MATHEMATICAL EXPECTATION

EXPECTATION OF A RANDOM VARIABLE

The **mean**, **expected value**, or **expectation** of a random variable X is written as $\mathbb{E}(X)$ or μ_X. If we observe N random values of X, then the mean of the N values will be approximately equal to $\mathbb{E}(X)$ for large N. The expectation is defined differently for continuous and discrete random variables.

Definition: Let X be a **continuous** random variable with p.d.f. $f_X(x)$. The expected value of X is

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x f_X(x)\, dx.$$

Definition: Let X be a **discrete** random variable with probability function $f_X(x)$. The expected value of X is

$$\mathbb{E}(X) = \sum_x x f_X(x) = \sum_x x\mathbb{P}(X = x).$$

The expectation, as defined above, agrees with the logical/theoretical argument also as is illustrated in the following example.

Suppose, a fair coin is tossed twice, then answer to the question, "How many heads do we expect theoretically/logically in two tosses?" is obviously 1 as the coin is unbiased and hence we will undoubtedly expect one head in two tosses. Expectation actually means "what we get on an average"? Now, let us obtain the expected value of the above question using the formula.

Let X be the number of heads in two tosses of the coin and we are to obtain E(X), i.e. expected number of heads. As X is the number of heads in two tosses of the coin, therefore X can take the values 0, 1, 2 and its probability distribution is given as

$X:$	0	1	2
$p(x):$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

[Refer Unit 5 of MST-003]

$$\therefore\ E(X) = \sum_{i=1}^{3} x_i p_i$$

$$= x_1p_1 + x_2p_2 + x_3p_3$$

$$= (0)\left(\frac{1}{4}\right) + (1)\left(\frac{1}{2}\right) + (2)\left(\frac{1}{4}\right) = 0 + \frac{1}{2} + \frac{1}{2} = 1$$

Example 1: If it rains, a rain coat dealer can earn Rs 500 per day. If it is a dry day, he can lose Rs 100 per day. What is his expectation, if the probability of rain is 0.4?

Solution: Let X be the amount earned on a day by the dealer. Therefore, X can take the values Rs 500, − Rs 100 ($\because$ loss of Rs 100 is equivalent to negative of the earning of Rs100).

$\therefore$ Probability distribution of X is given as

	Rainy Day	Dry day
$X(\text{in Rs.})$:	500	−100
$p(x)$:	0.4	0.6

Hence, the expectation of the amount earned by him is

$$E(X) = \sum_{i=1}^{2} x_i p_i = x_1 p_1 + x_2 p_2$$

$$= (500)(0.4) + (-100)(0.6) = 200 - 60 = 140$$

Thus, his expectation is Rs 140, i.e. on an overage he earns Rs 140 per day.

Example 2: A player tosses two unbiased coins. He wins Rs 5 if 2 heads appear, Rs 2 if one head appears and Rs1 if no head appears. Find the expected value of the amount won by him.

Solution: In tossing two unbiased coins, the sample space, is

$S = \{HH, HT, TH, TT\}$.

$$\therefore\ P[2\text{ heads}] = \frac{1}{4},\quad P(\text{one head}) = \frac{2}{4},\quad P(\text{no head}) = \frac{1}{4}.$$

Let X be the amount in rupees won by him

$\therefore$ X can take the values 5, 2 and 1 with

$$P[X=5]=P(2\text{heads})=\frac{1}{4},$$

$$P[X=2]=P[1\,\text{Head}]=\frac{2}{4}, \text{ and}$$

$$P[X=1]=P[\text{no Head}]=\frac{1}{4}.$$

$\therefore$ Probability distribution of X is

$X:$	5	2	1
$p(x)$	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$

Expected value of X is given as

$$E(X)=\sum_{i=1}^{3} x_i p_i = x_1 p_1 + x_2 p_2 + x_3 p_3$$

$$= 5\left(\frac{1}{4}\right)+2\left(\frac{2}{4}\right)+1\left(\frac{1}{4}\right) = \frac{5}{4}+\frac{4}{4}+\frac{1}{4}=\frac{10}{4}=2.5.$$

Thus, the expected value of amount won by him is Rs 2.5.

Example 3: Find the expectation of the number on an unbiased die when thrown.

Solution: Let X be a random variable representing the number on a die when thrown.

