Step by step notes.
We are going to solve
$\Omega_R \coloneqq (-R,R)\times\Sigma$

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \Omega_R \setminus \overline{\mathcal{D}} \\ \Delta u + \varepsilon_{\mathrm{r}} k^2 u = 0 & \text{in } \mathcal{D} \\ u^+ = u^- & \text{on } \partial\mathcal{D} \\ \nabla u^+ \cdot \mathbf{n} = \nabla u^- \cdot \mathbf{n} & \text{on } \partial\mathcal{D} \\ \nabla u \cdot \mathbf{n} = 0 & \text{on } (-R,R)\times\partial\Sigma \\ u - u^{\mathrm{i}} = \mathcal{N}^- \left(\left(\nabla u - \nabla u^i \right) \cdot \mathbf{n} \right) & \text{on } \{-R\}\times\Sigma \\ u - u^{\mathrm{i}} = \mathcal{N}^+ \left(\left(\nabla u - \nabla u^i \right) \cdot \mathbf{n} \right) & \text{on } \{R\}\times\Sigma \end{cases}$$

where $\mathcal{N}^+$ and $\mathcal{N}^-$ are two Neumann to Dirichlet operators defined on the right semiinfinite and left semiinfinite domains. That is $\mathcal{N}^+(f) = w|_{x_1=R}$ where w_f solves

$$\begin{cases} \Delta w + k^2 w = 0 & \text{in } (R,\infty)\times\Sigma \\ \nabla w \cdot \mathbf{n} = 0 & \text{on } (R,\infty)\times\partial\Sigma \\ \nabla w \cdot \mathbf{n} = f & \text{on } \{R\}\times\Sigma \\ w \text{ ratiates to the right} & \text{as } x_1 \to \infty \end{cases}$$

The solutions to BLA can be computed in a close form. They can be expressed as

$$w(x_1, \hat{\mathbf{x}}) = \sum_{n=0}^{\infty} w_n e^{i\beta_n x_1} \theta_n(\hat{\mathbf{x}})$$

where $\{\theta_n\}$ are orthonormal eigenfunctions of the laplace operator in Σ associated to eigenvalues k_n and $\beta_n = \sqrt{k - k_n}$ such that $\Im(\beta_n) \geq 0$. As can be seen all the eigenfunctions are either planewaves radiating towards the right or exponentially decaing towards the right.

Taking into account that $\{\theta_n\}$ form an orthonormal basis for $H^1(\Sigma)$ we have that

$$f = \sum_{n=0}^{\infty} f_n \theta_n$$

with

$$f_n = \int_\Sigma f(\hat{\mathbf{x}})\, \theta_n(\hat{\mathbf{x}})\, \mathrm{d}\hat{\mathbf{x}}$$

so, imposing the boundary condition

$$\nabla w \cdot \mathbf{n}|_{x_1=R} = \sum_{n=0}^{\infty} i\beta_n w_n e^{i\beta_n R} \theta_n(\hat{\mathbf{x}}) = \sum_{n=0}^{\infty} \int_\Sigma f(\hat{\mathbf{z}})\, \theta_n(\hat{\mathbf{z}})\, \mathrm{d}\hat{\mathbf{z}} \theta_n(\hat{\mathbf{x}})$$

that is

$$w_n = \frac{e^{-i\beta_n R}}{i\beta_n} \int_\Sigma f(\hat{\mathbf{z}})\, \theta_n(\hat{\mathbf{z}})\, \mathrm{d}\hat{\mathbf{z}}$$

so

$$w(R, \hat{\mathbf{x}}) = \sum_{n=0}^{\infty} \frac{1}{i\beta_n} \int_\Sigma f(\hat{\mathbf{z}})\, \theta_n(\hat{\mathbf{z}})\, \mathrm{d}\hat{\mathbf{z}} \theta_n(\hat{\mathbf{x}})$$

that is

$$\mathcal{N}^+(f) = \sum_{n=0}^{\infty} \frac{1}{i\beta_n} \int_\Sigma f(\hat{\mathbf{z}})\, \theta_n(\hat{\mathbf{z}})\, \mathrm{d}\hat{\mathbf{z}} \theta_n(\hat{\mathbf{x}})$$

and for the numerical implementation:

$$\mathcal{N}_N^+(f) = \sum_{n=0}^{N} \frac{1}{i\beta_n} \int_\Sigma f(\hat{\mathbf{z}})\, \theta_n(\hat{\mathbf{z}})\, \mathrm{d}\hat{\mathbf{z}} \theta_n(\hat{\mathbf{x}})$$

We define $\mathcal{N}^-$ in a similar manner, that is $\mathcal{N}^-(f) = w|_{x_1=-R}$ where w_f solves

$$\begin{cases} \Delta w + k^2 w = 0 & \text{in } (-\infty, -R) \times \Sigma \\ \nabla w \cdot \mathbf{n} = 0 & \text{on } (-\infty, -R) \times \partial\Sigma \\ \nabla w \cdot \mathbf{n} = f & \text{on } \{-R\} \times \Sigma \\ w \text{ ratiates to the left} & \text{as } x_1 \to -\infty \end{cases}$$

The solutions to BLA can be computed in a close form. They can be expressed as

$$w(x_1, \hat{\mathbf{x}}) = \sum_{n=0}^{\infty} w_n e^{-i\beta_n x_1} \theta_n(\hat{\mathbf{x}})$$

where $\{\theta_n\}$ are orthonormal eigenfunctions of the laplace operator in Σ associated to eigenvalues k_n and $\beta_n = \sqrt{k - k_n}$ such that $\Im(\beta_n) \geq 0$. As can be seen all the eigenfunctions are either planewaves radiating towards the left or exponentially decaing towards the left.

Taking into account that $\{\theta_n\}$ form an orthonormal basis for $H^1(\Sigma)$ we have that

$$f = \sum_{n=0}^{\infty} f_n \theta_n$$

with

$$f_n = \int_\Sigma f(\hat{\mathbf{x}})\, \theta_n(\hat{\mathbf{x}})\, \mathrm{d}\hat{\mathbf{x}}$$

so, imposing the boundary condition

$$\nabla w \cdot \mathbf{n}|_{x_1=-R} = \sum_{n=0}^{\infty} i\beta_n w_n e^{i\beta_n R}\theta_n(\hat{\mathbf{x}}) = \sum_{n=0}^{\infty} \int_{\Sigma} f(\hat{\mathbf{z}})\,\theta_n(\hat{\mathbf{z}})\,\mathrm{d}\hat{\mathbf{z}}\theta_n(\hat{\mathbf{x}})$$

that is

$$w_n = \frac{1}{i\beta_n} e^{-i\beta_n R} \int_{\Sigma} f(\hat{\mathbf{z}})\,\theta_n(\hat{\mathbf{z}})\,\mathrm{d}\hat{\mathbf{z}}$$

so

$$w(-R, \hat{\mathbf{x}}) = \sum_{n=0}^{\infty} \frac{1}{i\beta_n} \int_{\Sigma} f(\hat{\mathbf{z}})\,\theta_n(\hat{\mathbf{z}})\,\mathrm{d}\hat{\mathbf{z}}\theta_n(\hat{\mathbf{x}})$$

that is

$$\mathcal{N}^-(f) = \sum_{n=0}^{\infty} \frac{1}{i\beta_n} \int_{\Sigma} f(\hat{\mathbf{z}})\,\theta_n(\hat{\mathbf{z}})\,\mathrm{d}\hat{\mathbf{z}}\theta_n(\hat{\mathbf{x}})$$

and for the numerical implementation:

$$\mathcal{N}_N^+(f) = \sum_{n=0}^{N} \frac{1}{i\beta_n} \int_{\Sigma} f(\hat{\mathbf{z}})\,\theta_n(\hat{\mathbf{z}})\,\mathrm{d}\hat{\mathbf{z}}\theta_n(\hat{\mathbf{x}})$$

Funny enought we have that $\mathcal{N}^+ = \mathcal{N}^- = \mathcal{N}$. And so the formulation of the problem is

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \Omega_R \setminus \overline{\mathcal{D}} \\ \Delta u + \varepsilon_{\mathrm{r}} k^2 u = 0 & \text{in } \mathcal{D} \\ u^+ = u^- & \text{on } \partial\mathcal{D} \\ \nabla u^+ \cdot \mathbf{n} = \nabla u^- \cdot \mathbf{n} & \text{on } \partial\mathcal{D} \\ \nabla u \cdot \mathbf{n} = 0 & \text{on } (-R, R) \times \partial\Sigma \\ u = u^{\mathrm{i}} + \mathcal{N}\left(\left(\nabla u - \nabla u^i\right) \cdot \mathbf{n}\right) & \text{on } \{-R\} \times \Sigma \\ u = u^{\mathrm{i}} + \mathcal{N}\left(\left(\nabla u - \nabla u^i\right) \cdot \mathbf{n}\right) & \text{on } \{R\} \times \Sigma \end{cases}$$

We define a conforming triangulation as one such that no triangle belongs both to $\mathcal{D}$ and $\Omega_R \setminus \overline{\mathcal{D}}$.

For a triangle K, we define the local Trefftz space

$$V(K) = \left\{ v \in H^1(K) \,:\, \Delta v + k^2 v = 0 \right\} \quad \text{if } K \subset \Omega \setminus \overline{\mathcal{D}}$$

$$V(K) = \left\{ v \in H^1(K) \,:\, \Delta v + \varepsilon_{\mathrm{r}} k^2 v = 0 \right\} \quad \text{if } K \subset \mathcal{D}$$

We have that, for $K \subset \Omega \setminus \overline{\mathcal{D}}$

$$\int_K \left(\Delta u + k^2 u\right) \overline{v}\,\mathrm{d}x = 0$$

$$\int_K \Delta u \overline{v} \, \mathrm{d}x + \int_K k^2 u \overline{v} \, \mathrm{d}x = 0$$

and applying the divergence theorem twice:

$$\int_K \operatorname{div}\left(\nabla u \overline{v}\right) \, \mathrm{d}x - \int_K \nabla u \cdot \nabla \overline{v} \, \mathrm{d}x + \int_K k^2 u \overline{v} \, \mathrm{d}x = 0$$

$$\int_K \operatorname{div}\left(\nabla u \overline{v}\right) \, \mathrm{d}x - \int_K \operatorname{div}\left(u \nabla \overline{v}\right) \, \mathrm{d}x + \int_K u \Delta \overline{v} \, \mathrm{d}x + \int_K k^2 u \overline{v} \, \mathrm{d}x = 0$$

that is

$$\int_{\partial K} \left(\nabla u \overline{v} - u \overline{\nabla v}\right) \cdot \mathbf{n} \, \mathrm{d}S_x + \int_K u \overline{\left(\Delta v + k^2 v\right)} \, \mathrm{d}x = 0$$

where $\mathbf{n}$ is the outward facing normal. But recalling that $v \in V(K)$ we have that

$$\int_{\partial K} \left(\nabla u \cdot \mathbf{n} \overline{v} - u \overline{\nabla v} \cdot \mathbf{n}\right) \, \mathrm{d}S_x = 0$$

which is also true for $K \subset \mathcal{D}$.

We define the global Trefftz space with respect to the triangulation $\mathcal{T}$as

$$V(\mathcal{T}) = \bigtimes_{K \in \mathcal{T}} V_K$$

or

$$V(\mathcal{T}) = \left\{ v \in L^2(\Omega_R) \, : \, v|_K \in V(K) \right\}$$

functions in $V(\mathcal{T})$ do not need to be continuous along the triangle sides. Indeed, if we define the broken Sobolev space:

$$H^1(\mathcal{T}) = \left\{ v \in L^2(\Omega_R) \, : \, v|_K \in H^1(K) \right\}$$

we have that:

$$H^1(\Omega_R) \hookrightarrow V(\mathcal{T}) \hookrightarrow H^1(\mathcal{T})$$

At an inner edge E with normal $\mathbf{n}$ we define

$$w^+(\mathbf{x}) = \lim_{\epsilon \to 0^+} w(\mathbf{x} - \epsilon \mathbf{n}), \quad w^-(\mathbf{x}) = \lim_{\epsilon \to 0^+} w(\mathbf{x} + \epsilon \mathbf{n})$$

and the normal flux and average:

$$[[w]]_{\mathbf{n}} = \left(w^+ - w^-\right) \mathbf{n}, \quad \{\{w\}\} = \frac{w^+ + w^-}{2}$$

and remind that those quantities are independent of the choice of $\mathbf{n}$. For a vector quantity $\boldsymbol{\tau}$ we have

$$[[\boldsymbol{\tau}]]_{\mathbf{n}} = \left(\boldsymbol{\tau}^+ - \boldsymbol{\tau}^-\right) \cdot \mathbf{n}, \quad \{\{\boldsymbol{\tau}\}\} = \frac{\boldsymbol{\tau}^+ + \boldsymbol{\tau}^-}{2}$$

and the inverses:

$$w^+ = \{\{w\}\} + \frac{1}{2}[[w]]_{\mathbf{n}} \cdot \mathbf{n}, \quad w^- = \{\{w\}\} - \frac{1}{2}[[w]]_{\mathbf{n}} \cdot \mathbf{n}$$

and

$$\boldsymbol{\tau}^+ \cdot \mathbf{n} = \{\{\boldsymbol{\tau}\}\} \cdot \mathbf{n} + \frac{1}{2}[[\boldsymbol{\tau}]]_{\mathbf{n}}, \quad \boldsymbol{\tau}^- \cdot \mathbf{n} = \{\{\boldsymbol{\tau}\}\} \cdot \mathbf{n} - \frac{1}{2}[[\boldsymbol{\tau}]]_{\mathbf{n}}$$

