Pool Models

An Introduction

Markus Müller, Holger Metzler, Carlos Sierra

August 13, 2019

Max Planck Institute
for Biogeochemistry

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Outline

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1. Example applications of pool models
2. Reducing model complexity
   - The carbon cycle example
   - Asking simpler questions
   - Answer questions more simply

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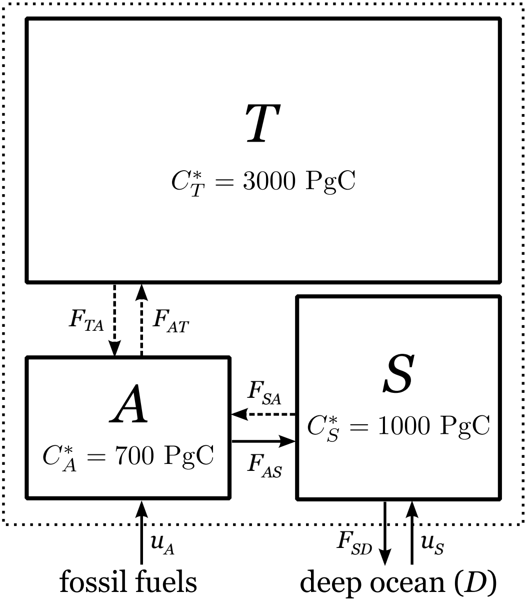
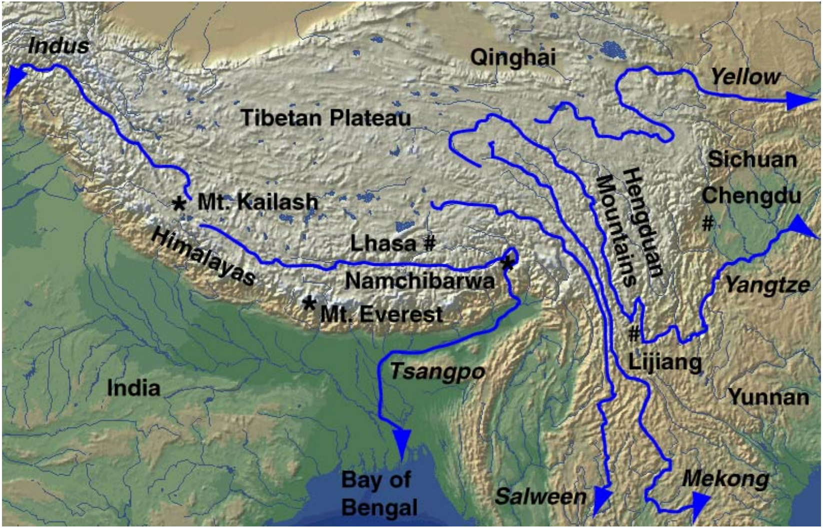
1. Example applications of pool models
2. Reducing model complexity
   - The carbon cycle example
   - Asking simpler questions
   - Answer questions more simply
3. Compartmental systems
   - Introduction
   - Linear autonomous
   - Generalization to nonlinear nonautonomous systems
   - Example application
4. Generalization to non Well Mixed Systems
   - Population Dynamics
   - Hydrology

Element cycling, e.g. C

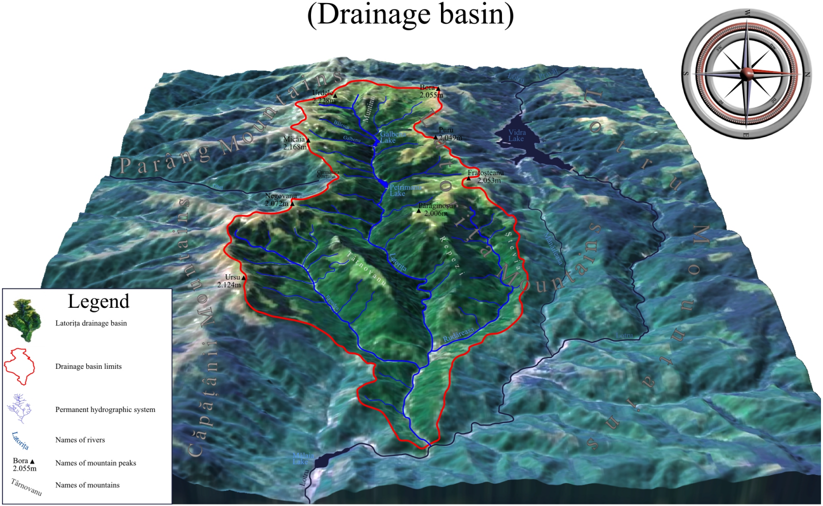


Hydrology, catchments

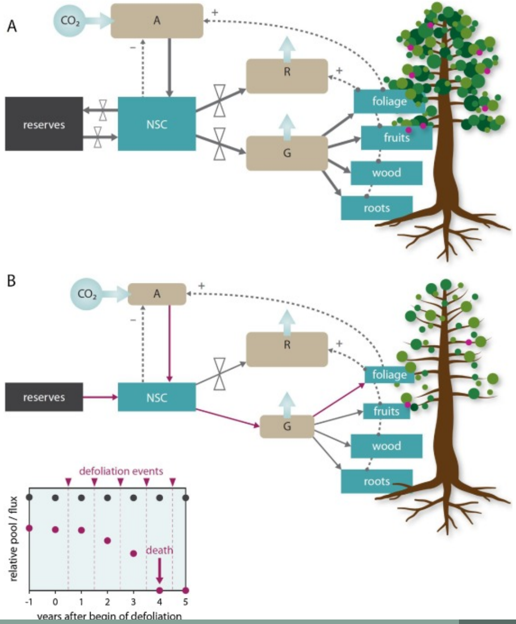


Latorița River, tributary of the Lotru River

(Drainage basin)

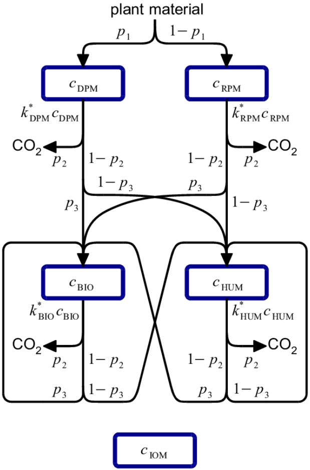


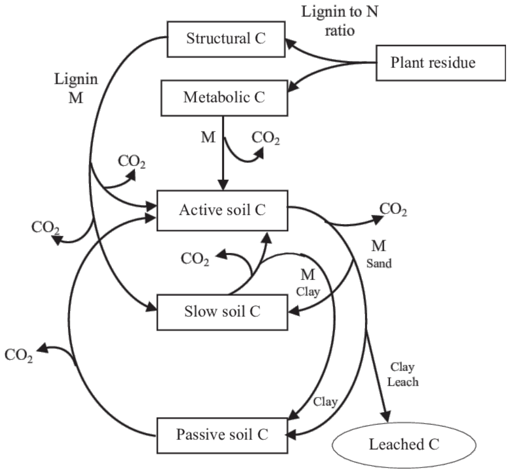
Plant physiology, carbon allocation



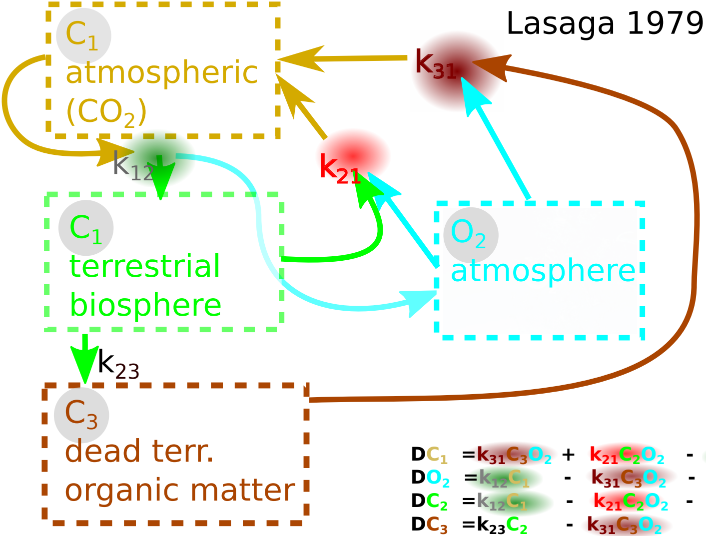
Hartmann et al. [2018]

