Non-naive version

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$$\sum_{E\in\mathcal{E}}\int_E \left(\hat{u}\left[\left[\overline{\nabla v}\right]\right]_{\mathbf{n}} - ik\boldsymbol{\sigma}\cdot[[\overline{v}]]_{\mathbf{n}}\right)\mathrm{d}S_x = 0 \quad \forall v\in V_h\left(\mathcal{T}\right)$$

Insead of approximating u at the edge by $\{\{u\}\}$ we will use instead:
if we are at Γ:

$$\hat{u}\to u^+ + \frac{d_1}{ik}\nabla u_{\mathbf{n}}^+$$

$$ik\boldsymbol{\sigma}\to 0$$

So the term is now:

$$\int_E\left(u^+ + \frac{d_1}{ik}\frac{\partial u}{\partial\mathbf{n}}^+\right)\overline{\frac{\partial v}{\partial\mathbf{n}}}^+\,\mathrm{d}S_x$$

that is

$$-\int_E\left(e^{ik\mathbf{d}_n\cdot\mathbf{x}} + d_1e^{ik\mathbf{d}_n\cdot\mathbf{x}}\mathbf{d}_n\cdot\mathbf{n}\right)e^{-ik\mathbf{d}_m\cdot\mathbf{x}}ik\mathbf{d}_m\cdot\mathbf{n}\,\mathrm{d}S_x$$

$$-\left(1+d_1\mathbf{d}_n\cdot\mathbf{n}\right)ik\mathbf{d}_m\cdot\mathbf{n}\int_E e^{ik(\mathbf{d}_n-\mathbf{d}_m)\cdot\mathbf{x}}\,\mathrm{d}S_x$$

$$-\left(1+d_1\mathbf{d}_n\cdot\mathbf{n}\right)ik\mathbf{d}_m\cdot\mathbf{n}\int_0^1 e^{ik(\mathbf{d}_n-\mathbf{d}_m)\cdot(\mathbf{p}+t(\mathbf{q}-\mathbf{p}))}l\,\mathrm{d}t$$

$$-\left(1+d_1\mathbf{d}_n\cdot\mathbf{n}\right)ik\mathbf{d}_m\cdot\mathbf{n}le^{ik(\mathbf{d}_n-\mathbf{d}_m)\cdot\mathbf{p}}\int_0^1 e^{ik(\mathbf{d}_n-\mathbf{d}_m)\cdot(\mathbf{q}-\mathbf{p})t}\,\mathrm{d}t$$

If $\mathbf{d}_n=\mathbf{d}_m$ then

$$-ikl\left(1+d_1\mathbf{d}_n\cdot\mathbf{n}\right)\mathbf{d}_m\cdot\mathbf{n}$$

else

$$-\left(1+d_1\mathbf{d}_n\cdot\mathbf{n}\right)ik\mathbf{d}_m\cdot\mathbf{n}le^{ik(\mathbf{d}_n-\mathbf{d}_m)\cdot\mathbf{p}}\frac{1}{ik\left(\mathbf{d}_n-\mathbf{d}_m\right)\cdot(\mathbf{q}-\mathbf{p})}\left(e^{ik(\mathbf{d}_n-\mathbf{d}_m)\cdot(\mathbf{q}-\mathbf{p})}-1\right)$$

$$-\frac{(1+d_1\mathbf{d}_n\cdot\mathbf{n})\,\mathbf{d}_m\cdot\mathbf{n}}{(\mathbf{d}_n-\mathbf{d}_m)\cdot\boldsymbol{\ell}}\left(e^{ik(\mathbf{d}_n-\mathbf{d}_m)\cdot\mathbf{q}}-e^{ik(\mathbf{d}_n-\mathbf{d}_m)\cdot\mathbf{p}}\right)$$

If we are in an Inner edge:

$$u\to\{\{u\}\}+\frac{b}{ik}\left[[\nabla u]\right]_{\mathbf{n}}$$

$$ik\boldsymbol{\sigma}\to\{\{\nabla u\}\}+aik\left[[u]\right]_{\mathbf{n}}$$

and the term is:

$$\int_E\left(\left(\{\{u\}\}+\frac{b}{ik}\left[[\nabla u]\right]_{\mathbf{n}}\right)\left[\left[\overline{\nabla v}\right]\right]_{\mathbf{n}}-\left(\{\{\nabla u\}\}+aik\left[[u]\right]_{\mathbf{n}}\right)\cdot\left[[\overline{v}]\right]_{\mathbf{n}}\right)\,\mathrm{d}S_x$$

This concerns test functions and trial functions in two different triangles. Lets compute the term for ϕ_n^+ and ψ_m^+, i.e. both on the same triangle upstream of $\mathbf{n}$.

$$\int_E\left(\left(\frac{\phi_n^+}{2}+\frac{b}{ik}\frac{\partial\phi_n^+}{\partial\mathbf{n}}\right)\overline{\frac{\partial\psi_m^+}{\partial\mathbf{n}}}-\left(\frac{1}{2}\frac{\partial\phi_n^+}{\partial\mathbf{n}}+aik\phi_n^+\right)\overline{\psi_m^+}\right)\,\mathrm{d}S_x$$

Lets assume that we are in an edge wich is completely contained in the background:

$$\frac{-ik}{2}\left(\mathbf{d}_m\cdot\mathbf{n}+\mathbf{d}_n\cdot\mathbf{n}+2b\mathbf{d}_n\cdot\mathbf{n}\mathbf{d}_m\cdot\mathbf{n}+2a\right)\int_E\left(e^{ik(\mathbf{d}_n-\mathbf{d}_m)\cdot\mathbf{x}}\right)\,\mathrm{d}S_x$$

if $\mathbf{d}_m=\mathbf{d}_n$ then (actually no, if $\mathbf{d}_n\cdot\boldsymbol{\ell}=\mathbf{d}_{\mathbf{n}}\cdot\boldsymbol{\ell}$)

$$-\frac{ikl}{2}\left(\mathbf{d}_m\cdot\mathbf{n}+\mathbf{d}_n\cdot\mathbf{n}+2b\mathbf{d}_n\cdot\mathbf{n}\mathbf{d}_m\cdot\mathbf{n}+2a\right)$$

else:

$$-\frac{1}{2}\frac{\left(\mathbf{d}_m\cdot\mathbf{n}+\mathbf{d}_n\cdot\mathbf{n}+2b\mathbf{d}_n\cdot\mathbf{n}\mathbf{d}_m\cdot\mathbf{n}+2a\right)}{(\mathbf{d}_n-\mathbf{d}_m)\cdot\boldsymbol{\ell}}\left(e^{ik(\mathbf{d}_n-\mathbf{d}_m)\cdot\mathbf{q}}-e^{ik(\mathbf{d}_n-\mathbf{d}_m)\cdot\mathbf{p}}\right)$$

the other 3 types of terms, are the corresponding to ϕ_n^+ and ψ^-:

$$-\int_E\left(\left(\frac{\phi_n^+}{2}+\frac{b}{ik}\frac{\partial\phi_n^+}{\partial\mathbf{n}}\right)\overline{\frac{\partial\psi_m^-}{\partial\mathbf{n}}}-\left(\frac{1}{2}\frac{\partial\phi_n^+}{\partial\mathbf{n}}+aik\phi_n^+\right)\overline{\psi_m^-}\right)\,\mathrm{d}S_x$$

that is, if $\mathbf{d}_n=\mathbf{d}_m$ then

$$\frac{ikl}{2}\left(\mathbf{d}_m\cdot\mathbf{n}+\mathbf{d}_n\cdot\mathbf{n}+2b\mathbf{d}_n\cdot\mathbf{n}\mathbf{d}_m\cdot\mathbf{n}+2a\right)$$

else:

$$\frac{1}{2}\frac{(\mathbf{d}_m\cdot\mathbf{n}+\mathbf{d}_n\cdot\mathbf{n}+2b\mathbf{d}_n\cdot\mathbf{n}\mathbf{d}_m\cdot\mathbf{n}+2a)}{(\mathbf{d}_n-\mathbf{d}_m)\cdot\boldsymbol{\ell}}\left(e^{ik(\mathbf{d}_n-\mathbf{d}_m)\cdot\mathbf{q}}-e^{ik(\mathbf{d}_n-\mathbf{d}_m)\cdot\mathbf{p}}\right)$$

the one corresponding to ϕ^- and ψ^+

$$\int_E\left(\left(\frac{\phi_n^-}{2}-\frac{b}{ik}\frac{\partial\phi_n^-}{\partial\mathbf{n}}\right)\overline{\frac{\partial\psi_m^+}{\partial\mathbf{n}}}-\left(\frac{1}{2}\frac{\partial\phi_n^-}{\partial\mathbf{n}}-aik\phi_n^+\right)\overline{\psi_m^+}\right)\mathrm{d}S_x$$

that is, if $\mathbf{d}_n=\mathbf{d}_m$ then

$$-\frac{ikl}{2}\left(\mathbf{d}_m\cdot\mathbf{n}+\mathbf{d}_n\cdot\mathbf{n}-2b\mathbf{d}_n\cdot\mathbf{n}\mathbf{d}_m\cdot\mathbf{n}-2a\right)$$

else:

$$-\frac{1}{2}\frac{(\mathbf{d}_m\cdot\mathbf{n}+\mathbf{d}_n\cdot\mathbf{n}-2b\mathbf{d}_n\cdot\mathbf{n}\mathbf{d}_m\cdot\mathbf{n}-2a)}{(\mathbf{d}_n-\mathbf{d}_m)\cdot\boldsymbol{\ell}}\left(e^{ik(\mathbf{d}_n-\mathbf{d}_m)\cdot\mathbf{q}}-e^{ik(\mathbf{d}_n-\mathbf{d}_m)\cdot\mathbf{p}}\right)$$

and finally the one corresponding to ϕ^- and ψ^-

$$-\int_E\left(\left(\frac{\phi_n^-}{2}-\frac{b}{ik}\frac{\partial\phi_n^-}{\partial\mathbf{n}}\right)\overline{\frac{\partial\psi_m^+}{\partial\mathbf{n}}}-\left(\frac{1}{2}\frac{\partial\phi_n^-}{\partial\mathbf{n}}-aik\phi_n^+\right)\overline{\psi_m^+}\right)\mathrm{d}S_x$$

that is, if $\mathbf{d}_n=\mathbf{d}_m$ then

$$\frac{ikl}{2}\left(\mathbf{d}_m\cdot\mathbf{n}+\mathbf{d}_n\cdot\mathbf{n}-2b\mathbf{d}_n\cdot\mathbf{n}\mathbf{d}_m\cdot\mathbf{n}-2a\right)$$

else:

$$\frac{1}{2}\frac{(\mathbf{d}_m\cdot\mathbf{n}+\mathbf{d}_n\cdot\mathbf{n}-2b\mathbf{d}_n\cdot\mathbf{n}\mathbf{d}_m\cdot\mathbf{n}-2a)}{(\mathbf{d}_n-\mathbf{d}_m)\cdot\boldsymbol{\ell}}\left(e^{ik(\mathbf{d}_n-\mathbf{d}_m)\cdot\mathbf{q}}-e^{ik(\mathbf{d}_n-\mathbf{d}_m)\cdot\mathbf{p}}\right)$$

If the edge is contained in the scatterer I would use exactly the same formulas but with $k=k^{\mathrm{i}}$, and if the edge is in the boundary of the scatterer I'm not sure. However that's not a problem for the first tests without scatterer.

The last term, the one on the Σ_{Left} and Σ_{Right} we use

$$\hat{u}=\mathcal{N}\left(\nabla u\cdot\mathbf{n}\right)+u^{\mathrm{inc}}-\mathcal{N}\left(\nabla u^{\mathrm{inc}}\cdot\mathbf{n}\right)$$

$$-ikd_2\mathcal{N}^*\left(\mathcal{N}\left(\nabla u\cdot\mathbf{n}\right)-\mathcal{N}\left(\nabla u^{\mathrm{inc}}\cdot\mathbf{n}\right)-u+u^{\mathrm{inc}}\right)$$

$$ik\boldsymbol{\sigma}=\nabla u$$

$$+ikd_2\left(\mathcal{N}\left(\nabla u\cdot\mathbf{n}\right)-\mathcal{N}\left(\nabla u^{\mathrm{inc}}\cdot\mathbf{n}\right)-u+u^{\mathrm{inc}}\right)\mathbf{n}$$