So (Magic DG formula):

$$\int_E w^+ \boldsymbol{\tau}^+ \cdot \mathbf{n}\, \mathrm{d}S_x - \int_E w^- \boldsymbol{\tau}^- \cdot \mathbf{n}\, \mathrm{d}S_x =$$

$$= \int_E \{\{w\}\}[[\boldsymbol{\tau}]]_{\mathbf{n}}\, \mathrm{d}S_x + \int_E [[w]]_{\mathbf{n}} \cdot \mathbf{n}\{\{\boldsymbol{\tau}\}\} \cdot \mathbf{n}\, \mathrm{d}S_x =$$

$$= \int_E \{\{w\}\}[[\boldsymbol{\tau}]]_{\mathbf{n}}\, \mathrm{d}S_x + \int_E [[w]]_{\mathbf{n}} \cdot \{\{\boldsymbol{\tau}\}\}\, \mathrm{d}S_x$$

$$\int_E \left(\{\{w\}\}[[\boldsymbol{\tau}]]_{\mathbf{n}} + [[w]]_{\mathbf{n}} \cdot \{\{\boldsymbol{\tau}\}\}\right) \mathrm{d}S_x$$

and adding BLA for all triangles in $\mathcal{T}$ we have:

$$\sum_{K \in \mathcal{T}} \int_{\partial K} \left(\nabla u \cdot \mathbf{n}\overline{v} - u\overline{\nabla v} \cdot \mathbf{n}\right) \mathrm{d}S_x = 0$$

$$\sum_{E \in \mathcal{E}_I} \int_E \left([[\nabla u]]_{\mathbf{n}} \overline{\{\{v\}\}} + \{\{\nabla u\}\} \cdot \overline{[[v]]_{\mathbf{n}}}\right) \mathrm{d}S_x$$

$$- \sum_{E \in \mathcal{E}_I} \int_E \left(\{\{u\}\} \overline{[[\nabla v]]_{\mathbf{n}}} + [[u]]_{\mathbf{n}} \cdot \overline{\{\{\nabla v\}\}}\right) \mathrm{d}S_x$$

$$+ \sum_{E \in \mathcal{E} \setminus \mathcal{E}_I} \int_E \left(\nabla u \cdot \mathbf{n}\overline{v} - u\overline{\nabla v} \cdot \mathbf{n}\right) \mathrm{d}S_x = 0$$

OR IF WE MULTIPLY EVERYTHING BY -1:

$$\sum_{K \in \mathcal{T}} \int_{\partial K} \left(-\nabla u \cdot \mathbf{n}\overline{v} + u\overline{\nabla v} \cdot \mathbf{n}\right) \mathrm{d}S_x = 0$$

$$\sum_{E \in \mathcal{E}_I} \int_E \left(\{\{u\}\} \overline{[[\nabla v]]_{\mathbf{n}}} + [[u]]_{\mathbf{n}} \cdot \overline{\{\{\nabla v\}\}}\right) \mathrm{d}S_x$$

$$- \sum_{E \in \mathcal{E}_I} \int_E \left([[\nabla u]]_{\mathbf{n}} \overline{\{\{v\}\}} + \{\{\nabla u\}\} \cdot \overline{[[v]]_{\mathbf{n}}}\right) \mathrm{d}S_x$$

$$+\sum_{E\in\mathcal{E}\setminus\mathcal{E}_I}\int_E\left(-\nabla u\cdot\mathbf{n}\overline{v}+u\overline{\nabla v}\cdot\mathbf{n}\right)\,\mathrm{d}S_x=0$$

If we sustitute the exact solutions u and ∇u we get that

$$\sum_{E\in\mathcal{E}_I}\int_E\left(\nabla u\cdot\overline{[[v]]_{\mathbf{n}}}\right)\,\mathrm{d}S_x-\sum_{E\in\mathcal{E}_I}\int_E\left(u\overline{[[\nabla v]]_{\mathbf{n}}}\right)\,\mathrm{d}S_x$$

$$+\sum_{E\in\mathcal{E}\setminus\mathcal{E}_I}\int_E\left(\nabla u\cdot\mathbf{n}\overline{v}-u\overline{\nabla v}\cdot\mathbf{n}\right)\,\mathrm{d}S_x=0\quad\forall v\in V\left(\mathcal{T}\right)$$

OR .... -1:

$$\sum_{E\in\mathcal{E}_I}\int_E\left(u\overline{[[\nabla v]]_{\mathbf{n}}}\right)\,\mathrm{d}S_x-\sum_{E\in\mathcal{E}_I}\int_E\left(\nabla u\cdot\overline{[[v]]_{\mathbf{n}}}\right)\,\mathrm{d}S_x$$

$$+\sum_{E\in\mathcal{E}\setminus\mathcal{E}_I}\int_E\left(u\overline{\nabla v}\cdot\mathbf{n}-\nabla u\cdot\mathbf{n}\overline{v}\right)\,\mathrm{d}S_x=0\quad\forall v\in V\left(\mathcal{T}\right)$$

why do we say that? No clue.s

However, for the numerical approximation of $V\left(\mathcal{T}\right)$, $V^{\mathbf{p}}\left(\mathcal{T}\right)$ we won't be able to have continuity accross boundaries, hence, in general we don't have a unique definition of u at E.

Geting rid of this "nonsense":

$$\sum_{K\in\mathcal{T}}\int_{\partial K}\left(u\overline{\nabla v}\cdot\mathbf{n}-\nabla u\cdot\mathbf{n}\overline{v}\right)\,\mathrm{d}S_x=0$$

$$\sum_{K\in\mathcal{T}}\sum_{E\in K}\int_E\left(u\overline{\nabla v}\cdot\mathbf{n}_{E,K}-\nabla u\cdot\mathbf{n}_{E,K}\overline{v}\right)\,\mathrm{d}S_x=0$$

If the edge belongs to Γ_R then the integral takes the special form:

$$\int_E u\overline{\nabla v}\cdot\mathbf{n}_{E,K}\,\mathrm{d}S_x=0$$

and if it belongs to Σ_R or Σ_{-R} it is:

$$\int_E\left(\left(u^{\mathrm{i}}+\mathcal{N}\left(\left(\nabla u-\nabla u^{i}\right)\cdot\mathbf{n}\right)\right)\overline{\nabla v}\cdot\mathbf{n}_{E,K}-\nabla u\cdot\mathbf{n}_{E,K}\overline{v}\right)\,\mathrm{d}S_x=0$$

which can be written as

$$\int_E\left(\left(\mathcal{N}\left(\nabla u\cdot\mathbf{n}\right)\right)\overline{\nabla v}\cdot\mathbf{n}_{E,K}-\nabla u\cdot\mathbf{n}_{E,K}\overline{v}\right)\,\mathrm{d}S_x=\int_E\left(\left(\mathcal{N}\left(\nabla u^{i}\cdot\mathbf{n}\right)-u^{\mathrm{i}}\right)\overline{\nabla v}\cdot\mathbf{n}_{E,K}\right)\,\mathrm{d}S_x$$

The naive version is just to compute it like that. However they say it's not very stable, so in the stabilized one they compute:

$$\sum_{K\in\mathcal{T}}\sum_{E\in K}\int_E\left(\hat{u}\overline{\nabla v}\cdot\mathbf{n}_{E,K}-\boldsymbol{\sigma}\cdot\mathbf{n}_{E,K}\overline{v}\right)\,\mathrm{d}S_x=0$$

the naive version does not make sense, each triangle is decoupled.

Lets do least squares on the jumps of the function and the gradient

$$J\left(u^-,u^+\right)=\frac{1}{2}\int_E\left(u^+-u^-\right)^2+k^2\left(\left(\nabla u^+-\nabla u^-\right)\cdot\mathbf{n}\right)^2\,\mathrm{d}S_x$$

$$J\left(u^-+v^-,u^++v^+\right)-J\left(u^-,u^+\right)=$$

$$\int_E\left(u^+-u^-\right)\overline{\left(v^+-v^-\right)}+k^2\left(\left(\nabla u^+-\nabla u^-\right)\cdot\mathbf{n}\overline{\left(\nabla v^+-\nabla v^-\right)\cdot\mathbf{n}}\right)\,\mathrm{d}S_x$$

———–

Ok, lets pause for a bit. The equation:

$$\int_{\partial K}\left(u\overline{\nabla v}\cdot\mathbf{n}-\nabla u\cdot\mathbf{n}\overline{v}\right)\,\mathrm{d}S_x=0$$

has nothing to do with u being the solution to the BVP. Is just a consequence of u and v belonging to $V(K)$, and it would be fulfilled also if v and u were to belong to $V_h(K)\subsetneq V(K)$.

If we add the contribution from two triangles on the same edge we obtain (the magic DG formula):

$$\int_E\left(u_{K_i}\overline{\nabla v}_{K_i}\cdot\mathbf{n}-\nabla u_{K_i}\cdot\mathbf{n}\overline{v}_{K_i}\right)\,\mathrm{d}S_x-\int_E\left(u_{K_j}\overline{\nabla v}_{K_j}\cdot\mathbf{n}-\nabla u_{K_j}\cdot\mathbf{n}\overline{v}_{K_j}\right)\,\mathrm{d}S_x=0$$

that is:

$$\int_E\left([[u]]_{\mathbf{n}}\cdot\left\{\left\{\overline{\nabla v}\right\}\right\}+\{\{u\}\}\left[\left[\overline{\nabla v}\right]\right]_{\mathbf{n}}\right)\,\mathrm{d}S_x-\int_E\left(\{\{\nabla u\}\}\cdot[[\overline{v}]]_{\mathbf{n}}+[[\nabla u]]_{\mathbf{n}}\{\{\overline{v}\}\}\right)\,\mathrm{d}S_x=0$$

In particular, the u which solves BLA, has no jumps, so it solves:

$$\int_E\left(u\left[\left[\overline{\nabla v}\right]\right]_{\mathbf{n}}-\nabla u\cdot[[\overline{v}]]_{\mathbf{n}}\right)\,\mathrm{d}S_x=0$$

One way of weakly enforcing no jumps would be to ask for each internal edge

$$\int_E\left(\{\{u\}\}\left[\left[\overline{\nabla v}\right]\right]_{\mathbf{n}}-\{\{\nabla u\}\}\cdot[[\overline{v}]]_{\mathbf{n}}\right)\,\mathrm{d}S_x=0$$

to be true, as, combining it with the other relation, which we know is true for every $u,v\in V(\mathcal{T})$ gives

$$\int_E \left([[u]]_{\mathbf{n}} \cdot \left\{\left\{\overline{\nabla v}\right\}\right\}\right) \mathrm{d}S_x - \int_E \left([[\nabla u]]_{\mathbf{n}} \left\{\left\{\overline{v}\right\}\right\}\right) \mathrm{d}S_x = 0 \quad \forall v \in V\left(\mathcal{T}\right)$$

(It would be interesting to write this as the variation of a functional to check that we indeed are minizing the $\mathbf{L^2}$ norm or something)

Naive implementation

Take $\{\{u\}\}$ to mean exactly what it means, that is, no strange numerical fluxes that is

$$\{\{u\}\} = \frac{u^+ + u^-}{2} \neq \frac{u^+ + u^-}{2} + \frac{b}{ik}\left(\nabla u^+ - \nabla u^-\right) \cdot \mathbf{n}$$

$$\{\{\nabla u\}\} = \frac{\nabla u^+ + \nabla u^-}{2}$$

unless we are on the boundary edges, in wich case, if it is in Γ then

$$\{\{u\}\} = u^+$$

$$\{\{\nabla u\}\} = 0$$

and if it is at Σ_R then

$$\{\{u\}\} = u^{\mathrm{i}} + \mathcal{N}\left(\left(\nabla u^+ - \nabla u^{\mathrm{i}}\right) \cdot \mathbf{n}\right)$$

$$\{\{\nabla u\}\} = \nabla u^+$$

then we are looking for $u \in V_h\left(\mathcal{T}\right)$ such that

$$\sum_{E \in \mathcal{E}} \int_E \left(\{\{u\}\} \left[\left[\overline{\nabla v}\right]\right]_{\mathbf{n}} - \{\{\nabla u\}\} \cdot [[\overline{v}]]_{\mathbf{n}}\right) \mathrm{d}S_x = 0 \quad \forall v \in V_h\left(\mathcal{T}\right)$$

That it $\forall v \in V_h\left(\mathcal{T}\right)$:

$$\sum_{E \in \mathcal{E}} \int_E \left(\{\{u\}\} \left[\left[\overline{\nabla v}\right]\right]_{\mathbf{n}} - \{\{\nabla u\}\} \cdot [[\overline{v}]]_{\mathbf{n}}\right) \mathrm{d}S_x +$$

$$+ \sum_{E \subset \Gamma} \int_E u \overline{\nabla v} \cdot \mathbf{n} \, \mathrm{d}S_x$$

$$+ \sum_{E \subset \Sigma_{R^+}} \int_E \left(\left(u^{\mathrm{i}} + \mathcal{N}\left(\left(\nabla u - \nabla u^{\mathrm{i}}\right) \cdot \mathbf{n}\right)\right) \overline{\nabla v} \cdot \mathbf{n} - \nabla u \cdot \mathbf{n} \overline{v}\right) \mathrm{d}S_x$$

$$+ \sum_{E \subset \Sigma_{R^-}} \int_E \left(\left(u^{\mathrm{i}} + \mathcal{N}\left(\left(\nabla u - \nabla u^{\mathrm{i}}\right) \cdot \mathbf{n}\right)\right) \overline{\nabla v} \cdot \mathbf{n} - \nabla u \cdot \mathbf{n} \overline{v}\right) \mathrm{d}S_x = 0$$

which can be reorganized as:

$$\mathcal{A}_h(u,v) = \ell_h(v) \quad \forall v \in V_h(\mathcal{T})$$

where

$$\mathcal{A}_h(u,v) =$$

$$\sum_{E\in\mathcal{E}} \int_E \left(\{\{u\}\} \left[\left[\overline{\nabla v}\right]\right]_{\mathbf{n}} - \{\{\nabla u\}\} \cdot [[\overline{v}]]_{\mathbf{n}}\right) \mathrm{d}S_x +$$

$$+ \sum_{E\subset\Gamma} \int_E u\overline{\nabla v} \cdot \mathbf{n} \, \mathrm{d}S_x$$

$$+ \sum_{E\subset\left(\Sigma_{R^+}\cup\Sigma_{R^-}\right)} \int_E \left(\mathcal{N}\left(\nabla u \cdot \mathbf{n}\right) \overline{\nabla v} \cdot \mathbf{n} - \overline{v} \nabla u \cdot \mathbf{n}\right) \mathrm{d}S_x$$

and

$$\ell_h(v) = \sum_{E\subset\left(\Sigma_{R^+}\cup\Sigma_{R^-}\right)} \int_E \left(\left(\mathcal{N}\left(\nabla u^{\mathrm{i}} \cdot \mathbf{n}\right) - u^{\mathrm{i}}\right) \overline{\nabla v} \cdot \mathbf{n}\right) \mathrm{d}S_x$$

Finite dimensional basis

If $u = \sum u_n \phi_n$ with $\mathbf{x} = (u_n)^{\mathsf{T}}$ and the same for $v = \sum v_m \psi_m$ then

$$\mathcal{A}_h(u,v) = \ell_h(v)$$

becomes

$$\sum_n u_n \sum_m \overline{v}_m \mathcal{A}_h(\phi_n, \psi_m) = \sum_m \overline{v_m} \ell_h(\psi_m)$$

which can be rewritten as

$$\mathbf{v}^*\mathbf{A}\mathbf{x} = \mathbf{v}^*\mathbf{b}$$

that is

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

where

$$b_m = \ell_h(\psi_m)$$

and

$$A_{mn} = \mathcal{A}_h(\phi_n, \psi_m)$$

We are going to try to assemble the matrix A column-wise, that is. For each n we have to look for all the "interacting" ϕ_m which will be the ones in its element and the neighbouring ones.

