3-1.(1) 首先对状态波函数做归一化.

(a) $N=\sqrt{\frac{30}{a^5}}$ (b) $N=\sqrt{\frac{8}{3a}}$

(a): 能量算符对应的本征波函数组: $\psi_n(x)=\sqrt{\frac{2}{a}}\sin\frac{n\pi}{a}x$

$$C_n=\int_0^a \frac{2\sqrt{15}}{a^3}x(x-a)\sin\frac{n\pi}{a}x\,dx=\begin{cases}\frac{4a^3}{n^3\pi^3}\cdot\frac{2\sqrt{5}}{a^3} & n\text{为奇数}\\ 0 & n\text{为偶数}\end{cases}$$

$$\therefore C_n=\begin{cases}\frac{8\sqrt{15}}{n^3\pi^3} & n\text{为奇数}\\ 0 & n\text{为偶数}\end{cases}\Rightarrow |C_n|^2=\begin{cases}960/\pi^6 n^6, & n\text{为奇数}\\ 0 & n\text{为偶数}\end{cases}$$

(b) $C_n=\int_0^a\sqrt{\frac{8}{3a}}\cdot\sqrt{\frac{2}{a}}\sin^2\left(\frac{\pi x}{a}\right)\sin\frac{n\pi x}{a}dx\to$ 和差化积公式

$$=\begin{cases}\frac{4}{\sqrt{3}a}\cdot\frac{4}{\pi n(n^2-4)} & n=1,3,5\cdots\\ 0 & n=2,4,6\cdots\end{cases}$$

~~n为奇数~~

$$|C_n|^2=\begin{cases}\frac{256}{3\pi^2n^2(n^2-4)^2} & n=1,3,5\\ 0 & n=2,4,6\end{cases}$$

(2) (a). $\bar{E}=\int\psi^*\hat{E}\psi\,dx=\frac{5\hbar^2}{ma^2}$

$E_n=\frac{\hbar^2\pi^2n^2}{2ma^2}$ $\therefore\bar{E}=\frac{10}{\pi^2}E_1$

(b) $\bar{E}=\int\psi^*\hat{E}\psi\,dx=\frac{2\pi^2\hbar^2}{3ma^2}$

$\therefore\bar{E}=\frac{4}{3}E_1$

3-2 $\psi(x) = A(\sin^2 kx + \frac{1}{2}\cos kx)$

$= \frac{A}{2} - \frac{A}{2}e^{2ikx} + \frac{A}{2}e^{ikx}$

$\bar{p} = \int \psi^* (-i\hbar \nabla) \psi \, dx$ 若 $\bar{p} \neq 0$ 则 $\bar{p}$ 是虚数

$\therefore \bar{p} = 0$

$\bar{T} = \frac{1}{4}\frac{\hbar^2}{2m} \times 4k^2 + \frac{1}{4} \cdot \frac{\hbar^2}{2m} \cdot k^2$

$= \frac{5k^2\hbar^2}{8m}$

3-3 一维 δ势阱.

$V(x) = -\alpha\delta(x) = \begin{cases} 0, & if\ x \neq 0 \\ \infty, & if\ x = 0 \end{cases}$

① $x<0$ $\psi(x) = \cancel{Ae^{-kx}} + Be^{kx}$ (peel off Ae^{-kx}, $x \to -\infty$. $\psi(x) \to 0$)

② $x>0$ $\psi(x) = Ce^{-kx}$

ψ 第一类边界条件 ∴ $B = C$

∴ $\psi(x) = \begin{cases} Be^{kx} & x \leq 0 \\ Be^{-kx} & x \geq 0 \end{cases}$

讨论 $x=0$ 点的波函数.

