MPRI
Foundations of proof systems 1
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Exercise 1 *Consider a language with three sorts of terms:* point, line *and* scalar*, two predicate symbols* $=$ *with arity* $\langle \text{scalar}, \text{scalar} \rangle$ *and* $\in$ *with arity* $\langle \text{point}, \text{line} \rangle$ *and two function symbols* d*,* distance*, with arity* $\langle \text{point}, \text{point}, \text{scalar} \rangle$ *and* b*,* bisector*, with arity* $\langle \text{point}, \text{point}, \text{line} \rangle$*. Let* Γ *be the set containing the propositions*

$$\forall x \forall y \forall z \; (x \in b(y,z) \Leftrightarrow d(x,y) = d(x,z))$$

$$\forall x \forall y \forall z \; ((x = y \wedge y = z) \Rightarrow x = z)$$

and A *a proposition stating that if two bisectors of the triangle* xyz *intersect at a point* w*, then the three bisectors intersect at this point:*

$$\forall w \forall x \forall y \forall z \; ((w \in b(x,y) \wedge w \in b(y,z)) \Rightarrow w \in b(x,z))$$

Write a proof of the sequent $\Gamma \vdash A$*.*

Exercise 2 *1. Prove that if the sequents* $\Gamma \vdash B$ *is provable in Natural Deduction, then so is the sequent* $\Gamma, A \vdash B$*.*

2. Prove that if the sequents $\Gamma, A \vdash B$ *and* $\Gamma \vdash A$ *are provable in Natural Deduction, then so is the sequent* $\Gamma \vdash B$*.*

3. Prove that if the sequent $\Gamma, A, B \vdash C$ *is provable, then the sequent* $\Gamma, A \wedge B \vdash C$ *is provable.*

Exercise 3 *Eliminate the cuts in the proof*

$$\dfrac{\dfrac{\overline{\exists x\,(P(x) \Rightarrow P(x)) \vdash \exists x\,(P(x) \Rightarrow P(x))}\;\text{axiom}}{\vdash \exists x\,(P(x) \Rightarrow P(x)) \Rightarrow \exists x\,(P(x) \Rightarrow P(x))}\;\Rightarrow\text{-intro} \qquad \dfrac{\dfrac{\overline{P(c) \vdash P(c)}\;\text{axiom}}{\vdash P(c) \Rightarrow P(c)}\;\Rightarrow\text{-intro}}{\vdash \exists x\,(P(x) \Rightarrow P(x))}\;\exists\text{-intro}}{\vdash \exists x\,(P(x) \Rightarrow P(x))}\;\Rightarrow\text{-elim}$$

Exercise 4 *Find a proof* π *that contains a single cut but such that eliminating this cut in* π *creates other cuts.*

Exercise 5 *1. Prove that a proof that is constructive, cut-free and without any axioms, ends with an introduction rule.*

2. Show that each hypothesis is necessary.

Exercise 6 *Consider a set* E *and the set* $R = \{x \in E \mid \neg x \in x\}$*. Consider the rule*

$$x \in R \longrightarrow x \in E \wedge \neg x \in x$$

1. Prove the sequent $R \in R \vdash \bot$*,*

2. *prove the sequent* $\vdash \neg R \in R$,
3. *prove the sequent* $R \in E \vdash R \in R$,
4. *prove the sequent* $R \in E \vdash \bot$,
5. *prove the sequent* $\vdash \neg R \in E$.
6. *Eliminate the cuts in this proof.*

Exercise 7 1. *Find a model where the proposition* $P(a)$ *is not valid.*

2. *Is the proposition* $P(a)$ *provable?*
3. *Find a model where the proposition* $\neg P(a)$ *is not valid.*
4. *Is the proposition* $\neg P(a)$ *provable?*
5. *Find a model valued in a Boolean algebra, where the proposition neither* $P(a)$ *nor* $\neg P(a)$ *are valid.*

Exercise 8 *Find a model valued in a Heyting algebra, where the proposition* $P(a) \vee \neg P(a)$ *is not valid.*