Sheet 9

Group 26 Tu 16:00

Ming Qu

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Assignment 2.

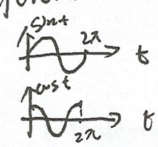
a) $p(t) = (R\cos(t), R\sin(t))$, $t \in [0, 2\pi]$

b) $T(t) = p'(t) = (R\cos'(t)^*, R\sin'(t)) = (-R\sin(t), R\cos(t))$

$T(\frac{\pi}{4}) = (-2\sin\frac{\pi}{4}, 2\cos\frac{\pi}{4}) = (-\sqrt{2}, \sqrt{2})$

c) Yes. $T(t) = p'(t) = (-R\sin t, R\cos t)$,

so p is continuously differentiable.

for all $t \in [0, 2\pi]$,  , $p'(t) \neq 0$.

Thus, this curve is regular.

d) No. $\|p'(t)\| = \sqrt{R^2(\sin^2 t + \cos^2 t)} = R = 2$.

Since a curve P is arc length parametrized, if $\|P'(t)\| = 1$.

So, this curve is not arc length parametrized

Assignment 3.

a) $P(t) = \sum_{i=0}^{n} b_i B_i^n(t)$ $t \in [0,1]$ $b_i \in \mathbb{R}^d$

where $B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$, $i = 0, \dots n$

The properties of Berstein polynomials:

① $\sum_{i=0}^{n} B_i^n(t) = 1$ Partition of unity

② $B_i^n(t) \geq 0$, $t \in [0,1]$ Nonnegativity

③ $B_i^n(t) = B_{n-i}^n(1-t)$ Symmetry

④ $B_i^n(t) = t\, B_{i-1}^{n-1}(t) + (1-t)\, B_i^{n-1}(t)$ Recursive definition

b) Definition:

A Bezier curve is a parametric curve that uses the Berstein polynomials as a basis. A Bézier curve of degree n (order $n+1$) is represented by $P(t) = \sum_{i=0}^{n} b_i B_i^n(t)$ $0 \leq t \leq 1$.

The coefficients b_i are control points or Bézier points and together with the basis function $B_i^n(t)$ determine the shape of the curve. Lines drawn between consecutive control points of the curve form the control polygon. The Bézier curve approximates the control polygon, it doesn't interpolate it.

Properties:

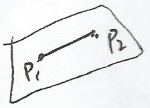
① Geometry invariance property: since $\sum_{i=0}^{n} B_i^n(t) = 1$, curve is invariant under affine transform

② Convex hull property: since $B_i^n(t) \geq 0$, the curve is contained in the convex hull of the control points

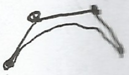
③ End points geometry property:

- The first and the last control points are end points of the curve. $b_0 = P(0)$, $b_n = P(1)$
- The tangents of the curve in the points b_0 and b_n are the first and last segments of the control polygon

c) Proof: A domain D is convex if for any two points P_1 and P_2 in the domain, the segment $\overline{P_1P_2}$ is entirely contained in the domain D. [1]



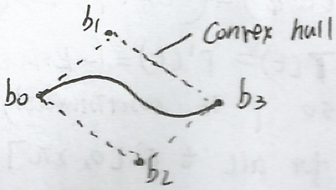
It can be shown that the intersection of convex domains is a convex domain. The convex hull of a set of points P is the boundary of the smallest convex domain containing P.



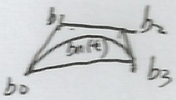
Using the above definitions and facts, the convex hull of a Bezier curve is the boundary of the intersection of all the convex sets containing all vertices or the intersection of the half spaces generated by taking three vertices at a time to construct a plane and having all vertices on one side.

The entire curve is contained within the convex hull of the control points.

~~by de casteljau's algorithm~~



By barycentric combination: $b^n(t) := \sum_{i=0}^{n} \alpha_i(t)\, b_i \Rightarrow$



since $\begin{cases} \sum_{i=0} \alpha_i = 1 \\ \alpha_i \geq 0 \\ \alpha_i(t) = B_i^n(t) \end{cases}$ recall $\sum_{i=0}^{n} B_i^n(t) = 0$, $B_i^n(t) \geq 0$ for $t \in [0,1]$

Reference [1] F. P. Preparata and M. I. Shamos. Computational Geometry: An Introduction. Springer-Verlag, New York, 1985.