Organic matter decomposition , soil models



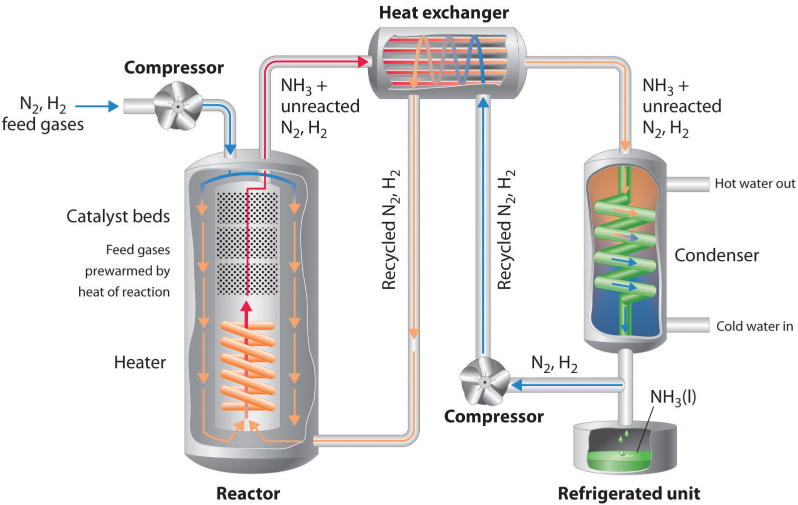


Lasaga 1979

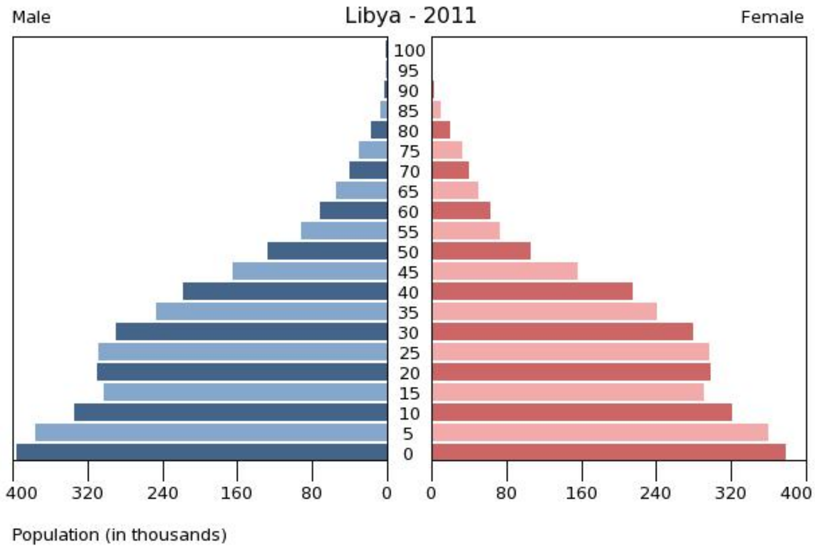


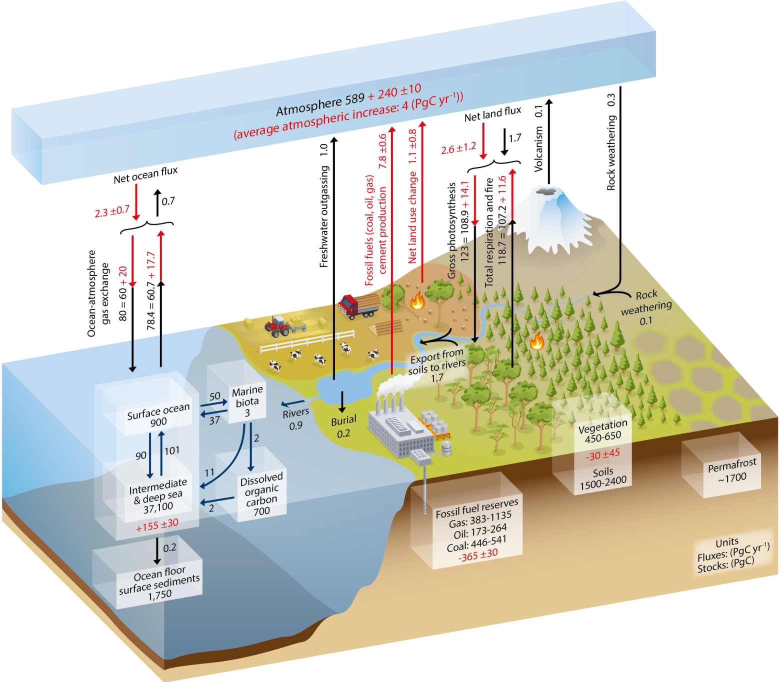
$$
\begin{aligned}
DC_1 &= k_{31}C_3O_2 + k_{21}C_2O_2 - k_{12}C_1\\
DO_2 &= k_{12}C_1 - k_{31}C_3O_2 - k_{21}C_2O_2\\
DC_2 &= k_{12}C_1 - k_{21}C_2O_2 - k_{23}C_2\\
DC_3 &= k_{23}C_2 - k_{31}C_3O_2
\end{aligned}
$$

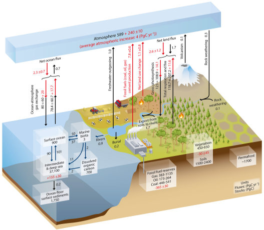
Chemical reactors

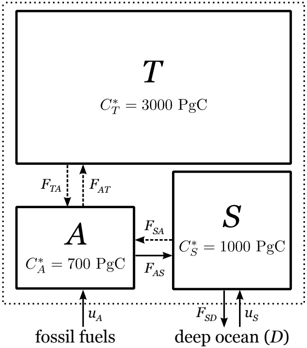


Population dynamics

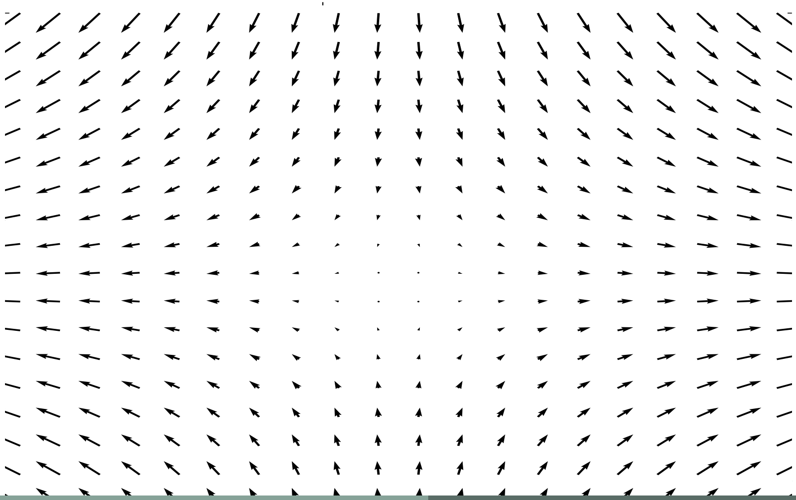


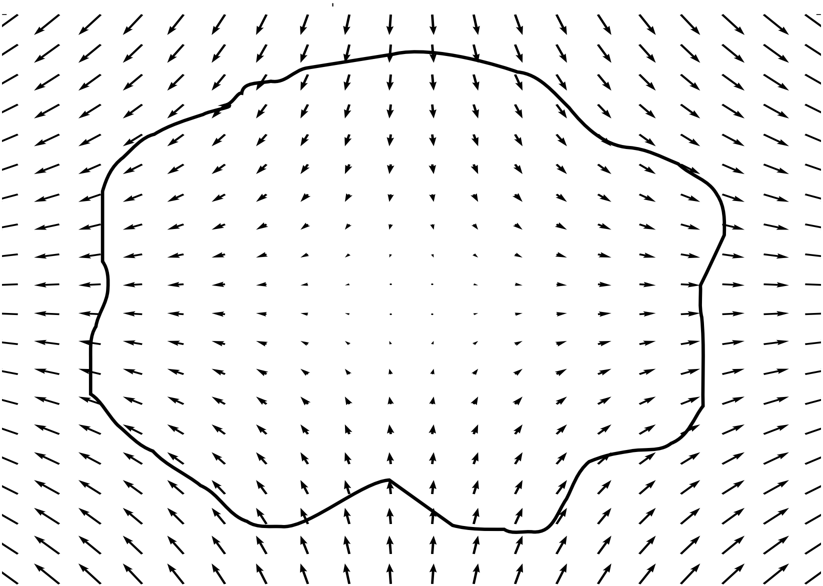

Atmosphere 589 + 240 ±10
(average atmospheric increase: 4 (PgC yr⁻¹))
Net ocean flux
2.3 ±0.7
0.7
Ocean-atmosphere gas exchange
80 = 60 + 20
78.4 = 60.7 + 17.7
Freshwater outgassing 1.0
Fossil fuels (coal, oil, gas) cement production 7.8 ±0.6
Net land use change 1.1 ±0.8
Net land flux
2.6 ±1.2
1.7
Gross photosynthesis 123 = 108.9 + 14.1
Total respiration and fire 118.7 = 107.2 + 11.6
Volcanism 0.1
Rock weathering 0.3
Rock weathering 0.1
Export from soils to rivers 1.7
Surface ocean 900
Marine biota 3
50
37
Rivers 0.9
Burial 0.2
90
101
2
11
Intermediate & deep sea 37,100
+155 ±30
Dissolved organic carbon 700
2
0.2
Ocean floor surface sediments 1,750
Vegetation 450-650
-30 ±45
Soils 1500-2400
Permafrost ~1700
Fossil fuel reserves
Gas: 383-1135
Oil: 173-264
Coal: 446-541
-365 ±30
Units
Fluxes: (PgC yr⁻¹)
Stocks: (PgC)

