$$\int_E \left(\hat{u}\overline{\nabla v\cdot\mathbf{n}} - ik\hat{\boldsymbol{\sigma}}\cdot\mathbf{n}\overline{v}\right) dS$$

but

$$\hat{u} = u^{\mathrm{i}} + \mathcal{N}\left(\nabla u\cdot\mathbf{n} - \nabla u^{\mathrm{i}}\cdot\mathbf{n}\right) - d_2 ik...$$

$$ik\hat{\boldsymbol{\sigma}} = \nabla u - d_2\dots$$

lets assume, for a moment that we set $d_2 = 0$.

$$\int_E \left(\left(u^{\mathrm{i}} + \mathcal{N}\left(\nabla u\cdot\mathbf{n} - \nabla u^{\mathrm{i}}\cdot\mathbf{n}\right)\right)\overline{\nabla v\cdot\mathbf{n}} - \nabla u\cdot\mathbf{n}\overline{v}\right) dS = 0$$

$$\int_E \left(\mathcal{N}\left(\nabla u\cdot\mathbf{n}\right)\overline{\nabla v\cdot\mathbf{n}} - \nabla u\cdot\mathbf{n}\overline{v}\right) dS = \int_E \left(\mathcal{N}\left(\nabla u^{\mathrm{i}}\cdot\mathbf{n}\right) - u^{\mathrm{i}}\right)\overline{\nabla v\cdot\mathbf{n}} dS$$

A_{mn} is

$$\int_E \left(\mathcal{N}\left(\nabla\phi_n\cdot\mathbf{n}\right)\overline{\nabla\psi_m\cdot\mathbf{n}} - \nabla\phi_n\cdot\mathbf{n}\overline{\psi_m}\right) dS$$

while b_m is

$$\int_E \left(\mathcal{N}\left(\nabla u^{\mathrm{i}}\cdot\mathbf{n}\right) - u^{\mathrm{i}}\right)\overline{\nabla\psi_m\cdot\mathbf{n}} dS$$

so... here we go again:
A_{mn} if Σ_R

$$\int_{-H}^{H} \left(\mathcal{N}\left(\nabla\phi_n\cdot\mathbf{n}\right)\overline{\nabla\psi_m\cdot\mathbf{n}} - \nabla\phi_n\cdot\mathbf{n}\overline{\psi_m}\right) dy$$

has two terms. The second is:

$$-\int_{-H}^{H} \nabla\phi_n\cdot\mathbf{i}\overline{\psi_m} dS = -\int_{-H}^{H} ikd_{n,x}e^{ik\mathbf{d}_n\cdot\mathbf{x}}e^{-ik\mathbf{d}_m\mathbf{x}}dy$$

$$= -ikd_{n,x}\int_{-H}^{H} e^{ik(d_{n,x}x + d_{n,y}y)}e^{-ik(d_{m,x}x + d_{m,y}y)}dy$$

$$= -ikd_{n,x}e^{ik(d_{n,x} - d_{m,x})R}\int_{-H}^{H} e^{ik(d_{n,y} - d_{m,y})y}dy$$

if $d_{n,y} = d_{m,y}$ then

$$-\int_{-H}^{H} \nabla\phi_n\cdot\mathbf{i}\overline{\psi_m} dS = -ikd_{n,x}e^{ik(d_{n,x} - d_{m,x})R}2H$$

$$= -2ikHd_{n,x}e^{ikH(d_{n,x}-d_{m,x})\frac{R}{H}}$$

else

$$-\int_{-H}^{H}\nabla\phi_n\cdot\mathbf{i}\overline{\psi_m}dS = -2ikHd_{n,x}e^{ikH(d_{n,x}-d_{m,x})\frac{R}{H}}\frac{\sin\left(kH\left(d_{n,y}-d_{m,y}\right)\right)}{kH\left(d_{n,y}-d_{m,y}\right)}$$