Say we are at evaluating the colum n of A which corresponds to ϕ_n. Then ϕ_n "appears" in three edges where the equations are:

$$\frac{1}{2}\int_E \left(\phi_n \left[\left[\overline{\nabla v}\right]\right]_{\mathbf{n}} - \nabla\phi_n \cdot [[\overline{v}]]_{\mathbf{n}}\right) \mathrm{d}S_x, \quad E \in \mathcal{E}_I$$

$$\int_E \phi_n \overline{\nabla v} \cdot \mathbf{n}\, \mathrm{d}S_x, \quad E \subset \Gamma$$

$$\int_E \left(\mathcal{N}\left(\nabla\phi_n \cdot \mathbf{n}\right) \overline{\nabla v} \cdot \mathbf{n} - \overline{v}\nabla\phi_n \cdot \mathbf{n}\right) \mathrm{d}S_x, \quad E \subset \Sigma_{R^+} \cup \Sigma_{R^-}$$

Now let say we are at an edge $E \subset \Gamma$, then the only rows involved are the ones corresponding to ϕ_m coming from the same triangle.

Am I intersed into that? another option is to evaluate all the terms that contain ϕ_m and ϕ_n directly. Well that's not very intelligent because I will have to do $\mathrm{N}^{\mathbf{2}}$ checks. Is it?

I still think that the u-edge is the most ordered way. Lets see what is the shape of this term

$$\int_E \phi_n \overline{\frac{\partial \psi_m}{\partial \mathbf{n}}}\, \mathrm{d}S_x$$

where

$$\phi_n = e^{ik_n \mathbf{d}_n \cdot \mathbf{x}}$$

$$\psi_m = e^{ik_m \mathbf{d}_m \cdot \mathbf{x}}$$

$$\overline{\frac{\partial \psi_m}{\partial \mathbf{n}}} = -\left(ik_m \mathbf{d}_m \cdot \mathbf{n}\right) e^{-ik_m \mathbf{d}_m \cdot \mathbf{x}}$$

$$\phi_n \overline{\frac{\partial \psi_m}{\partial \mathbf{n}}} = -\left(ik_m \mathbf{d}_m \cdot \mathbf{n}\right) e^{i(k_n \mathbf{d}_n - k_m \mathbf{d}_m)\cdot \mathbf{x}} \quad \text{(most of the time k\_n =k\_m)}$$

$$\int_E \phi_n \overline{\frac{\partial \psi_m}{\partial \mathbf{n}}}\, \mathrm{d}S_x = -\left(ik_m \mathbf{d}_m \cdot \mathbf{n}\right) \int_E e^{i(k_n \mathbf{d}_n - k_m \mathbf{d}_m)\cdot \mathbf{x}}\, \mathrm{d}S_x$$

$$\boldsymbol{\gamma}(t) = \mathbf{p} + (\mathbf{q} - \mathbf{p})\, t = \mathbf{p} + \mathbf{l}t, \ \|\boldsymbol{\gamma}'\| = \|\mathbf{q} - \mathbf{p}\| = l$$

$$-\left(ik_m \mathbf{d}_m \cdot \mathbf{n}\right) \int_E e^{i(k_n \mathbf{d}_n - k_m \mathbf{d}_m)\cdot \mathbf{x}}\, \mathrm{d}S_x = -\left(ik_m \mathbf{d}_m \cdot \mathbf{n}\right) l \int_0^1 e^{i(k_n \mathbf{d}_n - k_m \mathbf{d}_m)\cdot \boldsymbol{\gamma}(t)}\, \mathrm{d}t$$

$$= -\left(ik_m\mathbf{d}_m\cdot\mathbf{n}\right) le^{i(k_n\mathbf{d}_n-k_m\mathbf{d}_m)\cdot\mathbf{p}}\int_0^1 e^{i(k_n\mathbf{d}_n-k_m\mathbf{d}_m)\cdot\mathbf{l}t}\,\mathrm{d}t$$

$$= -\left(ik_m\mathbf{d}_m\cdot\mathbf{n}\right) le^{i(k_n\mathbf{d}_n-k_m\mathbf{d}_m)\cdot\mathbf{p}}\frac{1}{i\left(k_n\mathbf{d}_n-k_m\mathbf{d}_m\right)\cdot\mathbf{l}}\left(e^{i(k_n\mathbf{d}_n-k_m\mathbf{d}_m)\cdot\mathbf{l}}-1\right)$$

now lets clean it a little bit

$$= -\left(k_m\mathbf{d}_m\cdot\mathbf{n}\right) l\frac{1}{\left(k_n\mathbf{d}_n-k_m\mathbf{d}_m\right)\cdot\mathbf{l}}\left(e^{i(k_n\mathbf{d}_n-k_m\mathbf{d}_m)\cdot\mathbf{q}}-e^{i(k_n\mathbf{d}_n-k_m\mathbf{d}_m)\cdot\mathbf{p}}\right)$$

$$= \frac{k_m\mathbf{d}_m\cdot\mathbf{n}l}{\left(k_n\mathbf{d}_n-k_m\mathbf{d}_m\right)\cdot\mathbf{l}}\left(e^{i(k_n\mathbf{d}_n-k_m\mathbf{d}_m)\cdot\mathbf{p}}-e^{i(k_n\mathbf{d}_n-k_m\mathbf{d}_m)\cdot\mathbf{q}}\right)$$

If we define $\boldsymbol{\ell}$ such that $\mathbf{l} = l\boldsymbol{\ell}$

$$\frac{k_m\mathbf{d}_m\cdot\mathbf{n}}{\left(k_n\mathbf{d}_n-k_m\mathbf{d}_m\right)\cdot\boldsymbol{\ell}}\left(e^{i(k_n\mathbf{d}_n-k_m\mathbf{d}_m)\cdot\mathbf{p}}-e^{i(k_n\mathbf{d}_n-k_m\mathbf{d}_m)\cdot\mathbf{q}}\right)$$

for every $n \neq m$, and for $n = m$

$$\int_E \phi_m\overline{\frac{\partial\psi_m}{\partial\mathbf{n}}}\,\mathrm{d}S_x = -\left(ik_m\mathbf{d}_m\cdot\mathbf{n}\right)\int_E 1\,\mathrm{d}S_x = -ik_m\mathbf{d}_m\cdot\mathbf{n}l$$

Lets go with the interior edge term:

$$\frac{1}{2}\int_E \left(\phi_n\left[\left[\overline{\nabla v}\right]\right]_{\mathbf{n}} - \nabla\phi_n\cdot[[\overline{v}]]_{\mathbf{n}}\right)\,\mathrm{d}S_x,\quad E\in\mathcal{E}_I$$

if $v = \psi_m$ belongs to the same triangle

$$\frac{1}{2}\int_E \left(\phi_n\overline{\frac{\partial\psi_m}{\partial\mathbf{n}}} - \frac{\partial\phi_n}{\partial\mathbf{n}}\overline{\psi_m}\right)\,\mathrm{d}S_x$$

so, if we are carefull:

$$\frac{1}{2}\int_E \left(\phi_n\overline{\frac{\partial\psi_m}{\partial\mathbf{n}}} - \frac{\partial\phi_n}{\partial\mathbf{n}}\overline{\psi_m}\right)\,\mathrm{d}S_x =$$

$$= \frac{1}{2}\frac{k_m\mathbf{d}_m\cdot\mathbf{n}}{\left(k_n\mathbf{d}_n-k_m\mathbf{d}_m\right)\cdot\boldsymbol{\ell}}\left(e^{i(k_n\mathbf{d}_n-k_m\mathbf{d}_m)\cdot\mathbf{p}}-e^{i(k_n\mathbf{d}_n-k_m\mathbf{d}_m)\cdot\mathbf{q}}\right)$$

$$+\frac{1}{2}\frac{k_n\mathbf{d}_n\cdot\mathbf{n}}{\left(k_n\mathbf{d}_n-k_m\mathbf{d}_m\right)\cdot\boldsymbol{\ell}}\left(e^{i(k_n\mathbf{d}_n-k_m\mathbf{d}_m)\cdot\mathbf{p}}-e^{i(k_n\mathbf{d}_n-k_m\mathbf{d}_m)\cdot\mathbf{q}}\right)$$

$$= \frac{1}{2}\frac{\left(k_m\mathbf{d}_m+k_n\mathbf{d}_n\right)\cdot\mathbf{n}}{\left(k_n\mathbf{d}_n-k_m\mathbf{d}_m\right)\cdot\boldsymbol{\ell}}\left(e^{i(k_n\mathbf{d}_n-k_m\mathbf{d}_m)\cdot\mathbf{p}}-e^{i(k_n\mathbf{d}_n-k_m\mathbf{d}_m)\cdot\mathbf{q}}\right)$$

for $m \neq n$ and

$$-ik_m\mathbf{d}_m\cdot\mathbf{n}l$$

if $v=\psi_m$ belongs to the another triangle the term is just

$$-\frac{1}{2}\int_E\left(\phi_n\overline{\frac{\partial\psi_m}{\partial\mathbf{n}}}-\frac{\partial\phi_n}{\partial\mathbf{n}}\overline{\psi_m}\right)\,\mathrm{d}S_x=$$

$$-\frac{1}{2}\frac{(k_m\mathbf{d}_m+k_n\mathbf{d}_n)\cdot\mathbf{n}}{(k_n\mathbf{d}_n-k_m\mathbf{d}_m)\cdot\boldsymbol{\ell}}\left(e^{i(k_n\mathbf{d}_n-k_m\mathbf{d}_m)\cdot\mathbf{p}}-e^{i(k_n\mathbf{d}_n-k_m\mathbf{d}_m)\cdot\mathbf{q}}\right)$$

Finally, the Σ_R terms...

$$\int_E\left(\mathcal{N}\left(\nabla\phi_n\cdot\mathbf{n}\right)\overline{\frac{\partial\psi_m}{\partial\mathbf{n}}}-\frac{\partial\phi_n}{\partial\mathbf{n}}\overline{\psi_m}\right)\,\mathrm{d}S_x$$

the second part is

$$-\int_E\frac{\partial\phi_n}{\partial\mathbf{n}}\overline{\psi_m}\,\mathrm{d}S_x=\frac{k_n\mathbf{d}_n\cdot\mathbf{n}}{(k_n\mathbf{d}_n-k_m\mathbf{d}_m)\cdot\boldsymbol{\ell}}\left(e^{i(k_n\mathbf{d}_n-k_m\mathbf{d}_m)\cdot\mathbf{p}}-e^{i(k_n\mathbf{d}_n-k_m\mathbf{d}_m)\cdot\mathbf{q}}\right)$$

for $n\neq m$ and

$$-\int_E\frac{\partial\phi_n}{\partial\mathbf{n}}\overline{\psi_m}\,\mathrm{d}S_x=-ik_n\mathbf{d}_n\cdot\mathbf{n}l$$

but the first one...

