{at time } t \text{ with age} = \\ a\end{array} \begin{cases} \begin{array}{l}\text{remainings from input}\\ \text{at time } t-a\end{array}, & a < t \\ \\ , & a \geq t \end{cases}$$

Splitting the system

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$$= \begin{cases} \Phi(t, t-a)\,\mathbf{u}(t-a), & a < t, \\ \\ \end{cases}$$

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- $\mathbf{p}^0$ is a given age distribution of the initial content vector $\mathbf{x}^0$

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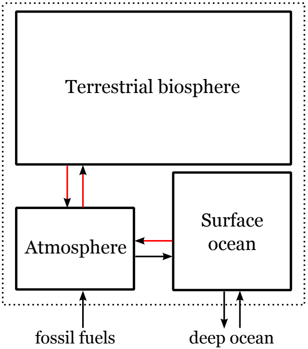
A global carbon cycle model



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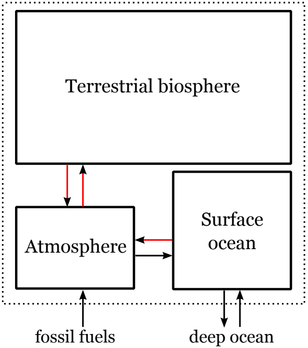
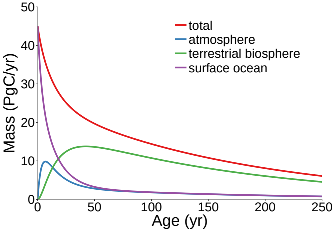


Equilibrium age densities in 1765



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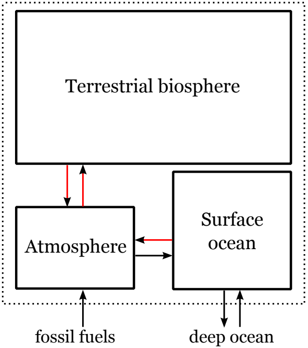

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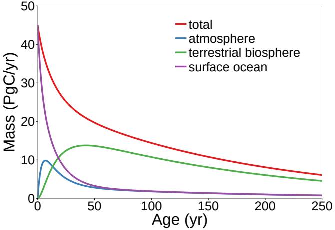
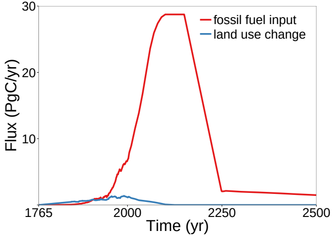
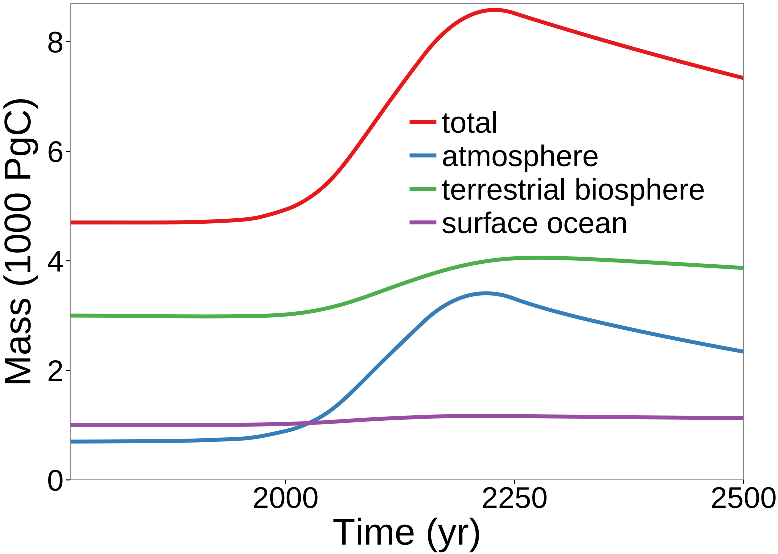