(that is, it coincides with the limit, which is nice)
if Σ_{-R}

$$\int_{-H}^{H}\left(\mathcal{N}\left(\nabla\phi_n\cdot\mathbf{n}\right)\overline{\nabla\psi_m\cdot\mathbf{n}}-\nabla\phi_n\cdot\mathbf{n}\overline{\psi_m}\right)dy$$

has two terms. The second is:

$$\int_{-H}^{H}\nabla\phi_n\cdot\mathbf{i}\overline{\psi_m}dS = \int_{-H}^{H}ikd_{n,x}e^{ik\mathbf{d}_n\cdot\mathbf{x}}e^{-ik\mathbf{d}_m\mathbf{x}}dy$$

$$= ikd_{n,x}\int_{-H}^{H}e^{ik(d_{n,x}x+d_{n,y}y)}e^{-ik(d_{m,x}x+d_{m,y}y)}dy$$

$$= ikd_{n,x}e^{-ik(d_{n,x}-d_{m,x})R}\int_{-H}^{H}e^{ik(d_{n,y}-d_{m,y})y}dy$$

if $d_{n,y}=d_{m,y}$ then

$$\int_{-H}^{H}\nabla\phi_n\cdot\mathbf{i}\overline{\psi_m}dS = ikd_{n,x}e^{ik(d_{n,x}-d_{m,x})R}2H$$

$$= 2ikHd_{n,x}e^{ikH(d_{n,x}-d_{m,x})\frac{R}{H}}$$

else

$$\int_{-H}^{H}\nabla\phi_n\cdot\mathbf{i}\overline{\psi_m}dS = 2ikHd_{n,x}e^{ikH(d_{n,x}-d_{m,x})\frac{R}{H}}\frac{\sin\left(kH\left(d_{n,y}-d_{m,y}\right)\right)}{kH\left(d_{n,y}-d_{m,y}\right)}$$

So for the second term

$$\boxed{-\int_{-H}^{H}\nabla\phi_n\cdot\mathbf{n}\overline{\psi_m}dS = \begin{cases}-2ikH\left(\mathbf{n}\cdot\mathbf{d}_n\right)e^{ikH(d_{n,x}-d_{m,x})\frac{x}{H}} & \text{if } d_{n,y}=d_{m,y}\\ -2ikH\left(\mathbf{n}\cdot\mathbf{d}_n\right)e^{ikH(d_{n,x}-d_{m,x})\frac{x}{H}}\frac{\sin(kH(d_{n,y}-d_{m,y}))}{kH(d_{n,y}-d_{m,y})} & \text{else}\end{cases}}$$

First term:

$$\int_{-H}^{H}\left(\mathcal{N}\left(\nabla\phi_n\cdot\mathbf{i}\right)\overline{\nabla\psi_m\cdot\mathbf{i}}\right)dy =$$

$$-\int_{-H}^{H}\left(\mathcal{N}\left(ikd_{n,x}e^{ik\mathbf{d}_n\cdot\mathbf{x}}\right)ikd_{m,x}e^{-ik\mathbf{d}_m\cdot\mathbf{x}}\right)dy =$$

$$k^2 d_{n,x}d_{m,x}e^{ikH(d_{n,x}-d_{m,x})\frac{R}{H}}\int_{-H}^{H}\left(\mathcal{N}\left(e^{ikd_{n,y}y}\right)e^{-ikd_{m,y}y}\right)dy =$$

with

$$\mathcal{N}\left(e^{ikd_{n,y}y}\right) = \frac{1}{ik}\left(\int_{-H}^{H}e^{ikd_{n,y}\eta}\frac{1}{\sqrt{2H}}d\eta\right)\frac{1}{\sqrt{2H}}+\sum_{s=1}^{\infty}\frac{1}{i\sqrt{k^2-\left(s\frac{\pi}{H}\right)^2}}\left(\int_{-H}^{H}e^{ikd_{n,y}\eta}\frac{\cos\left(s\pi\frac{\eta}{H}\right)}{\sqrt{H}}d\eta\right)\frac{\cos\left(s\pi\right.}{\sqrt{H}}$$

