2*eme* Cours : Cographes
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Which class of graphs ? First example the Cographs

Complement-reducible graphs

S. Seinsche. On a property of the class of n-colorable graphs. Journal on Combinatorial Theory (B), 16 :191–193, 1974.

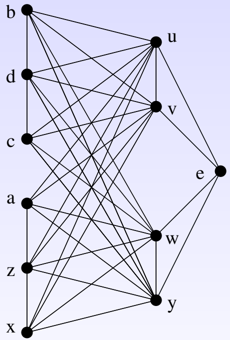
Recursive Definition

The class of cographs is the smallest class of graphs containing $G_0 = K_1$ and closed under series and parallel compositions. They can be represented via a tree (called a cotree) using these operations, the leaves being the vertices.

Computing the series or par allel operations

- parallel = connected components
- series = co-connected components (i.e., connected components of the complement)
- Consequences : the naive recognition algorithm in $O(n(n+m))$.

An example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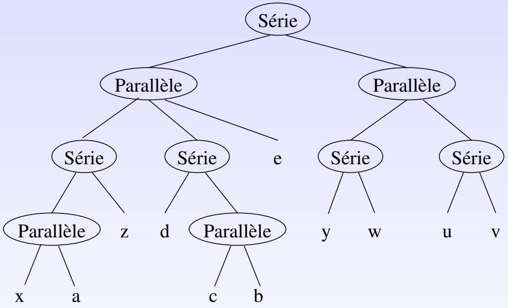
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Cographs properties

Characterisation Theorem : Seinsche 1974

A graph is a cograph iff it does not contain a P_4 (path of length 3 with 4 vertices) as an induced subgraph.

A proof is needed

[a]

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a. The result holds in the infinite case

1. A P_4 is not a cograph, i.e. ; no series or parallel operation can be applied to decompose a P_4.
2. Using the series and parallel one cannot create a P_4.
3. G is a cograph iff $\overline{G}$is a cograph.
4. A graph that contains a P_4 is not a cograph.
5. But why a graph with no P_4 is a cograph ?

The difficult part

Lemma

If G connected does not contain a P_4 then G admits a series composition.

Proof of the lemma

1. Suppose that G does not contain any P_4 (as induced subgraph) and is not decomposable with the series and parallel operations.
2. Therefore because 3 of the previous slide, $\overline{G}$ does not contain any P_4.
3. Consider a vertex x. $(\{x\}, N(x), \overline{N(x)})$ is a partition of the vertices of G and moreover since G and $\overline{G}$ are connected, $N(x) \neq \emptyset$ and $\overline{N(x)} \neq \emptyset$.
4. Let $A_1, \ldots A_k$ be the co-connected components of $G(N(x))$, and $B_1, \ldots B_h$ the connected components of $G(\overline{N(x)})$.
5. Since G is connected every set B_j has at least one edge $b_j a_i$ to some A_i.

1. If there exist $z \in B_j$, $zb_j \in E$. But then : (x, a_i, b_j, z) is a P_4 in G, except if $za_i \in E$.
   Following the paths of the connected component B_j, we conclude that a_i is connected to all vertices in B_j.
2. Symmetrically, if there exist $t \in A_i$, $ta_i \notin E$. But then : (x, b_j, t, a_i) is a P_4 in $\overline{G}$, except if $tb_j \in E$.
   Following the paths of the co-connected component A_i, we conclude that b_j is connected to all vertices in A_i.
3. Therefore between two sets A_i and B_j either there is no edge or it is a complete bipartite.

1. To finish the proof, let us consider the bipartite graph $B(G)$ generated by the sets $A'_i s$ and $B'_j s$. We can contract these sets to one vertex, because all vertices inside have the same neighborhood.
2. $B(G)$ has no parallel edge (two edges xy, zt such that $xt \notin E$ and $zy \notin E$).
   Suppose $b_j a_i$ and $b_q a_p$ are parallel edges. But since the $A'_i s$ are coconnected components necessarily $a_i a_p \in E$ and therefore we hare a P_4 in G, namely : (b_j, a_i, a_p, b_q)

1. Consider a bipartite with no parallel edges, then one can see easily that no two neighborhoods can either overlap or be disjoint. Every pair of neighborhood are comparable by inclusion. Therefore there is a total ordering of the neighborhoods.
2. Just take a set A with the biggest neighborhood in the $B's$. Necessarily A is connected to all $B's$.
3. And we have a series composition $G = G(A) \oplus G(\overline{A})$

Consequences :

Cographs is an hereditary class of graphs
(i.e., is G is a cograph, every induced subgraph of G is also a cograph).