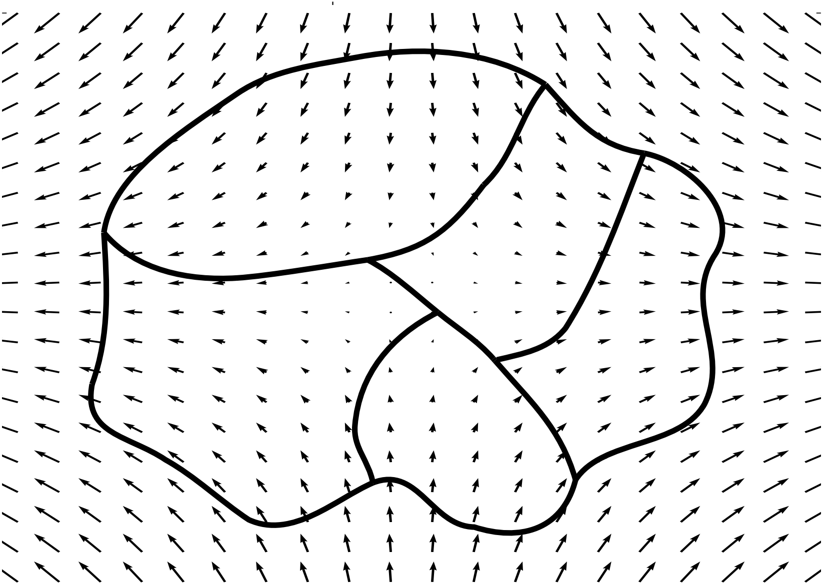


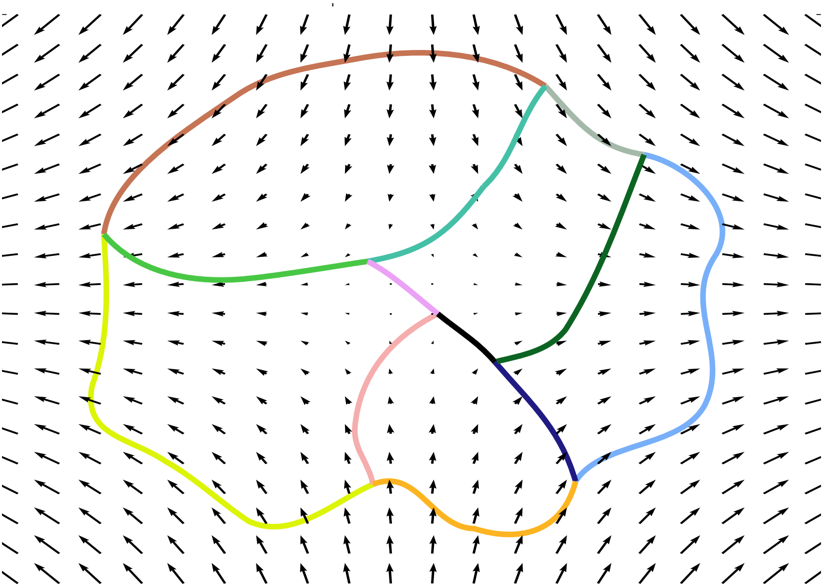


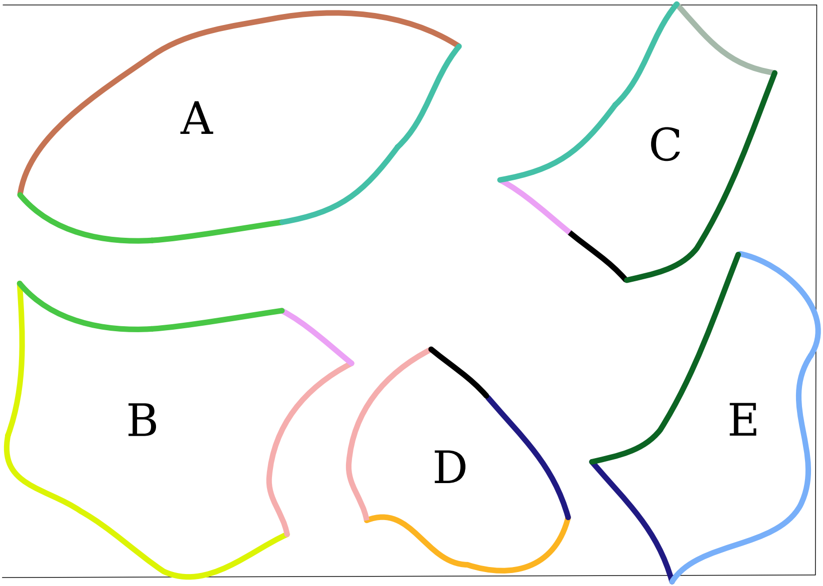
T
$C_T^* = 3000$ PgC
F_{TA}
F_{AT}
A
$C_A^* = 700$ PgC
F_{SA}
F_{AS}
S
$C_S^* = 1000$ PgC
u_A
fossil fuels
F_{SD}
u_S
deep ocean (D)

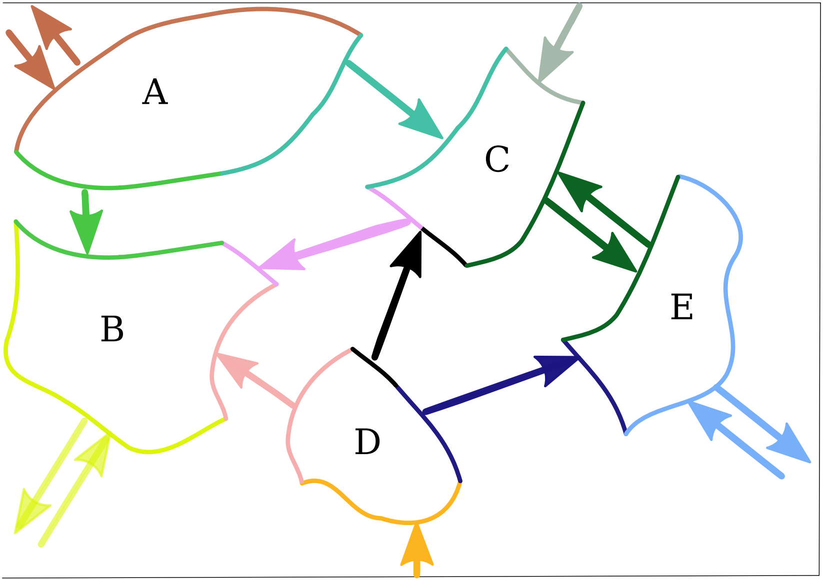


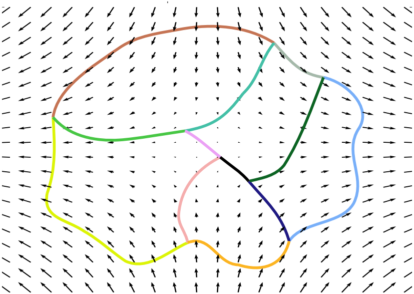


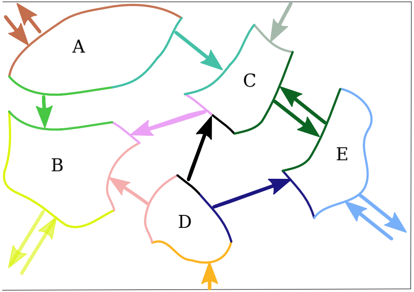


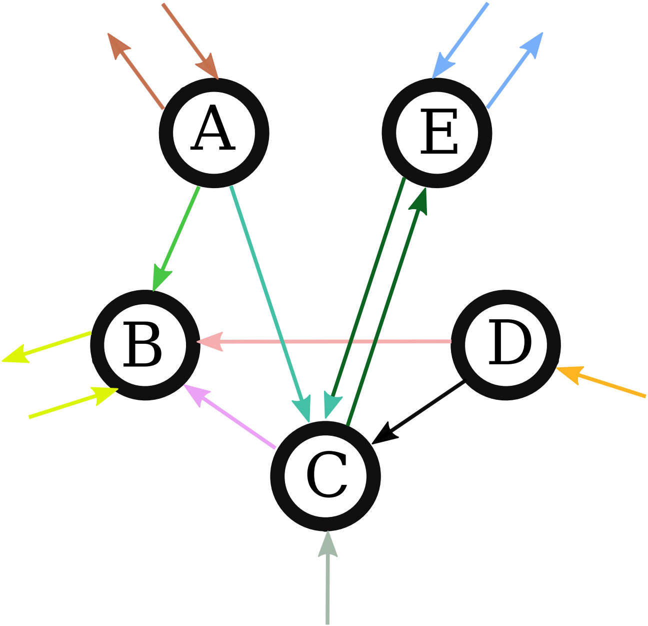


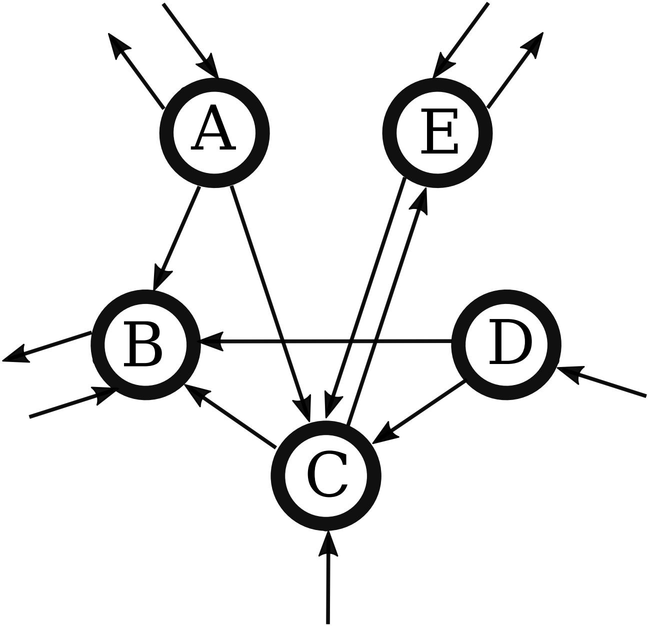

A
B
C
D
E

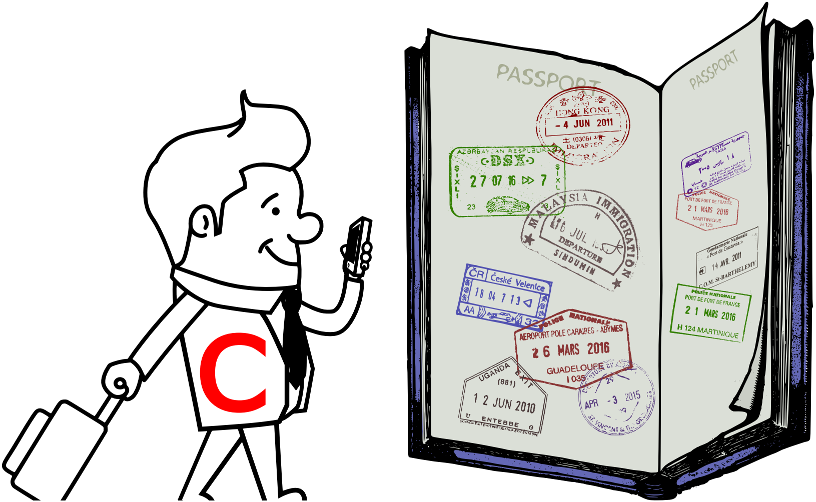

A
C
B
E
D

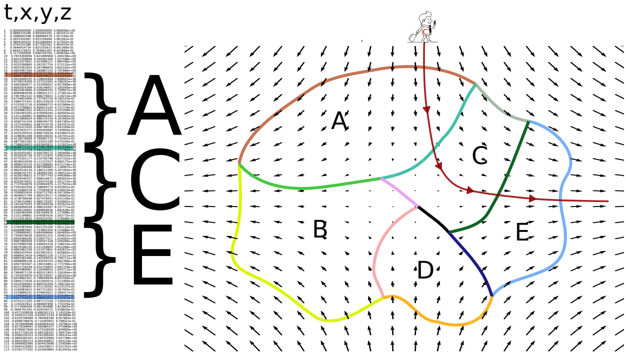



A
B
C
D
E


A
E
B
D
C


A
E
B
D
C

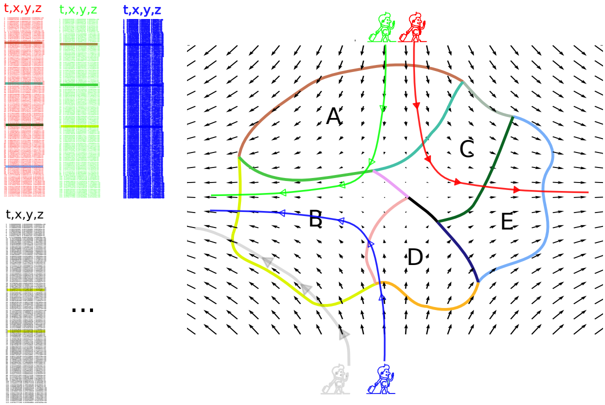

C
PASSPORT
PASSPORT
HONG KONG
- 4 JUN 2011
DEPARTED
DSX
27 07 16 7
MALAYSIA IMMIGRATION
DEPARTURE
SINDUMIN
ČR České Velenice
18 04 7 13
AA
POLICE NATIONALE
AEROPORT POLE CARAIBES - ABYMES
26 MARS 2016
GUADELOUPE
UGANDA
(881)
12 JUN 2010
ENTEBBE
APR -3 2015
EGYPT
PORT DE FORT DE FRANCE
21 MARS 2016
MARTINIQUE
Gendarmerie Nationale
« Port de Gustavia »
14 AVR. 2011
C.O.M. St-BARTHELEMY
POLICE NATIONALE
PORT DE FORT DE FRANCE
21 MARS 2016
H 124 MARTINIQUE



Pools arrival

1	A	16.45
2	C	42.45
3	E	68.56
4	Ext	94.23

...