if $d_{n,y}=0$ then

$$\mathcal{N}\left(e^{ikd_{n,y}y}\right) = \frac{1}{ik}$$

else

$$\mathcal{N}\left(e^{ikd_{n,y}y}\right) = \frac{1}{ik}\frac{\sin\left(kHd_{n,y}\right)}{kHd_{n,y}}+\sum_{s=1}^{\infty}\frac{1}{2i\sqrt{k^2-\left(s\frac{\pi}{H}\right)^2}}\left(\int_{-H}^{H}e^{i\left(kd_{n,y}+s\frac{\pi}{H}\right)\eta}+e^{i\left(kd_{n,y}-s\frac{\pi}{H}\right)\eta}d\eta\right)\frac{\cos\left(s\pi\frac{y}{H}\right)}{H}$$

and, if $\frac{kHd_{n,y}}{\pi}\neq s$ then

$$\mathcal{N}\left(e^{ikd_{n,y}y}\right) = \frac{1}{ik}\frac{\sin\left(kHd_{n,y}\right)}{kHd_{n,y}}+\sum_{s=1}^{\infty}\frac{1}{i\sqrt{k^2-\left(s\frac{\pi}{H}\right)^2}}\left(\frac{\sin\left(kHd_{n,y}+s\pi\right)}{kHd_{n,y}+s\pi}+\frac{\sin\left(kHd_{n,y}-s\pi\right)}{kHd_{n,y}-s\pi}\right)\cos\left(s\pi\frac{y}{H}\right)$$

so the term

$$k^2 d_{n,x}d_{m,x}e^{ikH(d_{n,x}-d_{m,x})\frac{R}{H}}\int_{-H}^{H}\left(\mathcal{N}\left(e^{ikd_{n,y}y}\right)e^{-ikd_{m,y}y}\right)dy$$

has four options

$d_{n,y}=0$ and $d_{m,y}=0$

$$k^2 d_{n,x}d_{m,x}e^{ikH(d_{n,x}-d_{m,x})\frac{R}{H}}\int_{-H}^{H}\frac{1}{ik}dy = -2ikHd_{n,x}d_{m,x}e^{ikH(d_{n,x}-d_{m,x})\frac{R}{H}}$$

$d_{n,y}=0$ and $d_{m,y}\neq 0$

$$-ikd_{n,x}d_{m,x}e^{ikH(d_{n,x}-d_{m,x})\frac{R}{H}}\int_{-H}^{H}e^{-ikd_{m,y}y}dy = -2ikHd_{n,x}d_{m,x}e^{ikH(d_{n,x}-d_{m,x})\frac{R}{H}}\frac{\sin\left(kHd_{m,y}\right)}{kHd_{m,y}}$$

$d_{n,y}\neq 0$ and $d_{m,y}=0$

$$k^2 d_{n,x} d_{m,x} e^{ikH(d_{n,x}-d_{m,x})\frac{R}{H}} \int_{-H}^{H} \left(\frac{1}{ik} \frac{\sin(kHd_{n,y})}{kHd_{n,y}} \right) dy = -2ikHd_{n,x} d_{m,x} e^{ikH(d_{n,x}-d_{m,x})\frac{R}{H}} \frac{\sin(kHd_{n,y})}{kHd_{n,y}}$$