$$\int_E\mathcal{N}\left(\frac{\partial\phi_n}{\partial\mathbf{n}}\right)\overline{\frac{\partial\psi_m}{\partial\mathbf{n}}}\,\mathrm{d}S_x$$

but

$$\mathcal{N}\left(f\right)\approx\sum_{n=0}^{N}\frac{1}{i\beta_n}f_n\theta_n\left(\hat{\mathbf{x}}\right)$$

with

$$f_n=\int_\Sigma f\left(\hat{\mathbf{z}}\right)\theta_n\left(\hat{\mathbf{z}}\right)\,\mathrm{d}\hat{\mathbf{z}}$$

$$\beta_n=\sqrt{k_n-k}$$

Rectangular waveguide

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \Omega_R \\ \nabla u \cdot \mathbf{n} = 0 & \text{on } (-R, R) \times \partial\Sigma \end{cases}$$

$$\Delta u + k^2 u = X''Y + XY'' + k^2 XY = 0$$

one option

$$\frac{X'' + k^2 X}{X} = -\frac{Y''}{Y} = k_n^2$$

$$\begin{cases} Y'' + k_n^2 Y = 0 & y \in (-H, H) \\ Y' = 0 & y = H \\ Y' = 0 & y = -H \end{cases}$$

$$Y(y) = A\cos(k_n y) + B\sin(k_n y)$$

$$Y'(y) = k_n(-A\sin(k_n y) + B\cos(k_n y))$$

$$Y'(H) = k_n(-A\sin(k_n H) + B\cos(k_n H)) = 0$$

$$Y'(-H) = k_n(A\sin(k_n H) + B\cos(k_n H)) = 0$$

adding them

$$B\cos(k_n H) = 0$$

substracting them

$$A\sin(k_n H) = 0$$

one option $A = B = 0$. More interesting option $B = 0$ and

$$k_n = n\frac{\pi}{H}$$

and

$$\theta_n(y) = \cos(k_n y) = \cos\left(n\pi\frac{y}{H}\right)$$

no, wait, thats not orthonormal

$$\int \cos\left(n\pi\frac{y}{H}\right)^2 \mathrm{d}y = \int \frac{1 + \cos\left(2n\pi\frac{y}{H}\right)}{2} \mathrm{d}y = \int_H^H \frac{1}{2}\mathrm{d}y = H, \quad n > 0$$

$$2H, \quad n = 0$$

so

$$\theta_0(y) = \frac{1}{\sqrt{2H}}$$

$$\theta_n(y) = \frac{\cos\left(n\pi\frac{y}{H}\right)}{\sqrt{H}}$$

for practical reasons we will define

$$c_n = \begin{cases} \sqrt{2H} & n = 0 \\ \sqrt{H} & n > 0 \end{cases}$$

so

$$\sum_{s=0}^{N} \frac{1}{i\beta_s c_s^2} \int_\Sigma \frac{\partial\phi_n}{\partial\mathbf{n}}(y) \overline{\cos\left(s\pi\frac{y}{H}\right)} \,\mathrm{d}y \int_E \cos\left(s\pi\frac{y}{H}\right) \overline{\frac{\partial\psi_m}{\partial\mathbf{n}}} \,\mathrm{d}S_x$$

we have two very similar integrals to compute:

$$\int_\Sigma \frac{\partial\phi_n}{\partial\mathbf{n}}(y) \overline{\cos\left(s\pi\frac{y}{H}\right)} \,\mathrm{d}y = \int_\Sigma \mathbf{n}\cdot\mathbf{d}_n e^{ik_n\mathbf{d}\cdot\mathbf{x}} \cos\left(s\pi\frac{y}{H}\right) \,\mathrm{d}y$$

$$= \frac{1}{2} ik_n\mathbf{n}\cdot\mathbf{d}_n \int_\Sigma e^{ik_n\mathbf{d}\cdot\mathbf{x}} \left(e^{is\pi\frac{y}{H}} + e^{-is\pi\frac{y}{H}}\right) \,\mathrm{d}y$$

$$= \frac{1}{2} ik_n\mathbf{n}\cdot\mathbf{d}_n \int_\Sigma e^{ik_n(d_x x + d_y y)} \left(e^{is\pi\frac{y}{H}} + e^{-is\pi\frac{y}{H}}\right) \,\mathrm{d}y$$

$$= \frac{1}{2} ik_n\mathbf{n}\cdot\mathbf{d}_n e^{ik_n d_x x} \int_\Sigma \left(e^{i\left(k_n d_y + \frac{s\pi}{H}\right)y} + e^{i\left(k_n d_y - \frac{s\pi}{H}\right)y}\right) \,\mathrm{d}y$$

$$= \frac{1}{2} ik_n\mathbf{n}\cdot\mathbf{d}_n e^{ik_n d_x x} \left(\frac{e^{i\left(k_n d_y + \frac{s\pi}{H}\right)y}}{i\left(k_n d_y + \frac{s\pi}{H}\right)} + \frac{e^{i\left(k_n d_y - \frac{s\pi}{H}\right)y}}{i\left(k_n d_y - \frac{s\pi}{H}\right)}\right)_{-H}^{H}$$

$$= ik_n\mathbf{n}\cdot\mathbf{d}_n e^{ik_n d_x x} \left(\frac{e^{i(k_n d_y H + s\pi)}}{2i\left(k_n d_y + \frac{s\pi}{H}\right)} + \frac{e^{i(k_n d_y H - s\pi)}}{2i\left(k_n d_y - \frac{s\pi}{H}\right)} - \frac{e^{-i(k_n d_y H + s\pi)}}{2i\left(k_n d_y + \frac{s\pi}{H}\right)} - \frac{e^{-i(k_n d_y H - s\pi)}}{2i\left(k_n d_y - \frac{s\pi}{H}\right)}\right)$$

$$= ik_n\mathbf{n}\cdot\mathbf{d}_n e^{ik_n d_x x} \left(\frac{\sin(k_n d_y H + s\pi)}{\left(k_n d_y + \frac{s\pi}{H}\right)} + \frac{\sin(k_n d_y H - s\pi)}{\left(k_n d_y - \frac{s\pi}{H}\right)}\right)$$

$$= ik_n H\mathbf{n}\cdot\mathbf{d}_n e^{ik_n d_{n,x} x} \left(\frac{\sin(k_n d_{n,y} H + s\pi)}{k_n d_{n,y} H + s\pi} + \frac{\sin(k_n d_{n,y} H - s\pi)}{k_n d_{n,y} H - s\pi}\right)$$

and this is only one of the terms products in that term... XD

$$\int_\Sigma \frac{\partial \phi_n}{\partial \mathbf{n}}(y)\overline{\cos\left(s\pi\frac{y}{H}\right)}\,\mathrm{d}y =$$

$$= ik_nH\mathbf{n}\cdot\mathbf{d}_n e^{ik_nd_{n,x}x}\left(\frac{\sin\left(k_nd_{n,y}H+s\pi\right)}{k_nd_{n,y}H+s\pi}+\frac{\sin\left(k_nd_{n,y}H-s\pi\right)}{k_nd_{n,y}H-s\pi}\right)$$

so

$$\int_E \cos\left(s\pi\frac{y}{H}\right)\overline{\frac{\partial\psi_m}{\partial\mathbf{n}}}\,\mathrm{d}S_x = \overline{\int_E \frac{\partial\psi_m}{\partial\mathbf{n}}\overline{\cos\left(s\pi\frac{y}{H}\right)}\,\mathrm{d}S_x} =$$

$$= -ik_m\mathbf{n}\cdot\mathbf{d}_m e^{-ik_md_{m,x}x}\left(\frac{e^{-i\left(k_md_{m,y}+\frac{s\pi}{H}\right)y}}{-2i\left(k_md_{m,y}+\frac{s\pi}{H}\right)}+\frac{e^{-i\left(k_md_{m,y}-\frac{s\pi}{H}\right)y}}{-2i\left(k_md_{m,y}-\frac{s\pi}{H}\right)}\right)_{-y_1}^{y_2}$$

in the special case that the edge takes the whole boundary, $y_2 = H$, $y_1 = -H$

$$= -ik_mH\mathbf{n}\cdot\mathbf{d}_m e^{-ik_md_{m,x}x}\left(\frac{\sin\left(k_md_{m,y}H+s\pi\right)}{\left(k_md_{m,y}H+s\pi\right)}+\frac{\sin\left(k_md_{m,y}H-s\pi\right)}{\left(k_md_{m,y}H-s\pi\right)}\right)$$

so in this special case, the term would look like:

$$\sum_{s=0}^{N}\frac{1}{i\beta_sc_s^2}\int_\Sigma\frac{\partial\phi_n}{\partial\mathbf{n}}(y)\overline{\cos\left(s\pi\frac{y}{H}\right)}\,\mathrm{d}y\int_\Sigma\cos\left(s\pi\frac{y}{H}\right)\overline{\frac{\partial\psi_m}{\partial\mathbf{n}}}\,\mathrm{d}S_x =$$

$$\sum_{s=0}^{N}\frac{k_mk_n}{i\beta_sc_s^2}H^2\mathbf{n}\cdot\mathbf{d}_n\mathbf{n}\cdot\mathbf{d}_m e^{i\left(k_nd_{n,x}-k_md_{m,x}\right)x}\left(\frac{\sin\left(k_nd_{n,y}H+s\pi\right)}{k_nd_{n,y}H+s\pi}+\frac{\sin\left(k_nd_{n,y}H-s\pi\right)}{k_nd_{n,y}H-s\pi}\right)\left(\frac{\sin\left(k_md_{m,y}H+s\pi\right)}{\left(k_md_{m,y}H+s\pi\right)}+\right.$$

SO for a mesh where Σ_R and Σ_{-R} coincide with elements sides, the last two terms in $\mathbf{A}$ are:

$$\int_E\left(\mathcal{N}\left(\nabla\phi_n\cdot\mathbf{n}\right)\overline{\frac{\partial\psi_m}{\partial\mathbf{n}}}-\frac{\partial\phi_n}{\partial\mathbf{n}}\overline{\psi_m}\right)\mathrm{d}S_x =$$

$$\sum_{s=0}^{N}\frac{k^2}{i\sqrt{k^2-\left(s\frac{\pi}{H}\right)^2}}\frac{H^2}{c_s^2}\mathbf{n}\cdot\mathbf{d}_n\mathbf{n}\cdot\mathbf{d}_m e^{i\left(kd_{n,x}-kd_{m,x}\right)x}\left(\frac{\sin\left(kd_{n,y}H+s\pi\right)}{kd_{n,y}H+s\pi}+\frac{\sin\left(kd_{n,y}H-s\pi\right)}{kd_{n,y}H-s\pi}\right)\left(\frac{\sin\left(kd_{m,y}H+s\pi\right.}{\left(kd_{m,y}H+s\pi\right)}\right.$$

$$+\frac{k\mathbf{d}_n\cdot\mathbf{n}}{\left(k\mathbf{d}_n-k\mathbf{d}_m\right)\cdot\boldsymbol{\ell}}\left(e^{i\left(k\mathbf{d}_n-k\mathbf{d}_m\right)\cdot\mathbf{p}}-e^{i\left(k\mathbf{d}_n-k\mathbf{d}_m\right)\cdot\mathbf{q}}\right)$$

the First term can be made prettier (CHECK EXPONENT X):

$$\frac{1}{ikH}\mathbf{n}\cdot\mathbf{d}_n\mathbf{n}\cdot\mathbf{d}_m e^{i\left(kd_{n,x}-kd_{m,x}\right)x}\frac{\sin\left(kd_{n,y}H\right)}{d_{n,y}}\frac{\sin\left(kd_{m,y}H\right)}{d_{m,y}}$$

$$+ (kH)^2\, \mathbf{n}\cdot\mathbf{d}_n \mathbf{n}\cdot\mathbf{d}_m e^{i(kd_{n,x}-kd_{m,x})x} \sum_{s=1}^{N} \frac{1}{i\sqrt{(kH)^2-(s\pi)^2}} \left(\frac{\sin(kd_{n,y}H+s\pi)}{kd_{n,y}H+s\pi} + \frac{\sin(kd_{n,y}H-s\pi)}{kd_{n,y}H-s\pi} \right) \left(\frac{\sin(kd_{m,}}{(kd_{m,y}}\right.$$

AND FINALLY, the b term. At least this has only one type of term:

$$\int_E \left(\left(\mathcal{N} \left(\nabla u^{\mathrm{i}} \cdot \mathbf{n} \right) - u^{\mathrm{i}} \right) \overline{\nabla v} \cdot \mathbf{n} \right) \mathrm{d}S_x =$$

$$\int_E \left(\mathcal{N} \left(\nabla u^{\mathrm{i}} \cdot \mathbf{n} \right) \overline{\frac{\partial \psi}{\partial \mathbf{n}}} - u^{\mathrm{i}} \overline{\frac{\partial \psi}{\partial \mathbf{n}}} \right) \mathrm{d}S_x$$

and this is difficult to evaluate, depending on u^{i}. Lets check with $u^{\mathrm{i}}(\mathbf{x}) = g_t^+(\mathbf{x})$ for som $t > 0$.

$$g_t^+(\mathbf{x}) = e^{i\beta_t x} \cos(k_t y) = e^{i\sqrt{k-\frac{t\pi}{H}}\,x} \frac{\cos\left(t\pi \frac{y}{H}\right)}{c_t}$$

we already have the second term:

$$-\int_{-H}^{H} u^{\mathrm{i}} \overline{\frac{\partial \psi}{\partial \mathbf{n}}} \,\mathrm{d}y =$$

$$-\frac{e^{i\sqrt{k-\frac{t\pi}{H}}\,x}}{c_t} \int_{-H}^{H} \cos\left(t\pi \frac{y}{H}\right) \overline{\frac{\partial \psi}{\partial \mathbf{n}}} \,\mathrm{d}y =$$

$$\frac{ik_m}{c_t} H \mathbf{n}\cdot\mathbf{d}_m e^{i\left(\sqrt{k-\frac{t\pi}{H}}-k_m d_{m,x}\right)x} \left(\frac{\sin(k_m d_{m,y}H + t\pi)}{(k_m d_{m,y}H + t\pi)} + \frac{\sin(k_m d_{m,y}H - t\pi)}{(k_m d_{m,y}H - t\pi)} \right)$$

