Discrete Mathematics

Graphs

10th Lecture

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Class: 1st stage.
Time:8:30AM-10:30AM

Graphs

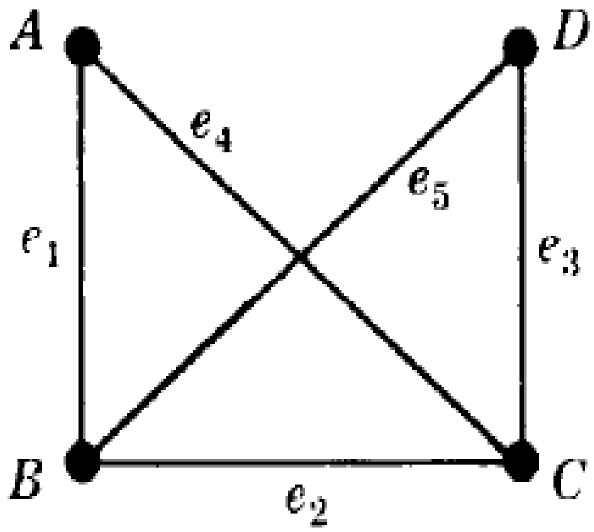
Graphs ***are discrete structures consisting of vertices and edges that connect these vertices***. There are different kinds of graphs, depending on whether edges have directions, whether multiple edges can connect the same pair of vertices, and whether loops are allowed. Problems in almost every conceivable discipline can be solved using graph models. In this lecture, many examples are given to illustrate how graphs are used as models in a variety of areas.

- A *graph* $G = (V, E)$ consists of V, a nonempty set of *vertices* (or *nodes*) and E, a set of *edges*.
- Each edge has either one or two vertices associated with it, called its *endpoints*.
- An edge is said to *connect* its endpoints.
- Vertices u and v are said to be *adjacent* or *neighbors* if there is an edge $e = \{u, v\}$.
- In such a case, u and v are called the *endpoints* of e, and e is said to *connect* u and v.
- Also, the edge e is said to be *incident on each* of its endpoints u and v.
- Graphs are pictured by diagrams in the plane in a natural way.
- Specifically, each vertex v in V is represented by a dot (or small circle), and each edge $e = \{v1, v2\}$ is represented by a curve which connects its endpoints $v1$ and $v2$.

EX_1: Fig. below represents the graph $G(V,E)$ where,

(i) V consists of vertices A, B, C, D.

(ii) E consists of edges $e_1 = \{A,B\}$, $e_2 = \{B,C\}$, $e_3 = \{C,D\}$, $e_4 = \{A,C\}$, $e_5 = \{B,D\}$.

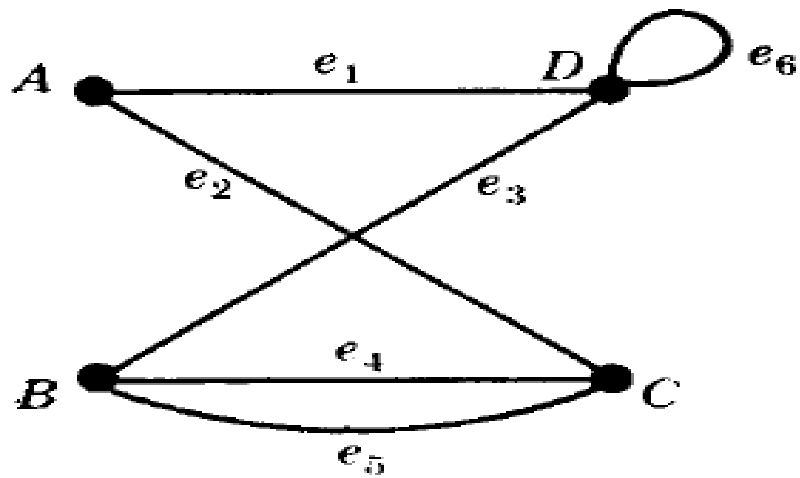


- A *directed graph* (or *digraph*) (V, E) consists of a nonempty set of vertices V and a set of *directed edges* (or *arcs*) E. Each directed edge is associated with an ordered pair of vertices. The directed edge associated with the ordered pair (u, v) is said to *start* at u and *end* at v.