$d_{n,y} \neq 0$ and $d_{m,y} \neq 0$

$$k^2 d_{n,x} d_{m,x} e^{ikH(d_{n,x}-d_{m,x})\frac{R}{H}} \int_{-H}^{H} \left(\frac{1}{ik} \frac{\sin(kHd_{n,y})}{kHd_{n,y}} + \sum_{s=1}^{\infty} \frac{1}{i\sqrt{k^2 - \left(s\frac{\pi}{H}\right)^2}} \left(\frac{\sin(kHd_{n,y}+s\pi)}{kHd_{n,y}+s\pi} + \frac{\sin(kHd_{n,y}}{kHd_{n,y} -} \right.\right.$$

$$k^2 d_{n,x} d_{m,x} e^{ikH(d_{n,x}-d_{m,x})\frac{R}{H}} \left(\frac{1}{ik} \frac{\sin(kHd_{n,y})}{kHd_{n,y}} \int_{-H}^{H} e^{-ikd_{m,y}y} dy + \sum_{s=1}^{\infty} \frac{1}{i\sqrt{k^2 - \left(s\frac{\pi}{H}\right)^2}} \left(\frac{\sin(kHd_{n,y}+s\pi)}{kHd_{n,y}+s\pi} + \right.\right.$$

$$-2ikHd_{n,x} d_{m,x} e^{ikH(d_{n,x}-d_{m,x})\frac{R}{H}} \left(\frac{\sin(kHd_{n,y})}{kHd_{n,y}} \frac{\sin(kHd_{m,y})}{kHd_{m,y}} + \frac{1}{2} \sum_{s=1}^{\infty} \frac{kH}{\sqrt{(kH)^2 - (s\pi)^2}} \left(\frac{\sin(kHd_{n,y}+s\pi)}{kHd_{n,y}+s\pi} \right.\right.$$

What about Σ_{-R} :

$$\int_{-H}^{H} \left(\mathcal{N} (\nabla \phi_n \cdot (-\mathbf{i})) \overline{\nabla \psi_m \cdot (-\mathbf{i})} \right) dy = \int_{-H}^{H} \left(\mathcal{N} (\nabla \phi_n \cdot \mathbf{i}) \overline{\nabla \psi_m \cdot \mathbf{i}} \right) dy$$

so in general, the First term

$$\int_{-H}^{H} \left(\mathcal{N} (\nabla \phi_n \cdot \mathbf{n}) \overline{\nabla \psi_m \cdot \mathbf{n}} \right) dy$$

has 4 options

$d_{n,y} = 0$ and $d_{m,y} = 0$

$$\boxed{-2ikHd_{n,x} d_{m,x} e^{ikH(d_{n,x}-d_{m,x})\frac{x}{H}}}$$

$d_{n,y} = 0$ and $d_{m,y} \neq 0$

$$\boxed{-2ikHd_{n,x} d_{m,x} e^{ikH(d_{n,x}-d_{m,x})\frac{x}{H}} \frac{\sin(kHd_{m,y})}{kHd_{m,y}}}$$

$d_{n,y} \neq 0$ and $d_{m,y} = 0$

$$\boxed{-2ikHd_{n,x} d_{m,x} e^{ikH(d_{n,x}-d_{m,x})\frac{x}{H}} \frac{\sin(kHd_{n,y})}{kHd_{n,y}}}$$

$d_{n,y} \neq 0$ and $d_{m,y} \neq 0$

$$\boxed{-2ikHd_{n,x}d_{m,x}e^{ikH(d_{n,x}-d_{m,x})\frac{x}{H}}\left(\frac{\sin\left(kHd_{n,y}\right)}{kHd_{n,y}}\frac{\sin\left(kHd_{m,y}\right)}{kHd_{m,y}}+\frac{1}{2}\sum_{s=1}^{\infty}\frac{kH}{\sqrt{\left(kH\right)^2-\left(s\pi\right)^2}}\left(\frac{\sin\left(kHd_{n,y}+s\pi\right.}{kHd_{n,y}+s\pi}\right.\right.}$$

whereas the RHS is:

$$b_m=\int_{-H}^{H}\left(\mathcal{N}\left(\nabla u^{\mathrm{i}}\cdot\mathbf{n}\right)-u^{\mathrm{i}}\right)\overline{\nabla\psi_m\cdot\mathbf{n}}dy$$

