1 UNIT

Introduction and Linear Programming

PART-1

Introduction, Basics of Operations Research.

CONCEPT OUTLINE : PART-1

Operations Research (OR) :

1. OR is the application of scientific methods, techniques and tools to problems involving the operations of systems so as to provide these in control of the operation with optimum solutions to the problem.
2. OR is an applied decision theory. It uses any scientific, mathematical or logical means to attempt to cope with the problems that confront the executive when he tries to achieve a thorough going rationality in dealing with his decision problems.
3. OR is the art of winning wars without actually fighting them.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 1.1. **What is OR ? What are the characteristics of OR techniques ?**

Answer

A. Operation Research (OR) :

1. OR is a scientific method of providing executive departments with a quantitative basis for decisions regarding the operations under their control.
2. OR is the application of scientific methods to problem arising from operations involving integrated systems of men, machines, and materials. It normally utilizes the knowledge and skill of an inter-disciplinary research team to provide the managers of such systems with optimum operating solutions.

B. Characteristics :

a. **Inter Disciplinary Approach :** The optimum solution of a problem is found by an inter disciplinary team comprising experts from different disciplines such as mathematics, statistics, economics, management, physics, computer science, engineering, etc.

b. **Systems Approach :** In operation research each problem is examined in its entirety. One part of organization (activity) has some effect on every other part of organization. Optimum operation of one part of the system may not be the optimum operation of all the activities. Therefore, manager must identify all possible interactions and determine their impact as a whole. The decision which is best for the organization as a whole is called an optimal decision.

Operation research utilizes scientific techniques to solve a problem.

Operation research tries to optimize total return by reducing the cost or loss and maximizing the profit.

In operation research, we cannot obtain perfect solution but the quality of decision is improved.

To solve complex problems computers are used.

Que 1.2. **Give an account of the origin and historical development of OR ?**

Answer

The main origin of operations research was during the second World War.

At that time, the military management in England called upon a team of scientists to study the strategic and tactical problems related to air and land defense of the country.

Since they were having very limited military resources, it was necessary to decide upon the most effective utilization of them, e.g., the efficient ocean transport, effective bombing, etc.

During World War II, the Military Commands of U.K. and U.S.A. engaged several inter-disciplinary teams of scientists to undertake scientific research into strategic and tactical military operations.

Their mission was to formulate specific proposals and plans for aiding the Military Commands to arrive at the decisions on optimal utilization of scarce military resources and efforts, and also to implement the decisions effectively.

As the name implies, 'Operations Research' was apparently invented because the team was dealing with research on (military) operations.

The work of this team of scientists was named as Operational Research in England.

The encouraging results obtained by the British OR teams quickly motivated the United States military management to start with similar activities.

Following the end of war, the success of military teams attracted the attention of industrial managers who were seeking solutions to their complex executive-type problems.

The most common problem was : what methods should be adopted so that the total cost is minimum or total profits maximum ?

11. The first mathematical technique in this field (called the simplex method or linear programming) was developed in 1947 by American mathematician, George B. Dantzig.
12. Since then, new techniques and applications have been developed through the efforts and cooperation of interested individuals in academic institutions and industry both.
13. Today, the impact of OR can be felt in many areas. A large number of management consulting firms are currently engaged in OR activities.
14. Apart from military and business applications, the OR activities include transportation system, libraries, hospitals, city planning, financial institutions, etc.

Que 1.3. **Why operations research is necessary in industry ?**

Answer

1. Operations research is necessary in industry due to the following reasons :

A. Complexity :

1. In a big industry, the number of factors influencing a decision has increased.
2. Situation has become big and complex because these factors interact with each other in a complicated fashion.
3. There is thus great uncertainty about the outcome of interaction of factors like technological, environmental, competitive, etc.
4. It is not easy to prepare a schedule which is both economical and realistic. This needs mathematical models, which, in addition to optimization, help to analyze the complex situation.
5. With such models, complex problems can be split up into simpler parts, each part can be analyzed separately and then the results can be synthesized to give insights into the problem.

B. Scattered Responsibility and Authority :

1. In a big industry responsibility and authority of decision making is scattered throughout the organization and thus the organization, if it is not conscious, may be following inconsistent goals.
2. Mathematical quantification of OR overcomes this difficulty to a great extent.

C. Uncertainty :

1. There is a great uncertainty about economic and general environment.
2. With economic growth, uncertainty is also increasing. This makes each decision costlier and time consuming.
3. OR is, thus, quite essential from reliability point of view.

D. Knowledge Explosion :

1. Knowledge is increasing at a very fast rate.

c. **In Agriculture :**
   i. OR approach needs to be equally developed on national or international basis.
   ii. The problem of optimal distribution of water from the various water resources is faced by each developing country and a good amount of scientific work can be done in this direction.

d. **Planning :**
   i. OR approach is equally applicable in both developing and developed economies.
   ii. In developing economies, there is a great scope of developing an OR approach towards planning.

e. **Public Utilities :**
   i. OR methods can also be applied in big hospitals to reduce waiting time of out-door patients and to solve the administrative problems.

Que 1.8. **Discuss the scope of OR in management.**

Answer

1. The areas of management where OR techniques have been successfully applied are :

A. **Allocation and Distribution :**
   a. Optimal allocation of limited resources such as men, machines, materials, time and money.
   b. Location and size of warehouses, distribution centres, retail depots, etc.
   c. Distribution policy.

B. **Production and Facility Planning :**
   a. Selection, location and design of production plants, distribution centres and retail outlets.
   b. Projects scheduling and allocation of resources.
   c. Preparation of forecasts for the various inventory items and computing economic order quantities and reorder levels.
   d. Determination of the number and size of the items to be produced.
   e. Maintenance policy and preventive maintenance.
   f. Scheduling and sequencing of production runs by proper allocation of machines.

C. **Procurement :**
   a. What, how and when to purchase at the minimum procurement cost.
   b. Bidding and replacement policies.
   c. Transportation planning and vendor analysis.

D. **Marketing :**
   a. Product selection, timing and competitive actions.
   b. Selection of advertising media.
   c. Demand forecasts and stock levels.
   d. Customer's preference for size, colour and packaging of various products.
   e. Best time to launch a new product.

E. **Finance :**
   a. Capital requirements, cash-flow analysis.
   b. Credit policies, credit risks, etc.
   c. Profit plan for the company.
   d. Determination of optimum replacement policies.
   e. Financial planning, dividend policies, investment and portfolio management, auditing etc.

F. **Personnel :**
   a. Selection of personnel, determination of retirement age and skills.
   b. Requirement policies and assignment of jobs.
   c. Wage / salary administration.

G. **Research and Development :**
   a. Determination of areas for research and development.
   b. Reliability and control of development projects.
   c. Selection of projects and preparation of their budgets.

Que 1.9. **"The hard problems are those for which models do not exist". Interpret this statement. Give some examples.**

UPTU 2011-12, Marks 05

Answer

1. Representation of the problem becomes difficult in the absence of model.
2. Analysis of the behaviour of the system for the purpose of improving its performance becomes more difficult.
3. New formulations without having any significant change in its frame for the problem takes more time and cost.
4. More assumptions are required for solving the problems in the absence of model.
5. In the absence of the model description of some aspects of a situation becomes more complex.
6. Without model 'if and what' type of questions, cannot the answered easily. Above points justify the statement that, "The hard problems are those for which models do not exist".

Que 1.10. **Define a model. List and describe various types of model used in operation research.** **UPTU 2012-13, Marks 10**

Answer

A. **Model :**

1. A model in OR is a simplified representation of an operation in which only the basic aspects of a typical problem under investigation are considered.
2. The objective of a model is to provide a means for analyzing the behaviour of the system for the purpose of improving its performance.
3. The main characteristics of a good model are given below :
   i. A good model should be capable of taking into account new formulations without having any significant change in its frame.
   ii. Assumptions made in the model should be as small as possible.
   iii. It should be simple and coherent. Number of variables used should be less.
   iv. It should be open to parametric type of treatment.
   v. It should not take much time in its construction for any problem.

B. **Classification of Model :**

a. **Models by Degree of Abstraction :** These models are based on the past data of the problems under consideration and can be categorised into :
   i. Language models, and
   ii. Case studies.

b. **Models by Functions :**
   i. **Descriptive Model :** It simply describes some aspects of a situation based on observation or survey. The result of an opinion poll represents a descriptive model.
   ii. **Predictive Model :** Such models can answer 'if and what' type of questions, *i.e.*, they make predictions regarding certain event.
   iii. **Normative Model :** Finally when a predictive model has been repeatedly successful, it can be used to prescribe source of action. Linear programming is a normative model because it prescribes what the managers ought to do.

c. **Model by Structure :**
   i. **Iconic Models :** These are pictorial representation of real systems and have the appearance or the real thing, *e.g.*, city maps, house blue prints, etc.
   ii. **Analogue Models :** These are more abstract than the iconic ones for there is no look-alike correspondence between these models and real life items. They are built by utilizing one set of properties to represent another set of properties *e.g.*, a network of pipes through which water is running.
   iii. **Mathematical Models :** These are most abstract in nature. They employ a set of mathematical symbols to represent the components of the real system. These are most general and precise.

d. **Models by Nature of Environment :**
   i. **Deterministic Models :** All the parameters and functional relationships are assumed to be known with certainty when the decision is to be made. Linear programming and break even models are the examples of this.
   ii. **Probabilistic Models :** Models in which at least one parameter or decision variable is a random variable are called the probabilistic or stochastic models.

e. **Models by the Extent of Generality :**
   i. **Specific Model :** When a model presents a system at some specific time, it is known as specific model. In these if the time factor is not considered they are termed as static and dynamic models.
   ii. **General Model :** Simulation and Heuristic models fall under this category. These models are mainly used to explore alternative strategies which have been overlooked previously.

Que 1.11. **What is meant by a mathematical model of a real situation ? Discuss the importance of models in the solution of OR problems.** **UPTU 2014-15, Marks 10**

Answer

A. **Mathematical Model of a Real Situation :**

1. It employs a set of mathematical symbols (letter, numbers, etc.) to represent the decision variables of the system under study.
2. These variables are related together by mathematical equation(s) / inequation (s) which describe the properties of the system.
3. A solution from the model is, then, obtained by applying well developed techniques.
4. The relationship between distance, velocity and acceleration is an example of mathematical model.

B. **Types of Mathematical Model :**

1. Mathematical techniques,
2. Statistical techniques,
3. Inventory models,

4. Allocation models.
5. Sequencing models.
6. Project scheduling by PERT and CPM.
7. Routing models.
8. Competitive models.
9. Queuing models, and
10. Simulation techniques and many more.

C. Mathematical Model of a Real Situation Based on Queuing Models :

1. Let us take an example, at what average rate must a clerk in a super market in order to ensure a probability of 0.90 that a customer will not have to wait longer than 12 min.
2. Customer arrives at the counter in Poisson fashion with mean rate of 15 per hour.
3. Service time has exponential distribution.

$$\lambda = 15/\text{hr} = \frac{15}{60} = \frac{1}{4} \text{ / minute}$$

4. Probability (waiting time $\geq$ 12) = 1 – 0.90 = 0.10

$$\therefore \int_{12}^{\infty} \frac{\lambda}{\mu} \cdot (\mu - \lambda) e^{-(\mu - \lambda)t} \, dt = 0.10$$

or $$\frac{\lambda}{\mu} (\mu - \lambda) \left[\frac{e^{-(\mu-\lambda)t}}{-(\mu - \lambda)} \right]_{12}^{\infty} = 0.10$$

or $$-\frac{1}{4\mu} [0 - e^{-(\mu - 0.25) 12}] = 0.10$$

or $$e^{3-12\mu} = 0.4\mu$$

or $$\mu = \frac{1}{2.48} \text{ / minute}$$

$\Rightarrow$ $$\mu = \frac{60}{2.48} = 24.2 \text{ customer/hour.}$$

D. Importance of Models in the Solution of OR Problem :

1. It provides a logical and systematic approach to the problem.
2. It indicates the scope as well as limitations of problem.
3. It helps in finding avenues for new research and improvement in a system.
4. It makes the overall structure of the problem more comprehensible and helps in dealing with the problem in its entirety.

PART-2

Linear Programming : Introduction and Scope, Problem Formation, Graphical Method, Simple Method, Primal and Dual Problem, and Sensitivity Analysis.

CONCEPT OUTLINE : PART-2

Linear Programming : The linear programming method is a technique for choosing the best alternative from a set of feasible alternatives, in situations in which the objective function as well as the constraints can be expressed as linear mathematical functions.

Graphical Method of Solution : Linear programming problem involving two decision variables can be solved by graphical method.

Simplex Method : It is an iterative method for finding out corner point solutions and taking them for optimality.

Principle of Duality : Associated with every LP problem there is another intimately related LP problem that is based upon the same data and having the same solution. The original LP problem is called primal and associated problem is called dual problem.

Sensitivity Analysis : It is undertaken to explore the effect of changes in LP parameters on the optimal solution.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 1.12. **What do you mean by linear programming problem (LPP) ? Describe the basic elements in a LPP.**

Answer

A. Linear Programming Problem :

1. Linear programming deals with the optimization (maximization or minimization) of a function of variables known as objective functions. It is subject to a set of linear equalities and / or inequalities known as constraints.
2. Linear programming is a mathematical technique, which involves the allocation of limited resources in an optimal manner, on the basis of a given criterion of optimality.
3. Linear programming is a technique for determining an optimum schedule of interdependent activities in view of the available resources.

4. The term programming means scheduling. Programming is just another word for 'Planning' and refers to the process of determining a particular plan of action from amongst several alternatives.
5. The word 'linear' indicates that all relationships involved in a particular problem are linear.

B. **Basic Elements in LPP :**

a. **Decision Variables and their Relationship :** The decision variables refer to candidates that are competing with one another for sharing the given limited resources.

b. **Well-defined Objective Function :** A linear programming problem must have a clearly defined objective function to optimize which may be either to maximize contribution by utilizing available resources or it may be to produce at the lowest possible cost by using a limited amount of productive factors.

c. **Presence of Constraints or Restrictions :** There must be limitations or constraints on the use of limited resources which are to be allocated among various competitive activities.

d. **Alternative Courses of Action :** There must be alternative courses of action, for example, it must be possible to make a selection between various combinations.

e. **Non Negative Restriction :** All decision variables must assume non-negative values as negative values of physical quantities is an impossible situation.
In other words, $x_1 \geq 0, x_2 \geq 0, \ldots\ldots, x_n \geq 0$.

Que 1.13. **How will you formulate a LPP mathematically ?**

Answer

Mathematically Formulation of the LPP : The procedure for mathematical formulation of a linear programming problem consists of the following major steps :

Step 1 : Study the given situation to find the key decisions to be made.

Step 2 : Identify the variables involved and designate them by symbols.

$$x_j \ (j = 1, 2, \ldots\ldots).$$

Step 3 : State the feasible alternatives which generally are : $x_j \geq 0$, for all j.

Step 4 : Identify the constraints in the problem and express them as linear inequalities or equations, LHS of which are linear functions of the decision variables.

Step 5 : Identify the objective function and express it as a linear function of the decision variables.

Que 1.14. ***XYZ* factory manufactures two articles *A* and *B*. To manufacture the article *A*, a certain machine has to be worked for 1.5 hours and in addition a craftsman has to work for 2 hours. To**

manufacture the article *B*, the machine has to be worked for 2.5 hours and in addition the craftsman has to work for 1.5 hours. In a week the factory can avail of 80 hours of machine time and 70 hours of craftsman's time. The profit on each article *A* is Rs. 50 and that on each article *B* is Rs. 40. If all the articles produce can be sold away, find how many of each kind should be produced to earn the maximum profit per week. Formulate the problem as LP model.

Answer

1. The data of the given problem can be summarized in the following tabular form :

Decision Variable	Article	Machine Time	Craftsman's Time	Profit Per Unit
x_1	A	1.5	2	Rs. 50
x_2	B	2.5	1.5	Rs. 40
Hours available (per week)		80 maximum	70 maximum	

a. **Formulation :** Let x_1 and x_2 be the number of articles A and articles B to be manufactured, respectively.

b. **Objective Function :**

1. Since total profit consists of profit derived from selling type A articles at Rs. 50 each plus the profit derived from selling type B articles at Rs. 40 each.
2. Thus, $50\,x_1$ is the profit earned by selling type A articles and $40\,x_2$ is the corresponding profit by selling type B articles.
3. As the factory wants to achieve the greatest possible profit (say Z), it can be stated algebraically by writing profit equation as :
$$\text{Maximize } Z = 50\,x_1 + 40\,x_2$$
4. In the form, the profit expression provides the objective function.

c. **Constraints :**

1. These are limitations or restrictions placed on availability or resources and can be constructed as follows :

i. Machine time for article A + Machine time for article B ≤ Available time on machine

i.e., $1.5\,x_1 + 2.5\,x_2 \leq 80$

ii. Craftsman time for article A + Craftsman time for article B ≤ Available craftsman time

i.e., $2\,x_1 + 1.5\,x_2 \leq 70$

iii. **Because, it is not possible to produce negative number of articles, we also include non-negative constraints.**

$$x_1 \geq 0, \quad x_2 \geq 0$$

2. Thus the appropriate mathematical formulation of the given problem as LP model is as shown below :

Maximize (total profit) $Z = 50\,x_1 + 40\,x_2$

Subject to the linear constraints

$$1.5\,x_1 + 2.5\,x_2 \leq 80$$
$$2\,x_1 + 1.5\,x_2 \leq 70$$
$$x_1, x_2 \geq 0$$

Que 1.15. **A firm can produce three types of cloth, say : *A*, *B* and *C*. Three kinds of wool are required for it, say : red, green and blue wool. One unit length of type *A* cloth needs 2 metres of red wool and 3 metres of blue wool ; one unit length of type *B* cloth needs 3 metres of red wool, 2 metres of green wool and 2 metres of blue wool ; and one unit of type *C* cloth needs 5 metres of green wool and 4 metres of blue wool. The firm has only a stock of 8 metres of red wool, 10 metres of green wool and 15 metres of blue wool. It is assumed that the income obtained from one unit length of type *A* cloth is Rs. 3.00, of type *B* cloth is Rs. 5.00, and of type *C* cloth is Rs. 4.00. Determine how the firm should use the available material so as to maximize the income from the finished cloth.**

Answer

1. The data of the problem can be summarised in the following tabular form :

Quality of wool	Type of cloth			Total quantity of wool available (in metres)
	A	*B*	*C*	
Red	2	3	0	8
Green	0	2	5	10
Blue	3	2	4	15
Income per unit length of cloth	Rs. 3.00	Rs. 5.00	Rs. 4.00	

2. Let x_1, x_2 and x_3 be the quantity (in metres) produced of cloth type *A*, *B* and *C* respectively.

3. Cloth *B* requires $3x_2$ metres of red wool and cloth *C* does not require red wool. Thus total quantity of red wool becomes :

$$2x_1 + 3x_2 + 0x_3 \text{ (red wool)}$$

4. Following similar agreements are for green and blue wools :

$$0x_1 + 2x_2 + 5x_3 \text{ (green wool)}$$
$$3x_1 + 2x_2 + 4x_3 \text{ (blue wool)}$$

5. Since not more than 8 metres of red, 10 metres of green and 15 metres of blue wool are available, the variable x_1, x_2, x_3 must satisfy the following restrictions :

$$2x_1 + 3x_2 \leq 8$$
$$2x_2 + 5x_3 \leq 10$$
$$3x_1 + 2x_2 + 4x_3 \leq 15$$

6. Also, negative quantities cannot be produced. Hence x_1, x_2, x_3 must satisfy the non negativity restrictions :

$$x_1 \geq 0, x_2 \geq 0, \text{ and } x_3 \geq 0$$

7. The total income from the finished cloth is given by

$$P = 3x_1 + 5x_2 + 4x_3$$

8. Thus the problem now becomes to find x_1, x_2, x_3 satisfying the restrictions and maximizing the profit function *P*.

Que 1.16. **How will you solve a LPP using graphical method ?**

Answer

1. Simple linear programming problems of two decision variable can be easily solved by using graphical method as follows :

Step 1 : Consider each inequality constraint as equation.

Step 2 : Plot each equation on the graph, as each one will geometrically represent a straight line.

Step 3 :

i. Shade the feasible region. Every point on the line will satisfy the equation of the line. If the inequality constraint corresponding to the line is $\leq$, then the region below the line lying in the first quadrant is shaded.

ii. The points lying in common region will satisfy all the constraints simultaneously. The common region thus obtain is called a feasible region.

Step 4 : Choose the convenient value of *Z* (say = 0) and plot the objective function line.

Step 5 : Pull the objective function line upto the extreme points of the feasible region. In the maximization case, this line will stop farthest from the origin and passing through at least one corner of the feasible region.

Step 6 : Read the coordinates of the extreme points selected in step 5, and find the maximum or minimum value of *Z*.

Que 1.17. **What are the limitations of graphical method for solving LPP ?**

Answer

A. **Limitations :**

1. For large problems having many limitations and constraints, the computational difficulties are enormous, even when assistance of large

digital computers is available. The approximations required to reduce such problems to meaningful sizes may yield the final results far different from the exact ones.

2. Another limitation of linear programming is that it may yield fractional valued answers for the decision variables, whereas it may happen that only integer values of the variables are logical. For instance, in finding how many lathes and milling machines to be produced, only integer values of the decision variables, say x_1 and x_2 are meaningful. Except when the variables have large values, rounding the solution values to the nearest integers will not yield an optimal solution. Such situations justify the use of special techniques like integer programming.

3. It is applicable to only static situations since it does not take into account the effect of time. The OR team must define the objective function and constraints which can change due to internal as well as external factors.

4. It assumes that the values of the coefficients of decision variables in the objective function as well as in all the constraints are known with certainty. Since in most of the business situations, the decision variable coefficients are known only probabilistically, it cannot be applied to such situations.

5. In some situations it is not possible to express both the objective function and constraints in linear form. For example, in production planning we often have non-linear constraints on production capacities like setup and takedown times which are often independent of the quantities produced.

Que 1.18. **Solve graphically the following LP problem :**

Maximize : $Z = 9x_1 + 3x_2$
subject to :

$$2x_1 + 3x_2 \le 13$$
$$2x_1 + x_2 \le 5$$
$$x_1, x_2 \ge 0$$

UPTU 2011-12, Marks 06

Answer

1. Maximize $Z = 9x_1 + 3x_2$
   Subject to constraints :

$$2x_1 + 3x_2 \le 13 \quad ...(1.18.1)$$
$$2x_1 + x_2 \le 5 \quad ...(1.18.2)$$
$$x_1, x_2 \ge 0$$

2. For $2x_1 + 3x_2 = 13$

x_1	0	6.5	2
x_2	4.3	0	3

3. For $2x_1 + x_2 = 5$

x_1	0	2.5	2
x_2	5	0	1

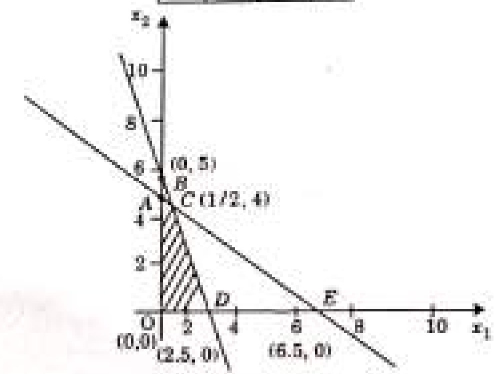

Fig. 1.18.1.

4. Intersection point of equations (1.18.1) and (1.18.2), we get

$$2x_1 + 3x_2 = 13$$
$$2x_1 + x_2 = 5$$
$$2x_2 = 8$$
$$x_2 = 4$$
$$x_1 = 1/2$$

covered area = $OACD$

Point (x_1, x_2)	Maximization Z, at corresponding points $Z = 9x_1 + 3x_2$
O (0, 0)	0
A (0, 4.3)	12.9
C (1/2, 4)	16.5
D (2.5, 0)	22.5

Que 1.19. **The manager of an oil refinery is to decide upon the optimal mix of two possible blending processes, of which the inputs and outputs per production run are as follows :**

	Input		Output	
Process	**Crude A**	**Crude B**	**Gasoline X**	**Gasoline Y**
1	5	3	5	8
2	4	5	4	4

The maximum amount available of crude A and B is 200 units and 150 units respectively. Markets requirements show that at least 100 units of gasoline X and 80 units of gasoline Y must be produced. The profits per production run from process 1 and 2 are Rs. 3 and

Rs. 4 respectively. Formulate the problem as a linear programming problem and solve the problem by graphical method.

UPTU 2012-13, 2013-14; Marks 10

Answer

1. Let the quantity for blending process 1 is x_1
   The quantity for blending process 2 is x_2
2. Then constraints : $5x_1 + 4x_2 \le 200$ ...(1.19.1)
   $3x_1 + 5x_2 \le 150$ ...(1.19.2)
   $5x_1 + 4x_2 \ge 100$ ...(1.19.3)
   $8x_1 + 4x_2 \ge 80$ ...(1.19.4)
   $\text{Max } Z = 3x_1 + 4x_2$
3. Now considering equality sign,
   $5x_1 + 4x_2 = 200$
   $3x_1 + 5x_2 = 150$
   $5x_1 + 4x_2 = 100$
   $8x_1 + 4x_2 = 80$
4. On plotting the graph as shown in Fig. 1.19.1 *ABCDEA* is a feasible region,

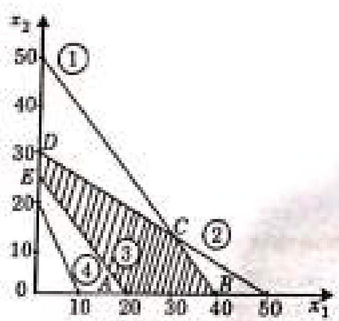

Fig. 1.19.1.

5. The optimum value of objective function occurs at one of the extreme (corner) points of the feasible region. The coordinates of extreme points are :

$$A = (20,0),\ B = (40,0),$$

$$C = \left(\frac{400}{13}, \frac{150}{13}\right),\ D = (0,30), \text{ and } E = (0,25).$$

6. Now, we compute the Z value corresponding to the extreme points :

Extreme Points	Coordinates (x_1, x_2)	Max $Z = 3x_1 + 4x_2$
A	(20, 0)	60
B	(40, 0)	120
C	(400/13, 150/13)	138.46 (Maximum)
D	(0,30)	120
E	(0,25)	100

7. Therefore max $Z = 138.46$ at $x_1 = \frac{400}{13}$, $x_2 = \frac{150}{13}$
8. Hence, to maximize the profit per production, the manager should run 400/13 blending of process 1 and 150/13 blending of process 2 per production.

Que 1.20. **Use the graphical method to solve the following LP problem.**
Maximize $Z = 3x_1 + 4x_2$
Subject to the constraints
i. $x_1 - x_2 = -1$ ii. $-x_1 + x_2 \le 0$ and $x_1, x_2 \ge 0$.

UPTU 2014-15, Marks 10

Answer

1. $$\text{Max. } Z = 3x_1 + 4x_2 \quad ...(1.20.1)$$

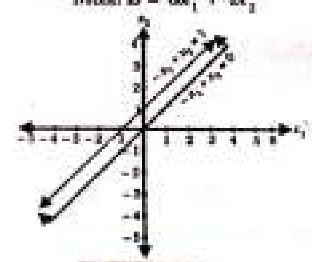

Fig. 1.20.1.

2. Given constraints : $x_1 - x_2 = -1$ ...(1.20.2)
   $-x_1 + x_2 \le 0$ ...(1.20.3)
3. From equation (1.20.2) and equation (1.20.3), on removing inequality sign and making right side positive.
   $-x_1 + x_2 = 1$ ...(1.20.4)
   $-x_1 + x_2 = 0$ ...(1.20.5)
4. It is clear from the Fig. 1.20.3 that two line ($x_1 - x_2 = -1$ and $-x_1 + x_2 \le 0$) will never cut each other.
5. Hence, the solution of equation (1.20.1) *i.e.*, Max. $Z = 3x_1 + 4x_2$ does not exist.

Que 1.21. **Enumerate the special cases that may arise while solving a linear programming problem ?**

Answer

A. **Infeasible Solution :**

i. When there is no value of the decision variables that satisfies all the constraints of LPP then it is said to have infeasible solution.

Example : Maximize $Z = 4x_1 + 2x_2$

subject to $2x_1 + 3x_2 \le 18$

$x_1 + x_2 \ge 10$

$x_1, x_2 \ge 0$

ii. **Solution :**

1. Line AB represents

$$2x_1 + 3x_2 = 18$$

and line CD represents

$$x_1 + x_2 = 10$$

2. In the Fig. 1.21.1 there is no point which satisfies both the constraints in first quadrant. Hence, problem cannot be solved or given LP problem has no feasible solution.

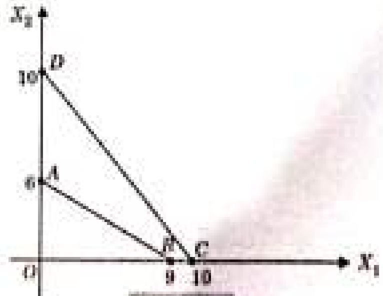

Fig. 1.21.1.

B. **Multiple Optimal Solution :** When more than one value of decision variables yields the same optimal value of objective function, such values of decision variables are known as alternate optimal solutions and LPP is said to have multiple optimal solutions.

C. **Redundancy :** A constraint in the given LPP is said to be redundant if the feasible region of the problem remains unchanged on deleting that constraint.

D. **Unbounded Solution :** When the value of objective function can be increased or decreased infinitely without any limitation, such solution of LPP is known as unbounded solution. In this situation, optimal solution may or may not exist.

Que 1.22. **Maximize $Z = 3x_1 - x_2$**

Subject to $2x_1 + x_2 \le 2$,

$x_1 + 3x_2 \ge 3$,

$x_2 \le 4$,

$x_1, x_2 \ge 0$.

UPTU 2012-13, Marks 05

Answer

1. Max $Z = 3x_1 - x_2$

Subject to $2x_1 + x_2 \le 2$

$x_1 + 3x_2 \ge 3$,

$x_2 \le 4$,

$x_1, x_2 \ge 0$

2. On removing inequality sign,

$$2x_1 + x_2 = 2$$

$$x_1 + 3x_2 = 3,$$

$$x_2 = 4$$

3. After plotting the values on graph, ABC is the feasible region as shown in Fig. 1.22.1

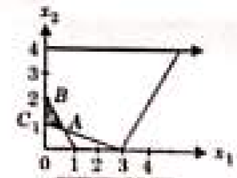

Fig. 1.22.1.

4. Value of maximum on extreme corners on triangle ABC,

$A\left(\frac{3}{5}, \frac{4}{5}\right)$, $Z = 1$

$B(0, 2)$, $Z = -2$

$C(0, 1)$, $Z = -1$

5. Hence, the maximum value of Max $Z = 1$ at point $A\left(\frac{3}{5}, \frac{4}{5}\right)$.

Que 1.23. **Define the following terms :**

a. **Slack variable,**
b. **Surplus variable,**
c. **Optimal solution,**
d. **Feasible solution,**
e. **Basic solution,**
f. **Basic feasible solution, and**
g. **Degenerate solution.**

Answer

a. **Slack Variable :** A variable added to the left hand side of a 'less than or equal to' constraint, to convert the constraint into an equality is called a slack variable. In economic terminology, the value of the non-negative variable can usually be interpreted as the amount of unused resources.

b. **Surplus Variable :** A variable subtracted from the left hand side of the 'greater than or equal to' constraint, to convert the constraint into an equality is called a surplus variable. The value of this variable can usually be interpreted as the amount over and above the required minimum level.

c. **Optimal Solution :** Any basic feasible solution which optimized (minimizes or maximizes) the objective function of a general LP problem is called an optimal basic feasible solution to the general LP problem.

d. **Feasible Solution :** Any solution that also satisfies the non negative restrictions of the general LP problem is called a feasible solution.

e. **Basic Solution :** For a set of m simultaneous equations in n unknowns $(n > m)$, a solution obtained by setting $(n - m)$ of the variables equal to zero and solving the remaining m equations in n unknowns is called a basic solution. The variables $(n - m)$ are called non-basic variables and remaining m variables are called basic variables and constitute a basic solution.

f. **Basic Feasible Solution :** A feasible solution to a general LP problem which is also basic solution is called a basic feasible solution.

g. **Degenerate Solution :** A basic solution to the system of equations is called degenerate solution if one or more of the basic variables become equal to zero.

Que 1.24. **Describe the standard form of a LPP.**

Answer

1. The standard form of the linear programming problem (LPP) is used to develop the procedure for solving general linear programming problems.
2. The characteristics of the standard form are explained in the following steps :

Step 1 :

a. All the constraints should be converted to equations except for the non negativity restrictions which remain as inequalities (≥ 0).

b. Constraints of the inequality type can be changed to equations by augmenting (adding or subtracting) the left side of each such constraints by non negative variables.

c. These new variables are called slack variables and are added if the constraints are ($\leq$) or subtracted if the constraints are ($\geq$).

d. Since in the case of ($\geq$) constraint, the subtracted variable represents the surplus of the left side over the right side, it is common to refer to it as surplus variable.

e. For example, consider the constraints :

$$3x_1 - 4x_2 \geq 7,\ x_1 - 2x_2 \leq 3$$

f. These constraints can be changed to equations by introducing slack variables x_3 and x_4 respectively. Thus, we get

$$3x_1 - 4x_2 - x_3 = 7$$
$$x_1 + 2x_2 + x_4 = 3, \text{ and}$$
$$x_3 \geq 0,\ x_4 \geq 0$$

Step 2 :

a. The right side elements of each constraint should be made non negative (if not).

b. The right side can be always made positive on multiplying both sides of the resulting equations by (-1) whenever it is necessary.

c. For example, consider the constraint as

$$3x_1 - 4x_2 \geq -4$$

Which can be written in the form of the equation $3x_1 - 4x_2 - x_3 = -4$ by introducing the surplus variable $x_3 \geq 0$.

d. Again, multiplying both sides by (-1), we get $-3x_1 + 4x_2 + x_3 = 4$, which is the constraints equation standard form.

Step 3 :

a. All variables must have non-negative values. A variable which is unrestricted in sign (*i.e.*, positive, negative or zero) is equivalent to the difference between two non-negative variable.

b. Thus, if x is unconstrained in sign, it can be replaced by $(x' - x'')$, where x' and x'' are both non-negative, that is $x' \geq 0$ and $x'' \geq 0$.

Step 4 :

a. The objective functions should be of maximization form.

b. The maximization of a function $f(x)$ is equivalent to the maximization of the negative expression of the function, $f(x)$, that is

$$\text{Min } f(x) = -\text{Max } [-f(x)]$$

c. For example, the linear objective function

$$\text{Min } Z = C_1x_1 + C_2x_2 + \ldots\ldots + C_nx_n$$

is equivalent to Max $(-Z)$, *i.e.*, Max $Z' = -C_1x_1 - C_2x_2 \ldots\ldots - C_nx_n$ with $\quad Z = -Z'$

d. Consequently, in any LP problem, the objective function can be put in the maximization form.

3. **Standard form of General LPP with '≤' Constraints :** Now applying all steps systematically to general form of LP problem with all (≤)

constraints following standard form is obtained. Also no difficulty will arise to convert the general LPP with constraints (≤, =, ≥).

$$\text{Max } Z = C_1x_1 + C_2x_2 + \ldots\ldots + C_nx_n + C_{n+1}x_{n+1} + \ldots\ldots + C_{n+m}x_{n+m}$$

Subject to

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + x_{n+1} = b_1$$
$$a_{21}x_2 + a_{22}x_2 + \ldots + a_{2n}x_n + x_{n+2} = b_2$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n + x_{n+m} = b_m$$

Where $x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0, x_{n+1} \geq 0, \ldots, x_{n+m} \geq 0$.

Que 1.25. **Write an algorithm to solve a linear programming problem using simplex method.**

Answer

1. For the solution of any LPP by using simplex method, the steps for the computation of an optimum solution are as follows :

Step 1 : Check whether the objective function of the given LPP is to be maximized by using the result, Minimum $Z = -$ Maximum $(-Z)$.

Step 2 : Check whether all b_i $(i = 1, 2, \ldots\ldots, m)$ are non negative. If any one of b_i is negative then multiply the corresponding inequation of the constraints by -1, so as to get all b_i $(i = 1, 2, \ldots\ldots, m)$ non negative.

Step 3 : Convert all the inequations of the constraints into equations by introducing slack and/or surplus variable in the constraints. Put the costs of these variable equal to zero.

Step 4 : Obtain an initial basic feasible solution to the problem in the form $x_B = B^{-1}b$ and put it in the first column of the simple table.

Step 5 : Compute the net evaluations $Z_j - C_j$ $(j = 1, 2, \ldots\ldots, n)$ by using the relation $Z_j - C_j = C_B y_j - C_j$.

Examine the sign of $Z_j - C_j$:

i. If all $(Z_j - C_j) \geq 0$ then the initial basic feasible solution X_B is an optimum basic feasible solution.

ii. If at least one $(Z_j - C_j) < 0$, proceed on to the next step.

Step 6 : If there are more than one negative $Z_j - C_j$, then choose the most negative of them. Let it be $Z_r - C_r$ for some $j = r$.

i. If all $Y_{ir} \leq 0$ $(i = 1, 2, \ldots\ldots, m)$ then there is an unbounded solution to the given problem.

ii. If at least one $Y_{ir} > 0$ $(i = 1, 2, \ldots\ldots, m)$ then the corresponding vector Y_r enters the basis Y_B.

Step 7 : Compute the ratios $\left\{\dfrac{X_{Bi}}{Y_{ir}},\ Y_{ir} > 0,\ \ i = 1, 2, \ldots\ldots, m\right\}$ and choose the minimum of them. Let the minimum of these ratios be X_{Bk}/Y_{kr}. Then the vector Y_k will leave the basis Y_B. The common element

Y_{kr}, which is in the k^{th} row and the r^{th} column is known as the leading element (or pivotal element) of the table.

Step 8 : Convert the leading element to unity by dividing its row by the leading element itself and all other elements in its column to zeros by making use of the relations :

$$\hat{Y}_{ij} = \hat{Y}_{ij} - \frac{Y_{kj}}{Y_{kr}} Y_{ir} \qquad i = 1, 2, \ldots\ldots, m+1 ; i \neq k$$

and

$$\hat{Y}_{kj} = \frac{Y_{kj}}{Y_{kr}} \qquad j = 0, 1, 2, \ldots\ldots, n$$

Step 9 : Go to step 5 and repeat the computational procedure, either an optimum solution is obtained or there is an indication of an unbounded solution.

Que 1.26. **Solve :**

$$\text{Max. } Z = x_1 + x_2 + x_3$$

Subject to :

$$4x_1 + 5x_2 + 3x_3 \leq 15$$
$$10x_1 + 7x_2 + x_3 \leq 12$$

and $x_1, x_2, x_3 \geq 0$

UPTU 2011-12, Marks 10

Answer

1. Given : Max. $Z = x_1 + x_2 + x_3$

Subject to
$$4x_1 + 5x_2 + 3x_3 \leq 15$$
$$10x_1 + 7x_2 + x_3 \leq 12$$
$$\text{and } x_1, x_2, x_3 \geq 0$$

2. After introducing slack variable, the constraint equations become :

$$4x_1 + 5x_2 + 3x_3 + S_1 = 15$$
$$10x_1 + 7x_2 + x_3 + S_2 = 12$$
$$\text{Max. } Z = x_1 + x_2 + x_3 + 0S_1 + 0S_2$$
$$x_1, x_2, x_3, S_1, S_2 \geq 0$$

3. **First Iteration :**

		C_j	1	1	1	1	1	
Basis Variables	C_B	X_B	x_1	x_2	x_3	S_1	S_2	**Min. Ratio X_B/x_j**
S_1	0	15	4	5	3	1	0	15/3 = 5 ←
S_2	0	12	10	7	1	0	1	12/1 = 12
$Z_j = \Sigma C_B a_{ij}$		0	0	0	0	0	0	← Δ_j
		$Z_j - C_j$	−1	−1	−1 ↑	0	0	

4. **Second Iteration :**

		C_j	1	1	1	1	1	
Basis Variables	C_B	X_B	x_1	x_2	x_3	S_1	S_2	Min. Ratio X_B/x_j
x_3	1	5	4/3	5/3	1	1/3	0	
S_2	0	12	26/3	16/3	0	–1/3	1	
$Z_j = \Sigma C_B a_{ij}$			1/3	3/2	0	1/3	0	

5. Since all $\Delta_j \geq 0$, therefore the optimum solution is obtain by the solution given $x_1 = 0, x_2 = 0, x_3 = 5$
Max $Z = 5$ is optimal.

Que 1.27. **Write an algorithm to solve a LPP using two phase method.**

Answer

A. Phase I :

1. In this phase, the simplex method is applied to a specially constructed auxiliary linear programming problem leading to a final simplex table containing a basic feasible solution to the original problem.

   Step 1 : Assign a cost –1 to each artificial variable and a cost 0 to all other variable (in place of their original cost) in the objective function.

   Step 2 : Construct the auxiliary linear programming problem in which the new objective function Z^* is to be maximized subject to the given set of constraints.

   Step 3 : Solve the auxiliary problem by simplex method until either of the following three possibilities arises :

   i. Max $Z^* < 0$ and at least one artificial vector appears in the optimum basis at a positive level. In this case given problem does not possess any feasible solution.

   ii. Max $Z^* = 0$ and at least one artificial vector appears in the optimum basis. In this case proceed to phase II.

   iii. Max $Z^* = 0$ and no artificial vector appears in the optimum basis. In this case proceed to phase II.