Pools arrival

1	A	16.45
2	C	42.45
3	E	68.56
4	Ext	94.23

Pools arrival

1	D	14.45
2	B	52.45
3	Ext	94.23

Pools arrival

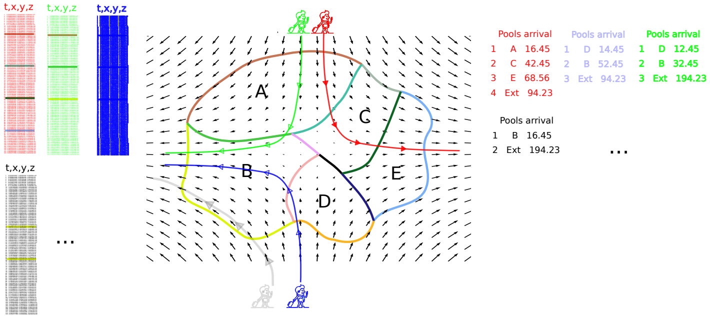
1	D	12.45
2	B	32.45
3	Ext	194.23

Pools arrival

1	B	16.45
2	Ext	194.23

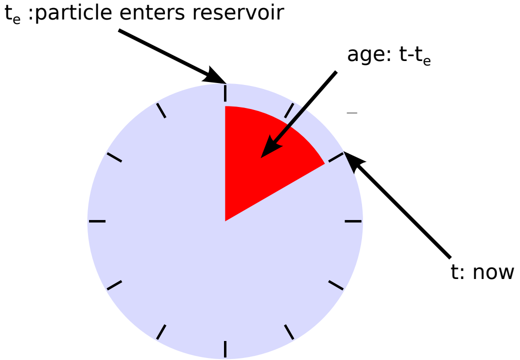
...

Possible descriptive statistics

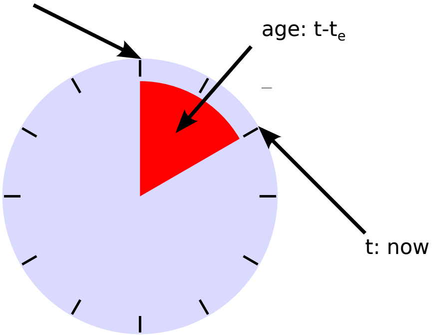


- number / mass of particles in pool A, B, ...
- average time spent in a pool A ,B...
- average time spent in the whole system
- average time of particles spent between pool C and E under the assumption of having entered by pool D (weird but possible...)
- deathrate of pool A.

Age of a Particle

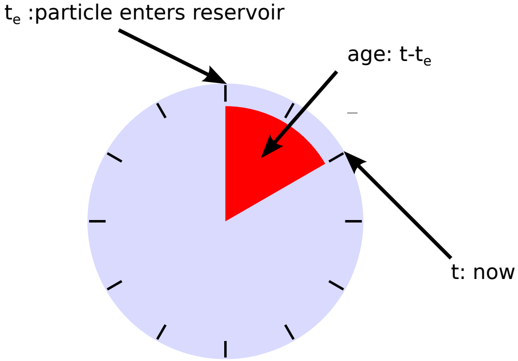


Age of a Particle



- The "age " is always defined in *context* of the reservoir

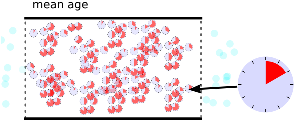
Age of a Particle



- The “age ” can not be negative!

Mean age

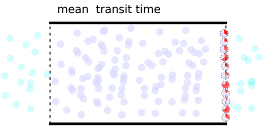
- Which set of particles to use for the average?
  proposition: *all* particles that are in the reservoir at the given time.
  $\rightarrow$ usually depends on input rates as well as the dynamics of the system.



$$\bar{a}(t) = \frac{a_1 + a_2 + \cdots + a_N}{N}$$

With $N = N(t)$ the number of all particles in the reservoir at time t.

Mean transit time



$$\bar{t}_r(t) = \frac{a_1 + a_2 + \cdots + a_{n_o}}{n_o}$$

With $n_o = n_o(t)$ the number of particles just leaving at time t

- Can be time dependent as well

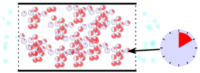
Mean transit time

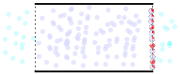
$$\bar{t}_r(t) = \frac{a_1 + a_2 + \cdots + a_{n_o}}{n_o}$$

With $n_o = n_o(t)$ the number of particles just leaving at time t

- Can be time dependent as well
- Includes only the subset of particles that are just leaving at the given time. (Can only be computed when there is an output stream)

Differences between mean age and mean transit time





- Includes all particles that are in the reservoir at the given time.
- Directly coupled to input rates

- Includes only the subset of particles that are just leaving at the given time.
- Indirectly coupled to inputs

Iteration over all particles

1. To compute the mean transit time we have to identify the particles just leaving.
2. Ask every leaving particle when it entered and compute its age.
3. Iterate over all particles and compute the average of their ages.

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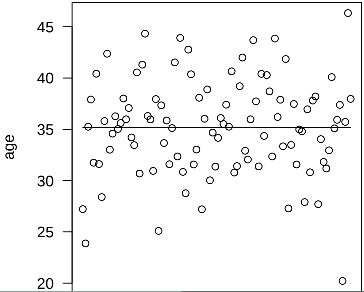
$$\bar{t}_r(t) = \frac{a_1 + a_2 + \cdots + a_{n_o}}{n_o}$$

With $n_o = n_o(t)$ the number of particles just leaving at time t



Iteration over all ages

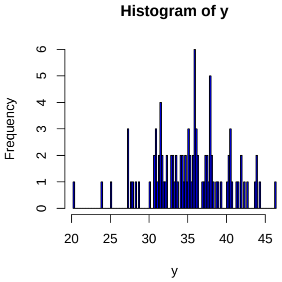
1. as above
2. as above + make a histogram of all ages
3. iterate over all ages and compute their weighted average

Iteration over all ages

1. as above
2. as above + make a histogram of all ages
3. iterate over all ages and compute their weighted average

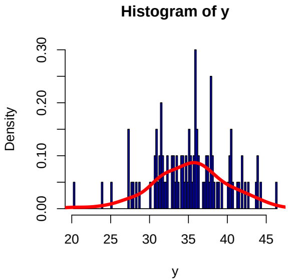
$$\bar{t}_r(t) = \frac{a_1 n_{a_1} + a_2 n_{a_2} + \cdots + a_n n_{a_n}}{n_o}$$

With $n_o = n_o(t) = n_{a_1} + n_{a_2} + \cdots + n_{a_n}$



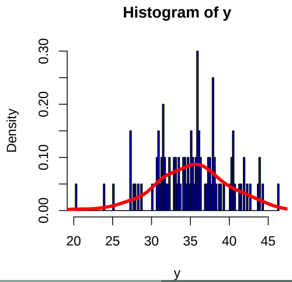
Integration over a density

$$
\begin{aligned}
\bar{t}_r(t) &= \lim_{n\to\infty} \frac{a_1 n_{a_1} + a_2 n_{a_2} + \cdots + a_n n_{a_n}}{n_o} \\
&= \lim_{n\to\infty} \sum_{minage}^{maxage} a \frac{n(a)}{n_o} da \\
&= \int_{minage}^{maxage} a\psi(a) da
\end{aligned}
$$

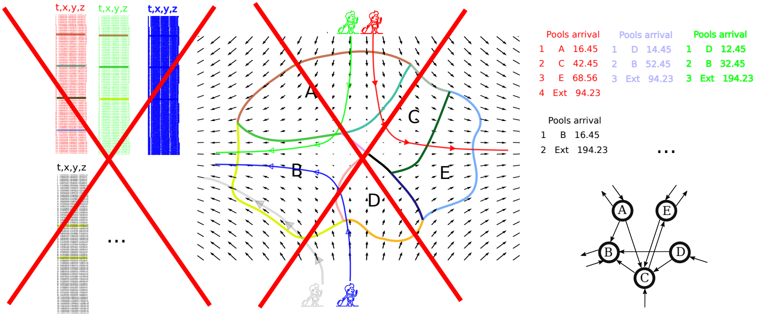


Same procedure for age density

$$
\begin{aligned}
\bar{a}(t) &= \lim_{n\to\infty} \frac{a_1 n_{a_1} + a_2 n_{a_2} + \cdots + a_n n_{a_n}}{n_p} \\
&= \lim_{n\to\infty} \sum_{minage}^{maxage} a \frac{n(a)}{n_p} da \\
&= \int_{minage}^{maxage} a\phi(a) da
\end{aligned}
$$



Possible Predictive Statistics ?



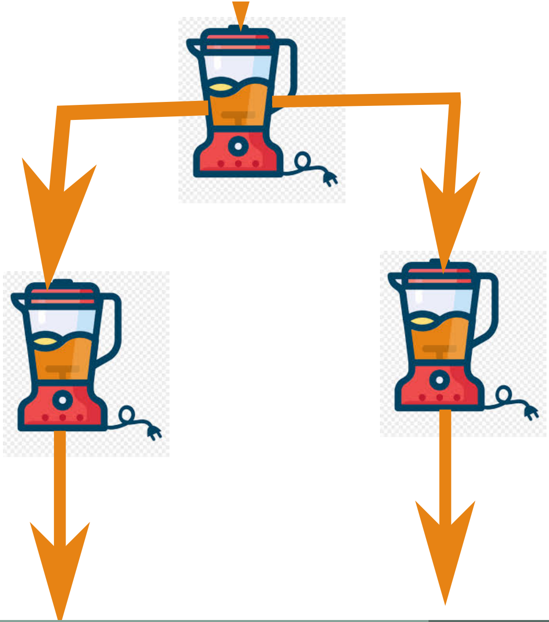
- Could we make a rule to predict the number of particles exiting from Pool E at time using only the particle passports? (Assuming that the exit from E is not recorded.)
- Could we make a rule to predict the age distribution of particles exiting from Pool E at time using only the particle passports? (Assuming again that the exit from E is not recorded.)

