Bild Stäbchen

There's experimente of Guazzelli and here is a picture of that experiment.

The particles are suspended in a fluid.

In an initially homogenous suspension, rods form clusters with higher particle concentrations which sediment faster than a single rod.

The clusters lead to downward streamers that are balanced by upward streamers.

Within a cluster, particles align in the direction of gravity most of the time, occasionally they flip.

Mathematical Model

This is a simplified model from the original model of Helzel&Tzavaras. It presents a sedimentation process in dilute suspension of rod-like particles. This model couples a kinetic eq. to a macroscopic flow eq. The first eq. is known as a Smochluchowski eq. And the macrospic flow is described in the third line by the Navier-Stokes eq. with an additional buoyancy term.

1. Term: **Temporal change of the particle distribution.** Zeitliche Änderung der Partikelverteilung. Zeitliche Ableitung der Verteilungsfunktion f. Beschreibt, wie sich die Verteilung der Partikel mit der Zeit ändert.
2. Term: **Convective transport by the macroscopic velocity field u. Konvektiver Transport durch das Strömungsfeld.** Räumliche Divergenz des konvektiven Flusses **u**f. Dieser Term beschreibt die Transportbewegung der Partikel durch das Geschwindigkeitsfeld u. Die Partikeln werden mit der Strömung transportiert.
3. Term: **Orientation of particles due to a macroscopic velocity gradient (\nabla_x).** Ausrichtung der Partikel durch Geschwindigkeitsgradienten. Einfluss der lokalen Geschwindigkeitsgradienten auf die Orientierung der Partikel.
   - ∇_x u ist der Geschwindigkeitsgradient des Strömungsfeldes.
   - Pn⊥ ist der Projektor auf die Ebene senkrecht zu n. Das bedeutet, dass nur Drehungen berücksichtigt werden, die senkrecht zur Partikelorientierung n wirken.
   - nf sorgt dafür, dass dieser Term mit der Anzahl der Partikel in einer bestimmten Richtung skaliert wird.
4. Term: **External forces (e.g., sedimentation) influencing the motion. The movement is influenced by the particle orientation due to gravity.** Zeigt, dass die Orientierung n eine Rolle spielt, also wie die Partikeln fallen hängt von der Orientierung ab: Partikel mit einer vertikalen Orientierung (d.h. n=(0,0,1)^T) sedimentieren schneller als solche mit einer horizontalen Ausrichtung.
 Äußere Kräfte (z. B. Sedimentation), die die Bewegung beeinflussen. Dieser Term beschreibt, wie sich Partikel unter einer äußeren Kraft (z. B. Gravitation) bewegen, wobei die Bewegung von der Partikelorientierung beeinflusst wird.
   - I+n⊗n ist eine Tensorstruktur, die die bevorzugte Bewegungsrichtung der Partikel aufgrund ihrer Orientierung beschreibt.
   - e3 ist der Einheitsvektor in z-Richtung, was darauf hinweist, dass dies ein Effekt wie Sedimentation oder eine externe Kraft in vertikaler Richtung sein könnte.
5. Term: **Rotational diffusion, describing random reorientations of the particles.** Rotationsdiffusion, die zufällige Drehungen der Partikel beschreibt. Rotationsdiffusion der Partikel. Dieser Term beschreibt die stochastische Rotation der Partikel aufgrund thermischer Bewegung oder zufälliger Wechselwirkungen mit der Umgebung.

Re = a Reynolds number based on the sedimentation velocity
Dr = the rotational diffusion coefficient
gamma = measures the relative importance of elastic forces over buoyancy forces.

Δnf ist der Laplace-Operator in der Orientierung n, der beschreibt, wie sich die Partikel im Orientierungsraum diffundieren.

The second and fourth term describe the movement of the rods through a macroscropic velocity field. The third term relates to a rotation of rod-like particles due to a macroscopic velocity gradient.

The term on the right hand side of the first equation models rotational diffusion.

Bei Navier Stokes eq: der letzte Term ist ein extra Term (Buoyancy term, da wirkt ein Kraftterm)

Bei original Model:
We consider the case where gamma=0, so we ignore two terms bc the buoyancy term is more important.

(in which the effects of translational Brownian motion of the rod-like particles are ignored.
So the effect of elastic forces in the flow equation is not that important compared to the buoyancy forces)

Status of project

There‘s a previous work from Dahm and Helzel. They considered a simplified model for the kinetic equation on S^1. For the simpler model, they derived a hierachy of moment equations.
Derive and approximate hierarchies of moment eqs. for the coupled kinetic-fluid model with f on S^2.

Motivation
The coupled system, which is a 5-dimensional problem, because it depends on space, time and on the orientations described on the sphere. The orientation can be described by two angles. We want to transform this high dimensional system into a lower dimensional system of moment equations, so that f will depend only on space.

To illustrate our ansatz we consider the example shear flow, but here f is still depending on space and the orientation.

Model Shear Flow

We consider the velocity of this form u, the velocity depends on the x-direction (x-Achse) and time. The flow occurs in z-direction. And in this case f is also a function of one variable.
The term in gray plays no role for this shear flow.

sin(theta)f ist die Erhaltungsgröße
\nabla_x(uf): dx(0)+dy(0)+dz(w(x)f), da f nicht von dz abhängt, daher fällt der komplette Term weg.

We assume that the velocity field only has a vertical component, given by u=(0,0,w(x,t))^T

Ansatz for the Moment Equations

We only consider the even order (bc the particles are symmetric).