RCP8.5 scenario



Time-dependent carbon contents



Questions of high scientific and societal interest

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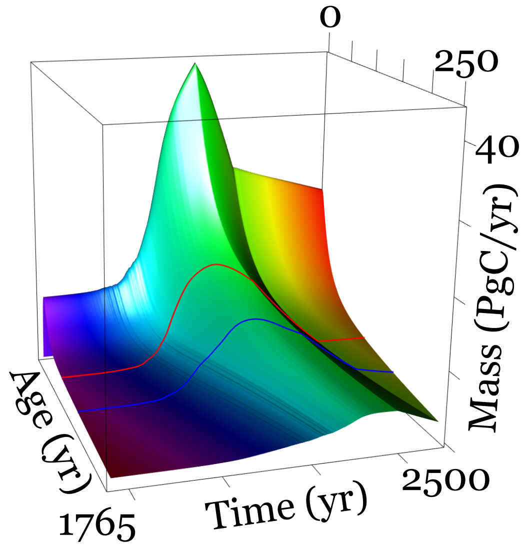

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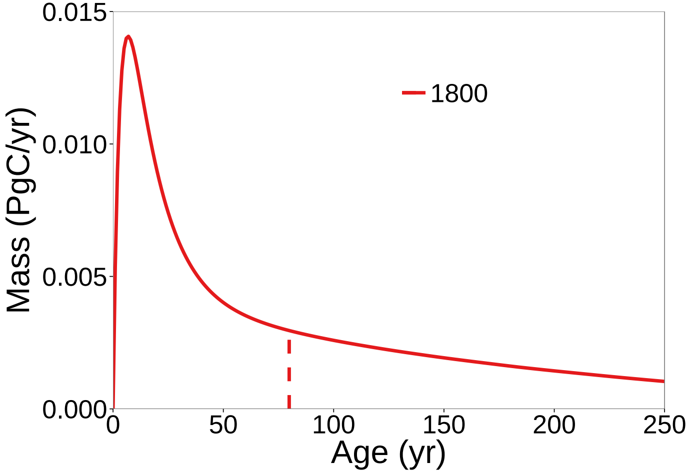

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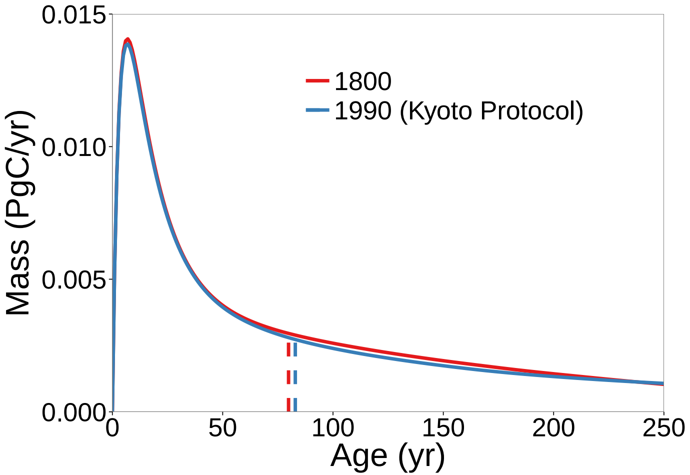

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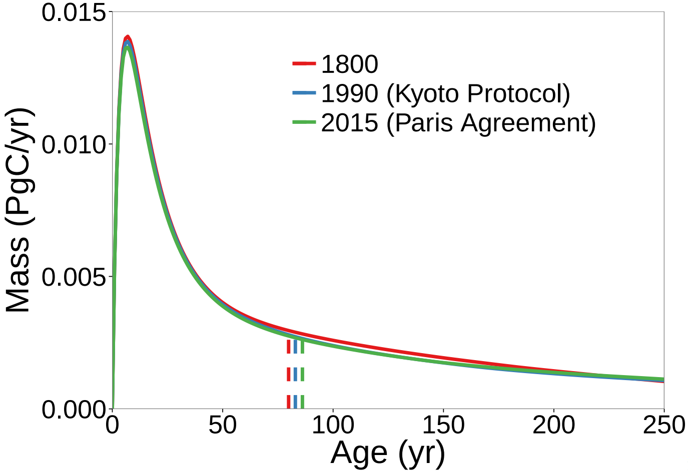