I THINK THIS SHOULD BE:

$$\hat{u} = \mathcal{N}\left(\nabla u \cdot \mathbf{n}\right) + u^{\mathrm{inc}} - \mathcal{N}\left(\nabla u^{\mathrm{inc}} \cdot \mathbf{n}\right)$$

$$+ikd_2 \mathcal{N}^* \left(\mathcal{N}\left(\nabla u \cdot \mathbf{n}\right) - \mathcal{N}\left(\nabla u^{\mathrm{inc}} \cdot \mathbf{n}\right) - u + u^{\mathrm{inc}}\right)$$

$$ik\boldsymbol{\sigma} = \nabla u$$

$$+ikd_2 \left(\mathcal{N}\left(\nabla u \cdot \mathbf{n}\right) - \mathcal{N}\left(\nabla u^{\mathrm{inc}} \cdot \mathbf{n}\right) - u + u^{\mathrm{inc}}\right)\mathbf{n}$$

so the term is

$$\int_E \left(\left(\mathcal{N}\left(\nabla u \cdot \mathbf{n}\right) + ikd_2 \mathcal{N}^* \left(\mathcal{N}\left(\nabla u \cdot \mathbf{n}\right) - u\right)\right) \overline{\nabla v} \cdot \mathbf{n} - \overline{v}\left(\nabla u + ikd_2\left(\mathcal{N}\left(\nabla u \cdot \mathbf{n}\right) - u\right)\mathbf{n}\right) \cdot \mathbf{n}\right) \mathrm{d}S_x$$

that is

$$\int_E \left(\mathcal{N}\left(\nabla u \cdot \mathbf{n}\right) \overline{\nabla v} \cdot \mathbf{n} + ikd_2 \mathcal{N}^* \left(\mathcal{N}\left(\nabla u \cdot \mathbf{n}\right) - u\right) \overline{\nabla v} \cdot \mathbf{n} - \overline{v} \nabla u \cdot \mathbf{n} - \overline{v} ikd_2 \left(\mathcal{N}\left(\nabla u \cdot \mathbf{n}\right) - u\right)\right) \mathrm{d}S_x$$

now, if $E = \Sigma_{\mathrm{Left}}$ and the adjoint is defined with respect to the $L^2\left(\Sigma_{\mathrm{Left}}\right)$ scalar product:

$$\int_{\Sigma_{\mathrm{Left}}} \left(\mathcal{N}\left(\nabla u \cdot \mathbf{n}\right) \overline{\nabla v \cdot \mathbf{n}} - \overline{v} \nabla u \cdot \mathbf{n} + ikd_2 \left(\mathcal{N}\left(\nabla u \cdot \mathbf{n}\right) - u\right) \overline{\mathcal{N}\left(\nabla v \cdot \mathbf{n}\right)} - \overline{v} ikd_2 \left(\mathcal{N}\left(\nabla u \cdot \mathbf{n}\right) - u\right)\right) \mathrm{d}S_x$$

$$\int_{\Sigma_{\mathrm{Left}}} \left(\mathcal{N}\left(\nabla u \cdot \mathbf{n}\right) \overline{\nabla v \cdot \mathbf{n}} - \overline{v} \nabla u \cdot \mathbf{n} + ikd_2 \left(\mathcal{N}\left(\nabla u \cdot \mathbf{n}\right) - u\right) \overline{\mathcal{N}\left(\nabla v \cdot \mathbf{n}\right) - v}\right) \mathrm{d}S_x$$

For ϕ_n and ψ_m that is

$$\int_{\Sigma_{\mathrm{Left}}} \left(-\mathcal{N}\left(ik\mathbf{d}_n \cdot \mathbf{n} e^{ik\mathbf{d}_n \cdot \mathbf{x}}\right) ik\mathbf{d}_m \cdot \mathbf{n} e^{-ik\mathbf{d}_m \cdot \mathbf{x}} - e^{-ik\mathbf{d}_m \cdot \mathbf{x}} ik\mathbf{d}_n \cdot \mathbf{n} e^{ik\mathbf{d}_n \cdot \mathbf{x}} + ikd_2 \left(\mathcal{N}\left(ik\mathbf{d}_n \cdot \mathbf{n} e^{ik\mathbf{d}_n \cdot \mathbf{x}}\right) - e^{ik\mathbf{d}}\right.\right.$$

with

$$\mathcal{N}(f) = \frac{1}{i\beta_0} f_0 \theta_0(y) + \sum_{s=1}^{\infty} \frac{1}{i\beta_s} f_s \theta_s(y)$$

$$\mathcal{N}(f) = \frac{1}{2ikH} \int_{\Sigma_{\mathrm{left}}} f(y)\, \mathrm{d}S_x + \sum_{s=1}^{\infty} \frac{1}{i\sqrt{k^2 - \left(s\frac{\pi}{H}\right)^2} H} \left(\int_{\Sigma_{\mathrm{left}}} f(y) \cos\left(s\pi \frac{y}{H}\right) \mathrm{d}S_x\right) \cos\left(s\pi \frac{y}{H}\right)$$

And they are a lot of terms in:

$ik \int_{\Sigma_{\text{Left}}} \left(-\mathcal{N}\left(ik\mathbf{d}_n \cdot \mathbf{n}e^{ik\mathbf{d}_n\cdot\mathbf{x}}\right)\mathbf{d}_m \cdot \mathbf{n}e^{-ik\mathbf{d}_m\cdot\mathbf{x}} - e^{-ik\mathbf{d}_m\cdot\mathbf{x}}\mathbf{d}_n \cdot \mathbf{n}e^{ik\mathbf{d}_n\cdot\mathbf{x}} + d_2\left(\mathcal{N}\left(ik\mathbf{d}_n \cdot \mathbf{n}e^{ik\mathbf{d}_n\cdot\mathbf{x}}\right) - e^{ik\mathbf{d}}\right.\right.$

lets compute first:

$$\mathcal{N}\left(ik\mathbf{d}_n \cdot \mathbf{n}e^{ik\mathbf{d}_n\cdot\mathbf{x}}\right) = \mathbf{d}_n\cdot\mathbf{n}ke^{-ikd_{n,x}R}\left(\frac{1}{2kH}\int_{-H}^{H} e^{ikd_{n,y}y}\,\mathrm{d}y + \sum_{s=1}^{\infty}\frac{1}{\sqrt{(kH)^2-(s\pi)^2}}\left(\int_{-H}^{H} e^{ikd_{n,y}y}\cos\left(s\pi\right.\right.\right.$$

