The Cubic Spline Interpolation

MA3105 Group project and Presentation

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Goal

- Presenting the cubic spline interpolation technique and the associated error analysis.
- Finding the spline interpolant of $f(x) = e^x$ on $[-1, 1]$ with three different meshes.
- Comparing the technique with linear and quadratic LIPs.

Spline interpolation

- Over an interval $\Omega = [a, b](a, b \in \mathbb{R})$, a few points and the corresponding functional values are given.
- $\Pi := \{a = x_0, x_1, \ldots, x_n = b\}$. $\forall x_i \in \Pi$, x_i is called a *knot*.
- Instead of fitting a polynomial with degree $\leq n$, we fit a polynomial s_i of degree m over every interval $[x_i, x_{i+1}], i = 0, 1, \ldots, n-1$.
- Each of $s_i, i = 0, 1, \ldots, n-1$ is called a *spline*.

A simple linear spline

- Connecting two consecutive points in the mesh Π with a straight line.
- $s_L(x) = \frac{x_i - x}{x_i - x_{i-1}} f(x_{i-1}) + \frac{x - x_{i-1}}{x_i - x_{i-1}} f(x_i)$
  $x \in [x_{i-1}, x_i], i = 1, \ldots, n$
- With $f \in C^2(\Omega)$ and $h := \max h_i$, $||f - s_L||_\infty \leq \frac{1}{8} h^2 (||f''||_\infty)$

Why spline interpolation

- Low computational cost
- Less possibility of over-fitting
- Customizability in terms of *smoothness*

The cubic spline

$f \in C(\omega)$ and the mesh is given by Π over the interval Ω.
Consider $\mathcal{S} := \{s | s \in C^2(\Omega)\}$ such that-

1. $s(x_i) = f(x_i), \forall x_i \in \Pi$
2. $deg(s) \leq 3$ on $[x_{i-1}, x_i], i = 1, \ldots, n$

Finding constraints

- On each $[x_{i-1}, x_i]$, $s_i := a_i x^3 + b_i x^2 + c_i x + d$
  $\implies 4n$ unknown coefficients to solve for.
- $2n$ constraints: $s_i(x_i) = f(x_i), s_i(x_{i+1}) = f(x_{i+1}); i = 0, \dots, n-1$
- $2(n-1)$ constraints:
  $s_i'(x_{i+1}) = s_{i+1}'(x_{i+1})$,
  $s_i''(x_{i+1}) = s_{i+1}''(x_{i+1}), i = 0, \dots, n-2$
- 2 boundary conditions: four major categories.

Different classes

- Natural cubic
  $$s''(x_0) = s''(x_n) = 0$$
- Clamped spline
  $$s'(x_0) = f'(x_0), s'(x_n) = f'(x_n)$$
  Or, $s''(x_0) = f''(x_0), s''(x_n) = f''(x_n)$
- Not-a-knot spline
  $$s'''_{[0,1]}(x_1) = s'''_{[1,2]}(x_1),$$
  $$s'''_{[n-2,n-1]}(x_{n-1}) = s'''_{[n-1,n]}(x_{n-1})$$
- Periodic spline
  $$s(x_0) = s(x_n), s'(x_0) = s'(x_n)$$

Deriving the equations

1. s be the spline interpolate over Ω.
2. $\sigma_i := s''(x_i)$ over the interval $[x_i, x_{i+1}]$
3. s'' is linear. Thus, $s''(x) = \dfrac{x_{i+1} - x}{h_i}\sigma_i + \dfrac{x - x_i}{h_i}\sigma_{i+1}, h_i := x_{i+1} - x_i$
4. s'' is integrated twice, with the constraints $s(x_i) = f(x_i), s(x_{i+1}) = f(x_{i+1})$.
5. s' has to be continuous over Ω $\implies s'_i(x_{i+1}) = s'_{i+1}(x_{i+1}), i = 0, \ldots, n-2$

Deriving the equations

The system of linear equations, given by-

$$h_{i+1}\sigma_i + 2(h_{i+2} + h_{i+1})\sigma_{i+1} + h_{i+2}\sigma_{i+2} = 6\Big(\frac{f(x_{i+2}) - f(x_{i+1})}{h_{i+2}} - \frac{f(x_{i+1}) - f(x_i)}{h_{i+1}}\Big)$$

For $i = 0, \ldots, n-2$

- a *tridiagonal system*, is to be solved.