Now, for the first one, assuming $x = R$

$$\int_{\Sigma_R} \mathcal{N}\left(\frac{\partial u^{\mathrm{i}}}{\partial x} \right) \overline{\frac{\partial \psi}{\partial \mathbf{n}}} \,\mathrm{d}S_x$$

$$\frac{i\beta_t}{c_t} e^{i\beta_t R} \int_{\Sigma_R} \mathcal{N}\left(\cos(k_t y) \right) \overline{\frac{\partial \psi}{\partial \mathbf{n}}} \,\mathrm{d}S_x$$

$$\sum_{s=0}^{N} \frac{1}{i\beta_s} \frac{i\beta_t}{c_t} e^{i\beta_t R} \frac{\int_\Sigma \cos(k_t y) \cos\left(s\pi \frac{y}{H}\right) \mathrm{d}y}{c_s^2} \int_E \cos\left(s\pi \frac{y}{H}\right) \overline{\frac{\partial \psi_m}{\partial \mathbf{n}}} \,\mathrm{d}S_x$$

$$\frac{e^{i\beta_t R}}{c_t} \int_{\Sigma_R} \cos\left(t\pi \frac{y}{H}\right) \overline{\frac{\partial \psi_m}{\partial \mathbf{n}}} \,\mathrm{d}y =$$

$$-\frac{ik_m H}{c_t}\mathbf{n}\cdot\mathbf{d}_m e^{i(\beta_t - k_m d_{m,x})R}\left(\frac{\sin\left(k_m d_{m,y}H + t\pi\right)}{k_m d_{m,y}H + t\pi} + \frac{\sin\left(k_m d_{m,y}H - t\pi\right)}{k_m d_{m,y}H - t\pi}\right)$$

SO FINALLY the b at Σ_R term are:

$$\int_E \left(\left(\mathcal{N}\left(\nabla u^{\mathrm{i}}\cdot\mathbf{n}\right) - u^{\mathrm{i}}\right)\overline{\nabla v}\cdot\mathbf{n}\right)\,\mathrm{d}S_x =$$

$$-\frac{ik_m H}{c_t}\mathbf{n}\cdot\mathbf{d}_m e^{i(\beta_t - k_m d_{m,x})R}\left(\frac{\sin\left(k_m d_{m,y}H + t\pi\right)}{k_m d_{m,y}H + t\pi} + \frac{\sin\left(k_m d_{m,y}H - t\pi\right)}{k_m d_{m,y}H - t\pi}\right)$$

$$+\frac{ik_m}{c_t}H\mathbf{n}\cdot\mathbf{d}_m e^{i\left(\sqrt{k-\frac{t\pi}{H}} - k_m d_{m,x}\right)R}\left(\frac{\sin\left(k_m d_{m,y}H + t\pi\right)}{\left(k_m d_{m,y}H + t\pi\right)} + \frac{\sin\left(k_m d_{m,y}H - t\pi\right)}{\left(k_m d_{m,y}H - t\pi\right)}\right)$$
$$= 0$$

are...0? It may be the case. However, for Σ_{-R} they are

$$\int_E \left(\left(\mathcal{N}\left(\nabla u^{\mathrm{i}}\cdot\mathbf{n}\right) - u^{\mathrm{i}}\right)\overline{\nabla v}\cdot\mathbf{n}\right)\,\mathrm{d}S_x =$$

$$+\frac{ik_m H}{c_t}\mathbf{n}\cdot\mathbf{d}_m e^{-i(\beta_t - k_m d_{m,x})R}\left(\frac{\sin\left(k_m d_{m,y}H + t\pi\right)}{k_m d_{m,y}H + t\pi} + \frac{\sin\left(k_m d_{m,y}H - t\pi\right)}{k_m d_{m,y}H - t\pi}\right)$$

$$+\frac{ik_m}{c_t}H\mathbf{n}\cdot\mathbf{d}_m e^{-i\left(\sqrt{k^2-\left(\frac{t\pi}{H}\right)^2} - k_m d_{m,x}\right)R}\left(\frac{\sin\left(k_m d_{m,y}H + t\pi\right)}{\left(k_m d_{m,y}H + t\pi\right)} + \frac{\sin\left(k_m d_{m,y}H - t\pi\right)}{\left(k_m d_{m,y}H - t\pi\right)}\right)$$

$$= 2\frac{ik_m}{c_t}H\mathbf{n}\cdot\mathbf{d}_m e^{-i\left(\sqrt{k^2-\left(\frac{t\pi}{H}\right)^2} - k_m d_{m,x}\right)R}\left(\frac{\sin\left(k_m d_{m,y}H + t\pi\right)}{\left(k_m d_{m,y}H + t\pi\right)} + \frac{\sin\left(k_m d_{m,y}H - t\pi\right)}{\left(k_m d_{m,y}H - t\pi\right)}\right)$$

that is

$$= 2\frac{ik_m}{c_t}H\mathbf{n}\cdot\mathbf{d}_m e^{-i\left(\sqrt{k^2-\left(\frac{t\pi}{H}\right)^2} - k_m d_{m,x}\right)R}\left(\frac{\sin\left(k_m d_{m,y}H + t\pi\right)}{\left(k_m d_{m,y}H + t\pi\right)} + \frac{\sin\left(k_m d_{m,y}H - t\pi\right)}{\left(k_m d_{m,y}H - t\pi\right)}\right)$$

need to come up with a different notation for k_m when refering to "k evaluated at the triangle corresponding to basis function ψ_m" as it collides with k_n i.e. the laplacian eigenvalue. I think the superscripts "+" and "-" are a good choice.

$$= \frac{4i}{\sqrt{2H}}\mathbf{n}\cdot\mathbf{d}_m e^{-ik(1-d_{m,x})R}\frac{\sin\left(kHd_{m,y}\right)}{d_{m,y}}$$

for $t=0$ and for $t>0$

$$2ik_m\sqrt{H}\mathbf{n}\cdot\mathbf{d}_m e^{-i\left(\sqrt{k^2-\left(\frac{t\pi}{H}\right)^2}-kd_{m,x}\right)R}\left(\frac{\sin\left(k_m d_{m,y}H+t\pi\right)}{\left(k_m d_{m,y}H+t\pi\right)}+\frac{\sin\left(k_m d_{m,y}H-t\pi\right)}{\left(k_m d_{m,y}H-t\pi\right)}\right)$$

Well it seems that the Naive version works. Lets see the Non Naive one:

Non-naive version:

Insead of approximating u at the edge by $\{\{u\}\}$ we will use instead:

if we are at Γ:

$$u\to u^+ + \frac{d_1}{ik}\nabla u^+_{\mathbf{n}}$$

$$ik\boldsymbol{\sigma}\to 0$$

So the term is now:

$$\int_E \left(u^+ + \frac{d_1}{ik}\frac{\partial u}{\partial \mathbf{n}}^+\right)\overline{\frac{\partial v}{\partial \mathbf{n}}}^+\,\mathrm{d}S_x$$

that is

$$-\int_E\left(e^{ik\mathbf{d}_n\cdot\mathbf{x}}+d_1 e^{ik\mathbf{d}_n\cdot\mathbf{x}}\mathbf{d}_n\cdot\mathbf{n}\right)e^{-ik\mathbf{d}_m\cdot\mathbf{x}}ik\mathbf{d}_m\cdot\mathbf{n}\,\mathrm{d}S_x$$

$$-\left(1+d_1\mathbf{d}_n\cdot\mathbf{n}\right)ik\mathbf{d}_m\cdot\mathbf{n}\int_E e^{ik(\mathbf{d}_n-\mathbf{d}_m)\cdot\mathbf{x}}\,\mathrm{d}S_x$$

$$-\left(1+d_1\mathbf{d}_n\cdot\mathbf{n}\right)ik\mathbf{d}_m\cdot\mathbf{n}\int_0^1 e^{ik(\mathbf{d}_n-\mathbf{d}_m)\cdot(\mathbf{p}+t(\mathbf{q}-\mathbf{p}))}l\,\mathrm{d}t$$

$$-\left(1+d_1\mathbf{d}_n\cdot\mathbf{n}\right)ik\mathbf{d}_m\cdot\mathbf{n}le^{ik(\mathbf{d}_n-\mathbf{d}_m)\cdot\mathbf{p}}\int_0^1 e^{ik(\mathbf{d}_n-\mathbf{d}_m)\cdot(\mathbf{q}-\mathbf{p})t}\,\mathrm{d}t$$

if $\mathbf{d}_n=\mathbf{d}_m$ then

$$-ikl\left(1+d_1\mathbf{d}_n\cdot\mathbf{n}\right)\mathbf{d}_m\cdot\mathbf{n}$$

else

$$-\left(1+d_1\mathbf{d}_n\cdot\mathbf{n}\right)ik\mathbf{d}_m\cdot\mathbf{n}le^{ik(\mathbf{d}_n-\mathbf{d}_m)\cdot\mathbf{p}}\frac{1}{ik\left(\mathbf{d}_n-\mathbf{d}_m\right)\cdot(\mathbf{q}-\mathbf{p})}\left(e^{ik(\mathbf{d}_n-\mathbf{d}_m)\cdot(\mathbf{q}-\mathbf{p})}-1\right)$$

$$-\frac{\left(1+d_1\mathbf{d}_n\cdot\mathbf{n}\right)\mathbf{d}_m\cdot\mathbf{n}}{\left(\mathbf{d}_n-\mathbf{d}_m\right)\cdot\boldsymbol{\ell}}\left(e^{ik(\mathbf{d}_n-\mathbf{d}_m)\cdot\mathbf{q}}-e^{ik(\mathbf{d}_n-\mathbf{d}_m)\cdot\mathbf{p}}\right)$$

If we are in an Inner edge:

$$u \to \{\{u\}\} + \frac{b}{ik} [[\nabla u]]_{\mathbf{n}}$$

$$ik\boldsymbol{\sigma} \to \{\{\nabla u\}\} + aik\, [[u]]_{\mathbf{n}}$$

and the term is:

$$\int_E \left(\left(\{\{u\}\} + \frac{b}{ik} [[\nabla u]]_{\mathbf{n}} \right) \left[\left[\overline{\nabla v} \right] \right]_{\mathbf{n}} - \left(\{\{\nabla u\}\} + aik\, [[u]]_{\mathbf{n}} \right) \cdot [[\overline{v}]]_{\mathbf{n}} \right) \mathrm{d}S_x$$

This concerns test functions and trial functions in two different triangles. Lets compute the term for ϕ_n^+ and ψ_m^+, i.e. both on the same triangle upstream of $\mathbf{n}$.

$$\int_E \left(\left(\frac{\phi_n^+}{2} + \frac{b}{ik} \frac{\partial \phi_n^+}{\partial \mathbf{n}} \right) \overline{\frac{\partial \psi_m^+}{\partial \mathbf{n}}} - \left(\frac{1}{2} \frac{\partial \phi_n^+}{\partial \mathbf{n}} + aik\phi_n^+ \right) \overline{\psi_m^+} \right) \mathrm{d}S_x$$

Lets assume that we are in an edge wich is completely contained in the background:

$$\frac{-ik}{2} \left(\mathbf{d}_m \cdot \mathbf{n} + \mathbf{d}_n \cdot \mathbf{n} + 2b\mathbf{d}_n \cdot \mathbf{n} \mathbf{d}_m \cdot \mathbf{n} + 2a \right) \int_E \left(e^{ik(\mathbf{d}_n - \mathbf{d}_m)\cdot \mathbf{x}} \right) \mathrm{d}S_x$$

if $\mathbf{d}_m = \mathbf{d}_n$ then

$$-\frac{ikl}{2} \left(\mathbf{d}_m \cdot \mathbf{n} + \mathbf{d}_n \cdot \mathbf{n} + 2b\mathbf{d}_n \cdot \mathbf{n} \mathbf{d}_m \cdot \mathbf{n} + 2a \right)$$

else:

$$-\frac{1}{2} \frac{\left(\mathbf{d}_m \cdot \mathbf{n} + \mathbf{d}_n \cdot \mathbf{n} + 2b\mathbf{d}_n \cdot \mathbf{n} \mathbf{d}_m \cdot \mathbf{n} + 2a \right)}{(\mathbf{d}_n - \mathbf{d}_m) \cdot \boldsymbol{\ell}} \left(e^{ik(\mathbf{d}_n - \mathbf{d}_m)\cdot \mathbf{q}} - e^{ik(\mathbf{d}_n - \mathbf{d}_m)\cdot \mathbf{p}} \right)$$

the other 3 types of terms, are the corresponding to ϕ_n^+ and ψ^-:

$$-\int_E \left(\left(\frac{\phi_n^+}{2} + \frac{b}{ik} \frac{\partial \phi_n^+}{\partial \mathbf{n}} \right) \overline{\frac{\partial \psi_m^-}{\partial \mathbf{n}}} - \left(\frac{1}{2} \frac{\partial \phi_n^+}{\partial \mathbf{n}} + aik\phi_n^+ \right) \overline{\psi_m^-} \right) \mathrm{d}S_x$$

that is, if $\mathbf{d}_n = \mathbf{d}_m$ then

$$\frac{ikl}{2} \left(\mathbf{d}_m \cdot \mathbf{n} + \mathbf{d}_n \cdot \mathbf{n} + 2b\mathbf{d}_n \cdot \mathbf{n} \mathbf{d}_m \cdot \mathbf{n} + 2a \right)$$

else:

$$\frac{1}{2} \frac{\left(\mathbf{d}_m \cdot \mathbf{n} + \mathbf{d}_n \cdot \mathbf{n} + 2b\mathbf{d}_n \cdot \mathbf{n} \mathbf{d}_m \cdot \mathbf{n} + 2a \right)}{(\mathbf{d}_n - \mathbf{d}_m) \cdot \boldsymbol{\ell}} \left(e^{ik(\mathbf{d}_n - \mathbf{d}_m)\cdot \mathbf{q}} - e^{ik(\mathbf{d}_n - \mathbf{d}_m)\cdot \mathbf{p}} \right)$$

the one corresponding to ϕ^- and ψ^+

$$\int_E \left(\left(\frac{\phi_n^-}{2} - \frac{b}{ik} \frac{\partial \phi_n^-}{\partial \mathbf{n}} \right) \overline{\frac{\partial \psi_m^+}{\partial \mathbf{n}}} - \left(\frac{1}{2} \frac{\partial \phi_n^-}{\partial \mathbf{n}} - aik\phi_n^+ \right) \overline{\psi_m^+} \right) \mathrm{d}S_x$$