B. Phase II :

1. Now assign the actual costs to the variable in the objective function and a zero cost to every artificial variable that appears in the basis at the zero level.
2. The new objective function is now maximized by simplex method subject to the given constraints.

3. That is, simplex method is applied to the modified simplex table obtained at the end of Phase I until an optimum basic feasible solution (if exists) has been attained.
4. The artificial variables which are non basic at the end of Phase I are removed.

Que 1.28. **Use the two-phase simplex method to :**

Maximize $Z = 5x_1 - 4x_2 + 3x_3$
Subject to $2x_1 + x_2 - 6x_3 = 20$
$6x_1 + 5x_2 + 10x_3 \leq 76$
$8x_1 - 3x_2 + 6x_3 \leq 50,$
$x_1, x_2, x_3 \geq 0$

UPTU 2013-14, Marks 10

Answer

1. The given constraints, after the introducing of slack and artificial variable, the constraint equations become :

$$2x_1 + x_2 - 6x_3 + A_1 = 20,$$
$$6x_1 + 5x_2 + 10x_3 + S_2 = 76,$$
$$8x_1 - 3x_2 + 6x_3 + S_3 = 50,$$
$$x_1, x_2, x_3, S_2, S_3, A_1 \geq 0$$

2. The new artificial objective function is

$$\text{minimize } W = A_1$$

3. The new objective function is also called the infeasibility form or auxiliary (dummy) objective functions. Thus the problem in standard form can be written as

$$\text{minimize } W = 0x_1 + 0x_2 + 0x_3 + 0S_2 + 0S_3 + A_1$$

Subject to the constraints,

$$2x_1 + x_2 - 6x_3 + 0S_2 + 0S_3 + A_1 = 20$$
$$6x_1 + 5x_2 + 10x_3 + S_2 + 0S_3 + 0A_1 = 76$$
$$8x_1 - 3x_2 + 6x_3 + 0S_2 + S_3 + 0A_1 = 50$$
$$x_1, x_2, x_3, S_2, S_3, A_1 \geq 0$$

Two-Phase Table

4. **First Iteration :**

		C_j	0	0	0	0	0	1	
Basis Variables	C_B	X_B	x_1	x_2	x_3	S_2	S_3	A_1	Min. Ratio
A_1	1	20	2	1	–6	0	0	1	10
S_2	0	76	6	5	10	1	0	0	38/3
S_3	0	50	$\boxed{8}$	–3	6	0	1	0	25/4 ←
$x_1 = x_2 = x_3 = 0$	Z_j	20	2	1	–6	0	0	1	
	C_j-Z_j		–2 ↑	–1	6	0	0	0	

5. Since $C_j - Z_j$ is negative under some variable column, so it is not optimal.

6. **Second Iteration :**

Basis Variables	C_B	C_j / X_B	0 / x_1	0 / x_2	0 / x_3	0 / S_2	0 / S_3	1 / A_1	Min. Ratio
A_1	1	15/2	0	(7/4)	(– 15/2)	0	(-1/4)	1	30/7 ←
S_2	0	77/2	0	(29/4)	(11/2)	1	(-3/4)	0	154/29
x_1	0	25/4	1	(– 3/8)	(3/4)	0	(1/8)	0	– 50/3
$S_1 = 0$	Z_j	15/2	0	7/4	– 15/2	0	– 1/4	1	
$x_2 = x_3 = 0$	C_j-Z_j		0	–7/4 ↑	15/2	0	1/4	0	

7. Since Again $C_j - Z_j$ is negative under some variable column, so it is not optimal.

8. **Third Iteration :**

Basis Variables	C_B	C_j / X_B	0 / x_1	0 / x_2	0 / x_3	0 / S_2	0 / S_3	1 / A_1	Min. Ratio
x_2	0	30/7	0	1	– 30/7	0	– 1/7	4/7	
S_2	0	52/7	0	0	256/7	1	2/7	– 29/7	
x_1	0	55/7	1	0	– 6/7	0	1/14	3/14	
$S_2 = A_1 = 0$	Z_j	0	0	0	0	0	0	0	
$x_3 = 0$	C_j-Z_j		0	0	0	0	0	1	

9. $C_j - Z_j$ is non negative under all column, it is optimal. Also min $W = 0$ and no artificial variable appears in the basis, this gives a basic feasible solution to the original problem.

10. **Phase - II :** The original objective function is

$$\text{maximize } Z = 3x_1 + 2x_2 + 2x_3$$

11. We are to maximize it subject to the original constraints. Using the solution of above table as starting solution for phase II and carrying out computations to get optimal results in the following table :

Basis Variables	C_B	C_j / X_B	5 / x_1	–4 / x_2	3 / x_3	0 / S_2	0 / S_3	Min. Ratio
x_2	– 4	30/7	0	1	– 30/7	0	–1/7	
S_2	0	52/7	0	0	256/7	1	2/7	

x_1	5	55/7	1	0	– 6/7	0	1/14	
$x_3 = 0$	Z_j	155/7	5	– 4	90/7	0	13/14	
S_1	C_j-Z_j		0	0	– 69/7	0	–13/14	

12. $C_j - Z_j$ is either negative or zero under all variable columns, it is optimal basic feasible solution for the original problem.

13. The optimal solution is

$$x_1 = \frac{55}{7}, x_2 = \frac{30}{7}, x_3 = 0 ; Z_{max} = \frac{155}{7} = 22\frac{1}{7}$$

Que 1.29. **Define duality. Describe how a dual can be obtained from a given primal problem.**

Answer

1. Every linear programming problem has associated with it's another linear programming problem.
2. The original problem can be considered the primal while the remaining problem it's dual.

a. **General Rules for Converting any Primal into its Dual :** If the system of constraints is a given LPP consists of a mixture of equations inequalities (≤ or ≥) non negative variables or unrestricted variables, then the dual of the given problem can be obtained by reducing it to standard primal form by adopting the following algorithm :

Step 1 : First convert the objective function to maximization form, if not.

Step 2 : If a constraint has inequality sign ≥, then multiply both sides by – 1 and make the inequality sign ≤.

Step 3 : If a constraint has an equality sign (=), then it is replaced by two constraints involving the equalities going in opposite directions, simultaneously.

For example an equation is replaced by two opposite inequalities (≤ and ≥) constraints :

$$x_1 + 2x_2 \le 4 \quad \text{and} \quad x_1 + 2x_2 \ge 4$$

The second inequality with ≥ sign, can be further written as $-x_1 - 2x_2 \le -4$.

Step 4 : Every unrestricted variable is replaced by the difference of two non negative variables.

Step 5 : We get the standard primal form, of given LPP in which :

a. All the constraints have '≤' sign, where the objective function is of maximization form; or

b. All the constraints have '≥' sign, where the objective function is of minimizing form.

Step 6 : Finally, the dual of the given problem is obtained by :

a. Transposing the rows and the columns of constraints coefficients.
b. Transposing the coefficients $(C_1, C_2, \ldots\ldots C_n)$ of the objective functions and the right side constraints $(b_1, b_2, \ldots\ldots\ldots b_m)$.
c. Changing the inequalities from '≤' to '≥' sign.
d. Minimizing the objective function instead of maximizing it.

Que 1.30. **Write the dual to the following primal LP problem :**

Maximize $Z = 20x_1 + 17x_2 + 18x_3 + 12x_4$

Subject to :

$4x_1 - 3x_2 + 8x_3 + 3x_4 \le 60$

$x_1 + x_2 + x_3 = 25$

$-x_2 + 4x_3 + 7x_4 \ge 35$

and x_4 is unrestricted in sign. **UPTU 2011-12, Marks 05**

Answer

1. Since x_4 is unrestricted, let it be replaced by $(x_4' - x_4'')$. Then the given problem becomes

$$\text{Maximize } Z = 20x_1 + 17x_2 + 18x_3 + 12(x_4' - x_4''),$$

Subject to, $4x_1 - 3x_2 + 8x_3 + 3(x_4' - x_4'') \ge 60.$

$x_1 + x_2 + x_3 = 25$

$-x_2 + 4x_3 + 7x_4' - 7x_4'' \ge 35$

$x_1, x_2, x_3, x_4', x_4'' \ge 0$

2. Since it is maximization problem, the third constraints is multiplied by -1 on both sides to give

$x_2 - 4x_3 - 7x_4' + 7x_4'' \le -35$

3. The equation $x_1 + x_2 + x_3 = 25$ can be expressed as a pair of inequalities.

$x_1 + x_2 + x_3 \ge 25$ and $x_1 + x_2 + x_3 \le 25$

4. Thus the given problem becomes,

$$\text{Maximize } Z = 20x_1 + 17x_2 + 18x_3 + 12(x_4' - x_4'')$$

Subject to $4x_1 - 3x_2 + 8x_3 + 3x'_4 - 3x_4'' \ge 60$

$x_1 + x_2 + x_3 \ge 25$

$x_2, x_1, x_3, x_4', x_4'' \ge 0$

5. Let $y_1, y_2, y_3, y_4, y_4''$ be the associated non negative dual variables.
6. Then the dual of this problem is

Minimize $\quad W = 60y_1 + 25y_2 - 35y_3$

Subject to

$4y_1 + y_2 + 0y_3 \le 20$

$-3y_1 + y_2 + y_3 \le 17$

$8y_1 + y_2 - 4y_3 \le 18$

$3y_1 + 0y_2 - 7y_3 \le 12$

$y_1, y_2 \ge 0, y_3$ is unrestricted.

Que 1.31. **Solve the following LP problem by using the two phase simplex method.**

Minimize $Z = x_1 - 2x_2 - 3x_3$

Subject to the constraints :

i. $-2x_1 + x_2 + 3x_3 = 2$

ii. $2x_1 + 3x_2 + 4x_3 = 1$ and $x_1, x_2, x_3 \ge 0$ **UPTU 2014-15, Marks 05**

Answer

A. **Phase I :**

Step 1 : Set up the problem in the standard form.

Phase 1 problem, in standard form can be expressed as

Minimize, $\quad W = 0x_1 + 0x_2 + 0x_3 + A_1 + A_2,$

subjected to $\quad -2x_1 + x_2 + 3x_3 + A_1 + 0A_2 = 2$

$2x_1 + 3x_2 + 4x_3 + 0A_1 + A_2 = 1$

and $\quad x_1, x_2, x_3, A_1, A_2 \ge 0$

Where A_1 and A_2 are artificial variables.

Step 2 : Find an initial basic feasible solution.

Substituting $x_1 = x_2 = x_3 = 0$ in the constraint equations.

We get, $A_1 = 2, \quad A_2 = 1$ as the initial basic feasible solution.

		C_j	0	0	0	1	1	
Basis	C_B	X_B	x_1	x_2	x_3	A_1	A_2	Min. Ratio
A_1	1	2	−2	1	3	1	0	2/3
A_2	1	1	2	3	$\boxed{4}$	0	1	1/4 ←
	$Z_j = \Sigma C_B a_{ij}$	3	0	4	7	1	1	
	$C_j - Z_j$		0	−4	−7 ↑	0	0	

Step 3 : Perform optimality test.

i. Since, $C_j - Z_j$ is negative under some columns (minimization problem), solution is not optimal.

ii. Since, $\quad W = A_1 + A_2 = 2 + 1 = 3 (> 0)$

Artificial variables A_1 and A_2 in the basis at a positive level ($A_1 = 2$, $A_2 = 1$), the problem does not possesses a feasible solution and the procedure stops.

iii. Hence, this problem has an infeasible solution.

Que 1.32. **Write down the working procedure to solve a LPP using dual simplex method.**

Answer

1. Dual simplex method is applicable to those linear programming problem that starts with infeasible but otherwise optimum solution. The method may be summarized as follows :

Step 1 : Write the given linear programming problem in its standard form and obtain a starting basic solution.

Step 2 :

a. If the current basic solution is feasible, use simplex method to obtain an optimum solution.

b. If the current basic solution is infeasible, *i.e.*, value of basis variables are ≤ 0 go to the next step.

Step 3 : Check whether the solution is optimum.

a. If the solution is not optimum add an artificial constraint in such a way that condition of optimality is satisfied.

b. If the solution is optimum, go to next step.

Step 4 : Select the basis variable having the most negative value. This basis variable becomes the leaving variable and the row corresponding to it becomes the key row.

Step 5 :

a. Obtain the ratios of the net evaluations to the corresponding coefficients in the key row.

b. Ignore the ratios associated with positive and zero denominators.

c. The entering vector is the one with the smallest absolute value of the ratios.

d. Columns corresponding to the entering vector become the key column.

Step 6 : Reduce the leading element into unity and all other entries of the key column to zero by elementary row operations.

Step 7 : Go to step 2 and repeat the procedure until an optimum basic feasible solution is attained.

Que 1.33. **Use dual simplex method to solve :**

$$\text{Min } Z = 3x_1 + x_2$$

Subject to : $x_1 + x_2 \geq 1,\ 2x_1 + 3x_2 \geq 2,\ x_1 \text{ and } x_2 \geq 0$

Answer

1. The given problem can be written as

$$\text{Max } Z' = -3x_1 - x_2, \qquad (Z' = -Z)$$

Subject to
$$-x_1 - x_2 \leq -1$$
$$-2x_1 - 3x_2 \leq -2$$
$$x_1, x_2 \geq 0$$

2. Adding the slack variables x_3 and x_4 to each constraint, respectively, we get

$$-x_1 - x_2 + x_3 = -1;\ -2x_1 - 3x_2 + x_4 = -2$$

3. Writing the constraint equations in matrix form, we have

$$\begin{bmatrix} -1 & -1 & 1 & 0 \\ -2 & -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

4. Now the starting dual simplex table can be constructed as

		C_j	-3	-1	0	0	
Basis Variables	C_B	X_B	x_1	x_2	x_3 (B_1)	x_4 (B_2)	
x_3	0	-1	-1	-1	1	0	
x_4	0	-2	-2	$\boxed{-3}$	0	1	$\leftarrow$
$Z = \Sigma C_B a_{ij}$		0	3	1 $\uparrow$	0	0 $\downarrow$	$\leftarrow \Delta_j$

5. The solution : $x_1 = 0, x_2 = 0, x_3 = -1$ and $x_4 = -2$ is the starting basic solution which is infeasible but optimal, we now start with the first iteration of dual simplex method.

6. **First Iteration :**

a. To determine the leaving vector (B_r) :
Since $X_{Br} = \min (x_{Bi}, x_{Bi} < 0) = \min (x_{B1}, x_{B2}) = \min (-1, -2) = -2 = x B_2$
Hence $r = 2$, so we must remove the vector B_2 (marked).

b. To determine the entering vector a_k for predetermined value of r $(= 2)$.

$$\text{Since, } \frac{\Delta k}{x_{2k}} = \max \left[\frac{\Delta_1}{x_{21}}, \frac{\Delta_1}{x_{22}}, \text{ For } x_{21} < 0,\ x_{22} < 0\right] = \max \left[\frac{3}{-2}, \frac{1}{-3}\right]$$

Hence $k = 2$.

c. So we must remove the vector a_2 corresponding to which x_2 is already given. To find the transformed table : Here the key element is (-3).

d. In the usual manner, we can get the transformed table as

		C_j	-3	-1	0	0	
Basis Variables	C_B	X_B	x_1	x_2 (B_2)	x_3 (B_1)	x_4	
x_3	0	$-1/3$	$-1/3$	0	1	$-1/3$	
x_2	-1	$2/3$	$2/3$	1	0	$-1/3$	$\leftarrow \Delta_j$
$Z' = \Sigma C_B a_{ij}$ $\therefore Z = 2/3$		$-2/3$	$7/3$	0	0 $\downarrow$	$1/3$ $\uparrow$	

e. Even now the corresponding solution is infeasible but optimal. So we proceed to second iteration.

7. **Second Iteration :**

a. To find the leaving vector (B_r) :
Since $XB_r = \min |x_{B1}|$ (x_{B2} is ignored because it is not negative)
Therefore, $r = 1$.
We must remove the vector B_1.

b. To find the entering vector a_k for predetermined value of r (= 1) :

Since $\frac{\Delta_k}{x_{rk}} = \frac{\Delta_k}{x_{1k}} = \max\left[\frac{\Delta_1}{x_{11}}, \frac{\Delta_4}{x_{14}}\right]$ (because only x_{11} and x_{14} are negative.)

$= \max\left[\frac{7/3}{-1/3}, \frac{1/3}{-1/3}\right] = \frac{-1}{1} = \frac{\Delta_4}{\Delta_{14}}$

Hence $k = 4$.

c. So we must enter the vector a_4 corresponding to which vector x_4 is given in table. The key element is $\left(-\frac{1}{3}\right)$.

d. In usual manner, we can get the following transformed table :

		C_j	-3	-1	0	0	
Basis Variables	C_B	X_B	x_1	x_2 (B_2)	x_3 (B_1)	x_4	
x_4	0	1	1	0	-3	1	
x_2	-1	1	1	1	-1	0	$\leftarrow \Delta_j$
$Z = \Sigma C_B a_{ij}$ $\therefore Z = 1$			2	0	1	0	

$$\Delta_1 = C_B x_1 - C_1 = (0, -1)(1, 1) + 3 = 2$$
$$\Delta_3 = C_B x_3 - C_3 = (0, -1)(-3, -1) - 0 = 1$$

e. At this stage, the solution $x_1 = 0$, $x_2 = 1$, $x_3 = 0$, $x_4 = 1$ becomes feasible and hence it is the required optimal solution with max $Z = 1$.

f. The solution of the dual problem can be obtained from the final table of primal solution as the value of j corresponding to slack variables of the primal.

i.e., $w_1 = 1$, $w_2 = 0$.

Que 1.24. **What do you mean by degeneracy in a LPP ? How will you resolve degeneracy ?**

Answer

1. At the stage of improving the solution during simplex procedure, minimum ratio X_B/x_k ($x_k > 0$) is determined in the last. But, sometimes this ratio may not be unique *i.e.*, the key element is not uniquely determined at the very first iteration, the variables value (one or more) in the X_B column become equal to zero, this causes the problem of degeneracy.

2. **Method to Resolve Degeneracy :** The following systematic procedure can be utilized to avoid cycling due to degeneracy in LPP :

Step 1 : First pick up the rows for which the minimum non negative ratio is same (tied). To be definite, suppose such rows are first, third, etc.

Step 2 : Now rearrange the columns of the usual simplex table so that columns forming the original unit matrix come first in proper order.

Step 3 : Then find the minimum of the ratio :

$$\left[\frac{\text{Elements of first columns of unit matrix}}{\text{Corresponding elements of key column}}\right]$$

Only for the rows for which minimum ratio was not unique. That is, for the rows first, third, etc., as picked up in step 1.

i. If this minimum is attained for third row (say), then this row will determine the key element by intersecting the key column.

ii. If this minimum is also not unique, then go to next step.

Step 4 : Now compute the minimum of the ratio :

$$\left[\frac{\text{Elements of second columns of unit matrix}}{\text{Corresponding elements of key column}}\right]$$

Only for the rows for which minimum ratio was not unique in step 3.

i. If this minimum ratio is unique for the first row (say), then this row will determine the key element by intersecting the key columns.

ii. If this minimum is still not unique then go to next step.

Step 5 : Next compute the minimum of the ratio :

$$\left[\frac{\text{Elements of third columns of unit matrix}}{\text{Corresponding elements of key column}}\right]$$

For the rows for which minimum ratio was not unique in step 4.

i. If this minimum ratio is unique for the third row (say), then this row will determine the key element by intersecting the key column.

ii. If this minimum ratio is still not unique, then go on repeating the above outlined procedure till the unique minimum, ratio is obtained to resolve the degeneracy. After the resolution of this tie, Simplex method is applied to obtain the optimum solution.

Que 1.25. **Define sensitivity analysis.**

OR

Discuss the role of sensitivity analysis in linear programming.

UPTU 2011-12, Marks 05

Answer

A. Sensitivity Analysis :

1. The investigation that deals with changes in the optimal solution due to changes in the parameters (a_{ij}, b_i and c_j) is called sensitivity analysis or post optimality analysis.
2. The objective of sensitivity analysis is to reduce the additional computational effort considerably which arise in solving a new problem.
3. The changes in the linear programming problem which are usually studied by sensitivity analysis include :
   a. Changes in objective function coefficients, c_j's, *i.e.*, profit (or cost) per unit associated with decision variables.
   b. Changes in b_i values (or right hand side constants of the constraints), *i.e.*, available resources.
   c. Changes in the coefficients of variables on the left hand side of the constraints, a_{ij}'s, *i.e.*, consumption of resources per unit of decision variables x_j.
   d. Structural changes due to the addition and subtraction of some variables.
   e. Structural changes due to the addition and subtraction of some linear constraints. In general, these changes may results in one of the following three cases :

Case I : The optimal solution remains unchanged, *i.e.*, the basic variables and their values remain essentially unchanged.

Case II : The basic variables remain the same, but their values are changed.

Case III : The basic solution changes completely.

Que 1.26. **Given the linear programming problem :**

Maximize $Z = 3x_1 + 5x_2$

Subject to the constraints :

$x_1 \le 4$, $x_2 \le 6$, $3x_1 + 2x_2 \le 18$; $x_1, x_2 \ge 0$.

Discuss the effect on the optimality of the solution, when the objective function is changed to $3x_1 + x_2$.

Answer

1. After introducing the slack variables $S_1 \ge 0$, $S_2 \ge 0$ and $S_3 \ge 0$ in the constraints of the given LPP by simplex method, the optimum simplex table is :

y_B	C_B	X_B	y_1	y_2	y_3	y_4	y_5
y_3	0	2	0	0	1	2/3	– 1/3
y_2	5	6	0	1	0	1	0
y_1	3	2	1	0	0	– 2/3	1/3
$Z - \Sigma C_B a_j$			3	5	0	3	1

2. Since, the objective function is changed to $3x_1 + x_2$, C_2 has been changed to 1 keeping C_1 fixed. So, we find the variation in C_2.
3. **Variation in C_2 :** Since $C_2 = C_B$, the range of ΔC_2 is given by

$$\underset{y_{2j} > 0}{\text{Minimum}} \left\{ \frac{-(Z_j - C_j)}{y_{2j}} \right\} \le \Delta C_2 \le \underset{y_{2j} < 0}{\text{Maximum}} \left\{ \frac{-(Z_j - C_j)}{y_{2j}} \right\}$$

i.e., $\text{Minimum} \left\{ \frac{-3}{1} \right\} \le \Delta C_2 < \infty$ or $-3 \le \Delta C_2 \le \infty$

$5 - 3 \le C_2 < 5 + \infty$ or $2 \le C_2 < \infty$

4. This indicates that if C_2 is changed to 1, the optimum solution obtained above does not remain optimal.
5. To find the new optimal solution :

When the objective function $Z = 3x_1 + 5x_2$ is changed to $Z^* = 3x_1 + x_2$,

$$Z_1 - C_1 = Z_2 - C_2 = Z_3 - C_3 = 0;$$

$$Z_4 - C_4 = C_B y_4 - C_4 = -1 \text{ and } Z_5 - C_5 = C_B y_5 - C_5 = 1$$

This shows that y_4 enters the basis and y_3 leaves the basis in the next iteration.

6. Thus, we have the following optimum simplex table :

y_B	C_B	X_B	y_1	y_2	y_3	y_4	y_5
y_4	0	3	0	0	3/2	1	– 1/2
y_2	1	3	0	1	– 3/2	0	1/2
y_1	3	4	1	0	1	0	0
$Z^* - \Sigma C_B a_j$			0	0	3/2	0	1/2

7. The optimum solution of the revised LPP is

$x_1 = 4$

$x_2 = 3$

Maximum $Z^* = 15$.

UNIT 2

Transportation and Assignment Problems

PART-1

Transportation Problem, Method of Obtaining Initial and Optimal Solution.

CONCEPT OUTLINE : PART-1

Transportation Problem : The transportation problem is to transport various amounts of a single homogeneous commodity, that are initially stored at various origins, to different destination in such a way that the total transportation cost is minimum.

The necessary as well as sufficient condition for a transportation model to have feasible solution.

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

Where,
a_i = Number of supply units,
b_j = Number of demand units,
m = Sources, and
n = Destinations.

Types of Transportation Problems :

1. Unbalanced transportation problem.
2. Maximization problem.
3. Different production costs.
4. No allocation in a particular cell/cells, and
5. Overtime production.

Feasible Solution : A feasible solution to a transportation problem is a set of non-negative allocations, x_{ij} that satisfies the row and column restrictions.

Basic Feasible Solution : A feasible solution to a transportation problem is said to be basic feasible solution, if it contains not more than $(m + n - 1)$ non-negative allocations, where m is the number of rows and n is the number of columns of the transportation problem.

Optimal Solution : A feasible solution that minimizes the transportation cost is called an optimal solution. In case of profit, it will maximize that.

The Condition for Optimality Check :

a. Number of allocations is $(m + n - 1)$, where m is the number of rows and n number of columns.
b. These $(m + n - 1)$ allocations should be in independent positions.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 2.1. Define transportation problem.

Answer

1. The transportation problem deals with the transportation of a product manufactured at different plants or factories (supply origins) to a number of different warehouses (demand destinations).
2. The objective is to satisfy the destination requirements within the plant's capacity constraints at the minimum transportation cost.
3. Thus transportation problems typically arise in situations involving physical movement of goods from plants to warehouses, warehouses to wholesalers, wholesalers to retailers and retailers to customers.
4. Solution of the transportation problem requires the determination of how many units should be transported from each supply origin to each demand destination in order to satisfy all the destination demands while minimizing the total associated cost of transportation.

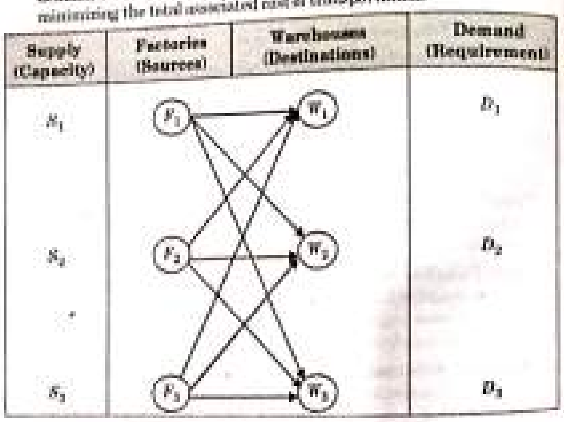

Fig. 2.1.1. Transportation problem chart.

Que 2.2. Write mathematical model for general transportation problem.

Answer

1. The classical transportation problem can be stated mathematically as follows :

   Let a_i = Quantity of product available at origin i,

   b_j = Quantity of product required at destination j,

   C_{ij} = The cost of transporting one unit of product from origin i to destination j, and

   x_{ij} = The quantity transported from origin i to destination j.

2. Assume that $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$ by which means that the total quantity available at the origin is precisely equal to the total amount required at the destinations.

3. With these, the problem can be stated as a linear programming problem as :

   Minimize total cost $Z = \sum_{i=1}^{m} \sum_{j=1}^{n} C_{ij} x_{ij}$

   Subject to $\sum_{j=1}^{n} x_{ij} = a_i$ for $i = 1, 2, \ldots, m$

   $\sum_{i=1}^{m} x_{ij} = b_j$ for $j = 1, 2, \ldots, n$

   and $x_{ij} \geq 0$ for all $i = 1, 2, \ldots, m$, and $j = 1, 2, \ldots, n$.

Que 2.3. What do you mean by transshipment problem ?

Answer

1. A transportation problem in which available commodity frequently moves from one source to another source or destination before reaching its actual destination is called a transshipment problem.
2. Such a problem cannot be solved by the usual transportation algorithm, but slight modification is required before applying it to the transshipment problem :
   a. The number of sources and destinations in the transportation problem are m and n respectively. In transshipment study, however, we have $m + n$ sources and destinations.
   b. If S_i denotes the i^{th} source and D_j denotes the j^{th} destination then commodity can move along the routes $S_i \to D_i \to D_j$, $S_i \to S_j \to D_j$, $S_i \to D_i \to S_i \to D_j$, or in various other ways, clearly, transportation cost from S_i to S_i is zero and the transportation costs from S_i to S_j or S_i to D_j do not have to be symmetrical, *i.e.*, in general $S_i \to S_j \neq S_j \to S_i$.

c. In solving the transshipment problem we first find the optimum solution to the transportation problem and then proceed in the same fashion as in solving transportation problem.

d. The basic feasible solution contains $2m + 2n - 1$ basic variables, if we omit the variables appearing in the $m + n$ diagonal cells, we are left with $m + n - 1$ basic variables.

Que 2.4. **Write short notes on :**

A. North-West corner method,
B. Least-Cost method, and
C. Vogel's approximation method (or Penalty method).

Answer

A. North-West Corner Method :

1. It is a simple and efficient method to obtain an initial basic feasible solution.
2. Various steps of the method are :

Step 1 :

Select the north-west (upper left hand) corner cell of the transportation table and allocate as much as possible so that either the capacity of the first row is exhausted or the destination requirement of the first column is satisfied, *i.e.*, $x_{11} = \min(a_1, b_1)$, where b_1 is the capacity of the first row and a_1 is the requirement of first column.

Step 2 :

1. If $b_1 > a_1$, we move down vertically to the second row and make the second allocation of magnitude $x_{21} = \min(a_2, b_1 - x_{11})$ in the cell (2, 1).
2. If $b_1 < a_1$, we move right horizontally to the second column and make the second allocation of magnitude $x_{12} = \min(a_1 - x_{11}, b_2)$ in the cell (1, 2).
3. If $b_1 = a_1$, there is a tie for the second allocation. One can make the second allocation of magnitude

$$x_{12} = \min(a_1 - a_1, b_1) = 0 \text{ in the cell } (1, 2).$$
$$x_{21} = \min(a_2, b_1 - b_1) = 0 \text{ in the cell } (2, 1).$$

Step 3 :

Repeat step 1 and 2 moving down towards the lower/right corner of the transportation table until all the requirements are satisfied.

B. Least-Cost Method :

1. This method takes into account the minimum unit cost and can be summarized as follows :

Step 1 :

Determine the smallest cost in the cost matrix of the transportation table. Let it be C_{ij} allocate $x_{ij} = \min(a_i, b_j)$ in the cell (i, j).

Step 2 :

2. If $x_{ij} = a_i$ cross-off the i^{th} row of the transportation table and decrease b_j by a_i. Go to step 3.
3. If $x_{ij} = b_j$ cross-off the j^{th} column of the transportation table and decrease a_i by b_j. Go to step 3.
4. If $x_{ij} = a_i = b_j$ cross-off the i^{th} row or j^{th} column but not both.

Step 3 :

1. Repeat steps 1 and 2 for the resulting reduced transportation table until all the requirements are satisfied. Whenever the minimum cost is not unique, make an arbitrary choice among the minima.

C. Vogel's Approximation Method (or Penalty Method) :

1. The Vogel's approximation method takes into account not only the least cost C_{ij} but also the costs that just exceed C_{ij}. The steps of the method are given below :

Step 1 :

1. For each row of the transportation table identify the smallest and the next to smallest costs. Determine the difference between them for each row.
2. Display them alongside the transportation table by enclosing them in parenthesis against the respective rows. Similarly, compute the differences for each column.

Step 2 :

1. Identify the row or column with the largest difference among all the rows and columns. If a tie occurs, use any arbitrary tie-breaking choice.
2. Let the greatest difference correspond to i^{th} row and let C_{ij} be the smallest cost in the i^{th} row.
3. Allocate the maximum feasible among $x_{ij} = \min(a_i, b_j)$ in the $(i, j)^{th}$ cell and cross-off either the i^{th} row or the j^{th} column in the usual manner.

Step 3 :

Recompute the column and row differences for the reduced transportation table and go to step 2. Repeat the procedure until all the rim requirements are satisfied.

Que 2.5. **Write the steps involved in solution of transportation problems using**

A. Row minima method, and
B. Column minima method.

Answer

A. Row Minima Method :

1. This method consists in allocating as much as possible in the lowest cost cell of the first row so that either the capacity of the first plant is exhausted or the requirement at j^{th} distribution centre is satisfied or both.

2. In case of tie among the cost, select arbitrarily.
3. Following three cases arise :
   a. If the capacity of the first plant is completely exhausted, cross off the first row and proceed to the second row.
   b. If the requirement at j^{th} distribution centre is satisfied, cross off the j^{th} column and reconsider the first row with the remaining capacity.
   c. If the capacity of the first plant as well as the requirement at j^{th} distribution centre is completely satisfied, make a zero allocation in the second lowest cost cell of the first row. Cross off the row as well as the j^{th} column and move down to the second row.

B. Column Minima Method :

1. This method consists in allocating as much as possible in the lowest cost cell of the first column so that either the demand of the first distribution centre is satisfied or the capacity of the plant is exhausted or both.
2. In case of tie among the lowest cost cells in the column, select arbitrarily.
3. Following three cases arise :
   a. If the requirement of the first distribution centre is satisfied, cross off the first column and move right to the second column.
   b. If the capacity of i^{th} plant is satisfied, cross off i^{th} row and reconsider the first column with the remaining requirement.
   c. If the requirement of the first distribution centre as well as the capacity of the i^{th} plant is completely satisfied, make a zero allocation in the second lowest cost cell of the first column. Cross off the first column as well as the i^{th} row and move right to the second column.

Que 2.6. **Determine an initial basic feasible solution to the following transportation problem using :**
A. North-West Corner rule, and
B. Least cost method.

		To A	B	C	Supply
From	a	2	7	4	5
	b	3	3	1	8
	c	5	4	7	7
	d	1	6	2	14
	Demand	7	9	18	

Answer

A. North-West Corner Rule :

1. Select the north-west (upper left hand) corner cell of the transportation table and allocating as much as possible,

	A	B	C	
a	2(5)	7	4	5/0
b	3	3	1	8
c	5	4	7	7
d	1	6	2	14
	7/2	9	18	

2. For second allocation selecting the north-west corner cell of the remaining transportation table and allocating as much as possible,

	A	B	C	
a	2(5)	7	4	0
b	3(2)	3	1	8/6
c	5	4	7	7
d	1	6	2	14
	2/0	9	18	

3. Similarly, selecting the north-west corner cell and allocating maximum possible,

	A	B	C	
a	2(5)	7	4	0
b	3(2)	3(6)	1	6/0
c	5	4	7	7
d	1	6	2	14
	0	9/3	18	

4. Similarly, selecting the north-west corner cell and allocating maximum possible,

	A	B	C	
a	2(5)	7	4	0
b	3(2)	3(6)	1	0
c	5	4(3)	7	7/4
d	1	6	2	14
	0	3/0	18	

5. Selecting and allocating the remaining part so that the capacity and destination requirement is fulfilled.

	A	B	C	
a	2(5)	7	4	0
b	3(2)	3(6)	1	0
c	5	4(3)	7(4)	4/0
d	1	6	2(14)	14/0
	0	0	18/0	

6. Hence, final allocation are shown below :

	A	B	C	
a	2(5)	7	4	5
b	3(2)	3(6)	1	8
c	5	4(3)	7(4)	7
d	1	6	2(14)	14
	7	9	18	

7. Total transportation cost

$$= (2 \times 5) + (3 \times 2) + (3 \times 6) + (4 \times 3) + (7 \times 4) + (2 \times 14)$$
$$= 10 + 6 + 18 + 12 + 28 + 28 = 102 \text{ units.}$$

B. Least Cost Method :

1. Selecting the smallest cost in the cost matrix of the transportation table and allocating maximum possible allocation :

	A	B	C	
a	2	7	4	5
b	3	3	1(8)	8/0
c	5	4	7	7
d	1(7)	6	2	7
	0	9	18/10	

2. For second allocation selecting smallest cost in the remaining cost transportation cost and allocating them,

	A	B	C	
a	2	7	4	5
b	3	3	1(8)	8
c	5	4	7	7
d	1(7)	6	2	14/7
	7/0	9	18	

3. Similarly, selecting cost in the cost table and allocating them.

	A	B	C	
a	2	7	4	5
b	3	3	1(8)	0
c	5	4	7	7
d	1(7)	6	2(7)	7/0
	0	9	10/3	

4. Similarly, selecting smallest cost in the cost table and allocating them.

	A	B	C	
a	2	7	4(3)	5/2
b	3	3	1(8)	0
c	5	4(7)	7	7/0
d	1(7)	6	2(7)	0
	0	9/2	3/0	

5. Selecting and allocating the remaining part so that the capacity and requirement is fulfilled.

	A	B	C	
a	2	7(2)	4(3)	2/0
b	3	3	1(8)	0
c	5	4(7)	7	0
d	1(7)	6	2(7)	0
	0	2/0	3/0	

6. Final allocation is given by,

	A	B	C	
a	2	7(2)	4(3)	5
b	3	3	1(8)	8
c	5	4(7)	7	7
d	1(7)	6	2(7)	14
	7	9	18	

7. Total transportation cost,

$= (7 \times 2) + (4 \times 3) + (1 \times 8) + (4 \times 7) + (1 \times 7) + (2 \times 7)$

$= 14 + 12 + 8 + 28 + 7 + 14 = 83$ units.

Que 2.7. **Use Vogel's approximation method (VAM) to find initial basic feasible solution to the transportation problem.**

	D_1	D_2	D_3	D_4	Supply
S_1	19	30	50	10	7
S_2	70	30	40	60	9
S_3	40	8	70	20	18
Demand	5	8	7	14	34

Answer

1. Checking the balancing of transportation problem.

	D_1	D_2	D_3	D_4	Supply
S_1	19	30	50	10	7 +
S_2	70	30	40	60	9 +
S_3	40	8	70	20	18 = 34
Demand	5 +	8 +	7 +	14 = 34	

2. Since, total demand = total supply
Therefore, this is a balanced transportation problem.
3. Assign penalties for each row and column. Then allocate highest penalty with maximum possible allocation.

	D_1	D_2	D_3	D_4	Supply	Penalty
S_1	19	30	50	10	7	19 – 10 = 9
S_2	70	30	40	60	9	40 – 30 = 10
S_3	40	8(8)	70	20	18/10	20 – 8 = 12
Demand	5	8/0	7	14		
Penalty	40 – 19 = 21	30 – 8 = 22	50 – 40 = 10	20 – 10 = 10		

4. Calculating the penalty for remaining transportation problem.
5. Assign penalties for each row and column. Then allocate highest penalty with maximum possible allocation.

	D_1	D_3	D_4	Supply	Penalty
S_1	19(5)	50	10	7/2	19 – 10 = 9
S_2	70	40	60	9	60 – 40 = 20
S_3	40	70	20	10	40 – 20 = 20
Demand	5/0	7	14		
Penalty	21	10	10		

6. Similarly, calculating and assigning penalties for remaining transportation problem and allocating highest penalty with maximum possible allocation.

	D_3	D_4	Supply	Penalty
S_1	50	10	2	40
S_2	40	60	9	20
S_3	70	20(10)	10/0	50 ←
Demand	7	14/4		
Penalty	10	10		

7. Similarly, repeating the above step for remaining transportation problem and allocating highest penalty with maximum possible allocation,

	D_1	D_4	Supply	Penalty
S_1	50	10(2)	20	40
S_2	40(7)	60(2)	9	20
Demand	7	4/2		
Penalty	10	50 ↑		

8. Final allocation of transportation is given,

	D_1	D_2	D_3	D_4	Supply
S_1	19(5)	30	50	10(2)	7
S_2	70	30	40(7)	60(2)	9
S_3	40	8(8)	70	20(10)	18
Demand	5	8	7	14	

9. Total cost $= (19 \times 5) + (10 \times 2) + (40 \times 7) + (60 \times 2) + (8 \times 8) + (20 \times 10)$

$= 95 + 20 + 280 + 120 + 64 + 200 = 779$

Que 2.8. **What is MODI method ? Write down the steps involved in solution of transportation problem using MODI method.**

Answer

A. **MODI Method :** The modified distribution method or MODI method is an efficient method of checking the optimality of the initial feasible solution.

B. **Steps for solving transportation problem using MODI method are :**

Step 1 : Find the initial basic feasible solution.

Step 2 : Check the number of occupied cells. If there are less than $m + n - 1$, there exists degeneracy and we introduce a very small positive assignment of ϵ (≈ 0) in suitable independent positions, so that the number of occupied cells is exactly equal to $m + n - 1$.

Step 3 : For each occupied cell in the current solution, solve the system of equations $u_i + v_j = c_{ij}$ starting initially with some $u_i = 0$ or $v_j = 0$ and entering successively the values of u_i and v_j in the transportation table margins.

Step 4 : Compute the net evaluations $z_{ij} - c_{ij} = u_i + v_j - c_{ij}$ for all unoccupied basic cells and enter them in the lower left corners of the basics.

Step 5 :

1. Let the unoccupied cell (r, s) enter the basics. Allocate an unknown quantity, say θ, to the cell (r, s). Identify a loop that start and ends at the cell (r, s) and connects some of the basic cells.
2. Add and subtract interchangeably, θ to and from the transition cells of the loop in such a way that the requirements remain satisfied.

Step 6 :

1. Assign a maximum value to θ in such a way that the value of one basic variable becomes zero and the other basic variable remains non-negative.
2. The basic cell where allocation has been reduced to zero leaves the basis.

Step 7 : Return to step 3 and repeat the process until an optimum basic feasible solution has been obtained.

Que 2.9. **Find the optimum solution to the following transportation problem in which the cells contain the transportation cost in rupees.**

Warehouses	1	2	3	4	5	Supply
1	7	6	4	5	9	40
2	8	5	6	7	8	30
3	6	8	9	6	5	20
4	5	7	7	8	6	10
Demand	30	30	15	20	5	

UPTU 2013-14, Marks 10

Answer

1. Checking the balance of transportation problem,

	1	2	3	4	5	Supply
1	7	6	4	5	9	40
2	8	5	6	7	8	30
3	6	8	9	6	5	20
4	5	7	7	8	6	10
Demand	30	30	15	20	5	100 (Total)

2. **Initial Feasible Solution :** We shall use Vogel's approximation method to find initial feasible solution.