$\therefore$ X can take the values 1, 2, 3, 4, 5, 6 with

$$P[X=1]=P[X=2]=P[X=3]=P[X=4]=P[X=5]=P[X=6]=\frac{1}{6}.$$

Thus, the probability distribution of X is given by

X:	1	2	3	4	5	6
p(x):	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Hence, the expectation of number on the die when thrown is

$$E(X)=\sum_{i=1}^{6}x_i p_i =1\times\frac{1}{6}+2\times\frac{1}{6}+3\times\frac{1}{6}+4\times\frac{1}{6}+5\times\frac{1}{6}+6\times\frac{1}{6} = \frac{21}{6}=\frac{7}{2}$$

Example 5: For a continuous distribution whose probability density function is given by:

$f(x)=\frac{3x}{4}(2-x),\ 0\le x\le 2,$ find the expected value of X.

Solution: Expected value of a continuous random variable X is given by

$$E(X)=\int_{-\infty}^{\infty}xf(x)\,dx =\int_0^2 x\frac{3x}{4}(2-x)\,dx=\frac{3}{4}\int_0^2 x^2(2-x)\,dx$$

$$=\frac{3}{4}\int_0^2(2x^2-x^3)\,dx = \frac{3}{4}\left[2\frac{x^3}{3}-\frac{x^4}{4}\right]_0^2=\frac{3}{4}\left[2\frac{(2)^3}{3}-\frac{(2)^4}{4}-0\right]$$

$$=\frac{3}{4}\left[\frac{16}{3}-\frac{16}{4}\right]=\frac{3}{4}\times\frac{16}{12}=1$$

Now, you can try the following exercises.

E1) You toss a fair coin. If the outcome is head, you win Rs 100; if the outcome is tail, you win nothing. What is the expected amount won by you?

E2) A fair coin is tossed until a tail appears. What is the expectation of number of tosses?

E3) The distribution of a continuous random variable X is defined by

$$f(x) = \begin{cases} x^3 & , \ 0 < x \le 1 \\ (2-x)^3 & , \ 1 < x \le 2 \\ 0 & , \ \text{elsewhere} \end{cases}$$

Obtain the expected value of X.

PROPERTIES OF EXPECTATION OF ONE-DIMENSIONAL RANDOM VARIABLE

Properties of mathematical expectation of a random variable X are:

1. E(k) = k, where k is a constant
2. E(kX) = kE(X), k being a constant.
3. E(aX + b) = aE(X) + b, where a and b are constants

Proof:

<u>Discrete case</u>:

Let X be a discrete r.v. which takes the values $x_1, x_2, x_3, \ldots$ with respective probabilities $p_1, p_2, p_3, \ldots$

1. $E(k) = \sum_i k\,p_i$ [By definition of the expectation]

$$= k\sum_i p_i$$

$$= k(1) = k$$ $\left[\because \text{sum of probabilities of all the possible value of r.v. is 1}\right]$

2. $E(kX) = \sum_i (kx_i)\,p_i$ [By def.]

$$= k\sum_i x_i\,p_i$$

$$= k\,E(X)$$

3. $E(aX+b) = \sum_i (ax_i + b)\,p_i$ [By def.]

$$= \sum_i (ax_ip_i + bp_i) = \sum_i ax_ip_i + \sum_i bp_i = a\sum_i x_ip_i + b\sum_i p_i$$

$$= aE(X) + b(1) = aE(X) + b$$

Continuous Case:

Let X be continuous random variable having f(x) as its probability density function. Thus,

1. $E(k)=\int_{-\infty}^{\infty} kf(x)dx$ [By def.]

$$=k\int_{-\infty}^{\infty} f(x)dx$$

$$=k(1)=k \qquad \left[\begin{array}{l}\because \text{integral of the p.d.f. over} \\ \text{the entire range is 1}\end{array}\right]$$

2. $E(kX)=\int_{-\infty}^{\infty}(kx)f(x)dx$ [By def.]

$$=k\int_{-\infty}^{\infty} xf(x)dx = kE(X)$$

3. $E(aX+b)=\int_{-\infty}^{\infty}(ax+b)f(x)dx = \int_{-\infty}^{\infty}(ax)f(x)dx+\int_{-\infty}^{\infty} bf(x)dx$

$$=a\int_{-\infty}^{\infty} xf(x)dx+b\int_{-\infty}^{\infty} f(x)dx = aE(X)+b(1) = aE(X)+b$$