Solving a tridiagonal system

- Algorithm:
  - Thomas algorithm.
  - $\mathcal{O}(n)$
- Non-singularity:
  - For $n \geq 3$, let $T \in \mathbb{R}^{n\times n}$ be tridiagonal.
  - T is non-singular, only when $|b_i| \geq |a_i| + |c_i|, i = 2, \ldots, n-1$ and $|b_1| > |c_1|, |b_n| > |a_n|$.[1]
  - Justification: U with no zero on its diagonal $\implies b_i^n \geq \left||b_i| - \left|\frac{a_i c_{i-1}}{b_{i-1}^n}\right|\right| > 0$

[1] Thm 3.4. Suli, Mayers

Error bound

- Natural cubic
  - $||f(x) - s_N(x)||_\infty \leq \mathcal{K}_N h^2$
  - $h := \max_\Omega(x_{i+1} - x_i), i = 0, \ldots, n-1$
  - $\mathcal{K}_N$ is dependent on $||f^{(4)}(x)||_\infty$.
  - The rate of convergence increases for clamped/ complete cubic.
- Clamped cubic
  - $||f(x) - s_C(x)||_\infty \leq \mathcal{K}_C h^4$
  - The optimum value[2] for $\mathcal{K}_C = \frac{5}{384}||f^{(4)}||_\infty$

[2] Theorem 5. Hall, Meyer (1976)

A different constraint

-The Hermite spline

Definition:

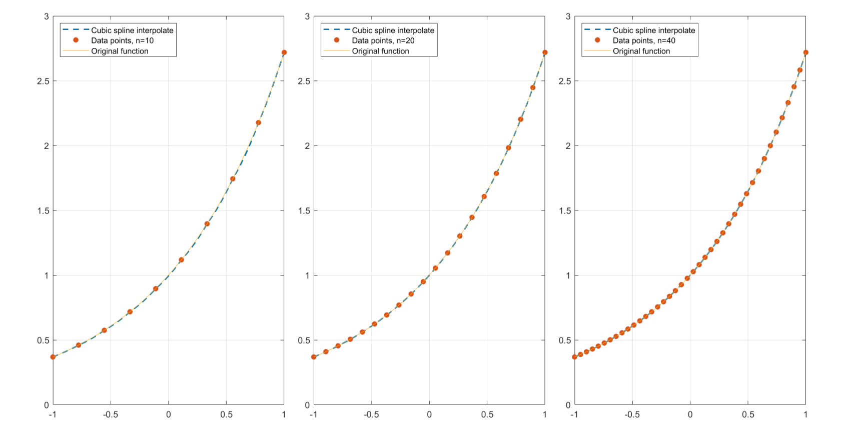
- $f, s \in C^1(\Omega)$
- $s(x_i) = f(x_i), s'(x_i) = f'(x_i), i = 0, \ldots, n$
- $deg(s_{[x_i, x_{i+1}]}) \leq 3, i = 0, \ldots, n-1$

Computation:

- $s_{[x_i, x_{i+1}]} := c_0 + c_1(x - x_i) + c_2(x - x_i)^2 + c_3(x - x_i)^3, i = 0, \ldots, n-1$
- Possible to directly compute the coefficients, e.g. $c_0 = f(x_i)$, $c_1 = f'(x_i)$ etcetera.
- $||f - s_H||_\infty \leq \dfrac{h^4 ||f^{(4)}||_\infty}{384}$