有. $-\frac{\hbar^2}{2m}\int_{-\varepsilon}^{+\varepsilon} \frac{d^2\psi}{dx^2}dx + \int_{-\varepsilon}^{+\varepsilon} V(x)\psi(x)\,dx = E\int_{-\varepsilon}^{+\varepsilon}\psi(x)\,dx$

$\Delta\left(\frac{d\psi}{dx}\right) = -\frac{2m\alpha}{\hbar^2}\psi(0)$

∴ $\psi(0) = B$ $k = \frac{m\alpha}{\hbar^2}$ $E = -\frac{m\alpha^2}{2\hbar^2}$

$\bar{T} = p^2/2m = (k\hbar)^2/2m = \frac{m\alpha^2}{2\hbar^2}$

$\bar{V} = \bar{E} - \bar{T} = -\frac{m\alpha^2}{\hbar^2}$

3-4 (1) 阶梯算符：$a_\pm \equiv \frac{1}{\sqrt{2\hbar m\omega}}(\mp ip + m\omega x)$

∴ $x = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-)$ $\quad p = i\sqrt{\frac{\hbar m\omega}{2}}(a_+ - a_-)$

∵ $\langle V\rangle = \langle \frac{1}{2} m\omega^2 x^2\rangle = \frac{1}{2} m\omega^2\int_{-\infty}^{\infty}\psi_n^* x^2\psi_n dx$

$x^2 = \frac{\hbar}{2m\omega}[(a_+)^2 + (a_+a_-) + (a_-a_+) + (a_-)^2]$

$\langle V\rangle = \frac{\hbar\omega}{4}\int\psi_n^*[(a_+)^2 + (a_+a_-) + (a_-a_+) + (a_-)^2]\psi_n dx$

$= \frac{\hbar\omega}{4}(n+n+1) = \frac{1}{2}\hbar\omega(n+\frac{1}{2})$

∴ $\overline{x^2} = (n+\frac{1}{2})\cdot\frac{\hbar}{m\omega}$

$\bar{x} = \int\psi_n^*\sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-)\psi_n^* dx = 0$ 同理 $\bar{p}_x = 0$.

$p^2 = -\frac{\hbar m\omega}{2}\cdot[(a_+)^2 - (a_+a_-) - (a_-a_+) + (a_-)^2]$

∴ $\overline{p^2} = \int -\frac{\hbar m\omega}{2}\psi_n^*[(a_+)^2 - (a_+a_-) - (a_-a_+) + (a_-)^2]\psi_n dx$

$= (n+\frac{1}{2})m\hbar\omega$

∴ $\overline{(\Delta x)^2} = (n+\frac{1}{2})\frac{\hbar}{m\omega}$ $\quad \overline{(\Delta p)^2} = (n+\frac{1}{2})m\hbar\omega$

∴ $\overline{(\Delta x)^2}\cdot\overline{(\Delta p)^2} = (n+\frac{1}{2})^2\hbar^2$

3-4 (2) $\sqrt{\frac{1}{2}}^2+\sqrt{\frac{1}{5}}^2+C^2=1 \Rightarrow C=\sqrt{\frac{3}{10}}$

$E_n=(n+\frac{1}{2})\hbar\omega$

$\therefore E_0=\frac{1}{2}\hbar\omega$; $E_1=\frac{3}{2}\hbar\omega$; $E_2=\frac{5}{2}\hbar\omega$

对于概率 $|C_0|^2=\frac{1}{2}$ $|C_1|^2=\frac{1}{5}$ $|C_2|^2=\frac{3}{10}$

$\overline{E}=\sum_i E_i|C_i|^2=(\frac{1}{4}+\frac{3}{10}+\frac{15}{20})\hbar\omega=\frac{13}{10}\hbar\omega.$

3-5 (1) $\hat{H}=-\frac{\hbar^2}{2m}\nabla^2+V$

$=\frac{i\cdot i\hbar}{2m}\nabla\cdot\nabla+V$

$=\frac{1}{2m}(\hat{p}\cdot\hat{p})+V$

这里要将 $\hat{H}$ 拆开以便于在积分号内运用厄米算符的性质

$\therefore \overline{E}=\int\psi^*\hat{H}\psi\,d\tau=\int\psi^*[\frac{1}{2m}(\hat{p}\cdot\hat{p})+V]\psi^*\,d\tau.$