Intermediate summary

1. Pool descriptions condense complex information to a time series of pool changes.(A series of stamps in the passport)
2. There are many possible statistics on sets of these time series, usually related to numbers of particles and times. (e.g. the number of particles in a pool,
3. Pool Models predict = model some of these exclusively with respect to the information obtainable from (all) passports.
4. The most common even disregard most of the information in the passports.

The simplest pool systems

The simplest pool systems are well mixed



Only the number of particles per Pool counts ...

	Pools	arrival
1	A	16.45
2	C	42.45
3	E	68.56
4	Ext	94.23

	Pools	arrival
1	D	14.45
2	B	52.45
3	Ext	94.23

	Pools	arrival
1	D	12.45
2	B	32.45
3	Ext	194.23

	Pools	arrival
1	B	16.45
2	Ext	194.23

...

Only the number of particles per Pool counts ...

	Pools	arrival
1	A	16.45
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	Pools	arrival
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3	Ext	194.23

	Pools	arrival
1	B	16.45
2	Ext	194.23

...

Only the number of particles per Pool counts ...

Pools	arrival	
1	A	16.45
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3	E	68.56
4	Ext	94.23

Pools	arrival	
1	D	14.45
2	B	52.45
3	Ext	94.23

Pools	arrival	
1	D	12.45
2	B	32.45
3	Ext	194.23

Pools	arrival	
1	D	12.45
2	B	32.45
3	Ext	44.23

leaves too early

Pools	arrival	
1	B	66.45
2	Ext	194.23

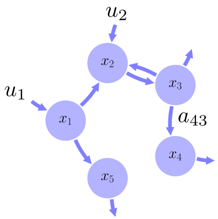
} arrives too late

...

How many particles are in pool B at time t =60?

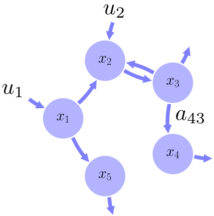
Linear autonomous compartmental models

$$\frac{d}{dt}\,\mathbf{x}(t) = \mathrm{A}\,\mathbf{x}(t) + \mathbf{u}$$



Linear autonomous compartmental models

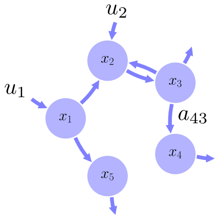
$$\frac{d}{dt}\,\mathbf{x}(t) = \mathrm{A}\,\mathbf{x}(t) + \mathbf{u}$$



$\mathbf{x}(t)$ vector of compartment content (e.g. C) at time t

Linear autonomous compartmental models

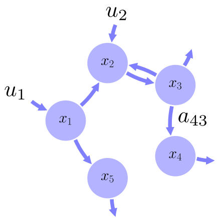
$$\frac{d}{dt}\,\mathbf{x}(t) = \mathrm{A}\,\mathbf{x}(t) + \mathbf{u}$$



$\mathbf{x}(t)$	vector of compartment content (e.g. C) at time t
$\mathbf{u}$	constant input vector

Linear autonomous compartmental models

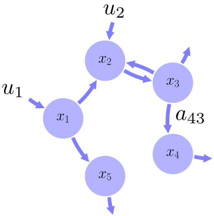
$$\frac{d}{dt}\,\mathbf{x}(t) = \mathrm{A}\,\mathbf{x}(t) + \mathbf{u}$$



$\mathbf{x}(t)$	vector of compartment content (e.g. C) at time t
$\mathbf{u}$	constant input vector
$\mathrm{A} = (a_{ij})$	compartmental matrix

Linear autonomous compartmental models

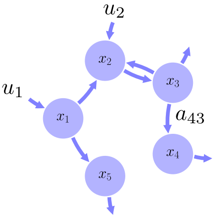
$$\frac{d}{dt}\,\mathbf{x}(t) = \mathrm{A}\,\mathbf{x}(t) + \mathbf{u}$$



$\mathbf{x}(t)$	vector of compartment content (e.g. C) at time t
$\mathbf{u}$	constant input vector
$\mathrm{A} = (a_{ij})$	compartmental matrix
$a_{ij}\,(i \neq j)$	fractional transfer coefficients,
	rate of flow from compartment j to compartment i

Linear autonomous compartmental models

$$\frac{d}{dt}\,\mathbf{x}(t) = \mathrm{A}\,\mathbf{x}(t) + \mathbf{u}$$



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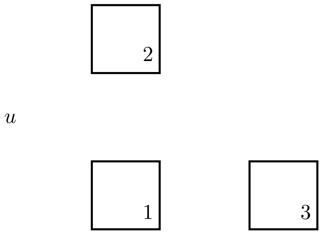
$\mathrm{A} = (a_{ij})$	compartmental matrix
$a_{ij}\ (i \neq j)$	fractional transfer coefficients,
	rate of flow from compartment j to compartment i
$-a_{ii} > 0$	rate of flow out of compartment i

A particle travels

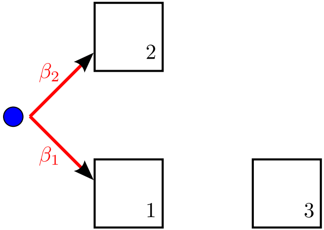
$$\frac{d}{dt}\,\mathbf{x}(t) = \mathrm{A}\,\mathbf{x}(t) + \mathbf{u}$$



$$\begin{array}{cc} \mathrm{A} & \mathbf{u} \\ \begin{pmatrix} -a_{11} & a_{12} & 0 \\ a_{21} & -a_{22} & 0 \\ a_{31} & 0 & -a_{33} \end{pmatrix} & \begin{pmatrix} u_1 \\ u_2 \\ 0 \end{pmatrix} \end{array}$$

$$-\sum \quad 0 \quad > 0 \quad > 0 \qquad \|\mathbf{u}\|$$

$$\frac{d}{dt}\mathbf{x}(t) = \mathrm{A}\,\mathbf{x}(t) + \mathbf{u}$$



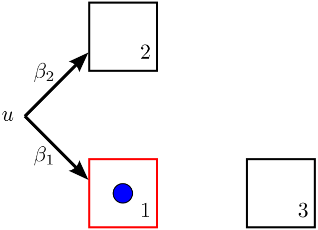
$$\begin{array}{cc} \mathrm{A} & \mathbf{u} \\ \begin{pmatrix} -a_{11} & a_{12} & 0 \\ a_{21} & -a_{22} & 0 \\ a_{31} & 0 & -a_{33} \end{pmatrix} & \begin{pmatrix} {\color{red}u_1} \\ {\color{red}u_2} \\ 0 \end{pmatrix} \end{array}$$

$$-\sum \quad 0 \quad > 0 \quad > 0 \qquad \|\mathbf{u}\|$$

$$\beta_1 = \frac{u_1}{u_1+u_2} \qquad \beta_2 = \frac{u_2}{u_1+u_2}$$

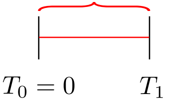
$T_0 = 0$

A particle travels $\frac{d}{dt}\mathbf{x}(t) = \mathrm{A}\,\mathbf{x}(t) + \mathbf{u}$



$$
\begin{array}{ccc}
\mathrm{A} & & \mathbf{u} \\
 & \begin{pmatrix} -a_{11} & a_{12} & 0 \\ a_{21} & -a_{22} & 0 \\ a_{31} & 0 & -a_{33} \end{pmatrix} & \begin{pmatrix} u_1 \\ u_2 \\ 0 \end{pmatrix} \\
-\sum & \begin{matrix} 0 & > 0 & > 0 \end{matrix} & \|\mathbf{u}\|
\end{array}
$$

$T_1 \sim \mathrm{Exp}(a_{11})$



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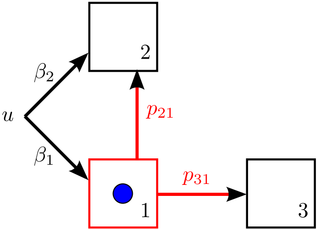
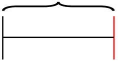


$$\begin{array}{cccc} & \mathrm{A} & & \mathbf{u} \\ & \begin{pmatrix} -a_{11} & a_{12} & 0 \\ a_{21} & -a_{22} & 0 \\ a_{31} & 0 & -a_{33} \end{pmatrix} & & \begin{pmatrix} u_1 \\ u_2 \\ 0 \end{pmatrix} \\ -\sum & 0 \quad\quad > 0 \quad\quad > 0 & & \|\mathbf{u}\| \end{array}$$

$$p_{21} = \frac{a_{21}}{a_{11}} \qquad p_{31} = \frac{a_{31}}{a_{11}}$$

$T_1 \sim \mathrm{Exp}(a_{11})$



$T_0 = 0$ $\quad$ T_1

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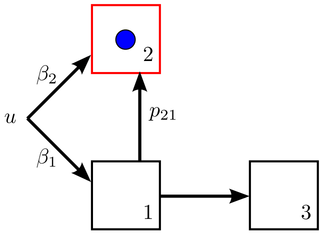

$$
\begin{array}{cc}
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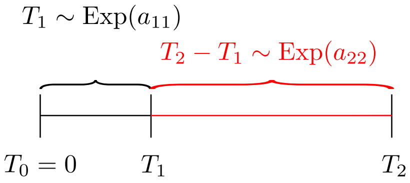
\begin{pmatrix} -a_{11} & a_{12} & 0 \\ a_{21} & -a_{22} & 0 \\ a_{31} & 0 & -a_{33} \end{pmatrix} & \begin{pmatrix} u_1 \\ u_2 \\ 0 \end{pmatrix}
\end{array}
$$

$$-\sum \quad 0 \quad > 0 \quad > 0 \qquad \|\mathbf{u}\|$$



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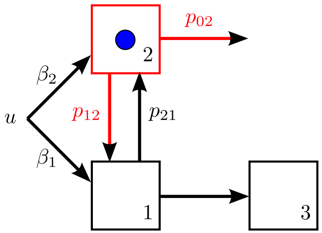