Multigraphs

Consider the diagram in Fig. below, The edges e_4 and e_5 are called *multiple edges* since they connect the same endpoints, and the edge e_6 is called a *loop* since its endpoints are the same vertex. Such a diagram is called a *multigraph*; the formal definition of a graph permits neither multiple edges nor loops. Thus a graph may be defined to be a multigraph without multiple edges or loops.

Remark: Some texts use the term *graph* to include multigraphs and use the term *simple graph* to mean a graph without multiple edges and loops.



Degree of a Vertex

The *degree* of a vertex *v* in a graph *G*, written **deg *(v)*, is equal to the number of edges in *G* which contain *v*, that is, which are incident on *v***. Since each edge is counted twice in counting the degrees of the vertices of *G*, we have the following simple but important result.

- The sum of the degrees of the vertices of a graph *G* is equal to twice the number of edges in *G*.

EX_2: Consider the graph in EX_1 We have:

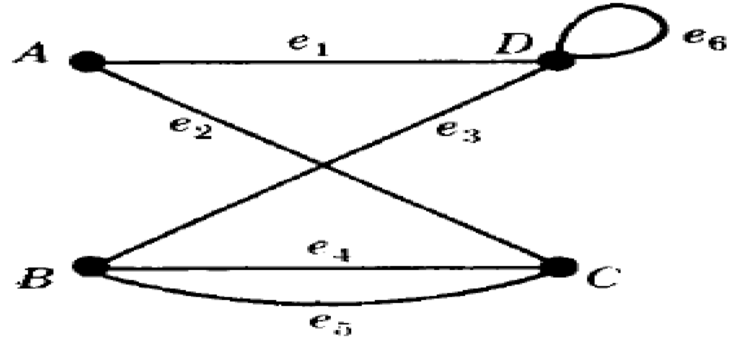
$$\deg(A) = 2,\ \deg(B) = 3,\ \deg(C) = 3,\ \deg(D) = 2.$$

The sum of the degrees equals 10 which, as expected, is twice the number of edges.

- A vertex is said to be *even* or *odd* according as its degree is an even or an odd number. Thus *A* and *D* are even vertices whereas *B* and *C* are odd vertices.

- The above bullet is also holds for multigraphs where a loop is counted twice toward the degree of its endpoint.
- A vertex of degree zero is called an *isolated* vertex.

EX_3: As shown in Fig. below, we have deg*(D)* = 4 since the edge e_6 is counted twice; hence *D* is an even vertex.



PATHS, CONNECTIVITY

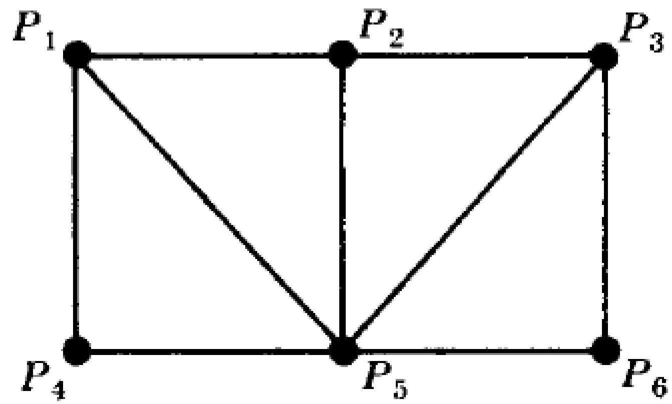
A *path* in a multigraph G consists of an alternating sequence of vertices and edges of the form

$$v_0, \quad e_1, \quad v_1, \quad e_2, \quad v_2, \quad \ldots, \quad e_{n-1}, \quad v_{n-1}, \quad e_n, \quad v_n$$

where each edge e_i contains the vertices v_{i-1} and v_i (which appear on the sides of e_i in the sequence). The number n of edges is called the *length* of the path. When there is no ambiguity, we denote a path by its sequence of vertices $(v_0, v_1, \ldots, v_n)$. The path is said to be *closed* if $v_0 = v_n$. Otherwise, we say the path is from v_0, to v_n or *between* v_0 and v_n, or *connects* v_0 to v_n.