Example 6: Given the following probability distribution:

X	−2	−1	0	1	2
$p(x)$	0.15	0.30	0	0.30	0.25

Find i) E(X)

ii) E(2X + 3)

iii) $E(X^2)$

iv) E(4X – 5)

Solution

i) $E(X)=\sum_{i=1}^{5} x_i p_i = x_1p_1 + x_2p_2 + x_3p_3 + x_4p_4 + x_5p_5$

$$=(-2)(0.15)+(-1)(0.30)+(0)(0)+(1)(0.30)+(2)(0.25)$$

$$=-0.3-0.3+0+0.3+0.5=0.2$$

ii) $E(2X + 3) = 2E(X)+3$ [Using property 3 of this section]

$= 2(0.2) + 3$ [Using solution (i) of the question]

$= 0.4 + 3 = 3.4$

iii) $E(X^2)=\sum_{i=1}^{5} x_i^2 p_i$ [By def.]

$$= x_1^2 p_1 + x_2^2 p_2 + x_3^2 p_3 + x_4^2 p_4 + x_5^2 p_5$$

$$= (-2)^2 (0.15) + (-1)^2 (0.30) + (0)^2 (0) + (1)^2 (0.30) + (2)^2 (0.25)$$

$$= (4)(0.15) + (1)(0.30) + (0) + (1)(0.30) + (4)(0.25)$$

$$= 0.6 + 0.3 + 0 + 0.3 + 1 = 2.2$$

iv) $E(4X-5) = E\left[4X + (-5)\right]$

$$= 4E(X) + (-5) \qquad \text{[Using property 3 of this section]}$$

$$= 4(0.2) - 5$$

$$= 0.8 - 5 = -4.2$$

Variance

Variance of a random variable X is second order central moment and is defined as

$$\mu_2 = V(X) = E[X-\mu]^2 = E\left[X - E(X)\right]^2$$

Also, we know that

$$V(X) = \mu_2' - (\mu_1')^2$$

where μ_1', μ_2' be the moments about origin.

$\therefore$ We have $V(X) = E\left(X^2\right) - \left[E(X)\right]^2$

$$\left[\because \mu_1' = E[X-0]^1 = E(X), \text{ and } \mu_2' = E[X-0]^2 = E\left(X^2\right)\right]$$

Theorem 8.1: If X is a random variable, then $V(aX+b)=a^2V(X)$, where a and b are constants.

Proof: $V(aX+b)=E\left[(aX+b)-E(aX+b)\right]^2$ [By def. of variance]

$$= E\left[aX+b-\left(aE(X)+b\right)\right]^2 \quad \text{[Using property 3 of Sec. 8.3]}$$

$$= E\left[aX+b-aE(X)-b\right]^2$$

$$= E\left[a\left\{X-E(X)\right\}\right]^2$$

$$= E\left[a^2\left(X-E(X)\right)^2\right]$$

$$= a^2E\left[X-E(X)\right]^2 \quad \text{[Using property 2 of section 8.3]}$$

$$= a^2V(X) \quad \text{[By definition of Variance]}$$

Cor. (i) $V(aX)=a^2V(X)$

(ii) $V(b) = 0$

(iii) $V(X + b) = V(X)$

Covariance

For a bivariate frequency distribution, you have already studied in Unit 6 of MST-002 that covariance between two variables X and Y is defined as

$$\text{Cov}(X, Y) = \frac{\sum_i f_i (x_i - \bar{x})(y_i - \bar{y})}{\sum_i f_i}$$

$\therefore$ For a bivariate probability distribution, Cov (X, Y) is defined as

$$\text{Cov}(X, Y) = \begin{cases} \sum_i p_{ij} (x_i - \bar{x})(y_i - \bar{y}), \text{ if } (X, Y) \text{ is two-dimensional discrete r.v.} \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \bar{x})(y - \bar{y}) f(x, y)\, dy dx, \text{ if } (X, Y) \text{ is two dimensional continuous r.v.} \end{cases}$$

where $p_{ij} = P\left[X = x_i, Y = y_j\right]$

$$= E\left(X - \bar{X}\right)\left(Y - \bar{Y}\right) \qquad \text{[By definition of expectation]}$$

$$= E\left[\left(X - E(X)\right)\left(Y - E(Y)\right)\right] \quad \left[\begin{array}{l} \because E(X) = \text{Mean of X i.e. } \bar{X}, \\ E(Y) = \text{Mean of Y i.e. } \bar{Y} \end{array}\right]$$

On simplifying,

Cov(X, Y) = E(XY) – E(X) E(Y).

