23.12 RLC Series AC Circuits

Learning Objectives

By the end of this section, you will be able to:

- Calculate the impedance, phase angle, resonant frequency, power, power factor, voltage, and/or current in a RLC series circuit.
- Draw the circuit diagram for an RLC series circuit.
- Explain the significance of the resonant frequency.

Impedance

When alone in an AC circuit, inductors, capacitors, and resistors all impede current. How do they behave when all three occur together? Interestingly, their individual resistances in ohms do not simply add. Because inductors and capacitors behave in opposite ways, they partially to totally cancel each other's effect. Figure 23.46 shows an RLC series circuit with an AC voltage source, the behavior of which is the subject of this section. The crux of the analysis of an RLC circuit is the frequency dependence of X_L and X_C, and the effect they have on the phase of voltage versus current (established in the preceding section). These give rise to the frequency dependence of the circuit, with important "resonance" features that are the basis of many applications, such as radio tuners.
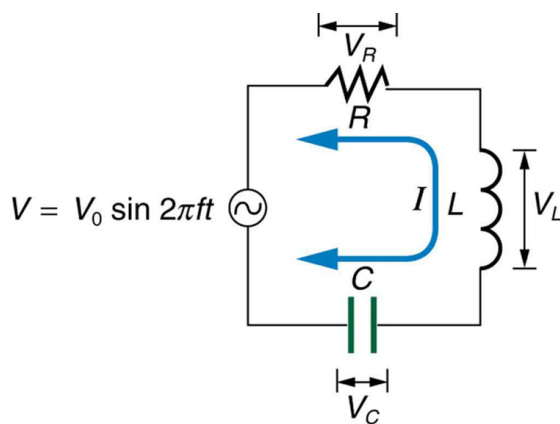

Figure 23.46 An RLC series circuit with an AC voltage source.

The combined effect of resistance R, inductive reactance X_L, and capacitive reactance X_C is defined to be impedance, an AC analogue to resistance in a DC circuit. Current, voltage, and impedance in an RLC circuit are related by an AC version of Ohm's law:

$$I_0 = \frac{V_0}{Z} \text{ or } I_{\text{rms}} = \frac{V_{\text{rms}}}{Z}.$$

23.63

Here I_0 is the peak current, V_0 the peak source voltage, and Z is the impedance

of the circuit. The units of impedance are ohms, and its effect on the circuit is as you might expect: the greater the impedance, the smaller the current. To get an expression for Z in terms of R, X_L, and X_C, we will now examine how the voltages across the various components are related to the source voltage. Those voltages are labeled V_R, V_L, and V_C in Figure 23.46.

Conservation of charge requires current to be the same in each part of the circuit at all times, so that we can say the currents in R, L, and C are equal and in phase. But we know from the preceding section that the voltage across the inductor V_L leads the current by one-fourth of a cycle, the voltage across the capacitor V_C follows the current by one-fourth of a cycle, and the voltage across the resistor V_R is exactly in phase with the current. Figure 23.47 shows these relationships in one graph, as well as showing the total voltage around the circuit $V = V_R + V_L + V_C$, where all four voltages are the instantaneous values. According to Kirchhoff's loop rule, the total voltage around the circuit V is also the voltage of the source.

You can see from Figure 23.47 that while V_R is in phase with the current, V_L leads by 90º, and V_C follows by 90º. Thus V_L and V_C are 180º out of phase (crest to trough) and tend to cancel, although not completely unless they have the same magnitude. Since the peak voltages are not aligned (not in phase), the peak voltage V_0 of the source does *not* equal the sum of the peak voltages across R, L, and C. The actual relationship is

$$V_0 = \sqrt{V_{0R}{}^2 + (V_{0L} - V_{0C})^2},$$

23.64

where V_{0R}, V_{0L}, and V_{0C} are the peak voltages across R, L, and C, respectively. Now, using Ohm's law and definitions from Reactance, Inductive and Capacitive, we substitute $V_0 = I_0 Z$ into the above, as well as $V_{0R} = I_0 R$, $V_{0L} = I_0 X_L$, and $V_{0C} = I_0 X_C$, yielding

$$I_0 Z = \sqrt{I_0^2 R^2 + (I_0 X_L - I_0 X_C)^2} = I_0 \sqrt{R^2 + (X_L - X_C)^2}.$$