	1	2	3	4	5	Supply	Penalty			
1	7(5)	6	4(15)	5(20)	9	40/25/5/0	1	1	[2]	2
2	8	5(30)	6	7	8	30/0	1	[2]	—	—
3	6(15)	8	9	6	5(5)	20/15/10	1	1	1	[1]
4	5(10)	7	7	8	6	10/0	1	1	1	1
Demand	30/25/15/0	30/0	15/0	20/0	5/0					
	1	1	[2]	1	1					
	1	1	—	1	1					
Penalties	1	—	—	1	1					
	1	—	—	—	1					

A. Optimality Test :

1. To test the basic feasible solution for optimality total number of allocation should be $(m + n - 1)$ *i.e.*, $(5 + 4 - 1) = 8$.
2. But the total allocations are 7, so there is degeneracy in the problem.
3. Thus we allocate the unknown quantity ϵ to the cell (1,2) or (2,3).
4. Let us allocate it to cell (2,3), so that the number of allocate cells become 8.

	1	2	3	4	5
1	7(5)	6	4(15)	5(20)	9
2	8	5(30)	6(ϵ)	7	8
3	6(15)	8	9	6	5(5)
4	5(10)	7	7	8	6

5. The optimality test can be performed by the following step :

a. Matrix of (u_i, v_j) for allocated cells.

u_i \ v_j	0	−4	−3	−2	−1
7	7		4	5	
9		5	6		
6	6				5
5	5				

b. Matrix with cell value of $(u_i + v_j)$ for empty cells.

u_i \ v_j	0	−4	−3	−2	−1
7	.	3	.	.	6
9	9	.	.	7	8
6	.	2	3	4	.
5	.	1	2	3	4

i. Let $v_1 = 0$

ii. Now calculating u_i and v_j for all occupied cell.

$$\begin{aligned}
u_1 + v_1 &= C_{11} &\Rightarrow\quad u_1 + 0 = 7 &\Rightarrow u_1 = 7\\
u_3 + v_1 &= C_{31} &\Rightarrow\quad u_3 + 0 = 6 &\Rightarrow u_3 = 6\\
u_4 + v_1 &= C_{41} &\Rightarrow\quad u_4 + 0 = 5 &\Rightarrow u_4 = 5\\
u_1 + v_3 &= C_{13} &\Rightarrow\quad 7 + v_3 = 4 &\Rightarrow v_3 = -3\\
u_1 + v_4 &= C_{14} &\Rightarrow\quad 7 + v_4 = 5 &\Rightarrow v_4 = -2\\
u_3 + v_5 &= C_{35} &\Rightarrow\quad 6 + v_5 = 5 &\Rightarrow v_5 = -1\\
u_2 + v_3 &= C_{23} &\Rightarrow\quad u_2 - 3 = 6 &\Rightarrow u_2 = 9\\
u_2 + v_2 &= C_{22} &\Rightarrow\quad 9 + v_2 = 5 &\Rightarrow v_2 = -4
\end{aligned}$$

iii. Now to calculate the unoccupied cell and put it on the cell evaluation.

$$\begin{aligned}
z_{12} &= u_1 + v_2 - C_{12} = 3\\
z_{15} &= u_1 + v_5 - C_{15} = 3\\
z_{21} &= u_2 + v_1 - C_{21} = 7\\
z_{24} &= u_2 + v_4 - C_{24} = 0\\
z_{25} &= u_2 + v_5 - C_{25} = 0\\
z_{32} &= u_3 + v_2 - C_{32} = 0\\
z_{33} &= u_3 + v_3 - C_{33} = 0
\end{aligned}$$

− 5		15 +	20	
+	30	− ϵ		
15				5
10				

Initial feasible solution with closed path.

5−ϵ		15+ϵ	20	
ϵ	30			
15				5
10				

→

5		15	20	
ϵ	30			
15				5
10				

2nd feasible solution

Step II : Optimality Test :

1. Repeating step I, we get

v_j →	0	–3	–3	–2	–1
u_i = 7	7		4	5	
8	8				
6	6				5
5	5				

Matrix of (u_i, v_j) for allocated cells.

v_j →	0	–3	–3	–2	–1
u_i = 7	·	3	·	·	6
8	·	·	5	6	7
6	·	3	3	4	·
5	·	2	2	3	4

Matrix with cell value of $(u_i + v_j)$ for empty cells.

Cell evaluation matrix $(u_i + v_j - C_{ij})$

·	2	·	·	6
·	·	1	1	1
·	5	6	2	·
·	5	5	5	2

2. All cell values are positive, the second feasible solution is optimal.
3. Therefore optimal transportation is,

Warehouses	1	2	3	4	5	Supply
1	7(5)	6	4(15)	5(20)	9	40
2	8(ε)	5(30)	6	7	8	30
3	6(15)	8	9	6	5(5)	20
4	5(10)	7	7	8	6	10
Demand	30	30	15	20	5	

4. Therefore, total transportation cost is,
$= \text{Rs. } (7 \times 5) + (4 \times 15) + (20 \times 5) + (8 \times \varepsilon) + (5 \times 30) + (6 \times 15) + (5 \times 5) + (10 \times 5)$
$= \text{Rs. } [35 + 60 + 100 + 0 + 150 + 90 + 25 + 50] = \text{Rs. } 510.$

PART-2

Degeneracy in Transportation Problems, Unbalanced Transportation Model.

CONCEPT OUTLINE : PART-2

Degeneracy in Transportation Problem : A basic feasible solution for the general transportation problem must consist of $(m + n - 1)$ occupied cells. The basic solution will be called degenerate when the number of occupied cells is less than the required number $(m + n - 1)$. Degeneracy can occur in initial solution or it may arise in some subsequent iterations.

Unbalanced Transportation Problem : The two sets of constraints will be consistent *i.e.*, the system will be balanced if,

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

If it is not so in any problem then it will be known as an unbalanced transportation problem.

It follows that if many problem demand and supply does not balance then we have to add dummy source / destination to convert it in balanced form.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 2.10. **What is degeneracy ? How does problem of degeneracy arise in transportation problem ? How can we deal with this problem ? Solve the following transportation problem for maximum profit.**

Warehouse	Market A	B	C	D
X	12	18	6	25
Y	8	7	10	18
Z	14	3	11	20

Available units at warehouses : X : 200, Y : 500, Z : 300.
Demand in market : A : 180, B : 320, C : 100, D : 400 units.

Answer

A. Degeneracy in Transportation Problem :

1. A basic feasible solution for the general transportation problem must consist of $(m + n - 1)$ occupied cells.
2. The basic solution will be called degenerate when the number of occupied cells is less than the required number $(m + n - 1)$.
3. Degeneracy can occur in initial solution or it may arise in some subsequent iterations.

B. Problem Arises and Solution in Degeneracy :

1. We have seen that in case of simplex algorithm, the basic feasible solution may become degenerate at the initial stage or at some intermediate stage of computation.

2. In a transportation problem with m origins and n destinations if a basic feasible solution has less than $m + n - 1$ allocation (occupied cells), the problem is said to be a degenerate transportation problem.
3. Degeneracy can occur in the initial solution or during some subsequent iteration.

i. Degeneracy in the Initial Solution :

1. Normally, while finding the initial solution (by any of the methods), any allocation made either satisfies supply or demand, but not both.
2. If, however both supply and demand are satisfied simultaneously, row as well as column are cancelled simultaneously and the number of allocations become one less than $m + n - 1$.
3. If this phenomenon occurs twice, the number of allocations becomes two less than $m + n - 1$ and so on.
4. This degeneracy is resolved or the above degenerate solution is made non-degenerate in the following manner :
   a. First of all the requisite number of vacant cells with least unit costs are chosen so that (in case of tie choose arbitrarily) :
      i. These cells plus the existing number of allocations are equal to $m + n - 1$.
      ii. These $m + n - 1$ cells are in independent positions *i.e.*, no closed path (loop) can be formed among them. If a loop is formed cell / cells with next lower cost is /are chosen so that no loop is formed among them. This can always be done if the solution we start with contains allocated cells in independent positions.
   b. Now allocate an infinitesimally small but positive value ϵ (Greek letter epsilon) to each of the chosen cells. Subscripts are used when more than one such letter is required (*e.g.*, ϵ_1, ϵ_2, etc.).
   c. These ϵ's are then treated like any other positive basic variable and are kept in the transportation array (matrix) until temporary degeneracy is removed or until the optimal solution is reached, whichever occurs first. At that point we set each $\epsilon = 0$. Notice that ϵ is infinitesimally small and hence its effect can be neglected when it is added to or subtracted from a positive value (e.g., $10 + \epsilon = 10$, $5 - \epsilon = 5$, $\epsilon + \epsilon = 2\epsilon$, $\epsilon - \epsilon = 0$).
   d. Consequently, they do not appreciably alter the physical nature of the original set of allocations but do help in carrying out further computations such as optimality test.

ii. Degeneracy During some Subsequent Iteration :

1. Sometimes even if the starting feasible solution is non-degenerate, degeneracy may develop later at some subsequent iteration.
2. This happens when the selection of the entering variable (least value in the closed path that has been assigned a negative sign), causes two or more current basic variables (allocated cell values) to become zero.

3. In this case we allocate ϵ to recently vacated cell with least cost so that there are exactly $m + n - 1$ allocated cells in independent positions and the procedure can then be continued in the usual manner and apply the common difference method for this allocation.

C. Numerical :

1. Convert transportation problem into minimization type problem.

	A	B	C	D	S	Penalty			
X	13	7(200)	19	0	200/0	7	—	—	—
Y	17	18(100)	15	7(400)	500/100/0	8	[8]	2	3
Z	11(180)	22(20)	14(100)	5	300/120/20/0	6	6	3	[6]
D	180/0	320/120/0	100/0	400/0					
	2	[11]	1	5					
Penalty	6	4	1	2					
	[6]	4	1	—					
	—	4	1	—					

2. All allocation in minimization table.

	A	B	C	D	Supply
X	13	7(200)	19	0	200
Y	17	18(100)	15	7(400)	500
Z	11(180)	22(20)	14(100)	5	300
Demand	180	320	100	400	

3. Actual transportation table : This is optimal feasible solution because $m + n - 1 = 6$ in maximum profit table.

	A	B	C	D	Supply
X	12	18(200)	6	25	200
Y	8	7(100)	10	18(400)	500
Z	14(180)	3(20)	11(100)	20	300
Demand	180	320	100	400	

4. Maximum profit of this transportation is

= (200 × 18) + (100 × 7) + (400 × 18) + (180 × 14) + (20 × 3) + (100 × 11)
= Rs. 15180

Que 2.11. **Solve the transportation problem for optimum solution :**

Stores

Warehouses	1	2	3	4	5	6	Supply
1	9	12	9	6	9	10	5
2	7	3	7	7	5	5	6
3	6	5	9	11	3	11	2
4	6	8	11	2	2	10	9
Demand	4	4	6	2	4	2	

UPTU 2012-13, Marks 10

Answer

1. The basic feasible solution is,

Stores

Warehouses	1	2	3	4	5	6	Supply	Penalties						
1	9	12	[5] 9	6	9	10	5/0	3	3	0	0	0	0	[9]
2	7	[4] 3	7	7	5	[2] 5	6/4/0	2	2	2	[4]	—	—	—
3	[1] 6	5	[1] 9	11	3	11	2/1/0	2	2	2	1	3	3	9
4	[3] 6	8	11	[2] 2	[4] 2	10	9/7/3/0	0	0	[4]	2	[5]	—	—
Demand	4/1/0	4/0	6/1/0	2/0	4/0	2/0								

Penalties					
0	2	2	4	1	[5]
0	2	2	[4]	1	—
0	2	2	—	1	—
0	2	2	—	—	—
0	—	0	—	—	—
[3]	—	0	—	—	—

2. Since the number of allocations is 8 which is not equal to 9 $(m + n - 1 = 4 + 6 - 1)$.
3. Therefore the initial solution is degenerate,
4. So an infinitesimally small amount *i.e.*, $\in$ is allocated to it.

9	12	9(5)	6	9	10
7	3(4)	7	7	5(∈)	5(2)
6(1)	5	9(1)	11	3	11
6(3)	8	11	2(2)	4(2)	10

5. Now optimality test can be performed,

Step 1 : Develop u_i and v_j matrix.

	v_1	v_2	v_3	v_4	v_5	v_6
u_1			9			
u_2		3			5	5
u_3	6		9			
u_4	6			2	2	

Putting $v_1 = 0$
$u_3 = 6$ $u_1 = 6$
$u_4 = 6$ $u_2 = 9$
$v_3 = 3$ $v_2 = -6$
$v_4 = -4$
$v_5 = -4$
$v_6 = -4$

Step 2 : Develop $u_i + v_j$ matrix for unallocated cells.

6	0		2	2	2
9		12	5		
	0		2	2	2
	0	9			2

Step 3 : Subtract values of $u_i + v_j$ matrix from original cell value.

3	12		4	7	8
–2 +		–5 –	2		
+	5	–	9	1	9
–	8	2		+	8

6. So new allocations (only show location) are :

	1	2	3	4	5	6
1			5			
2		4	+ ∈		3 – ∈	2
3	1 + ∈		1 – ∈			
4	3 – ∈			2	4 + ∈	

→

	1	2	3	4	5	6
1			5			
2		4	∈			2
3	1		1			
4	3			2	4	

7. Again performing optimality test by repeating step 1, step 2 and step 3 we get, same optimal solution.

Step 4 : Test for Optimality :

a. Number of allocations are $m + n - 1$ (= 9)
b. These $m + n - 1$ allocations are in independent positions.
8. Therefore, optimality test can be applied.

u_i \ v_j	0	–1	3	–4	–4	1
6			9			
4		3	7			5
6	6		9			
6	6			2	2	

(u_i, v_j) matrix for occupied cells

v_j	0	–1	3	–4	–4	1
u_i 6	6	5	·	2	2	7
4	4	·	·	0	0	·
6	·	5	·	2	2	7
6	·	5	9	·	·	7

$(u_i + v_j)$ matrix for vacant cells

3	7	·	4	7	3
3	·	·	7	5	·
·	0	·	9	1	4
·	3	2	·	·	3

Cell evaluation matrix

9. Since all the cell values are positive, the 2nd feasible solution is an optimal solution.
10. Since the above matrix contains a zero entry, there exist alternative optimal solutions.
11. Thus the optimal solution for problem is,

		1	2	3	4	5	6	Supply
Warehouses	1	9	12	9 (5)	6	9	10	5
	2	7	3 (4)	7	7	5	5 (2)	6
	3	6 (1)	5	9 (1)	11	3	11	2
	4	6 (3)	8	11	2 (2)	2 (4)	10	9
Demand		4	4	6	2	4	2	

12. Total cost of transportation,
$= (9 \times 5) + (3 \times 4) + (5 \times 2) + (6 \times 1) + (9 \times 1) + (6 \times 3) + (2 \times 2) + (2 \times 4)$
= Rs. 112

Que 2.12. **What is unbalanced transportation problem ?**

Answer

1. In the problems, if the total availability from all the origins was equal to the total demand at all the destinations *i.e.*, $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$, then such problems are called balanced transportation problems.
2. In many real life situations, however, the total availability may not be equal to the total demand *i.e.*, $\sum_{i=1}^{m} a_i \neq \sum_{j=1}^{n} b_j$; such problems are called

unbalanced transportation problems. In these problems either some available resources will remain unused or some requirements will remain unfilled.

2. Since a feasible solution exists only for a balanced problem, it is necessary that the total availability be made equal to the total demand.
3. If total capacity or availability is more than the demand and if there are no costs associated with the failure to use the excess capacity, we add a dummy (fictitious) destination to take up the excess capacity and the costs of shipping to this destination are set equal to zero.
4. The zero cost cells are treated the same way as real cost cells and the problem is solved as a balanced problem.
5. If there is, however, a cost associated with unused capacity (e.g., maintenance cost) and it is linear, it too can be easily treated.
6. In case the total demand is more than the availability, we add a dummy origin (source) to "fill" the balance requirement and the shipping costs are again set equal to zero.
7. However, in real life, the cost of unfilled demand is seldom zero since it may be more involved.

Que 2.13. **A manufacturer wants to ship 22 loads of his products as shown below. The matrix gives the kilometers from sources of supply to the destinations.**

	D_1	D_2	D_3	D_4	D_5	Supply
S_1	5	8	6	6	3	8
S_2	4	7	7	6	5	5
S_3	8	4	6	6	4	9
Demand	4	4	5	4	8	22 / 25

The shipping cost is Rs. 10 per load per km. What shipping schedule should be used in order to minimize the total transportation cost ?

UPTU 2014-15, Marks 10

Answer

1. Here, demand ≠ supply
So we add a dummy row with supply = 3 units.
Hence, demand = supply
2. Now,

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	1	2	3	4	5	Supply	Penalty			
1	5	8	6	6	3(8)	8/0	2	2	[2]	—
2	4(4)	7	7	6(1)	5	5/1/0	1	1	1	0
3	8	4(4)	6(2)	6(3)	4	9/5/2/0	2	0	0	4
4	0	0	0(3)	0	0	3/0	0	—	—	—
Demand	4/0	4/0	5/2/0	4/3/0	8/0					
	4	4	[6]	6	3					
	1	[3]	—	0	1					
Penalties	1	—	0	0	1					
	[4]	—	—	—	0					

3. **Allocation List :**

$$X_{15} = 8, X_{24} = 1, X_{33} = 2, X_{43} = 3$$
$$X_{21} = 4, X_{32} = 4, X_{34} = 3$$

Allocation cost $= 8 \times 3 + 4 \times 4 + 1 \times 6 + 4 \times 4 + 2 \times 6 + 3 \times 6 + 3 \times 0$
$= 92$ units

4. Total shipping cost = Rs. $92 \times 22 \times 10$ = Rs. 20,240

Que 2.14. **Solve the following transportation problem and find the minimum transportation cost :**

Destination

	D_1	D_2	D_3	D_4	D_5	
S_1	16	16	13	22	17	50
S_2	14	14	12	19	15	60
S_3	19	19	20	23	14	50
	30	20	70	30	60	

UPTU 2011-12, Marks 10

Answer

1. Checking the balancing of transportation problem,

Destination

	D_1	D_2	D_3	D_4	D_5	Supply
S_1	16	16	13	22	17	50 +
S_2	14	14	12	19	15	60 +
S_3	19	19	20	23	14	50 = 160
Demand	30 +	20 +	70 +	30 +	60 = 210	

2. This is an unbalanced transportation problem converting it into balanced transportation problem by adding a dummy row.

A. **By VAM Method :**
Step-1 :

	D_1	D_2	D_3	D_4	D_5	Supply	Penalty			
S_1	16	16	13(50)	22	17	50/0	3	3	3	[3]
S_2	14(10	14(20)	12(20)	19	15(10)	60/0	2	2	2	2
S_3	19	19	20	23	14(50)	50/0	5	5	[5]	—
S_4	0(20)	0	0	0(30)	0	50/20/0	0	0	—	—
Demand	30/10	20	70/20	30/0	60/10					
	14	14	12	[19]	14					
	[14]	14	12	—	14					
Penalties	2	2	1	—	1					
	2	2	1	—	2					

3. Now, By using VAM, we get final allocation

	D_1	D_2	D_3	D_4	D_5
S_1	16	16	13(50)	22	17
S_2	14(10)	14(20)	12(20)	19	15(10)
S_3	19	19	20	23	14(50)
S_4	0(20)	0	0	0(30)	0

4. So number of allocations $= m + n = 8$, this is optimal.
5. Therefore all $d_{ij} \geq 0$, hence the basic feasible solution show by the table and optimal transportation cost is

$= (13 \times 50) + (14 \times 10) + (14 \times 20) + (12 \times 20) + (15 \times 10) + (14 \times 50) + (0 \times 20) + (0 \times 30)$

= Rs. 2160.

PART-3
Assignment Model.

CONCEPT OUTLINE : PART-3

Assignment : The assignment problem may be defined as : "Given n facilities and n jobs and given the effectiveness of each facility for each job, the problem is to assign each facility to one and only one job so as to optimize the given measure of effectiveness."

Ways of Process Job Through Machines :

1. Processing of n job through one machine.

2. Processing of n job through two machine.
3. Processing of n job through three machine.
4. Processing of two job through m machine.
5. Processing of n job through m machine.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 2.15. **What do you mean by assignment problem ? How w you formulate it mathematically ?**

Answer

A. **Assignment Problem :**

The assignment problem is a specific case of the transportation prob in which the objective is to assign a number of resources to equ number of activities at a minimum cost (or maximum profit).

B. **Mathematical Formulation of Assignment Problem :**

1. Consider a problem of assignment of n resources (workers) to n activ (jobs) so as to minimize the overall cost or time in such a way that ea resource can associate with one and only one job.
2. The cost (or effectiveness) matrix (c_{ij}) is given as under :

$$\begin{array}{cc|cccc|c} & & \multicolumn{4}{c|}{\text{Activity}} & \text{Available} \\ & & A_1 & A_2 & \cdots & A_n & \\ \hline & R_1 & c_{11} & c_{12} & \cdots & c_{1n} & 1 \\ & R_2 & c_{21} & c_{22} & \cdots & c_{2n} & 1 \\ \text{Resource} & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & R_n & c_{n1} & c_{n2} & \cdots & c_{nn} & 1 \\ \hline \text{Required} & & 1 & 1 & \cdots & 1 & \end{array}$$

3. The cost matrix is same as that of a transportation problem except th availability at each of the resources and the requirement at each of t destinations is unity (due to the fact that assignments are made o one-to-one basis).
4. Let x_{ij} denote the assignment of i^{th} resource to j^{th} activity, such that

$$x_{ij} = \begin{cases} 1, \text{ if resource } i \text{ is assigned to activity } j \\ 0, \text{ otherwise} \end{cases}$$

Then, the mathematical formulation of the assignment problem is

$$\text{Minimize} \quad z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}\, x_{ij} \text{ subject to the constraints :}$$

$$\sum_{i=1}^{n} x_{ij} = 1 \text{ and } \sum_{j=1}^{n} x_{ij} = 1 \,;\; x_{ij} = 0 \text{ or } 1$$

for all $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$.

c_{ij} is the cost associated with assigning to i^{th} activity.

Que 2.16. **Explain the variations of the assignment problem.**

Answer

1. There are two variations of the assignment problem :

A. **The Maximal Assignment Problem :**

1. Sometimes, the assignment problem deals with the maximization of an objective function rather than to minimize it. For example, it may be required to assign persons to jobs in such a way that the expected profit is maximum.
2. Such problem may be solved easily by first converting it to a minimization problem and then applying the usual procedure of assignment algorithm.
3. This conversion can be very easily done by subtracting from the highest element, all the elements of the given profit matrix, or equivalently, by placing minus sign before each element of the profit matrix in order to make it cost matrix.

B. **Restrictions on Assignment :**

1. Sometimes technical, legal or other restrictions do not permit the assignment of a particular facility to a particular job.
2. Such difficulty can be overcome by assigning a very high cost to the corresponding cell, so that the activity will be automatically excluded from the optimal solution.

Que 2.17. **Write brief notes on :**

A. **Degenerate Transportation Problem, and**
B. **Hungarians method for assignment problems.**

UPTU 2013-14, Marks 10

Answer

A. **Degenerate Transportation Problem :** Refer Q. 2.10, Page 59B, Unit-2.

B. **Hungarian Assignment Method (Minimization Case) :**

1. The Hungarian assignment method provides us with an efficient means of finding of the optimal solution without having to make a direct comparison of every option.

2. It operates on principle of matrix reduction. This just means that by subtracting and adding appropriate numbers in the cost table or matrix we can reduce the problem to matrix of opportunity costs.

3. If we can reduce the matrix to the point where there is one zero element in each row and column, it will then be possible to make optimal assignments *i.e.*, in which all the opportunity costs are zero.

4. Hungarian method of assignment problem can be summarized in the following steps :

Step 1 : Find the opportunity cost table by :

a. Subtracting the smallest number in each row of the original cost table matrix from every number in that row.

b. Then subtracting the smallest number in each column of the table obtained [part(*a*)] from every number in that column.

Step 2 : Make assignments in the opportunity cost matrix in the following ways :

a. Examine the rows successively until a row with exactly one unmarked zero found. Enclose this zero in a box (□) as an assignment will be made there and (×) all other zeros appearing in the corresponding column as they will not be considered for future assignment. Proceed in this way until all the rows have been examined.

b. After examining all the rows completely, examine the columns successively until a column with exactly one unmarked zero is found. Make an assignment to the single zero by putting square (□) around it and cross out (×) all other zeros appearing in the corresponding row as they will not be used to make any other assignment in that row. Proceed in this manner until all columns have been examined.

c. Repeat the operations (*a*) and (*b*) successively until one of the following situation arises :

i. All the zeros in rows or columns are either marked (□) or crossed (×) and there is exactly one assignment in each row and in each column. In such a case optimal assignment policy for the given problem is obtained.

ii. There may be some rows (or columns) without assignment, *i.e.*, the total number of marked zeros is less than the order of the matrix.

Step 3 : Develop the new revised opportunity cost table :

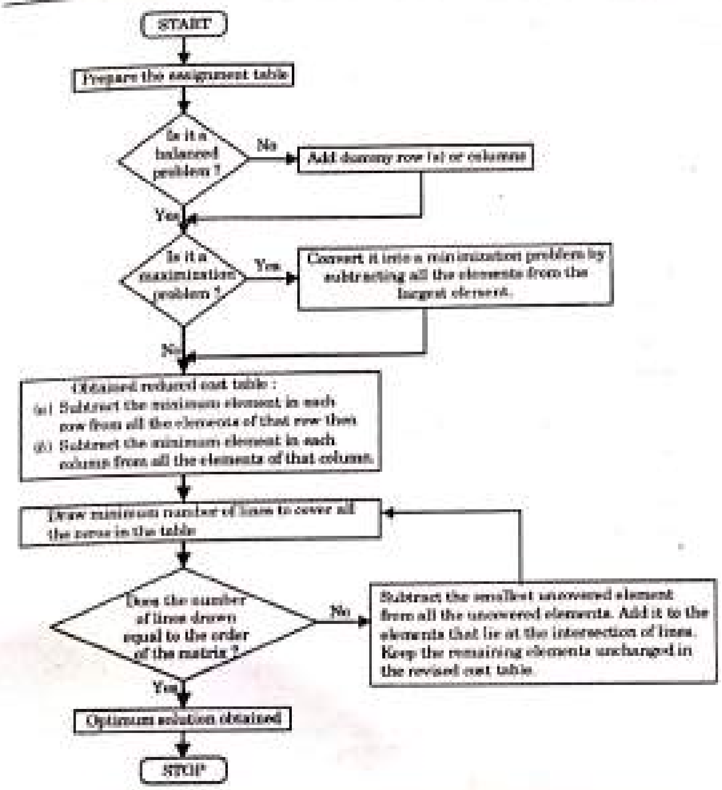


Table. 2.17.1. Flow chart for the Hungarian method.

a. An optimal solution is seldom obtained from the initial opportunity cost table.

b. Often we will need to revised the table in order to shift one (or more) of the zero costs from its present location to a new uncovered location to emerge with a new zero opportunity cost.

c. This is accomplished by subtracting the smallest number not covered by a line from a number not covered by the straight line.

d. This same smallest number is then added to every number (including zeros) lying at the intersection of any two lines.

Step 4 : Repeat step 2 to step 3 until an optimal solution is obtained.

Que 2.18. **A machine tool company decides to make four subassemblies through four contractors. Each contractor is to receive only one subassembly. The cost of each subassembly is**

determined by the bids submitted by each contractor and is shown in table in hundreds of rupees.

		Contractors			
		1	2	3	4
Subassemblies	1	15	13	14	17
	2	11	12	15	13
	3	13	12	10	11
	4	15	17	14	16

A. Formulate the mathematical model for the assignment problem.
B. Assign the different subassemblies to contractors so as to minimize the total cost.

UPTU 2012-13, Marks 10

Answer

A. Formulation of the Model :

Step I : Key decision is what to whom *i.e.*, which subassembly be assigned to which contractor or what are the 'n' optimum assignments on 1-1 basis.

Step II : Possible alternatives are $n!$ possible arrangements for $n \times n$ assignment situation. In the given situation there are 4! different arrangements.

Step III : Objective is to minimize the total cost involved.

i.e., Minimize $\quad Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$.

Step IV : Constraints :

a. Constraints on subassemblies are :

$x_{11} + x_{12} + x_{13} + x_{14} = 1$
$x_{21} + x_{22} + x_{23} + x_{24} = 1$
$x_{31} + x_{32} + x_{33} + x_{34} = 1$
$x_{41} + x_{42} + x_{43} + x_{44} = 1$

b. Constraints on contractors are :

$x_{11} + x_{21} + x_{31} + x_{41} = 1$
$x_{12} + x_{22} + x_{32} + x_{42} = 1$
$x_{13} + x_{23} + x_{33} + x_{43} = 1$
$x_{14} + x_{24} + x_{34} + x_{44} = 1$

B. Assignment :

Step 1 : Subtracting the smallest element of each row from every element of the corresponding row, we get the reduced matrix.

2	0	1	4
0	1	4	2
3	2	0	1
1	3	0	2

Step 2 : Subtracting the smallest element of each column of the reduced matrix from every element of the corresponding column, we get the following reduced matrix :

2	0	1	3
0	1	4	1
3	2	0	0
1	3	0	1

Step 3 : Starting with row 1, we enrectangle (□) (*i.e.*, make assignment) a single zero, if any, and cross (×) all other zeros in the column so marked. Thus, we get

2	$\boxed{0}$	1	3
$\boxed{0}$	1	4	1
3	2	$\cancel{0}$	$\boxed{0}$
1	3	$\boxed{0}$	1

The optimal assignment is :
$1 \to 2, 2 \to 1, 3 \to 4$, and $4 \to 3$.
Therefore minimum total cost = (13 + 11 + 11 +14) × 100 = Rs. 4900.

Que 2.19. Five wagons are available at stations 1,2,3,4, and 5. These are required at five stations *I, II, III, IV*, and *V*. The mileages between various stations are given by the table below. How should the wagons be assigned so as to minimize the total mileage covered ?

	I	II	III	IV	V
1	10	5	9	18	11
2	13	9	6	12	14
3	3	2	4	4	5
4	18	9	12	17	15
5	11	6	14	19	10

UPTU 2013-14, Marks 10

Answer

Step I : Subtracting the smallest element of each row from every element of the corresponding row, we get the reduced matrix.

	I	II	III	IV	V
1	5	0	4	13	6
2	7	3	0	6	8
3	1	0	2	2	3
4	9	0	3	8	6
5	5	0	8	13	4

Step II : Subtracting the smallest element of each column of the reduced matrix from every element of the corresponding column, we get the following reduced matrix :

	I	II	III	IV	V
1	4	0	4	11	3
2	6	3	0	4	5
3	0	0	2	0	0
4	8	0	3	6	3
5	4	0	8	11	1

Step III : Starting with row 1, we enrectangle (□) (*i.e.*, make, assignment) a single zero, if any, and cross (×) other zeros in the column so marked. Thus, we get

4	$\boxed{0}$	4	11	3
6	3	$\boxed{0}$	4	5
$\boxed{0}$	×	2	×	×
8	×	3	6	3
4	×	8	11	1

Step IV :

i. Since rows 4 and 5 do not have any assignment, we mark these rows (√).

ii. Now there is a zero in the column of the marked rows. So, we mark second column (√).

iii. Further there is an assignment in the first row of the ticked column. So, we marked first row (√).

iv. Draw straight lines through all unmarked rows and marked columns. Thus we have.

	I	II	III	IV	V	
1	4	$\boxed{0}$	4	11	3	✓
2	6	3	$\boxed{0}$	4	5	
3	$\boxed{0}$	×	2	×	×	
4	8	×	3	6	3	✓
5	4	×	8	11	1	✓
		✓				

v. In last step, we observe that the minimum number of lines so drawn is 3, which is less than the order of the cost matrix, indicating that the current assignment is not optimum.

vi. To increase the minimum number of lines, we generate new zeros in the modified matrix.

vii. The smallest element not covered by the lines is 1.

viii. Subtracting this element from all the uncovered elements and adding the same to all the elements at the intersection of the lines, we obtain the following new reduced matrix.

	I	II	III	IV	V
1	3	0	3	10	2
2	6	4	0	4	5
3	0	1	2	0	0
4	7	0	2	5	2
5	3	0	7	10	0

Step V : Repeating the step III and IV, for the optimal solution.

	I	II	III	IV	V	
1	3	$\boxed{0}$	3	10	2	✓
2	6	4	$\boxed{0}$	4	5	
3	$\boxed{0}$	1	2	×	×	
4	7	×	2	5	2	✓
5	3	×	7	10	$\boxed{0}$	
		✓				

	I	II	III	IV	V
1	1	0	1	8	0
2	6	6	0	4	5
3	0	3	2	0	0
4	5	0	0	3	0
5	3	2	7	10	0

Step VI : This is not optimal further check the optimal solution repeating the step III and VI.

	I	II	III	IV	V	
1	1	$\boxed{0}$	1	8	×	✓
2	6	6	$\boxed{0}$	4	5	✓
3	$\boxed{0}$	3	2	×	×	✓
4	5	×	×	3	×	✓
5	3	2	7	10	$\boxed{0}$	✓
		✓	✓		✓	

	I	II	III	IV	V
1	0	0	1	7	0
2	5	6	0	3	5
3	0	4	3	0	1
4	4	0	0	2	0
5	2	2	7	9	0

Assignment the above matrix, we get,

	I	II	III	IV	V
1	$\boxed{0}$	$\times$	1	7	$\times$
2	5	6	$\boxed{0}$	3	5
3	$\times$	4	3	$\boxed{0}$	1
4	4	$\boxed{0}$	$\times$	2	$\times$
5	2	2	7	9	$\boxed{0}$

Now, since each row and each column has one and only one assignment, an optimal solution is reached. Thus, the optimum assignment is :

$1 \to I$, $2 \to III$, $3 \to IV$, $4 \to II$, and $5 \to V$

The minimum total mileage associated with this solution is

$$Z_{min} = 10 + 6 + 4 + 9 + 10 = 39$$

Que 2.20. A marketing manager has five salesman and five sales districts. Considering the capabilities of the salesman and the nature of the districts, the marketing manager estimates that the sales per month (in hundred rupees) for each salesman in each district would be as follows :

		Districts				
		A	*B*	*C*	*D*	*E*
Sales man	1	32	38	40	28	40
	2	40	24	28	21	36
	3	41	27	33	30	37
	4	22	38	41	36	36
	5	29	33	40	35	39

Find the assignment of salesman to district that will remain in maximum sales.

UPTU 2014-15, Marks 10

Answer

Same as Q. 2.19, Page 73B, Unit-2.

After using changed values,

For maximum sales, allocate these located zeroes to the original matrix. i.e.,

		Districts				
		A	*B*	*C*	*D*	*E*
Salesman	1	32	$\boxed{38}$	40	28	40
	2	$\boxed{40}$	24	28	21	36
	3	41	27	33	30	$\boxed{37}$
	4	22	38	$\boxed{41}$	36	36
	5	29	33	40	$\boxed{35}$	39

Now, the maximum sale = 38 + 40 + 37 + 41 + 35 = 191

Hence, the maximum sales per month = 191 × 100

= Rs. 19100

Que 2.21. Alpha Corporation has four plants, each of which can manufacture any one of four products *A*, *B*, *C* or *D*. Production costs differ from one plant to another and so do the sales revenue. Given the revenue and the cost data below, determine which product should each plant produce to maximize profit.

Sales Revenue (in Rs.1,000)

		Plant			
		1	2	3	4
Product	*A*	50	68	49	61
	B	60	70	50	75
	C	55	67	52	70
	D	58	65	55	69

Production cost in (Rs.1,000)

		Plant			
		1	2	3	4
Product	*A*	49	60	45	60
	B	54	63	46	49
	C	51	62	48	68
	D	55	63	49	66

UPTU 2011-12, Marks 10

Answer

1. We know that

Profit = Sales revenue − Product cost

2. So, according to this,

Profit matrix

$$
\text{Product}\quad
\begin{array}{c|cccc|}
 & \multicolumn{4}{c}{\text{Plant}} \\
 & 1 & 2 & 3 & 4 \\
\hline
A & 1 & 8 & 4 & 1 \\
B & 6 & 7 & 4 & 26 \\
C & 4 & 5 & 4 & 2 \\
D & 3 & 2 & 6 & 3 \\
\hline
\end{array}
$$

3. This is the maximization problem so we have to need to convert this problem to minimization problem.

4. For minimization conversion, we should subtract each element of the matrix from the greatest element (26) of the matrix so we obtain the matrix,

$$
\begin{array}{c|cccc|}
 & \multicolumn{4}{c}{\text{Plant}} \\
 & 1 & 2 & 3 & 4 \\
\hline
A & 26-1 & 26-8 & 26-4 & 26-1 \\
B & 26-6 & 26-7 & 26-4 & 26-26 \\
C & 26-4 & 26-5 & 26-4 & 26-2 \\
D & 26-3 & 26-2 & 26-6 & 26-3 \\
\hline
\end{array}
\Rightarrow
\begin{array}{c|cccc|}
 & 1 & 2 & 3 & 4 \\
\hline
A & 25 & 18 & 22 & 25 \\
B & 20 & 19 & 22 & 0 \\
C & 22 & 21 & 22 & 24 \\
D & 23 & 24 & 20 & 23 \\
\hline
\end{array}
$$

5. Subtract the minimum element of each row from all element of the corresponding row.

$$
\begin{array}{c|cccc|c}
 & 1 & 2 & 3 & 4 & \text{Min} \\
\hline
A & 25 & 18 & 22 & 25 & 18 \\
B & 20 & 19 & 22 & 0 & 0 \\
C & 22 & 21 & 22 & 24 & 21 \\
D & 23 & 24 & 20 & 23 & 20 \\
\hline
\end{array}
\Rightarrow
\begin{array}{c|cccc|}
 & 1 & 2 & 3 & 4 \\
\hline
A & 7 & 0 & 4 & 7 \\
B & 20 & 19 & 22 & 0 \\
C & 1 & 0 & 1 & 3 \\
D & 3 & 4 & 0 & 3 \\
\hline
\end{array}
$$

6. Subtract the minimum element of each column from all element of the corresponding column.

$$
\begin{array}{c|cccc|}
 & 1 & 2 & 3 & 4 \\
\hline
A & 7 & 0 & 4 & 7 \\
B & 20 & 19 & 22 & 0 \\
C & 1 & 0 & 1 & 3 \\
D & 3 & 4 & 0 & 3 \\
\hline
\text{Column Min} & 1 & 0 & 0 & 0
\end{array}
\Rightarrow
\begin{array}{c|cccc|}
 & 1 & 2 & 3 & 4 \\
\hline
A & 6 & 0 & 4 & 7 \\
B & 19 & 19 & 22 & 0 \\
C & 0 & 0 & 1 & 3 \\
D & 2 & 4 & 0 & 3 \\
\hline
\end{array}
$$

$$
\text{Or}\quad
\begin{array}{c|cccc|}
 & 1 & 2 & 3 & 4 \\
\hline
A & 6 & \boxed{0} & 4 & 7 \\
B & 19 & 19 & 22 & \boxed{0} \\
C & \boxed{0} & \cancel{0} & 1 & 3 \\
D & 2 & 4 & \boxed{0} & 3 \\
\hline
\end{array}
$$

7. Since each row and column is having only one assignment, the optimal assignment is reached and is given by :
$A \to 2, B \to 4, C \to 1, D \to 3$
Hence, the optimal maximum profit is
= Rs (8 + 26 + 4 + 6)
= Rs. 44

☺☺☺

3

UNIT

Decision and Game Theory

PART-1

Game Theory, Two Person Zero Sum Game, Solution with / without Saddle Point, and Dominance Rule.

CONCEPT OUTLINE : PART-1

Game : Game is defined as an activity, between two or more persons, involving activities by each person according to a set of rules, at the end of which each person receives some benefits or satisfaction or suffers loss (negative benefit).

Payoff : A quantitative measure of satisfaction a person gets at the end of each play is called payoff. It is a real valued function of variables in the game.

Rectangular Game : The game whose payoff matrix is of the order $m \times n$ ($m \neq n$), is termed as rectangular game.

Strategy : It can be viewed as a rule for decision making in advance of all the games by which the player decides the activities he should adopt.

a. **Pure Strategy :** If a player knows exactly what the other player is going to do, like deterministic situation.
b. **Mixed Strategy :** If a player is guessing as to which activity is to be selected by the other player on any particular occasion, like probabilistic situation.

Minimax (Maximin) Criterion : It states that if a player lists the worst possible outcome of all his potential strategies, he will choose that strategy to be the most suitable for him which corresponds to the best of these worst outcomes. Such a strategy is called an optimal strategy.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 3.1. Write a short note on game theory and give its characteristics.

Answer

A. **Game Theory :**

1. The theory of games (game theory or competitive strategies) is a mathematical theory that deals with the general features of competitive situations.