$=\int[\frac{1}{2m}\psi^*\hat{p}\cdot\hat{p}\psi+\psi^*V\psi]\,d\tau$

$=\int[\frac{\hbar^2}{2m}\nabla\psi^*\cdot\nabla\psi+\psi^*V\psi]\,d\tau.$

(2) 证明: $\frac{\partial w}{\partial t}+\nabla\cdot\vec{S}=0$ 其中, $\overline{E}=\int w\,d\tau$ $\vec{S}\equiv-\frac{\hbar^2}{2m}[\frac{\partial\psi^*}{\partial t}\nabla\psi+\frac{\partial\psi}{\partial t}\nabla\psi^*]$

证: $\frac{\partial w}{\partial t}=\frac{\hbar^2}{2m}\nabla\frac{\partial}{\partial t}\psi^*\cdot\nabla\psi+\frac{\hbar^2}{2m}\nabla\psi^*\cdot\nabla\frac{\partial}{\partial t}\psi+\frac{\partial}{\partial t}\psi^*V\psi+\psi^*V\frac{\partial}{\partial t}\psi$

由薛定谔方程: $(-\frac{\hbar^2}{2m}\nabla^2+V)\psi=+i\hbar\frac{\partial}{\partial t}\psi$

$\therefore -\frac{\hbar^2}{2m}\nabla^2\psi+V\psi=i\hbar\frac{\partial}{\partial t}\psi$ ①

$-\frac{\hbar^2}{2m}\nabla^2\psi^*+V\psi^*=-i\hbar\frac{\partial}{\partial t}\psi^*$ ②

$\frac{\partial}{\partial t}\psi^*①+\frac{\partial}{\partial t}\psi② \Rightarrow \frac{\partial}{\partial t}\psi^*V\psi+\psi^*V\frac{\partial}{\partial t}\psi=+\frac{\hbar^2}{2m}\nabla^2\psi\cdot\frac{\partial}{\partial t}\psi^*+\frac{\hbar^2}{2m}\nabla^2\psi^*\frac{\partial}{\partial t}\psi$

$\therefore \frac{\partial w}{\partial t}=\frac{\hbar^2}{2m}\nabla\frac{\partial}{\partial t}\psi^*\cdot\nabla\psi+\frac{\hbar^2}{2m}\nabla\psi^*\cdot\nabla\frac{\partial}{\partial t}\psi+\frac{\hbar^2}{2m}\nabla^2\psi\frac{\partial}{\partial t}\psi^*+\frac{\hbar^2}{2m}\nabla^2\psi^*\frac{\partial}{\partial t}\psi$

$\nabla\cdot\vec{S}=-\frac{\hbar^2}{2m}[\frac{\partial\psi^*}{\partial t}\nabla^2\psi+\frac{\partial\psi}{\partial t}\nabla^2\psi^*]$

$\therefore \frac{\partial w}{\partial t}+\nabla\cdot\vec{S}=\frac{\hbar^2}{2m}\nabla\frac{\partial}{\partial t}\psi^*\cdot\nabla\psi+\frac{\hbar^2}{2m}\nabla\psi^*\cdot\nabla\frac{\partial}{\partial t}\psi$

$=\frac{\hbar^2}{2m}[\nabla(\hat{H}\psi^*)\cdot\nabla\psi+\nabla\psi^*\cdot\nabla\hat{H}\psi]$

$=\frac{\hbar^2}{2m}[-\lambda\nabla\psi^*\cdot\nabla\psi+\lambda\nabla\psi^*\cdot\nabla\psi]=0$

即 $\frac{\partial w}{\partial t}+\nabla\cdot\vec{S}=0.$

3-6. (1) $[\hat{L}_x, y]\psi = y^2\frac{\hbar}{i}\frac{\partial}{\partial z}\psi - z\frac{\hbar}{i}\psi - yz\frac{\hbar}{i}\frac{\partial}{\partial y}\psi$

