Auxiliar problem

$$\begin{cases} \Delta u_f + k^2 u_f = 0 & \text{in } \mathbb{R}^2 \setminus \overline{\mathrm{B}_R} \\ \nabla u_f \cdot \mathbf{n} = f & \text{on } \partial \mathrm{B}_R \\ \frac{\partial u_f}{\partial r} - iku_f = o\left(\frac{1}{r}\right) & \text{on } r \to \infty, \text{ uniformly on } \theta \end{cases}$$

Neumann to Dirichled operator:

$$\mathcal{N} : L^2\left(\partial \mathrm{B}_R\right) \to L^2\left(\partial \mathrm{B}_R\right)$$

$$u_f = \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{2}{k} \frac{a_n}{H_{n-1}^{(1)}(kR) - H_{n+1}^{(1)}(kR)} H_n^{(1)}(kr) e^{in\theta}$$

$$\nabla u_f \cdot \mathbf{n} = \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{2}{k} \frac{a_n}{H_{n-1}^{(1)}(kR) - H_{n+1}^{(1)}(kR)} \frac{\partial H_n^{(1)}(kr)}{\partial r} e^{in\theta}$$

but

$$\frac{dH_\alpha^{(1)}(x)}{dx} = \frac{H_{\alpha-1}^{(1)}(x) - H_{\alpha+1}^{(1)}(x)}{2}$$

so

$$\nabla u_f \cdot \mathbf{n} = \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} a_n \frac{H_{n-1}^{(1)}(kr) - H_{n+1}^{(1)}(kr)}{H_{n-1}^{(1)}(kR) - H_{n+1}^{(1)}(kR)} e^{in\theta}$$

$$\nabla u_f \cdot \mathbf{n}\big|_{r=R} = \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} a_n e^{in\theta}$$

so

$$\nabla u_f \cdot \mathbf{n}\big|_{r=R} = f$$

implies that a_n are the coefficients of the Fourier expansion of $f(\theta)$ and the Neuman to Dirichlet map can be evaluated as:

$$\begin{aligned} \mathcal{N}(f) = u_f\big|_{r=R} &= \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{2}{k} \frac{a_n}{H_{n-1}^{(1)}(kR) - H_{n+1}^{(1)}(kR)} H_n^{(1)}(kR) e^{in\theta} \\ &= \sum_{n=-\infty}^{\infty} \frac{2}{k} \frac{H_n^{(1)}(kR)}{H_{n-1}^{(1)}(kR) - H_{n+1}^{(1)}(kR)} a_n \frac{e^{in\theta}}{\sqrt{2\pi}} \end{aligned}$$

where a_n are the coefficients of the Fourier expansion of $f(\theta)$ (it was nicer on the waveguide)

$$a_n =$$

$$\mathcal{N}\left(f\right)=\sum_{n=-\infty}^{\infty}\frac{1}{\pi k}\frac{H_n^{(1)}\left(kR\right)}{H_{n-1}^{(1)}\left(kR\right)-H_{n+1}^{(1)}\left(kR\right)}\left(\int_0^{2\pi}f\left(\beta\right)e^{-in\beta}d\beta\right)e^{in\theta}$$

Fluxes:

$$-\int_{\Sigma}\left(\mathcal{N}_{k,\Sigma}\left(\nabla_h w\cdot\mathbf{n}\right)\nabla_h\overline{v}\cdot\mathbf{n}-\nabla_h w\cdot\mathbf{n}\overline{v}-dik\left(\mathcal{N}_{k,\Sigma}\left(\nabla_h w\cdot\mathbf{n}\right)-w\right)\overline{\left(\mathcal{N}_{k,\Sigma}\left(\nabla_h v\cdot\mathbf{n}\right)-v\right)}\right)dS$$

vamos a probar primero con los términos fáciles XD, los que no tienen d, por ejemplo

$$\int_{\Sigma}\mathcal{N}_{k,\Sigma}\left(\nabla_h w_n\cdot\mathbf{n}\right)\nabla_h\overline{v}_m\cdot\mathbf{n}$$

$w_n=e^{ik\mathbf{d}_n\cdot\mathbf{x}}$, $v_m=e^{ik\mathbf{d}_m\cdot\mathbf{x}}$

$$k^2\mathbf{d}_m\cdot\mathbf{n}\mathbf{d}_n\cdot\mathbf{n}\int_{\Sigma}\mathcal{N}_{k,\Sigma}\left(e^{ik\mathbf{d}_n\cdot\mathbf{x}}\right)e^{-ik\mathbf{d}_m\cdot\mathbf{x}}=$$

$$k^2\mathbf{d}_m\cdot\mathbf{n}\mathbf{d}_n\cdot\mathbf{n}\int_{\Sigma}\sum_{n=-\infty}^{\infty}\left(\frac{1}{\pi k}\frac{H_n^{(1)}\left(kR\right)}{H_{n-1}^{(1)}\left(kR\right)-H_{n+1}^{(1)}\left(kR\right)}\left(\int_0^{2\pi}f\left(\beta\right)e^{-in\beta}d\beta\right)e^{in\theta}\right)e^{-ik\mathbf{d}_m\cdot\mathbf{x}}\,dS_x=$$

$$k\frac{1}{\pi}\frac{H_n^{(1)}\left(kR\right)}{H_{n-1}^{(1)}\left(kR\right)-H_{n+1}^{(1)}\left(kR\right)}\mathbf{d}_m\cdot\mathbf{n}\mathbf{d}_n\cdot\mathbf{n}\int_{\Sigma}\sum_{n=-\infty}^{\infty}\left(\int_0^{2\pi}f\left(\beta\right)e^{-in\beta}d\beta\right)e^{in\theta}e^{-ik\mathbf{d}_m\cdot\mathbf{x}}\,dS_x=$$