we have to distinguish to cases: $\mathbf{d}_n \cdot \mathbf{j} = 0$ that is, $d_{n,y} = 0$. Then

$$\mathcal{N}\left(ik\mathbf{d}_n \cdot \mathbf{n}e^{ik\mathbf{d}_n\cdot\mathbf{x}}\right) = \mathbf{d}_n \cdot \mathbf{n}e^{-ikd_{n,x}R}$$

if not:

$$\mathcal{N}\left(ik\mathbf{d}_n \cdot \mathbf{n}e^{ik\mathbf{d}_n\cdot\mathbf{x}}\right) = \mathbf{d}_n\cdot\mathbf{n}e^{-ikd_{n,x}R}\left(\frac{\sin\left(d_{n,y}kH\right)}{d_{n,y}kH} + \sum_{s=1}^{\infty}\frac{kH}{\sqrt{(kH)^2-(s\pi)^2}}\left(\frac{\sin\left(d_{n,y}kH+s\pi\right)}{d_{n,y}kH+s\pi} + \frac{\sin\left(d_{n,y}\right.}{d_{n,y}k}\right.\right.$$

wich can be rewritten , if wanted, as

$$\mathcal{N}\left(ik\mathbf{d}_n \cdot \mathbf{n}e^{ik\mathbf{d}_n\cdot\mathbf{x}}\right) =$$

$$= \mathbf{d}_n \cdot \mathbf{n}e^{-ikd_{n,x}R}\left(\sum_{s=-\infty}^{\infty}\frac{kH}{\sqrt{(kH)^2-(s\pi)^2}}\frac{\sin\left(d_{n,y}kH+s\pi\right)}{d_{n,y}kH+s\pi}\cos\left(s\pi\frac{y}{H}\right)\right)$$

Now:

first term:

$$-ik\int_{\Sigma_{\text{Left}}}\mathcal{N}\left(ik\mathbf{d}_n \cdot \mathbf{n}e^{ik\mathbf{d}_n\cdot\mathbf{x}}\right)\mathbf{d}_m \cdot \mathbf{n}e^{-ik\mathbf{d}_m\cdot\mathbf{x}}\,\mathrm{d}S_x$$

$$-ike^{ikd_{m,x}R}\mathbf{d}_m \cdot \mathbf{n}\int_{-H}^{H}\mathcal{N}\left(ik\mathbf{d}_n \cdot \mathbf{n}e^{ik\mathbf{d}_n\cdot\mathbf{x}}\right)e^{-ikd_{m,y}y}\,\mathrm{d}y$$

If $\mathbf{d}_n \cdot \mathbf{j} = 0$ then and $\mathbf{d}_m \cdot \mathbf{j} = 0$ then

$$-2ikHe^{i(d_{m,x}-d_{n,x})kR}\mathbf{d}_m \cdot \mathbf{n}\mathbf{d}_n \cdot \mathbf{n}$$

If $\mathbf{d}_n \cdot \mathbf{j} = 0$ then and $\mathbf{d}_m \cdot \mathbf{j} \neq 0$ then

$$-2ikHe^{i(d_{m,x}-d_{n,x})kR}\mathbf{d}_m \cdot \mathbf{n}\mathbf{d}_n \cdot \mathbf{n}\frac{\sin\left(d_{m,y}kH\right)}{d_{m,y}kH}$$

If $\mathbf{d}_n \cdot \mathbf{j} \neq 0$ then and $\mathbf{d}_m \cdot \mathbf{j} = 0$ then

$$-2ikHe^{i(d_{m,x}-d_{n,x})kR}\mathbf{d}_m\cdot\mathbf{n}\mathbf{d}_n\cdot\mathbf{n}\frac{\sin\left(d_{n,y}kH\right)}{d_{n,y}kH}$$

If $\mathbf{d}_n\cdot\mathbf{j}\neq 0$ then and $\mathbf{d}_m\cdot\mathbf{j}\neq 0$ then

$$-ikHe^{i(d_{m,x}-d_{n,x})kR}\mathbf{d}_m\cdot\mathbf{n}\mathbf{d}_n\cdot\mathbf{n}\left(2\frac{\sin\left(d_{n,y}kH\right)}{d_{n,y}kH}\frac{\sin\left(d_{m,y}kH\right)}{kHd_{m,y}}+\right.$$

$$+\sum_{s=1}^{\infty}\frac{kH}{\sqrt{\left(kH\right)^2-\left(s\pi\right)^2}}\left(\frac{\sin\left(d_{n,y}kH+s\pi\right)}{d_{n,y}kH+s\pi}+\frac{\sin\left(d_{n,y}kH-s\pi\right)}{d_{n,y}kH-s\pi}\right)\left(\frac{\sin\left(s\pi+kHd_{m,y}\right)}{s\pi+kHd_{m,y}}+\frac{\sin\left(s\pi-kHd_{m,y}\right)}{s\pi-kHd_{m,y}}\right.$$

and this is only the first therm, and there are like five or six like it.

Second term

$$-ik\int_{\Sigma_{\text{Left}}}e^{-ik\mathbf{d}_m\cdot\mathbf{x}}\mathbf{d}_n\cdot\mathbf{n}e^{ik\mathbf{d}_n\cdot\mathbf{x}}\,\mathrm{d}S_x$$

$$-ike^{ik(d_{m,x}-d_{n,x})R}\mathbf{d}_n\cdot\mathbf{n}\int_{-H}^{H}e^{ik(d_{n,y}-d_{m,y})y}\,\mathrm{d}y$$

if $d_{n,y}=d_{m,y}$ then

$$-ik\int_{\Sigma_{\text{Left}}}e^{-ik\mathbf{d}_m\cdot\mathbf{x}}\mathbf{d}_n\cdot\mathbf{n}e^{ik\mathbf{d}_n\cdot\mathbf{x}}\,\mathrm{d}S_x=-2ikH\mathbf{d}_n\cdot\mathbf{n}e^{ik(d_{m,x}-d_{n,x})R}$$

else

$$-ik\int_{\Sigma_{\text{Left}}}e^{-ik\mathbf{d}_m\cdot\mathbf{x}}\mathbf{d}_n\cdot\mathbf{n}e^{ik\mathbf{d}_n\cdot\mathbf{x}}\,\mathrm{d}S_x=$$

$$-2ikH\mathbf{d}_n\cdot\mathbf{n}e^{ik(d_{m,x}-d_{n,x})R}\frac{\sin\left(\left(d_{n,y}-d_{m,y}\right)kH\right)}{\left(d_{n,y}-d_{m,y}\right)kH}$$

