**Lemma 0.1.** *For a positive interval, $I_i^+$ defined on $\psi$, the eventually statement ($\Diamond_{[a,b]}\psi$) is true starting from start timestamp, $t_r^{\Diamond_{[a,b]}\psi}$, up to finish timestamp, $t_f^{\Diamond_{[a,b]}\psi}$, if the width of next negative interval ($I_{i+1}^-$) is greater than $b-a$ where:* $t_r^{\Diamond_{[a,b]}\psi} = t_r^{\psi} - b$ *and* $t_f^{\Diamond_{[a,b]}\psi} = t_f^{\psi} - a$

*Proof.* According to the eventually definition, $\Diamond_{[a,b]}\psi$ is true if there exists a time $t' \in [t+a, t+b]$ such that $\psi$ is satisfied.

Assume $t_r^{\psi}$ is the first time that $\psi$ is true. Then, the following condition is always true:

$$t_r^{\Diamond_{[a,b]}\psi} + a < t_r^{\psi} < t_r^{\Diamond_{[a,b]}\psi} + b$$

Since we looking for the earliest time that the condition is met, the rising time of $\Diamond_{[a,b]}\psi$ is equal to: $t_r^{\Diamond_{[a,b]}\psi} = t_r^{\psi} - b$. Similarly, assume $t_f^{\psi}$ is the last instance that $\psi$ is true. Then,

$$t_f^{\Diamond_{[a,b]}\psi} + a < t_f^{\psi} < t_f^{\Diamond_{[a,b]}\psi} + b$$

is always true.

Since we are looking for the latest time that the condition is met, the falling time of $\Diamond_{[a,b]}\psi$ is equal to:

$$t_f^{\Diamond_{[a,b]}\psi} = t_f^{\psi} - a$$

Since the width of the next negative interval is greater than $b-a$, we can show that the computed rising timestamp corresponding to the next positive interval is not less than the falling timestamp corresponding to the current interval (i.e. $t_{f_i}^{\Diamond_{[a,b]}\psi} < t_{r_{i+1}}^{\Diamond_{[a,b]}\psi}$):

$${}^{\psi}_{f_i} - a < t_{r_{i+1}}^{\psi} - b$$

is always true since the width of the next negative interval is greater than $b-a$ ($t_{r_{i+1}}^{\psi} - t_{f_i}^{\psi} > b-a$). $\square$

**Lemma 0.2.** *For a positive interval, $I_i^+$ on $\psi$, the statement $\Box_{[a,b]}\psi$ is true starting from start timestamp, $t_r^{\Box_{[a,b]}\psi}$, up to finish timestamp, $t_f^{\Box_{[a,b]}\psi}$, if the width of $I_i^+$ is greater than $b-a$, where $t_r^{\Box_{[a,b]}\psi} = t_r^{\psi} - a$ and $t_f^{\Box_{[a,b]}\psi} = t_f^{\psi} - b$.*

*Proof.* According to the globally definition, $\Box_{[a,b]}\psi$ is true if for all time instances $t' \in [t+a, t+b]$, $\psi$ is satisfied. In order to find the rising timestamp of $\Box_{[a,b]}\psi$, we are looking for the first time that $\psi$ is true.

Assume $t_r^{\psi}$ is the first time that $\psi$ is true. Then,

$$t_r^{\psi} < t_r^{\Box_{[a,b]}\psi} + a$$

is true.

Since we are looking for the earliest time, the rising time of $\Box_{[a,b]}\psi$ is equal to:

$$t_r^{\Box_{[a,b]}\psi} = t_r^{\psi} - a$$

.

Similarly, assume $t_f^{\psi}$ is the last instance that $\psi$ is true. Then,

$$t_f^{\psi} > t_f^{\Box_{[a,b]}\psi} + b$$

is true.

Since we are looking for the latest time, the falling time of $\Box_{[a,b]}\psi$ is equal to:

$$t_f^{\Box_{[a,b]}\psi} = t_f^{\psi} - b$$

.