23.65

I_0 cancels to yield an expression for Z:

$$Z = \sqrt{R^2 + (X_L - X_C)^2},$$

23.66

which is the impedance of an RLC series AC circuit. For circuits without a resistor, take $R = 0$; for those without an inductor, take $X_L = 0$; and for those without a capacitor, take $X_C = 0$.
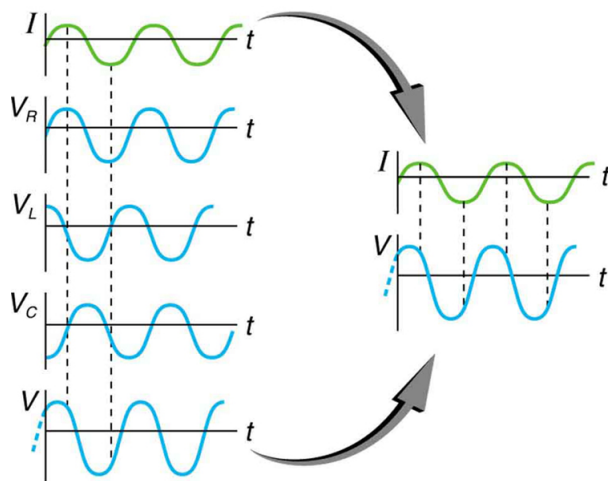
Figure 23.47 This graph shows the relationships of the voltages in an RLC circuit to the current. The voltages across the circuit elements add to equal the voltage of the source, which is seen to be out of phase with the current.

Example 23.12

Calculating Impedance and Current   An RLC series circuit has a 40.0 Ω resistor, a 3.00 mH inductor, and a 5.00 F capacitor. (a) Find the circuit's impedance at 60.0 Hz and 10.0 kHz, noting that these frequencies and the values for L and C are the same as in Example 23.10 and Example 23.11. (b) If the voltage source has $V_{\text{rms}} = 120$ V, what is I_{rms} at each frequency?

Strategy   For each frequency, we use $Z = \sqrt{R^2 + (X_L - X_C)^2}$ to find the impedance and then Ohm's law to find current. We can take advantage of the results of the previous two examples rather than calculate the reactances again.

Solution for (a)   At 60.0 Hz, the values of the reactances were found in Example 23.10 to be $X_L = 1.13$ Ω and in Example 23.11 to be $X_C = 531$ Ω. Entering these and the given 40.0 Ω for resistance into $Z = \sqrt{R^2 + (X_L - X_C)^2}$ yields

$$
\begin{aligned}
Z &= \sqrt{R^2 + (X_L - X_C)^2} \\
&= \sqrt{(40.0\ \Omega)^2 + (1.13\ \Omega - 531\ \Omega)^2} \\
&= 531\ \Omega\ \text{at 60.0 Hz.}
\end{aligned}
$$

23.67

Similarly, at 10.0 kHz, $X_L = 188$ Ω and $X_C = 3.18$ Ω, so that

3

$$Z = \sqrt{(40.0\ \Omega)^2 + (188\ \Omega - 3.18\ \Omega)^2}$$
$$= 190\ \Omega \text{ at } 10.0 \text{ kHz.}$$

23.68

Discussion for (a) In both cases, the result is nearly the same as the largest value, and the impedance is definitely not the sum of the individual values. It is clear that X_L dominates at high frequency and X_C dominates at low frequency.

Solution for (b) The current I_{rms} can be found using the AC version of Ohm's law in Equation $I_{\text{rms}} = V_{\text{rms}}/Z$:

$I_{\text{rms}} = \frac{V_{\text{rms}}}{Z} = \frac{120\text{ V}}{531\ \Omega} = 0.226$ A at 60.0 Hz

Finally, at 10.0 kHz, we find

$I_{\text{rms}} = \frac{V_{\text{rms}}}{Z} = \frac{120\text{ V}}{190\ \Omega} = 0.633$ A at 10.0 kHz

Discussion for (a) The current at 60.0 Hz is the same (to three digits) as found for the capacitor alone in Example 23.11. The capacitor dominates at low frequency. The current at 10.0 kHz is only slightly different from that found for the inductor alone in Example 23.10. The inductor dominates at high frequency.