And the next 4 terms are combinations of these ones. They are:

$$-ikd_2\int_{\Sigma_{\text{Left}}}\mathcal{N}\left(ik\mathbf{d}_n\cdot\mathbf{n}e^{ik\mathbf{d}_n\cdot\mathbf{x}}\right)e^{-ik\mathbf{d}_m\cdot\mathbf{x}}\,\mathrm{d}S_x$$

but we had already computed it, so there are 4 options:

If $\mathbf{d}_n\cdot\mathbf{j}=0$ then and $\mathbf{d}_m\cdot\mathbf{j}=0$ then

$$-2ikHe^{i(d_{m,x}-d_{n,x})kR}d_2\mathbf{d}_n\cdot\mathbf{n}$$

If $\mathbf{d}_n\cdot\mathbf{j}=0$ then and $\mathbf{d}_m\cdot\mathbf{j}\neq 0$ then

$$-2ikHe^{i(d_{m,x}-d_{n,x})kR}d_2\mathbf{d}_n\cdot\mathbf{n}\frac{\sin\left(d_{m,y}kH\right)}{d_{m,y}kH}$$

If $\mathbf{d}_n\cdot\mathbf{j}\neq 0$ then and $\mathbf{d}_m\cdot\mathbf{j}=0$ then

$$-2ikHe^{i(d_{m,x}-d_{n,x})kR}d_2\mathbf{d}_n\cdot\mathbf{n}\frac{\sin\left(d_{n,y}kH\right)}{d_{n,y}kH}$$

If $\mathbf{d}_n\cdot\mathbf{j}\neq 0$ then and $\mathbf{d}_m\cdot\mathbf{j}\neq 0$ then

$$-2ikHe^{i(d_{m,x}-d_{n,x})kR}d_2\mathbf{d}_n\cdot\mathbf{n}\frac{\sin\left(d_{n,y}kH\right)}{d_{n,y}kH}\frac{\sin\left(d_{m,y}kH\right)}{kHd_{m,y}}$$

next we have the symmetric term:

$$-d_2ik\int_{\Sigma_{\text{Left}}}\left(e^{ik\mathbf{d}_n\cdot\mathbf{x}}\overline{\mathcal{N}\left(ik\mathbf{d}_m\cdot\mathbf{n}e^{ik\mathbf{d}_m\cdot\mathbf{x}}\right)}\right)\,\mathrm{d}S_x$$

wich can be expressed as:

$$\overline{d_2ik\int_{\Sigma_{\text{Left}}}\left(\mathcal{N}\left(ik\mathbf{d}_m\cdot\mathbf{n}e^{ik\mathbf{d}_m\cdot\mathbf{x}}\right)e^{-ik\mathbf{d}_n\cdot\mathbf{x}}\right)\,\mathrm{d}S_x}$$

so we have 4 options again:

If $\mathbf{d}_n\cdot\mathbf{j}=0$ then and $\mathbf{d}_m\cdot\mathbf{j}=0$ then

$$-2ikHe^{i(d_{m,x}-d_{n,x})kR}d_2\mathbf{d}_m\cdot\mathbf{n}$$

If $\mathbf{d}_n\cdot\mathbf{j}=0$ then and $\mathbf{d}_m\cdot\mathbf{j}\neq 0$ then

$$-2ikHe^{i(d_{m,x}-d_{n,x})kR}d_2\mathbf{d}_m\cdot\mathbf{n}\frac{\sin\left(d_{m,y}kH\right)}{d_{m,y}kH}$$

If $\mathbf{d}_n\cdot\mathbf{j}\neq 0$ then and $\mathbf{d}_m\cdot\mathbf{j}=0$ then

$$-2ikHe^{i(d_{m,x}-d_{n,x})kR}d_2\mathbf{d}_m\cdot\mathbf{n}\frac{\sin\left(d_{n,y}kH\right)}{d_{n,y}kH}$$

If $\mathbf{d}_n\cdot\mathbf{j}\neq 0$ then and $\mathbf{d}_m\cdot\mathbf{j}\neq 0$ then

$$-2ikHe^{i(d_{m,x}-d_{n,x})kR}d_2\mathbf{d}_m\cdot\mathbf{n}\frac{\sin\left(d_{n,y}kH\right)}{d_{n,y}kH}\frac{\sin\left(d_{m,y}kH\right)}{kHd_{m,y}}$$

Then we have the term

$$ikd_2\int_{\Sigma_{\text{Left}}}e^{ik\mathbf{d}_n\cdot\mathbf{x}}e^{-ik\mathbf{d}_m\cdot\mathbf{x}}\,\mathrm{d}S_x$$

which is like the second one, that is:

If $d_{n,y}=d_{m,y}$ then

$$2ikHd_2e^{ik(d_{m,x}-d_{n,x})R}$$

else

$$2ikHd_2 e^{ik(d_{m,x}-d_{n,x})R} \frac{\sin\left(\left(d_{n,y}-d_{m,y}\right)kH\right)}{\left(d_{n,y}-d_{m,y}\right)kH}$$

Finally, the last term in this contribution is:

$$ikd_2 \int_{\Sigma_{\text{Left}}} \left(\mathcal{N}\left(ik\mathbf{d}_n \cdot \mathbf{n} e^{ik\mathbf{d}_n\cdot\mathbf{x}}\right) \overline{\mathcal{N}\left(ik\mathbf{d}_m \cdot \mathbf{n} e^{ik\mathbf{d}_m\cdot\mathbf{x}}\right)} \right) \mathrm{d}S_x$$

which is a "new" term. Luckily both functions on the integrand are expressed in an orthogonal basis:

If $\mathbf{d}_n \cdot \mathbf{j} = 0$ then and $\mathbf{d}_m \cdot \mathbf{j} = 0$ then

$$2ikHd_2\mathbf{d}_n \cdot \mathbf{n}\mathbf{d}_m \cdot \mathbf{n} e^{ik(d_{m,x}-d_{n,x})R}$$

If $\mathbf{d}_n \cdot \mathbf{j} = 0$ then and $\mathbf{d}_m \cdot \mathbf{j} \neq 0$ then

$$2ikHd_2\mathbf{d}_n \cdot \mathbf{n}\mathbf{d}_m \cdot \mathbf{n} e^{ik(d_{m,x}-d_{n,x})R} \frac{\sin\left(d_{m,y}kH\right)}{d_{m,y}kH}$$