Now, if X and Y are independent random variables then, by multiplication theorem,

E(XY) = E(X)E(Y) and hence in this case Cov(X, Y) = 0.

Remark 2:

i) If X and Y are independent random variables, then

$V(X + Y) = V(X) + V(Y)$.

ii) If X and Y are independent random variables, then

$V(X - Y) = V(X) + V(Y)$.

iii) If X and Y are independent random variables, then

$V(aX + bY) = a^2V(X) + b^2V(Y)$.

Example 7: Considering the probability distribution given in Example 6, obtain

i) V(X)

ii) V(2X + 3).

Solution:

(i) $V(X) = E(X^2) - [E(X)]^2$

$= 2.2 - (0.2)^2$ $\left[\begin{array}{l}\text{The values have already been obtained} \\ \text{in the solution of Example 6}\end{array}\right]$

$= 2.2 - 0.04 = 2.16$

(ii) $V(2X + 3) = (2)^2 V(X)$ [Using the result of Theorem 8.1]

$= 4V(X) = 4(2.16) = 8.64$

Theorem 8.2: If X and Y are random variables, then $E(X+Y)=E(X)+E(Y)$

Proof:

Discrete case:

Let (X, Y) be a discrete two-dimensional random variable which takes up the values (x_i, y_j) with the joint probability mass function $p_{ij} = P\left[X = x_i \cap Y = y_j\right]$.

Then, the probability distribution of X is given by

$$p_i = p(x_i) = P[X = x_i]$$

$$= P[X = x_i \cap Y = y_1] + P[X = x_i \cap Y = y_2] + \ldots \quad \begin{bmatrix} \because \text{event } X = x_i \text{ can happen with} \\ Y = y_1 \text{ or } Y = y_2 \text{ or } Y = y_3 \text{ or} \ldots \end{bmatrix}$$

$$= p_{i1} + p_{i2} + p_{i3} + \ldots$$

$$= \sum_j p_{ij}$$

Similarly, the probability distribution of Y is given by

$$p_j' = p(y_j) = P\left[Y = y_j\right] = \sum_i p_{ij}$$

$$\therefore\ E(X) = \sum_i x_i p_i,\ E(Y) = \sum_j y_j p_j' \text{ and } E(X+Y) = \sum_i \sum_j (x_i + y_j) p_{ij}$$

$$\text{Now } E(X+Y) = \sum_i \sum_j (x_i + y_j) p_{ij}$$

$$= \sum_i \sum_j x_i p_{ij} + \sum_i \sum_j y_j p_{ij}$$

$$= \sum_i x_i \sum_j p_{ij} + \sum_j y_j \sum_i p_{ij}$$

[$\because$ in the first term of the right hand side, x_i is free from j and hence can be taken outside the summation over j; and in second term of the right hand side, y_j is free from i and hence can be taken outside the summation over i.]

$$\therefore\ E(X+Y) = \sum_i x_i p_i + \sum_j y_j p_j' = E(X) + E(Y)$$

Continuous Case:

Let (X, Y) be a bivariate continuous random variable with probability density function $f(x,y)$. Let $f(x)$ and $f(y)$ be the marginal probability density functions of random variables X and Y respectively.

$$\therefore\ E(X)=\int_{-\infty}^{\infty} x\ f(x)dx,\ E(Y)=\int_{-\infty}^{\infty} y\ f(y)dy,$$

$$\text{and } E(X+Y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x+y)f(x,y)dy\,dx.$$

$$\text{Now, } E(X+Y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x+y)f(x,y)dy\,dx$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x\,f(x,y)dy\,dx+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}y\,f(x,y)dy\,dx$$

$$=\int_{-\infty}^{\infty}x\left(\int_{-\infty}^{\infty}f(x,y)dy\right)dx+\int_{-\infty}^{\infty}y\left(\int_{-\infty}^{\infty}f(x,y)dx\right)dy$$

[$\because$ in the first term of R.H.S., x is free from the integral w.r.t. y and hence can be taken outside this integral. Similarly, in the second term of R.H.S, y is free from the integral w.r.t. x and hence can be taken outside this integral.]