that is, if $\mathbf{d}_n = \mathbf{d}_m$ then

$$-\frac{ikl}{2} \left(\mathbf{d}_m \cdot \mathbf{n} + \mathbf{d}_n \cdot \mathbf{n} - 2b\mathbf{d}_n \cdot \mathbf{n} \mathbf{d}_m \cdot \mathbf{n} - 2a \right)$$

else:

$$-\frac{1}{2} \frac{\left(\mathbf{d}_m \cdot \mathbf{n} + \mathbf{d}_n \cdot \mathbf{n} - 2b\mathbf{d}_n \cdot \mathbf{n} \mathbf{d}_m \cdot \mathbf{n} - 2a \right)}{\left(\mathbf{d}_n - \mathbf{d}_m \right) \cdot \boldsymbol{\ell}} \left(e^{ik(\mathbf{d}_n - \mathbf{d}_m) \cdot \mathbf{q}} - e^{ik(\mathbf{d}_n - \mathbf{d}_m) \cdot \mathbf{p}} \right)$$

and finally the one corresponding to ϕ^- and ψ^-

$$-\int_E \left(\left(\frac{\phi_n^-}{2} - \frac{b}{ik} \frac{\partial \phi_n^-}{\partial \mathbf{n}} \right) \overline{\frac{\partial \psi_m^+}{\partial \mathbf{n}}} - \left(\frac{1}{2} \frac{\partial \phi_n^-}{\partial \mathbf{n}} - aik\phi_n^+ \right) \overline{\psi_m^+} \right) \mathrm{d}S_x$$

that is, if $\mathbf{d}_n = \mathbf{d}_m$ then

$$\frac{ikl}{2} \left(\mathbf{d}_m \cdot \mathbf{n} + \mathbf{d}_n \cdot \mathbf{n} - 2b\mathbf{d}_n \cdot \mathbf{n} \mathbf{d}_m \cdot \mathbf{n} - 2a \right)$$

else:

$$\frac{1}{2} \frac{\left(\mathbf{d}_m \cdot \mathbf{n} + \mathbf{d}_n \cdot \mathbf{n} - 2b\mathbf{d}_n \cdot \mathbf{n} \mathbf{d}_m \cdot \mathbf{n} - 2a \right)}{\left(\mathbf{d}_n - \mathbf{d}_m \right) \cdot \boldsymbol{\ell}} \left(e^{ik(\mathbf{d}_n - \mathbf{d}_m) \cdot \mathbf{q}} - e^{ik(\mathbf{d}_n - \mathbf{d}_m) \cdot \mathbf{p}} \right)$$

If the edge is contained in the scatterer I would use exactly the same formulas but with $k = k^{\mathrm{i}}$, and if the edge is in the boundary of the scatterer I'm not sure. However that's not a problem for the first tests without scatterer.

The last term, the one on the Σ_{Left} and Σ_{Right} we use

$$\hat{u} = \mathcal{N} \left(\nabla u \cdot \mathbf{n} \right) + u^{\mathrm{inc}} - \mathcal{N} \left(\nabla u^{\mathrm{inc}} \cdot \mathbf{n} \right)$$

$$-ikd_2 \mathcal{N}^* \left(\mathcal{N} \left(\nabla u \cdot \mathbf{n} \right) - \mathcal{N} \left(\nabla u^{\mathrm{inc}} \cdot \mathbf{n} \right) - u + u^{\mathrm{inc}} \right)$$

$$ik\boldsymbol{\sigma} = \nabla u$$

$$+ikd_2 \left(\mathcal{N} \left(\nabla u \cdot \mathbf{n} \right) - \mathcal{N} \left(\nabla u^{\mathrm{inc}} \cdot \mathbf{n} \right) - u + u^{\mathrm{inc}} \right) \mathbf{n}$$

I THINK THIS SHOULD BE:

$$\hat{u} = \mathcal{N} \left(\nabla u \cdot \mathbf{n} \right) + u^{\mathrm{inc}} - \mathcal{N} \left(\nabla u^{\mathrm{inc}} \cdot \mathbf{n} \right)$$

$$+ikd_2\mathcal{N}^*\left(\mathcal{N}\left(\nabla u\cdot\mathbf{n}\right)-\mathcal{N}\left(\nabla u^{\mathrm{inc}}\cdot\mathbf{n}\right)-u+u^{\mathrm{inc}}\right)$$

$$ik\boldsymbol{\sigma}=\nabla u$$

$$+ikd_2\left(\mathcal{N}\left(\nabla u\cdot\mathbf{n}\right)-\mathcal{N}\left(\nabla u^{\mathrm{inc}}\cdot\mathbf{n}\right)-u+u^{\mathrm{inc}}\right)\mathbf{n}$$

so the term is

$$\int_E\left(\left(\mathcal{N}\left(\nabla u\cdot\mathbf{n}\right)+ikd_2\mathcal{N}^*\left(\mathcal{N}\left(\nabla u\cdot\mathbf{n}\right)-u\right)\right)\overline{\nabla v}\cdot\mathbf{n}-\overline{v}\left(\nabla u+ikd_2\left(\mathcal{N}\left(\nabla u\cdot\mathbf{n}\right)-u\right)\mathbf{n}\right)\cdot\mathbf{n}\right)\,\mathrm{d}S_x$$

that is

$$\int_E\left(\mathcal{N}\left(\nabla u\cdot\mathbf{n}\right)\overline{\nabla v}\cdot\mathbf{n}+ikd_2\mathcal{N}^*\left(\mathcal{N}\left(\nabla u\cdot\mathbf{n}\right)-u\right)\overline{\nabla v}\cdot\mathbf{n}-\overline{v}\nabla u\cdot\mathbf{n}-\overline{v}ikd_2\left(\mathcal{N}\left(\nabla u\cdot\mathbf{n}\right)-u\right)\right)\,\mathrm{d}S_x$$

now, if $E=\Sigma_{\mathrm{Left}}$ and the adjoint is defined with respect to the $L^2\left(\Sigma_{\mathrm{Left}}\right)$ scalar product:

$$\int_{\Sigma_{\mathrm{Left}}}\left(\mathcal{N}\left(\nabla u\cdot\mathbf{n}\right)\overline{\nabla v\cdot\mathbf{n}}-\overline{v}\nabla u\cdot\mathbf{n}+ikd_2\left(\mathcal{N}\left(\nabla u\cdot\mathbf{n}\right)-u\right)\overline{\mathcal{N}\left(\nabla v\cdot\mathbf{n}\right)}-\overline{v}ikd_2\left(\mathcal{N}\left(\nabla u\cdot\mathbf{n}\right)-u\right)\right)\,\mathrm{d}S_x$$

$$\int_{\Sigma_{\mathrm{Left}}}\left(\mathcal{N}\left(\nabla u\cdot\mathbf{n}\right)\overline{\nabla v\cdot\mathbf{n}}-\overline{v}\nabla u\cdot\mathbf{n}+ikd_2\left(\mathcal{N}\left(\nabla u\cdot\mathbf{n}\right)-u\right)\overline{\mathcal{N}\left(\nabla v\cdot\mathbf{n}\right)-v}\right)\,\mathrm{d}S_x$$

For ϕ_n and ψ_m that is

$$\int_{\Sigma_{\mathrm{Left}}}\left(-\mathcal{N}\left(ik\mathbf{d}_n\cdot\mathbf{n}e^{ik\mathbf{d}_n\cdot\mathbf{x}}\right)ik\mathbf{d}_m\cdot\mathbf{n}e^{-ik\mathbf{d}_m\cdot\mathbf{x}}-e^{-ik\mathbf{d}_m\cdot\mathbf{x}}ik\mathbf{d}_n\cdot\mathbf{n}e^{ik\mathbf{d}_n\cdot\mathbf{x}}+ikd_2\left(\mathcal{N}\left(ik\mathbf{d}_n\cdot\mathbf{n}e^{ik\mathbf{d}_n\cdot\mathbf{x}}\right)-e^{ik\mathbf{d}_n\cdot\mathbf{x}}\right)\right.$$

with

$$\mathcal{N}\left(f\right)=\frac{1}{i\beta_0}f_0\theta_0\left(y\right)+\sum_{s=1}^{\infty}\frac{1}{i\beta_s}f_s\theta_s\left(y\right)$$

$$\mathcal{N}\left(f\right)=\frac{1}{2ikH}\int_{\Sigma_{\mathrm{left}}}f\left(y\right)\,\mathrm{d}S_x+\sum_{s=1}^{\infty}\frac{1}{i\sqrt{k^2-\left(s\frac{\pi}{H}\right)^2}H}\left(\int_{\Sigma_{\mathrm{left}}}f\left(y\right)\cos\left(s\pi\frac{y}{H}\right)\,\mathrm{d}S_x\right)\cos\left(s\pi\frac{y}{H}\right)$$

And they are a lot of terms in:

$ik\int_{\Sigma_{\mathrm{Left}}}\left(-\mathcal{N}\left(ik\mathbf{d}_n\cdot\mathbf{n}e^{ik\mathbf{d}_n\cdot\mathbf{x}}\right)\mathbf{d}_m\cdot\mathbf{n}e^{-ik\mathbf{d}_m\cdot\mathbf{x}}-e^{-ik\mathbf{d}_m\cdot\mathbf{x}}\mathbf{d}_n\cdot\mathbf{n}e^{ik\mathbf{d}_n\cdot\mathbf{x}}+d_2\left(\mathcal{N}\left(ik\mathbf{d}_n\cdot\mathbf{n}e^{ik\mathbf{d}_n\cdot\mathbf{x}}\right)-e^{ik\mathbf{d}_n\cdot\mathbf{x}}\right)\right.$

lets compute first:

$$\mathcal{N}\left(ik\mathbf{d}_n\cdot\mathbf{n}e^{ik\mathbf{d}_n\cdot\mathbf{x}}\right)=\mathbf{d}_n\cdot\mathbf{n}ke^{-ikd_{n,x}R}\left(\frac{1}{2kH}\int_{-H}^{H}e^{ikd_{n,y}y}\,\mathrm{d}y+\sum_{s=1}^{\infty}\frac{1}{\sqrt{(kH)^2-(s\pi)^2}}\left(\int_{-H}^{H}e^{ikd_{n,y}y}\cos\left(s\pi\frac{y}{H}\right)\right.\right.$$

we have to distinguish to cases: $\mathbf{d}_n\cdot\mathbf{j}=0$ that is, $d_{n,y}=0$. Then

$$\mathcal{N}\left(ik\mathbf{d}_n\cdot\mathbf{n}e^{ik\mathbf{d}_n\cdot\mathbf{x}}\right)=\mathbf{d}_n\cdot\mathbf{n}e^{-ikd_{n,x}R}$$

if not:

$$\mathcal{N}\left(ik\mathbf{d}_n\cdot\mathbf{n}e^{ik\mathbf{d}_n\cdot\mathbf{x}}\right)=\mathbf{d}_n\cdot\mathbf{n}e^{-ikd_{n,x}R}\left(\frac{\sin\left(d_{n,y}kH\right)}{d_{n,y}kH}+\sum_{s=1}^{\infty}\frac{kH}{\sqrt{(kH)^2-(s\pi)^2}}\left(\frac{\sin\left(d_{n,y}kH+s\pi\right)}{d_{n,y}kH+s\pi}+\frac{\sin\left(d_{n,y}kH\right.}{d_{n,y}kH-}\right.\right.$$

wich can be rewritten , if wanted, as

$$\mathcal{N}\left(ik\mathbf{d}_n\cdot\mathbf{n}e^{ik\mathbf{d}_n\cdot\mathbf{x}}\right)=$$

$$=\mathbf{d}_n\cdot\mathbf{n}e^{-ikd_{n,x}R}\left(\sum_{s=-\infty}^{\infty}\frac{kH}{\sqrt{(kH)^2-(s\pi)^2}}\frac{\sin\left(d_{n,y}kH+s\pi\right)}{d_{n,y}kH+s\pi}\cos\left(s\pi\frac{y}{H}\right)\right)$$

Now:

first term:

$$-ik\int_{\Sigma_{\text{Left}}}\mathcal{N}\left(ik\mathbf{d}_n\cdot\mathbf{n}e^{ik\mathbf{d}_n\cdot\mathbf{x}}\right)\mathbf{d}_m\cdot\mathbf{n}e^{-ik\mathbf{d}_m\cdot\mathbf{x}}\,\mathrm{d}S_x$$

$$-ike^{ikd_{m,x}R}\mathbf{d}_m\cdot\mathbf{n}\int_{-H}^{H}\mathcal{N}\left(ik\mathbf{d}_n\cdot\mathbf{n}e^{ik\mathbf{d}_n\cdot\mathbf{x}}\right)e^{-ikd_{m,y}y}\,\mathrm{d}y$$

If $\mathbf{d}_n\cdot\mathbf{j}=0$ then and $\mathbf{d}_m\cdot\mathbf{j}=0$ then

$$-2ikHe^{i(d_{m,x}-d_{n,x})kR}\mathbf{d}_m\cdot\mathbf{n}\mathbf{d}_n\cdot\mathbf{n}$$

If $\mathbf{d}_n\cdot\mathbf{j}=0$ then and $\mathbf{d}_m\cdot\mathbf{j}\neq0$ then

$$-2ikHe^{i(d_{m,x}-d_{n,x})kR}\mathbf{d}_m\cdot\mathbf{n}\mathbf{d}_n\cdot\mathbf{n}\frac{\sin\left(d_{m,y}kH\right)}{d_{m,y}kH}$$

If $\mathbf{d}_n\cdot\mathbf{j}\neq0$ then and $\mathbf{d}_m\cdot\mathbf{j}=0$ then

$$-2ikHe^{i(d_{m,x}-d_{n,x})kR}\mathbf{d}_m\cdot\mathbf{n}\mathbf{d}_n\cdot\mathbf{n}\frac{\sin\left(d_{n,y}kH\right)}{d_{n,y}kH}$$

If $\mathbf{d}_n\cdot\mathbf{j}\neq0$ then and $\mathbf{d}_m\cdot\mathbf{j}\neq0$ then

$$-ikHe^{i(d_{m,x}-d_{n,x})kR}\mathbf{d}_m\cdot\mathbf{n}\mathbf{d}_n\cdot\mathbf{n}\left(2\frac{\sin\left(d_{n,y}kH\right)}{d_{n,y}kH}\frac{\sin\left(d_{m,y}kH\right)}{kHd_{m,y}}+\right.$$

$$\left.+\sum_{s=1}^{\infty}\frac{kH}{\sqrt{(kH)^2-(s\pi)^2}}\left(\frac{\sin\left(d_{n,y}kH+s\pi\right)}{d_{n,y}kH+s\pi}+\frac{\sin\left(d_{n,y}kH-s\pi\right)}{d_{n,y}kH-s\pi}\right)\left(\frac{\sin\left(s\pi+kHd_{m,y}\right)}{s\pi+kHd_{m,y}}+\frac{\sin\left(s\pi-kHd_{m,y}\right)}{s\pi-kHd_{m,y}}\right)\right)$$

and this is only the first therm, and there are like five or six like it.

