Sheet 2 - Transformations

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Assigment 2

Let $a, b, c \in \mathbb{R}^3$.

a) show that $a \times (b+c) = a \times b + a \times c$

Proof: $a \times (b+c) = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \times \begin{pmatrix} b_x + c_x \\ b_y + c_y \\ b_z + c_z \end{pmatrix}$

$$= \begin{pmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{pmatrix} \cdot \begin{pmatrix} b_x + c_x \\ b_y + c_y \\ b_z + c_z \end{pmatrix}$$

$$= \begin{pmatrix} -(b_y+c_y)a_z + (b_z+c_z)a_y \\ (b_x+c_x)a_z - (b_z+c_z)a_x \\ -(b_x+c_x)a_y + (b_y+c_y)a_x \end{pmatrix}$$

$$a \times b + a \times c = \begin{pmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{pmatrix} \cdot \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} + \begin{pmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{pmatrix} \cdot \begin{pmatrix} c_x \\ c_y \\ c_z \end{pmatrix}$$

$$= \begin{pmatrix} -a_z b_y + a_y b_z \\ a_z b_x - a_x b_z \\ -a_y b_x + a_x b_y \end{pmatrix} + \begin{pmatrix} -a_z c_y + a_y c_z \\ a_z c_x - a_x c_z \\ -a_y c_x + a_x c_y \end{pmatrix}$$

$$= \begin{pmatrix} -(b_y+c_y)a_z + (b_z+c_z)a_y \\ (b_x+c_x)a_z - (b_z+c_z)a_x \\ -(b_x+c_x)a_y + (b_y+c_y)a_x \end{pmatrix}$$

so $a \times (b+c) = a \times b + a \times c$

b) Show that $a \cdot (a \times b) = 0$

Proof: set that $a = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}$ $\quad b = \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix}$

$$a \times b = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \times \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} = \begin{pmatrix} -a_z b_y + a_y b_z \\ a_z b_x - a_x b_z \\ -a_y b_x + a_x b_y \end{pmatrix}$$

$$a \cdot (a \times b) = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \begin{pmatrix} -a_z b_y + a_y b_z \\ a_z b_x - a_x b_z \\ -a_y b_x + a_x b_y \end{pmatrix}$$

$$= -a_x a_z b_y + a_x a_y b_z + a_y a_z b_x - a_x a_y b_z -$$
$$a_z a_y b_x + a_z a_x b_y$$

$$= 0$$

Therefore $a \cdot (a \times b) = 0$.

c) Proof:

① $$|a\times b| = |(a_1e_x + a_2e_y + a_3e_z)\times(b_1e_x + b_2e_y + b_3e_z)|$$

$$= a_1b_1\, e_x\times e_x + a_1b_2\, e_x\times e_y + a_1b_3\, e_x\times e_z$$
$$+ a_2b_1\, e_y\times e_x + a_2b_2\, e_y\times e_y + a_2b_3\, e_y\times e_z$$
$$+ a_3b_1\, e_z\times e_x + a_3b_2\, e_z\times e_y + a_3b_3\, e_z\times e_z$$

since $e_x\times e_x = 0$ $e_y\times e_y = 0$ $e_z\times e_z = 0$

$e_x\times e_y = e_z$ $e_x\times e_z = -e_y$ $e_y\times e_z = e_x$

so $$|a\times b| = |(a_2b_3 - a_3b_2)e_x + (a_3b_1 - a_1b_3)e_y + (a_1b_2 - a_2b_1)e_z|$$

$$= \left|\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} e_x - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} e_y + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} e_z\right|$$

③ In formal case, the notation has the form of a determinate

Using cofactor expansion: $\begin{vmatrix} e_x & e_y & e_z \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} e_x - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} e_y + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} e_z$

The determinent can be computed, it expands to

$$|a\times b| = \left|\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} e_x - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} e_y + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} e_z\right|$$

Therefore $$|a\times b| = \begin{vmatrix} e_x & e_y & e_z \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

d) Proof: $a \times b = \begin{pmatrix} -a_z b_y + a_y b_z \\ a_z b_x - a_x b_z \\ -a_y b_x + a_x b_y \end{pmatrix}$

$$c \cdot (a \times b) = \begin{pmatrix} c_x \\ c_y \\ c_z \end{pmatrix} \cdot \begin{pmatrix} -a_z b_y + a_y b_z \\ a_z b_x - a_x b_z \\ -a_y b_x + a_x b_y \end{pmatrix}$$

$$= -a_z b_y c_x + a_y b_z c_x + a_z b_x c_y - a_x b_z c_y - a_y b_x c_z + a_x b_y c_z$$

$$b \times c = \begin{pmatrix} -b_z c_y + b_y c_z \\ b_z c_x - b_x c_z \\ -b_y c_x + b_x c_y \end{pmatrix}$$

$$a \cdot (b \times c) = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \cdot \begin{pmatrix} -b_z c_y + b_y c_z \\ b_z c_x - b_x c_z \\ -b_y c_x + b_x c_y \end{pmatrix}$$