Since the width of positive interval is greater than $b-a$, we can show that the computed falling timestamp is always greater than the falling one (i.e. $t_f^{\Box_{[a,b]}\psi} > t_r^{\Box_{[a,b]}\psi}$):

$$t_f^{\psi} - b > t_r^{\psi} - a$$

which is always true since the width of the positive interval is greater than $b - a$ ($t_r^{\psi} - t_f^{\psi} > b - a$). $\Box$

**Definition 0.1.** Overlapped Intervals, A positive interval on a boolean signal, $I^+_{\psi_{1_i}}$ overlaps with another positive interval on another boolean signal, $I^+_{\psi_{2_j}}$ if and only if
$\exists t' \in I^+_{\psi_{1_i}}, \exists t'' \in I^+_{\psi_{2_j}}\ s.t.\ (s,t') \models \psi_1 \wedge (s,t'') \models \psi_2 \wedge t' = t''$

**Lemma 0.3.** *An until statement ($\psi_1 \mathcal{U} \psi_2$) is false for non-overlapped intervals of $\psi_1$ and $\psi_2$.*

*Proof.* If $I^+_{\psi_{1_i}}$ and $I^+_{\psi_{2_j}}$ does not overlap, it means there does not exist a time that both $\psi_1$ and $\psi_2$ are satisfied (i.e. $\nexists t' \in I^+_{\psi_{1_i}} \cup I^+_{\psi_{1_i}}\ s.t.\ (s,t) \models \psi_1 \wedge \psi_2$.)

According to until operator definition, there should be a point that $\psi_2$ is satisfied and $\psi_1$ is satisfied up to that point. Therefore, the result of evaluation is false if there does not exist a pair of intervals that overlap. $\Box$

**Lemma 0.4.** *Assume that $I_1$ and $I_2$ are two positive intervals defined on boolean signals $\psi_1$ and $\psi_2$ respectively. If the length of overlapping interval of $I_1$ and $I_2$ is greater than a (i.e. $I^+_{\psi_{1_i}} \cap I^+_{\psi_{2_j}} > a$), $\psi_1 \mathcal{U}_{[a,b]} \psi_2$ is true starting from $t_r^{\mathcal{U}}$ and up to $t_f^{\mathcal{U}}$ so, we have:*

$$t_r^{\mathcal{U}} = max(t_{r_i}^{\psi_1}, t_{r_j}^{\psi_2} - b)$$

*and*

$$t_f^{\mathcal{U}} = min(t_{f_i}^{\psi_1}, t_{f_i}^{\psi_2}) - a$$

*where*

$t_{r_i}^{\psi_1}$ *and* $t_{r_j}^{\psi_2}$ *are timestamps corresponding to rising edges of positive intervals of* $\psi_1$ *and* $\psi_2$*, and* $t_{f_i}^{\psi_1}$ *and* $t_{f_j}^{\psi_2}$ *are timestamps corresponding to falling edges of positive intervals of* $\psi_1$ *and,* $\psi_2$ *respectively.*

*Proof.* To ensure $\psi_1 \mathcal{U}_{[a,b]} \psi_2$ is true starting from $t_r^{\mathcal{U}}$ until $t_f^{\mathcal{U}}$, three conditions must be satisfied:

i) computed $t_r^{\mathcal{U}}$ must be less than $t_f^{\mathcal{U}}$ to indicate a positive interval and

ii) $max(t_{r_i}^{\psi_1}, t_{r_j}^{\psi_2} - b)$ is the start time and iii) $min(t_{f_i}^{\psi_1}, t_{f_i}^{\psi_1}) - a$ is the finish time.