If $\mathbf{d}_n \cdot \mathbf{j} \neq 0$ then and $\mathbf{d}_m \cdot \mathbf{j} = 0$ then

$$2ikHd_2\mathbf{d}_m \cdot \mathbf{n}\mathbf{d}_n \cdot \mathbf{n} e^{ik(d_{m,x}-d_{n,x})R} \frac{\sin\left(d_{n,y}kH\right)}{d_{n,y}kH}$$

If $\mathbf{d}_n \cdot \mathbf{j} \neq 0$ then and $\mathbf{d}_m \cdot \mathbf{j} \neq 0$ then

$$ikHd_2\mathbf{d}_m \cdot \mathbf{n}\mathbf{d}_n \cdot \mathbf{n} e^{ik(d_{m,x}-d_{n,x})R} \left(2\frac{\sin\left(d_{n,y}kH\right)}{d_{n,y}kH} \frac{\sin\left(d_{m,y}kH\right)}{d_{m,y}kH} + \right.$$

$$\left. + \sum_{s=1}^{\infty} \frac{(kH)^2}{(kH)^2-(s\pi)^2} \left(\frac{\sin\left(d_{n,y}kH+s\pi\right)}{d_{n,y}kH+s\pi} + \frac{\sin\left(d_{n,y}kH-s\pi\right)}{d_{n,y}kH-s\pi} \right) \left(\frac{\sin\left(d_{m,y}kH+s\pi\right)}{d_{m,y}kH+s\pi} + \frac{\sin\left(d_{m,y}kH-s\pi\right)}{d_{m,y}kH-s\pi} \right) \right)$$

Lets try to agroup terms in terms like the first and like the second one.

First-like terms:

If $\mathbf{d}_n \cdot \mathbf{j} = 0$ then and $\mathbf{d}_m \cdot \mathbf{j} = 0$ then

$$-2ikHe^{i(d_{m,x}-d_{n,x})kR} \left(\mathbf{d}_m \cdot \mathbf{n}\mathbf{d}_n \cdot \mathbf{n} + d_2 \left(\mathbf{d}_n \cdot \mathbf{n} + \mathbf{d}_m \cdot \mathbf{n} - \mathbf{d}_n \cdot \mathbf{n}\mathbf{d}_m \cdot \mathbf{n} \right) \right)$$

If $\mathbf{d}_n \cdot \mathbf{j} = 0$ then and $\mathbf{d}_m \cdot \mathbf{j} \neq 0$ then

$$-2ikHe^{i(d_{m,x}-d_{n,x})kR} \left(\mathbf{d}_m \cdot \mathbf{n}\mathbf{d}_n \cdot \mathbf{n} + d_2 \left(\mathbf{d}_n \cdot \mathbf{n} + \mathbf{d}_m \cdot \mathbf{n} - \mathbf{d}_n \cdot \mathbf{n}\mathbf{d}_m \cdot \mathbf{n} \right) \right) \frac{\sin\left(d_{m,y}kH\right)}{d_{m,y}kH}$$

If $\mathbf{d}_n \cdot \mathbf{j} \neq 0$ then and $\mathbf{d}_m \cdot \mathbf{j} = 0$ then

$$-2ikHe^{i(d_{m,x}-d_{n,x})kR}\left(\mathbf{d}_m\cdot\mathbf{n}\mathbf{d}_n\cdot\mathbf{n}+d_2\left(\mathbf{d}_n\cdot\mathbf{n}+\mathbf{d}_m\cdot\mathbf{n}-\mathbf{d}_n\cdot\mathbf{n}\mathbf{d}_m\cdot\mathbf{n}\right)\right)\frac{\sin\left(d_{n,y}kH\right)}{d_{n,y}kH}$$

If $\mathbf{d}_n\cdot\mathbf{j}\neq 0$ then and $\mathbf{d}_m\cdot\mathbf{j}\neq 0$ then

$$-2ikHe^{i(d_{m,x}-d_{n,x})kR}\Bigg(\left(\mathbf{d}_m\cdot\mathbf{n}\mathbf{d}_n\cdot\mathbf{n}+d_2\left(\mathbf{d}_n\cdot\mathbf{n}+\mathbf{d}_m\cdot\mathbf{n}-\mathbf{d}_n\cdot\mathbf{n}\mathbf{d}_m\cdot\mathbf{n}\right)\right)\frac{\sin\left(d_{n,y}kH\right)}{d_{n,y}kH}\frac{\sin\left(d_{m,y}kH\right)}{kHd_{m,y}}$$

$$+\frac{(1-d_2)}{2}\mathbf{d}_m\cdot\mathbf{n}\mathbf{d}_n\cdot\mathbf{n}\sum_{s=1}^{\infty}\frac{kH}{\sqrt{(kH)^2-(s\pi)^2}}\left(\frac{\sin\left(d_{n,y}kH+s\pi\right)}{d_{n,y}kH+s\pi}+\frac{\sin\left(d_{n,y}kH-s\pi\right)}{d_{n,y}kH-s\pi}\right)\left(\frac{\sin\left(s\pi+kHd_{m,y}\right.}{s\pi+kHd_{m,y}}\right.$$

and the second-like terms:
if $d_{n,y}=d_{m,y}$ then

$$-2ikH\left(\mathbf{d}_n\cdot\mathbf{n}-d_2\right)e^{ik(d_{m,x}-d_{n,x})R}$$

else

$$-2ikH\left(\mathbf{d}_n\cdot\mathbf{n}-d_2\right)e^{ik(d_{m,x}-d_{n,x})R}\frac{\sin\left(\left(d_{n,y}-d_{m,y}\right)kH\right)}{\left(d_{n,y}-d_{m,y}\right)kH}$$

The terms on Σ_{Right} are the same but changing $-R$ for R. (in fact they should be written as a single expresion depending on x, with d_n going first)

