Discrete Mathematics

Proof Techniques

6th Lecture

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Class: 1st stage.
Time:8:30AM-10:30AM

Proof Techniques

- A **proof** is a valid argument that establishes the truth of a mathematical statement.
- A **theorem** is a statement that can be shown to be true.
- A **direct proof** of a conditional statement $p \rightarrow q$ is constructed when the first step is the assumption that p is true; subsequent steps are constructed using rules of inference, with the final step showing that q must also be true.
- The integer n is *even* if there exists an integer k such that $n = 2k$.
- The integer n is *odd* if there exists an integer k such that $n = 2k + 1$.
- Two integers have the *same parity* when both are even or both are odd; they have *opposite parity* when one is even and the other is odd.

Note: *The parity of an integer is its attribute of being even or odd. Thus, it can be said that 6 and 14 have the same parity (since both are even), whereas 7 and 12 have opposite parity (since 7 is odd and 12 is even).*

EX_1: Give a direct proof of the theorem "If n is an odd integer, then n^2 is odd."

Sol: By the definition of an odd integer, it follows that $n = 2k + 1$, where k is some integer.

We want to show that n^2 is also odd. We can square both sides of the equation $n = 2k + 1$ to obtain a new equation that expresses n^2. When we do this, we find that $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$. By the definition of an odd integer, we can conclude that n^2 is an odd integer (it is one more than twice an integer). Consequently, we have proved that if n is an odd integer, then n^2 is an odd integer.

EX_2: Give a direct proof that if *m* and *n* are both perfect squares, then *nm* is also a perfect square. (An integer *a* is a **perfect square** if there is an integer *b* such that $a = b^2$.)

Sol:

By the definition of a perfect square, it follows that there are integers *s* and *t* such that $m = s^2$ and $n = t^2$. The goal of the proof is to show that *mn* must also be a perfect square when *m* and *n* are; looking ahead we see how we can show this by substituting s^2 for *m* and t^2 for *n* into *mn*. This tells us that $mn = s^2t^2$.

Hence, $mn = s^2t^2 = (ss)(t\,t) = (st)(st) = (st)^2$, using commutatively and associativity of multiplication. By the definition of perfect square, it follows that *mn* is also a perfect square, because it is the square of *st*, which is an integer. We have proved that if *m* and *n* are both perfect squares, then *mn* is also a perfect square.

Proof by Contraposition:

- An extremely useful type of indirect proof is known as **proof by contraposition.**
- Proofs by contraposition make use of the fact that the conditional statement $p \rightarrow q$ is equivalent to its contrapositive, $\neg q \rightarrow \neg p$.
- This means that the conditional statement $p \rightarrow q$ can be proved by showing that its contrapositive, $\neg q \rightarrow \neg p$, is true.
- In a proof by contraposition of $p \rightarrow q$, we take $\neg q$ as a premise, and using axioms, definitions, and previously proven theorems, together with rules of inference, we show that $\neg p$ must follow.

EX_3: Prove that if n is an integer and $3n + 2$ is odd, then n is odd.

Sol: We first attempt a direct proof. To construct a direct proof, we first assume that $3n + 2$ is an odd integer. This means that $3n + 2 = 2k + 1$ for some integer k. Can we use this fact to show that n is odd? We see that $3n + 1 = 2k$, but there does not seem to be any direct way to conclude that n is odd. Because our attempt at a direct proof failed, we next try a proof by contraposition.

The first step in a proof by contraposition is to assume that the conclusion of the conditional statement "If $3n + 2$ is odd, then n is odd" is false; namely, assume that n is even. Then, by the definition of an even integer, $n = 2k$ for some integer k. Substituting $2k$ for n, we find that $3n + 2 = 3(2k) + 2 = 6k + 2 = 2(3k + 1)$. This tells us that $3n + 2$ is even (because it is a multiple of 2), and therefore not odd. This is the negation of the premise of the theorem. Because the negation of the conclusion of the conditional statement implies that the hypothesis is false, the original conditional statement is true. Our proof by contraposition succeeded; we have proved the theorem "If $3n + 2$ is odd, then n is odd."

VACUOUS “فارغ” AND TRIVIAL “تافه” PROOFS

- We can quickly prove that a conditional statement $p \rightarrow q$ is true when we know that p is false, because $p \rightarrow q$ must be true when p is false.
- Consequently, if we can show that p is false, then we have a proof, called a **vacuous proof**, of the conditional statement $p \rightarrow q$.
- Vacuous proofs are often used to establish special cases of theorems that state that a conditional statement is true for all positive integers [i.e., a theorem of the kind $\forall n P(n)$, where $P(n)$ is a propositional function].

EX_4: Show that the proposition $P(0)$ is true, where $P(n)$ is “If $n > 1$, then $n^2 > n$” and the domain consists of all integers.

Sol*:* Note that $P(0)$ is “If $0 > 1$, then $0^2 > 0$”
We can show $P(0)$ using a vacuous proof. Indeed, the hypothesis $0 > 1$ is false. This tells us that $P(0)$ is automatically true.

- We can also quickly prove a conditional statement $p \to q$ if we know that the conclusion q is true.
- By showing that q is true, it follows that $p \to q$ must also be true.
- A proof of $p \to q$ that uses the fact that q is true is called a **trivial proof**.
- Trivial proofs are often important when special cases of theorems are proved

EX_5: Let *P(n)* be "If a and b are positive integers with $a \geq b$, then $a^n \geq b^n$," where the domain consists of all nonnegative integers. Show that *P*(0) is true.

Sol: The proposition *P*(0) is "If $a \geq b$, then $a^0 \geq b^0$." Because $a^0 = b^0 = 1$, the conclusion of the conditional statement "If $a \geq b$, then $a^0 \geq b^0$" is true. Hence, this conditional statement, which is *P*(0), is true. This is an example of a trivial proof.

- The real number r is *rational* if there exist integers p and q with $q \neq 0$ such that $r = p/q$.
- A real number that is not rational is called *irrational*.

EX_6: Prove that the sum of two rational numbers is rational. (Note that if we include the implicit quantifiers here, the theorem we want to prove is "For every real number r and every real number s, if r and s are rational numbers, then $r + s$ is rational.)

Sol:We first attempt a direct proof. To begin, suppose that r and s are rational numbers. From the definition of a rational number, it follows that there are integers p and q, with $q \neq 0$, such that $r = p/q$, and integers t and u, with $u \neq 0$, such that $s = t/u$. Can we use this information to show that $r + s$ is rational? The obvious next step is to add $r = p/q$ and $s = t/u$, to obtain:

$$r + s = \frac{p}{q} + \frac{t}{u} = \frac{pu + qt}{qu}$$

Because $q \neq 0$ and $u \neq 0$, it follows that $qu \neq 0$. Consequently, we have expressed $r + s$ as the ratio of two integers, $pu + qt$ and qu, where $qu \neq 0$. This means that $r + s$ is rational. We have proved that the sum of two rational numbers is rational; our attempt to find a direct proof succeeded.

THANK YOU