**Part i)** To prove the computed rising time is less than the falling one, we need to show that

$$t_r^{\mathcal{U}} < t_f^{\mathcal{U}}$$

or

$$max(t_{r_i}^{\psi_1}, t_{r_j}^{\psi_2} - b) < min(t_{f_i}^{\psi_1}, t_{f_i}^{\psi_2}) - a$$

Assuming

$$t_{r_i}^{\psi_1} > t_{r_j}^{\psi_2} - b$$

we have

$$t_{r_i}^{\psi_1} < min(t_{f_i}^{\psi_1}, t_{f_i}^{\psi_2}) - a$$

If $t_{f_i}^{\psi_1} < t_{f_i}^{\psi_2}$, it yields $t_{r_i}^{\psi_1} < t_{f_i}^{\psi_1} - a$

which is always true since the width of the positive interval of $\psi_1$ is greater than $a$.

If $t_{f_i}^{\psi_2} < t_{f_i}^{\psi_1}$, it yields $t_{r_i}^{\psi_1} < t_{f_i}^{\psi_2} - a$

which is always true since the width of overlapping interval of $\psi_1$ and $\psi_2$ is greater than $a$.

Assuming

$$t_{r_j}^{\psi_2} - b > t_{r_i}^{\psi_1}$$

we have

$$t_{r_j}^{\psi_2} - b < min(t_{f_i}^{\psi_1}, t_{f_i}^{\psi_2}) - a$$

If $t_{f_i}^{\psi_1} < t_{f_i}^{\psi_2}$, it yields

$$t_{r_j}^{\psi_2} - b < t_{f_i}^{\psi_1} - a$$

or

$$t_{f_i}^{\psi_1} - t_{r_j}^{\psi_2} > a - b$$

It's obvious that $a - b < 0$ and $t_{f_i}^{\psi_1} - t_{r_j}^{\psi_2} > 0$ since the minimum width of overlapping interval of $\psi_1$ and $\psi_2$ is greater than 0.

If $t_{f_i}^{\psi_2} < t_{f_i}^{\psi_1}$, it yields

$$t_{r_j}^{\psi_2} - b < t_{f_i}^{\psi_2} - a$$

or

$$t_{f_i}^{\psi_2} - t_{r_j}^{\psi_2} > a - b$$

which is true since the width positive interval of $\psi_2$ is greater than 0 and $a - b < 0$.

**Part ii)**: We need to show $t_r^{\mathcal{U}}$ is the first instance of time that $\psi_1 \mathcal{U}_{[a,b]} \psi_2$ is true. Assume $t_{r_2}^{\mathcal{U}}$ is the first time that $\psi_2$ is satisfied. So, $t_{r_2}^{\mathcal{U}} \in [t_r^{\mathcal{U}} + a, t_r^{\mathcal{U}} + b]$. As a result, $t_r^{\mathcal{U}} + a < t_{r_2}^{\mathcal{U}} < t_r^{\mathcal{U}} + b$. Since we look for the first time that $\psi_1 \mathcal{U}_{[a,b]} \psi_2$ is satisfied, so:

$$t_r^{\mathcal{U}} > t_{r_2}^{\mathcal{U}} - b \tag{1}$$

Besides, $\psi_1$ must be true from $t_r^{\mathcal{U}}$ until $t_{r_2}^{\mathcal{U}}$. So, we have:

$$t_r^{\mathcal{U}} > t_{r_1}^{\mathcal{U}} \tag{2}$$

$$t_{r_2}^{\mathcal{U}} > t_{f_1}^{\mathcal{U}} \tag{3}$$

Condition 3 is always true since the minimum overlapping interval of $\psi_1$ and $\psi_2$ is greater that 0. Combining 1 and 2, one can compute the first time (rising time) that $\psi_1 \mathcal{U}_{[a,b]} \psi_2$ is satisfied as: $t_r^{\mathcal{U}} = max(t_{r_i}^{\psi_1}, t_{r_j}^{\psi_2} - b)$

**Part iii)**: We need to show that $t_f^{\mathcal{U}}$ is the final time that $\psi_1 \mathcal{U}_{[a,b]} \psi_2$ is true.