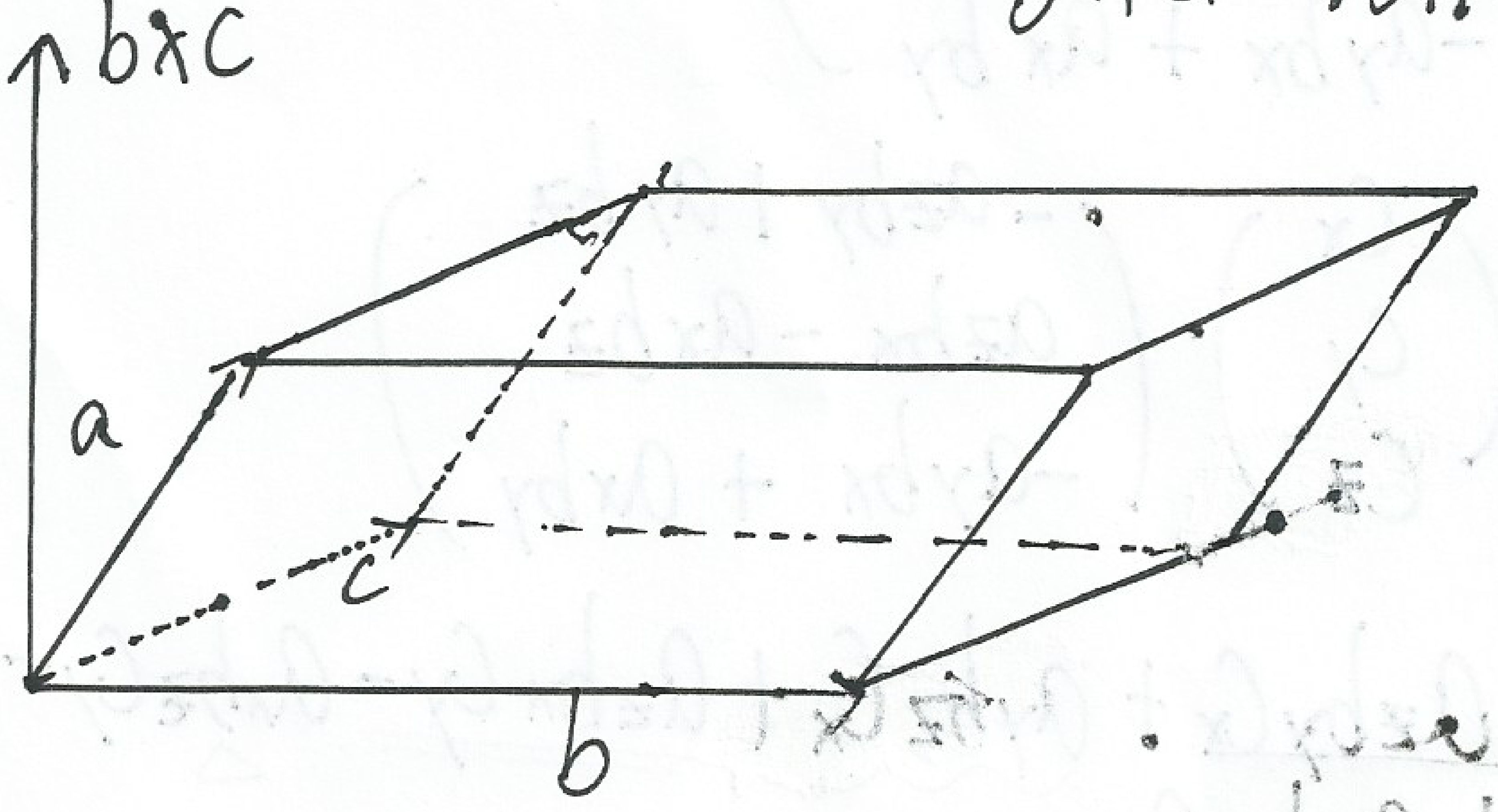
$$= -a_x b_z c_y + a_x b_y c_z + a_y b_z c_x - a_y b_x c_z - a_z b_y c_x + a_z b_x c_y$$

$$a \cdot (b \times c) - c \cdot (a \times b) = 0$$

therefore $a \cdot (b \times c) = c \cdot (a \times b)$

(e) Proof

$b \times c = |b||c| \cdot \sin\theta$



$(b \times c)$ is the area of the parallelogram base

the direction of $(b \times c)$ is perpendicular to the base

The height of the paralelepiped is the component of a in the direction normal to the base.

The height is $|a|\cos\phi$, where ϕ is the angle between a and $b \times c$

The volume is therefore: $\text{Volume} = (b \times c)|a| \cdot \cos\phi$

$= a \cdot (b \times c)$