0. at Σ_R and

$$-2\int_{-H}^{H}u^{\mathrm{i}}\overline{\nabla\psi_m\cdot\mathbf{n}}dy$$

at Σ_{-R} which is

$$2ikH\mathbf{d}_m\cdot\mathbf{n}e^{i\left(\sqrt{(kH)^2-(t\pi)^2}-kHd_{m,x}\right)\frac{x}{H}}\int_{-H}^{H}\frac{1}{H}\cos\left(t\pi\frac{y}{H}\right)e^{-ikd_{m,y}y}dy$$

The last integral:

if $d_{m,y}=0$ then

$$\begin{cases}2 & t=0\\ 0 & t>0\end{cases}$$

else

$$\begin{cases}2\frac{\sin(kHd_{m,y})}{kHd_{m,y}} & t=0\\ \frac{\sin(t\pi+kHd_{m,y})}{t\pi+kHd_{m,y}}+\frac{\sin(t\pi-kHd_{m,y})}{t\pi-kHd_{m,y}} & t>0\end{cases}$$

Terms of d_2.

the term is

$$\hat{u}=\ldots d_2\left(-ik\mathcal{N}^*\left(\mathcal{N}\left(\nabla u\cdot\mathbf{n}-\nabla u^{\mathrm{i}}\cdot\mathbf{n}\right)-\left(u-u^{\mathrm{i}}\right)\right)\right)$$

$$i\kappa\hat{\boldsymbol{\sigma}}=\cdots-d_2ik\left(\mathcal{N}\left(\nabla u\cdot\mathbf{n}-\nabla u^{\mathrm{i}}\cdot\mathbf{n}\right)-\left(u-u^{\mathrm{i}}\right)\right)\mathbf{n}$$

so it is d_2 times

$$\int_E\left(\left(-ik\mathcal{N}^*\left(\mathcal{N}\left(\nabla u\cdot\mathbf{n}-\nabla u^{\mathrm{i}}\cdot\mathbf{n}\right)-\left(u-u^{\mathrm{i}}\right)\right)\right)\overline{\nabla v\cdot\mathbf{n}}+ik\left(\mathcal{N}\left(\nabla u\cdot\mathbf{n}-\nabla u^{\mathrm{i}}\cdot\mathbf{n}\right)-\left(u-u^{\mathrm{i}}\right)\right)\overline{v}\right)dS=0$$

$\mathcal{N}^*$ is the operator such that

$$\left\langle\mathcal{N}\left(f\right),w\right\rangle=\left\langle f,\mathcal{N}^*\left(w\right)\right\rangle$$

that is

$$\int_{\Sigma_{+R}} \mathcal{N}(f)\,\overline{w}\,\mathrm{d}S = \int_{\Sigma_{+R}} f\overline{\mathcal{N}^*(w)}\,\mathrm{d}S$$

$$\int_{-H}^{H} \mathcal{N}(f)(y)\,\overline{w(y)}\,\mathrm{d}y = \int_{-H}^{H} f(y)\,\overline{\mathcal{N}^*(w)(y)}\,\mathrm{d}y$$

$$\int_{-H}^{H}\left(\frac{1}{ik}\frac{1}{2H}\int_{-H}^{H} f(\eta)\ \mathrm{d}\eta + \frac{1}{H}\sum_{s=1}^{\infty}\frac{1}{i\sqrt{k^2-\left(s\frac{\pi}{H}\right)^2}}\left(\int_{-H}^{H} f(\eta)\cos\left(s\pi\frac{\eta}{H}\right)\ \mathrm{d}\eta\right)\cos\left(s\pi\frac{y}{H}\right)\right)\overline{w(y)}\,\mathrm{d}y = \int$$

assume that

$$w(y) = \frac{w_0}{\sqrt{2H}} + \frac{1}{\sqrt{H}}\sum_{s=1}^{\infty} w_s\cos\left(s\pi\frac{y}{H}\right)$$