There are two cases that indicates a falling timestamp:

a) there does not exist a time $t' > t_{f_2}^{\mathcal{U}} \in [t_f^{\mathcal{U}} + a, t_f^{\mathcal{U}} + b]$ such that $\psi_2$ is true and

b) there exists a time $t' > t_{f_2}^{\mathcal{U}} \in [t_f^{\mathcal{U}} + a, t_f^{\mathcal{U}} + b]$ such that $\psi_2$ is true <u>but</u> there exists a time $t'' > t_{f_2}^{\mathcal{U}} \in [t_f^{\mathcal{U}} + a, t_f^{\mathcal{U}} + b]$ such that $\psi_1$ is not true.

For the first case, assume $t_{f_2}^{\mathcal{U}}$ is the final time that $\psi_2$ is satisfied. So, $t_{f_2}^{\mathcal{U}} \in [t_f^{\mathcal{U}} + a, t_f^{\mathcal{U}} + b]$.

As a result, $t_f^{\mathcal{U}} + a < t_{f_2}^{\mathcal{U}} < t_f^{\mathcal{U}} + b$. As we look for final time that $\psi_1 \mathcal{U}_{[a,b]} \psi_2$ is satisfied, so:

$$t_f^{\mathcal{U}} < t_{f_2}^{\mathcal{U}} - a \tag{4}$$

In second case, $\psi_1$ should be true from $t_f^{\mathcal{U}}$ until $t_{f_2}^{\mathcal{U}}$. Then:

$$t_f^{\mathcal{U}} > t_{r_1}^{\mathcal{U}} \tag{5}$$

$$t_{f_1}^{\mathcal{U}} > t_{f_2}^{\mathcal{U}} \tag{6}$$

Since we are looking for the final time that $\psi_2$ is satisfied but $\psi_1$ is not, we can replace $t_{f_2}$ with $t_f + a$. As a result,

$$t_f^{\mathcal{U}} > t_{f_1}^{\mathcal{U}} - a \tag{7}$$

Combining 4 and 7, one can compute the final time that $\psi_1 \mathcal{U}_{[a,b]} \psi_2$ is satisfied as $t_f^{\mathcal{U}} = min(t_{f_i}^{\psi_1}, t_{f_i}^{\psi_2}) - a$.

Condition 5 is satisfied if we replace $t_f^{\mathcal{U}}$ with $min(t_{f_i}^{\psi_1}, t_{f_i}^{\psi_2}) - a$.

So, we have $min(t_{f_i}^{\psi_1}, t_{f_i}^{\psi_2}) - a > t_{r_1}^{\mathcal{U}}$ which is always true because both the width of positive interval of $\psi_1$ and the minimum overlapping interval of $\psi_1$ and $\psi_2$ are greater than $a$. □

**Lemma 0.5.** *If the width of overlapping interval of $\psi_1$ and $\psi_2$ is less than a (a computed falling timestamp is less than previous computed rising one), we can cancel them out and $\psi_1 \mathcal{U}_{[a,b]} \psi_2$ remains false in that interval ($[t_{f_i}^{\mathcal{U}}, t_{r_i}^{\mathcal{U}}]$).*

*Proof.* According to Lemma 0.4, $t_{r_i}^{\mathcal{U}}$ is the first time that $\psi_1 \mathcal{U}_{[a,b]} \psi_2$ is true. So, for all $t_{f_{i-1}}^{\mathcal{U}} < t < t_{r_i}^{\mathcal{U}}$, $\psi_1 \mathcal{U}_{[a,b]} \psi_2$ is false. Besides, $t_{f_i}^{\mathcal{U}}$ is the final time that $\psi_1 \mathcal{U}_{[a,b]} \psi_2$ is true. So, for all $t_{f_i}^{\mathcal{U}} < t < t_{r_{i+1}}^{\mathcal{U}}$, $\psi_1 \mathcal{U}_{[a,b]} \psi_2$ is false. As a result, for all times $t \in [t_{f_i}^{\mathcal{U}}, t_{r_i}^{\mathcal{U}}]$, $\psi_1 \mathcal{U}_{[a,b]} \psi_2$ is false. □