A *simple path* is a path in which all vertices are distinct. (A path in which all edges are distinct will be called a *trail*.) A *cycle* is a closed path of length 3 or more in which all vertices are distinct except $v_0 = v_n$. A cycle of length k is called a *k-cycle*.

EX_4: Consider the graph *G* in the fig. below Consider the following sequences:



$$\alpha = (P_4, P_1, P_2, P_5, P_1, P_2, P_3, P_6), \quad \beta = (P_4, P_1, P_5, P_2, P_6),$$

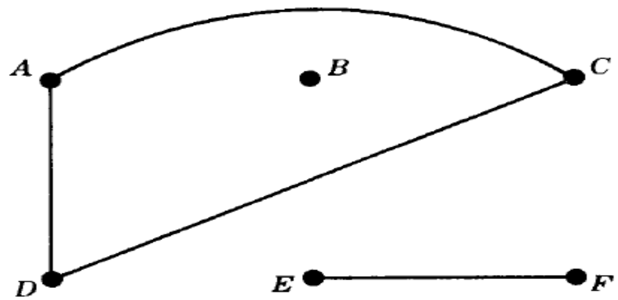
$$\gamma = (P_4, P_1, P_5, P_2, P_3, P_5, P_6), \qquad \delta = (P_4, P_1, P_5, P_3, P_6).$$

The sequence α is a path from P_4 to P_6; but it is not a trail since the edge $\{P_1, P_2\}$ is used twice. The sequence β is not a path since there is no edge $\{P_2, P_6\}$. The sequence γ is a trail since no edge is used twice; but it is not a simple path since the vertex P_5 is used twice. The sequence δ is a simple path from P_4 to P_6; but it is not the shortest path (with respect to length) from P_4 to P_6. The shortest path from P_4 to P_6 is the simple path (P_4, P_5, P_6) which has length 2.

By eliminating unnecessary edges, it is not difficult to see that any path from a vertex u to a vertex v can be replaced by a simple path from u to v.

onnectivity, Connected Components

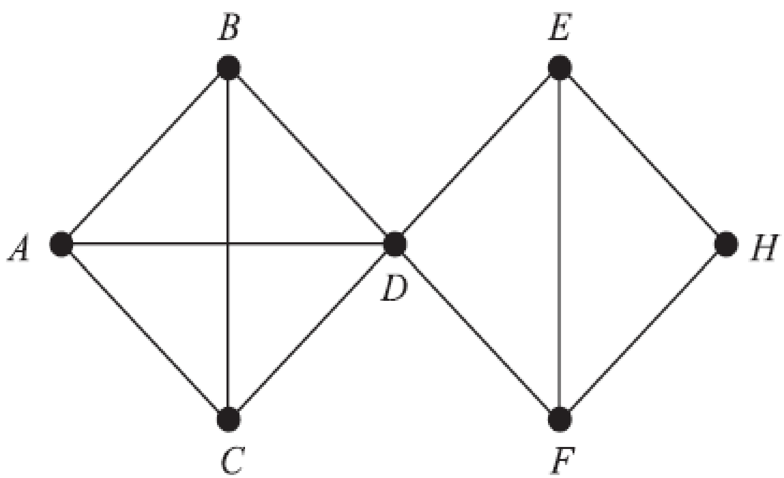
- A graph G is *connected* if there is a path between any two of its vertices.
- The graph in Fig. of EX_4 is connected, but the graph in Fig. below is not connected since, for example, there is no path between vertices D and E.