$\quad - y^2\frac{\hbar}{i}\frac{\partial}{\partial z}\psi + yz\frac{\hbar}{i}\frac{\partial}{\partial y}\psi$

$\quad = -z\frac{\hbar}{i}\psi$

∴ $[\hat{L}_x, y] = i\hbar z$. 同理 $[\hat{L}_x, z] = -i\hbar y$ $[\hat{L}_x, x] = 0$

∴有 $[\hat{L}_\alpha, \hat{x}_\beta] = \varepsilon_{\alpha\beta\gamma} i\hbar \hat{x}_\gamma$

∴同理可得. $[\hat{L}_\alpha, \hat{p}_\beta] = \varepsilon_{\alpha\beta\gamma} i\hbar \hat{p}_\gamma$

(2) $[\hat{L}_\alpha, \hat{r}^2] = [\hat{L}_\alpha, \hat{x}_\alpha^2] + [\hat{L}_\alpha, \hat{x}_\beta^2] + [\hat{L}_\alpha, \hat{x}_\gamma^2]$

$= [\hat{L}_\alpha, \hat{x}_\alpha]\hat{x}_\alpha + \hat{x}_\alpha[\hat{L}_\alpha, \hat{x}_\alpha]$

$+ [\hat{L}_\alpha, \hat{x}_\beta]\hat{x}_\beta + \hat{x}_\beta[\hat{L}_\alpha, \hat{x}_\beta]$

$+ [\hat{L}_\alpha, \hat{x}_\gamma]\hat{x}_\gamma + \hat{x}_\gamma[\hat{L}_\alpha, \hat{x}_\gamma]$

$= 0$

同理可得. $[\hat{L}_\alpha, \hat{r}^2] = [\hat{L}_\alpha, \hat{p}^2] = [\hat{L}_\alpha, \hat{p}\cdot\hat{r}] = 0$

(3) $[\hat{L}_\alpha, (\hat{p}\cdot\hat{r})\hat{p}] = (\hat{p}\cdot\hat{r})[\hat{L}_\alpha, \hat{p}] + [\hat{L}_\alpha, \hat{p}\cdot\hat{r}]\hat{p}$

$= (\hat{p}\cdot\hat{r})[\hat{L}_\alpha, \hat{p}]$

$= (\hat{p}\cdot\hat{r})\varepsilon_{\alpha\beta\gamma} i\hbar \hat{p}_\gamma - (\hat{p}\cdot\hat{r})\varepsilon_{\alpha\beta\gamma} i\hbar \hat{p}_\beta$

同理有: $[\hat{L}_\alpha, (a\hat{r} + b\hat{p})] = \varepsilon_{\alpha\beta\gamma} i\hbar (a\hat{r} + b\hat{p})_\gamma - \varepsilon_{\alpha\beta\gamma} i\hbar (a\hat{r} + b\hat{p})_\beta$

(4) $[\hat{L}_\alpha, \hat{x}_\beta \hat{x}_\gamma] = \hat{x}_\beta [\hat{L}_\alpha, \hat{x}_\gamma] + [\hat{L}_\alpha, \hat{x}_\beta] \hat{x}_\gamma$

$= i\hbar (\hat{x}_\gamma \hat{x}_\gamma - \hat{x}_\beta \hat{x}_\beta) = i\hbar (\hat{x}_\gamma^2 - \hat{x}_\beta^2)$

同理有: $[\hat{L}_\alpha, \hat{p}_\beta \hat{p}_\gamma] = i\hbar (\hat{p}_\gamma^2 - \hat{p}_\beta^2)$

$[\hat{L}_\alpha, \hat{x}_\beta \hat{p}_\gamma] = i\hbar (\hat{x}_\gamma \hat{p}_\gamma - \hat{x}_\beta \hat{p}_\beta)$

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