Politecnico di Milano

Financial Engineering

Final Project

Structured Products: Normal and Shifted LMM

Group 11

Ignazio Bovo
Matteo Paggiaro

June 14, 2019

In recent times in the Eurozone the financial crisis and the politics of the ECB made the existence of negative interest rate possible. Clearly all classica interest rate model are based on the assumption of positive interest rate, hence to circumvent this problem new models for pricing interest rate derivatives have been introduced.
In this project report we will discuss the results obtained in pricing a portfolio of interest rate derivatives (such as caps, swaps and digital floor) using two different model for the underlying forward 3-month LIBOR rate of interest: the normal model (also known as the Bachelier Model) and the shifted Black Model. First we are going to briefly present the two models and discuss their pourpose and peculiarity, next we report the result obtained by pricing the instruments. Next we present the calibration process to obtain the term structure of implied volatility. Finally we discuss different hedging strategies to manage risk.

1 Portfolio Description

We are given the following portfolio

- Payer swap with settlement date `18-Sep-15`, maturity in `10y` and strike .6%. Premia paid semestrally
- Receiver swap with settlement `4-Sep-15`, maturity in `8y` and strike 1.0%. Premia paid semestrally
- Short Cap with settlement `11-Aug-15`, maturity in `5y` and strike .3$. Premia paid semestrally
- Long Cap with settlement `22-Jul-15` maturity in `6.5y` and strike −.35%. Premia paid semestrally
- Long Floor with settlement `9-Jul-15`, maturity in `10y` and strike −1%. Premia paid quarterly
- Long digital floor with settlement `13-Jul-15`, maturity in `2.5y` strike −.24%. Premia paid semestrally

The instruments considered have various settlement dates, however the evaluation date is `8-Apr-2016`. To price the instruments in the portfolio, we ignore the amounts paid before the evaluation date.
Moreover since the libor for the first payment has already been fixed at evaluation date, we need to update our pricing formula to account for that.

2 Models Description

Denote by $L_i(t)$ the forward 3-month Libor at evaluation date t, for the time period $T_i : T_i+$ 3 months.

The Normal model proposes the following time evolution:

$$dL_i(t) = \sigma_i^N(t) \cdot dW(t) \tag{2.1}$$

Where $\sigma_i^N(t)$ is the volatility associated to the normal model for the Libor from T_i up to T_i+ 3 months.
The shifted black model instead proposes the following dynamic

$$dL_i(t) = [L_i(t) + \alpha_i]\sigma_i^S(t) \cdot dW(t) \tag{2.2}$$

Where $\sigma_i^S(t)$ is the volatility associated to the shifted black model for the Libor from T_i up to T_i+ 3 months.
For notation pourposes define the following quantities The two models provides the following risk neutral-formula for caps/floor, swaps and digital floor. The two models respectively given the following formulas

	Normal	Shifted Black
Caplet	$\delta_i B(0, T_i + \delta_i)[F_i(0)\Phi(d) + K\phi(d)]$	$\delta_i B(0, T_i + \delta_i)[(F_i(0) + \alpha)\Phi(d_1) - (K + \alpha)\phi(d_2)]$
Floorlet	$\delta_i B(0, T_i + \delta_i)[K(0)\Phi(-d) + F\phi(d)]$	$\delta_i B(0, T_i + \delta_i)[(K(0) + \alpha)\Phi(-d_1) - (F_i + \alpha)\phi(-d_2)]$

Where

$$d_{1,2} = \frac{\log((F_i(0) + \alpha)/(K + \alpha)) \pm \frac{1}{2}\sigma T}{\sigma^S\sqrt{T}}$$
$$d = \frac{F_i(0) - K}{\sigma^N\sqrt{T}}$$

(a) Net Present Value of the portfolio

Model Calibration

We need to calibrate the term structure of volatility

$$\{\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_N\}$$

Associated to each libor payment date $T_1, \ldots, T_N$, for each of the two models
Since each volatility $\{\sigma_i\}$ is associated to a caplet having underlying interest rate $L_i(t)$, we need to strip the caplet volatility (also known as spot volatilities) from the cap volatilities (also known as flat volatilities) provided in the dataset.
The technique used is the one explained during class, imposing linear interpolation between caplet volatilities, We summarize the volatility surface obtained in the following tables where the volatilities in the same row are associated to caplets with the same payment date and volatilities in the same column are associated to caplets with the same price. **TABLES HERE**