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Assignment 4.

a) From computational: Splines are piecewise polynomial functions, where each polynomial segment has only a limited degree. But the Bezier curve's degree is tied to the number of control points. So, the order (splines) < the order (Bezier curve)

From practical: Splines are more controlable and flexible than Bezier curve. If we change some control points, Bezier curve will be changed entirely, but splines only change related segments.

b) Definition:

Spline is a numeric function that is defined by polynomial functions, and which possesses a high degree of smoothness at the places where the polynomial pieces connect (which are known as knots).

The intervals $[t_i, t_{i+1}]$, $i=0, \dots n-1$ form a knot vector $T=(t_0, t_1 \dots t_n)$ where $t_0 \le t_1 \le \dots \le t_n$. Each interval maps a polygon segment $P_i: [t_{i-1}, t_i] \to R$ (spline segment)

Set the spline S:

Then

$$S_1(t) = P_1(t), \quad t_0 \le t \le t_1$$
$$S_2(t) = P_2(t), \quad t_1 \le t \le t_2$$
$$\vdots$$
$$S_n(t) = P_n(t), \quad t_{n-1} \le t \le t_n$$
$$S(t) = S_1(t) + S_2(t) + \dots + S_n(t).$$

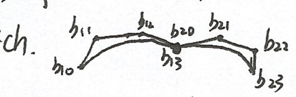
c) Let $q_1: [a_1, b_1] \to \mathbb{R}^3$, $q_2: [a_2, b_2] \to \mathbb{R}^3$ be two n times continuously differentiable regular curves

C^n-continuous means that at the points b_1, a_2, $q_1^{(k)}(b_1) = q_2^{(k)}(a_2)$ for all $k=0, \dots, n$

G^n-continuous means that at the points b_1, a_2, there exists a re-parameterization $\tilde{q}_1: [\tilde{a}_1, \tilde{b}_1] \to \mathbb{R}^3$ that $\tilde{q}_1$ and q_2 C^n continuous.

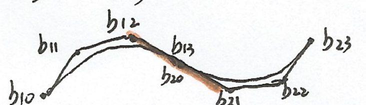
$$\tilde{q}_1^{(k)}(\tilde{b}_1) = q_2^{(k)}(a_2) \text{ for all } k=0, \dots n$$

d) C^0 condition: $P_1(b_1) = P_2(a_2)$: the end points must match.



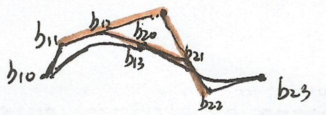
$b_{13} = b_{20}$

C^1 condition: $P_1'(b_1) = P_2'(a_2)$ and $P_1(b_1) = P_2(a_2)$: the endpoints and tangent vectors must match.



$b_{13} = b_{20}$
$b_{13} - b_{12} = b_{21} - b_{20}$

C^2 condition: C^1 condition and $P_1''(b_1) = P_2''(a_2)$: the endpoints, and tangent vectors, and curvature vectors match.



$b_{13} = b_{23}$
$b_{13} - b_{12} = b_{21} - b_{20}$
$b_{12} - (b_{12} - b_{11}) = b_{21} + (b_{21} - b_{22})$

e) B-splines: B-spline is a generalization of the Bezier curves. Let a vector known as the knot vector be defined $T = \{t_0, t_1, \dots t_m\}$, and define control points $P_0, \dots P_n$ and the degree as $p = m+n-1$.

The "knots" $t_{p+1}, \dots t_{m-p-1}$ are called internal knots

Define the basis function as

$$N_{i,0}(t) = \begin{cases} 1 & t \in [t_i, t_{i+1}],\ t_i < t_{i+1} \\ 0 & \text{o.w} \end{cases}$$

$$N_{i,j}(t) = \frac{t - t_i}{t_{i+j} - t_i} N_{i,j-1}(t) + \frac{t_{i+j+1} - t}{t_{i+j+1} - t_{i+1}} N_{i+1,j-1}(t). \quad \text{where } j = 1, 2, \dots p.$$

Then, the curve is defined by

$$C(t) = \sum_{i=0}^{n} P_i N_{i,p}(t)$$ is B-spline.

Properties:

1. Geometry invariance property
2. End points geometric property
3. Convex hull property
4. Local control