This proof can be generalized to study related classes of graphs such as :

- Prime graphs under modular decomposition
- P_4-sparse graphs (A graph is P_4-sparse if any set of five vertices induces at most one graph P_4).
- P_4-connected graphs
- ... P_4-extensible

Cographs an interesting class of graphs

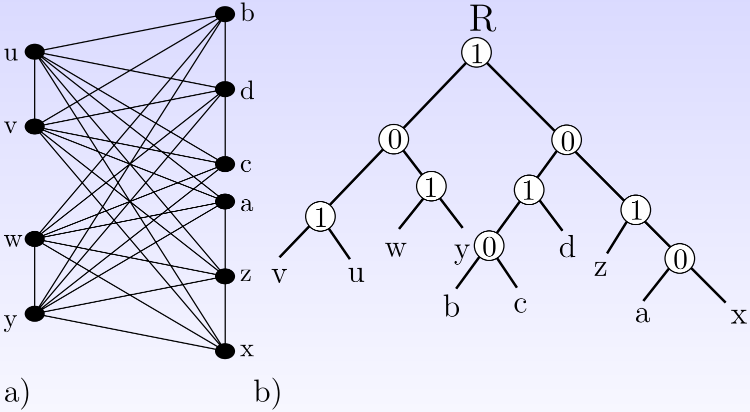

FIGURE: a) A cograph G. b) An embedding of the cotree T_G of G .

Properties of the cotree

- Vertices of the cotree can be labelled with 0 (parallel) and 1 (series).
- From G to $\overline{G}$ just exchange 0's and 1's in the cotree. So one extra bit is enough to encode both of them.
- $xy \in E$ iff $LCA(x, y)$ in the cotree is labelled with 1
- The cotree provides an exact coding of the graph in $O(|V(G)|)$. And the query $xy \in E(G)$? can be answered in $O(1)$ using LCA operations.

Twins

Twins

$x, y \in V$ are false- (resp. true-) **twins** if $N(x) = N(y)$ (resp. $N(x) \cup \{x\} = N(y) \cup \{y\}$.
x, y are false twins in G iff x, y are true twins in $\overline{G}$.

Elimination scheme

G is a cograph iff there exists an ordering of the vertices s.t. x_i has a twin (false or true) in $G\{x_{i+1}, \ldots x_n\}$

Cograph applications

- Fork, Join operations.
- Series parallel electrical networks
- Series-parallel orders (applications to scheduling)
- Quantum physics : "Two-colorable graph states with maximal Schmidt measure" Simone Severini1, University of York, U.K. 2005

Other applications

1. Redondancy elimination in graphs
2. Applications of quasi-twins :
   data compression in bipartite graphs,
   Identifying customers : if you change your phone card but keep the same set of correspondants
   (FBI . . .)

Not so easy algorithmic questions

- How to recognize and certify in linear time, if a graph is a cograph ?
  Yes case, build a cotree.
  No case, exhibit a P_4.
- How to compute in linear time the classes of (false) twins ?

Eventually the class of cograph has :

- A forbidden induced subgraph characterization
- A recursive definition and a tree structure
- An efficient encoding
- An elimination scheme

Using cotrees one can plynomially solve on cographs, NP-complete problems in the general case :

- Maximum clique
- Coloration
- If G is connected then $Diameter(G) \leq 2$
- Eighenvalues ...

Using the cotree in a bottom up way

- Max clique
  Consider the cotree as an expression to evaluate with the following rules :
  put a 1 on a leaf
  interpret a 1 (resp. 0) node of the cotree as a $+$ (resp. max)
- Min coloration
  same rules
- Therefore Max clique $=$ Min Coloration
  $\omega(G) = \chi(G)$ and cographs are perfect graphs

If the cotree is given, Max clique and Min coloration can be computed in $O(|V(G)|)$ for a cograph G.
Else we need to compute the cotree and the algorithm requires $O(|V(G)| + |E(G)|)$.

But they are not so simple (a cograph may have an exponential number of maximal cliques !).
This is why last year we had 2 internships introducing and studying extensions of cographs, namely "switch cographs" and "k-cographs".
Keeping the tree-structure but allowing new operations.

Exercices and problems

1. (Research problem) Find efficient algorithms to compute quasi-twins and generalize to community detections in social networks, in a dynamic settings.
2. How to certify some cograph elimination scheme.
3. Find a polynomial algorithm for graph isomorphism for cographs