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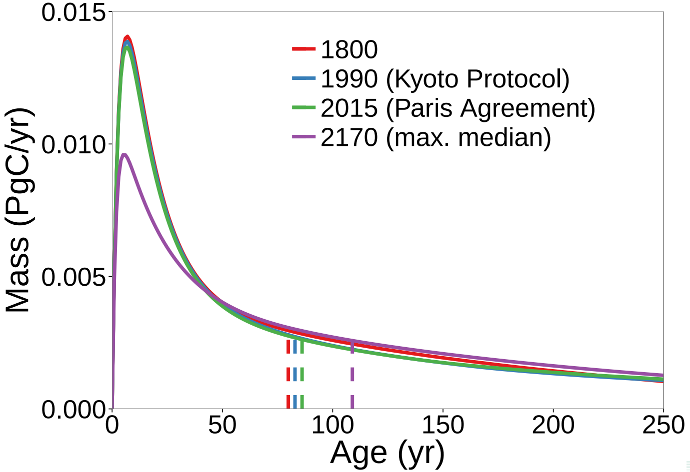

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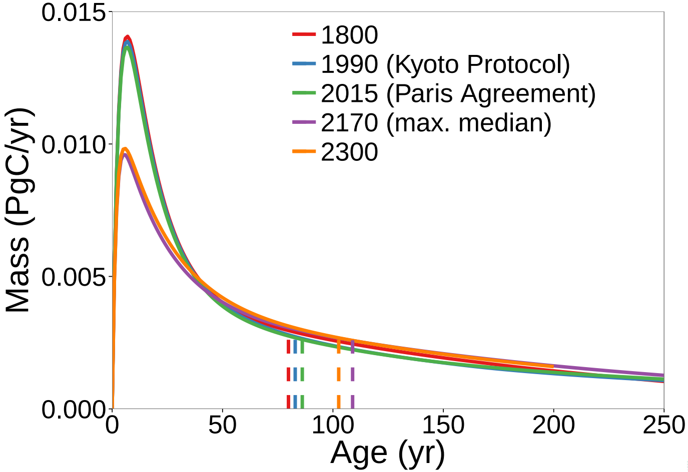

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- **two Python packages**
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    - `http://github.com/goujou/LAPM`
  - (numerical) computation of age and transit time properties for nonlinear nonautonomous systems

System age

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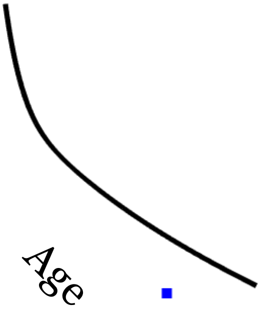

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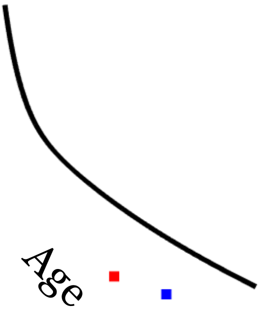

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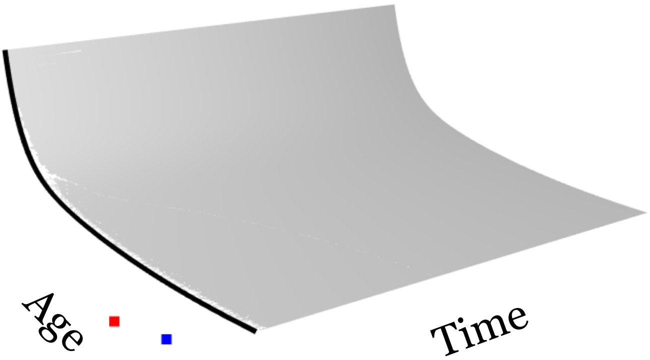