then

$$\int_{-H}^{H}\left(\frac{1}{ik}\frac{1}{2H}\int_{-H}^{H} f(\eta)\ \mathrm{d}\eta + \frac{1}{H}\sum_{s=1}^{\infty}\frac{1}{i\sqrt{k^2-\left(s\frac{\pi}{H}\right)^2}}\left(\int_{-H}^{H} f(\eta)\cos\left(s\pi\frac{\eta}{H}\right)\ \mathrm{d}\eta\right)\cos\left(s\pi\frac{y}{H}\right)\right)\overline{w(y)}\,\mathrm{d}y =$$

$$\int_{-H}^{H}\left(\frac{1}{ik}\frac{1}{2H}\int_{-H}^{H} f(\eta)\ \mathrm{d}\eta + \frac{1}{H}\sum_{s=1}^{\infty}\frac{1}{i\sqrt{k^2-\left(s\frac{\pi}{H}\right)^2}}\left(\int_{-H}^{H} f(\eta)\cos\left(s\pi\frac{\eta}{H}\right)\ \mathrm{d}\eta\right)\cos\left(s\pi\frac{y}{H}\right)\right)\left(\frac{\overline{w_0}}{\sqrt{2H}} + \frac{1}{\sqrt{}}\right.$$

$$\frac{1}{ik}f_0\overline{w_0} + \sum_{s=1}^{\infty}\frac{1}{i\sqrt{k^2-\left(s\frac{\pi}{H}\right)^2}}f_s\overline{w_s} = f_0\overline{\left(-\frac{w_0}{ik}\right)} + \sum_{s=1}^{\infty} f_s\overline{\left(-\frac{w_s}{i\sqrt{k^2-\left(s\frac{\pi}{H}\right)^2}}\right)} =$$

$$\int_{-H}^{H} f(y)\,\overline{\mathcal{N}^*(w)(y)}\,\mathrm{d}y$$

so

$$\overline{\mathcal{N}^*(w)(y)} = \overline{\left(-\frac{w_0}{ik}\right)}\frac{1}{\sqrt{2H}} + \sum_{s=1}^{\infty}\overline{\left(-\frac{w_s}{i\sqrt{k^2-\left(s\frac{\pi}{H}\right)^2}}\right)}\frac{\cos\left(s\pi\frac{y}{H}\right)}{\sqrt{H}}$$

that is

$$\mathcal{N}^*(w)(y) = -\frac{w_0}{ik}\frac{1}{\sqrt{2H}} - \sum_{s=1}^{\infty}\frac{w_s}{i\sqrt{k^2-\left(s\frac{\pi}{H}\right)^2}}\frac{\cos\left(s\pi\frac{y}{H}\right)}{\sqrt{H}}$$

or more explicitly:

$$\mathcal{N}^*(w)(y) = -\frac{1}{ik}\left(\int_{-H}^{H} w(\eta)\frac{1}{\sqrt{2H}}\,\mathrm{d}\eta\right)\frac{1}{\sqrt{2H}} - \sum_{s=1}^{\infty}\frac{w_s}{i\sqrt{k^2-\left(s\frac{\pi}{H}\right)^2}}\left(\int_{-H}^{H} w(\eta)\frac{\cos\left(s\pi\frac{\eta}{H}\right)}{\sqrt{H}}\,\mathrm{d}\eta\right)\frac{\cos\left(s\pi\frac{y}{H}\right)}{\sqrt{H}}$$