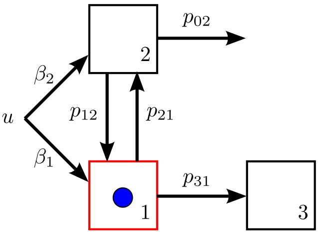
$$
\begin{array}{ccc}
 & \mathrm{A} & \mathbf{u} \\
 & \begin{pmatrix} -a_{11} & a_{12} & 0 \\ a_{21} & -a_{22} & 0 \\ a_{31} & 0 & -a_{33} \end{pmatrix} & \begin{pmatrix} u_1 \\ u_2 \\ 0 \end{pmatrix} \\
-\sum & \begin{matrix} 0 & \quad > 0 & \quad > 0 \end{matrix} & \|\mathbf{u}\|
\end{array}
$$

$$p_{12} = \frac{a_{12}}{a_{22}} \qquad p_{02} = 1 - \frac{a_{12}}{a_{22}}$$

$$\underbrace{}_{T_1 \sim \mathrm{Exp}(a_{11})}\underbrace{}_{T_2 - T_1 \sim \mathrm{Exp}(a_{22})}$$

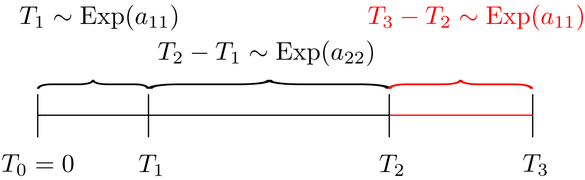
$T_0 = 0 \qquad T_1 \qquad T_2$

A particle travels $\frac{d}{dt}\,\mathbf{x}(t) = \mathrm{A}\,\mathbf{x}(t) + \mathbf{u}$



$$
\begin{array}{cc}
\mathrm{A} & \mathbf{u} \\
\begin{pmatrix} {\color{red}-a_{11}} & a_{12} & 0 \\ a_{21} & -a_{22} & 0 \\ a_{31} & 0 & -a_{33} \end{pmatrix} & \begin{pmatrix} u_1 \\ u_2 \\ 0 \end{pmatrix}
\end{array}
$$

$$
-\sum \quad 0 \quad > 0 \quad > 0 \qquad \|\mathbf{u}\|
$$



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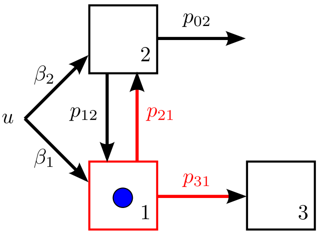
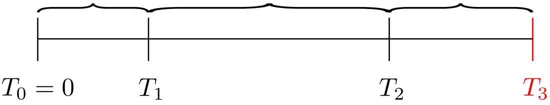


$$
\begin{array}{ccc}
 & \mathrm{A} & \mathbf{u} \\
 & \begin{pmatrix} -a_{11} & a_{12} & 0 \\ {\color{red}a_{21}} & -a_{22} & 0 \\ {\color{red}a_{31}} & 0 & -a_{33} \end{pmatrix} & \begin{pmatrix} u_1 \\ u_2 \\ 0 \end{pmatrix} \\
-\sum & \begin{matrix} {\color{red}0} & \quad > 0 & \quad > 0 \end{matrix} & \|\mathbf{u}\|
\end{array}
$$

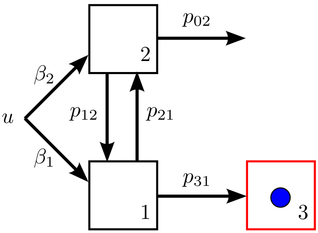
$$
{\color{red} p_{21} = \frac{a_{21}}{a_{11}} \qquad p_{30} = \frac{a_{31}}{a_{11}}}
$$

$T_1 \sim \mathrm{Exp}(a_{11})$ $\quad T_3 - T_2 \sim \mathrm{Exp}(a_{11})$

$T_2 - T_1 \sim \mathrm{Exp}(a_{22})$



A particle travels $\frac{d}{dt}\mathbf{x}(t) = \mathrm{A}\,\mathbf{x}(t) + \mathbf{u}$



$$
\begin{array}{ccc}
 & \mathrm{A} & \mathbf{u} \\
 & \begin{pmatrix} -a_{11} & a_{12} & 0 \\ a_{21} & -a_{22} & 0 \\ a_{31} & 0 & {\color{red}-a_{33}} \end{pmatrix} & \begin{pmatrix} u_1 \\ u_2 \\ 0 \end{pmatrix} \\
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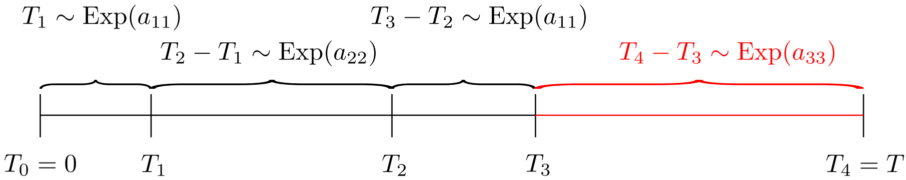
\end{array}
$$



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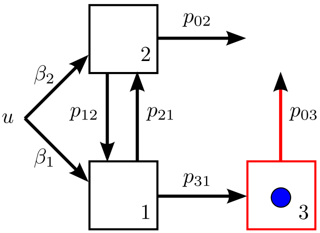


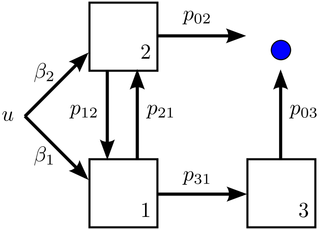
$$
\begin{array}{cc}
 & \mathrm{A} \qquad\qquad\qquad \mathbf{u} \\
 & \begin{pmatrix} -a_{11} & a_{12} & \color{red}{0} \\ a_{21} & -a_{22} & \color{red}{0} \\ a_{31} & 0 & -a_{33} \end{pmatrix} \quad \begin{pmatrix} u_1 \\ u_2 \\ 0 \end{pmatrix} \\
-\sum & \quad 0 \qquad > 0 \qquad \color{red}{> 0} \qquad\quad \|\mathbf{u}\|
\end{array}
$$

$\color{red}{p_{03} = 1}$

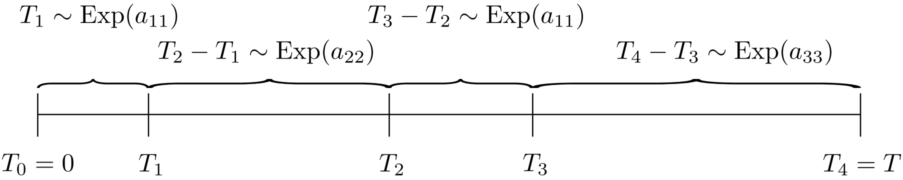
$T_1 \sim \mathrm{Exp}(a_{11})$ $\quad T_2 - T_1 \sim \mathrm{Exp}(a_{22})$ $\quad T_3 - T_2 \sim \mathrm{Exp}(a_{11})$ $\quad T_4 - T_3 \sim \mathrm{Exp}(a_{33})$

$T_0 = 0 \qquad T_1 \qquad T_2 \qquad T_3 \qquad \color{red}{T_4 = T}$

A particle travels $\quad \frac{d}{dt}\mathbf{x}(t) = \mathrm{A}\,\mathbf{x}(t) + \mathbf{u}$



$$\begin{array}{ccc} & \mathrm{A} & \mathbf{u} \\ & \begin{pmatrix} -a_{11} & a_{12} & 0 \\ a_{21} & -a_{22} & 0 \\ a_{31} & 0 & -a_{33} \end{pmatrix} & \begin{pmatrix} u_1 \\ u_2 \\ 0 \end{pmatrix} \\ -\sum & \begin{matrix} 0 & \quad > 0 & \quad > 0 \end{matrix} & \|\mathbf{u}\| \end{array}$$



Transit time distribution

- transit time computation:

$$T = \underbrace{(T_1 - T_0)}_{\sim \mathrm{Exp}(a_{11})} + \underbrace{(T_2 - T_1)}_{\sim \mathrm{Exp}(a_{22})} + \underbrace{(T_3 - T_2)}_{\sim \mathrm{Exp}(a_{11})} + \underbrace{(T_4 - T_3)}_{\sim \mathrm{Exp}(a_{33})}$$

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- mixture of exponential distributions

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$\rightarrow$ T follows phase-type distribution with parameters β, A

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$\beta = \frac{\mathbf{u}}{\|\mathbf{u}\|}$

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$\beta = \frac{\mathbf{u}}{\|\mathbf{u}\|}$

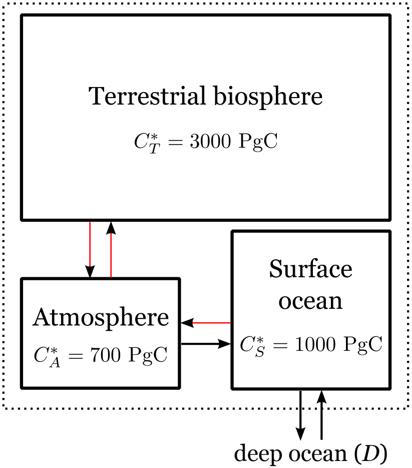
→ $T \sim \mathrm{PH}(\beta, \mathrm{A})$

General, simple, explicit formulas

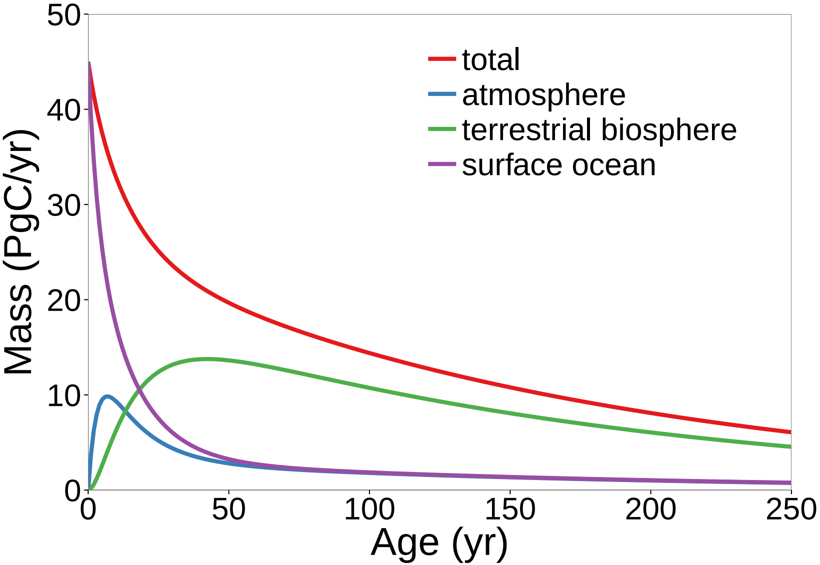
- phase-type distribution is well known:
  - probability density
  - cumulative distribution function
  - quantiles
  - mean and higher order moments
    - $\mathbb{E}[T] = \|\mathrm{A}^{-1}\,\beta\| = \frac{\|\mathbf{x}^*\|}{\|\mathbf{u}\|}$ (mean transit time)
- system age is also phase-type distributed
  - parameters $\eta := \frac{\mathbf{x}^*}{\|\mathbf{x}^*\|}, \mathrm{A}$
- probability density of compartmental age
  - $\mathbf{f_a}(y) = (\mathrm{X}^*)^{-1}\, e^{y\,\mathrm{A}}\, \mathbf{u}$