2. This theory is helpful when two or more individuals or organiza[illegible] with conflicting objectives try to make decisions.
3. In such situations, a decision made by one decision-maker affec[illegible] decision made by one or more of the remaining decision makers and final outcome depend upon the decision of all the parties.
4. Such situations often arise in the fields of business, industry, econ[illegible] sociology and military training.
5. This theory is applicable to a wide variety of situations such as players struggling to win at chess, candidates fighting an election, enemies planning war tactics, firms struggling to maintain their ma[illegible] shares, launching advertisement campaigns by companies marke[illegible] competing product, negotiations between organizations and unions.

B. Characteristics of Game Theory :

1. There are finite numbers of participants or competitors. If the nu[illegible] of participants is 2, the game is called two-person game; for nu[illegible] greater than two, it is called *n*-person game.
2. Each participant has available to him a list of finite number of pos[illegible] courses of action. The list may not be same for each participant.
3. Each participant knows all the possible choices available to others [illegible] does not know which of them is going to be chosen by them.
4. A play is said to occur when each of the participants chooses one of [illegible] courses of action available to him. The choices are assumed to be m[illegible] simultaneously so that no participant knows the choices made by ot[illegible] until he has decided his own.
5. Every combination of courses of action determines an outcome wh[illegible] results in gains to the participants. The gain may be positive, negati[illegible] zero. Negative gain is called a loss.
6. The gain of a participant depends not only on his own actions but [illegible] those of others.
7. The gains (payoffs) for each and every play are fixed and specifie[illegible] advance and are known to each player. Thus each player knows [illegible] the information contained in the payoff matrix.
8. The players make individual decisions without direct communica[illegible]

Que 3.2. **Write any four limitations of game theory.**

Answer

1. The environment in which management decisions are made is rar[illegible] two-person; the government or society is often an external party inv[illegible] in decision making.
2. Sum of the gains and losses of the opponents may not be zero, resul[illegible] in non-zero-sum games.

3. In real life situations, it is very rare that both parties will have equal information and intelligence.
4. It is not easy to find the values of the payoff matrix accurately. Inaccurate values in the matrix will yield misleading results. To establish that one outcome is better than the other may not be difficult, but it is quite another thing to establish exactly how much better.

Que 3.3. **Describe the following :**

A. Two person zero sum game,

B. Payoff matrix.

Answer

A. Two Person Zero Sum Game :

1. When there are two competitors playing a game, it is called a 'two-person game'.
2. Games having the 'zero-sum' character *i.e.*, the algebraic sum of gains and losses of all the players is zero are called zero-sum games.
3. The play does not add a single paisa to the total wealth of all the players; it merely results in a new distribution of initial money among them.
4. Zero-sum games with two players are called two person zero sum games. In this case the loss (gain) of one player is exactly equal to the gain (loss) of the other.

B. Payoff Matrix :

1. When the players select their particular strategies, the payoffs (gains or losses) can be represented in the form of a matrix called the payoff matrix.
2. Since the game is zero-sum, therefore gain of one player is equal to the loss of other and vice-versa.
3. In other words, one player's payoff table would contain the same amounts in payoff table of other player with the sign changed. Thus, it is sufficient to construct payoff only for one of the players.
4. Let player A have m strategies $A_1, A_2, ..., A_m$ and player B have n strategies $B_1, B_2, ..., B_n$.
5. It is assumed that each player has his choices among the pure strategies. Also it is assumed that player A is always the gainer and player B is always the loser. It means, all payoffs are assumed in terms of player A.
6. Let a_{ij} be the payoff which player A gains from player B if player A chooses strategy A_i and player B chooses strategy B_j. Then the payoff matrix to player A is :

So,

For A optimal strategy = II

For B optimal strategy = II

2. Game value for $A = 0$
3. Game value for $B = 0$

Que 3.7. **Find the range of values of p and q which will render the entry (2, 2) a saddle point for the game.**

Player B

	2	4	5
Player A	10	7	q
	4	p	6

Answer

1. First ignoring the values of p and q we determine the maximin and minimax values of payoff matrix as follows :

	B_1	B_2	B_3	Row Minima
A_1	2*	4	5	2
A_2	10'	7*'	q	7
A_3	4*	p	6'	4
Column Maxima	10	7	6	

2. Since the entry (2, 2) is saddle point for the game,

$\therefore$ Maximin value = 7

Minimax value = 7

3. This imposes the condition on p as $p \le 7$ and on q as $q \ge 7$. Hence the range of p and q will be $p \le 7$, $q \ge 7$.

Que 3.8. **Solve the following game by using the principle of dominance :**

Player A	Player B					
	B_1	B_2	B_3	B_4	B_5	B_6
A_1	4	2	0	2	1	1
A_2	4	3	1	3	2	2
A_3	4	3	7	−5	1	2
A_4	4	3	4	−1	2	2
A_5	4	3	3	−2	2	2

Find the best strategy for each player and the value of the game.

UPTU 2013-14, Marks 10

Answer

1. The given payoff matrix has no saddle point. From player A's point of view, row 1(A_1) is dominated by row 2 (A_2) and row 5 (A_5) is dominated by row 4 (A_4). Accordingly, row1 (A_1) and 5(A_5) are deleted. The following reduced matrix results.

Player B

		B_1	B_2	B_3	B_4	B_5	B_6
Player A	A_2	4	3	1	3	2	2
	A_3	4	3	7	−5	1	2
	A_4	4	3	4	−1	2	2

2. From player B's point of view, column B_1 and B_2 are dominated by column B_3, B_4 and B_5, also column B_6 is dominated by column B_5.

Therefore, columns B_1, B_2 and B_6 are deleted, resulting in

Player B

		B_3	B_4	B_5
	A_2	1	3	2
Player A	A_3	7	−5	1
	A_4	4	−1	2

3. Now none of single row (or column) dominates another row (or column). However, column 5 (B_5) is dominated by the average of column B_3, B_4, which is

$$\begin{bmatrix} \frac{1+3}{2} \\ \frac{7-5}{2} \\ \frac{4-1}{2} \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3/2 \end{bmatrix}$$

4. Accordingly, column B_5 is deleted and the following matrix is obtained :

Player B

		B_3	B_4
	A_2	1	3
Player A	A_3	7	−5
	A_4	4	−1

5. Further, row A_4 is dominated by the average of row A_2 and A_3.

Hence row A_4 is deleted. The resulting 2 × 2 game is obtained.

		Player B B_3	B_4
Player A	A_2	1	3
	A_3	7	–5

6. Solving this game by arithmetic method, we get

		Player B B_3	B_4	
Player A	A_2	1	3	12 6 6/7
	A_3	7	–5	2 1 1/7
		8	6	
		4	3	
		4/7	3/7	

7. Therefore, optimal strategy for player A : (0, 6/7, 1/7, 0, 0)
   Optimal strategy for player B : [0, 0, 4/7, 3/7, 0, 0]
8. Game value : $\dfrac{1\times 8+3\times 6}{8+6} = \dfrac{26}{14} = \dfrac{13}{7}$

Que 3.9. **Reduce the following game by dominance property and solve it :**

		Player B 1	2	3	4	5
Player A	I	1	3	2	7	4
	II	3	4	2	5	6
	III	6	5	7	6	5
	IV	2	0	6	3	1

Answer

1. Here, row IV is dominated by row III. Deleting row IV we get

		Player B 1	2	3	4	5
Player A	I	1	3	2	7	4
	II	3	4	2	5	6
	III	6	5	7	6	5

2. Now, column 4 is dominated by columns 1 and 2, also column 5 is dominated by column 2. Therefore, deleting columns 4 and 5 we get

		Player B 1	2	3
Player A	I	1	3	2
	II	3	4	1
	III	6	5	7

3. In the above matrix, row I as well as II are dominated by row III. Therefore, we delete rows I and II and get

		Player B 1	2	3
Player A	III	6	5	7

4. Out of the three strategies available to player B, he will use no. 2 in order to minimize his losses. Therefore, the solution to the problem is optimal strategy for A : III, optimal strategy for B : 2; game value (for A): 5.
5. **Remark :** Reduction of this problem by the rules of dominance has resulted in saddle point (III, 2). This is because the original problem has the saddle point. It was reduced as per the statement of the problem; otherwise it should have been directly solved using the saddle point method.

Que 3.10. **Solve the following game whose payoff matrix is given by :**

	B_1	B_2	B_3	B_4	B_5
A_1	3	–1	4	6	7
A_2	–1	8	2	4	12
A_3	16	8	6	14	12
A_4	1	11	–4	2	1

UPTU 2011-12, Marks 05

Answer

1. Pay off matrix is

		Player B B_1	B_2	B_3	B_4	B_5	Row minimum
Player A	A_1	3	-1^*	4	6	7	–1
	A_2	-1^*	8	2	4	12^+	–1
	A_3	16^+	8	$\boxed{6^*}$	14^+	12	6
	A_4	1	11	-4^*	2	1	–4
Column max		16	11	6	14	12	

2. Since the entry (A_3, B_3) *i.e.*, (III, III) is saddle point for the game,

$\therefore$ Maximin = 5

Minimax = 5

Value of game = 5

Strategy for $A : A_3$: III row

Strategy for $B : B_3$: III column

3. The value of game for player A = 6 and – 6 for player B.

Que 3.11. **For what value of λ, the game with following payoff matrix is strictly determinable ?**

Player A	Player B		
	B_1	B_2	B_3
A_1	λ	6	2
A_2	– 1	λ	– 7
A_3	– 2	4	λ

UPTU 2014-15, Marks 10

Answer

Player A	Player B			Row minima
	B_1	B_2	B_3	
A_1	λ	6	2	2
A_2	–1	λ	–7	– 7
A_3	–2	4	λ	– 2
Column maxima	–1	6	2	

$\therefore$ Maximin value = 2 and minimax value = – 1

$\therefore$ Value of the game lies between – 1 and 2 *i.e.*, $-1 \le V \le 2$

$\therefore$ For strictly determinable game, since,

maximum value = minimax value, we must have $-1 \le \lambda \le 2$.

Que 3.12. **In a game of matching coins, player A wins Rs. 2 if there are two heads, wins nothing if there are two tails and loses Re. 1 when there are one head and one tail. Determine the payoff matrix, best strategies for each player and the value of game to A.**

Answer

1. The payoff matrix for A will be

Player A	Player B	
	H	T
H	2	– 1
T	– 1	0

2. Since, there is no saddle point, the optimal strategies will be mixed strategies. Using the steps described above, we get

Player A	Player B			
	H	T		
H	2	– 1	1	1/4 = 0.25
T	– 1	0	3	3/4 = 0.75
	1	3		
	0.25	0.75		

3. Thus for optimum gains, player A should use strategy H for 25 % of the time and strategy T for 75 % of the time, while player B should use strategy H 25 % of the time and strategy T 75 % of the time.

4. To obtain the value of the game any of the following expressions may be used :

5. **Using A's oddments :**

B plays H; value of the game, $V = \text{Rs.}\left(\dfrac{1\times 2 - 3\times 1}{3+1}\right) = \text{Rs.}\left(-\dfrac{1}{4}\right)$

B plays T; value of the game, $V = \text{Rs.}\left(\dfrac{1\times(-1) + 3\times 0}{3+1}\right) = \text{Rs.}\left(-\dfrac{1}{4}\right)$

6. **Using B's oddments :**

A plays H; value of the game, $V = \text{Rs.}\left(\dfrac{1\times 2 - 1\times 3}{3+1}\right) = \text{Rs.}\left(-\dfrac{1}{4}\right)$

A plays T; value of the game, $V = \text{Rs.}\left(\dfrac{-1\times 1 + 0\times 3}{1+3}\right) = \text{Rs.}\left(-\dfrac{1}{4}\right)$

7. The above values of V are equal only if sum of the oddments vertically and horizontally are equal. Cases in which it is not so are treated later.

Thus the full solution of the game is

$A(1, 3)$; $B(1, 3)$; V = Rs. (– 1/4)

8. This is the value of the game to A *i.e.*, A gains Rs. (–1/4) *i.e.*, he loses Rs. 1/4 which B, in turn, gets. Arithmetic method is easier than algebraic method but it cannot be applied to larger games.

PART-2

Different Methods Like Algebraic, Graphical, Linear Programming

CONCEPT OUTLINE : PART-2

Graphical Solution of 2 × n or m × 2 Games : The problem may originally be a 2 × n or a m × 2 game or a problem might have been reduced to such size after applying the dominance rule. In either case, we can use graphical method to solve the problem. By using the graphical approach, it is aimed to reduce a game to the order of 2 × 2 by identifying and eliminating the dominated strategies, and then solve it by the analytical method used for solving such games. The resultant solution is also the solution to the original problem.

Method of Linear Programming : It is the most general method of solving any 2 person zero sum game. If there is no saddle point, dominance is unsuccessful in reducing the game and the method of matrices also fails then linear programming offers the best method of solution.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 3.13. Use graphical method to solve the game whose payoff matrix is :

		Player B			
		I	II	III	IV
Player A	I	2	1	0	-2
	II	1	0	3	2

Answer

1. The problem does not possess a saddle point. Let the player A play the mixed strategy $S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix}$ where $p_2 = 1 - p_1$, against B. Then A's expected payoffs against B's pure moves are given by,

B's pure move	A's expected payoff $E(p_1)$
I	$E_1(p_1) = p_1 + 1$
II	$E_2(p_1) = p_1$
III	$E_3(p_1) = -3p_1 + 3$
IV	$E_4(p_1) = -4p_1 + 2$

2. These expected payoff equations are then plotted as functions of p_1 as shown in Fig. 3.13.1, which shows the payoffs of each column represented point on two vertical axis 1 and 2, unit distance apart.
3. Thus line I joins the first payoff element 2 in the first column represented by +2 on axis 2, and the second payoff element 1 in the first column represented by +1 on axis 1. Similarly, lines II, III and IV join the corresponding representation of payoff elements in the second, third and fourth columns.
4. Since the player 1 wishes to maximize its minimum expected payoff we consider the highest point of intersection H on the lower envelope of 1's expected payoff equations.
5. This point H represents the maximum expected value of the game for 1. The line II and IV, passing through H, define the two relevant moves II and IV that alone 2 needs to play. The solution to the original 2 × 4 game, therefore, boils down that of the simpler game with the 2 × 2 payoff matrix :

Now if $S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix}$ and $S_B = \begin{bmatrix} II & IV \\ q_2 & q_4 \end{bmatrix}$

be the optimum strategies for A and B, then we have

$$p_1 = \frac{2-0}{1+2-(-2)} = 2/5, p_2 = 1 - p_1 = 3/5,$$

$$q_2 = \frac{2-(-2)}{1+2-(-2)} = 4/5, q_4 = 1 - q_2 = 1/5.$$

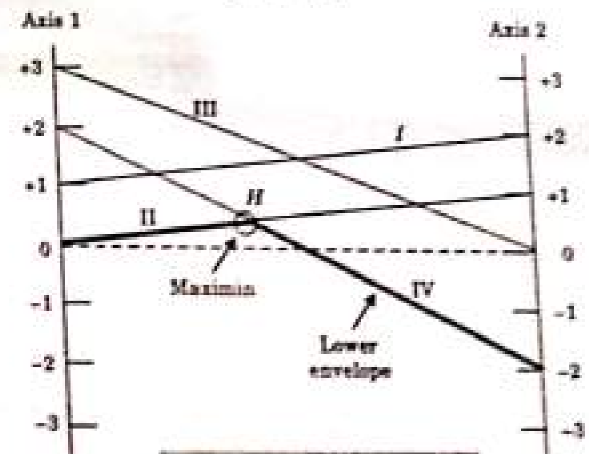


Fig. 3.13.1. The maximin value.

Hence, the solution to the game is

i. The optimum strategy for A is $S_A = \begin{bmatrix} A_1 & A_2 \\ 2/5 & 3/5 \end{bmatrix}$.

ii. The optimum strategy for B is $S_B = \begin{bmatrix} I & II & III & IV \\ 0 & 4/5 & 0 & 1/5 \end{bmatrix}$ and

iii. The expected value of the game is $V = \dfrac{2 \times 1 - 0 \times (-2)}{1 + 2 - (0 - 2)} = \dfrac{2}{5}$.

Que 3.14. **Obtain the optimal strategies for both players and the value of the game for two person zero sum game whose payoff matrix is as follows :**

Player A	Player B	
	B_1	B_2
A_1	1	−3
A_2	3	5
A_3	−1	6
A_4	4	1
A_5	2	2
A_6	−5	0

UPTU 2012-13, Marks 10

Answer

1. Given :

Player A	Player B	
	B_1	B_2
A_1	1	−3
A_2	3	5
A_3	−1	6
A_4	4	1
A_5	2	2
A_6	−5	0

2. Add 5 in each element.

Player A	Player B	
	B_1	B_2
A_1	6	2
A_2	8	10
A_3	4	11
A_4	9	6
A_5	7	7
A_6	0	5

3. A's pure move — B's expected payoff $E(q_1)$

A's pure move	B's expected payoff $E(q_1)$
A_1	$E_1(q_1) = 4q_1 + 2$
A_2	$E_2(q_1) = -2q_1 + 10$
A_3	$E_3(q_1) = -7q_1 + 11$
A_4	$E_4(q_1) = 3q_1 + 6$
A_5	$E_5(q_1) = +7$
A_6	$E_6(q_1) = 5 - 5q_1$

4. 2 × 2 payoff matrix for player A = $\begin{array}{c c} & \begin{array}{cc} B_2 & B_4 \end{array} \\ \begin{array}{c} A_1 \\ A_4 \end{array} & \begin{bmatrix} 3 & 5 \\ 4 & 1 \end{bmatrix} \end{array}$

$$p_1 = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{1 - 4}{3 + 1 - (5 + 4)} = \frac{3}{5}$$

$$p_2 = \frac{2}{5}$$

$$q_1 = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{1 - 5}{-5} = \frac{4}{5}$$

$$q_2 = \frac{1}{5}$$

$$V = \frac{a_{11}\,a_{22} - a_{21}\,a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{3 \times 1 - 4 \times 5}{-5} = \frac{17}{5}$$

$$S_A = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 \\ 0 & 3/5 & 0 & \dfrac{2}{5} \end{bmatrix}, S_B = \begin{bmatrix} q_1 & q_2 \\ \dfrac{4}{5} & \dfrac{1}{5} \end{bmatrix}$$

Que 3.15. **Describe the procedure for solution of $m \times n$ game as an LPP.**

Answer

1. Let us consider an $m \times n$ payoff matrix (a_{ij}) for player A. Let

$$S_m = \begin{bmatrix} A_1, \ldots, A_m \\ p_1, \ldots, p_m \end{bmatrix} \text{ and } S_n = \begin{bmatrix} B_1, \ldots, B_n \\ q_1, \ldots, q_n \end{bmatrix}$$

Where $\sum_{i=1}^{m} p_i = \sum_{j=1}^{n} q_j = 1$, be the mixed strategies for the two players respectively.

2. Then, the expected gains g_j ($j = 1, 2, \ldots, n$) of player A against B's pure strategies will be

$$g_1 = a_{11}p_1 + a_{21}p_2 + \ldots + a_{m1}p_m$$
$$g_2 = a_{12}p_1 + a_{22}p_2 + \ldots + a_{m2}p_m$$
$$\vdots$$
$$g_n = a_{1n}p_1 + a_{2n}p_2 + \ldots + a_{mn}p_m$$

and expected losses l_i ($i = 1, 2, \ldots, m$) of players B against A's pure strategies will be

$$l_1 = a_{11}q_1 + a_{12}q_2 + \ldots + a_{1n}q_n$$
$$l_2 = a_{21}q_1 + a_{22}q_2 + \ldots + a_{2n}q_n$$
$$\vdots$$
$$l_m = a_{m1}q_1 + a_{m2}q_2 + \ldots + a_{mn}q_n$$

3. The objective of player A is to select p_i ($i = 1, 2, \ldots, m$) such that he can maximize his minimum expected gains; and the player B desires to select q_j ($j = 1, 2, \ldots, n$) that will minimize his expected losses.
4. Thus, if we let $u = \min_j \sum_{i=1}^{m} a_{ij}p_i$ ($j = 1, 2, \ldots, n$) and $v = \max_i \sum_{j=1}^{n} a_{ij}q_j$ ($i = 1, 2, \ldots, m$); the problem of two players could be written as :

a. Player A

Maximize u = Minimize $\frac{1}{u} = \sum_{i=1}^{m} \frac{p_i}{u}$ subject to the constraints :

$$\sum_{i=1}^{m} a_{ij}p_i \geq u \text{ and } \Sigma p_i = 1, p_i \geq 0 \; (i = 1, 2, \ldots, m)$$

b. Player B

Minimize v = Maximize $\frac{1}{v} = \sum_{j=1}^{n} \frac{q_j}{v}$ subject to the constraints :

$$\sum_{j=1}^{n} a_{ij}q_j \leq v \text{ and } \Sigma q_j = 1, q_j \geq 0 \; (j = 1, 2, \ldots, n)$$

5. Assuming that $u > 0$ and $v > 0$, we introduce new variables defined by $p'_i = p_i / u$ and $q'_j = q_j / v$ ($i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$).
6. Then the pair of linear programming problems can be re-written as :

a. Player A

Minimize $p_0 = p'_1 + p'_2 + \ldots + p'_m$ subject to the constraints :

$$a_{1j}p'_1 + a_{2j}p'_2 + \ldots + a_{mj}p'_m \geq 1$$
$$p'_i \geq 0 \qquad (i = 1, 2, \ldots, m \; ; j = 1, 2, \ldots, n)$$

b. Player B

Maximize $q_0 = q'_1 + q'_2 + \ldots + q'_n$ subject to the constraints :

$$a_{i1}q'_1 + a_{i2}q'_2 + \ldots + a_{in}q'_n \leq 1$$
$$q'_j \geq 0 \qquad (i = 1, 2, \ldots, m \; ; j = 1, 2, \ldots, n)$$

Que 3.16. **In the following two-person zero-sum game the payoff table for player 1 is given :**

		Player 2		
Strategy		1	2	3
Player 1	I	0	–2	2
	II	5	4	–3
	III	2	3	–4

Is any of the strategies dominated by any other ? If so, reduce the table. Does the game have a stable solution ? Show with the help of minimax and maximin values. Use mixed strategies and formulate the problem as a linear programming problem.

Answer

1. Given :

		Player 2		
Strategy		1	2	3
Player 1	I	0	–2	2
	II	5	4	–3
	III	2	3	–4

i. Yes.

ii. If no then,

i. Row III[rd] is dominant by row II[nd], so deleting row III[rd].

		Player 2		
		1	2	3
Player 1	I	0	–2	2
	II	5	4	–3

2. Column 1 is dominated by column 2, so deleting column 1.

		Player 2		Row minimum
		2	3	
Player 1	I	– 2*	2'	– 2
	II	4'	– 3*	– 3
Column maximum		4	2	

3. Since, there is no saddle point, hence there is no stable solution, value of game lies between – 2 to 2.

Maximin value = –2
Minimax value = 2

4. **Formulation of LPP :**

Player 2

		2	3
Player 1	I	-2	2
	II	4	-3

5. Since two of the entries of the payoff matrix are negative, we shall add a constant say 4, to each of the values, by which each of them would become a positive value.

Player 2

		2	3
Player 1	I	2	6
	II	8	1

6. Let x_1 and x_2 represent the probabilities with which player 1 chooses strategies I and II respectively, while y_1 and y_2 be the probabilities in respect of player 2 choosing strategies 2 and 3.

7. From 1st player point of view :

Minimize : $\frac{1}{U} = X_1 + X_2$

Subject to : $2X_1 + 8X_2 \geq 1$

$6X_1 + X_2 \geq 1$

$X_1, X_2 \geq 0$

Where, $X_1 = \frac{x_1}{U}$ and $X_2 = \frac{x_2}{U}$

8. Similarly for player 2 :

Maximize : $\frac{1}{V} = Y_1 + Y_2$

Subject to : $2Y_1 + 6Y_2 \leq 1$

$8Y_1 + Y_2 \leq 1$

$Y_1, Y_2 \geq 0$

Where, $Y_i = \frac{y_i}{V}$ $(i = 1, 2)$

Que 3.17. **For the following payoff matrix, find the value of the game and strategies using linear programming :**

Player B

Player A	Strategy	1	2	3
	1	3	-1	4
	2	6	7	-2

Answer

1. Since two of the entries in the payoff matrix are negative, we shall add a constant say 3, to each of the values, by which each of them would become a positive value. The payoff matrix then becomes as

Player B

		1	2	3
Player A	1	6	2	7
	2	9	10	1

2. Now let x_1 and x_2 represent the probabilities with which A choose strategies 1 and 2 respectively, while y_1, y_2 and y_3 be the probabilities in respect of B choosing strategies 1, 2 and 3.

3. From A's point of view, the problem is

Minimize : $\frac{1}{U} = X_1 + X_2$

Subject to : $6X_1 + 9X_2 \geq 1$

$2X_1 + 10X_2 \geq 1$

$7X_1 + 1X_2 \geq 1$

$X_1, X_2 \geq 0$

Where $X_1 = \frac{x_1}{U}$ and $X_2 = \frac{x_2}{U}$

4. Similarly from B's point of view, the problem is

Maximize : $\frac{1}{V} = Y_1 + Y_2 + Y_3$

Subject to :

$6Y_1 + 2Y_2 + 7Y_3 \leq 1$

$9Y_1 + 10Y_2 + Y_3 \leq 1$

$Y_1, Y_2, Y_3 \geq 0$

Where $Y_i = \frac{y_i}{V}$

5. Simplex table : Optimal solution

	C_j	1	1	1	0	0	0
C_B	Basis	Y_1	Y_2	Y_3	S_1	S_2	
1	Y_3	42/68	0	1	5/34	–1/34	2/17
1	Y_2	57/68	1	0	–1/68	7/68	3/34
	Solution	0	3/34	2/17	0	0	
	Δ_j	–31/68	0	0	–9/68	–5/68	

The optimal values of Y_1, Y_2 and Y_3 are 0, $\frac{3}{34}$ and $\frac{2}{17}$ respectively.

6. Hence,

$$\frac{1}{V} = 0 + \frac{3}{34} + \frac{2}{17} = \frac{7}{34}$$

7. Therefore, $V = \frac{34}{7}$. Since a value 3 was added to the original pay... values, the game value is equal to $V - 3$ or $\frac{34}{7} - 3 = \frac{13}{7}$. Further, sin... $y_i = Y_i V$, we have $y_1 = 0 \times \frac{34}{7} = 0, y_2 = \frac{3}{34} \times \frac{34}{7} = \frac{3}{7}$

and $$y_3 = \frac{2}{17} \times \frac{34}{7} = \frac{4}{7}.$$

8. The value of the dual variables X_1 and X_2 can be read from the Δ_j row ... the simplex table. From this $X_1 = \frac{9}{68}$ and $X_2 = \frac{5}{68}$. From these,

$$\frac{1}{U} = \frac{9}{68} + \frac{5}{68} = \frac{7}{34}.$$

Therefore $$U = V = \frac{34}{7}.$$

Thus, $$x_1 = \frac{9}{68} \times \frac{34}{7} = \frac{9}{14} \text{ and } x_2 = \frac{5}{68} \times \frac{34}{7} = \frac{5}{14}.$$

9. The solution to the problem therefore is :

Player A		Player B		
Strategy	Probability	Strategy	Probability	Game Value
1	9/14	1	0	$\frac{13}{7}$
2	5/14	2	3/4	
		3	4/7	

PART-3

Sequencing, Basic Assumptions, n Job Through Two/ Three Machines, and 2 Jobs on m Machines.

CONCEPT OUTLINE : PART-3

Sequencing : Suppose there are n jobs (1, 2, 3, ..., n), each of which has to be processed one at a time at each of m machines A, B, C,... The order of processing each job through machines is given. The time that each job must require on each machine is known. The problem is to find a sequence among $(n!)^m$ number of all possible sequences for processing the jobs so that the total elapsed time for all the jobs will be minimum.

Total Elapsed Time : This is the time between starting the first job and completing the last job. This also includes idle time, if exists.

No Passing Rule : This rule means that passing is not allowed *i.e.*, the same order of jobs is maintained over each machine. If each of the n jobs is to be processed through two machines A and B in the order AB, then this rule means that each job will go to machine A first and then to B.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 3.18. **Define the problem of sequencing.**

Answer

1. The selection of an appropriate order for a series of jobs to be done on a finite number of service facilities, in some pre-assigned order, is called sequencing.
2. Suppose there are n jobs (1, 2, 3, ..., n) each of which has to be processed one at a time at each of m machines A, B, C, ...
3. The order of processing each job through machines is given.
4. The time that each job must require on each machine is known.
5. The problem is to find a sequence among $(n!)^m$ number of all possible sequences (or orders) for processing the jobs so that the total elapsed time for all jobs will be minimum.
6. Mathematically, let

   A_i = Time for job i on machine A.

B_i = Time for job i on machine B.

T = Time from start of first job to completion of the last job.

Then, the problem is to determine for each machine a sequence of jobs $i_1, i_2, i_3 ... i_n$. Where $(i_1, i_2, i_3 ... i_n)$ is the permutation of the integers which will minimize T.

Que 3.19. **Explain the basic terms used in sequencing.**

Answer

A. Number of Machines : It refers to the number of service facilities through which a job must pass before it is assumed to be completed.

B. Processing Order : It refers to the order (sequence) in which given machines are required for completing the job.

C. Processing Time : It indicates the time required by a job on each machine.

D. Total Elapsed Time : It is the time interval between starting the first job and completing the last job including the idle time (if any) in a particular order by the given set of machines.

E. Idle Time on a Machine : It is the time for which a machine does not have a job to process, *i.e.*, time from the end of job $(i - 1)$ to the start of job i.

F. No Passing Rule : It refers to the rule of maintaining the order in which jobs are to be processed on given machine. For example, if n jobs are to be processed through three machines, M_1, M_2, M_3 in the order M_1, M_2, M_3 then this rule will mean that each job will go to machine M_1 first, then to M_2 and lastly to M_3.

Que 3.20. **What are the assumptions used in a sequencing problem ?**

Answer

1. The processing times on different machines are independent of the order of the job in which they are to be processed.
2. Only one job can be processed on a given machine at a time.
3. The time taken by the jobs in moving from one machine to another is very negligible and is taken as equal to zero.
4. Each job once started on a machine is to be performed upto the completion on that machine.
5. Machines to be used are of different types.
6. All jobs are known and are ready for processing before the period under consideration begins.
7. Processing time is given and do not change.

8. The order of completion of the jobs is independent of the sequence of jobs.

Que 3.21. **How will you process n-jobs through two machines ?**

Answer

A. Processing n-Jobs through Two Machines :

1. The simplest possible sequencing decision problem is that n-jobs two machine sequencing problem wherein we want to determine the sequence in which n-jobs should be processed through two machines so as to minimize the total elapsed time.
2. This type of problem can be completely described as :
   a. Only two machines A and B are involved.
   b. Each job is processed in the order A, B, and
   c. The expected processing times, $A_1, A_2, ..., A_n$; $B_1, B_2, ..., B_n$ are known, as given below :

Job	i	1	2	3	...	n
	A_i	A_1	A_2	A_3	...	A_n
	B_i	B_1	B_2	B_3	...	B_n

3. The procedure for the solution of the above problem was developed by Johnson and Bellman (1953) and use the following steps :

a. Select the smallest processing time occurring in the list, $A_1, A_2, ..., A_n$ and $B_1, B_2, ..., B_n$. If there is a tie, select either of the smallest processing time.

b. i. If the smallest processing time is A_r, do the r^{th} job first and place it at the beginning of the sequence.

ii. If it is B_s, do the s^{th} job last of all and place it at the end of the sequence. The decision will apply to both machine A and B.

iii. a. If there is a tie for min $A_r = B_s$, process the r^{th} job first and the s^{th} one in last.

b. If there is a tie for minimum among A_r then do any one of these jobs for which there is a tie, first.

c. If there is a tie for minimum among B_s's then do any of these jobs in the last.

c. There are now $(n - 1)$ jobs left to be ordered. Repeat steps 1 and 2 to the reduced set of processing times obtained by deleting the processing times for both the machines corresponding to the job already assigned.

d. Continue the process placing the job next to first or next to last, and so on till all jobs have been assigned a position what is called as 'optimal sequence'.

e. After finding the optimal sequence as stated above, we can find the overall or total elapsed time and also the idle times on machines A and ... as under :

i. **Total Elapsed Time :** The time between starting the first job in the optimal sequence on machine A and B completing the last job in the optimal sequence on machine B.

ii. **Idle Time on Machine A** = (Time when the last job in the optimal sequence is completed on machine B) – (Time when the last job in the optimal sequence completed on machine A)

iii. **Idle Time on Machine B** = (Time when first job in the optimal sequence complete on machine A) + (Time when k^{th} job starts on machine B) – (Time when $(k - 1)^{th}$ job finished on machine B)

Que 3.22. **How will you process n jobs through three machines?**

Answer

1. Only three machines A, B and C are involved.
2. Each job is processed in the prescribed order ABC (first on machine A, then on B and thereafter on C).
3. No passing of jobs is permitted (*i.e.*, the same order over each machine is maintained).
4. The actual or expected processing times $A_1, A_2, ..., A_n; B_1, B_2, ..., B_n$ and $C_1, C_2, ..., C_n$ are known and represented by a table of the type shown below.

Machine times for n jobs and three machines

Job	A	B	C
1	A_1	B_1	C_1
2	A_2	B_2	C_2
3	A_3	B_3	C_3
⋮	⋮	⋮	⋮
i	A_i	B_i	C_i
⋮	⋮	⋮	⋮
n	A_n	B_n	C_n

5. The problem, again, is to find the optimum sequence of jobs which minimize T.
6. No general solution is available at present for such a case. However, the method of n jobs through two machines can be extended to cover the special cases where either one or both of the following conditions hold good (if neither of the conditions holds good, the method fails and the optimal sequence has to be found by enumerating all the sequences).

a. The minimum time on machine A is ≥ maximum time on machine B.

b. The minimum time on machine C is ≥ maximum time on machine B.

7. The method, described here without proof, is to replace the problem by an equivalent problem involving n jobs and two machines. These two (fictitious) machines are denoted by G and H and their corresponding processing times are given by

$$G_i = A_i + B_i, H_i = B_i + C_i$$

8. If this new problem with the prescribed order GH is solved by the method of n jobs through 2 machines the resulting optimal sequence will also be optimal for the original problem.

Que 3.23. **How will you process 2 jobs on m machines ?**

Answer

1. Let us consider the following situation :
   a. There are m machines, denoted by A, B, C, ..., K.
   b. Only two jobs are to be performed : job 1 and job 2.
   c. The technological ordering of each of the two jobs through m machines is known. This ordering may or may not be the same for both jobs. Alternative ordering is not permissible for either job.
   d. The actual or expected processing times $A_1, B_1, C_1, ..., K_1; A_2, B_2, C_2, ..., K_2$ are known.
   e. Each machine can work only one job at a time and storage space for in-process inventory is available.
2. The problem is to minimize the total elapsed time T *i.e.*, to minimize the time from the start of first job to the completion of the second job.
3. Such a problem can be solved by graphical method which is simple and provides good (though not necessarily optimal) results.

Que 3.24. **In a factory, there are six jobs to perform, each of which should go through two machines A and B in the order A, B. The processing timings (in hours) for the job are given here. You are required to determine the sequence for performing the jobs that would minimize the total elapsed time, T. What is the value of T ?**

Job	J_1	J_2	J_3	J_4	J_5	J_6
Machine A	1	3	8	5	6	3
Machine B	5	6	3	2	2	10

Answer

1. The smallest processing time in the given problem is 1 on machine *A*. So, perform J_1 in the beginning as shown below :

J_1					

2. The reduced set of processing times becomes :

Job	J_2	J_3	J_4	J_5	J_6
Machine *A*	3	8	5	6	3
Machine *B*	6	3	2*	2*	10

3. The minimum processing time in this reduced problem is 2 which corresponds to J_4 and J_5 both on machine *B*. Since the corresponding processing time of J_5 on machine *A* is larger than the corresponding processing time of J_4 on machine *A*. J_5 will be processed in the last and J_4 shall be penultimate. The updated job sequence is

J_1				J_4	J_5

4. The remaining processing times are :

Job	J_2	J_3	J_6
Machine *A*	3	8	3
Machine *B*	6	3	10

5. Now there is a tie among 3 jobs for the smallest processing time in this reduced problem. These correspond to J_2 and J_6 on machine *A*, and to J_1 on machine *B*. As the corresponding processing time of J_6 on machine *B* is larger than the corresponding processing time of J_2 on machine *B*, J_6 will be processed next to J_1. Now J_2 should be placed next. The updated job sequence is

J_1	J_6	J_2	J_3	J_4	J_5

6. This sequence is the optimum one. The total elapsed time is calculated below :

Job	Machine *A*		Machine *B*		Idle time on *B*
	In	Out	In	Out	
J_1	0	1	1	6	1
J_6	1	4	6	16	—
J_2	4	7	16	22	—
J_3	7	15	22	25	—
J_4	15	20	25	27	—
J_5	20	26	27	29	—

7. From the above information, we get $T = 29$ hours. Idle time of machine *A* is $(29 - 26) = 3$ hours and that for machine *B* is one hour.

Que 3.25. **A readymade garment manufacturer has to process 6 items through two stages of production, viz. cutting and sewing. The time taken for each of these items at the different stages is given below in appropriate units :**

Item	1	2	3	4	5	6
Cutting time	30	120	50	20	90	110
Sewing time	80	100	90	60	30	10

Find the order in which these items are to be processed through these stages so as to minimize the total processing time. Also calculate total elapsed time and idle times.

UPTU 2012-13, Marks 10

Answer

1. Applying steps 1 and 2 of solution procedure, we observe that the smallest processing time is 10 units for item 6 on sewing machine. Therefore we shall schedule item 6 in the last as shown below : The reduced set of processing time becomes :

					6

Item	1	2	3	4	5
Cutting time	30	120	50	20*	90
Sewing time	80	100	90	60	30

2. Again the smallest processing time in the reduced list is 20 units for item 4 on cutting machine. Thus, we shall do job 4 first and list the elements as follows :

4					6

3. After assigning items 4 and 6, we are now left with 4 items and two machines with the reduced set of processing times as follows :

Item	1	2	3	5
Cutting time	30*	120	50	90
Sewing time	80	100	90	30*

4. There are two equal minimal values : cutting time of 30 unit for item 1 and sewing time of 30 unit for item 5. According to the rules, item 1 is scheduled next to item 4 and 6 next to item 5, yielding the sequence as follows :

Scanned by CamScanner

4	1			5	6

5. The reduced set of processing times now becomes :

Item	2	3
Cutting time	120	50*
Sewing time	100	90

6. Here, since smallest cutting time is 50 unit for item 3, we place item in the third cell and the remaining item 2 in the fourth cell and we get the optimal sequence as :

4	1	3	2	5	6

7. The minimum elapsed time from the start of the first item to the completion of the last item corresponding the optimal sequence is computed as shown in the following table :

Item	Cutting machine		Sewing machine		Idle time
	Time in	Time out	Time in	Time out	Cutting machine
4	0	20	20	80	20
1	20	50	80	160	0
3	50	100	160	250	0
2	100	220	250	350	0
5	220	310	350	380	0
6	310	420	420	430	40

8. From the above table, we get $T = 430$ unit. Idle time for cutting machine is 10 hours (from 420 unit to 430 unit) and for sewing machine is $20 + 40 = 60$ unit (from 0 – 20 unit and 380 – 420 unit).

Que 3.26. There are seven jobs, each of which has to go through the machines *A* and *B* in the order *AB*.
Processing times in hours are as follows :

Job	1	2	3	4	5	6	7
Machine *A*	3	12	15	6	10	11	9
Machine *B*	8	10	10	6	12	1	3

Determine a sequence of these jobs that will minimize the total elapsed time *T*. Also find *T* and idle time for machine *A* and *B*.

UPTU 2013-14, 2014-15; Marks 10

Answer

1. By examining the processing times we find the smallest value. It is 1 hour for job 6 on machine *B*. Thus we schedule job 6 last on machine *A* as shown below :

					6

2. The reduced set of processing times becomes

Job	1	2	3	4	5	7
Machine *A*	3	12	15	6	10	9
Machine *B*	8	10	10	6	12	3

3. There are two equal minimal values : processing time of 3 hours for job 1 on machine *A* and processing time of 3 hours for job 7 on machine *B*. According to rules, job 1 is scheduled first and job 7 next to job 6 as shown below :

1					7	6

4. The reduced set of processing times becomes ;

Job	2	3	4	5
Machine *A*	12	15	6	10
Machine *B*	10	10	6	12

5. Again there are two equal minimal values ; processing time of 6 hours for job 4 on machine *A* as well as on machine *B*. We may choose arbitrarily to process (schedule) job 4 next to job 1 or next to job 7 as shown below :

1	4				7	6

or

1				4	7	6

6. The reduced set of processing times becomes ;

Job	2	3	5
Machine *A*	12	15	10
Machine *B*	10	10	12

7. There are three equal minimal values : processing time of 10 hours for job 5 on machine *A* and for jobs 2 and 3 on machine *B*. According to rules, job 5 is scheduled next to job 4 in the first schedule or next to job 1 in the second schedule. Job 2 then is scheduled next to job 7 in the first schedule or next to job 4 in the second schedule. The optimal sequences are shown below :

1	4	5	3	2	7	6

or

1	5	3	2	4	7	6

8. Now we can calculate the elapsed time corresponding to either of the optimal sequences, using the individual processing times given in the problem. The elapsed time and idle times for machines *A* and *B* will be

same for either sequence. The details corresponding to the first schedule are shown in table 3.26.1

Table 3.26.1

Job	Machine A		Machine B		Idle time for machine B
	Time in	Time out	Time in	Time out	
1	0	3	3	11	3
4	3	9	11	17	0
5	9	19	19	31	2
3	19	34	34	44	3
2	34	46	46	56	2
7	46	55	58	59	0
6	55	66	66	67	7

8. Thus the minimum elapsed time is 67 hours. Idle time for machine A is 1 hour (66th-67th hour) and for machine B is 17 hours.