SO

$$-ik\int_{-H}^{H}\left(\left(\mathcal{N}\left(\nabla u\cdot\mathbf{n}-\nabla u^{\mathrm{i}}\cdot\mathbf{n}\right)-\left(u-u^{\mathrm{i}}\right)\right)\overline{\left(\mathcal{N}\left(\nabla v\cdot\mathbf{n}\right)-v\right)}\right)dy$$

that is

$$-ik\int_{-H}^{H}\left(\mathcal{N}\left(\nabla u\cdot\mathbf{n}\right)-u\right)\overline{\left(\mathcal{N}\left(\nabla v\cdot\mathbf{n}\right)-v\right)}dy = -ik\int_{-H}^{H}\left(\mathcal{N}\left(\nabla u^{\mathrm{i}}\cdot\mathbf{n}\right)-u^{\mathrm{i}}\right)\overline{\left(\mathcal{N}\left(\nabla v\cdot\mathbf{n}\right)-v\right)}dy$$

lets go first with the RHS for $u^{\mathrm{i}}(x,y) = e^{i\beta_t x}\cos\left(t\pi\frac{y}{H}\right) = e^{i\sqrt{k^2-\left(t\frac{\pi}{H}\right)^2}x}\cos\left(t\pi\frac{y}{H}\right)$, in particular, for $t=0$ $u^i(x,y)=e^{ikx}$ is a plane wave.

if we are in Σ_R then $\mathcal{N}\left(\nabla u^{\mathrm{i}}\cdot\mathbf{n}\right)=u^{\mathrm{i}}$ so

$$-ik\int_{-H}^{H}\left(\mathcal{N}\left(\nabla u^{\mathrm{i}}\cdot\mathbf{n}\right)-u^{\mathrm{i}}\right)\overline{\left(\mathcal{N}\left(\nabla v\cdot\mathbf{n}\right)-v\right)}dy = 0$$

if we are in Σ_{-R} then $\mathcal{N}\left(\nabla u^{\mathrm{i}}\cdot\mathbf{n}\right)=-u^{\mathrm{i}}$ so

$$-ik\int_{-H}^{H}\left(\mathcal{N}\left(\nabla u^{\mathrm{i}}\cdot\mathbf{n}\right)-u^{\mathrm{i}}\right)\overline{\left(\mathcal{N}\left(\nabla v\cdot\mathbf{n}\right)-v\right)}dy = 2ik\int_{-H}^{H}u^{\mathrm{i}}\overline{\left(\mathcal{N}\left(\nabla v\cdot\mathbf{n}\right)-v\right)}dy$$

so the b_m is

$$2ik\int_{-H}^{H}u^{\mathrm{i}}\overline{\left(\mathcal{N}\left(\nabla\psi_m\cdot\mathbf{n}\right)-\psi_m\right)}dy = 2ik\int_{-H}^{H}u^{\mathrm{i}}\overline{\left(\mathcal{N}\left(e^{ik\mathbf{d}_m\cdot\mathbf{x}}ik\mathbf{d}_m\cdot\mathbf{n}\right)-e^{ik\mathbf{d}_m\cdot\mathbf{x}}\right)}dy$$

$$= 2ikHe^{i\left(kHd_{m,x}-\sqrt{(kH)^2-(t\pi)^2}\right)\frac{R}{H}}\frac{1}{H}\int_{-H}^{H}\cos\left(t\pi\frac{y}{H}\right)\overline{\left(ik\mathbf{d}_m\cdot\mathbf{n}\mathcal{N}\left(e^{ikd_{m,y}y}\right)-e^{ikd_{m,y}y}\right)}dy$$

lets break it in two

$$= -\frac{1}{H}\int_{-H}^{H}\cos\left(t\pi\frac{y}{H}\right)\overline{e^{ikd_{m,y}y}}dy = -\frac{1}{H}\int_{-H}^{H}\cos\left(t\pi\frac{y}{H}\right)e^{-ikd_{m,y}y}dy$$

if $d_{m,y}=0$

$$
= \begin{cases} -2 & t = 0 \\ 0 & t > 0 \end{cases}
$$

else

$$
-\frac{1}{H}\int_{-H}^{H}\cos\left(t\pi\frac{y}{H}\right)e^{-ikd_{m,y}y}dy
$$

$$
= -\frac{1}{H}\int_{-H}^{H}\frac{\left(e^{it\pi\frac{y}{H}}+e^{-it\pi\frac{y}{H}}\right)}{2}e^{-ikHd_{m,y}\frac{y}{H}}dy = -\frac{1}{H}\int_{-H}^{H}\frac{\left(e^{i(t\pi-kHd_{m,y})\frac{y}{H}}+e^{-i(t\pi+kHd_{m,y})\frac{y}{H}}\right)}{2}dy
$$