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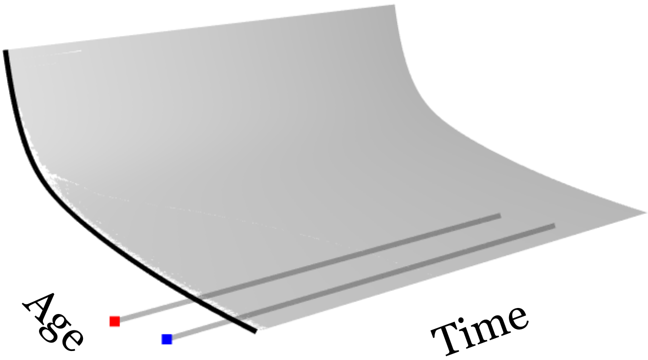

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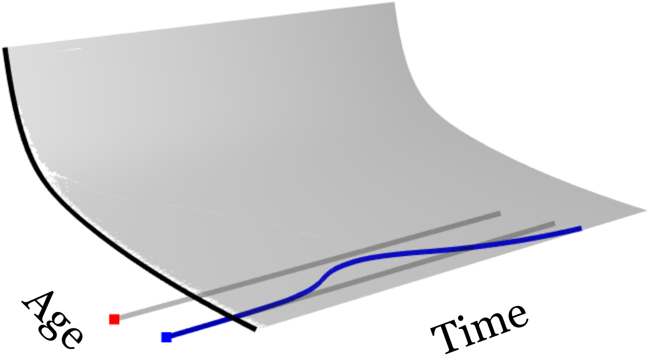

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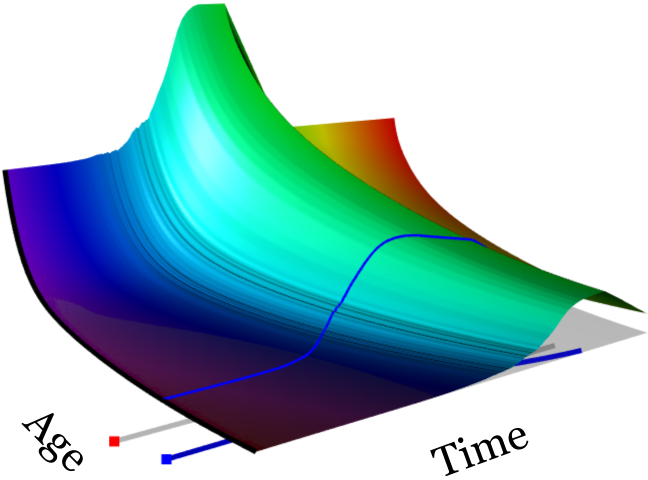

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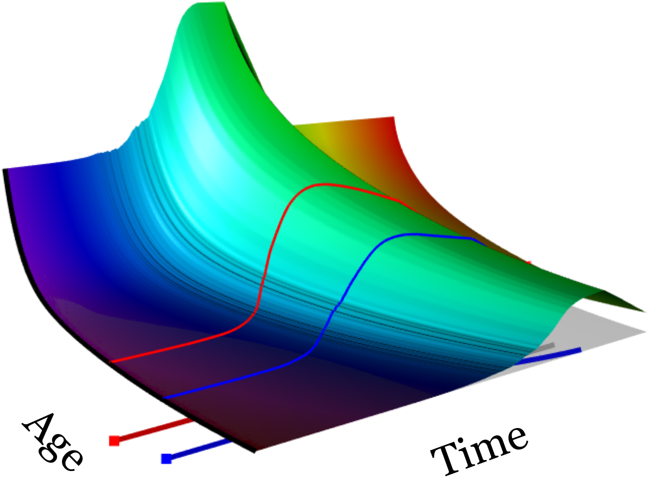

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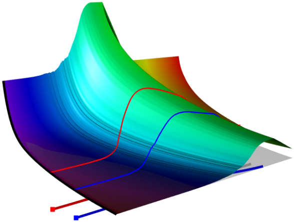