Player A \ Player B	B_1	B_2	B_3	Row minima
A_1	λ	6	2	2
A_2	−1	λ	−7	−7
A_3	−2	4	λ	−2
Column maxima	−1	6	2	

∴ Maximin value = 2 and minimax value = − 1

∴ Value of the game lies between − 1 and 2 *i.e.*, $-1 \le V \le 2$

∴ For strictly determinable game, since,

maximum value = minimax value, we must have $-1 \le \lambda \le 2$.

Que 3.27. There are five jobs, each of which must go through machine A, B and C in the order ABC. Processing times are given in the table :

Job *i*	Processing times		
	A_i	B_i	C_i
1	8	5	9
2	10	6	9
3	6	2	8
4	7	3	6
5	11	4	5

Determine a sequence for five jobs that will minimize the elapsed time T.

Answer

1. Here, min A_i = 6, max B_i = 6, min C_i = 4. Since one of two condition is satisfied by min A_i = max B_i, so the procedure adopted as follows :

a. The equivalent problem, involving five jobs and two fictitious machine G and H, becomes :

Job *i*	Processing times	
	G_i (= A_i + B_i)	H_i (= B_i + C_i)
1	13	14
2	16	15
3	8	10
4	10	9
5	15	9

b. This new problem can be solved by the procedure described earlier. Because of ties, possible optimal sequences are :

i. 3 | 2 | 1 | 4 | 5

ii. 3 | 2 | 4 | 1 | 5

iii. 3 | 2 | 4 | 5 | 1

iv. 3 | 2 | 5 | 4 | 1

v. 3 | 2 | 1 | 5 | 4

vi. 3 | 2 | 5 | 1 | 4

2. It is possible to calculate the minimum elapsed time for first sequence as shown in the table below :

Job	Machine A		Machine B		Machine C	
	Time in	Time out	Time in	Time out	Time in	Time out
3	0	6	6	8	8	16
2	6	16	16	22	22	31
1	16	24	24	29	31	35
4	24	31	31	34	35	41
5	31	42	42	46	46	51

3. Thus, any of the sequence from i to vi may be used to order the jobs through machine A, B and C and they all will give a minimum elapsed time of 51 hrs. Idle time for machine A is 9 hrs, for B 31 hrs and for C 19 hrs

Que 3.28. **Determine the optimal sequence of jobs that minimize the total elapsed time based on the following information :**

	Job	A	B	C	D	E	F	G
Machine	M_1	3	8	7	4	9	8	7
	M_2	4	3	2	5	1	4	3
	M_3	6	7	5	11	5	6	12

UPTU 2011-12, Marks 10

Answer

1. From the elapsed time given on three machines is clear that minimum time elapsed on machine 3 is hours, *i.e.*, equal to the maximum elapsed time on machine 2 hence solution is possible.
2. Now we should add the processing times of different jobs on machine M_1 and M_2 machines M_2 and M_3 respectively as follows ;

Step-I

Job	Processing times	
	$G_i = (M_1 + M_2)$	$H_i = (M_2 + M_3)$
A	$3 + 4 = 7$	$4 + 6 = 10$
B	$8 + 3 = 11$	$3 + 7 = 10$
C	$7 + 2 = 9$	$2 + 5 = 7$
D	$4 + 5 = 9$	$5 + 11 = 16$
E	$9 + 1 = 10$	$1 + 5 = 6$
F	$8 + 4 = 12$	$4 + 6 = 10$
G	$7 + 3 = 10$	$3 + 12 = 15$

Step-II

Job	Processing times	
	G_i	H_i
A	7(3)	10
B	11	10(5)
C	9	7(2)
D	9(4)	16
E	10	6(1)
F	12	10 (6)
G	10(7)	15

3. Now, the minimum time elapsed among the two combinations of machine is 6 hours job E in the last combination. Hence, job E processed last

$G_i \rightarrow$ Job sequence $\leftarrow H_i$

						E

4. Next minimum elapsed time is 7 hours for job A on combination 1 so, next job to be processed is job A. In this way, we get the following sequence :

A	D	G	B	F	C	E

5. The total elapsed time may be calculated in the following way :

Job	Machine M_1		Machine M_2		Machine M_3	
	Time in	Time out	Time in	Time out	Time in	Time out
A	00	03	03	07	07	13
D	03	07	07	12	13	24
G	07	14	14	17	24	36
B	14	22	22	25	36	43
F	14	22	22	25	36	43
C	30	37	37	39	49	54
E	37	46	46	47	54	59

Total elapsed time
Machine M_1 — 46 hours
Machine M_2 — 47 hours
Machine $M_3 \rightarrow$ 59 hours
Total processing time is 59 hours.

Que 3.29. **A machine operator has to perform three operations : turning, threading and knurling on a number of different jobs. The time required to perform these operations (in minutes) for each job is known. Determine the order in which the jobs should be processed in order to minimize the total time required to turn out all the jobs. Also find the idle times for the times for the three operations.**

Job	Time for turning (minutes)	Time for threading (minutes)	Time for knurling (minutes)
1	3	8	13
2	12	6	14
3	5	4	9
4	2	6	12
5	9	3	8
6	11	1	13

Answer

1. Here, min $A_i = 2$, max $B_i = 8$, min $C_i = 8$. Since C_i = max B_i
The equivalent problem involving 6 jobs and two fictitious operation G and H.
2. Processing times for 6 jobs and two fictitious operations :

Job	G_i = Turning + Threading (min)	H_i = Threading + Knurling (min)
1	11	21
2	18	20
3	9	13
4	8	18
5	12	11
6	12	14

3. Examining the column G_i and H_i, we find that the smallest value under operation G_i in row 4. Thus we schedule job 4 first (on machine and thereafter on H_i) as shown below :

4					

4. The reduced set of processing time becomes :

Job	G_i	H_i
1	11	21
2	18	20
3	9	13
5	12	11
6	12	14

5. The next smallest value is 9 under column G_i for job 3. Hence schedule job 3 as shown below :

4	3				

6. The reduced set of processing time becomes,

Job	G_i	H_i
1	11	21
2	18	20
3	9	13
5	12	11
6	12	14

7. There are two equal minimal values : processing time of 11 minutes under column G_i for job 1 and processing time of 11 minutes under column H_i for job 5. According to the rules, job 1 is scheduled next to job 3 and 5 is scheduled last as shown below ;

4	3	1			5

8. The reduced set of processing time becomes,

Job	G_i	H_i
2	18	20
6	12	14

9. The smallest value is 12 under column G_i for job 6. Hence we schedule job 6 next to job 1 and the optimal sequence becomes,

4	3	1	6	2	5

10. Now we may calculate the elapsed time corresponding to the optimal sequence, using the individual processing time given in the problem.

	Turning operation		Threading operation		Knurling operation	
Job	Time in	Time out	Time in	Time out	Time in	Time out
4	0	2	2	8	8	20
3	2	7	8	12	20	29
1	7	10	12	20	29	42
6	10	21	21	22	42	55
2	21	33	33	39	55	69
5	33	42	42	45	69	77

11. Thus the minimum elapsed time is 77 minutes. Idle time for turning operation is 77 – 42 = 35 minutes, for threading operation is 2 + 1 + 11 + 3 + (77 – 45) = 49 minutes and for knurling operation is 3 minutes.

Que 3.30. Use graphical method to minimize the time required to process the following jobs on the machines. For each machine specify the job which should be done first. Also calculate the total elapsed time to complete both jobs.

Machines

Job 1 Sequence	A	B	C	D	E
Time (hr)	6	8	4	12	4
Job 2 Sequence	B	C	A	D	E
Time (hr)	10	8	6	4	12

Answer

Step 1 : Draw two axes at right angles to each other. Represent processing time on job 1 along horizontal axis and processing time on job 2 along vertical axis. Scale used must be same for both jobs.

Step 2 : Layout the machine times for the two jobs on corresponding axis in the given technological order.

Step 3 : Machine A requires 6 hrs for job 1 and 6 hrs for job 2. A square *PQRS* is constructed for machine *A*. Similarly rectangles are constructed for machine *B*, *C*, *D* and *E*.

Step 4 : Make a program by starting with zero time (origin O) and moving through the various stages of completion (points) till the point marked 'finish' is reached. Choose path consisting only of horizontal, vertical and 45° lines. A horizontal line represent work on job 1 while job 2 remains idle; a vertical line represents work on job 2 while job 1 remains idle and a 45° line to the base represents simultaneous work on both the jobs.

Step 5 : Find the optimal path (program). An optimal path is one that minimize idle time on job 1 as well as job 2. Obviously, the optimal path is one which coincides with 45° line to the maximum extent. Further both jobs cannot be processed simultaneously on one machine. Graphically, this means that diagonal movement through the blacked out areas is not allowed. A good path, accordingly, is chosen by eye and drawn on the graph (path OUVWXYZ).

6. **Step 6 :** Find the elapsed time. It is obtained by adding the idle time for either job to the processing time for that job.

The idle time for the chosen path is found to be 4 + 6 = 10 hours for job 1 and 4 hours for job 2.

∴ Total elapsed time = 34 + 10 = 44 hrs (considering job 1)

= 40 + 4 = 44 hrs (considering job 2)

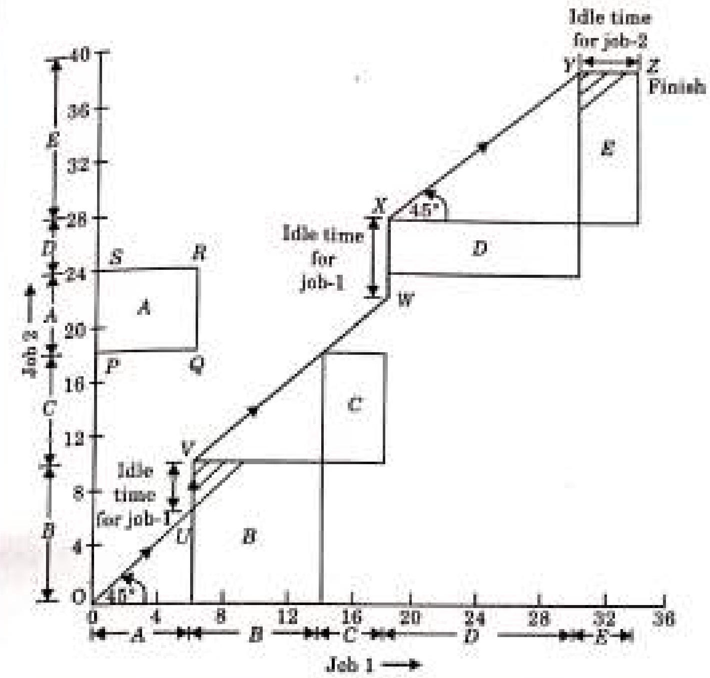


Fig. 3.30.1. Graphical solution of 2 jobs and 5 machines problem.

7. **Step 7 :** The optimal schedule corresponding to the chosen path is shown in Fig. 3.30.2.

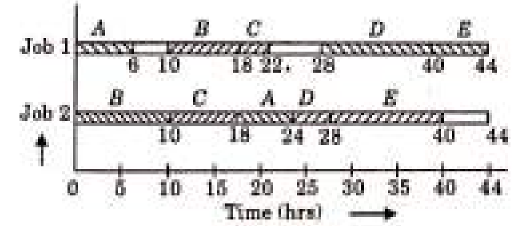


Fig. 3.30.2.

4 UNIT

Stochastic Inventory Models and Simulations

PART-1

Single and Multi Period Models with Continuous and Discrete Demands, Service Level and Reorder Policy.

CONCEPT OUTLINE : PART-1

Inventory : It consists of usable but idle resources. The resources may be of any type-men, materials, machines, etc. The term is generally used to indicate raw materials in process, finished product, packaging, spares and others; stock in order to meet an expected demand or distribution in future.

Inventory Control Methods : A fundamental objective of a good inventory control system is to determine 'what to order', 'how much to order', 'when to order', 'how much to carry in stock' so as to get economy in purchasing, storing, manufacturing and selling. The measures and necessary steps taken in this direction are formulated as inventory control methods :

a. JIT (Just In Time) method.
b. ABC (Always Better Control) method.
c. VED (Vital Essential Desirable) method, etc.

EOQ : Economic order quantity (EOQ) is that size of order which minimizes the total annual cost of carrying inventory and cost of ordering.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 4.1. **Define inventory. What are the advantages and disadvantages of having inventories ?** **UPTU 2012-13, Marks**

Answer

A. **Inventory :**
1. Inventory may be defined as the stock of goods, commodities or other economic resources that are stored or reserved in order to ensure smooth and efficient running of business affairs.
2. Goods may be kept in the following form :
   a. **Raw Material Inventory :** Raw materials which are kept in stock to be used in the production of goods.
   b. **Work-in-Process Inventory :** Semifinished goods or goods in process which are stored during the production process.

   c. **Finished Goods Inventory :** Finished goods awaiting shipment from the factory.
   d. Inventory also includes furniture, machinery, fixtures, etc.

B. **Classification of Inventory :** The term inventory may be classified into two categories :

a. **Direct Inventory :**
1. The items which play a direct role in manufacturing and become an integral part of finished goods are included in the category of direct inventory.
2. It includes the following :
   i. Raw material inventory.
   ii. Work in progress inventory.
   iii. Finished goods inventory.
   iv. Spare parts.

b. **Indirect Inventory :**
1. Indirect inventory include those items which are necessarily required for manufacturing but do not become the component of finished product, like oil, grease, lubricants, petrol, office material, maintenance material, etc.

C. **Advantages :**
1. Ensure an adequate supply of items to the customers and avoid the shortages as far as possible at the minimum cost.
2. Makes use of available capital in a most effective way and avoids an unnecessary expenditure on high inventories, etc.
3. Reduces the risk of loss due to the changes in price of items stocked at the time of making the stock.
4. Provides cushion between work-centres thereby assures a smooth and efficient running of the organization.
5. Takes advantages of quantity discounts on bulk purchases.
6. Serves as a buffer stock in case of delayed deliveries by the suppliers.
7. Eliminates the possibility of duplicate ordering.
8. Helps in minimizing the loss due to deterioration, obsolescence, damages or pilferage of goods, etc.
9. Helps in maintaining economy by absorbing some of the fluctuations when the demand for an item fluctuates or is seasonal.
10. Controls and minimizes accumulation and build-up of surplus stock, and eliminates the dead movable surplus stock as far as practicable.
11. Utilizes the benefits of price fluctuations.
12. The firm ensures smooth functioning of its various departments by maintaining reasonable stocks with the help of inventory control.

D. **Disadvantages :**

1. Too often inventories are often used wrongly as a substitute for management. For example, if there are large finished goods inventories, inaccurate sales forecasting by marketing department may never be apparent.
2. Similarly, a production foreman who has large-in-process inventories may be able to hide his poor planning since there is always something to manufacture.
3. Furthermore, inventory means unproductive 'tied-up' capital of the enterprise. The capital could be usefully utilised in other ventures as well.
4. With large inventory, there is always likelihood of obsolescence too.
5. Also maintenance of inventory costs additional money to be spent on personnel equipment, insurance etc. Thus excess inventories are not at all desirable.

Que 4.2. **What are the objectives of inventory management ?**

Answer

A. **Objectives of Inventory Management :**

1. **Service to the Customers :** Sufficient stock of **finished** product is to be maintained to meet the requirements of **customers.**
2. **Effective use of Capital :** The system should **enable the** management to make effective use of its capital, *i.e.*, **lock-up of capital** should be minimum.
3. **Continuity of Productive Operations : The** system should be made to ensure the continuity of productive **operation** by ensuring uniform flow of material and eliminating **the possibility** of **stock outs.**
4. **Reduction of Administrative Work Load :** The administrative work load on the purchasing, receiving, **inspection, stores, accounts** and other related departments should **be minimum.**
5. **Economy in Purchasing : The system should enable** the management to gain economy in bulk **purchasing and take advantage** of price discount.
6. **Minimization of Risk Obsolescence and Deterioration :**
   a. The possibility of the **risk of loss** on account of obsolescence and deterioration should be **minimized.**
   b. In-built checks in the **system should enable the** managements to weed out obsolete **and non-moving items periodically** and automatically.
7. **Up-to-date Accurate Records :** In order to enable the company to prepare periodical financial **statements, minimizing** of discrepancies between physical stock **and book balance** and up-to-date stock records must be maintained.

Que 4.3. **Represent in the form of a table the various types inventory control models.**

UPTU 2013-14, Marks 05

OR

Derive a single period probabilistic inventory model with instantaneous and continuous demand and no setup cost.

UPTU 2011-12, Marks 05

Answer

A. **Table of Inventory Models :**

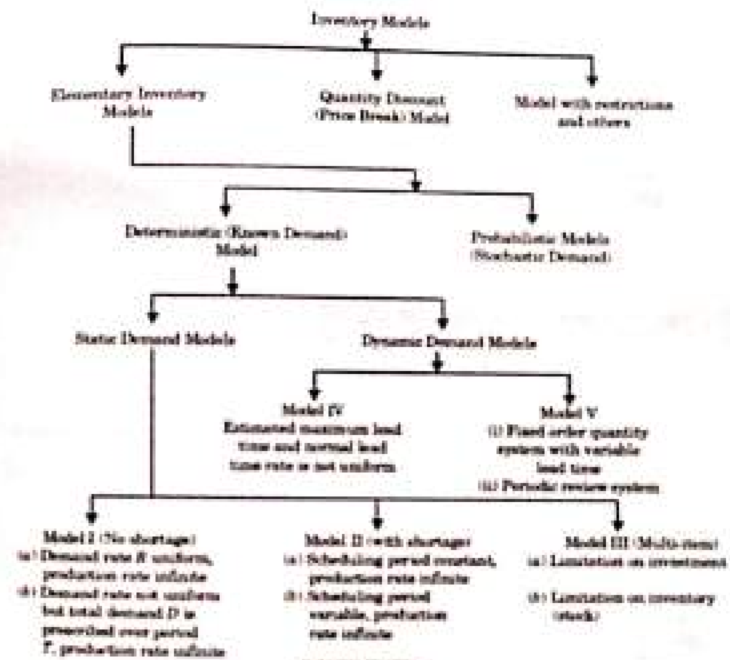


Fig. 4.3.1.

B. **Single Period Inventory Method (Probabilistic Inventory Model) :**

1. In most situations demand is probabilistic since only probability distribution of future demand, rather than the exact value of demand itself, is known.
2. The probability distribution of future demand is usually determined from the data collected from past experience.

3. In such situations we choose policies that minimize the expected costs rather than the actual costs.
4. Expected costs are obtained by multiplying the actual cost for a particular situation with the probability of occurrence of that situation and then either summing or integrating accordingly as the probability distribution is discrete or continuous.
5. Costly spare parts, perishable goods, seasonal items and fashion goods are examples of probabilistic models.
6. Replacement orders are either not possible or become abnormally expensive and uneconomical. The decision is of the one-short type.

C. Mathematical Expression :

1. Let $p(d \geq Q)$ = Probability that the demand is for Q units or more
MP = Marginal profit per unit sale.
ML = Marginal loss from each unit that is left unsold.
2. It is assumed that the inventory carrying costs for the season are fixed and independent of the quantity purchased and further that the ordering cost is negligible.
3. Now,
Expected marginal profit ≥ Expected marginal loss

$$E(MP) \geq E(ML)$$

or $$p(d \geq Q)[MP] \geq [1 - p(d \geq Q)](ML)$$

or $$p(d \geq Q)[MP] + p(d \geq Q)[ML] \geq ML$$

or $$p(d \geq Q) \geq \frac{ML}{MP + ML}$$

4. The decision rule will be :
"Buy the maximum quantity Q such that the probability of setting demand d exceeding Q is greater than the ratio $\frac{ML}{MP + ML}$."
5. Sometimes the following formula can also be used :

$$p(d \geq Q) \geq \frac{C_h}{C_h + C_s}$$

Que 4.4. **Explain the following terms :**

A. Lead time **B. Re-order point**
C. EOQ **D. Buffer stock**

Answer

A. Lead Time or Delivery Lag :

1. The time interval between placement of an order and receipt of goods against it is called lead time.
2. It is normally short in case of local supplier or off the shelf items and greater for made to order or out-station supplier.

3. For the same item and the same supplier it can vary from stage to stage.
4. Lead time may, therefore, be a stochastic variable.
5. This complicates the problem of accumulating the forecasts of demand over the period encompassed by the lead time.

B. Reorder Point :

1. In a firm, there should be enough stock for each item so that customer's order can be reasonably met from this stock until replenishment.
2. This stock level is known as reorder level, has therefore, to be determined for each item.
3. It is determined by balancing the cost of maintaining these stocks and the disservice to the customer if his orders are not filled in time.
4. For example :
i. if the lead time is 2 months, we will have to accumulate the next 2 months forecasts in order to find the forecast lead time demand.
ROL = Lead time demand (LTD)
ii. When the demand pattern is almost stationary and depicts no trend or seasonal variations,
Lead time demand = Lead time × Average demand ($LT \times R$)
∴ ROL = Lead time × Average demand
iii. If the safety stock is provided, ROL is given by,
ROL = Lead time demand + Safety stock
iv. Further, if time T is required for reviewing the system, then

$$ROL = LTD + SS + \frac{RT}{2}$$

v. If there is a provision of safety stock,
Maximum inventory = $q + SS$

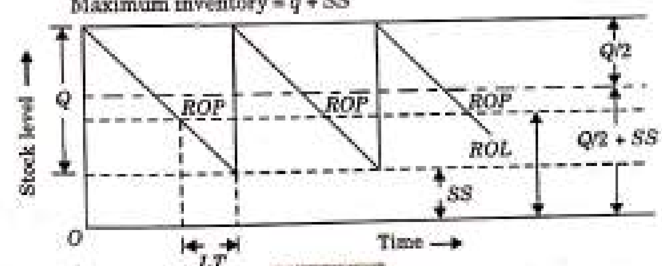


Fig. 4.4.1.

Minimum inventory = SS

Average inventory = $\frac{SS + (q + SS)}{2} = \frac{q}{2} + SS$

vi. Fig. 4.4.1 represents an ideal inventory model wherein the actual demand is same as expected demand and there are no variations from the forecasts.

vii. Here, the moment when stock level reaches the *ROL*, order of the fixed size ($q = EOQ$) is placed; these points have been marked as reorder points (*ROP*).

viii. As soon as the stock level reaches the safety stock, the supplies are received, the stock level, thereby, reaching the maximum level $q + SS$.

C. Economic Order Quantity :

1. By the 'order quantity' we mean the quantity produced or procured during one production cycle.
2. When the size of order increases, the ordering costs (cost of purchasing, inspection etc.) will decrease whereas the inventory carrying costs (cost of storage, insurance, etc.) will increase.
3. Thus in the production process there are two opposite costs, one encourages the increase in the order size and the other discourages.
4. Economic order quantity (*EOQ*) is that size of order which minimizes total annual costs of carrying inventory and cost of ordering.
5. The two opposite costs can be shown graphically by plotting them against the order size.
6. It is evident from Fig. 4.4.2 that the minimum total cost occurs at the point where the ordering costs and inventory carrying costs are equal.

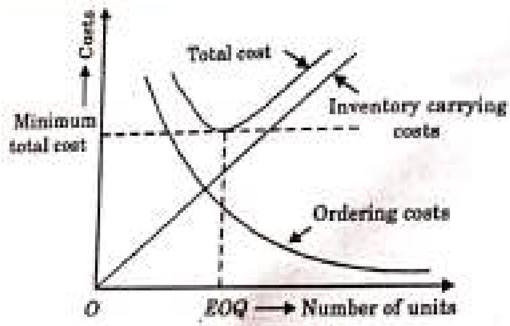


Fig. 4.4.2.

D. Buffer Stock :

1. They are required as protection against the uncertainties of supply and demand.
2. A company may well know the average demand of an item that it needs, however, the actual demand may turn out to be quite different–it may well exceed the average value.
3. Similarly the average delivery period (lead time) may be known but due to some unforeseen reasons, the actual delivery period could be much more.
4. Such situations require extra stock of item to reduce the number of stock-outs or back-orders.

5. This extra stock in excess of the average demand during the lead time is called buffer stock (or safety stock or cushion stock).

Que 4.5. **Define and explain :**

A. Re-order point.

B. Stochastic multi-period model with setup cost.

Answer

A. Re-Order Point : Refer Q. 4.4, Page 124B, Unit-4.

B. Stochastic Multi-Period Model with Setup Cost :

1. The introduction of a fixed setup cost K which is incurred when ordering, often adds more realism to the model.
2. However, the mathematics becomes very complex, and the results available are primarily those that characterise the form of the optimal policy.
3. In particular, if the ordering cost is assumed to be $K + CZ$ for $Z > 0$ and zero for $Z = 0$, and if $L(y)$ is strictly convex, then the optimal policy has the form :

At the beginning of period i $(i = 1, 2, \ldots n)$,

$$\begin{cases} \text{order upto } S_i (\text{order } S_i - x_i), & \text{if } x_i < s_i^4 \\ \text{do not order,} & \text{if } x_i < s_i \end{cases}$$

i. However, exact computations of s_i and S_i for the finite or infinite horizon model are extremely difficult.

ii. But, the importance of this result cannot be minimized.

iii. Even if the exact s_i and S_i are unknown, it is important to know that one should consider using policies of this form rather than a policy from another class.

Que 4.6. **Write short-note on :**

a. Ordering cost,

b. Holding cost, and

c. Shortage cost.

Answer

a. Ordering Cost :

1. This is the cost associated with ordering of raw material for production purposes.
2. Advertisement, consumption stationary and postage, telephone charges and travelling expenditures incurred, etc., constitute the ordering cost.

b. Holding Cost :

1. This is also known as carrying cost and is the cost associated with the goods in stock is known as holding cost.

2. Holding cost is assumed to vary directly with the size of inventory as well as the time for which the item is held in stock.

c. **Shortage Cost :**

1. The penalty costs that are incurred as a result of running out of stock are known as shortage or stock-out costs.
2. These are denoted by C_2 or C_s per unit of goods for a specified period. It results in loss of sales.

Que 4.7. **What is order quantity ?**

Answer

1. Order quantity means the quantity produced or procured during one production cycle.
2. Following assumptions are made for deriving the relation :
   i. Demand is known and uniform.
   ii. Let D denotes the total number of units purchased/produced or supplied per time period and Q denotes the lot size in each production run.
   iii. Shortages are not permitted, *i.e.*, as soon as the level of the inventory reaches zero, the inventory is replenished.
   iv. Production or supply of commodity is instantaneous (abundant availability).
   v. Lead time is zero.
   vi. Set-up cost (or ordering cost) per production run of procurement cost is C_s (or A).
   vii. Holding cost is C_1 per unit inventory for a unit, *i.e.*, $C_1 = IC$, where C is the unit cost, I is called inventory carrying charge expressed as a percentage of the value of the average inventory.
3. This fundamental situation can be shown on an inventory-time diagram with Q on the vertical axis and time on the horizontal axis. The total time period (one year) is divided into n parts.

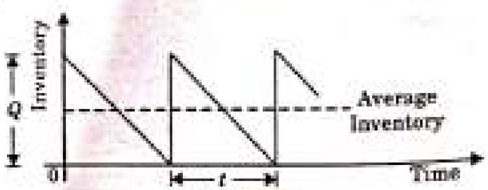


Fig. 4.7.1.

4. Here, it is assumed that after each time t, the quantity Q is produced/purchased or supplied throughout the entire time period, say one year.

5. Now, if n denotes the total number of runs of the quantity produced or purchased during the year, then clearly we have
$$1 = nt \text{ and } D = nQ.$$
6. It may be clear that the average amount of inventory at hand on any day is then $\frac{1}{2}Q$, as shown by dotted line in Fig. 4.7.1. Total inventory over the time period t days is the area of the first triangle $\left(=\frac{1}{2}Qt\right)$. Thus, the average inventory at any time on any given day in the t period is $\frac{1}{2}\frac{Qt}{t} = \frac{1}{2}Q$.
7. Now, since each of the triangles in Fig. 4.7.1 over a year period looks the same, $\frac{1}{2}Q$ remains the average amount of inventory in each interval of length t during the entire period. Annual inventory holding cost is therefore given by
$$f(Q) = \frac{1}{2}QC_1$$
8. Annual costs associated with runs of size Q are given by
$$g(Q) = nC_s = \frac{D}{Q}C_s,\text{ since } n = \frac{D}{Q}.$$
Since the minimum total cost occurs at the point where ordering cost and the total inventory carrying cost are equal, we must have
$$f(Q) = g(Q)$$
This implies, that $\frac{1}{2}QC_1 = \frac{D}{Q}C_s$
9. Hence, the optimum value of Q is
$$Q^o = \sqrt{2DC_s/C_1}$$
This is known as the economic (optimum) lot size formula due to R.H. Wilson.
10. The above EOQ formula can also be expressed in terms of the economic order value as :
$$Q^o = \sqrt{\frac{2AD}{IC}}$$

Que 4.8. **Determine the re-order level and buffer stock for dynamic demand models.**

Answer

1. To determine the buffer stock, we approximate the estimated maximum lead time and normal lead time for a particular time.

2. Let L_d = Difference between maximum and normal lead times,
R = Consumption rate during the lead time.

Then, the buffer stock is given by the formula,

$$B = L_d \times R$$

i.e., Buffer stock = (Maximum lead time – Normal time) × Consumption rate

For example, let the monthly consumption for a certain item be 300 units, the normal lead time be 15 days, and the maximum lead time is estimated as 3 months. Then, the buffer stock is given by :

$$B = (3 - 1/2) \times 300 = 750 \text{ units}$$

3. Now three situations may arise :
   i. If we do not maintain a buffer stock, then the total requirements for inventory during the lead time will become $L_d R$. This implies that as soon as the stock reaches a level $L_d R$, we place an order for q units.
   We call this the 'reorder point' or 'reorder level', which is given by
   $$ROL = L_d R$$
   ii. In order to avoid the shortage, we have to maintain a buffer stock 'B' and place an order when stock level reaches $B + L_d R$. Hence the reorder level is given by
   $$ROL = B + L_d R$$
   For example, if the monthly consumption rate for items is 150 units, the normal lead time is 15 days, and the buffer stock is 200 units, then
   $$ROL = 200 + 1/2\,(150) = 275 \text{ units}$$
   iii. If we take t days for reviewing the reorder system, then on the assumption of uniform consumption rate, we get
   $$ROL = B + L_d R + Rt/2$$

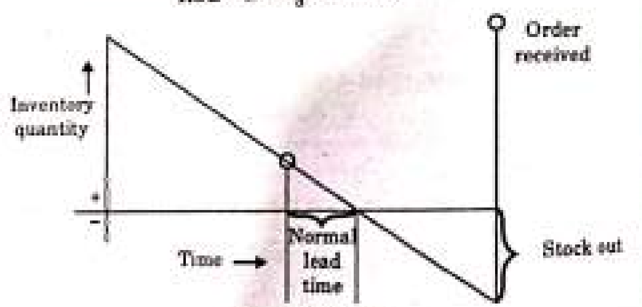


Fig. 4.8.1.

Que 4.9. **XYZ Company wants to provide a 95 percent service level to its customers. Using the past history of demand, the following data is available. Daily demand follows normal distribution with an average daily demand of 20 units and the**

standard deviation of 5 units. The lead time for procurement is 4 days. The cost of placing an order is Rs. 12 and inventory carrying cost is Rs. 1.20 per unit per year. There are no stockout costs and unfilled orders are supplied after the items are received. What should be the inventory policy for the company ? **UPTU 2011-12, Marks 10**

Answer

1. **Given :** Average daily demand = 20 units, Lead time = 4 days
Standard derivation, $\sigma = 5$
Order cost / year C_3 = Rs 12/order
Carrying cost, C_1 = Rs 1.2 per unit/year
Annual demand = average daily demand × no. of day in year
$(R) = 20 \times 365 = 7300$ units

2. We know that
$$EOQ\ (Q^*) = \sqrt{\frac{2C_3R}{C_1}} = \sqrt{\frac{2 \times 12\ \ 7300}{1.2}}$$
$$= \sqrt{2 \times 10 \times 7300} = 382 \text{ units}$$

3. Expected demand during lead time
μ = average demand per day × lead time
$= d \times L = 20 \times 4 = 80$ units.

4. Variable demand during lead time,
σ^2 = variance of the demand for each day with lead time.

∴ Sum of variance of the demand for each day with lead time.
$$= \sum_{i=1}^{4} \sigma^2 = 4(5)^2 = 100$$

So, $\overline{S.D} = \sqrt{\text{sum of variance of the demand for each day}}$
= 10 units

5. For 95 % service level from normal distribution table, point z will be find.
Hence stock = $1.645 \times \overline{S.D} = 1.645 \times 10 = 16.45$

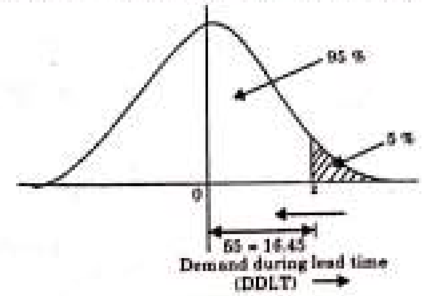


Fig. 4.9.1.

6. ROL = Average demand during lead time + safety s...
$= 80 + 16.45 = 96.45$ units

7. Reorder level x, corresponding to a service level 95 % can be obtain... following way :

$$z = \frac{x - \mu}{\sigma}$$ (The value of z corresponding to 9... service is 1.645)

$$1.645 = \frac{x - 80}{10}$$

$$x = 16.45 + 80$$

$$y = 96.45 \text{ units}$$

Que 4.10. **A stockist has to supply 400 units of a product ev... Monday to his customers. He gets the product at Rs. 50 per u... from the manufacturer. The cost of ordering and transporta... from the manufacturer is Rs. 75 per order. The cost of carry... inventory is 7.5 % per year of the cost of the product. Find :**

i. The economic lot size.
ii. The total optimal cost (including the capital cost).
iii. The total weekly profit if the item is sold for Rs. 55 per unit.

UPTU 2012-13, Marks...

Answer

Given :

R = 400 units/week,
C_3 = Rs 75/order,
C_1 = 7.5 % per year of the cost of product

$= \text{Rs.} \left(\frac{7.5}{100} \times 50\right)$ per unit per year

$= \text{Rs.} \left(\frac{7.5}{100} \times \frac{50}{52}\right)$ per unit per week

$= \text{Rs.} \ \frac{3.75}{52}$ per unit per week

i. $$EOQ = \sqrt{\frac{2C_3 R}{C_1}} = \sqrt{\frac{2 \times 75 \times 400 \times 52}{3.75}}$$
$= 912.14$ units/order

ii. Total optimal cost, C_0 = Capital cost + $\sqrt{2 C_1 C_3 R}$

$$= 400 \times 50 + \sqrt{\frac{2 \times 3.75}{52} \times 75 \times 400}$$
$= 20,000 + 65.80$
$=$ Rs. 20,065.80 per week

iii. Total profit, $P = 55 \times 400 - C_0$

$= 22,000 - 20,065.80$
$=$ Rs. 1934.20 per week.

Que 4.11. **A manufacturing company needs 2500 units of a particular component every year. The company buys it at the rate of Rs. 30 per unit. The order-processing cost for this part is estimated at Rs. 15 and the cost of carrying a part in stock comes to about Rs. 4 per year.**
The company can manufacture this part internally. In that case, it saves 20 % of the price of the product. However, it estimates a setup cost of Rs. 250 per production run. The annual production rate would be 9,800 units. However, the inventory holding costs remain unchanged.

a. Determine the EOQ and the optimal number of orders placed in a year.
b. Determine the optimum production lot size and the average duration of the production run.
c. Should the company manufacture the component internally or continue to purchase it from the supplier ?

Answer

a. Given : R = 2500 units, C_3 = Rs. 15/order and
C_1 = Rs. 4/unit/yr,

We have, $$EOQ = \sqrt{\frac{2C_3 R}{C_1}} = \sqrt{\frac{2 \times 15 \times 2500}{4}} = 137 \text{ units}$$

Optimal number of orders $= \frac{R}{EOQ} = \frac{2500}{137} = 18$

b. Economic lot size (ELS) $= \sqrt{\frac{2RS}{C_1}} \cdot \sqrt{\frac{P}{P - R}}$

$$= \sqrt{\frac{2 \times 2500 \times 250}{4}} \cdot \sqrt{\frac{9800}{9800 - 2500}} = 648 \text{ units}$$

Average duration of the production run $= \frac{648}{2500} = 0.26$ yr

c. i. When item purchased from outside :

$$\text{Total cost} = R \cdot C + \frac{R}{EOQ} \cdot A + \frac{EOQ}{2} \cdot C_1$$

$$= 2500 \times 30 + \frac{2500}{137} \times 15 + \frac{137}{2} \times 4 = \text{Rs. } 75548$$

ii. When item is produced internally :
Cost per unit = 80 % of Rs. 30 = Rs. 24
Setup cost = Rs. 250 per setup

$$\text{Total cost} = RC + \frac{R}{ELS}\cdot S + \frac{ELS}{2}\cdot\frac{P-R}{P}\cdot C_1$$

$$= 2500 \times 24 + \frac{2500}{648} \times 250 + \frac{9800-2500}{9800} \times \frac{648}{2} \times$$

$$= \text{Rs. } 61930$$

Evidently, the company should manufacture the product internally.

Que 4.12. **Describe the importance of ABC analysis as a selective approach for inventory control.**

UPTU 2011-12, Marks

Answer

1. The ABC analysis is based on Pareto's law that a few high usage value items constitute a major part of the capital invested in inventories whereas bulk of items having low usage value constitute insignificant part of the capital.
2. It contemplates to classify all the inventory items into three categories based on their usage values. Items of high usage value but small in number are classified as 'A' items and would be under strict control of top level management. 'C' items are large in number but require little capital and would be under simple control.
3. Items of moderate value and size are classified as 'B' items and would attract reasonable attention of the middle level management. ABC analysis (Always Better Control analysis) is also known as 'control by importance and exception' and 'proportional value analysis'.
4. In most organizations normal inventory items show the following distribution patterns :

   A : 5 % to 10 % of the total number of items accounting for 70 % to 80 % of the annual usage value.

   B : 10 % to 20 % accounting for about 15 % to 20 % of the annual usage value, and

   C : 70 % to 80 % of the number of items accounting for 5 % to 15 % of the annual usage value.
5. The following steps are performed for the ABC analysis :
   i. Find the annual usage value of every item in the sample by multiplying the annual requirement by its unit cost.
   ii. Arrange these items in descending order of usage value computed above.
   iii. Accumulate the total number of items and usage value.
   iv. Convert the accumulated total number of items and usage value into percentage of the grand total.
   v. Plot the two percentage on the graph paper. The curve obtained is called ABC distribution curve or Pareto curve or Lorentz curve or the curve of maldistribution.

   vi. Mark the cut-off points X and Y where the curve changes its slope, dividing it into three segments A, B and C. These segments A, B and C for the sample are then generalized over the entire quantity of stock items.

Que 4.13. **How will you control the inventories of a manufacturing organization ? Discuss the various inventory costs associated with this organization.**

UPTU 2013-14, Marks 05

Answer

A. Inventory Controlling of Manufacturing Organization : The inventory control problem consists of determination of three basic factors :

a. When to order (produce or purchase) ?
b. How much to order ?
c. How much safety stock should be kept ?

a. When to Order :

1. This is related to the lead time (also called delivery lag) of an item.
2. Lead time may be defined as the time interval between the placement of an order for an item and its receipt in stock.
3. It may be replenishment order on an outside firm or within the works.
4. There should be enough stock for each item so that customer's orders can be reasonably met from this stock until replenishment.
5. This stock level, known as reorder level, has, therefore, to be determined for each item.
6. It is determined by balancing the cost of maintaining these stocks and the disservice to the customer if his orders are not filled in time.

b. How much to Order :

1. As already discussed, each order has associated with it the ordering cost or acquisition cost.
2. To keep it low, the number of orders should be as few as possible *i.e.*, the order size should be large.
3. But large order size would imply high inventory cost.
4. Thus the problem of how much to order is solved by compromising between the acquisition costs and inventory carrying costs.

c. How much Should be the Safety Stock :

1. This is important to avoid overstocking while ensuring that no stock-outs take place.

2. The inventory control policy of an organization depends, upon the demand characteristics.
3. The demand for an item may be independent or dependent.
4. For instance, the demand for the different models of television sets manufactured by a company does not depend upon the demand of any other item, while the demand of its various components will depend upon the demand of the television sets and may be arithmetically computed from the latter.
5. The independent demand is usually ascertained by extrapolating the past demand history *i.e.*, by forecasting.
6. The order level can be fixed from the forecasts and the lead time.
7. Thus while in the case of dependent demand, simple arithmetic computations are enough to ascertain requirement of the components, in the case of independent demand items, statistical forecasting techniques have to be employed.
8. The family tree drawn in the next section gives an idea of the various inventory control policies.