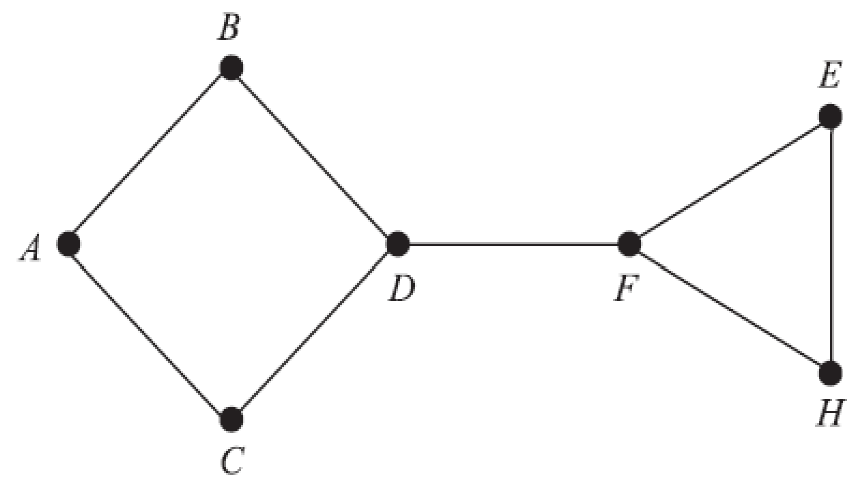
- Suppose G is a graph. A connected subgraph H of G is called a *connected component* of G if H is not contained in any larger connected subgraph of G. It is intuitively clear that any graph G can be partitioned into its connected components. For example, the graph G in the Fig. above has three connected components, the subgraphs induced by the vertex sets $\{A,C,D\}$, $\{E,F\}$, and $\{B\}$.
- The vertex B in the Fig. above is called an *isolated vertex* since B does not belong to any edge or, in other words, $\deg(B) = 0$. Therefore, as noted, B itself forms a connected component of the graph.

Distance and Diameter

- Consider a connected graph G.
- The *distance* between vertices u and v in G, written $d(u, v)$, is the length of the shortest path between u and v.
- The *diameter* of G, written diam(G), is the maximum distance between any two points in G.
- For example, in Fig *(a)*, $d(A,F) = 2$ and diam$(G) = 3$
- whereas in Fig. *(b)*, $d(A,H) = 4$ and diam$(G) = 5$.

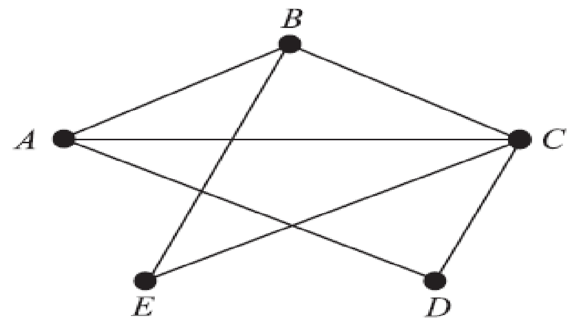


(a)



(b)

EX_5: Consider the graph G in the Fig. below:



(a) Describe G formally, that is, find the set $V(G)$ of vertices of G and the set $E(G)$ of edges of G.

(b) Find the degree of each vertex and verify that "The sum of the degrees of the vertices of a graph G is equal to twice the number of edges in G".

Ans:

(a) There are five vertices so $V(G) = \{A, B, C, D, E\}$. There are seven pairs $\{x, y\}$ of vertices where the vertex x is connected with the vertex y, hence

$$E(G) = [\{A, B\}, \{A, C\}, \{A, D\}, \{B, C\}, \{B, E\}, \{C, D\}, \{C, E\}]$$

(b) The degree of a vertex is equal to the number of edges to which it belongs; e.g., $\deg(A) = 3$ since A belongs to the three edges $\{A, B\}, \{A, C\}, \{A, D\}$. Similarly,

$$\deg(B) = 3,\ \deg(C) = 4,\ \deg(D) = 2,\ \deg(E) = 2$$

The sum of the degrees is $3 + 3 + 4 + 2 + 2 = 14$ which does equal twice the number of edges.

THANK YOU