Resonance in RLC Series AC Circuits

How does an RLC circuit behave as a function of the frequency of the driving voltage source? Combining Ohm's law, $I_{\text{rms}} = V_{\text{rms}}/Z$, and the expression for impedance Z from $Z = \sqrt{R^2 + (X_L - X_C)^2}$ gives

$I_{\text{rms}} = \frac{V_{\text{rms}}}{\sqrt{R^2+(X_L-X_C)^2}}.$

23.69

The reactances vary with frequency, with X_L large at high frequencies and X_C large at low frequencies, as we have seen in three previous examples. At some intermediate frequency f_0, the reactances will be equal and cancel, giving $Z = R$ —this is a minimum value for impedance, and a maximum value for I_{rms} results. We can get an expression for f_0 by taking

$X_L = X_C.$

23.70

Substituting the definitions of X_L and X_C,

$2\pi f_0 L = \frac{1}{2\pi f_0 C}.$

23.71

Solving this expression for f_0 yields

4

$$f_0 = \frac{1}{2\pi\sqrt{LC}},$$

23.72

where f_0 is the resonant frequency of an RLC series circuit. This is also the *natural frequency* at which the circuit would oscillate if not driven by the voltage source. At f_0, the effects of the inductor and capacitor cancel, so that $Z = R$, and I_{rms} is a maximum.

Resonance in AC circuits is analogous to mechanical resonance, where resonance is defined to be a forced oscillation—in this case, forced by the voltage source—at the natural frequency of the system. The receiver in a radio is an RLC circuit that oscillates best at its f_0. A variable capacitor is often used to adjust f_0 to receive a desired frequency and to reject others. Figure 23.48 is a graph of current as a function of frequency, illustrating a resonant peak in I_{rms} at f_0. The two curves are for two different circuits, which differ only in the amount of resistance in them. The peak is lower and broader for the higher-resistance circuit. Thus the higher-resistance circuit does not resonate as strongly and would not be as selective in a radio receiver, for example.
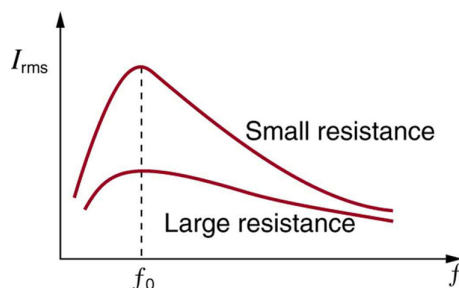

Figure 23.48 A graph of current versus frequency for two RLC series circuits differing only in the amount of resistance. Both have a resonance at f_0, but that for the higher resistance is lower and broader. The driving AC voltage source has a fixed amplitude V_0.

Example 23.13

Calculating Resonant Frequency and Current   For the same RLC series circuit having a 40.0 Ω resistor, a 3.00 mH inductor, and a 5.00 F capacitor: (a) Find the resonant frequency. (b) Calculate I_{rms} at resonance if V_{rms} is 120 V.

Strategy   The resonant frequency is found by using the expression in $f_0 = \frac{1}{2\pi\sqrt{LC}}$. The current at that frequency is the same as if the resistor alone were in the circuit.

Solution for (a)   Entering the given values for L and C into the expression given for f_0 in $f_0 = \frac{1}{2\pi\sqrt{LC}}$ yields

$$f_0 = \frac{1}{2\pi\sqrt{\text{LC}}}$$

$$= \frac{1}{2\pi\sqrt{(3.00\times10^{-3}\text{ H})(5.00\times10^{-6}\text{ F})}} = 1.30\text{ kHz.}$$

23.73

Discussion for (a)   We see that the resonant frequency is between 60.0 Hz and 10.0 kHz, the two frequencies chosen in earlier examples. This was to be expected, since the capacitor dominated at the low frequency and the inductor dominated at the high frequency. Their effects are the same at this intermediate frequency.

Solution for (b)   The current is given by Ohm's law. At resonance, the two reactances are equal and cancel, so that the impedance equals the resistance alone. Thus,

$I_{\text{rms}} = \frac{V_{\text{rms}}}{Z} = \frac{120\text{ V}}{40.0\ \Omega} = 3.00\text{ A.}$

23.74

Discussion for (b)   At resonance, the current is greater than at the higher and lower frequencies considered for the same circuit in the preceding example.