Derivation of the Moment Equations

We derive the moment equations by inserting.. and projecting to the basis (Multiply consecutively with all basis functions used in the ansatz and integrate the resulting equations over the sphere)

This is the general form of the hyperpolic system w. source term.

Derivation: A Closer Look

Term Laplace-Operator:
From the properties this term
corresponds to the Laplace-Beltrami operator*sin(theta). So after we insert the ansatz for f into this term, we get sin()*eigenvalue*ansatz. And if we then multiply with this corresponding ansatz, then almost all terms will be omitted. And as the result Dr*eigenvalue*coefficient.

Term [1]:
When we multiply term 1 with P0 then we get the time derivative for
f0. If we multiply it with P22, then we get the time derivative for c22. So after we insert the ansatz for f in term [1], we get the time derivative of these coefficients.
Bc the function c doesnt depend on \phi and \theta, we take this out of the integral. And this expression is the L2-inner product.

<u>Bc of the time I will only explain this term [2] in details.</u>

Term [2]:
Short: The term cos(phi)cos(theta)… is a multiple one of the second order basis functions. That is the reason why we get in the first row and in the first column here an entry and other entries are zero.

For N=1, the matrix A has the form. The term cos φ cos θ sin θ is a multiple one of the second order basis functions. Let‘s consider the projection of products of spherical harmonics to spherical harmonic basis function

The first row means that we multiply the ansatz with P00. And we will get the entry [1,3]. We get here an entry bc we insert the ansatz w this term   cos φ cos θ sin θ and project it to P00.

Assume: Set f^N to P00. Then multiply this ansatz with the basisfct up to order two, we will get this entry [3,1].

For N=2:

Block A_{1,2}: If we use in our ansatz the basis functions up to order 4, and then multiply them by the basis functions of up to order 2, we will get the Block A_{1,2}.

Block $A_{1,2}^T$: If we use up to order 2 in order Ansatz and multiply it w basis function order 4, we get the Block $A_{1,2}^T$, which is the transport of $A_{1,2}$.

If more base functions are used, we always have three additional blocks.
This pattern will continue bc

Note that the number of moments depends on the size N via $(N+1)(2N+1)$.

Moreover the Matrix A is symmetric, therefore the system is hyperbolic.

Remaining terms:

We follow the same procedure. We insert our Ansatz and then project onto all the basis functions used in the Ansatz. In this case, however, the calculations are more extensive because we need to compute the derivative of θ. This leads to the matrix D*Q.

Splitting Alg

Step 1, 5 and 9, we use a Strang Splitting Algorithm for our hyperbolic moment equations and between these steps we use the Strang Splitting for the second equation. For the hyperbolic system in step 5 we use wave propagation.
In between those substeps the velocity is evolved. For the evolution of w we used simple second order accurate finite difference methods. The hyperbolic system of moment equations (i.e., line 5) is evolved in time using LeVeque's high resolution wave propagation algorithm. The evolution of the source term (i.e., line 1 and 9) is performed with a higher-order Runge-Kutta method.

Numerical Simulation

First Picture Pfeilen
The picture shows the alignment of rod-like particles under the influence of a vertical flow field. The arrows represent the velocity field, with the largest arrows showing stronger flow areas (higher concentration) and smaller arrows showing weaker flow.

Near the central area, the arrows curve downward, and it shows how particles fall downwards by gravity and the velocity gradient.

The arrows pointing upwards could indicate particles that are still rotating or orienting themselves and thus experiencing a brief upward movement.
Die Pfeile nach oben könnten Partikel zeigen, die sich noch drehen oder orientieren und dadurch eine kurzzeitige Aufwärtsbewegung erfahren.

The spheres illustrate the distribution of rod-like particle orientations under the influence of the flow.

We see that they tend to orient along the main shear direction.

2. Bild: we can see that the cluster become thinner and more pronounced.

Solution on the sphere

For this 1st subproblem we want to take a closer look at the numerical solution for different Dr.
We consider here the numerical solution for a fixed w_x.

Blau = niedrige Dichte (weniger Partikel sind in diese Richtung ausgerichtet).
Rot = hohe Dichte (mehr Partikel sind in diese Richtung ausgerichtet).

For a larger Dr=1, by using N=1 moment equations we can see that the solution structure has already a good approximation as N=6.

N = 1: 6 moment equations
N = 6: 91 moment equations
N = 7: 120 moment equations

Accuracy Study (Konvergenzstudie mit ref sol. N=7)

Here we focus on the coupled system.

In the following example we use N=7 moment equations for the reference solution.
This accuracy study fit in with what we have shown previously for the first subproblem.

Cluster Formation

We started with almost constant density at some random noise and with these pictures we want to illustrate how the cluster form. Here we can see the solution of c^0_0 (corresponds to the first component, means the density of the particles) at different times.
The Figure shows that using N=1 moment eqs, compared to the ref. sol. w. N=6 moment eqs., already provides a good approximation.

Rectilinear Flow

The whole approach can also be transferred to a more general form, e.g. Rectilinear Flow.
Here we get a hyperbolic system again.
Gleichung (6): Here the source term didnt change, on the left hand side we get additional a Matrix B.

The pictures show the clustering behavior at time=100 and time=200 for two different cases N=1&N=7.
The overall structure appears similiar in both cases for t=100. But over time we can see some different. Like in this picture we can see slightly more pronounced peaks.