$$
= -\left(\frac{\sin\left(t\pi-kHd_{m,y}\right)}{t\pi-kHd_{m,y}}+\frac{\sin\left(t\pi+kHd_{m,y}\right)}{t\pi+kHd_{m,y}}\right)
$$

wich for $t = 0$ gives

$$
= -2\frac{\sin\left(kHd_{m,y}\right)}{kHd_{m,y}}
$$

and the other term:

$$
-ik\mathbf{d}_m\cdot\mathbf{n}\frac{1}{H}\int_{-H}^{H}\cos\left(t\pi\frac{y}{H}\right)\overline{\mathcal{N}\left(e^{ikd_{m,y}y}\right)}dy
$$

$$
-ik\mathbf{d}_m\cdot\mathbf{n}\frac{1}{H}\int_{-H}^{H}\cos\left(t\pi\frac{y}{H}\right)\overline{\left(\frac{1}{2ikH}\int_{-H}^{H}e^{ikd_{m,y}\eta}\,\mathrm{d}\eta+\frac{1}{H}\sum_{s=1}^{\infty}\frac{1}{i\sqrt{(kH)^2-(s\pi)^2}}\int_{-H}^{H}e^{ikd_{m,y}\eta}\cos\left(s\pi\frac{\eta}{H}\right)\mathrm{d}\right.}
$$

so, if $d_{m,y} = 0$ then

$$
= -ik\mathbf{d}_m\cdot\mathbf{n}\frac{1}{H}\int_{-H}^{H}\cos\left(t\pi\frac{y}{H}\right)\overline{\left(\frac{1}{ik}\right)}dy = \mathbf{d}_m\cdot\mathbf{n}\frac{1}{H}\int_{-H}^{H}\cos\left(t\pi\frac{y}{H}\right)dy
$$

$$
= \begin{cases} 2\mathbf{d}_m\cdot\mathbf{n} & t = 0 \\ 0 & t > 0 \end{cases}
$$

else

$$
-ik\mathbf{d}_m\cdot\mathbf{n}\frac{1}{H}\int_{-H}^{H}\cos\left(t\pi\frac{y}{H}\right)\overline{\left(\frac{1}{2ikH}\int_{-H}^{H}e^{ikd_{m,y}\eta}\,\mathrm{d}\eta+\frac{1}{H}\sum_{s=1}^{\infty}\frac{1}{i\sqrt{(kH)^2-(s\pi)^2}}\int_{-H}^{H}e^{ikd_{m,y}\eta}\cos\left(s\pi\frac{\eta}{H}\right)\mathrm{d}\right.}
$$

$$
= \begin{cases} \mathbf{d}_m\cdot\mathbf{n}\frac{\sin(kHd_{m,y})}{kHd_{m,y}} & t = 0 \\ \mathbf{d}_m\cdot\mathbf{n}\frac{kH}{\sqrt{(kH)^2-(t\pi)^2}}\left(\frac{\sin(kHd_{m,y}+t\pi)}{kHd_{m,y}+t\pi}+\frac{\sin(kHd_{m,y}-t\pi)}{kHd_{m,y}-t\pi}\right) & t > 0 \end{cases}
$$

and FINALLY the LHS

A_{mn}

$$-ik\int_{-H}^{H}\left(\mathcal{N}\left(\nabla\phi_n\cdot\mathbf{n}\right)-\phi_n\right)\overline{\left(\mathcal{N}\left(\nabla\psi_m\cdot\mathbf{n}\right)-\psi_m\right)}dy$$