$$=\int_{-\infty}^{\infty}x\,f(x)dx+\int_{-\infty}^{\infty}y\,f(y)dy\quad\left[\begin{array}{l}\text{Refer to the definition of marginal density}\\ \text{function given in Unit 7 of this course}\end{array}\right]$$

$$=E(X)+E(Y)$$

Theorem 8.3: If X and Y are independent random variables, then

E(XY) = E(X) E(Y)

Proof:

Discrete Case:

Let (X, Y) be a two-dimensional discrete random variable which takes up the values (x_i, y_j) with the joint probability mass function

$p_{ij} = P[X = x_i \cap Y = y_j]$. Let p_i and $p_j^{'}$ be the marginal probability mass functions of X and Y respectively.

$$\therefore \; E(X) = \sum_i x_i p_i, \; E(Y) = \sum_j y_j p_j^{'}, \text{ and}$$

$$E(XY) = \sum_i \sum_j (x_i y_j) p_{ij}$$

But as X and Y are independent,

$$\therefore \; p_{ij} = P[X = x_i \cap Y = y_j]$$

$$= P[X = x_i] P[Y = y_j] \quad \left[\begin{array}{l} \because \text{ if events A and B are independent,} \\ \text{then } P(A \cap B) = P(A) P(B) \end{array} \right]$$

$$= p_i p_j^{'}$$

$$\text{Hence, } E(XY) = \sum_i \sum_j (x_i y_j) p_i p_j^{'}$$

$$= \sum_i \sum_j x_i y_j p_i p_j^{'}$$

$$= \sum_i \sum_j (x_i p_i y_j p_j^{'})$$

$$= \sum_i x_i p_i \sum_j y_j p_j^{'} \quad \left[\begin{array}{l} \because \; x_i p_i \text{ is free from j and hence can be} \\ \text{taken outside the summation over j} \end{array} \right]$$

$$= E(X) \, E(Y)$$

Continuous Case:

Let (X, Y) be a bivariate continuous random variable with probability density function f(x, y). Let f(x) and f(y) be the marginal probability density function of random variables X and Y respectively.

$$\therefore\ E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx,\ \ E(Y) = \int_{-\infty}^{\infty} y\, f(y)\, dy,$$

and $E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x, y)\, dy\, dx$.

Now $E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x, y)\, dy\, dx$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x)\, f(y)\, dy\, dx \quad \left[\begin{array}{l}\because \text{X and Y are independent, } f(x,y)=f(x)f(y) \\ \text{(see Unit 7 of this course)}\end{array}\right]$$

$$= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} (x\, f(x))(y f(y))\right) dy\, dx$$

$$= \left(\int_{-\infty}^{\infty} x\, f(x)\, dx\right)\left(\int_{-\infty}^{\infty} y\, f(y)\, dy\right)$$

$$= E(X)E(Y)$$

Example 8: If X and Y are independent random variables with variances 2 and 3 respectively, find the variance of 3X + 4Y.

Solution: $V(3X + 4Y) = (3)^2\, V(X) + (4)^2\, V(Y)$ [By Remark 3 of Section 8.4]

$$= 9(2) + 16(3) = 18 + 48 = 66$$

SUMMARY

The following main points have been covered in this unit:

1) Expected value of a random variable X is defined as

$$E(X) = \sum_{i=1}^{n} x_i p_i \text{ , if X is a discrete random variable}$$

$$= \int_{-\infty}^{\infty} x f(x)\, dx \text{ , if X is a continuous random variable.}$$

2) Important properties of expectation are:

i) $E(k) = k$, where k is a constant.

ii) $E(kX) = kE(X)$, k being a constant.

iii) $E(aX + b) = aE(X) + b$, where a and b are constants

iv) Addition theorem of Expectation is stated as:

If X and Y are random variables, then $E(X+Y) = E(X) + E(Y)$.

v) Multiplication theorem of Expectation is stated as:

If X and Y are independent random variables, then

$E(XY) = E(X)E(Y)$.