==================================

The b term should be:
(OBTAIN)

$$=\int_{\Sigma_{\text{Left}}}\left(\left(\mathcal{N}\left(\nabla u^{\text{inc}}\cdot\mathbf{n}\right)-u^{\text{inc}}\right)\overline{\nabla v}\cdot\mathbf{n}+ikd_2\left(\mathcal{N}\left(\nabla u^{\text{inc}}\cdot\mathbf{n}\right)-u^{\text{inc}}\right)\overline{\left(\mathcal{N}\left(\nabla v\cdot\mathbf{n}\right)-v\right)}\right)\mathrm{d}S_x$$

$$=\int_{\Sigma_{\text{Left}}}\left(\left(\mathcal{N}\left(\nabla u^{\text{inc}}\cdot\mathbf{n}\right)-u^{\text{inc}}\right)\left(\overline{\nabla v\cdot\mathbf{n}}+ikd_2\overline{\left(\mathcal{N}\left(\nabla v\cdot\mathbf{n}\right)-v\right)}\right)\right)\mathrm{d}S_x$$

For a g_t^+ incident field we can compute closed forms:

$$u_t^{\text{inc}}=e^{i\sqrt{k^2-\left(t\frac{\pi}{H}\right)^2}x}\cos\left(t\pi\frac{y}{H}\right)$$

The term for ψ_m is then (first on Σ_{Right})

$$\int_{\Sigma_{\text{Left}}}\left(0\left(\overline{\nabla v\cdot\mathbf{n}}+ikd_2\overline{\left(\mathcal{N}\left(\nabla v\cdot\mathbf{n}\right)-v\right)}\right)\right)\mathrm{d}S_x=0$$

because g_t^+ functions are outgoing radiating functions for Σ_{Right}, that is

$$g_t^+ = \mathcal{N}\left(\nabla g_t^+ \cdot \mathbf{n}\right) \quad \text{on } \Sigma_{\text{Right}}$$

we can check it if you dont believe me:

$$\mathcal{N}\left(\nabla g_t^+ \cdot \mathbf{i}\right) = \mathcal{N}\left(i\sqrt{k^2 - \left(t\frac{\pi}{H}\right)^2} g_t^+\right) =$$

$$= \frac{i\sqrt{k^2 - \left(t\frac{\pi}{H}\right)^2} e^{i\sqrt{k^2-\left(t\frac{\pi}{H}\right)^2}R}}{ik2H} \int_{-H}^{H} \cos\left(t\pi\frac{y}{H}\right)\,\mathrm{d}y + i\sqrt{k^2 - \left(t\frac{\pi}{H}\right)^2} e^{i\sqrt{k^2-\left(t\frac{\pi}{H}\right)^2}R} \sum_{s=1}^{\infty} \frac{1}{i\sqrt{k^2 - \left(s\frac{\pi}{H}\right)^2}} \left(\int_{-H}^{H}\right.$$

$$e^{i\sqrt{k^2-\left(t\frac{\pi}{H}\right)^2}R} \cos\left(t\pi\frac{y}{H}\right) = g_t^+\big|_{x=R}$$

on the other hand, on Σ_{Left}

$$\mathcal{N}\left(-\nabla g_t^+ \cdot \mathbf{i}\right) = -\mathcal{N}\left(i\sqrt{k^2 - \left(t\frac{\pi}{H}\right)^2} g_t^+\right) = -\,g_t^+\big|_{x=-R}$$

(lets check it...)

$$-\nabla g_t^+ \cdot \mathbf{i}\big|_{x=-R} = -i\sqrt{k^2 - \left(t\frac{\pi}{H}\right)^2} e^{-i\sqrt{k^2-\left(t\frac{\pi}{H}\right)^2}R} \cos\left(t\pi\frac{y}{H}\right)$$

$$\mathcal{N}\left(-\nabla g_t^+ \cdot \mathbf{i}\right) =$$

$$= \frac{-i\sqrt{k^2 - \left(t\frac{\pi}{H}\right)^2} e^{-i\sqrt{k^2-\left(t\frac{\pi}{H}\right)^2}R}}{k2H} \int_{-H}^{H} \cos\left(t\pi\frac{y}{H}\right)\,\mathrm{d}y + \frac{-i\sqrt{k^2 - \left(t\frac{\pi}{H}\right)^2} e^{-i\sqrt{k^2-\left(t\frac{\pi}{H}\right)^2}R}}{H} \sum_{s=1}^{\infty} \frac{1}{i\sqrt{k^2 - \left(s\frac{\pi}{H}\right)^2}}$$

$$= 0 - e^{-i\sqrt{k^2-\left(t\frac{\pi}{H}\right)^2}R} \cos\left(t\pi\frac{y}{H}\right) = -\,g_t^+\big|_{x=-R}$$

So now, the Σ_{Left} term would be:

$$\int_{\Sigma_{\text{Left}}} \left(\left(\mathcal{N}\left(\nabla u^{\text{inc}} \cdot \mathbf{n}\right) - u^{\text{inc}}\right)\left(\overline{\nabla v \cdot \mathbf{n}} + ikd_2\overline{\left(\mathcal{N}\left(\nabla v \cdot \mathbf{n}\right) - v\right)}\right)\right) \mathrm{d}S_x$$

$$e^{i\left(kd_{m,x} - \sqrt{k^2-\left(t\frac{\pi}{H}\right)^2}\right)R} \int_{\Sigma_{\text{Left}}} \left(\left(-i\sqrt{k^2 - \left(t\frac{\pi}{H}\right)^2}\mathcal{N}\left(\cos\left(t\pi\frac{y}{H}\right)\right) - \cos\left(t\pi\frac{y}{H}\right)\right)\left(ik\left(\left(d_{m,x} - d_2\right)e^{-ikd_{m,}}\right.\right.\right.$$