Power in RLC Series AC Circuits

If current varies with frequency in an RLC circuit, then the power delivered to it also varies with frequency. But the average power is not simply current times voltage, as it is in purely resistive circuits. As was seen in Figure 23.47, voltage and current are out of phase in an RLC circuit. There is a phase angle ϕ between the source voltage V and the current I, which can be found from

$\cos \phi = \frac{R}{Z}.$

23.75

For example, at the resonant frequency or in a purely resistive circuit $Z = R$, so that $\cos \phi = 1$. This implies that $\phi = 0$ and that voltage and current are in phase, as expected for resistors. At other frequencies, average power is less than at resonance. This is both because voltage and current are out of phase and because I_{rms} is lower. The fact that source voltage and current are out of phase affects the power delivered to the circuit. It can be shown that the *average power* is

$P_{\text{ave}} = I_{\text{rms}}V_{\text{rms}}\cos \phi,$

23.76

Thus cos ϕ is called the power factor, which can range from 0 to 1. Power factors near 1 are desirable when designing an efficient motor, for example. At the resonant frequency, cos $\phi = 1$.

Example 23.14

Calculating the Power Factor and Power   For the same RLC series circuit having a 40.0 Ω resistor, a 3.00 mH inductor, a 5.00 F capacitor, and a voltage source with a V_{rms} of 120 V: (a) Calculate the power factor and phase angle for $f = 60.0$Hz. (b) What is the average power at 50.0 Hz? (c) Find the average power at the circuit's resonant frequency.

Strategy and Solution for (a)   The power factor at 60.0 Hz is found from

$$\cos \phi = \frac{R}{Z}.$$

23.77

We know $Z= 531$ Ω from Example 23.12, so that

$$\cos \phi = \frac{40.0\ \Omega}{531\ \Omega} = 0.0753 \text{ at } 60.0 \text{ Hz}.$$

23.78

This small value indicates the voltage and current are significantly out of phase. In fact, the phase angle is

$$\phi = \cos^{-1} 0.0753 = 85.7^{\text{o}} \text{ at } 60.0 \text{ Hz}.$$

23.79

Discussion for (a)   The phase angle is close to 90º, consistent with the fact that the capacitor dominates the circuit at this low frequency (a pure RC circuit has its voltage and current 90º out of phase).

Strategy and Solution for (b)   The average power at 60.0 Hz is

$$P_{\text{ave}} = I_{\text{rms}} V_{\text{rms}} \cos \phi.$$

23.80

I_{rms} was found to be 0.226 A in Example 23.12. Entering the known values gives

$$P_{\text{ave}} = (0.226 \text{ A})(120 \text{ V})(0.0753) = 2.04 \text{ W at } 60.0 \text{ Hz}.$$

23.81

Strategy and Solution for (c)   At the resonant frequency, we know cos $\phi = 1$, and I_{rms} was found to be 6.00 A in Example 23.13. Thus,

$$P_{\text{ave}} = (3.00 \text{ A})(120 \text{ V})(1) = 360 \text{ W at resonance } (1.30 \text{ kHz})$$

Discussion Both the current and the power factor are greater at resonance, producing significantly greater power than at higher and lower frequencies.

Power delivered to an RLC series AC circuit is dissipated by the resistance alone. The inductor and capacitor have energy input and output but do not dissipate it out of the circuit. Rather they transfer energy back and forth to one another, with the resistor dissipating exactly what the voltage source puts into the circuit. This assumes no significant electromagnetic radiation from the inductor and capacitor, such as radio waves. Such radiation can happen and may even be desired, as we will see in the next chapter on electromagnetic radiation, but it can also be suppressed as is the case in this chapter. The circuit is analogous to the wheel of a car driven over a corrugated road as shown in Figure 23.49. The regularly spaced bumps in the road are analogous to the voltage source, driving the wheel up and down. The shock absorber is analogous to the resistance damping and limiting the amplitude of the oscillation. Energy within the system goes back and forth between kinetic (analogous to maximum current, and energy stored in an inductor) and potential energy stored in the car spring (analogous to no current, and energy stored in the electric field of a capacitor). The amplitude of the wheels' motion is a maximum if the bumps in the road are hit at the resonant frequency.
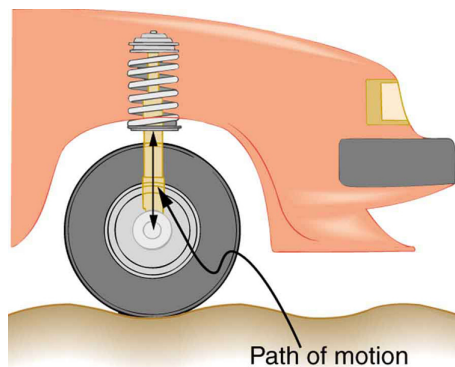