Second term

$$-ik\int_{\Sigma_{\text{Left}}}e^{-ik\mathbf{d}_m\cdot\mathbf{x}}\mathbf{d}_n\cdot\mathbf{n}e^{ik\mathbf{d}_n\cdot\mathbf{x}}\,\mathrm{d}S_x$$

$$-ike^{ik(d_{m,x}-d_{n,x})R}\mathbf{d}_n\cdot\mathbf{n}\int_{-H}^{H}e^{ik(d_{n,y}-d_{m,y})y}\,\mathrm{d}y$$

if $d_{n,y}=d_{m,y}$ then

$$-ik\int_{\Sigma_{\text{Left}}}e^{-ik\mathbf{d}_m\cdot\mathbf{x}}\mathbf{d}_n\cdot\mathbf{n}e^{ik\mathbf{d}_n\cdot\mathbf{x}}\,\mathrm{d}S_x=-2ikH\mathbf{d}_n\cdot\mathbf{n}e^{ik(d_{m,x}-d_{n,x})R}$$

else

$$-ik\int_{\Sigma_{\text{Left}}}e^{-ik\mathbf{d}_m\cdot\mathbf{x}}\mathbf{d}_n\cdot\mathbf{n}e^{ik\mathbf{d}_n\cdot\mathbf{x}}\,\mathrm{d}S_x=$$

$$-2ikH\mathbf{d}_n\cdot\mathbf{n}e^{ik(d_{m,x}-d_{n,x})R}\frac{\sin\left(\left(d_{n,y}-d_{m,y}\right)kH\right)}{\left(d_{n,y}-d_{m,y}\right)kH}$$

And the next 4 terms are combinations of these ones. They are:

$$-ikd_2\int_{\Sigma_{\text{Left}}}\mathcal{N}\left(ik\mathbf{d}_n\cdot\mathbf{n}e^{ik\mathbf{d}_n\cdot\mathbf{x}}\right)e^{-ik\mathbf{d}_m\cdot\mathbf{x}}\,\mathrm{d}S_x$$

but we had already computed it, so there are 4 options:

If $\mathbf{d}_n\cdot\mathbf{j}=0$ then and $\mathbf{d}_m\cdot\mathbf{j}=0$ then

$$-2ikHe^{i(d_{m,x}-d_{n,x})kR}d_2\mathbf{d}_n\cdot\mathbf{n}$$

If $\mathbf{d}_n\cdot\mathbf{j}=0$ then and $\mathbf{d}_m\cdot\mathbf{j}\neq 0$ then

$$-2ikHe^{i(d_{m,x}-d_{n,x})kR}d_2\mathbf{d}_n\cdot\mathbf{n}\frac{\sin\left(d_{m,y}kH\right)}{d_{m,y}kH}$$

If $\mathbf{d}_n\cdot\mathbf{j}\neq 0$ then and $\mathbf{d}_m\cdot\mathbf{j}=0$ then

$$-2ikHe^{i(d_{m,x}-d_{n,x})kR}d_2\mathbf{d}_n\cdot\mathbf{n}\frac{\sin\left(d_{n,y}kH\right)}{d_{n,y}kH}$$

If $\mathbf{d}_n\cdot\mathbf{j}\neq 0$ then and $\mathbf{d}_m\cdot\mathbf{j}\neq 0$ then

$$-2ikHe^{i(d_{m,x}-d_{n,x})kR}d_2\mathbf{d}_n\cdot\mathbf{n}\frac{\sin\left(d_{n,y}kH\right)}{d_{n,y}kH}\frac{\sin\left(d_{m,y}kH\right)}{kHd_{m,y}}$$

next we have the symmetric term:

$$-d_2ik\int_{\Sigma_{\mathrm{Left}}}\left(e^{ik\mathbf{d}_n\cdot\mathbf{x}}\overline{\mathcal{N}\left(ik\mathbf{d}_m\cdot\mathbf{n}e^{ik\mathbf{d}_m\cdot\mathbf{x}}\right)}\right)\,\mathrm{d}S_x$$

wich can be expressed as:

$$\overline{d_2ik\int_{\Sigma_{\mathrm{Left}}}\left(\mathcal{N}\left(ik\mathbf{d}_m\cdot\mathbf{n}e^{ik\mathbf{d}_m\cdot\mathbf{x}}\right)e^{-ik\mathbf{d}_n\cdot\mathbf{x}}\right)\,\mathrm{d}S_x}$$

so we have 4 options again:

If $\mathbf{d}_n\cdot\mathbf{j}=0$ then and $\mathbf{d}_m\cdot\mathbf{j}=0$ then

$$-2ikHe^{i(d_{m,x}-d_{n,x})kR}d_2\mathbf{d}_m\cdot\mathbf{n}$$

If $\mathbf{d}_n\cdot\mathbf{j}=0$ then and $\mathbf{d}_m\cdot\mathbf{j}\neq 0$ then

$$-2ikHe^{i(d_{m,x}-d_{n,x})kR}d_2\mathbf{d}_m\cdot\mathbf{n}\frac{\sin\left(d_{m,y}kH\right)}{d_{m,y}kH}$$

If $\mathbf{d}_n\cdot\mathbf{j}\neq 0$ then and $\mathbf{d}_m\cdot\mathbf{j}=0$ then

$$-2ikHe^{i(d_{m,x}-d_{n,x})kR}d_2\mathbf{d}_m\cdot\mathbf{n}\frac{\sin\left(d_{n,y}kH\right)}{d_{n,y}kH}$$

If $\mathbf{d}_n\cdot\mathbf{j}\neq 0$ then and $\mathbf{d}_m\cdot\mathbf{j}\neq 0$ then

$$-2ikHe^{i(d_{m,x}-d_{n,x})kR}d_2\mathbf{d}_m\cdot\mathbf{n}\frac{\sin\left(d_{n,y}kH\right)}{d_{n,y}kH}\frac{\sin\left(d_{m,y}kH\right)}{kHd_{m,y}}$$

Then we have the term

$$ikd_2\int_{\Sigma_{\mathrm{Left}}}e^{ik\mathbf{d}_n\cdot\mathbf{x}}e^{-ik\mathbf{d}_m\cdot\mathbf{x}}\,\mathrm{d}S_x$$

which is like the second one, that is:

If $d_{n,y}=d_{m,y}$ then

$$2ikHd_2e^{ik(d_{m,x}-d_{n,x})R}$$

else

$$2ikHd_2e^{ik(d_{m,x}-d_{n,x})R}\frac{\sin\left(\left(d_{n,y}-d_{m,y}\right)kH\right)}{\left(d_{n,y}-d_{m,y}\right)kH}$$

Finally, the last term in this contribution is:

$$ikd_2 \int_{\Sigma_{\text{Left}}} \left(\mathcal{N}\left(ik\mathbf{d}_n \cdot \mathbf{n} e^{ik\mathbf{d}_n \cdot \mathbf{x}}\right) \overline{\mathcal{N}\left(ik\mathbf{d}_m \cdot \mathbf{n} e^{ik\mathbf{d}_m \cdot \mathbf{x}}\right)} \right) \mathrm{d}S_x$$

which is a "new" term. Luckily both functions on the integrand are expressed in an orthogonal basis:

If $\mathbf{d}_n \cdot \mathbf{j} = 0$ then and $\mathbf{d}_m \cdot \mathbf{j} = 0$ then

$$2ikHd_2\mathbf{d}_n \cdot \mathbf{n}\mathbf{d}_m \cdot \mathbf{n} e^{ik(d_{m,x}-d_{n,x})R}$$

If $\mathbf{d}_n \cdot \mathbf{j} = 0$ then and $\mathbf{d}_m \cdot \mathbf{j} \neq 0$ then

$$2ikHd_2\mathbf{d}_n \cdot \mathbf{n}\mathbf{d}_m \cdot \mathbf{n} e^{ik(d_{m,x}-d_{n,x})R} \frac{\sin\left(d_{m,y}kH\right)}{d_{m,y}kH}$$

If $\mathbf{d}_n \cdot \mathbf{j} \neq 0$ then and $\mathbf{d}_m \cdot \mathbf{j} = 0$ then

$$2ikHd_2\mathbf{d}_m \cdot \mathbf{n}\mathbf{d}_n \cdot \mathbf{n} e^{ik(d_{m,x}-d_{n,x})R} \frac{\sin\left(d_{n,y}kH\right)}{d_{n,y}kH}$$

If $\mathbf{d}_n \cdot \mathbf{j} \neq 0$ then and $\mathbf{d}_m \cdot \mathbf{j} \neq 0$ then

$$ikHd_2\mathbf{d}_m \cdot \mathbf{n}\mathbf{d}_n \cdot \mathbf{n} e^{ik(d_{m,x}-d_{n,x})R} \left(2\frac{\sin\left(d_{n,y}kH\right)}{d_{n,y}kH} \frac{\sin\left(d_{m,y}kH\right)}{d_{m,y}kH} + \right.$$

$$\left. + \sum_{s=1}^{\infty} \frac{(kH)^2}{(kH)^2 - (s\pi)^2} \left(\frac{\sin\left(d_{n,y}kH + s\pi\right)}{d_{n,y}kH + s\pi} + \frac{\sin\left(d_{n,y}kH - s\pi\right)}{d_{n,y}kH - s\pi} \right) \left(\frac{\sin\left(d_{m,y}kH + s\pi\right)}{d_{m,y}kH + s\pi} + \frac{\sin\left(d_{m,y}kH - s\pi\right)}{d_{m,y}kH - s\pi} \right) \right)$$

Lets try to agroup terms in terms like the first and like the second one.

First-like terms:

If $\mathbf{d}_n \cdot \mathbf{j} = 0$ then and $\mathbf{d}_m \cdot \mathbf{j} = 0$ then

$$-2ikHe^{i(d_{m,x}-d_{n,x})kR} \left(\mathbf{d}_m \cdot \mathbf{n}\mathbf{d}_n \cdot \mathbf{n} + d_2\left(\mathbf{d}_n \cdot \mathbf{n} + \mathbf{d}_m \cdot \mathbf{n} - \mathbf{d}_n \cdot \mathbf{n}\mathbf{d}_m \cdot \mathbf{n}\right)\right)$$

If $\mathbf{d}_n \cdot \mathbf{j} = 0$ then and $\mathbf{d}_m \cdot \mathbf{j} \neq 0$ then

$$-2ikHe^{i(d_{m,x}-d_{n,x})kR} \left(\mathbf{d}_m \cdot \mathbf{n}\mathbf{d}_n \cdot \mathbf{n} + d_2\left(\mathbf{d}_n \cdot \mathbf{n} + \mathbf{d}_m \cdot \mathbf{n} - \mathbf{d}_n \cdot \mathbf{n}\mathbf{d}_m \cdot \mathbf{n}\right)\right) \frac{\sin\left(d_{m,y}kH\right)}{d_{m,y}kH}$$

If $\mathbf{d}_n \cdot \mathbf{j} \neq 0$ then and $\mathbf{d}_m \cdot \mathbf{j} = 0$ then

$$-2ikHe^{i(d_{m,x}-d_{n,x})kR} \left(\mathbf{d}_m \cdot \mathbf{n}\mathbf{d}_n \cdot \mathbf{n} + d_2\left(\mathbf{d}_n \cdot \mathbf{n} + \mathbf{d}_m \cdot \mathbf{n} - \mathbf{d}_n \cdot \mathbf{n}\mathbf{d}_m \cdot \mathbf{n}\right)\right) \frac{\sin\left(d_{n,y}kH\right)}{d_{n,y}kH}$$

If $\mathbf{d}_n \cdot \mathbf{j} \neq 0$ then and $\mathbf{d}_m \cdot \mathbf{j} \neq 0$ then

$$-2ikHe^{i(d_{m,x}-d_{n,x})kR}\left((\mathbf{d}_m\cdot\mathbf{n}\mathbf{d}_n\cdot\mathbf{n}+d_2\left(\mathbf{d}_n\cdot\mathbf{n}+\mathbf{d}_m\cdot\mathbf{n}-\mathbf{d}_n\cdot\mathbf{n}\mathbf{d}_m\cdot\mathbf{n}\right)\right)\frac{\sin\left(d_{n,y}kH\right)}{d_{n,y}kH}\frac{\sin\left(d_{m,y}kH\right)}{kHd_{m,y}}$$

$$+\frac{(1-d_2)}{2}\mathbf{d}_m\cdot\mathbf{n}\mathbf{d}_n\cdot\mathbf{n}\sum_{s=1}^{\infty}\frac{kH}{\sqrt{\left(kH\right)^2-\left(s\pi\right)^2}}\left(\frac{\sin\left(d_{n,y}kH+s\pi\right)}{d_{n,y}kH+s\pi}+\frac{\sin\left(d_{n,y}kH-s\pi\right)}{d_{n,y}kH-s\pi}\right)\left(\frac{\sin\left(s\pi+kHd_{m,y}\right)}{s\pi+kHd_{m,y}}+\right.$$

and the second-like terms:

if $d_{n,y}=d_{m,y}$ then

$$-2ikH\left(\mathbf{d}_n\cdot\mathbf{n}-d_2\right)e^{ik(d_{m,x}-d_{n,x})R}$$

else

$$-2ikH\left(\mathbf{d}_n\cdot\mathbf{n}-d_2\right)e^{ik(d_{m,x}-d_{n,x})R}\frac{\sin\left(\left(d_{n,y}-d_{m,y}\right)kH\right)}{\left(d_{n,y}-d_{m,y}\right)kH}$$