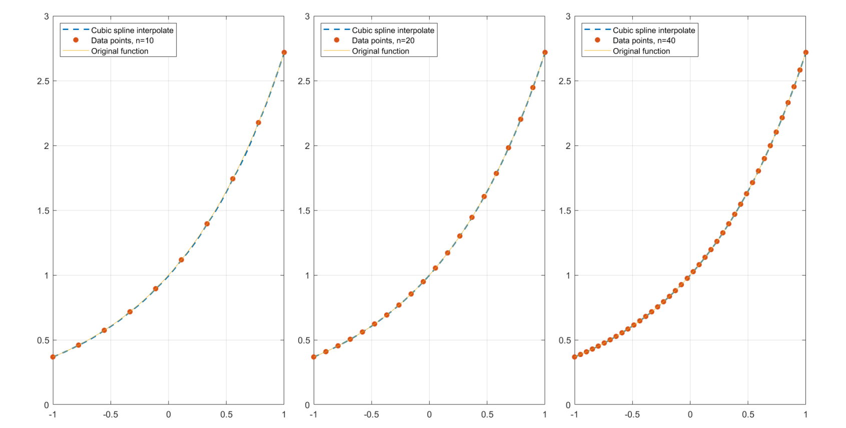
Plots and errors

-Natural cubic spline



Plots and errors

-Clamped cubic spline



Plots and errors

L_∞ norm of the error:

n	Natural	Clamped
10	0.0155	0.0138
20	0.0039	0.0036
40	9.76e-4	8.95e-4

L_2 norm of the error:

n	Natural	Clamped
10	0.0762	0.0783
20	0.0182	0.0184
40	0.0044	0.0045

The linear and quadratic LIP

Linear, over $[x_0, x_1]$:

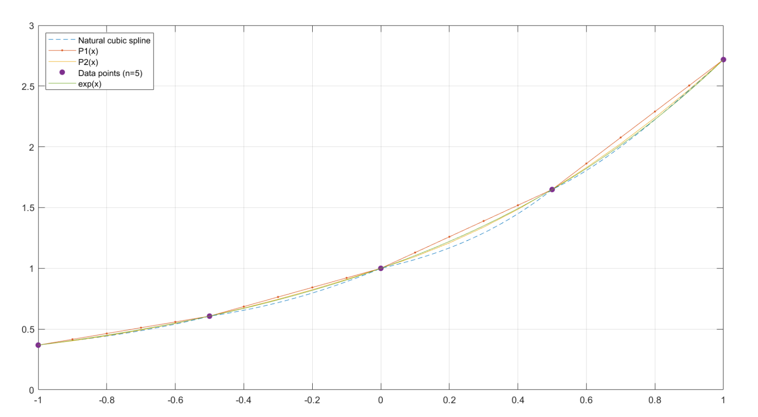
$$P_1(x) = \frac{(x - x_1)}{(x_0 - x_1)} y_0 + \frac{(x - x_0)}{(x_1 - x_0)} y_1$$

Quadratic, over $[x_0, x_2]$:

$$P_2(x) =$$

$$y_0 \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} + y_1 \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} + y_2 \frac{(x - x_0)(x - x_2)}{(x_2 - x_0)(x_2 - x_1)}$$

Comparison with Lagrangian



Comparison with Lagrangian

-With different mesh density

L_∞ norm of the error

n	$P_1(x)$	$P_2(x)$	$s_N(x)$
5	0.064917	0.014416	0.059208
10	0.014825	0.001189	0.015507
15	0.005478	0.000391	0.007013
20	0.000690	0.000075	0.003943
25	0.000000	0.000047	0.002547
30	0.000000	0.000047	0.001778
35	0.000000	0.000031	0.001295
40	0.000000	0.000005	0.000976

Comparison with the Lagrangian

-With different mesh density

L_2 norm of the error

n	$P_1(x)$	$P_2(x)$	$s_N(x)$
5	0.135386	0.032530	0.337271
10	0.026198	0.002383	0.076190
15	0.010887	0.000749	0.032801
20	0.001568	0.000132	0.018169
25	0.000000	0.000087	0.011525
30	0.000000	0.000094	0.007955
35	0.000000	0.000068	0.005819
40	0.000000	0.000012	0.004442

Remark

- The error bound of a Lagrangian interpolate converges faster than a cubic spline interpolate.
- On a uniform mesh with n points, $||f - p|| \leq \mathcal{K}_{\mathcal{L}} |\pi_{n+1}(x)| = \mathcal{O}(h^{n+1})$, where $h := \frac{b-a}{n-1}$.

References

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4. Hall, C. A., & Meyer, W. W. (1976b). *Optimal error bounds for cubic spline interpolation*. Journal of Approximation Theory, 16(2), 105–122.