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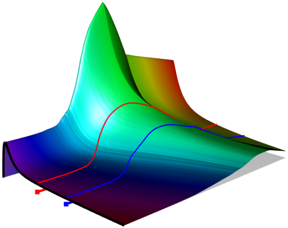
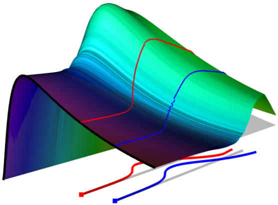
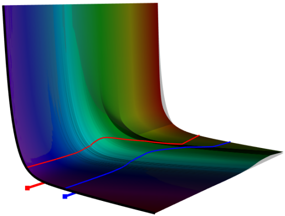


Terrestrial biosphere



Surface ocean



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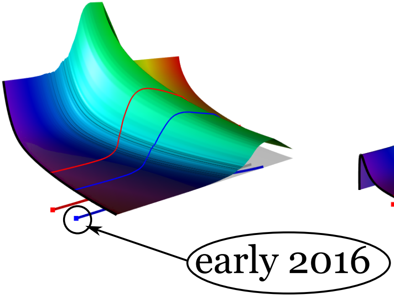

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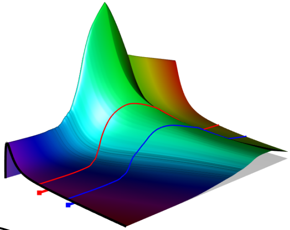

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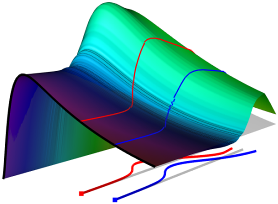

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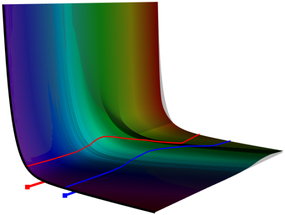


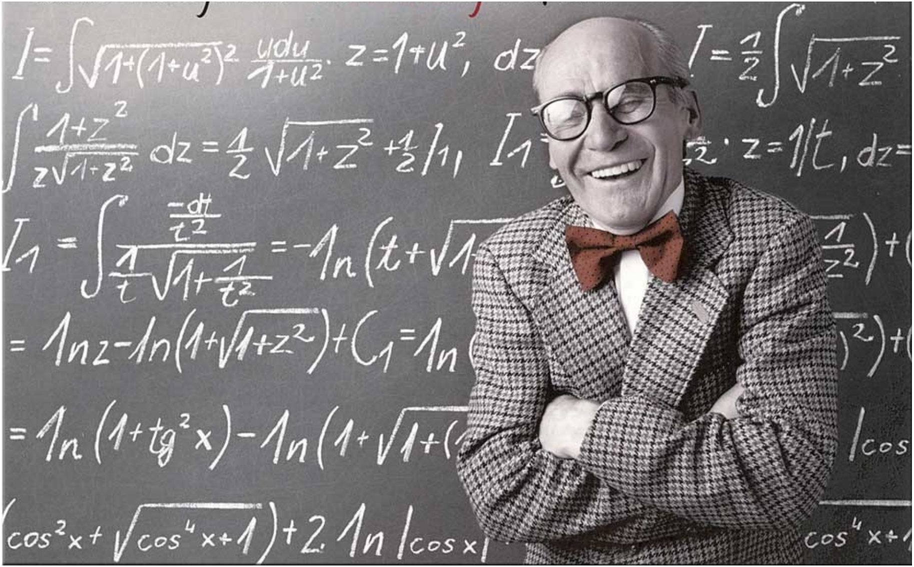
Thanks

Funding

Emmy Noether-Programm

DFG Deutsche Forschungsgemeinschaft

Thank you for your attention



bibliography

Henrik Hartmann, Henry D. Adams, William M. Hammond, Günter Hoch, Simon M. Landhäusser, Erin Wiley, and Sönke Zaehle. Identifying differences in carbohydrate dynamics of seedlings and mature trees to improve carbon allocation in models for trees and forests. *Environmental and Experimental Botany*, 152:7 – 18, 2018. doi: https://doi.org/10.1016/j.envexpbot.2018.03.011. URL `http://www.sciencedirect.com/science/article/pii/S0098847218303915`.