B. Cost Involved in Inventory Management :

1. The cost associated with carrying or holding the goods in stock is known as holding or carrying cost which is usually denoted by C_1 or C_h per unit of goods for a unit of time.
2. The penalty costs that are incurred as a result of running out of stock are known as shortage or stock-out costs. These are denoted by C_2 or C_s per unit of goods for a specified period.
3. These costs arises due to shortage of goods, sales may be lost, goodwill may be lost either by a delay in meeting the demand or being quite unable to meet the demand at all.
4. In the case where the unfilled demand for the goods can be satisfied at a latter data (backlog case), these costs are usually assumed to vary directly with the storage quantity and the delaying time both.
5. On the other hand, if the unfilled demand is lost, shortage cost becomes proportional to shortage quantity only. These includes the fixed cost associated with obtaining goods through placing of an order or purchasing or manufacturing or setting up a machinery before starting production.
6. So, they include costs of purchase, requisition, follow-up, receiving the goods, quality control, etc. These are also called order costs or replenishment costs, usually denoted C_3 or C_0 per production run (cycle). They are assumed to be independent of the quantity order or produced.

Que 4.14. **The demand for an item in a company is 18,000 units per year, and the company can produce the item at a rate of 3,000 per**

month. The cost of one set up is Rs. 500 and the holding cost of one unit per month is 15 paise. The shortage cost of one unit is Rs. 240 per year. Determine the optimum manufacturing quantity and the number of shortages. Also determine the manufacturing time and the time between set-ups.

UPTU 2014-15, Marks 10

Answer

1. **Given :**

Demand, $D = 18{,}000$ units/year

$d = 3{,}000$/month

$\Rightarrow$ $d = 36000$/year

Set up cost, $C_0 = $ Rs. 500

Holding cost, $C_1 = 15$ paise/unit/month

$= \text{Rs. } \frac{15}{100} \times 12$ /unit/year

$C_1 = $ Rs. 1.8/unit/year

Shortage cost, $C_2 = $ Rs. 240/year

2. Optimum manufacturing quantity

$$Q^* = \sqrt{\frac{2DC_0}{C_1}\left(\frac{C_2 + C_1}{C_2}\right)}\sqrt{\frac{d}{d-D}}$$

$$Q^* = \sqrt{\frac{2 \times 18000 \times 500}{1.8} \times \left(\frac{240 + 1.8}{240}\right)} \cdot \sqrt{\frac{36000}{36000 - 18000}}$$

$= 3174.11405 \times 1.414213$

$= 4488.875$

$Q^* = 4489$ unit (approx)

3. Number of shortage

$$Q_s^* = \frac{C_1}{C_1 + C_2} Q^* \left(1 - \frac{D}{d}\right)$$

$$= \frac{1.8}{1.8 + 240} \times 4489 \times \left(1 - \frac{18000}{36000}\right) = 16.708$$

$Q_s^* = 17$ (approx)

4. Manufacturing time $= \frac{Q^*}{d} = \frac{4489}{36000}$ year

$= \frac{4489}{36000} \times 12$ month

$= 1.5$ month

5. Time between set-ups $= \frac{Q^*}{D} = \frac{4489}{18000}$ year

$= \frac{4489}{18000} \times 12$ months

$= 3$ months

Que 4.15. A company uses Rs. 10,000 worth of an item during the year. The ordering costs are Rs. 25 per order and carrying charges are 12.5 % of the average inventory value. Find the economic order quantity, number of orders per year, time period per order and the total cost.

UPTU 2013-14, Marks 05

Answer

1. Given : CR = Rs. 10,000, C_3 = Rs. 25 per order, $I = 12.5\,\% = 0.125$

2. $$\text{EOQ, } q_0 = \sqrt{\frac{2\,C_3R}{C_1}} = \sqrt{\frac{2\,C_3R}{CI}}$$

$\therefore$ $$\text{EOQ in rupees} = Cq_0 = C\sqrt{\frac{2C_3R}{CI}} = \sqrt{\frac{2C_3\,CR}{I}} = \sqrt{\frac{2 \times 25 \times 10{,}000}{0.125}} = \text{Rs. } 2{,}000.$$

3. $$\text{Number of orders/year}, n_0 = \frac{R}{q_0} = \frac{CR}{Cq_0} = \frac{10{,}000}{2{,}000} = 5$$

4. $$\text{Time period per order}, t_0 = \frac{1}{n_0} = \frac{1}{5}\text{ year} = 73 \text{ days.}$$

5. $$\text{Annual variable cost} = \sqrt{2C_3C_1R} = \sqrt{2C_3\,CIR} = \sqrt{2\,C\,ICR}$$
$$= \sqrt{2 \times 25 \times 0.125 \times 10{,}000} = \text{Rs. } 250$$

$\therefore$ $$\text{Total annual cost} = CR + \sqrt{2C_3\,C_1R} = \text{Rs. } (10{,}000 + 250) = \text{Rs. } 10{,}250.$$

Que 4.16. A manufacturing company has determined from an analysis of its accounting and production data for a certain part that (*i*) its demand is 9000 units per annum and is uniformly distributed over the year, (*ii*) its cost price is Re. 1 per unit, (*iii*) its ordering cost is Rs. 30 per order, (*iv*) the inventory carrying charge is 5 % of the inventory value.
Further, it is known that the lead time is uniform and equals to 6 working days, and that the total working days in a year are 300.
Determine :

a. The optimum number of orders per annum.
b. The total ordering and holding cost associated with the policy of ordering an amount equal to EOQ.

c. The re-order level.
d. The length of inventory cycle.
e. The amount of savings that would be possible by switching to the policy of ordering EOQ determined in (*a*) from the present policy of ordering the requirements of this part thrice a year; and
f. The increase in the total cost associated with ordering (i) 25 % more, and (ii) 30 % less than the EOQ.

Answer

a. $$EOQ = q_0 = \sqrt{\frac{2C_3R}{C_1}} = \sqrt{\frac{2 \times 30 \times 9000}{\frac{5}{100} \times 1}} = 3286 \text{ units}$$

Optimum number of orders per year,
$$N = \frac{R}{EOQ} = \frac{9000}{3286} = 2.7 \approx 3$$

b. $$\text{Total variable cost} = \sqrt{2C_3C_1R} = \sqrt{2 \times 30 \times \frac{5}{100} \times 1 \times 9000} = \text{Rs. } 164.32$$

c. Re-order level = lead time in days × demand per day
$$= 6 \times \frac{9000}{300} = 180 \text{ units}$$

d. $$\text{Length of inventory cycle} = \frac{EOQ}{R} = \frac{3286}{9000} = 0.365 \text{ yrs}$$
$$= 0.365 \times 300 = 109.5 \text{ days}$$

e. For the present policy of an order quantity = 3000 units
Ordering cost = 30 × 3 = Rs. 90
$$\text{Holding cost} = \frac{3000}{1} \times \frac{5}{100} \times 1 = \text{Rs. } 150$$
$\therefore$ Total variable cost (3000) = 90 + 150 = Rs. 240
Thus, saving in cost = Rs. 240 – Rs. 164 = Rs. 76 per yr

f. i. Ordering 25 % more than EOQ :
$$\text{Order quantity} = \frac{125}{100} \times 3286 = 4108 \text{ units}$$
With EOQ = 3286 and EOQ = 4108
$$k = \frac{4108}{3286} = 1.3$$
$$\text{Ratio of variable costs} = \frac{1}{2}\left(\frac{1}{k} + k\right) = \frac{1}{2}\left(\frac{1}{1.3} + 1.3\right) = 1.03 = \frac{103}{100}$$
$$\text{Thus, the cost would increase by } \frac{3}{100}\text{th or } 164 \times \frac{3}{100}$$
$$= \text{Rs. } 4.92 \approx \text{Rs. } 5$$

ii. Ordering 30 % lesser than EOQ :
In such situation, $k = 0.7$ and

$$\text{Ratio of variable costs} = \frac{1}{2}\left(\frac{1}{0.7} + 0.7\right) = 1.06$$

$$= \frac{106}{100} = \frac{53}{50}$$

Thus, the increase in cost would be $\frac{3}{50}$th over the cost for EOQ, and would equal to $164 \times \frac{3}{50}$ = Rs. 9.84 ≈ Rs 10.

Que 4.17. **A Company uses 8000 units of a product as raw material, costing Rs. 10 per unit. The administrative cost per purchase is Rs. 40. The holding costs are 28 % of the average inventory. The company is following an optimal purchase policy and places orders according to the EOQ. It has been offered a quantity discount of one percent if it purchases its entire requirement only four times a year.**
Should the company accept the offer of quantity discount of one percent ? If not, what minimum discount should the company demand ?

Answer

1. $$EOQ, q_0 = \sqrt{\frac{2C_3R}{C_1}} = \sqrt{\frac{2 \times 40 \times 8000}{\frac{28}{100} \times 10}} \approx 478 \text{ units}$$

2. $$\text{Total cost} = RC + \frac{R}{q_o}C_3 + \frac{q_o}{2}C_1$$

$$= 8000 \times 10 + \frac{8000}{478} \times 40 + \frac{478}{2} \times \frac{28}{100} \times 10$$

$$= \text{Rs. } 81340$$

3. **Discount Offer :**
   i. With a 1 % discount offered, the unit price = 99 % of Rs. 10
   = Rs. 9.90
   ii. $$\text{Total cost} = 8000 \times 9.90 + \frac{8000}{2000} \times 40 + \frac{2000}{2} \times \frac{28}{100} \times 9.90$$
   $$= \text{Rs. } 82132$$
   iii. Since the total cost is higher when 1 % discount is offered on a condition, the company should not accept the offer.
4. **Determination of the Minimum Discount Acceptable :**
   i. Let x be the percentage minimum discount acceptable to the company. It may be determined by setting the total cost equal to the total cost associated with the policy of ordering EOQ. Accordingly,

$$8000 \times \left(\frac{100 - x}{100}\right) \times 10 + 4 \times 40 + \frac{2000}{2} \times 0.28 \times \left(\frac{100 - x}{100}\right) \times 10 = 81340$$

   ii. On simplification, $\left(\frac{100 - x}{100}\right) \times 82800 = 81340 - 160 = 81180$

$$\frac{100 - x}{100} = \frac{81180}{82800} = 0.98$$

$$x = 100 - 98 = 2$$

Thus, minimum discount acceptable to the company is 2 %.

Que 4.18. **A company is considering the feasibility of changing suppliers for coupling hardware company has an optimal purchasing policy with Ace Hardware at a discount of 1 %, current yearly purchaser are Rs. 85000 and the Administrative charges are Rs. 150 per purchase and the carrying charges are 30 % of the average inventory level. Bids received from the other suppliers are :**
Nutz Co. offers 5 % discount if ordered twice a year and Grabbers Co. offers 3 % discount if ordered four times a year. Should the company retain the present supplier or accept the proposed offers and, if so, which offer ?

Answer

1. $$EOQ, q_o = \sqrt{\frac{2C_3R}{C_1}} = \sqrt{\frac{2C_3R}{CI}}$$

2. For in rupees, $$q = C\sqrt{\frac{2C_3R}{CI}} = \sqrt{\frac{2CC_3R}{I}}$$

$$= \sqrt{\frac{2 \times 85000 \times 150}{0.30}} \quad (\because \text{Order value, } CR = 85000)$$

$$= 9219.54 = \text{Rs. } 9220$$

3. Total cost at 10 % discount,

$$= \frac{1}{2}C_1q + \frac{C_3R}{q} + CR$$

$$= \frac{1}{2}CIq + \frac{C_3R}{q} + CR$$

$$= \frac{1}{2} \times 9220 \times 0.30 \times 0.99 + 150 \times \frac{85000}{9220} + 85000 \times 0.99$$

$$= 1369.17 + 1382.86 + 84150 = 86902.03$$

$$= \text{Rs. } 86902$$

4. Total cost at 5 % discount from Nutz Co. offer

$$= \frac{q}{2}CI + \frac{C_3R}{q} + CR$$

$$= \frac{R}{2\times2}CI + \frac{C_3R}{R/2} + CR$$

$$= \frac{85000}{2\times2} \times 0.30\times0.95 + 2\times150 + 85000\times0.95$$

$$= 6056.25 + 300 + 80750$$

$$= \text{Rs. } 87106.25$$

5. Total cost at 3 % discount from Grabbers Co. offer,

$$= \frac{q}{2}CI + C_3\frac{R}{q} + CR = \frac{R}{4\times2}CI + \frac{C_3R}{R/4} + CR$$

$$= \frac{85000}{2\times4} \times 0.30\times0.97 + 4\times150 + 85000\times0.97$$

$$= 2964.375 + 500 + 82450 = \text{Rs. } 86141.875$$

6. The company should, therefore, accept the offer from Grabbers Co.

PART-2

Simulations, Applications, Advantages and Limitations, Monte-Carlo Simulation, Application to Queuing, Inventory and other Problems.

CONCEPT OUTLINE : PART-2

Simulation : There are many problems of real life, which cannot be represented mathematically due to the complexity and probabilistic nature of problems. Simulation is used when all other techniques fail. It determines the effect of alternative policies without disturbing the system.

i. **Analogue Simulation or Environment Simulation :** In this reality is simulated in physical form.
ii. **Computer Simulation or System Simulation :** For example problems approx mathematical model is developed by using computer program.

Monte-Carlo Simulation : The stochastic / probabilistic simulation model in which statistical distribution functions are created by using a sequence of random numbers.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 4.19. **What is simulation ? Explain elements and structure of simulation system ?**

Answer

A. **Simulation :**
1. There are many problems of real life, which cannot be represented mathematically due to the complexity and probabilistic nature or problems.
2. It determines the effect of alternative policies without disturbing the system.
3. Simulation is technique for carrying out experiments for analyzing the behaviour and evaluating the performance of real system in actual environment, on simplified system.
4. Simulation is a representation of reality through a use of a model or other device, which will react in the same manner as reality under a given set of conditions.
   Simulation is defined as the use of a system model that has the designed characteristics of reality in order to produce the essence of actual operation.

B. **Types of Simulation :**
A number of experiments are performed on simulated models in laboratories or on computer. Accordingly, Simulation is mainly of two types :

a. **Analogue Simulation or Environmental Simulation :** In this reality is simulated in physical form. e.g.,
   i. Testing an aircraft model in a wind tunnel,
   ii. Planetarium, and
   iii. Model of traffic system with signals and crossing for joy and ride of children.

b. **Computer Simulation or System Simulation :** For complex problems mathematical model is developed using computer program.

C. **Elements and Structure of Simulation System :**

a. **State Variables :**
   1. Depending upon the nature of the problem, a number of variables are required to describe the physical system.
   2. These variables are known as state variables e.g., for simulation of operations in a bank, the possible state variables are the numbers of busy counters, the numbers of customers waiting in a queue or being served and the arrival time of next customers.

b. **Random Variables :** The random variable is a real valued function defined over a sample space associated with the outcome of a conceptual chance experiment.

c. **Event :** An event is an occurrence, which causes change in the state variables as a result of which there is a change in the state of the system.

d. **Static and Dynamic Simulation Model :** A static simulation model studies the profile of a system at a particular point of time.

e. **Random Number :** It is a number in a sequence of numbers whose probability of occurrence is same as that of any other number in that sequence.

Que 4.20. **Write down short note on simulation process ?**

Answer

A. **Simulation Process :** All simulation vary in complexity from situation to situation. However in general, the process of simulating a system consists of the following steps :

Step 1 : Identify and clearly define the problem.

Step 2 : List the statement of objectives decision variables and decision rules of the problem.

Step 3 : Formulate an appropriate model of the given problem.

Step 4 : Test the model, compare its behaviour with the behaviour of the actual problem environment.

Step 5 : Identify and collect the data needed to test the model.

Step 6 : Run the simulation model.

Step 7 : Analyze the results of the simulation. If the simulation process complete, then select the best course of action otherwise make the desired changes in model decision variables, parameters and return to step 4.

Step 8 : Re-run the simulation to test the new solution.

Step 9 : Validate the simulation, *i.e.*, increase the chances that any inferences draw about the real situation from running the simulation will be valid.

Que 4.21. **Describe the simulation process. What are the advantages, limitations and applications of simulation ?**

UPTU 2013-14, Marks 05

OR

What is simulation ? Describe the simulation process. What are the advantages and limitations of simulation ? Also specify the areas where simulation can be used.

UPTU 2012-13, Marks 10

Answer

A. **Simulation :** Refer Q. 4.19, Page 143B, Unit-4.

B. **Simulation Process :** Refer Q. 4.20, Page 144B, Unit-4.

C. **Advantages :**

1. It is an efficient tool to solve the problems where it is expensive, difficult to conduct experiments on real system.
2. There is no sufficient time to allow the real system to operate extensively.
3. Simulation is useful in sharpening the managerial decision making skills through gaming without disturbing normal operations.
4. Simulation models can be used to conduct experiments without disrupting real system. Experimenting with real system can be very expensive.
5. It enables the manager to provide insight into certain managerial problems where analytical solution of a problem is not possible.
6. Non-technical manager can comprehend simulation more easily.
7. It allows experimentation with a model of the real life system rather than the actual system.

D. **Limitations :**

1. It does not generate optimal solutions. It is a trial-and-error approach that may produce different solutions in repeated runs.
2. Methods generally are not efficient.
3. It is a long complicated process to develop a model and expensive also.
4. Each solution model is unique and cannot be used for other similar problems.
5. It is descriptive process rather than optimization process.
6. It is difficult to quantify the variable.
7. Simulation generates a way of evaluating solutions but it does not generate solutions by itself.
8. Simulation models cannot evaluate all the situations.

E. **Applications of Simulation :**

1. Evaluating alternative investment opportunities and financial forecasting.
2. Testing an aircraft model in a wind tunnel.
3. It is used in banks to determine the number of tellers required to serve customers.
4. It is used to model hospitals, banks, educational institutions and urban systems.
5. Modelling, luggage handling in airports.

6. For systems that already exist, simulations can be used to test for minor design changes or to look at the control policies. For example, for bottling lines the rules used to control the flow or bottles can be investigated as well as the effects of minor breakdowns.
7. Simulation is commonly used in the area of developing new systems, particularly those that involve a high capital investment. For example, simulation can be used to test the performance of personal computer (PC) assembly lines to ascertain the throughput possible, the level of utilization of operators and any potential problems.
8. It can be used to assess the best position for work-in-programs (WIP), stores and the level to be used.

Que 4.22. **Write the application of various simulation languages.**

Answer

A. Applications of Various Simulation Languages :

1. The efficiency of programming and execution of a simulation project depends upon the programming language used.
2. In addition to the general purpose languages such as FORTRAN and PL1, a large number of specialized computer languages are used to write the simulation programmes.
3. The FORTRAN, being highly general in nature, can be used for any simulation project. Being well known and commonly available on computer system, FORTRAN is quite often used to write the simulation programmes.
4. It is generally considered to be more efficient in computer time and storage requirements. However, programming in FORTRAN is more difficult and time consuming, as compared to the special simulation languages.
5. When the complexities of the simulation project increase, the book keeping of the intricate details of the simulation becomes difficult and this makes the programming in FORTRAN harder.
6. Thus for realistic situations, GASP and SIMSCRIPT are two widely used general simulation languages, which can easily do the job of FORTRAN or PL1.
7. These are FORTRAN based languages and hence the knowledge of FORTRAN is a pre-requisite for learning GASP and SIMSCRIPT.
8. The most commonly used simulation language is GPSS (General Purpose Simulation System), which was developed by IBM. It is easy to learn and incorporates all the features which are unique to simulation.
9. GPSS is a problem oriented language, but has a wide range of applications.
10. It employs the next event incrementing time flow mechanism and uses integral time units.

11. The calculations in integer arithmetic help to keep the round off errors to minimum.
12. In GPSS, the system to be simulated is flow charted in the form of block diagrams, and the blocks are then written in GPSS statements.
13. The simulation programming languages are very economical in respect of the users programming time, though they take a slightly larger CPU time in execution of the programme.
14. Some of the many simulation languages are, DYNAMO, SIMPAC, SIMULATE, SIMULA, CSWP, GSD, ESP and CSL.

Que 4.23. **What are different simulation models ? Explain.**

Answer

A. Deterministic Model :

1. A deterministic model is used in that situation wherein the result is established straightforwardly from a series of condition.
2. In a situation wherein the cause-and-effect relationship is stochastically or randomly determined, the stochastic model is used.
3. A deterministic model has no stochastic elements and the entire input and output relation of the model is conclusively determined.
4. A dynamic model and a static model are included in the deterministic model.
5. Simulation by the deterministic model can be considered as one of the specific instances of simulation by the stochastic. In other words since there are no random elements in the deterministic model, simulation can well be done just once.
6. However, in case the initial conditions or the boundary conditions are to be varied, simulation has to be repeated by changing the data.

B. Probabilistic Model :

1. Probabilistic modeling is any form of modeling that utilizes the presumed probability distribution of certain input assumptions to calculate the implied probability distribution for chosen output matrices.
2. This differs from a standard deterministic model, say a typical Excel spreadsheet, where you can change the values of input assumptions at random and see the impact of those changes on the output.
3. However we make only a limited effort to determine just how likely it is that the assumptions and therefore the outputs will change.
4. Monte-Carlo simulation is most commonly used technique for probabilistic modeling.
5. The approach involves taking an underlying deterministic model and running it several thousand times. In each iteration's different values for the input assumptions are fed into the model.

5. The values for each assumption are sampled from probability distributions chosen as the basis for understanding the behaviour of that variable.
6. The histogram of all the output values aggregated across all the iterations then describes the likely probability distribution of output.

C. **Static Model :**

1. The model which does not consider the passage of time is known as a static model. For understanding vast and complex phenomenon there is a method wherein time is suppressed and the spatial existence substances or resources is comprehended as a distribution or layout problem at the specific time.
2. A static model is used when one attempts to look at the balance of variables in steady state rather than to understand behaviour under a transient state.
3. Equations of materials or energy balance and algebraic equations can be used as equations representing a static model.
4. Linear programming, non-linear programming, integer programming, etc. are the methods for finding a solution using a static model.
5. In a situation where a large-scale allocation problem is to be temporally traced, there is also a method whereby the spatial allocation is solved by linear programming and those results are taken as the starting values for the next period.
6. The temporal and spatial distribution plan is determined by solving the spatial distribution problem for each period by linear programming and thereafter clubbing them together temporally.
7. Simulation using a static model can also be regarded as one of the means of approximating the solution of a differential equation.

D. **Dynamic Model :**

1. All phenomenon's in the natural world change with the passage of time, for modeling natural phenomenon and physical phenomenon, a dynamic model, which exhibits temporal change, can be used.
2. A differential equation forms the basis for expression of a dynamic model.
3. In order to understand a differential equation accurately and properly it is necessary to finely divide it temporally and spatially and the analyst is compelled to battle, with the memory capacity and computation time.
4. As a means for curtailing the memory capacity and computation time, various approximate solutions of differential equations have been developed.
5. In simulation which uses a dynamic model, besides simulating the object by the differential equation itself, there are various other levels, such as simulating with an approximate solution of the differential equation and so on.

6. From the point of view of the usage conditions, some have been developed into a simulator for use in a specific field or field-oriented simulator, similar to the control-system simulator.

Que 4.24. **Explain the theory of Monte-Carlo simulation ?**

OR

What is Monte-Carlo simulation ? Discuss in brief.

UPTU 2011-12, Marks 05

OR

Explain the Monte-Carlo simulation using an example. Discuss its various applications.

UPTU 2012-13, Marks 10

Answer

A. **Monte-Carlo Simulation :**

1. The Monte-Carlo method of simulation was developed by the two mathematicians John Von Neumann and Stainslow Ulam. The technique provided an approximate but quite workable solution to the problem.
2. The technique employs random numbers and is used to solve problems that involve probability and wherein physical experimentation is impracticable and formulation of mathematical model is impossible. It is a method of simulation of sampling technique.
3. The steps involved in carrying out Monte-Carlo simulation are :
   a. Select the measure of effectiveness (objective function) of the problem. It is either to be maximized or minimized. For example, it may be idle time of service facility in a queuing problem or number of shortage or the total inventory cost in an inventory control problem.
   b. Identify the variables that affect the measure of effectiveness significantly. For example, number of service facilities in a queuing problem or demand, lead time and safety stock in inventory control problem.
   c. Determine the cumulative probability distribution of each variable selected in step 2. Plot these distributions with values of the variables along x-axis and cumulative probability values along the y-axis.
   d. Get a set of random numbers.
   e. Consider each random number as a decimal value of the cumulative probability distribution. Enter the cumulative distribution plot along the y-axis. Project this point horizontally till it meets the distribution curve. Then project the point of intersection down on the x-axis.
   f. Record the value (or values) generated in step 5. Substitute in the formula chosen for measure of effectiveness and find its simulated value.

6. Repeat step 5 and 6 until sample is large enough to the satisfaction of the decision maker.

B. Example : Value of π experimentally by simulation :

1. Draw the coordinate axes OX and OY. With center O, draw an arc PR of unit radius as shown in Fig. 4.24.1 and complete the square OPQR.
2. Equation of the circle is $x^2 + y^2 = 1$. From random number select any two random numbers, say 0.2068 and 0.7295 and let $x = 0.2068$ and $y = 0.7295$.
3. Plot the point P_1 (0.2068, 0.7295). Obviously, if $x^2 + y^2 = 1$, P_1 will lie inside or on the circle but if $x^2 + y^2 > 1$, the point P_1 will lie outside the circle but within the square.
4. In this manner, hundreds or thousands of pairs of random numbers are selected and it is ascertained whether the points representing them lie in/on the arc or beyond the arc but inside the square.
5. Suppose N is the total number of points considered, out of which n lie in on the arc. Then

$$\frac{n}{N} = \frac{\text{Area enclosed by the arc}}{\text{Area of the square}}$$

$$= \frac{\frac{\pi}{4}(1)^2}{1} = \frac{\pi}{4}$$

or

$$\frac{n}{N} = \frac{\pi}{4}, \text{ i.e., } \pi = \frac{4n}{N}$$

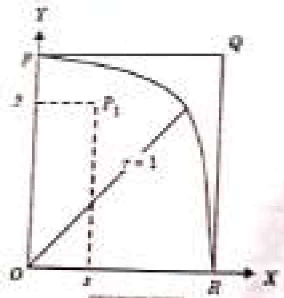


Fig. 4.24.1.

6. The above equation gives the experimental value of π. Obviously, the larger the sample size N, closer will be the obtained value to the true value of π.

C. Applications of Monte-Carlo Simulation :

1. To estimate parameters of a model or to construct the functional form of the model.
2. In situations where it is not possible to gain practical experience because one is dealing with problems which have not yet arisen.
3. To treat the course of action that cannot be formulated into a model.
4. To study transitional process.
5. For the cases where mathematical and statistical problems are too complex to be dealt with, alternative methods have to be devised.

Que 4.25. What are the advantages and disadvantages of Monte-Carlo simulation technique ?

Answer

A. Advantages :

1. This method helps in finding the solution of complex mathematical expression which otherwise is not possible.
2. This method avoids the expenses and difficulties of trial and error experimentation.

B. Disadvantages :

1. It is an expensive way of getting a solution of any problem.
2. The computations even in simple cases are quite complicated.
3. Monte-Carlo method does not provide optimal answers to the problems. The results are good only when the size of the sample is sufficiently large.

Que 4.26. What is Monte Carlo simulation ? Describe the idea of experimentation (random sampling) in simulation.

UPTU 2014-15, Marks 10

Answer

A. Monte Carlo Simulation : Refer Q. 4.24, Page 149B, Unit-4.

B. Monte-Carlo Simulation Technique with Respect to Random Sampling :

1. The principle behind this technique is replacement of actual statistical universe by a universe described by some assumed probability distribution and then sampling from the theoretical population by means of random numbers.

2. In case, it is not possible to describe a system in terms of standard probability distribution such as, normal, Poisson, exponential, gamma, etc., an empirical probability distribution can be constructed.
3. Essentially, above process is generation of simulated statistics (random variables) which can be explained in simple terms as picking up a random number and substituting this value in standard probability density function to obtain random variable or simulated statistic.
4. When probability density function is not standard for a given process, we build empirical probability density functions, along with the likely values or process parameters.
5. The random number is generated either on a computer or is picked up from a table and then the value is compared with cumulative probability and likely value of process parameter is obtained.

C. Stratified Sampling : The basic principle of this technique is divide the probability distribution into n intervals of equal probability, where n is the number of iterations that are to be performed on the model.

1. The probability distribution is split into n intervals of equal probability, where n is the number of iterations that are to be performed on the model. Fig. 4.26.1 illustrates an example of the stratification that is produced for 20 iterations of a normal distribution. The bands can be seen to get progressively wider towards the tails as the probability density drops away.

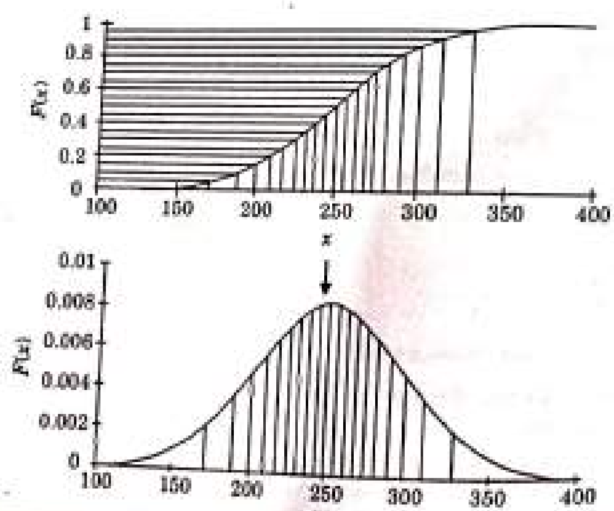


Fig. 4.26.1. The effect of stratification in Latin hypercube sampling.

2. In the first iteration, one of these intervals is selected using a random number.
3. A second random number is then generated to determine where, within that interval, $F(x)$ should lie. In practice, the second half of the first random number can be used for this purpose, reducing simulation time.
4. $x = G(F(x))$ is calculated for that value of $F(x)$.
5. The process is repeated for the second iteration, but the interval used in the first iteration is marked as having already been used and therefore will not be selected again.
6. This process is repeated for all of the iterations. Since the number of iterations n is also the number of intervals, each interval will only have been sampled once and the distribution will have been reproduced with predictable uniformity over the $F(x)$ range.

D. Random Number Generator :

1. A random number generator is a function object that can be used to generate a random sequence of integers.
2. If f is a random number generator and N is a positive integer, the $f(N)$ will return an integer less than N and greater than or equal to 0. If f is called many times with the same value of N, it will yield a sequence of numbers that is uniformly distributed in the range $[0, N]$.

Que 4.27. **A company manufactures 30 items per day. The sale of these items depends upon demand which has the following distribution :**

Sales (units)	Probability
27	0.10
28	0.15
29	0.20
30	0.35
31	0.15
32	0.05

The production cost and sale price of each unit are Rs. 40 and Rs. 50 respectively. Any unsold product is to be disposed of at a loss of Rs. 15 per unit. There is a penalty of Rs. 5 per unit if the demand is not met. Using the following random numbers estimate total profit/loss for the company for the next 10 days : 10, 99, 65, 99, 95, 01, 79, 11, 16, 20. If the company decided to produce 29 items per day, what is the advantage or disadvantage of company ?

Answer

1. First of all, random numbers 00-99 are allocated in proportion to the probabilities associated with the sale of the item as given below :

Sales (units)	Probability	Cumulative probability	Random numbers assigned
27	0.10	0.10	00-09
28	0.15	0.25	10-24
29	0.20	0.45	25-44
30	0.35	0.80	45-79
31	0.15	0.95	80-94
32	0.05	1.00	95-99

2. Let us now simulate the demand for next 10 days using the given numbers in order to estimate the total profit/loss for the company. Since the production cost of each item is Rs. 40 and sale price is Rs. 50, therefore the profit per unit of the sold item will be Rs. 10. There is loss of Rs. 15 per unit associated with each unsold unit and a penalty of Rs. 5 per unit if the demand is not met. Accordingly, the profit/loss for next 10 days are calculated in column (iv) of the table below if the company manufactures 30 items per day :

(i) Day	(ii) Random Number	(iii) Estimated sale	(iv) Profit/loss per day when production = 30 items per day	(v) Profit/loss per day when production = 29 items per day
1	10	28	(28×10) – (2×15) = Rs. 250	(28×10) – (1×15) = Rs. 265
2	99	32	(30×10) – (2×5) = Rs. 290	(29×10) – (3×5) = Rs. 275
3	65	30	(30×10) = Rs. 300	(29×10) – (1×5) = Rs. 285
4	99	32	(30×10) – (2×5) = Rs. 290	(29×10) – (3×5) = Rs. 275
5	95	32	(30×10) – (2×5) = Rs. 290	(29×10) – (3×5) = Rs. 275
6	01	27	(27×10) – (3×15) = Rs. 225	(27×10) – (2×15) = Rs. 240
7	79	30	(30×10) = Rs. 300	(29×10) – (1×5) = Rs. 285
8	11	28	(28×10) – (2×15) = Rs. 250	(28×10) – (1×15) = Rs. 265
9	16	28	(28×10) – (2×15) = Rs. 250	(28×10) – (1×15) = Rs. 265
10	20	28	(28×10) – (2×15) = Rs. 250	(28×10) – (1×15) = Rs. 265
Total Profit			Rs. 2695	Rs. 2695

3. The total profit for next 10 days will be Rs. 2695 if the company manufactures 30 items per day. In case, the company decides to produce 29 items per day, then the profit of the company for next 10 days is calculated in column (v) of the above table.

4. It is evident from this table that there is no additional profit or loss if the production is reduced to 29 items per day since the total profit remains unchanged *i.e.*, Rs. 2695.

Que 4.28. **Bright bakery keeps stock of a popular brand of cake. Previous experiences indicate the daily demand as given below :**

Daily demand	0	10	20	30	40	50
Probability	0.01	0.02	0.15	0.5	0.12	0.02

Consider the following sequence of random numbers.
R.No's : 48, 78, 19, 51; 56, 77, 15, 14, 68, 09. Using this sequence, simulate the demand for the next ten days. Find out the stock situation if the owner of the bakery decides to make 30 cakes on the basis of simulated data.

UPTU 2012-13, Marks 10

Answer

1. The simulated demand for the cakes for the next 10 days can be calculated as follows :
   a. Allocation of random numbers to demand of cakes

Demand	Probability	Cumulative probability	Random number interval	Random numbers fitted
0	0.01	0.01	00	
10	0.20	0.21	01–20	19(3), 15(7), 14(8), 9(10)
20	0.15	0.36	21 – 35	
30	0.50	0.86	36 – 85	48(1), 78(2), 51(4), 56(5), 77(6), 68(9)
40	0.12	0.98	86 – 97	
50	0.02	1.00	98 – 99	

∴ Number of cakes demanded in next 10 days are :

30, 30, 10, 30, 30, 30, 10, 10, 30, 10

b. The stock situation for various days if the decision is made to make 30 cakes every day is given in the table below :

Day	Demand	No. of cakes made (daily)	Stock
1	30	30	–
2	30	30	–
3	10	30	20
4	30	30	20
5	30	30	20
6	30	30	20
7	10	30	40
8	10	30	60
9	30	30	60
10	10	30	80

∴ Average daily demand

$$= \frac{1}{10}\,[30 + 30 + 10 + 30 + 30 + 30 + 10 + 10 + 30 + 10]$$

$$= \frac{220}{10} = 22$$

Note : We are using 0.2 instead of 0.02 for demand 10.

Que 4.29. **A company manufactures around 200 mopeds. Depending upon the availability of raw materials and other conditions, the daily production has been varying from 196 mopeds to 204 mopeds, whose probability distribution is as given below :**

Production / day	196	197	198	199	200	201	202	203	204
Probability	0.05	0.09	0.12	0.14	0.20	0.15	0.11	0.08	0.06

The finished mopeds are transported in a specially designed three-storeyed lorry that can accommodate only 200 mopeds. Using the following 15 random numbers 82, 89, 78, 24, 53, 61, 18, 45, 04, 23, 50, 77, 27, 54 and 10, simulate the process to find out :

i. What will be the average number of mopeds waiting in the factory ?

ii What will be the number of empty spaces in the lorry ?

UPTU 2013-14, Marks 10

Answer

1. The random numbers are established as in table below :

Table. 4.29.1.

Production/day	Probability	Cumulative probability	Random number interval
196	0.05	0.05	00 – 04
197	0.09	0.14	05 – 13
198	0.12	0.26	14 – 25
199	0.14	0.40	26 – 39
200	0.20	0.60	40 – 59
201	0.15	0.75	60 – 74
202	0.11	0.86	75 – 85
203	0.08	0.94	86 – 93
204	0.06	1.00	94 – 99

2. Based on the 15 random numbers given, we simulate the production per day in the table 4.29.2 :

Table. 4.29.2.

Day No.	Random number	Production per day	No. of mopeds waiting	Empty space in in the lorry
1	82	202	2	–
2	89	203	5	–
3	78	202	7	–
4	24	198	5	–
5	53	200	5	–
6	61	201	6	–
7	18	198	4	–
8	45	200	4	–
9	04	196	–	–
10	23	198	–	2
11	50	200	–	–
12	77	202	2	–
13	27	199	1	–
14	54	200	1	–
15	10	197	–	2

3. Average number of empty spaces in the lorry = $\frac{4}{15}$ = 0.27
4. Average number of mopeds waiting in the factory
   $= \frac{1}{15}[2+5+7+5+5+6+4+4+2+1+1] = 2.8$

Que 4.30. **Using random numbers to simulate a sample, find the probability that a packet of 6 products does not contain any defective product, when the production line produces 10 percent defective products. Compare the answer with the expected probability.**

UPTU 2014-15, Marks 10

Answer

1. Defective product = 10 %
   Non-defective product = 90 %
2. Let if we have 100 random numbers (0 to 99), then 90 (or 90 %) of them represent non-defective products and remaining 10 (or 10 %) of them represent defective products.
3. Thus, the random numbers 00 to 89 are assigned to variables representing non-defective products and 90 to 99 are assigned to variables representing defective products.
4. Now, if we choose a set of 2-digit random numbers in the range 00 to 99 to represent a packet of 6 products as shown in Table 4.30.1 then we would expect that 90 % of the time they would fall in the range 00 to 89.

Table 4.30.1.

S. No.	Sample number	Random number					
1.	A	86	02	22	57	51	68
2.	B	39	77	32	77	09	79
3.	C	28	06	24	25	93	22
4.	D	97	66	63	99	61	80
5.	E	69	30	16	09	05	53
6.	F	33	63	99	19	87	26
7.	G	87	14	77	43	96	43
8.	H	99	53	93	61	28	52
9.	I	92	86	52	77	65	15
10.	J	18	46	23	34	25	85

5. From the Table 4.30.1 it is clear that among ten simulated samples (from A to J), 6 samples (*i.e.*, C, D, F, G, H, and I) contain one or more defectives and 4 samples (*i.e.*, A, B, E and J) contain no defectives.
6. Thus the expected percentage of non-defective products is 40 %.
7. However, theoretically, the probability that a packet of six products containing no defective products while only 90 % products are non-defective is given by
   $$(0.9)^6 = 0.531441 = 53.14\ \%$$
8. Answer from expected probability gives more probability of getting non-defective product as compared to random number simulation of samples.

☺☺☺

5
UNIT

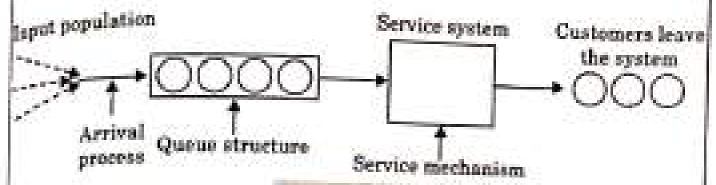
Queuing Models and Project Management

PART-1

Characteristics of Queuing Model, M/M/1 and M/M/S System, Cost Consideration.

CONCEPT OUTLINE : PART-1

Queuing System : Queue (waiting line) stands for a number of customers waiting to be serviced. The queue does not include the customer being served.



Queuing System.

Elements of Queuing Model : A Queuing system is specified completely by seven main elements :

1. Input or arrival (inter-arrival) distribution,
2. Output or departure (service) distribution,
3. Service channels,
4. Service discipline,
5. Maximum number of customer allowed in the system,
6. Calling source or population, and
7. Customer's behaviour.

Terms used in Waiting Line Problems :

i. **Balking :** A customer may leave the queue because the queue is too long and he has no time to wait, or there is sufficient waiting place.

ii. **Reneging :** This occur when a waiting customer leaves the queue due to impatient.

iii. **Jockeying :** Customers may jockey from one waiting line to another. It may be seen that this occurs in the supermarket.

Characteristics of a Queuing System :

1. **Queue Length (L_q) :** The average number of customer in the queue waiting to get service. This excludes the customers being served.
2. **System Length (L_s) :** The average number of customers in the system including those waiting as well as being served.

3. **Waiting Time in the Queue (W_q) :** The average time for which a customer has to wait in the queue to get service.
4. **Total Time in the System (W_s) :** The average total time spent by a customer in the system from the moment he arrives till leaves the system. It is taken to be the waiting time plus service time.
5. **Utilization Factor (ρ) :** It is the proportion of time a server actually spends with the customers. It is also called traffic intensity.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 5.1. **Discuss queuing models and give some important applications.**

OR

What is a "Queuing model" ? What are its objectives ? Giving basic structure explain (M/M/1) model.