Pricing

Once we obtained the volatility surface

$$(K, T) \mapsto \sigma(K, T)$$

For the set of dates T given by the caplet payment dates, and the cap strikes K, to find the values for σ for the dates and the strikes required by our portfolio instruments spline interpolation is used. The settlement date is before the valuation date (`8-Apr-2016`), hence we do not consider payment that occour before `8-Apr-2016`. To compute the first payment (whose amount is already know at ev. date) we use the libor 3m fixing from the table provided. The swaps are then priced using the bootstrap curve, and the caps floor and digital floor are priced according to the selected model (Normal or shifted Black).
Summing up the contributions of the prices of swaps, caps, floors and the digital floor, we obtain the following Net Present Value, here displayed according to the two different models:

Normal	Shifted
3.787 mln€	3.786 mln€

(b) Portfolio hedging

We use a delta-vega hedging strategy for our portfolio using buckets of `2y,4y,7y,10y`.
For hedging purposes we use:

- 4 ATM swaps, with maturity resp. of 2,4,7,10 and different constant notionals $x_1, \ldots x_4$
- 4 caps with maturity resp. $2, 4, 7, 10$ years and constant notionals respectively $x_5, \ldots, x_8$ with strikes equivalent to the corresponding swap rates

To require that the hedged portfolio is delta neutral is equivalent to solve the following linear system:

$$\begin{aligned}
\Delta_{2y} &= \Delta_{p,2y} + x_1\Delta_{1,2y} + \cdots + x_8\Delta_{8,2y} &&= 0 \\
\Delta_{4y} &= \Delta_{p,4y} + x_2\Delta_{2,4y} + \cdots + x_8\Delta_{8,4y} &&= 0 \\
\Delta_{7y} &= \Delta_{p,7y} + x_3\Delta_{3,7y} + \cdots + x_8\Delta_{8,7y} &&= 0 \\
\Delta_{10y} &= \Delta_{p,10y} + x_4\Delta_{4,10y} + \cdots + x_8\Delta_{8,10y} &&= 0
\end{aligned}$$

Speaking about vega-neutrality, we observe that the payoff of the IR swaps doesn't depend on the volatility. Hence the only objects that contribute to changing the vega are caps and floors.
For this reason, the linear system associated to vega-neutrality is smaller than the one associated to the delta:

$$\begin{aligned}
\mathcal{V}_{2y} &= \mathcal{V}_{p,2y} + x_5\mathcal{V}_{1,2y} + \cdots + x_8\mathcal{V}_{4,2y} &&= 0 \\
\mathcal{V}_{4y} &= \mathcal{V}_{p,4y} + x_6\mathcal{V}_{2,4y} + \cdots + x_8\mathcal{V}_{4,4y} &&= 0 \\
\mathcal{V}_{7y} &= \mathcal{V}_{p,7y} + x_7\mathcal{V}_{3,7y} + x_8\mathcal{V}_{4,7y} &&= 0 \\
\mathcal{V}_{10y} &= \mathcal{V}_{p,10y} + x_8\mathcal{V}_{4,10y} &&= 0
\end{aligned}$$

In order to compute the values for delta in each bucket we use the coarse-grained shift technique, shifting the forward rate curve by weights between 0 and 1 base points, with the +1 shift in correspondence of the bucket considered. Similarly to compute the vega we apply the coarse-grained shift to the spot volatility surface.

Preliminary remarks:

Observe that we can give an "a priori" estimation of the delta and gamma for the objects we own in our portfolio and for the ones we use for hedging.
For example, a payer IR swap surely has positive delta, while for a receiver IR swap the delta will be negative. For what it concerns caps and floors, the situation is similar. Suppose that we have a long position in a cap or floor. Then the delta of the cap will be positive, and the delta of the floor negative.
Regarding the vega, we have already explained why swaps don't give any contribution to it. On the other hand, holding a long position in either a cap or a floor will yield a positive vega. For this reason, the only way to have vega neutrality starting from a positive portfolio vega in a given bucket is to sell short either a cap or a floor.
In the case of our portfolio, the vega is positive in all four buckets. In fact, we are short of a cap with relatively small notional and early maturity, while being long of another cap with longer maturity, two floors and the digital floor.
Hence it seems natural that, in order to provide a proper vega hedging, we will have to sell short either caps or floors expiring in the four reference buckets.
This short-selling will have a double effect: while the risk associated with a shift in the volatility will be hedged as asked, it is likely that the global delta of the portfolio with four new short positions on caps or floors will have increased.
For this task we use swaps. Being ATM swaps, their price is null, while their delta isn't. Hence the price of our hedging will be negative: we will receive money from the short selling.