Application to a nonlinear carbon cycle model

Nonlinear model in steady state [Rodhe and Björkström, 1979] with three compartments



Equilibrium age densities



Linear nonautonomous systems

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$$\Rightarrow \underbrace{\Phi(t,t-a)\,\mathbf{u}(t-a)}_{\text{mass with age } a \text{ at time } t}$$

The state transition operator Φ

- solution to the matrix ordinary differential equation

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$$\Phi(t,s) = e^{(t-s)\,\mathrm{A}} \quad \text{(matrix exponential)}$$

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$\rightarrow$ describes the time evolution of mass in the system

$\rightarrow$ we can always obtain it (at least numerically)

Splitting the system

- we split the system into two parts

$$\begin{array}{l}\text{mass in the system}\\ \text{at time } t \text{ with age} =\\ a\end{array} \left\{ \begin{array}{l} \\ \\ \\ \end{array} \right.$$

Splitting the system

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$$\text{mass in the system at time } t \text{ with age } a = \begin{cases} \qquad , & a < t \\ \qquad , & a \geq t \end{cases}$$

Splitting the system

- we split the system into two parts

$$\text{mass in the system at time } t \text{ with age } a = \begin{cases} \text{remainings from input at time } t-a, & a < t \\ \\ , & a \geq t \end{cases}$$

Splitting the system

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Nonlinear nonautonomous systems

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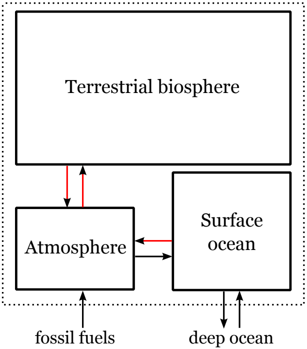
A global carbon cycle model



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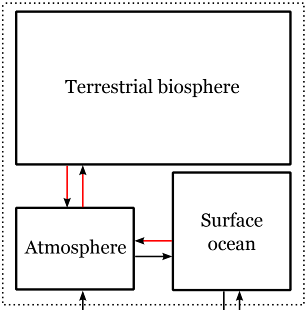
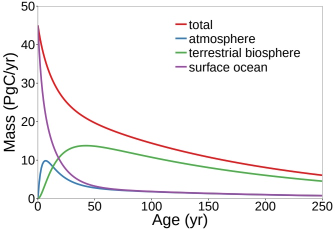


Equilibrium age densities in 1765



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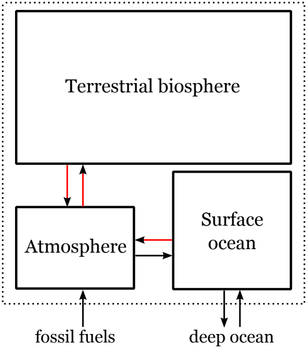

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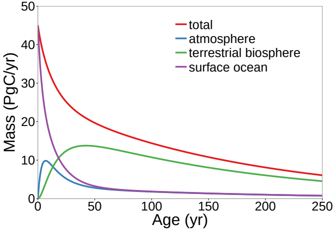
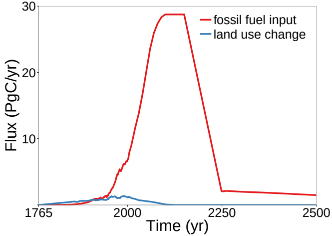
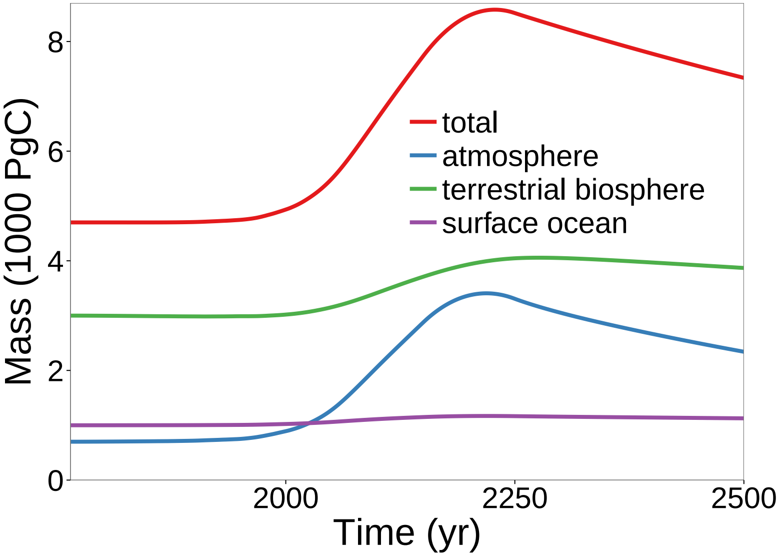


RCP8.5 scenario



Time-dependent carbon contents



Questions of high scientific and societal interest

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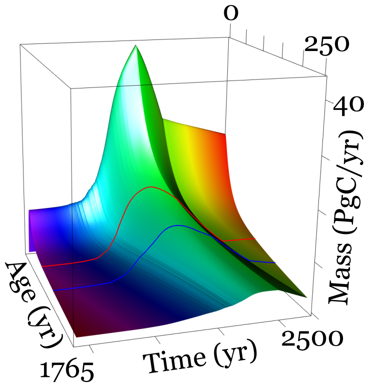
Time evolution of the atmosphere's carbon age density.



- median age
- mean age
- surface color constant along the time-age diagonal such that the color reflects the moment of entry into the system.
- left edge equilibrium age density of the atmosphere's carbon at (time $= 1765\,\mathrm{yr}$)
- lower edge (age $= 250\,\mathrm{yr}$) mass is in the system with age equal to $250\,\mathrm{yr}$

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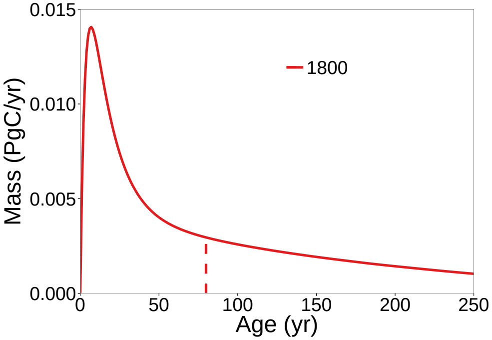

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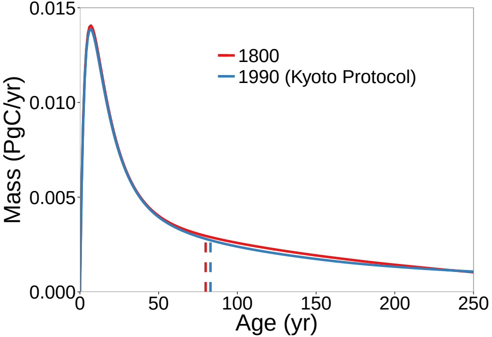

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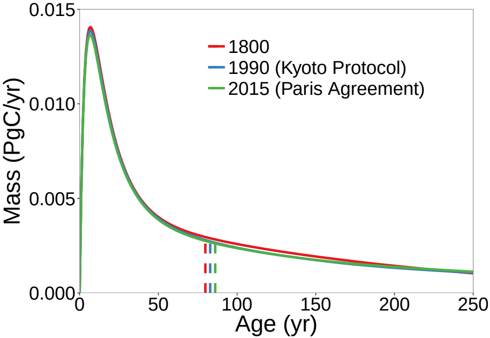

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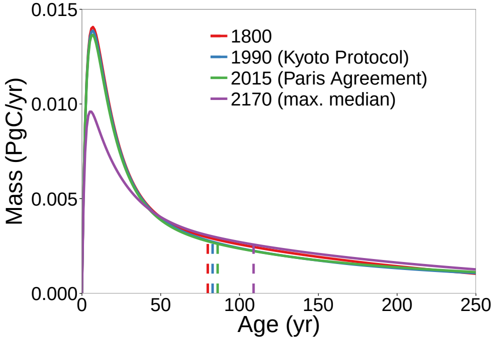

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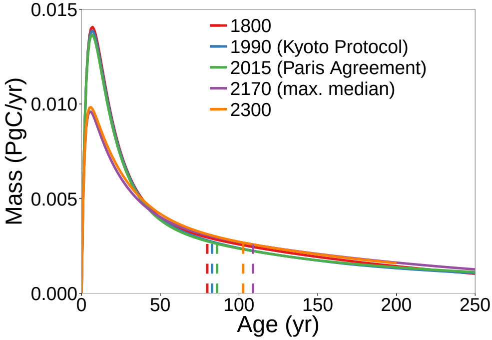

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- **two Python packages**
  - (symbolic) computation of age and transit time properties for autonomous systems in steady state
    - `http://github.com/goujou/LAPM`
  - (numerical) computation of age and transit time properties for nonlinear nonautonomous systems

System age

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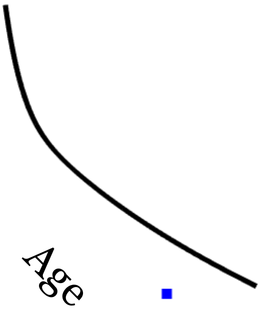

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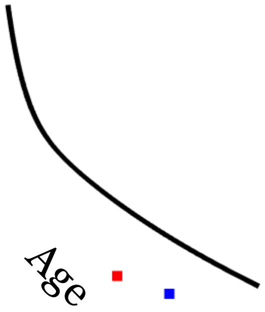