Path of motion

Figure 23.49 The forced but damped motion of the wheel on the car spring is analogous to an RLC series AC circuit. The shock absorber damps the motion and dissipates energy, analogous to the resistance in an RLC circuit. The mass and spring determine the resonant frequency.

A pure LC circuit with negligible resistance oscillates at f_0, the same resonant frequency as an RLC circuit. It can serve as a frequency standard or clock circuit—for example, in a digital wristwatch. With a very small resistance, only a very small energy input is necessary to maintain the oscillations. The circuit is analogous to a car with no shock absorbers. Once it starts oscillating, it continues at its natural frequency for some time. Figure 23.50 shows the analogy between an LC circuit and a mass on a spring.
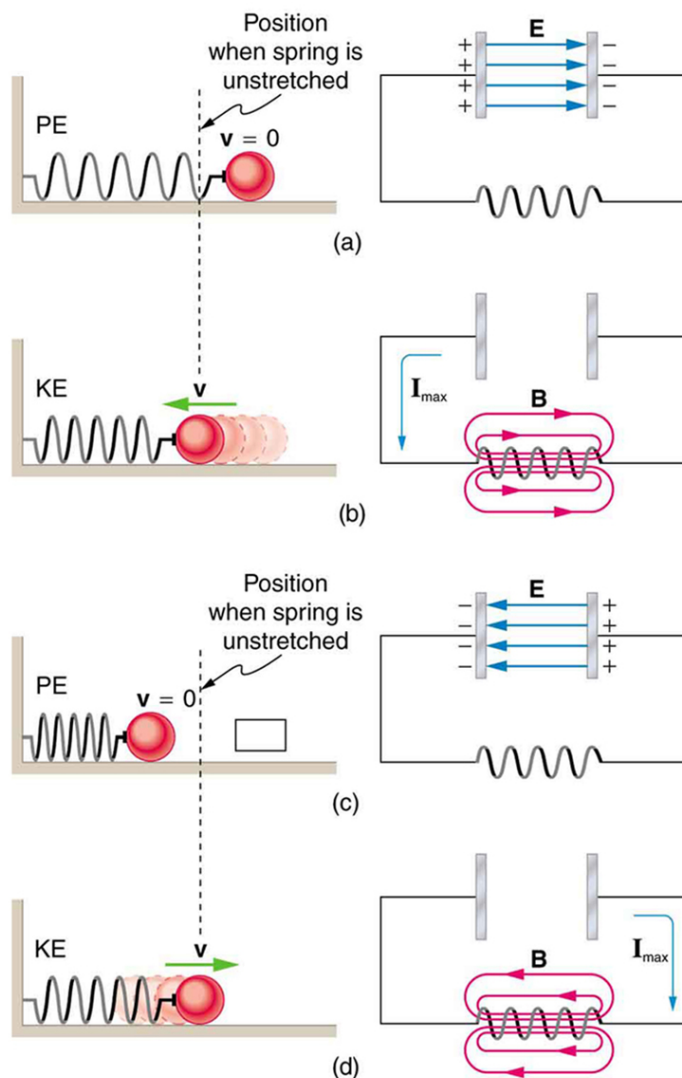
Figure 23.50 An LC circuit is analogous to a mass oscillating on a spring with no friction and no driving force. Energy moves back and forth between the inductor and capacitor, just as it moves from kinetic to potential in the mass-spring system.

PhET Explorations

Circuit Construction Kit (AC+DC), Virtual Lab Build circuits with capacitors, inductors, resistors and AC or DC voltage sources, and inspect them using lab instruments such as voltmeters and ammeters.

Click to view content