Results:

Solving the linear system yields the following results for nominals:

- Swap notionals (millions):

Bucket	Normal	Shifted
2y	178.8	178.7
4y	-14.0	-13.1
7y	-20.7	-18.6
10y	-6.27	-6.17

- Floor notionals (millions, short-sold):

Bucket	Normal	Shifted
2y	153.6	161.6
4y	69.6	70.1
7y	78.5	73.9
10y	11.5	10.1

From the results obtained, we see that the short selling of the four floors will ensure a value which is very close to the net present value of the original portfolio:

Normal	Shifted
3.99 mln€	3.78 mln€

In conclusion, we have guaranteed ourselves a hedge against possible future changes in market scenarios.

Which model should we prefer?

Given that with the normal LMM model the price in which we sell short the IR options is higher, we would choose this latter model for pricing and hedging.
However, numerical results show that the two models are equivalent. In fact, the implied volatilities of the two models are derived starting from the same market prices, giving thus two different points of view of the same market situation.

(c) Steepening of the rates curve

In the case we are expecting a **steepening** of the interest rates curve wrt the maturities, we can build a portfolio that replicates a neutral vega position and an exposed delta.
In particular, we want the delta to be negative in the 2 years bucket and positive in the 10 years, thus reflecting the expectation of the trader in the evolution of the market. In our case, for a 2-10y steepening position of $4k$ €per basis point, the linear system becomes the following:

$$\begin{aligned}
\Delta_{2y} &= \Delta_{p,2y} + x_1\Delta_{1,2y} + \cdots + x_8\Delta_{8,2y} &&= -4\cdot 10^4 \\
\Delta_{4y} &= \Delta_{p,4y} + x_2\Delta_{2,4y} + \cdots + x_8\Delta_{8,4y} &&= 0 \\
\Delta_{7y} &= \Delta_{p,7y} + x_3\Delta_{3,7y} + \cdots + x_8\Delta_{8,7y} &&= 0 \\
\Delta_{10y} &= \Delta_{p,10y} + x_4\Delta_{4,10y} + \cdots + x_8\Delta_{8,10y} &&= +4\cdot 10^k
\end{aligned}$$

$$\begin{aligned}
\mathcal{V}_{2y} &= \mathcal{V}_{p,2y} + x_5\mathcal{V}_{1,2y} + \cdots + x_8\mathcal{V}_{4,2y} &&= 0 \\
\mathcal{V}_{4y} &= \mathcal{V}_{p,4y} + x_6\mathcal{V}_{2,4y} + \cdots + x_8\mathcal{V}_{4,4y} &&= 0 \\
\mathcal{V}_{7y} &= \mathcal{V}_{p,7y} + x_7\mathcal{V}_{3,7y} + x_8\mathcal{V}_{4,7y} &&= 0 \\
\mathcal{V}_{10y} &= \mathcal{V}_{p,10y} + x_8\mathcal{V}_{4,10y} &&= 0
\end{aligned}$$

Observe that, since the vega hedging is not affected from the steepening of the interest rates curve, the notionals of the caps don't change from point (b), hence neither does the price received from the short selling. The only thing that changes is the distribution of swap notionals, that is more steepened than the classic delta hedging, as shown in the tab below:

Bucket	Normal	Shifted
2y	151.0	150.8
4y	2.31	3.22
7y	-51.3	-49.2
10y	14.6	14.7

(d) Hedging the digital risk

Theoretical background

Having hedged the portolio in the delta and vega risk measures in point (b), there still remains an exotic risk within the portfolio that hasn't been hedged yet. This risk is called **digital**, for the fact that it is correlated to the presence of digital options in our portfolio.
The digital risk measures the risk associated to an increase in the slope of the volatility surface around the strike rate of the digital option.

Observe that increasing the slope around the strike will affect significantly only the value of the digital option and not the vanilla IR options. This is due to the digital correction in the price of a digital option, namely the fact that its value has a linear dependence with the partial derivative of the volatility surface wrt the strike.

Hedging in practice

In order to hedge this risk, we buy a cap spread with a spread between the strikes equal to 1 basis point centered in the strike of the digital floor. The nominals of the cap spread will be the same of the ones of the digital floor, rescaled by the digital payoff and multiplied by 10^4 (the inverse of 1 basis point). When the slope of the volatility curve moves, the changes in value of the digital floor will be absorbed by opposite changes in value of the cap spread, thus resulting in a hedge of the digital risk.

The price of this digital hedging will be:

Normal	Shifted
641 k€	639 k€

3 Comments

We have chosen to utilize floors for hedging since they let us buy less swap contracts, since they do a better job in handling the delta-sensitivity in this particular portfolio.
Moreover the shifted black model is skewed, while the normal model is symmetric; hence, given the initial forward curve, the shifted model is more appropriate when one expects interest rates to appear more positive despite the initial forward curve; meanwhile the normal model is more appropriate when one expects the interest rates to increase or decrease with equal probability given the initial forward curve.