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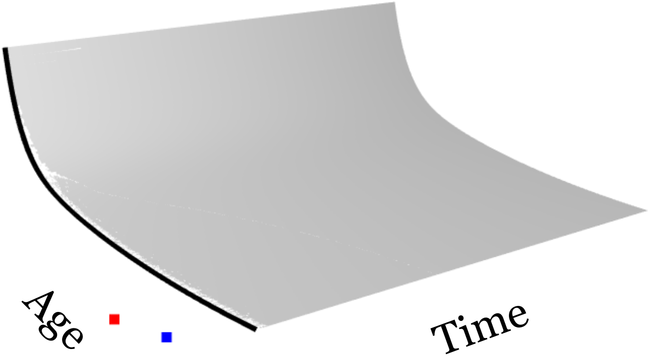

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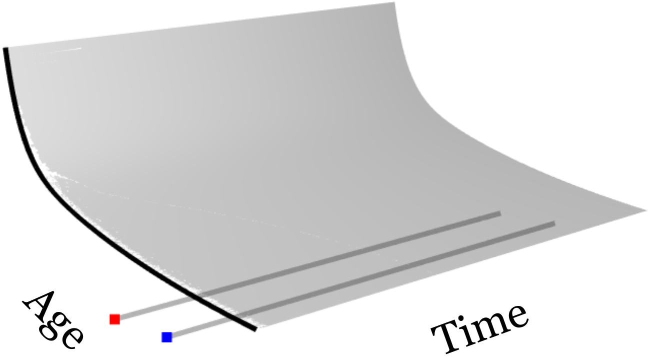

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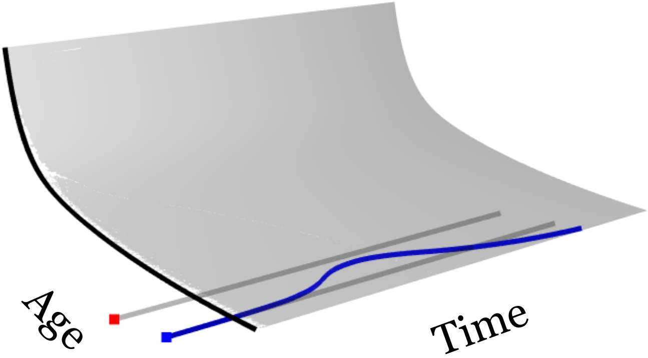

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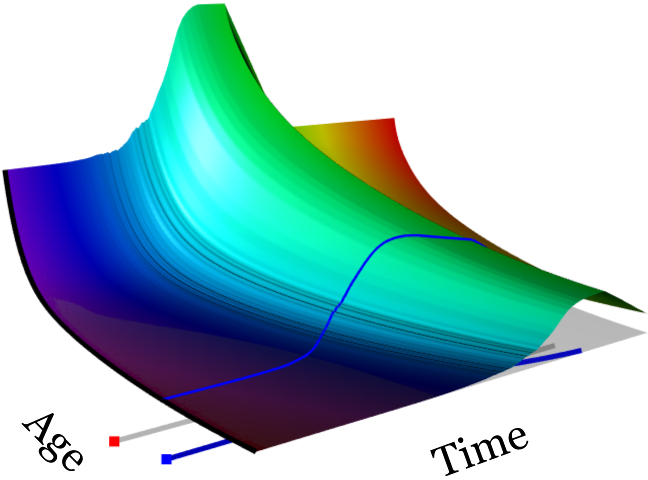

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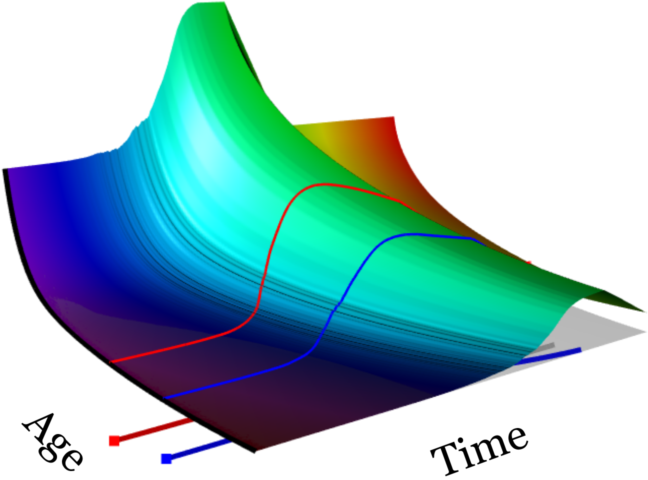

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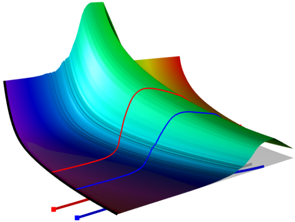
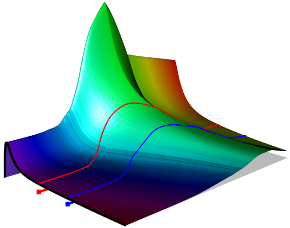
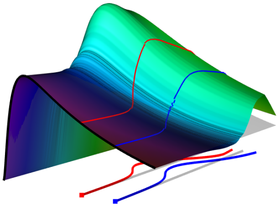


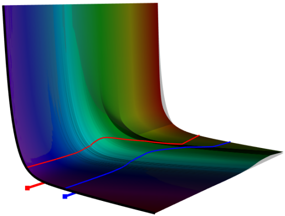
Atmosphere



Terrestrial biosphere



Surface ocean



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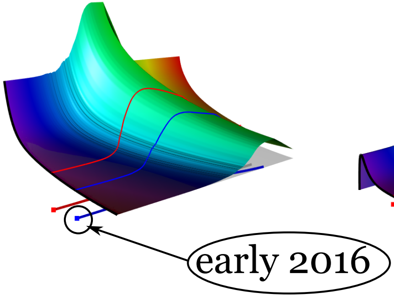

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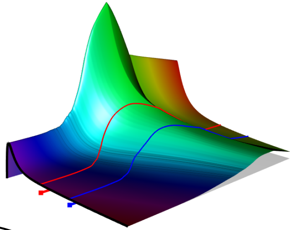

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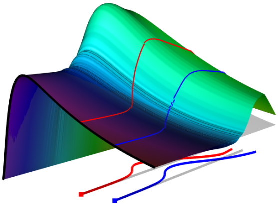

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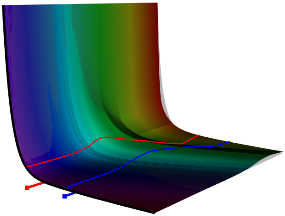
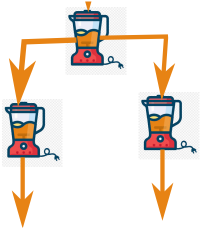
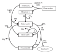
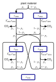
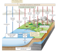
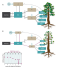
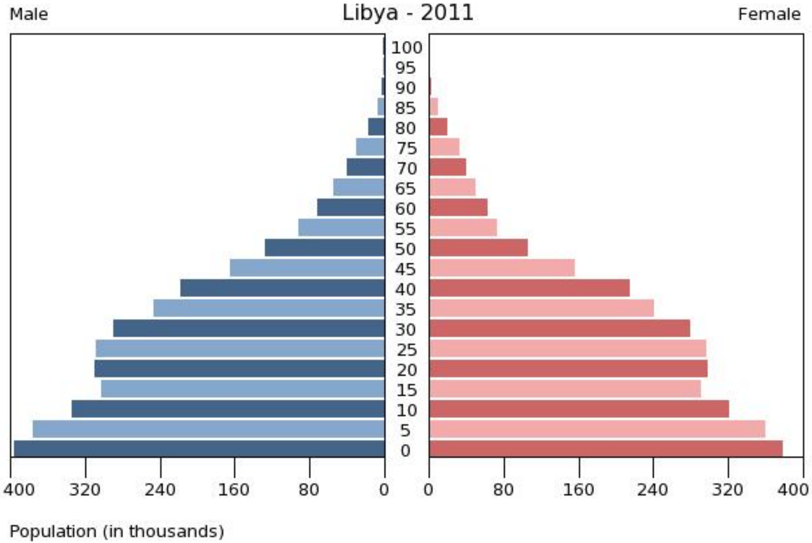


Examples for “Well-mixed Systems”



- Century
  
- RothC
  
- Element Cycling Models
  
- Reactors 
- Carbon Allocation 
- ...

Population Dynamics $\phi(a)$



Age distribution varies with time. $\phi(a) = \phi(a, t)$

McKendrick–von Foerster Equation

- original:

$$\frac{\partial \phi}{\partial a} + \frac{\partial \phi}{\partial t} = m(a)\phi$$

The “deathrate” $m(a)$ depends on the age a.

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- in compartmental systems

$$\frac{\partial \phi}{\partial a} + \frac{\partial \phi}{\partial t} = m(t)\phi$$

  $m(t)$ depends only on time t.

Age and transit-time distributions for catchments

- Single catchment
  - identical equations as for population dynamics
  - “deathrate” $\rightarrow$“age selection function”.
- multiple connected catchments
  - Possible: Prediction of age selection function for systems of well mixed pools. (from transit time and age distributions)
  - Challenge: Prediction of age selection function for systems of non well mixed pools. (global age selection functions from pool wise age selection functions)

Summary

- Pool models greatly simplify:
  - The questions (that can be) asked
  - The way questions are answered by exploiting symmetries

- Natural descriptive statistics are time related distributions (age and transit time , per pool per system ...)

- The simplest and most prominet pool models with the most theory are “well mixed ” or “compartmental systems” modeled by ordinary differential equations.

- Much more general pool models are possible including non constant deathrates (age selection functions) or delays, that have to be solved with different mathematical tools. (partial differential equations, integral equations, delay equations, Monte Carlo simulations ...) still much simpler than the underlying complete models.

Thanks

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DFG Deutsche Forschungsgemeinschaft

bibliography

Henrik Hartmann, Henry D. Adams, William M. Hammond, Günter Hoch, Simon M. Landhäusser, Erin Wiley, and Sönke Zaehle. Identifying differences in carbohydrate dynamics of seedlings and mature trees to improve carbon allocation in models for trees and forests. *Environmental and Experimental Botany*, 152:7 – 18, 2018. doi: https://doi.org/10.1016/j.envexpbot.2018.03.011. URL `http://www.sciencedirect.com/science/article/pii/S0098847218303915`.

Henning Rodhe and Anders Björkström. Some consequences of non-proportionality between fluxes and reservoir contents in natural systems. *Tellus*, 31(3):269–278, 1979.