Answer

A. Queuing Model : A queuing model can be completely described by

a. The input (or arrival pattern),
b. The service mechanism (or service pattern),
c. The queue discipline, and
d. Customer's behaviour.

a. The Input (or Arrival Pattern) :

1. The input, describes the way in which the customers arrive and join the system.
2. Generally, the customers arrive in a more or less random fashion which is not worth making the prediction.
3. Thus, the arrival pattern can be described in terms of probabilities and consequently the probability distribution for inter-arrival times (the time between two successive arrivals) or the distribution of number of customers arriving in unit time must be defined.
4. The present chapter is only dealt with those queuing systems in which the customers arrive in 'Poisson' or 'Completely random' fashion. Other types of arrival pattern may also be observed in practice that have been studied in queuing theory. Two such patterns are observed, when
   i. Arrivals are of regular intervals, and
   ii. There is general distribution (perhaps normal) of time between successive arrivals.

b. The Service Mechanism (or Service Pattern) :

1. It is specified when it is known how many customers can be served at a time, what the statistical distribution of service time is, and when service is available.
2. It is true in most situations that service time is a random variable with the same distribution for all arrivals, but cases occur where there are clearly two or more classes of customers (e.g., machines waiting for repairing) each with a different service time distribution.
3. Service time may be constant or a random variable.
4. Distributions of service time which are important in practice are 'negative exponential distribution' and the related 'Erlang (Gamma) distribution'.

c. The Queue Discipline :

1. The queue discipline is the rule determining the formation of the, queue the manner of the, customer's behaviour while waiting, and he manner in which they are chosen for service.
2. Properties of queuing system which are concerned with waiting times, in general, depend on queue discipline.
3. The following notations are used for describing the nature of service discipline :
   i. **FIFO** → First In, First Out or **FCFS** → First come, First Serve.
   ii. **LIFO** → Last In, First Out or **FILO** → First In, Last Out.
   iii. **SIRO** → Service in Random Order.

d. Customer's Behaviour : The customers generally behave in four ways :

i. **Balking :** A customer may leave the queue because the queue is too long and he has no time to wait, or there is not sufficient waiting space.
ii. **Reneging :** This occurs when a waiting customer leaves the queue due to impatience.
iii. **Priorities :** In certain applications, some customers are served before others regardless of their order of arrival. These customers have priority over others.
iv. **Jockeying :** Customers may jockey from one waiting line to another. It may be seen that this occurs in the supermarket.

B. General Structure of Queuing System :

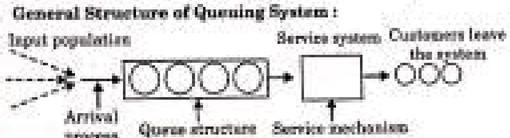


Fig. 5.1.1.

C. Applications of Queuing Models :

1. Scheduling of mechanical transport fleets.
2. Scheduling distribution of scarce war material.
3. Scheduling of jobs in production control.
4. Minimizing of congestion due to traffic delay at tool booths.
5. Solution of inventory control problems.

D. Model I (Erlang Model) :

1. This model is symbolically represented by (M/M/1) : (FCFS/∞/∞).
2. This represents Poisson arrival (exponential interarrival), Poisson departure (exponential service time), single server, first come first serve service discipline, infinite number of customers allowed in the system and infinite population.
3. Since the Poisson and exponential distributions are related to each other, both of them are denoted by the symbol 'M' due to Markovian property of exponential distribution.

Que 5.2. **What do you understand by a queue ? State the important distribution of arrival interval and service times. Giving elements of a queuing model.**

OR

State some of the important distribution of arrival and service time.

Answer

1. Queue (waiting time) stands for a number of customers waiting to be serviced. The queue does not include the customer being serviced.

A. Arrival Distribution :

1. It represents the pattern in which the number of customers arrives at the service facility.
2. Arrivals may also be represented by the inter-arrival time, which is the period between two successive arrivals.
3. Arrivals may be separated by equal intervals of time or by unequal but definitely known intervals of time or by unequal intervals of time whose probabilities are known; these are called random arrivals.
4. The rate at which customers arrive to be serviced, *i.e.*, number of customers arriving per unit of time is called arrival rate.
5. When the arrival rate is random, the customers arrive in no logical pattern or order over time. This represents most cases in the business world.
6. When arrivals are random, we have to know the probability distribution describing arrivals, specifically the time between arrivals.

7. Management scientists have demonstrated that random arrivals are often best described by the Poisson distribution.
8. Arrivals are not always Poisson, and we need to be certain that the assumption of Poisson distribution is appropriate before we use it.
9. Mean value of arrival rate is represented by λ.
10. It may be noted that the Poisson distribution with mean arrival rate λ is equivalent to the (negative) exponential distribution of inter-arrival times with mean inter-arrival time $1/\lambda$.

B. Service (Departure) Distribution :

1. It represents the pattern in which the number of customers leaves the service facility.
2. Departures may also be represented by the service (inter-departure) time, which is the time period between two successive services.
3. Service time may be constant or variable but known, or random (variable with only known probability).
4. If service times are randomly distributed, we have to find out what probability distribution best describes their behaviour.
5. In many cases where service times are random, management scientists have found that they are best described by exponential probability distribution.
6. If service times are exponentially distributed and arrivals Poisson distributed, the mathematics necessary to study waiting line behaviour is somewhat easier to develop and use.
7. Fig. 5.2.1, illustrates an exponential probability distribution of service times; from this we find that the probability of long service times is rather small.

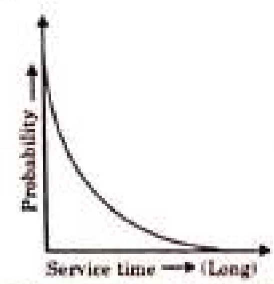


Fig. 5.2.1. Exponential distribution of service time.

8. The rate at which one service channel can perform the service, *i.e.*, number of customers served per unit of time is called service rate.
9. This rate assumes the service channel to be always busy, *i.e.*, no idle time is allowed. Mean value of service rate is represented by μ.

C. **Elements of Queuing Model :**

a. **Queue Discipline :**

i. It is a rule according to which customers are selected for service when a queue has been formed.

ii. The most common queue discipline is the "first come, first serve" (FCFS), or the "first in, first out" (FIFO) rule under which the customers are serviced in the strict order of their arrivals.

iii. Other queue discipline include : "last in, first out" (LIFO) rule according to which the last arrival in the system is serviced first.

iv. This discipline is practiced in most cargo handling situations where the last item loaded is removed first.

v. Besides these, other disciplines are : "selection for service in random order" (SIRO) rule according to which the arrivals are serviced randomly irrespective of their arrivals in the system, and a variety of priority schemes — according to which a customer's service is done in preference over some other customer.

b. **Input and Holding Times :**

i. It represents the pattern in which customers arrive at the system.

ii. Arrivals may also be represented by the inter-arrival time, which is the time period between two successive arrivals.

iii. Arrivals may be separated by equal intervals of time, or unequal but definitely known intervals of time, or by unequal intervals of time whose probabilities are known; these are called random arrivals.

iv. The rate at which customers arrive at the service station, that is the number of customers arriving per unit of time is called arrival rate.

v. The time elapsed from the commencement of service to its completion for a customer at a service facility is referred to the service time or holding time.

Que 5.3. **Explain the characteristics of Queuing Model. A person repairing radios finds that time spent on radio sets has exponential distribution with mean 20 minutes. If the radios are repaired in the order in which they come in and their arrival is approximately Poisson with an average rate of 15 for 8 hours a day, what is repairman's expected idle time each day ? How many jobs are ahead of the average set just brought in ?** **UPTU 2013-14, Marks 10**

Answer

A. **Characteristics of the Queuing System :**

a. **Queue Length (L_q) :** The average number of customer in the queue waiting to get service. This excludes the customers being served.

b. **System Length (L_s) :** The average number of customers in the system including those waiting as well as being served.

c. **Waiting Time in the Queue (W_q) :** The average time for which a customer has to wait in the queue to get service.

d. **Total Time in the System (W_s) :** The average total time spent by a customer in the system from the moment he arrives till leaves the system. It is taken to be the waiting time plus service time.

e. **Utilization Factor (ρ) :** It is the proportion of time a server actually spends with the customer.

B. **Numerical :**

1. **Given :** Arrival rate, $\lambda = \frac{15}{8 \times 60} = \frac{1}{32}$

Service rate, $\mu = \frac{1}{20}$ units/minute.

2. Number of jobs ahead of the set brought in = Average number of jobs in the system,

$$L_s = \frac{\lambda}{\mu - \lambda} = \frac{1/32}{1/20 - 1/32} = \frac{5}{3}$$

3. Number of hours for which the repairman remains busy in an 8-hour day,

$$= 8\ \frac{\lambda}{\mu} = 8 \times \frac{1/32}{1/20} = 8 \times \frac{20}{32} = 5 \text{ hours.}$$

4. Therefore, time for which repairman remains idle in an 8-hour day,

$$= 8 - 5 = 3 \text{ hours.}$$

Que 5.4. **Give classification of queuing models.**

Answer

The various types of queuing models can be classified as follows :

A. **Probabilistic Queuing Models :**

a. **Model I (Erlang Model) :**

1. This model is symbolically represented by (M/M/1) : (FCFS/∞/∞). This represents Poisson arrival (exponential inter-arrival), Poisson departure (exponential service time), single server, first come-first serve service discipline, infinite number of customers allowed in the system and infinite population.

2. Since the Poisson and exponential distributions are related to each other, both of them are denoted by the symbol 'M' due to Markovian property of exponential distribution.

b. **Model II (General Erlang Model) :**

1. Though this model is also represented by (M/M/1) : (FCFS/∞/∞), it is a general queuing model in which the arrival and service rates depend upon the length of the queue.
2. Some person desiring service may not join the queue since it is too long, thus affecting the arrival rate.
3. Similarly, service rate is also affected by the length of the queue.

c. **Model III :**

1. This model is represented by (M/M/1) : (SIRO/∞/∞).
2. It is essentially same as model I except that the service discipline is SIRO instead of FCFS.

d. **Model IV :**

1. This model is represented by (M/M/1) : (FCFS/N/∞).
2. In this model the capacity of the system is limited or finite, say N.
3. So the number of arrivals cannot exceed N.

e. **Model V :**

1. This model is represented by (M/M/1) : (FCFS/N/M).
2. It is finite-population or limited source model.
3. In this model the probability of an arrival depends upon the number of potential customers available to enter the system.

f. **Model VI :**

1. This model is represented by (M/M/C) : (FCFS/∞/∞).
2. This is same as model I except that there are C service channels working in parallel.

g. **Model VII :**

1. This model is represented by ($M/E_k/1$) : (FCFS/∞/∞).
2. In this model instead of exponential service time, there is Erlang service time with k phases.

h. **Model VIII :**

1. This model is represented by (M/M/1) : (GD/M/N), where $M \le N$.
2. It represents machine repair problem with a single repairman. N is the total numbers of machines out of which M are broken down and forming a queue. GD represents a general service discipline.

i. **Model IX :**

1. This model is represented by (M/M/C) : (GD/M/N), $M \le N$.
2. It is same as model VIII except that there are C repairmen, $C < N$.

j. **Model X :** This is called power supply model.

B. **Deterministic Model :**

a. **Model XI :**

1. This model is represented by (D/D/1) : (FCFS/∞/∞).
2. In this model inter-arrival time as well as service time are fixed and known with certainty.
3. The model is, therefore, called deterministic model.

C. **Mixed Queuing Model :**

a. **Model XII :**

1. This model is represented by (M/D/1) : (FCFS/∞/∞).
2. Here, arrival rate is Poisson distributed while the service rate is deterministic or constant.

Que 5.5. **What is meant by a M/M/S Model ? What happens in a M/M/1 model when the mean arrival rate exceeds the service rate ? What happens when they are equal ? Derive the expression for the expected number of customers in the system for the M/M/1 model (Assume the expression for the probability of 'n' customers in the system for $n = 0, 1, 2, ...$).**

Answer

A. **M/M/S Model :**

1. In this more realistic queuing system the customers arrive in a Poisson fashion with mean arrival rate λ.
2. There are S (fixed) number of counters (service stations) arranged in parallel, and a customer can go to any of the free counters for his service, where the service time at each counter is identical and follows the same exponential distribution law.
3. The mean service rate per busy server is μ. Therefore, overall service rate, when there are n units in the system, may be obtained in the following two situations :
   a. If $n \le S$, all the customers may be served simultaneously. There will be no queue $(S - n)$ number of servers may remain idle, and then
   $$\mu_n = n\mu, \; n = 0, 1, 2, ..., S.$$
   b. If $n \ge S$, all the servers are busy, maximum number of customers waiting in queue will be $(n - S)$, then $\mu_n = S\mu$. Now, this model may also be considered as a special case of Model II with
   $$\lambda_n = \lambda \text{ (for } n = 0, 1, 2, ...)$$
   $$\mu_n = \begin{cases} n\mu \text{ (for } n = 0, 1, 2, ..., S) \\ S\mu \text{ (for } n \ge S) \end{cases}$$
4. If the arrival rate of jobs to the system is larger than the system service capacity, the system is full with a relatively high proportion of the time.

5. This in turn leads to more jobs being turned away because of the full system.
6. In fact, the effective arrival rate (those jobs getting into the system) will necessarily be less than system's service capacity.
7. If the mean arrival rate is equal to the mean service rate, $\lambda = \mu$ for the M/M/1 system, then each probability is equal so that

$$p_0 = \ldots = p_3 = \frac{1}{4}.$$

8. The effective arrival rate is, thus, given by $\lambda_e = \lambda(1 - p_3) = \frac{3}{4}\lambda < \mu$.

B. Derivation :

1. For an infinite capacity model (M/M/1)

$$\lambda p_0 = \mu p_1$$
$$\lambda p_1 = \mu p_2$$
$$\lambda p_2 = \mu p_3$$
$$\vdots$$
$$\lambda p_n = \mu p_{n+1}$$
$$\vdots$$
$$\sum_{n=0}^{\infty} p_n = 1$$

2. The above system can be rewritten to obtain the following equivalent system :

$$p_1 = \frac{\lambda}{\mu} p_0$$
$$p_2 = \frac{\lambda}{\mu} p_1$$
$$p_3 = \frac{\lambda}{\mu} p_2$$
$$\vdots$$
$$p_n = \frac{\lambda}{\mu} p_{n-1}$$
$$\vdots$$

3. Using a successive substitution procedure, each p_n term can be written as function of p_0 to obtain

$$p_n = \left(\frac{\lambda}{\mu}\right)^n p_0, \text{ for } n = 0, 1, 2, \ldots$$

4. The final step is to substitute above equation into the Norming equation yielding,

$$p_0 + \left(\frac{\lambda}{\mu}\right) p_0 + \left(\frac{\lambda}{\mu}\right)^2 p_0 + \ldots + \left(\frac{\lambda}{\mu}\right)^n p_0 + \ldots = 1$$

5. Which can be solved to obtain an expression for p_0 as

$$p_0 = \frac{1}{\left(1 + \frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}\right)^2 + \ldots + \left(\frac{\lambda}{\mu}\right)^n + \ldots\right)}$$

6. The denominator is a geometric series that has a finite value if $\frac{\lambda}{\mu} < 1$.
7. Under the condition $\lambda < \mu$, this series sums to

$$p_0 = 1 - \frac{\lambda}{\mu}$$

8. And the general solution to the steady-state probabilities is

$$p_n = \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n \quad \text{(for } n = 0, 1, \ldots)$$

9. The utilization factor 'u' for the server is obtained from

$$u = 1 - p_0 = 1 - \left(1 - \frac{\lambda}{\mu}\right) = \frac{\lambda}{\mu}$$

10. Expected number of jobs :

$$W = E[N] = \sum_{n=0}^{\infty} n p_n = \sum_{n=0}^{\infty} n\left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n$$

$$= \left(1 - \frac{\lambda}{\mu}\right)\sum_{n=0}^{\infty} n\left(\frac{\lambda}{\mu}\right)^n = \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)\left(\frac{1}{1 - \frac{\lambda}{\mu}}\right)^2$$

$$W = \frac{\lambda/\mu}{(1 - \lambda/\mu)} = \frac{\rho}{1 - \rho} \qquad \left\{\rho = \frac{\lambda}{\mu}\right\}$$

Que 5.6. **What are the important assumptions of queuing models ?**

Answer

A. Important Assumptions of Queuing Models :

1. The customers arrive for service at a single service facility at random according to Poisson distribution with mean arrival rate λ or equivalently, the inter-arrival times follow exponential distribution.
2. The service time has exponential distribution with mean service rate μ.
3. The service discipline followed is first come, first served.

4. Customer behaviour is normal *i.e.*, customers desiring service join the queue, wait for their turn and leave only after getting served.
5. Service facility behaviour is normal. It serves the customers continuously, without break, as long as there is queue. Also it serves only one customer at a time.
6. The waiting space available for customers in the queue is infinite.
7. The calling source (population) has infinite size.
8. The elapsed time since the start of the queue is sufficiently long so that the system has attained a steady state or stable state.
9. The mean arrival rate λ is less than the mean service rate μ.

Que 5.7. **Discuss the cost associated with queuing system.**

Answer

1. In order to solve a queuing problem, service facility must be manipulated so that an optimum balance is obtained between the cost of waiting time and the cost of idle time.
2. The cost of waiting customers generally includes either the indirect cost of lost business (because people go somewhere else, buy less than they had intended to, or do not come again in future) or direct cost of idle equipment and persons; for example, cost of truck drivers and equipment waiting to be unloaded or cost of operating an airplane or ship waiting to land or dock.
3. The cost of idle service facilities is the payment to be made to the servers (engaged at the facilities) for the period for which they remain idle.
4. By increasing the investment in labour and equipment (service facilities), waiting time and the losses associated with it can be decreased. It is desirable, then, to obtain the minimum sum of these two costs; costs of investment and operation, and costs due to waiting.
5. This optimum balance of costs can be obtained by scheduling the flow of units requiring service and / or providing proper number of facilities.

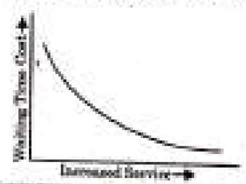


Fig. 5.7.1. Relationship between level of service and waiting time cost.

6. Fig. 5.7.1 illustrates the relationship between the level of service provided and the cost of waiting time.
7. It is observed that as the level of service in increased (as more servers are provided), the cost of waiting time decreases.

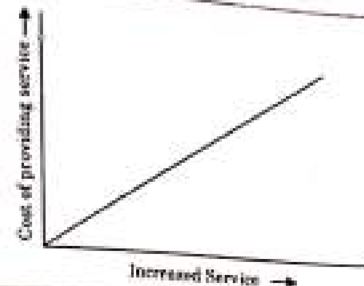


Fig. 5.7.2. Relationship between level of service and cost of providing service.

8. Fig. 5.7.2 illustrates the relationship between the level of service and the cost of providing that service.
9. It is observed that as the level of service increases, so does the cost of providing that increased service.
10. In Fig. 5.7.3 the waiting time cost is added to the cost of providing service to establish a total expected cost.

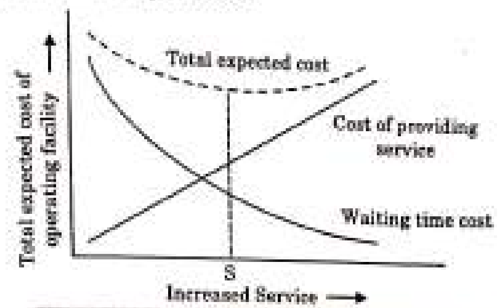


Fig. 5.7.3. Total cost of operating service facility.

11. We see that the total expected cost is minimum at a service level denoted by point S. Thus the objective of the techniques explained in the remainder of this chapter is really to determine that particular level of service which minimizes the total cost of providing service and waiting for that service.
12. Let C_W = Expected waiting cost/unit/unit time,
    L_S = Expected (average) number of units in the system, and
    C_f = Cost of servicing one unit.
13. The expected waiting cost per unit time (period) $= E_w \cdot L_s = C_w \cdot \frac{\lambda}{\mu - \lambda}$,
    and expected service cost per unit time (period) $= C_f \cdot \mu$.

14. Total cost, $C = C_W \frac{\lambda}{\mu - \lambda} + \mu C_f$

This will be minimum if $\frac{d}{d\mu}(C) = 0$

or if $-C_w \frac{\lambda}{(\mu - \lambda)^2} + C_f = 0$, which gives $\mu = \lambda \pm \sqrt{\frac{C_w}{C_f}}\,\lambda$.

20. Note that a plus and minus sign appear before the square root sign. A negative value of μ is not a possible answer in real life problems. μ given by the above equation is called minimum cost service rate.

Que 5.8. **Arrivals at a telephone booth are considered to be following Poisson distribution with an average time of 10 minutes between one arrival and the next. Length of a phone call is assumed to be distributed exponentially with mean 3 minutes; find,**

i. What is the probability that a person arriving at the booth will have to wait ?

ii. What is the average length of the queue that is formed time to time ?

iii. A new booth can be installed if a person has to wait for atleast 3 minutes for the phone. By how much must the flow of arrival be increased in order to justify a second booth ?

Answer

1. **Given :** Arrival rate $= \frac{1}{10}$ per min

Service rate $= \frac{1}{3}$ per min

i. Probability that a person will have to wait $= \frac{\lambda}{\mu} = \frac{1/10}{1/3} = 0.3$

ii. Average queue length $= \frac{\mu}{\mu - \lambda} = \frac{1/3}{\frac{1}{3} - \frac{1}{10}} = \frac{30}{7 \times 3} = 1.4$ persons

iii. Let λ_1 be the new (increased) arrival rate to justify the installation of the second telephone booth.

Average waiting time in the queue $= \frac{\lambda_1}{\mu(\mu - \lambda_1)}$

$$3 = \frac{\lambda_1}{\frac{1}{3}\left(\frac{1}{3} - \lambda_1\right)}$$

$$\frac{1}{3} - \lambda_1 = \lambda_1$$

$$\lambda_1 = \frac{1}{6} \text{ arrivals / min}$$

$\therefore$ Increase in flow rate of arrivals $= \frac{1}{6} - \frac{1}{10} = \frac{5-3}{30} = \frac{1}{15}$ per min

Que 5.9. **A single counter reservation office serves passengers with an average service time 5 minutes (exponentially distributed). Passenger comes for reservation in Poisson at an average rate of 8 per hour.**

i. What is average queue length ?

ii. What is probability that there are 3 passengers in the system ?

iii. Probability that a passenger shall wait for more than 10 minutes before getting served.

iv. What is average queue length if average number of passengers increases to 12 per hour ?

Answer

1. **Given :**

Arrival rate, $\lambda = \frac{8}{60}$ per min $= \frac{2}{15}$ per min

Service rate, $\mu = \frac{1}{5}$ passengers per min

i. Average queue length,

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{\left(\frac{2}{15}\right)^2}{\frac{1}{5}\left(\frac{1}{5} - \frac{2}{15}\right)}$$

$$= \frac{\frac{4}{225}}{\frac{1}{5} \times \frac{1}{15}} = \frac{4 \times 75}{225} = \frac{4}{3} = 1.33$$

ii.

$$p(n = 3) = P_0 + P_1 + P_2 + P_3$$

$$= \left(1 - \frac{\lambda}{\mu}\right) + \frac{\lambda}{\mu}\left(1 - \frac{\lambda}{\mu}\right) + \left(\frac{\lambda}{\mu}\right)^2\left(1 - \frac{\lambda}{\mu}\right) + \left(\frac{\lambda}{\mu}\right)^3\left(1 - \frac{\lambda}{\mu}\right)$$

$$= \left(1 - \left(\frac{2/15}{1/5}\right)\right) + \frac{2/15}{1/5}\left(1 - \left(\frac{2/15}{1/5}\right)\right)$$

$$+ \left(\frac{2/15}{1/5}\right)^2\left(1 - \left(\frac{2/15}{1/5}\right)\right) + \left(\frac{2/15}{1/5}\right)^3\left(1 - \left(\frac{2/15}{1/5}\right)\right)$$

$$= \frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \frac{8}{81}$$

$$= \frac{27 + 18 + 12 + 8}{81} = \frac{65}{81} = 0.80$$

iii. Probability [waiting time ≥ 10],

$$= \int_{10}^{\infty} \frac{\lambda}{\mu}(\mu-\lambda)e^{-(\mu-\lambda)t}\,dt = \frac{\lambda}{\mu}(\mu-\lambda)\left[\frac{e^{-(\mu-\lambda)t}}{-(\mu-\lambda)}\right]_{10}^{\infty}$$

$$= \frac{-\lambda}{\mu}[0-e^{-(\mu-\lambda)10}] = \frac{\lambda}{\mu}e^{-(\mu-\lambda)10}$$

$$= \frac{2/15}{1/5}e^{-\left(\frac{1}{5}-\frac{2}{15}\right)\times 10} = \frac{2}{3}e^{-\frac{2}{3}} = 0.34227$$

iv. $$L_q' = \frac{\lambda_1^2}{\mu(\mu-\lambda_1)} = \frac{\left(\frac{12}{60}\right)^2}{\frac{1}{5}\left(\frac{1}{5}-\frac{12}{60}\right)} = \infty$$

$\because$ $\lambda_1 = \mu$, so queue length is infinite.

Que 5.10. **A television repairman finds that the time spent on his jobs has an exponential distribution with a mean of 30 minutes. If he repairs sets in the order in which they came in and if the arrivals of sets follows a Poisson distribution approximately with an average rate of 10 per-8 hour day, what is the repairman's expected idle time each day ? How many jobs are ahead of the average set just brought in ?**

UPTU 2012-13, Marks 10

Answer

1. Given : Arrival rate, $\lambda = \frac{10}{8 \times 60} = \frac{1}{48}$ units/min

 Service rate, $\mu = \frac{1}{30}$ units/ min

2. Number of hours for which the repairman remains busy in an 8-hour day

$$= 8 \times \frac{\lambda}{\mu} = 8 \times \frac{\frac{1}{48}}{\frac{1}{30}} = 5 \text{ h}$$

3. So time for which repairman remains idle in an 8 hours day = 8 – 5 = 3 h

$$\rho = \frac{\lambda}{\mu} = \frac{30}{48} = 0.625$$

4. Number of jobs ahead of the set brought in = Average number of jobs in the system

$$= \frac{\rho}{1-\rho} = \frac{0.625}{1-0.625} = \frac{0.625}{0.375} = 1.67$$
$$= 2 \text{ TV sets}$$

Que 5.11. **In a railway marshalling yard, goods trains arrive at a rate of 30 trains per day. Assuming that the inter arrival time follows an exponential distribution and the service time (the time taken to hump a train) distribution is also exponential with an average of 36 minutes. Calculate :**

i. Expected queue size (line length), and

ii. Probability that the queue size exceeds 10.

If the input of trains increases to an average of 33 per day, what will be the change in (a) and (b) ?

UPTU 2014-15, Marks 10

Answer

1. Given : Arrival rate, λ = 30 trains/day

 Service time = 36 min

$$\text{Service rate, } \mu = \frac{60 \times 24}{36} = 40 \text{ trains/day}$$

a. Expected queue size, $L_q = \frac{\lambda^2}{\mu(\mu-\lambda)}$

$$\Rightarrow \quad L_q = \frac{(30)^2}{40 \times (40-30)}$$

$$\Rightarrow \quad L_q = \frac{900}{40 \times 10} = \frac{9}{4}$$

b. Probability that the queue size exceeds 10 : $n \geq 10$

$$P_n = \left(1-\frac{\lambda}{\mu}\right)\cdot\left(\frac{\lambda}{\mu}\right)^n$$

$$= \left(1-\frac{30}{40}\right)\cdot\left(\frac{30}{40}\right)^{10}$$

$$= \left(\frac{1}{4}\right)\cdot\left(\frac{3}{4}\right)^{10}$$

$$= .014$$

2. **For Case II :**

 λ_1 = 33 trains per day

 μ = 40 trains per day

i. Expected queue size, $L_q = \frac{\lambda^2}{\mu(\mu-\lambda)}$

$$\Rightarrow \quad \frac{(33)^2}{40\,(40-33)} = 3.889$$

ii. Probability that the queue size exceeds 10 : $n \geq 10$

$$P_n = \left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n$$

$$= \left(1-\frac{33}{40}\right)\left(\frac{33}{40}\right)^{10}$$

$$= (0.175)\,(0.825)^{10}$$

$$= 0.0255$$

PART-2

Basic Concept, Rules for Drawing the Network Diagram, Application of CPM and PERT Techniques in Project Planning and Control, Crashing of Operations, and Resource Allocation.

CONCEPT OUTLINE : PART-2

Project : A project is defined as a combination of inter related activities which must be executed in a certain order before the entire task is completed.

Project Management : It is a scientific way of planning, implementing and controlling various aspects of project such as time, money materials, manpower and other resources with the intention of achieving the basic objective or goals including technical specification, cost and time schedule.

PERT : It helps in designing, planning, coordinating and decision making to accomplish the project economically in the minimum available time with limited available resources.

CPM : CPM uses networking principle for planning and controlling projects. It is very widely used next to PERT model.

Crashing of Operations : In order to find optimum duration which will result in minimum cost, crashing of network is done.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 5.12. **What is a Project ? Explain in brief important characteristics of project.**

UPTU 2011-12, Marks 10

Answer

A. **Project :**

1. It is defined as an organised programme of pre determined group of activities that are non routine in nature and that must be completed using the available resources within the given time limit.
2. According to Harison, "A project can be defined as a non routine, non repetitive, one-off undertaking, normally with discrete time, financial and technical performance goals".

   For Example : The construction of a house is a project. The construction of a house consist of so many activities like digging of foundation pits, construction of foundation, construction of walls, fixing of doors and windows, fixing of sanitary fitting, wiring etc.

B. **Stages of a Project :**

1. Whether a project is large or small, the stages of a project are ultimately the same.
2. The main five stages of project are given as :
   a. The initiation,
   b. Planning,
   c. Scheduling or execution,
   d. Controlling, and
   e. Monitoring.

a. **The Initiation :**
   i. The initiation phase will start as soon as giving the assignment to the project team member.
   ii. It includes overall project goal and ask the client or project owner as many question as possible, so the team members can plan the project efficiently.

b. **Planning :**
   i. Once team members have initiated the project and gather all relevant information, they will begin planning the project.
   ii. The planning stage depends on the size of the project *i.e.*, how much information have to be organize and how large the team is.
   iii. The end result of planning should be a clear project plan or schedule from which everyone follows their assigned task.

c. **Scheduling or Execution :**
   i. After completion of planning, the team can begin executing the project against their assigned task.
   ii. This is the stage where everyone actually starts doing the work.

iii. The project is officially kicked off from the execution stage with person-l meeting to ensure everyone has all the necessary resources to begin execution of their part of the project.

d. **Monitor and Control :** While the project is in the execution stage, monitoring and controlling begins to ensure that project is moving along as planned.

C. **Characteristics of a Project :** The characteristics of a project are given as :

a. **Objective :** Every project has specific objective. The project ceases to exist after objectives have been achieved.

b. **Life Span :** The life span representing the start and end of a project are specified in the objective.

c. **Uniqueness :** Every project is unique and is one unit with one responsible authority. However, there are many participants in the project.

d. **Team Work :** A team is constituted with member drawn from different disciplines, specialization, organizations and may be countries. The success of project depends upon team work.

e. **Life Cycle :** A project has a life cycle consisting of conception, design, implementation and commissioning stages.

f. **Change :** A project is not rigid in life span. Changes occur throughout the life span of a project. Some of the changes may not have major impact. However, some changes can affect the earlier character or course of the project.

g. **Customer Specific :** A project is always customer specific. The customer gives various requirement and constraint within which the project has to be executed.

h. **Complexity :** A project is a complex set of activities relating to technology, machinery, material equipment and people, work culture and ethics, financial resources. Execution of the project in time by proper scheduling of different activities contributes to the complexity of project.

i. **Risk and Uncertainty :** Risk and uncertainty go hand and hand with the project. The degree of risk and uncertainty will depend upon the type of project *i.e.*, *R* and *D* projects and ill-defined project.

j. **Optimization :** Project management concept have evolved with the aim of achieving optimal utilization of available resources.

Que 5.13. **Describe the basic rules for drawing a network diagram.**

Answer

A. **Rules for Drawing a Network Diagram :**

1. Each activity is represented by one and only one arrow.

2. Each activity must be identified by its starting and end node which implies that :

a. Two activities should not be identified by the same completion events, and

b. Activities must be represented either by their symbols or by the corresponding ordered pair of starting completion events.

3. Nodes are numbered to identify an activity uniquely. Tail node should be lower than the head node of an activity.

4. Between any pair of nodes, there should be one and only one activity, however more than one activity may terminate to a node.

5. Arrow should be kept straight and not curved on bent.

6. The logical sequence between activities must follow the following rules :

a. An event cannot occur until all the incoming activities into it have been completed.

b. An activity cannot start unless all the preceding activities on which it depends have been completed.

c. Dummy activities should only be introduced if absolutely necessary.

Que 5.14. **Define critical path and its importance in network analysis.**

Answer

A. **Critical Path :**

1. The sequence of critical activities in a network is called critical path.

2. The critical path is longest path in the network from the starting event to ending event and defines the minimum time required to complete the project.

3. The critical path is denoted by darker or double lines to distinguish it from the other non-critical paths.

4. The critical path has two principle features :

a. First, if the project has to be shortened some of the activities on that path must be shortened.

b. The variation in actual performance from the expected activity duration time will be completely reflected in one to one fashion in the anticipated completion of the whole project.

B. **Importance of Critical Path in Network Analysis :**

1. Analysis and breakdown the project in terms of specific activities and / or events.

2. Determine the independence and sequence of specific activities and prepare a network.

3. Assign estimates of time, cost or both to all the activities of the network.
4. Identify the longest or critical path through the network.
5. Monitor, evaluate and control the progress of the project by replanning, rescheduling and reassignment of resources.
6. The critical path of a project can change during the course of the project, due to uncertainties in completing the activities as per the original plan. For this purpose the network needs to be updated from time to time from the start of the project till the end of the project.

Que 5.15. **Explain the terms : Total float, free float, independent float, slack, critical event, critical activities.**

Answer

A. Total Float :

1. It may be defined as "the amount of time by which completion of an activity could be delayed beyond the earliest expected completion time without affecting overall project duration time."
2. Total Float, $TF_{ij} = LS_{ij} - ES_{ij} = (L_j - t_{ij}) - E_i$

   $= L_j - E_i - t_{ij}$

   Where, E_i = Earliest expected completion time of tail event.

   = Earliest starting time for an activity (i, j).

   and L_j = Latest allowable completion time of head event.

   = Latest finish time of an activity (i, j).
3. Obviously, the total float of critical activities is always zero.

B. Free Float :

1. This is concerned with commencement of subsequent activity.
2. It may be defined as "the time by which the completion of an activity can be delayed beyond the earliest finish time without affecting the earliest start of a subsequent activity".
3. Using notations given earlier, the free float for activity (i, j) can be expressed as follows :

   FF_{ij} = Earliest event time for subsequent activity j – Earliest event time for activity i – Activity time for (i, j)

   $= (E_j - E_i) - t_{ij}$

   $= E_j - (E_i + t_{ij})$

   $= \text{Min}(ES_{ij}) - EF_{ij}, (i < j).$

C. Independent Float :

1. This is concerned with prior and subsequent activities.
2. It may be defined as "the amount of time by which the start of an activity can be delayed without affecting the earliest start time of any immediately successor activities, assuming that the preceding activity has finished at its latest finish time".
3. Independent float of an activity (i, j) is given by :

   Independent Float $(IF_{ij}) = (E_j - L_i) - t_{ij}$

D. Slack of an Event :

1. The basic difference between slack and float times is that slack is used for events only whereas float is applied for an activity.
2. For any given event, the event slack is defined as the difference between the latest event and earliest event times. For a given activity (i, j) let us define :

   Head slack $(HS) = L_j - E_j$ and

   Tail slack $(TS) = L_i - E_i$
3. We can represent all the floats defined earlier, in terms of head and tail slack as shown below :

   Total float $= L_j - E_i - t_{ij}$

   Free float = Total float – Head slack

   $= L_j - E_i - t_{ij} - (L_j - E_j)$
4. Independent float = Free float – Tail slack

   $= E_j - E_i - t_{ij} - (L_i - E_i)$

E. Critical Event :

1. The slack of an event is the difference between the latest and earliest event times or slack $(i) = L_i - E_i$.
2. The events with zero slack time are known as critical events.

F. Critical Activities :

1. The difference between the least start time and earliest start time of an activity will indicate the amount of time by which the activity can be delayed without affecting the total project duration.
2. This difference is usually called the total float. Activities with zero total float are known as critical activities.

Que 5.16. **Distinguish between PERT and CPM.**

Answer

S.No.	PERT	CPM
1.	A probability model with uncertainty in activity duration. The duration of each activity is normally computed from multiple times estimates with a view to take into account time uncertainty.	A deterministic model with well known activity times based upon the past experience. It therefore does not deal with uncertainty in time.
2.	It is said to be event oriented as the result of analysis are expressed in terms of events of distinct points in time indicative of progress.	It is activity oriented as the result of calculations is considered in terms of activities or operations of the project.
3.	The use of dummy activities is required for representing the proper sequencing.	The use of dummy activities is not necessary.
4.	PERT is generally used for those projects where time required to complete various activities is not known a priori.	CPM is commonly used for those projects that are repetitive in nature and where one has prior experience of handling similar projects.
5.	It is applied for widely planning and scheduling research programs and developing projects.	It is used for construction projects and business problems.
6.	PERT analysis does not usually consider costs.	CPM deals with costs of project schedules and their minimization. The concept of crashing is applied mainly to CPM as a controlling.
7.	PERT is an important control device too, for it assists the management in controlling a project by calling attention as a result of constant review to such delays in activities which might cause a delay in the project's completion date.	It is difficult to use CPM as a controlling device for the simple reason that one must repeat the entire evaluation of the project each time, the changes are introduced into the network.
8.	PERT helps the manager to schedule and coordinate various activities so that the project can be completed on scheduled time.	CPM places dual emphasis on time costs and evaluates the trade-off between projects cost and time. By deploying additional resources, it allows the project manager to manipulate project duration within certain limits so that project duration can be shortened at an optimal cost.

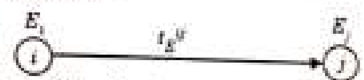
Que 5.17. **Explain the following terms in PERT / CPM.**

i. **Earliest time,**
ii. **Latest time,**
iii. **Total activity time,**
iv. **Event slack, and**
v. **Critical path.**

UPTU 2014-15, Marks 10

Answer

i. **Earliest Time :** It is the time at which an event is accepted to be completed at the earliest.



E_j = Maximum of all $[E_i + t_E^{ij}]$.

Where, E_i = The earliest expected time for event i, and

E_j = The earliest expected time for event j.

ii. **Latest Time :** It is the time at which a particular event must be completed at the latest.

L_i = Minimum of all $[L_j - t_E^{ij}]$.

Where, L_i = Latest allowable time for event i, and

L_j = Latest allowable time for event j.

iii. **Total Activity Time :** It is defined as the time at which the project completed all its activities.

iv. **Event Slack :** It is define as the difference between latest event and earliest event times. Mathematically, for a given activity (i, j),

Head event slack = $L_j - E_j$, Tail event slack = $L_i - E_i$

$$S_i = L_i - E_i$$
$$S_j = L_j - E_j$$

v. **Critical Path :** Refer Q. 5.14, Page, Unit-5.

Que 5.18. **Define PERT and relative terms.**

A. Project Evaluation and Review Technique (PERT) :

1. The main object in the analysis through PERT is to find out the completion for a particular event within specified date. If yes, what are the chances of completing the job ?
2. The PERT approach takes uncertainties into the account.
3. In this approach, three different time are associated with each activity : the optimistic time, the pessimistic time, and the most likely time.
4. These three times provide a measure of uncertainty associated with that activity.

B. Relative Terms :

a. The Optimistic Time :

1. It is the shortest possible time in which the activity can be finished. It assumes that everything goes very well.
2. This is denoted by t_o (or a).

b. The Most Likely Time :

1. This represents the longest and is the estimate of the normal time the activity would take. This assumes normal delays.
2. If a graph is plotted between the time of completion and the frequency of completion in that time period then the most likely time will represent the highest frequency of occurrence. This is denoted by t_m.

c. The Pessimistic Time :

1. This represents the longest time the activity could take if everything goes wrong.
2. As in optimistic estimate this value may be such that only one in hundred or one in twenty will take time longer than this value. This is denoted by t_p.

d. Expected Time :

1. This is the average time an activity will take if it were to be repeated on large number of times and is based on the assumption that the activity time follows beta distribution.
2. This is given by the formula :

$$t_e = (t_o + 4t_m + t_p)/6$$

e. Variance : The variance for the activity is given by the formula :

$$\sigma^2 = [(t_p - t_o)/6]^2$$

Where t_o is the optimistic time, t_p is the pessimistic time, t_m is the most likely time, t_e is the expected time and σ^2 is the variance.

Que 5.19. **Write short note on**
i. Resource levelling.
ii. Lowest cost schedule.

UPTU 2011-12, Marks 10

Answer

A. Resource Levelling :

1. In the process of resource levelling, whenever the availability of resource becomes less than its maximum requirement, the only alternative it to delay the activity having larger float.
2. In case, two or more activities require the same amount of resources, the activity with minimum duration is chosen for resource allocation. Resource levelling is done if the restriction is on the availability of resources.

B. Steps Involved in Resource Levelling :

1. Lower the peak requirement of the resources by staggering the resource-input on non-critical activities. If necessary, sub-critical and critical activities can also be tackled to bring peak demands below the specified levels. Thus, completion of work may be delayed due to resource constraints.
2. Either increase the duration of critical activities or place some of the concurrent activities in series to reduce the peak demands of the scarce resources. This will increase the duration of the project.
3. Rearrange the activities in descending order of the magnitude of the positive float, as resources can be conveniently diverted from the activities which possess large amount of float.