Lets compute first the Newman to Dirichlet operators:

$$\mathcal{N}\left(\cos\left(t\pi\frac{y}{H}\right)\right)=\begin{cases}\frac{1}{ik} & t=0\\ \frac{1}{i\sqrt{k^2-\left(t\frac{\pi}{H}\right)^2}}\cos\left(t\pi\frac{y}{H}\right) & t>0\end{cases}$$

$$\overline{\left(\mathcal{N}\left(e^{ikd_{m,y}y}\right)\right)}=\overline{\frac{1}{2ikH}\int_{-H}^{H}e^{ikd_{m,y}y}\,\mathrm{d}y+\sum_{s=1}^{\infty}\frac{1}{i\sqrt{k^2-\left(s\frac{\pi}{H}\right)^2}H}\left(\int_{-H}^{H}e^{ikd_{m,y}y}\cos\left(s\pi\frac{y}{H}\right)\,\mathrm{d}y\right)\cos\left(s\pi\frac{y}{H}\right)}$$

if $\mathbf{d}_m\cdot\mathbf{j}=0$ then

$$\overline{\left(\mathcal{N}\left(e^{ikd_{m,y}y}\right)\right)}=\overline{\left(\frac{1}{ik}\right)}=-\frac{1}{ik}$$

else

$$\overline{\left(\mathcal{N}\left(e^{ikd_{m,y}y}\right)\right)}=\overline{\frac{1}{ikH}\frac{\sin\left(d_{m,y}kH\right)}{kd_{m,y}}+\sum_{s=1}^{\infty}\frac{1}{i\sqrt{k^2-\left(s\frac{\pi}{H}\right)^2}}\left(\frac{\sin\left(kHd_{m,y}+s\pi\right)}{kHd_{m,y}+s\pi}+\frac{\sin\left(kHd_{m,y}-s\pi\right)}{kHd_{m,y}-s\pi}\right)\cos\left(s\right.}$$

$$=-\frac{1}{ikH}\frac{\sin\left(d_{m,y}kH\right)}{kd_{m,y}}-\sum_{s=1}^{\infty}\frac{1}{i\sqrt{k^2-\left(s\frac{\pi}{H}\right)^2}}\left(\frac{\sin\left(kHd_{m,y}+s\pi\right)}{kHd_{m,y}+s\pi}+\frac{\sin\left(kHd_{m,y}-s\pi\right)}{kHd_{m,y}-s\pi}\right)\cos\left(s\pi\frac{y}{H}\right)$$

so the b term is :
if $\mathbf{d}_m\cdot\mathbf{j}=0$

$$-2ike^{i\left(kd_{m,x}-\sqrt{k^2-\left(t\frac{\pi}{H}\right)^2}\right)R}\int_{-H}^{H}\left(\cos\left(t\pi\frac{y}{H}\right)\left(d_{m,x}-d_2-d_2d_{m,x}\right)\right)\,\mathrm{d}y$$

which is

$$\begin{cases}-4ikHe^{ikH\left(d_{m,x}-1\right)\frac{R}{H}}\left(d_{m,x}-d_2-d_2d_{m,x}\right) & t=0\\ 0 & t>0\end{cases}$$

else

$$-2ike^{i\left(kd_{m,x}-\sqrt{k^2-\left(t\frac{\pi}{H}\right)^2}\right)R}\int_{-H}^{H}\left(\cos\left(t\pi\frac{y}{H}\right)\left(\left(d_{m,x}-d_2\right)e^{-ikd_{m,y}y}-d_2d_{m,x}\left(\frac{\sin\left(d_{m,y}kH\right)}{d_{m,y}kH}+\sum_{s=1}^{\infty}\frac{}{\sqrt{(k\right.}}\right.\right.\right.$$

lets separate both operands:

$$-2ike^{i\left(kd_{m,x}-\sqrt{k^2-\left(t\frac{\pi}{H}\right)^2}\right)R}\left(d_{m,x}-d_2\right)\int_{-H}^{H}\left(\cos\left(t\pi\frac{y}{H}\right)e^{-ikd_{m,y}y}\right)\,\mathrm{d}y$$

$$+2ike^{i\left(kd_{m,x}-\sqrt{k^2-\left(t\frac{\pi}{H}\right)^2}\right)R}d_2d_{m,x}\int_{-H}^{H}\left(\cos\left(t\pi\frac{y}{H}\right)\left(\frac{\sin\left(d_{m,y}kH\right)}{d_{m,y}kH}+\sum_{s=1}^{\infty}\frac{kH}{\sqrt{(kH)^2-(s\pi)^2}}\left(\frac{\sin\left(kHd_{m,y}\right.}{kHd_{m,y}+}\right.\right.\right.$$

the last one is easy, as they are orthogonal:

$$\begin{cases} +4ikHe^{ikH(d_{m,x}-1)\frac{R}{H}}d_2d_{m,x}\frac{\sin(d_{m,y}kH)}{d_{m,y}kH} & t=0 \\ +2ikHe^{i\left(kHd_{m,x}-\sqrt{(kH)^2-(t\pi)^2}\right)\frac{R}{H}}d_2d_{m,x}\frac{kH}{\sqrt{(kH)^2-(t\pi)^2}}\left(\frac{\sin(kHd_{m,y}+t\pi)}{kHd_{m,y}+t\pi}+\frac{\sin(kHd_{m,y}-t\pi)}{kHd_{m,y}-t\pi}\right) & t>0 \end{cases}$$

the first one:

$$-2ike^{i\left(kd_{m,x}-\sqrt{k^2-\left(t\frac{\pi}{H}\right)^2}\right)R}\left(d_{m,x}-d_2\right)\int_{-H}^{H}\left(\cos\left(t\pi\frac{y}{H}\right)e^{-ikd_{m,y}y}\right)\,\mathrm{d}y=$$

$$\begin{cases} -4ikHe^{ikH(d_{m,x}-1)\frac{R}{H}}\left(d_{m,x}-d_2\right)\frac{\sin(kHd_{m,y})}{kHd_{m,y}} & t=0 \\ -2ikHe^{i\left(kd_{m,x}-\sqrt{k^2-\left(t\frac{\pi}{H}\right)^2}\right)R}\left(d_{m,x}-d_2\right)\left(\frac{\sin(t\pi-kHd_{m,y})}{t\pi-kHd_{m,y}}+\frac{\sin(t\pi+kHd_{m,y})}{t\pi+kHd_{m,y}}\right) & t>0 \end{cases}$$

(there is no need in such difference) so adding them

$$\begin{cases} -4ikHe^{ikH(d_{m,x}-1)\frac{R}{H}}\left(d_{m,x}-d_2-d_2d_{m,x}\right)\frac{\sin(kHd_{m,y})}{kHd_{m,y}} \\ -2ikHe^{-i\left(\sqrt{(kH)^2-(t\pi)^2}-kHd_{m,x}\right)\frac{R}{H}}\left(d_{m,x}-d_2-d_2d_{m,x}\frac{kH}{\sqrt{(kH)^2-(t\pi)^2}}\right)\left(\frac{\sin(kHd_{m,y}+t\pi)}{kHd_{m,y}+t\pi}+\frac{\sin(kHd_{m,y}-t\pi)}{kHd_{m,y}-t\pi}\right) \end{cases}$$