The terms on Σ_{Right} are the same but changing $-R$ for R. (in fact they should be written as a single expresion depending on x, with d_n going first)

==================================

The b term should be:

$$\int_{\Sigma_{\text{Left}}}\left(\left(\mathcal{N}\left(\nabla u^{\text{inc}}\cdot\mathbf{n}\right)-u^{\text{inc}}+ikd_2\mathcal{N}^*\left(\mathcal{N}\left(\nabla u^{\text{inc}}\cdot\mathbf{n}\right)-u^{\text{inc}}\right)\right)\overline{\nabla v}\cdot\mathbf{n}-\overline{v}\left(\mathcal{N}\left(\nabla u^{\text{inc}}\cdot\mathbf{n}\right)-u^{\text{inc}}\mathbf{n}\right)\cdot\mathbf{n}\right)\,\mathrm{d}S_x$$

that is

$$\int_{\Sigma_{\text{Left}}}\left(\left(\mathcal{N}\left(\nabla u^{\text{inc}}\cdot\mathbf{n}\right)-u^{\text{inc}}+ikd_2\left(\mathcal{N}\left(\nabla u^{\text{inc}}\cdot\mathbf{n}\right)-u^{\text{inc}}\right)\right)\overline{\mathcal{N}\left(\nabla v\cdot\mathbf{n}\right)-v}\right)\,\mathrm{d}S_x$$

For a g_t^+ incident field we can compute closed forms:

$$u_t^{\text{inc}}=e^{i\sqrt{k^2-\left(t\frac{\pi}{H}\right)^2}x}\cos\left(t\pi\frac{y}{H}\right)$$

The term for ψ_m is then (first on Σ_{Right})

$$\int_{\Sigma_{\text{Right}}}\left((0+ikd_2 0)\overline{\mathcal{N}\left(\nabla v\cdot\mathbf{n}\right)-v}\right)\,\mathrm{d}S_x=0$$

because g_t^+ functions are outgoing radiating functions for Σ_{Right}, that is

$$g_t^+=\mathcal{N}\left(\nabla g_t^+\cdot\mathbf{n}\right)\quad\text{on }\Sigma_{\text{Right}}$$

we can check it if you dont believe me:

$$\mathcal{N}\left(\nabla g_t^+ \cdot \mathbf{i}\right) = \mathcal{N}\left(i\sqrt{k^2-\left(t\frac{\pi}{H}\right)^2}g_t^+\right) =$$

$$= \frac{i\sqrt{k^2-\left(t\frac{\pi}{H}\right)^2}e^{i\sqrt{k^2-\left(t\frac{\pi}{H}\right)^2}R}}{ik2H}\int_{-H}^{H}\cos\left(t\pi\frac{y}{H}\right)\,\mathrm{d}y + i\sqrt{k^2-\left(t\frac{\pi}{H}\right)^2}e^{i\sqrt{k^2-\left(t\frac{\pi}{H}\right)^2}R}\sum_{s=1}^{\infty}\frac{1}{i\sqrt{k^2-\left(s\frac{\pi}{H}\right)^2}}\left(\int_{-H}^{H}\cos\right.$$

$$e^{i\sqrt{k^2-\left(t\frac{\pi}{H}\right)^2}R}\cos\left(t\pi\frac{y}{H}\right) = g_t^+\big|_{x=R}$$

on the other hand, on Σ_{Left}

$$\mathcal{N}\left(-\nabla g_t^+ \cdot \mathbf{i}\right) = -\mathcal{N}\left(i\sqrt{k^2-\left(t\frac{\pi}{H}\right)^2}g_t^+\right) = -\,g_t^+\big|_{x=-R}$$

(lets check it...)

$$-\nabla g_t^+ \cdot \mathbf{i}\big|_{x=-R} = -i\sqrt{k^2-\left(t\frac{\pi}{H}\right)^2}e^{-i\sqrt{k^2-\left(t\frac{\pi}{H}\right)^2}R}\cos\left(t\pi\frac{y}{H}\right)$$

$$\mathcal{N}\left(-\nabla g_t^+ \cdot \mathbf{i}\right) =$$

$$= \frac{-i\sqrt{k^2-\left(t\frac{\pi}{H}\right)^2}e^{-i\sqrt{k^2-\left(t\frac{\pi}{H}\right)^2}R}}{k2H}\int_{-H}^{H}\cos\left(t\pi\frac{y}{H}\right)\,\mathrm{d}y + \frac{-i\sqrt{k^2-\left(t\frac{\pi}{H}\right)^2}e^{-i\sqrt{k^2-\left(t\frac{\pi}{H}\right)^2}R}}{H}\sum_{s=1}^{\infty}\frac{1}{i\sqrt{k^2-\left(s\frac{\pi}{H}\right)^2}}\left(\int\right.$$

$$= 0 - e^{-i\sqrt{k^2-\left(t\frac{\pi}{H}\right)^2}R}\cos\left(t\pi\frac{y}{H}\right) = -\,g_t^+\big|_{x=-R}$$

So now, the Σ_{Left} term would be:

$$-2\left(1+ikd_2\right)e^{-i\sqrt{k^2-\left(t\frac{\pi}{H}\right)^2}R}e^{ik(d_{m,x}R)}\int_{-H}^{H}\left(\cos\left(t\pi\frac{y}{H}\right)\overline{\mathcal{N}\left(-ikd_{m,x}e^{ikd_{m,y}y}\right)-e^{ikd_{m,y}y}}\right)\,\mathrm{d}y$$

Lets compute both terms separately

$$\int_{-H}^{H}\left(\cos\left(t\pi\frac{y}{H}\right)\overline{-e^{ikd_{m,y}y}}\right)\,\mathrm{d}y = -\int_{-H}^{H}\left(\cos\left(t\pi\frac{y}{H}\right)e^{-ikd_{m,y}y}\right)\,\mathrm{d}y$$

if $\mathbf{d}_m \cdot \mathbf{j} = 0$ then

$$= -\int_{-H}^{H}\cos\left(t\pi\frac{y}{H}\right)\,\mathrm{d}y = \begin{cases}2H & t=0\\ 0 & t>0\end{cases}$$

else

$$= -\frac{1}{2}\int_{-H}^{H}\left(e^{i(t\pi - kHd_{m,y})\frac{y}{H}} + e^{-i(t\pi + kHd_{m,y})\frac{y}{H}}\right)\,\mathrm{d}y$$

$$= -\left(\frac{e^{i(t\pi - kHd_{m,y})} - e^{-i(t\pi - kHd_{m,y})}}{2i\left(t\pi - kHd_{m,y}\right)\frac{1}{H}} + \frac{e^{i(t\pi + kHd_{m,y})} - e^{-i(t\pi + kHd_{m,y})}}{2i\left(t\pi + kHd_{m,y}\right)\frac{1}{H}}\right)$$

$$= -H\left(\frac{\sin\left(t\pi - kHd_{m,y}\right)}{t\pi - kHd_{m,y}} + \frac{\sin\left(t\pi + kHd_{m,y}\right)}{t\pi + kHd_{m,y}}\right)$$

the other term is:

$$\int_{-H}^{H}\left(\cos\left(t\pi\frac{y}{H}\right)\overline{\mathcal{N}\left(-ikd_{m,x}e^{ikd_{m,y}y}\right)}\right)\,\mathrm{d}y =$$

$$d_{m,x}\int_{-H}^{H}\left(\cos\left(t\pi\frac{y}{H}\right)ik\overline{\mathcal{N}\left(e^{ikd_{m,y}y}\right)}\right)\,\mathrm{d}y =$$

but

$$ik\overline{\mathcal{N}\left(e^{ikd_{m,y}y}\right)} = -\frac{1}{2H}\int_{-H}^{H}e^{-ikd_{m,y}y}\,\mathrm{d}y - \sum_{s=1}^{\infty}\frac{k}{\sqrt{(kH)^2 - (s\pi)^2}}\left(\int_{-H}^{H}e^{-ikd_{m,y}y}\cos\left(s\pi\frac{y}{H}\right)\,\mathrm{d}y\right)\cos\left(s\pi\frac{y}{H}\right)$$

if $\mathbf{d}_m \cdot \mathbf{j} = 0$ then

$$ik\overline{\mathcal{N}\left(e^{ikd_{m,y}y}\right)} = -1$$

else

$$ik\overline{\mathcal{N}\left(e^{ikd_{m,y}y}\right)} = -\frac{\sin\left(d_{m,y}kH\right)}{d_{m,y}kH} - \sum_{s=1}^{\infty}\frac{kH}{\sqrt{(kH)^2 - (s\pi)^2}}\left(\frac{\sin\left(s\pi - kHd_{m,y}\right)}{s\pi - kHd_{m,y}} + \frac{\sin\left(s\pi + kHd_{m,y}\right)}{s\pi + kHd_{m,y}}\right)\cos\left(s\pi\frac{y}{H}\right)$$

so finally, the term:

$$d_{m,x}\int_{-H}^{H}\left(\cos\left(t\pi\frac{y}{H}\right)ik\overline{\mathcal{N}\left(e^{ikd_{m,y}y}\right)}\right)\,\mathrm{d}y$$

if $\mathbf{d}_m \cdot \mathbf{j} = 0$ then

$$-d_{m,x}\int_{-H}^{H}\cos\left(t\pi\frac{y}{H}\right)\,\mathrm{d}y = \begin{cases} -d_{m,x}2H & t = 0 \\ 0 & t > 0 \end{cases}$$

else

$$d_{m,x}\int_{-H}^{H}\left(\cos\left(t\pi\frac{y}{H}\right)ik\overline{\mathcal{N}\left(e^{ikd_{m,y}y}\right)}\right)\,\mathrm{d}y=\begin{cases}-d_{m,x}2H\frac{\sin(d_{m,y}kH)}{d_{m,y}kH} & t=0\\ -d_{m,x}H\frac{kH}{\sqrt{(kH)^2-(t\pi)^2}}\left(\frac{\sin(t\pi-kHd_{m,y})}{t\pi-kHd_{m,y}}+\frac{\sin(t\pi+kHd_{m,y})}{t\pi+kHd_{m,y}}\right) & t>0\end{cases}$$

and FINALLY the b term is:

$$-2\left(1+ikd_2\right)e^{-i\sqrt{k^2-\left(t\frac{\pi}{H}\right)^2}R}e^{ik(d_{m,x}R)}\int_{-H}^{H}\left(\cos\left(t\pi\frac{y}{H}\right)\overline{\mathcal{N}\left(-ikd_{m,x}e^{ikd_{m,y}y}\right)-e^{ikd_{m,y}y}}\right)\,\mathrm{d}y$$

if $\mathbf{d}_m\cdot\mathbf{j}=0$ then
$t=0$

$$-2\left(1+ikd_2\right)e^{-i\sqrt{k^2-\left(t\frac{\pi}{H}\right)^2}R}e^{ik(d_{m,x}R)}\left(1-d_{m,x}\right)2H$$

$t>0$

$$=-\int_{-H}^{H}\cos\left(t\pi\frac{y}{H}\right)\,\mathrm{d}y=\begin{cases}2H & t=0\\ 0 & t>0\end{cases}$$

else

$$=-\frac{1}{2}\int_{-H}^{H}\left(e^{i(t\pi-kHd_{m,y})\frac{y}{H}}+e^{-i(t\pi+kHd_{m,y})\frac{y}{H}}\right)\,\mathrm{d}y$$

$$=-\left(\frac{e^{i(t\pi-kHd_{m,y})}-e^{-i(t\pi-kHd_{m,y})}}{2i\left(t\pi-kHd_{m,y}\right)\frac{1}{H}}+\frac{e^{i(t\pi+kHd_{m,y})}-e^{-i(t\pi+kHd_{m,y})}}{2i\left(t\pi+kHd_{m,y}\right)\frac{1}{H}}\right)$$

$$=-H\left(\frac{\sin\left(t\pi-kHd_{m,y}\right)}{t\pi-kHd_{m,y}}+\frac{\sin\left(t\pi+kHd_{m,y}\right)}{t\pi+kHd_{m,y}}\right)$$

if $\mathbf{d}_m\cdot\mathbf{j}=0$ then

$$-d_{m,x}\int_{-H}^{H}\cos\left(t\pi\frac{y}{H}\right)\,\mathrm{d}y=\begin{cases}-d_{m,x}2H & t=0\\ 0 & t>0\end{cases}$$

else

$$d_{m,x}\int_{-H}^{H}\left(\cos\left(t\pi\frac{y}{H}\right)ik\overline{\mathcal{N}\left(e^{ikd_{m,y}y}\right)}\right)\,\mathrm{d}y=\begin{cases}-d_{m,x}2H\frac{\sin(d_{m,y}kH)}{d_{m,y}kH} & t=0\\ -d_{m,x}H\frac{kH}{\sqrt{(kH)^2-(t\pi)^2}}\left(\frac{\sin(t\pi-kHd_{m,y})}{t\pi-kHd_{m,y}}+\frac{\sin(t\pi+kHd_{m,y})}{t\pi+kHd_{m,y}}\right) & t>0\end{cases}$$