C. Lowest Cost Schedule :

1. From a managerial point of view, it is clear that there is an optimum project duration.
2. The characteristic of the optimum is that it represents an intermediate schedule between one with excessive direct costs for shortening activities without a significant reduction in project duration and one with excessive indirect costs resulting in working many of the critical activities at their normal rates.
3. The time-cost trade of analysis finds the optimum (lest cost) schedule.
4. The general procedure involves defining the cost-time relationships and assigning the activities of the project their normal durations.
5. The corresponding critical path is then computed and the associated costs are recorded.

6. The next step is to consider reducing the duration of the project. Since such a reduction can be effected only if the duration of a critical activity is reduced, attention must be paid to such activities alone.
7. In order to achieve a reduction in the duration at the least possible cost, one must compress as much as possible the critical activity having the smallest cost-time slope.
8. The amount by which an activity can be compressed is limited by its crash time. However, other limits must be taken into account before the exact compression amount can be determined.
9. The result of compressing an activity as a new schedule, perhaps with a new critical path.
10. The cost associated with the new schedule must be greater than the immediately preceding one.
11. The new schedule must now be considered for compression by selecting the (uncrashed) critical activity with the least cost-time slope.
12. The procedue is repeated until all critical activities are at their crash times.
13. The final result of the above calculation is a cost-time curve for the different schedules and their corresponding costs.
14. As the duration of the project increases, the indirect costs also increase. The sum of these two costs (direct + indirect) gives the total costs of the project.
15. The optimum schedule corresponds to the minimum total cost.

Que 5.20. **Describe in detail about crashing of operation.**

Answer

1. In crashing following steps should be followed one by one :

Step 1 : Find critical path and normal project duration.

Step 2 : Find cost slope of each activity.

Step 3 : Select the activity with minimum cost slope and crash it first. While crashing, it should be remembered that it should not be crash beyond limit such that other paths become critical.

Step 4 : Select next higher cost slope with critical activity. While doing so, there may be one or more than one critical path. Then, one activity from each path should be crashed simultaneously by same amount of duration.

Step 5 : Repeat the above steps unless total minimum cost and optimum duration is obtained.

Que 5.21. **Write short note on resource allocation.**

Answer

1. The project manager may face the situation that there is no constraint on project completion time but availability of resources is a constraint.
2. The extension of project completion time should be minimum to control the overhead expenses and also to satisfy the resource constraints.
3. There are many techniques available for handling the problem of resource allocation in project scheduling or fixed resource limit scheduling or resource smoothing. Heuristic (rule of thumb) solution is widely used.
4. Heuristics give a reasonably good solution which may, however, not ensure optimum resource allocation.
5. The essential rules of Heuristic solution are given below. The skill of applying the technique is by practice. Therefore, rules are better understood while solving a problem.
   a. The resources are to be allocated serially in time. Resource allocation should start on the first day. All possible jobs are to be scheduled for the first day before moving to the second day and so on.
   b. When more jobs complete for the same resource, preference is to be given to job/jobs with least float.
   c. Breaking of jobs is not allowed. Jobs once started should continue till they are finished.
   d. Wherever possible, non critical jobs are to be postponed so that critical job can be scheduled without increasing the project duration time.
   e. It must be ensured that resource constraint is not violated at any stage while performing the above exercise.

Que 5.22. **The following table lists the jobs of a network along with their time estimates :**

Jobs	Duration in days		
	Optimistic	Most likely	Pessimistic
01 to 02	3	6	15
02 to 03	6	12	30
02 to 04	5	11	17
03 to 04	3	9	27
03 to 05	1	4	7
05 to 06	2	5	8
03 to 06	4	19	28
04 to 06	2	5	14

i. **Draw the network and calculate the length and variance of critical path.**
ii. **What is the probability that jobs on critical path will be completed by the due date of 40 days ?**
iii. **What is your estimate of the probability that the entire project will be completed by the due date ?** **UPTU 2012-13, Marks 10**

Answer

Activity	t_o	t_m	t_p	$t_e = \frac{t_o + t_p + 4t_m}{6}$	$\sigma^2 = \left(\frac{t_p - t_o}{6}\right)^2$
1 – 2	3	6	15	7	4
2 – 3	6	12	30	14	16
2 – 4	5	11	17	11	4
3 – 4	3	9	27	11	16
3 – 5	1	4	7	4	1
5 – 6	2	5	8	5	1
3 – 6	4	19	28	18	16
4 – 6	2	5	14	6	4

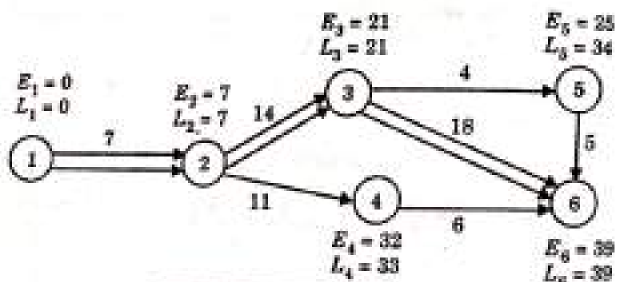


Fig. 5.22.1. Network diagram.

The network is shown in Fig. 5.22.1. The earliest and latest time of each have been computed and indicated on the network. With the help of latest times, the longest path 1 → 2 → 3 → 6 can be traced.
Length of critical path = 7 + 14 + 18 = 39 days.

1. Standard deviation for critical path,

$$\sigma = \sqrt{4 + 16 + 16} = \sqrt{36} = 6 \text{ days}$$

2. The probability to complete the project before due date

$$Z = \frac{\text{Due date - Expected date}}{\sqrt{\sigma^2}}$$

$$= \frac{40 - 39}{\sqrt{36}} = 0.5662 = 56.62 \%$$

3. Standard deviation of whole project

$$\sigma = \sqrt{4 + 16 + 4 + 16 + 1 + 1 + 16 + 4}$$

$$\sigma = \sqrt{62} \text{ days}$$

4. The probability that the entire project will be completed by the due date

$$Z = = \frac{40 - 39}{\sqrt{62}} = 0.127$$

$$= 0.5505 = 55.05\%$$

Que 5.23. **A small project is composed of 7 activities whose time estimates are listed below. Activities are being identified by their beginning (i) and ending (j) node numbers.**

Activity		Time		
i	j	t_o	t_m	t_p
1	2	1	1	7
1	3	1	4	7
1	4	2	2	8
2	5	1	1	1
3	5	2	5	14
4	6	2	5	8
5	6	3	6	15

1. **Draw the network,**
2. **Calculate the expected variances for each,**
3. **Find the expected project completed time,**
4. **Calculate the probability that the project will be completed at least 3 weeks before than expected, and**
5. **If the project due date is 18 weeks, what is the probability of not meeting the due data ?** **UPTU 2013-14, Marks 10**

Answer

Activity		Time			$t_e = \frac{t_o + 4t_m + t_p}{6}$	$\sigma^2 = \left(\frac{t_p - t_o}{6}\right)^2$
i	j	t_o	t_m	t_p		
1	2	1	1	7	2	1
1	3	1	4	7	4	1
1	4	2	2	8	3	1
2	5	1	1	1	1	0
3	5	2	5	14	6	4
4	6	2	5	8	5	1
5	6	3	6	15	7	4

1. Network diagram.

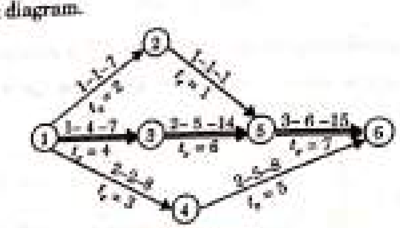


Fig. 5.23.1.

3. Length of the path $1-2-5-6 = 2+1+7 = 10$ weeks.
Length of the path $1-3-5-6 = 4+6+7 = 17$ weeks.
Lenth of path $1-4-6 = 3+5 = 8$ weeks.
Since $1-3-5-6$ has the longest duration, it is the critical path of the network.

∴ Expected project Completion time = 17 weeks.

2. Variance of the project length is the sum of the variance of the activities on the critical path,

$$V_{cp} = V_{1-3} + V_{3-5} + V_{5-6} = 1 + 4 + 4 = 9$$

$$\sigma = \sqrt{V_{cp}} = 3 \text{ weeks}$$

4. Probability that the project will be completed at least 3 weeks before than expected time.

Expected time = 17 weeks
Scheduled time = 17 − 3 = 14 weeks

∴ Standard normal deviate,

$$z = \frac{14-17}{3} = -1$$

For $z = -1$, probability is 0.1586 or 15.86 %, probability of completing the project at least 3 weeks earlier than expected time i.e., within 14 weeks.

5. When the project due date is 18 weeks

$$z = \frac{18-17}{3} = \frac{1}{3} = 0.333$$

for $z = 0.333$, $P = 63.04$ %

Probability of meeting due date is 63.07 % and probability of not meeting due date 36.95 %.

Que 5.24. **The following tasks has to be performed periodically on the heat exchangers in a refinery :** **UPTU 2014-15, Marks 10**

Task	Immediate predecessors	Time (days)
A	–	14
B	*A*	22
C	*B*	10
D	*B*	16
E	*B*	12
F	*C*	10
G	*C*	6
H	*F*, *G*	8
I	*D*, *E*, *H*	24
J	*I*	16

i. **Draw a network diagram of activities for the project.**
ii. **Indentify the critical path. What is its length ?**

Answer

i. Network diagram :

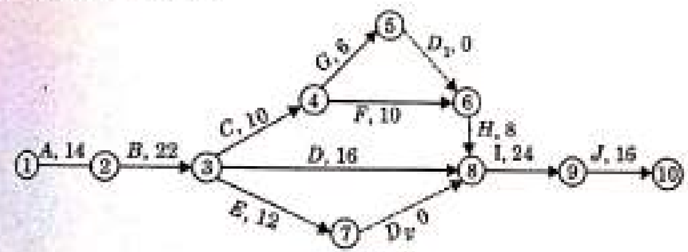


Fig. 5.24.1.

where, D_1 and D_2 are the dummy activities.

ii. Path $(1-2-3-4-5-6-8-9-10)$ duration

$$= 14 + 22 + 10 + 6 + 0 + 8 + 24 + 16$$
$$= 100 \text{ days}$$

Path (1 – 2 – 3 – 4 – 6 – 8 – 9 – 10) duration
= 14 + 22 + 10 + 10 + 8 + 24 + 16
= 104 days

Path (1 – 2 – 3 – 8 – 9 – 10) duration
= 14 + 22 + 16 + 24 + 16
= 92 days

Path (1 – 2 – 3 – 7 – 8 – 9 – 10) duration
= 14 + 22 + 12 + 0 + 24 + 16
= 88 days

Since, path (1 – 2 – 3 – 4 – 6 – 8 – 9 – 10) has maximum duration.
Hence, this is the critical path and duration = 104 days.

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1 UNIT Introduction and Linear Programming (2 Marks Questions)

Memory Based Questions

1.1. What is operations research ?

ANS. It is the application of scientific methods, techniques and tools to problems involving the operations of a systems so as to provide those in control of the operation with optimum solutions to the problems.

1.2. What are the characteristics of operations research ?

ANS. The essential characteristics of operations research are :

i. Its system orientation,
ii. The use of interdisciplinary teams,
iii. Application of scientific method, and
iv. Use of computer.

1.3. What are the different phases of scientific method in operations research ?

ANS. The scientific method in operation research consists of the following three phases :

i. The judgment phase,
ii. The research phase, and
iii. The action phase.

1.4. Why is operations research necessary in industries ?

ANS. Operations research is necessary in industries due to following reasons :

i. Complexity,
ii. Scattered responsibility and authority,
iii. Uncertainty, and
iv. Knowledge explosion.

1.5. Why do computers have a vital role in development of OR ?

ANS. It is because in most OR techniques, computations are so complex that these techniques would be of no real use in the absence of the computers. Most large scale applications of OR techniques which require only a few minutes on the computer, may take weeks, months, even years to yield the same results manually.

1.6. What are the scopes of OR ?

Ans. The scopes of OR are as follows :

i. In defense,
ii. In industry,
iii. In agriculture,
iv. Planning, and
v. Public utilities.

1.7. Define LPP.

Ans. The general LPP calls for optimizing a linear function of variables called the 'objective function' subject to a set of linear equations and / or inequalities called the 'constraints' or 'restrictions'.

1.8. How will you formulate a LPP mathematically ?

Ans. **Step 1 :** Study the given situation to find the key decisions to be made.

Step 2 : Identify the variables involved and designated them by symbols.

$$x_j (j = 1, 2, \ldots\ldots)$$

Step 3 : State the feasible alternatives which generally are : $x_j \geq 0$ for all j.

Step 4 : Identify the constraints in the problem and express them as linear inequalities or equations, LHS of which are linear functions of decision variables.

Step 5 : Identify the objective function and express it as a linear function of the decision variables.

1.9. What are the fundamental conditions of simple method ?

Ans. The basis of the simple method consists of two fundamental conditions :

i. The feasibility condition, and
ii. The optimality condition.

1.10. Define the following terms :

i. **Solution, and**
ii. **Feasible solution.**

Ans.

i. **Solution :** A set of variables $[x_1, x_2, \ldots, x_{n+m}]$ is called a solution to LP problem if it satisfies the constraints.
ii. **Feasible Solution :** A set of variables $[x_1, x_2, \ldots x_{n+m}]$ is called a feasible solution to LP problem if it satisfies the constraints as well as non-negativity restrictions.

1.11. Define basic solution :

Ans. A solution obtained by setting n variables (among $m + n$ variables) equal to zero and solving for remaining m variables is called a basic solution. These m variables are called basic variables and the remaining n variables that have been put equal to zero each are called non-basic variables.

1.12. What is basic feasible solution ?

Ans. It is a basic solution that also satisfies the non-negativity restrictions. All variables in a basic feasible solution are ≥ 0. Every basic feasible solution of a problem is an extreme point of the convex set of feasible solutions and every extreme point is a basic feasible solution of the set of constraints.

1.13. Define the following terms :

i. **Non-degenerate basic feasible solution.**
ii. **Degenerate basic feasible solution.**
iii. **Optimal basic feasible solution.**

Ans.

i. **Non-degenerate Basic Feasible Solution :** It is a basic feasible solution in which all the m basic variables are positive (> 0) and the remaining n variables are zero each.
ii. **Degenerate Basic Feasible Solution :** It is a basic feasible solution in which one or more of the m basic variables are equal to zero.
iii. **Optimal Basic Feasible Solution :** It is the basic feasible solution that also optimizes the objective function.

1.14. Define sensitivity analysis.

Ans. The investigation that deals with changes in the optimal solution due to changes in the parameter (a_{ij}, b_i and c_j) is called sensitivity analysis or post-optimality analysis.

1.15. Define duality.

Ans. Every linear programming problem has associated with its another linear programming problem. The original problem can be considered the primal while the remaining problem it's dual.

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UNIT 2

Transportation and Assignment Problems (2 Marks Questions)

Memory Based Questions

2.1. What is assignment problem ?

Ans. It is a special case of the transportation problem in which the objective is to assign a number of resources to the equal number of activities at a minimum cost.

2.2. Explain the reduced matrix method ?

Ans. **Reduced Matrix Method (Hungarian Method) :** It is based on the concept of opportunity cost. Opportunity cost show the relative penalties associated with assignment of resources to an activity as opposed to making the best or least cost assignment. If we can reduce the cost matrix to the extent of having at least one zero in each row and each column, then it will be possible to make optimal assignment.

2.3. Write the methods of solving assignment problem.

Ans. An assignment problem can be solved using the following four methods :

i. Complete enumeration method,
ii. Transportation method,
iii. Simplex method, and
iv. Hungarian method.

2.4. For a salesman, who has to visit n cities, what are the ways of his tour plan ?

Ans. $(n-1)!$.

2.5. The minimum number of lines covering all zeros in a reduced cost matrix of order n can be ?

Ans. At the most n.

2.6. In an assignment problem involving four workers and three jobs, what is the total no. of assignments possible ?

Ans. 3.

2.7. If there are n workers and n jobs, how many solutions are possible there ?

Ans. $n!$ solutions.

2.8. Why is assignment problem considered as a particular case of transportation problem ?

Ans. An assignment problem is considered as a particular case of transportation problem because :

i. All the rim conditions are 1.
ii. All x_{ij} are either 1 or 0.
iii. The number of rows equals to columns.

2.9. Why is an activity assigned to a resource with zero opportunity cost ?

Ans. While solving an assignment to a resource with zero opportunity cost because the objective is to minimize total cost of assignment.

2.10. How is a maximization assignment problem transformed into a minimization problem ?

Ans. Maximization assignment problem is transformed into a minimization problem by subtracting each element of the profit matrix from the highest element of the matrix.

2.11. What is the purpose of a dummy row or column in an assignment problem ?

Ans. The purpose of a dummy row or column in an assignment problem is to obtain balance between total activities and total resources.

2.12. Define transportation problem.

Ans. The transportation problem is to transport various amounts of a single homogeneous commodity that are initially stored at various origins, to different destinations in such a way that the total transportation cost is minimum.

2.13. Define loop.

Ans. In a transportation table, an ordered set of four or more cells is said to form a loop, if

i. Any two adjacent cells in the ordered set lie either in the same row or in the same column; and
ii. Any three or more adjacent cells in the ordered set do not lie in the same row or in the same column.

2.14. What are the methods of finding an optimal solution ?

Ans. Methods of finding an optimal solution of the transportation problem will consist of two main steps :

i. To find an initial basic feasible solution.
ii. To obtain an optimal solution by making successive improvements to initial basic feasible solution until no further decrease in the transportation cost is possible.

2.15. What are the methods of finding an initial basic feasible solution in transportation problem ?

Ans. There are following three methods :

i. North-west corner method,
ii. Least-cost method, and
iii. Vogel's approximation method (or penalty method).

2.16. What is transhipment problem ?

Ans. A transportation problem in which available commodity frequently moves from one source to another source or destination before reaching its actual destination is called transhipment problem.

2.17. What is the condition of balanced transportation problem ?

Ans. The transportation problem is balanced if total demand equals to total supply irrespective of the number of sources and destinations.

2.18. If the solution of a transportation problem with m-sources and n-destinations is feasible, then what is the number of allocations ?

Ans. $m + n - 1$.

2.19. What does occurrence of degeneracy mean in transportation problem ?

Ans. While solving a transportation problem, the occurrence of degeneracy means the solution so obtained is not feasible.

2.20. Why is dummy source or destination introduced in transportation problem ?

Ans. The dummy source or destination is introduced in transportation problem to satisfy the rim conditions.

2.21. What is the disadvantage of using North-West corner rule ?

Ans. The one disadvantage of using North-West corner rule is that it does not take into account the cost of transportation.

2.22. In order for a transportation matrix which has six rows and four columns not to be degenerate, how much the number of allocated cells must be in the matrix ?

Ans. Number of rows, $m = 6$

Number of columns, $n = 4$

Minimum number of allocated cell $= m + n - 1$

$= 6 + 4 - 1 = 9$

2.23. What is the condition of degeneracy, when there are 'm' rows and 'n' columns in a transportation problem ?

Ans. When the number of allocations is less than $m + n - 1$.

2.24. What is stage ?

Ans. Each point in the problem where a decision must be made for the shortest route problem, a stage may be group of cities with a common property. In the salesman allocation problem each territory represents a stage.

2.25. Define state.

Ans. Information describing the problem at each stage, generally in the form of specific values of state variable. In the shortest route problem, the state at any stage was a specific city. Each state in the salesman allocation problem had one state variable, the number of salesman still available. The state was the specific value of the number. Conceptually, the state variable links together the stages in a multistage decision problem.

2.26. Define optimal policy.

Ans. A policy which optimizes the value of a criterion, objective, or return function. Starting in any given state of any stage, the optimal policy depends only upon that state and not upon how it was reached. In other words, in accordance with the principle of optimality, the optimal decision at any stage is in no way dependent on the previous history of the system.

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3 UNIT Decision and Game Theory and Sequencing (2 Marks Questions)

Memory Based Questions

3.1. What is decision making ?

Ans. A decision, in general, may be defined as the selection by the decision maker of an act, considered to be best according to some pre-designated standard, from among the several available options.

3.2. What are the different types of decisions ?

Ans. In general, decisions can be classified into three categories :

i. Strategic decisions,
ii. Administrative decision, and
iii. Operating decision.

3.3. What are the main components of decision making ?

Ans. The various components of decision making are as follows :

i. The decision maker,
ii. Objectives,
iii. The system or environment,
iv. Alternative courses of action, and
v. Choices of having unequal efficiencies for the desired outcomes.

3.4. Define EMV.

Ans. **EMV (Expected Monetary Value) :** EMV for the specified course of action is the weighted average payoff, *i.e.*, the sum of product of the payoff for the several combination of courses of action and states of nature multiplied by the probability of occurrence of each outcome.

3.5. What is EVPI ?

Ans. **EVPI (Expected Value of Perfect Information) :** It may be defined as the maximum amount one would be willing to pay to obtain perfect information about the state of nature that would occur.

EVPI = EPPI - EMV

3.6. What is EPPI ?

Ans. **EPPI (Expected Profit with Perfect Information) :** It is the maximum attainable expected monetary value (EMV) based on perfect information about the state of nature that will occur. It is also defined as the sum of the product of best state of nature corresponding to each optimal course of action and its probability.

3.7. Define decision tree.

Ans. **Decision Tree :** It is a graphical representation of the decision process indicating decision alternatives, states of nature, probabilities attached to the states of nature and conditional benefits and losses.

3.8. Which criteria is not applicable to decision making under risk ?

Ans. Knowledge of likelihood occurrence of each state of nature is not applicable to decision making under risk.

3.9. What is the minimum expected opportunity loss ?

Ans. The minimum expected opportunity loss (EOL) is equal to EVPI and the minimum regret.

3.10. Which criteria is not used for decision making under uncertainty ?

Ans. Minimize expected loss criterion is not used for decision making under uncertainty.

3.11. What is game ?

Ans. It is defined as an activity between two or more persons involving activities by each person according to set of rules, at the end of which each person receives some benefits or satisfaction or suffers loss.

3.12. Write down the characteristics of game theory.

Ans. There are various types of games that can be classified on the basis of the following characteristics :

i. Chance of strategy,
ii. Number of persons,
iii. Number of activities,
iv. Number of alternatives available to each person, and
v. Payoff.

3.13. What is zero sum game ?

Ans. If the players make payments only to each other *i.e.*, the loss of one is the gain of others, and nothing comes from outside, the competitive game is said to be zero sum game.

3.14. Define strategy.

ANS A strategy for a player is defined as a set of rules or alternative courses of action available to him in advance, by which player decides the course of action that he should adopt.

3.15. What are the types of strategy ?

ANS A strategy may be of two types :

i. Pure strategy, and
ii. Mixed strategy.

3.16. Define minimax criterion and optimal strategy.

ANS The minimax criterion of optimality states that if a player lists the worst possible outcomes of all his potential strategies, he will choose that strategy to be the most suitable for him which corresponds to the best of these worst outcomes. Such a strategy is called an optimal strategy.

3.17. Define saddle point.

ANS A saddle point of a payoff matrix is the position of such an element in the payoff matrix which is minimum in its row and maximum in its column.

3.18. What are the rules for determining a saddle point ?

ANS Rules for determining a saddle point :

i. Select the minimum element of each row of the payoff matrix and mark them by 'O'.
ii. Select the greatest element of each column of the payoff matrix and mark them by '□'.
iii. If there appears an element in the payoff matrix marked by 'O' and '□' both the position of that element is a 'saddle point' of the payoff matrix.

3.19. What happens when maximin and minimax values of game are same ?

ANS Saddle point exists when the maximin and minimax values of game are the same.

3.20. When the sum of gains of one player is equal to the sum of losses to another player, then this situation is known as ?

ANS Zero sum game.

3.21. Which principle is used to reduce the size of the payoff matrix of a game ?

ANS Dominance principle is used to reduce the size of the payoff matrix of a game.

3.22. What happens if the payoff matrix of a game is transposed ?

ANS It the payoff matrix of a game is transposed, the saddle point of the game, if exists, changes.

3.23. When the game is said to be fair ?

ANS A game is said to be fair, if upper and lower values of the game are same and zero.

3.24. Define sequencing.

ANS Suppose there are n jobs (1, 2, 3,...., n), each of which has to be processed one at a time at each of m machines A, B, C,..... The order of processing each job through machines is given. The time that each job must require on each machine is known. The problem is to find a sequence among $(n!)^m$ number of all possible sequences for processing the jobs so that the total elapsed time for all the jobs will be minimum.

3.25. What is shortest processing time ?

ANS **Shortest Processing Time (SPT) :** Sequencing the jobs in a way that with least processing time is picked up first, followed by the one with the next smallest processing time and so on is known as SPT sequencing.

3.26. Define earliest due date rule ?

ANS **Earliest Due Date (EDD) Rule :** According to this rule jobs are sequenced in the order of non-decreasing due dates. This rule minimizes the maximum job lateness as well as maximum job tardiness. However, this rule tends to make more jobs tardy and increases the mean tardiness.

3.27. What is slack time for a job ?

ANS Slack time for a job is defined as the due date of the job minus its processing time.

3.28. What is slack time remaining (STR) rule ?

ANS Sequencing the jobs in such a way that the jobs with the least slack time are picked up first for processing followed by the one with the next smallest slack time and so on is called the slack time remaining rule.

3.29. A sequencing problem involving six jobs and three machines, what is the required evaluation ?

ANS $(6!)^3$ sequences.

3.30. How can we solved a sequencing problem involving processing of two jobs on 'n' machines ?

ANS The sequencing problem involving processing of two jobs on 'n' machines can be solved graphically.

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UNIT 4 Stochastic Inventory Models and Simulations (2 Marks Questions)

Memory Based Questions

4.1. Define inventory.

Ans. It may be defined as the stock of good, commodities or other economic resources that are stored or reserved in order to ensure smooth and efficient running of business affairs.

4.2. Define inventory control.

Ans. The function of directing the movement of goods through the entire manufacturing cycle from the requisitioning of raw materials to the inventory of finished goods orderly mannered to meet the objectives of maximum customer service with minimum investment and efficient plant operation.

4.3. What are the different types of inventories ?

Ans. There are following five types of inventories :
i. Movement inventories,
ii. Buffer inventories,
iii. Anticipation inventories,
iv. Decoupling inventories, and
v. Lot-size inventories.

4.4. What are the causes of poor inventory control ?

Ans. These are the following causes of poor inventory control :
i. Overbuying without regard to the forecast or proper estimate of demand to take advantage of favourable market.
ii. Overproduction or production of goods much before the customer requires them.
iii. Cancellation of orders and minimum quantity stipulations by the suppliers may also give rise to large inventories.

4.5. Why is it essential to carry inventories ?

Ans. The reasons for carrying inventories are as follows :
i. To take quantity discounts on bulk purchases,
ii. To ensure of an adequate supply of items to customers and avoid the shortage as far as possible at the minimum cost, and
iii. To carry buffer stocks in case of delayed deliveries by the suppliers.

4.6. What is the fundamental objective of an inventory control ?

Ans. A fundamental objective of a good inventory system is to determine 'what to order', 'how much to order', 'when to order', and 'how much to carry in stock' so as to gain economy in purchasing, storing, manufacturing and selling.

4.7. What are the different costs associated with inventory control ?

Ans. Various costs associated with inventory control are often classified as follows :
i. Set-up cost,
ii. Ordering cost,
iii. Purchase or production cost,
iv. Carrying or holding cost,
v. Shortage cost,
vi. Salvage cost, and
vii. Revenue cost.

4.8. What are the factors that affecting inventory control ?

Ans. The factors that affecting inventory control are as follows :
i. Demand,
ii. Lead time,
iii. Order cycle,
iv. Time horizon,
v. Re-order level, and
vi. Stock replenishment.

4.9. Define the following terms :
i. Under certainty
ii. Under risk
iii. Under uncertainty

Ans.
i. **Under Certainty :** When the future demand is known exactly, the inventory problem is said to be under certainty.
ii. **Under Risk :** If the probability distribution of future demand (from past records) is known, the inventory problem is said to be under risk.
iii. **Under Uncertainty :** If the levels of future demand are totally unknown, the inventory problem is termed to be under uncertainty.

4.10. Define re-order level.

Ans. **Re-order Level :** The level between maximum and minimum stock, at which purchasing activities must start for replenishment, is known as re-order level.

4.11. What is inventory turnover ?

Ans. **Inventory Turnover :** It is defined as the ratio of the value of materials consumed to the average investment in inventories for the same period.

$$\text{Inventory turnover} = \frac{\text{value of the material consumed}}{\text{value of average inventory}}$$

4.12. Define economic order quantity (EOQ).

Ans. **Economic Order Quantity :** It is that size of order which minimizes the total annual cost of carrying inventory and cost of ordering under the assumed conditions of certainty and that annual demands are known.

4.13. What is *ABC* analysis ?

Ans. ***ABC* Analysis :** It is based on Pareto's law that a few high usage value items constitute a major part of the capital invested in inventories, whereas bulk of items having low usage value constitute insignificant part of the capital.

4.14. A firm uses every year 12000 units of a raw material costing Rs. 1.25 per unit. Ordering cost is Rs. 15 per order and holding cost is 5 % per year of average inventory. Calculate economic order quantity (EOQ).

Ans.

$$EOQ = \sqrt{\frac{2C_3R}{C_1}} = \sqrt{\frac{2 \times 15 \times 12000}{\frac{5}{100} \times 1.25}}$$

$$= 2400 \text{ units.}$$

4.15. A particular item has a demand of 9,000 units / year. The cost of one procurement is Rs. 100 and the holding cost per unit is Rs. 2.40 per year. Determine the number of order per year and the time between orders.

Ans. Number of order per year,

$$n_o = \sqrt{\frac{C_1R}{2C_3}} = \sqrt{\frac{2.4 \times 9000}{2 \times 100}} = 10.4 \text{ order / year}$$

Time between orders, $t_o = \frac{1}{n_o} = \frac{1}{10.4} = 0.0962$ years

$= 1.15$ months between procurement.

4.16. Determine the total cost per year if the cost of one unit is Rs. 1 for an item has a demand of 11,000 per year. The cost of one procurement is Rs. 95 and the holding cost per unit is Rs. 3 per year. The replacement is instantaneous and shortages are allowed.

Ans.

$$C_0 = 11{,}000 + \sqrt{2C_1C_3R}$$

$$= 11{,}000 + \sqrt{2 \times 3 \times 95 \times 11000}$$

$$= 13{,}503.99 \text{ / year}$$

4.17. Define simulation.

Ans. It is a representation of reality through the use of a model or other device which will react in the same manner as reality under a given set of conditions.

4.18. What are the types of simulation ?

Ans. Simulation is mainly of two types :

i. Analogue simulation, and
ii. Computer simulation.

4.19. Give the classification of simulation model ?

Ans. The simulation models can be classified into following four categories :

i. Deterministic models,
ii. Stochastic model,
iii. Static model, and
iv. Dynamic model.

4.20. What are the different phases of simulation model ?

Ans. A simulation model mainly consists of two basic phases :

i. Data generation, and
ii. Book-keeping.

4.21. Define the following terms :

I. Random variable
II. Random variate

Ans.

I. **Random Variable :** It is a real valued function defined over a sample space associated with the outcome of a conceptual chance experiment.

II. **Random Variate :** It refers to a particular outcome of an experiment, *i.e.*, a numerical or sample value of a random variable.

4.22. Define the random number.

Ans. It refers to a uniform random variable or a numerical value assigned to a random variable following uniform probability density function. In other words, it is a sequence of numbers whose probability of occurrence is the same as that of any other number in that sequence.

4.23. What are pseudo-random numbers ?

Ans. Random numbers are called pseudo-random numbers when they are generated by some deterministic process but they qualify the pre-determined statistical test for randomness.

4.24. Where is the Monte-Carlo simulation used ?

Ans. Monte-Carlo simulation is generally used to solve problems which cannot be adequately represented by the mathematical models or where solution of the model is not possible by analytical method.

4.25. What are the applications of simulation models ?

Ans. The applications of simulation models include the following :

i. Financial studies involving risky investments.
ii. Military studies of logistics, support planning and weapon system effectiveness.
iii. Testing of decision rules for hospital admission and operating policies.
iv. Studies of individual and group behaviour.

4.26. Give the advantages of simulation.

Ans. Following are the advantages of simulation :

i. Simulation methods are easier to apply than pure analytical methods.
ii. The knowledge of a system obtained in designing and conducting the simulation.
iii. It enables us to assess the possible risks involved in a new policy before actually implementing it.

4.27. Give any three limitations of simulation.

Ans. Following are the limitations of simulation :

i. Simulation generates a way of evaluating solutions but it does not generate the solution techniques.
ii. Not all situations can be evaluated using simulation. Only simulations involving uncertainty are considered.
iii. Simulation is a time-consuming exercise.

4.28. Why is large complicated simulation models appreciated ?

Ans. Large complicated simulation models are appreciated, because their average costs are not well defined.

4.29. For which type of problem biased random sampling is used ?

Ans. Biased random sampling is used for which has unequal probability.

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5 UNIT

Queuing Models and Project Management (2 Marks Questions)

Memory Based Questions

5.1. Why does queuing problem arise ?

Ans. Queuing problems arise because :

i. There is too much demand on the facilities so that we say there is an excess of waiting time or inadequate number of service facilities.
ii. There is too less demand, in which case there is too much idle facility time or too many facilities.

5.2. Describe the queuing system.

Ans. A queuing system can be completely described by :

a. The input,
b. The service mechanism,
c. The queue discipline, and
d. Customer's behaviour.

5.3. Define the queue discipline.

Ans. The queue discipline is the rule of determining the formation of the queue, the manner of the customer's behaviour while waiting, and the manner in which they are chosen for service. The simplest discipline is "first come, first served".

5.4. What are the different behaviour of customers ?

Ans. The customers generally behave in four ways :

i. Balking,
ii. Reneging,
iii. Priorities, and
iv. Jockeying.

5.5. Define the following terms :

i. **Transient state**
ii. **Steady state**

Ans. i. **Transient State :** A system is said to be in "transient state" when its operating characteristics are dependent on time.

ii. **Steady State :** A steady state condition is said to prevail when the behaviour of the system becomes independent of time.

5.6. Give the operating characteristics of a queuing system.

Ans. Some of the operational characteristics of a queuing system are as follows :

i. Expected number of customers in the system.
ii. Expected number of customers in the queue.
iii. Expected waiting time in the system.
iv. Expected waiting time in queue.
v. The server utilization factor.

5.7. Define the following factors :
i. Queue length
ii. System length

Ans. i. **Queue Length :** The average number of customers in the queue waiting to get service. This excludes the customers being served.

ii. **System Length :** The average number of customers in the system including those waiting as well as those being served.

5.8. Define the following factors :
i. Waiting time in the queue (W_q)
ii. Total time in the system

Ans. i. **Waiting Time in the Queue :** The average time for which a customer has to wait in the queue to get service. It is denoted by W_q.

ii. **Total Time in the System :** The average total time spent by a customer in the system from the moment he arrives till he leaves the system. It is taken to be the waiting time plus the service time.

5.9. Define traffic intensity.

Ans. The ratio $\frac{\lambda}{\mu}$ is called the traffic intensity or the utilisation factor and it determines the degree to which the capacity of the service station is utilised.

5.10. What are the limitations of queuing model ?

Ans. Following are the limitations of queuing model :

i. The waiting space for the customers is usually limited.
ii. The arrival rate may be state dependent.
iii. The population of customers may not be infinite and the queuing discipline may not be first come, first served.

5.11. In which situation the queuing theory is best suited ?

Ans. Queuing theory is applied best in the situations where there is only one channel of arrival at random and the service time is constant.

5.12. If average arrival rate in a queue is 6 per hr. and the average service rate is 10 per hr, what is the average number of customers in the line, including the customer being served ?

Ans. $\lambda = 6, \mu = 10$

$$\text{Average number of customers} = \frac{\lambda}{(\mu - \lambda)}$$

$$= \frac{6}{(10-6)} = 1.5.$$

5.13. Give the applications of queuing models.

Ans.
1. Scheduling of mechanical transport fleets.
2. Scheduling distribution of scarce war material.
3. Scheduling of jobs in production control.
4. Minimizing of congestion due to traffic delay at tool booths.
5. Solution of inventory control problems.

5.14. Define network.

Ans. It consists of a set of points and a set of lines connecting different pairs of points. The points are called nodes and the lines are called links.

5.15. Define the following terms :
i. Path
ii. Tree
iii. Link capacity

Ans. i. **Path :** It is a sequence of distinct links that join nodes through some other nodes.
Example : $1 \to 2 \to 3 \to 5$, and $1 \to 4 \to 3 \to 5$.

ii. **Tree :** It is a connected network that may involve only a subset of all the nodes of the network, without having any cycle in between.

iii. **Link Capacity :** The maximum amount of flow that can pass through a directed link is called a link capacity.

5.16. Define activity.

Ans. Any individual operation, which utilises resources and has an end and a beginning, is called activity.

5.17. Give the classification of activity ?

Ans. Activities are classified into following four categories :

i. Predecessor activity.
ii. Successor activity.

iii. Concurrent activity, and
iv. Dummy activity.

5.18. Define network scheduling.

Ans. It is a technique used for planning and scheduling large projects in the fields of construction, maintenance, fabrication, purchasing, computer system installation, research and development designs, etc.

5.19. What are the types of errors observed in drawing network diagrams ?

Ans. There are three types of errors most commonly observed in drawing network diagrams :

i. Dangling,
ii. Looping, and
iii. Redundancy.

5.20. Define event float.

Ans. The float of an event is the difference between its latest time (L_i) and its earliest time (E_i),

i.e.,

Event float = $L_i - E_i$.

5.21. Define interfering float.

Ans. It can be defined as the difference between the latest finish time of the activity under consideration and the earliest start time of the following activity, or zero, whichever is larger.

5.22. Define critical path.

Ans. The sequence of critical activities in a network is called the critical path. The critical path is the longest path in the network from the starting event to ending event and defines the minimum time required to complete the project.

5.23. What are the main features of critical path ?

Ans. The critical path has two main features :

i. If the project has to be shortened, the some of the activities on that path must also be shortened. The application of additional resources on other activities will not give the desired result unless that critical path is shortened first.
ii. The variation in actual performance from the expected activity duration time will be completely reflected in one to one fashion in the anticipated completion of the whole project.

5.24. What do you mean by project evaluation and review technique (PERT) ?

Ans. It is the basic network technique which includes planning,

monitoring and control of projects. It is applied in planning and control of complex set of tasks, function and relationships.

5.25. What is the objective of PERT ?

Ans. The main objective in the analysis through PERT is to find out the completion for a particular event within specified date.

5.26. What is optimistic time ?

Ans. It is the shortest possible time in which the activity can be finished. It assumes that everything goes very well. It is denoted by t_o.

5.27. What do you mean by most likely time ?

Ans. It represents the time that would be expected to occur most often if the activities were frequently repeated under exactly the same conditions.

5.28. Define pessimistic time.

Ans. It represents the longest time, the activity could take to finish. It is the worst time estimate and denotes that time which activity would take if bad luck was faced.

5.29. What are the requirements for application of PERT ?

Ans. **Requirements for Application of PERT :**

i. PERT is applicable for project management which is basically new or of number repetitive nature.
ii. PERT is usually applicable to very large, complex, customized projects that consist of many inter-related activities to be performed either concurrently or sequentially.

5.30. In critical path analysis, what is the meaning of CPM ?

Ans. Critical path method.

5.31. The three exist estimates of a PERT activity are : optimistic time = 8 min, most likely time = 10 min and pessimistic time = 14 min. What would be the expected time of the activity ?

Ans. Expected time = $\dfrac{t_0 + 4t_m + t_p}{6} = \dfrac{8 + 40 + 14}{6} = 10.33$ min

5.32. In a network, what is total float equals to ?

Ans. $LFT_j = EST_i - T_{i-j}$

5.33. What does it imply, if an activity has zero slack ?

Ans. If an activity has zero slack, it implies that it lies on the critical path.

5.34. What is the objective of network analysis ?

Ans. The objective of network analysis is to minimize total project duration.

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B. Tech.
(SEM. VIII) EVEN SEMESTER THEORY EXAMINATION, 2011-12
OPERATIONS RESEARCH

Time : 3 Hours **Total Marks : 100**

Note :
1. Attempt all questions.
2. Be precise in your answers.
3. Assume suitable value of missing data if any.
4. Use of standard normal distribution table is permitted.

1. Attempt any **two** parts :

a. i. "The hard problems are those for which models do not exist." Interpret this statement. Give some examples. (5)

Ans. Refer Q. 1.9, Page 11B, Unit-1.

ii. Write the dual to the following primal LP problem :

Maximize $Z = 20x_1 + 17x_2 + 18x_3 + 12x_4$ subject to :

$4x_1 - 3x_2 + 8x_3 + 3x_4 \le 60$

$x_1 + x_2 + x_3 = 25$

$-x_2 + 4x_3 + 7x_4 \ge 35$

and x_4 is unrestricted in sign. (5)

Ans. Refer Q. 1.30, Page 34B, Unit-1.

b. i. Discuss the role of sensitivity analysis in linear programming. (4)

Ans. Refer Q. 1.35, Page 40B, Unit-1.

ii. Solve graphically the following LP problem.

Maximize : $Z = 9x_1 + 3x_2$

Subject to :

$2x_1 + 3x_2 \le 13$

$2x_1 + x_2 \le 5$

$x_1, x_2 \ge 0$ (6)

Ans. Refer Q. 1.18, Page 20B, Unit-1.

c. Solve :

Max. $Z = x_1 + x_2 + x_3$

Subject to :

$4x_1 + 5x_2 + 3x_3 \le 15$

$10x_1 + 7x_2 + x_3 \le 12$

and $x_1, x_2, x_3 \ge 0$ (10)