vi) If $X_1, X_2, ..., X_n$ be any n random variables and if $a_1, a_2, ..., a_n$ are any n constants, then

$$E(a_1X_1 + a_2X_2 + ... + a_nX_n) = a_1E(X_1) + a_2E(X_2) + ... + a_nE(X_n).$$

Variance of a random variable X is given as

$$V(X) = E[X-\mu]^2 = E[X - E(X)]^2$$

$$\text{Cov}(X,Y) = \begin{cases} \sum_i p_i(x_i - \bar{x})(y_i - \bar{y}), & \text{if } (X, Y) \text{ is discrete r.v.} \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-\bar{x})(y-\bar{y})f(x,y)\,dy\,dx, & \text{if } (X, Y) \text{ is continuous r.v.} \end{cases}$$

$$= E[(X - E(X))(Y - E(Y))]$$

$$= E(XY) - E(X)\,E(Y).$$

SOLUTIONS/ANSWERS

E1) Let X be the amount (in rupees) won by you.

$\therefore$ X can take the values 100, 0 with $P[X = 100] = P[\text{Head}] = \frac{1}{2}$, and $P[X = 0] = P[\text{Tail}] = \frac{1}{2}$.

$\therefore$ probability distribution of X is

X :	100	0
$p(x)$	$\frac{1}{2}$	$\frac{1}{2}$

and hence the expected amount won by you is

$$E(X) = 100 \times \frac{1}{2} + 0 \times \frac{1}{2} = 50.$$

E2) Let X be the number of tosses till tail turns up.

$\therefore$ X can take values 1, 2, 3, 4… with

$$P[X = 1] = P[\text{Tail in the first toss}] = \frac{1}{2}$$

$$P[X = 2] = P[\text{Head in the first and tail in the second toss}] = \frac{1}{2} \times \frac{1}{2} = \left(\frac{1}{2}\right)^2,$$

$$P[X = 3] = P[HHT] = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \left(\frac{1}{2}\right)^3, \text{ and so on.}$$

$\therefore$ Probability distribution of X is

X:	1	2	3	4	5…
$p(x)$	$\frac{1}{2}$	$\left(\frac{1}{2}\right)^2$	$\left(\frac{1}{2}\right)^3$	$\left(\frac{1}{2}\right)^4$	$\left(\frac{1}{2}\right)^5$ …

and hence

$$E(X) = 1 \times \frac{1}{2} + 2 \times \left(\frac{1}{2}\right)^2 + 3 \times \left(\frac{1}{2}\right)^3 + 4 \times \left(\frac{1}{2}\right)^4 + \ldots$$

E3) $E(X)=\int_{-\infty}^{\infty} x\,f(x)\,dx$

$$=\int_{-\infty}^{0} x\,f(x)\,dx+\int_{0}^{1} x\,f(x)\,dx+\int_{1}^{2} x\,f(x)\,dx+\int_{2}^{\infty} x\,f(x)\,dx$$

$$=\int_{-\infty}^{0} x(0)\,dx+\int_{0}^{1} x\left(x^{3}\right)dx+\int_{1}^{2} x(2-x)^{3}\,dx+\int_{2}^{\infty} x(0)\,dx$$

$$=0+\int_{0}^{1} x^{4}dx+\int_{1}^{2} x\left[8-x^{3}-6x(2-x)\right]dx+0$$

$$=\int_{0}^{1} x^{4}dx+\int_{1}^{2}\left(8x-x^{4}-12x^{2}+6x^{3}\right)dx$$

$$=\left[\frac{x^{5}}{5}\right]_{0}^{1}+\left[8\frac{x^{2}}{2}-\frac{x^{5}}{5}-12\frac{x^{3}}{3}+6\frac{x^{4}}{4}\right]_{1}^{2}$$

$$=\frac{1}{5}+\left[\left\{\frac{8(2)^{2}}{2}-\frac{(2)^{5}}{5}-\frac{12(2)^{3}}{3}+\frac{6(2)^{4}}{4}\right\}-\left\{\frac{8(1)^{2}}{2}-\frac{(1)^{5}}{5}-\frac{12(1)^{3}}{3}+\frac{6(1)^{4}}{4}\right\}\right]$$

$$=\frac{1}{5}+\left[\left\{16-\frac{32}{5}-32+24\right\}-\left\{4-\frac{1}{5}-4+\frac{3}{2}\right\}\right]$$

$$=\frac{1}{5}+\left[\frac{8}{5}-\frac{13}{10}\right]=\frac{1}{5}+\frac{3}{10}=\frac{1}{2}\ .$$