Discrete Mathematics

Functions & Relations

3rd Lecture

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Class: 1st stage.
Time:8:30AM-10:30AM

Functions

- Let $f1$ and $f2$ be functions from A to **R**. Then $f1 + f2$ and $f1f2$ are also functions from A to **R** defined for all $x \in A$ by

  $(f_1 + f_2)(x) = f_1(x) + f_2(x)$

  $(f_1 f_2)(x) = f_1(x)\, f_2(x)$.

EX_1: Let $f1$ and $f2$ be functions from **R** to **R** such that $f1(x) = x^2$ and $f2(x) = x - x^2$. What are the functions $f1 + f2$ and $f1f2$?

Sol: From the definition of the sum and product of functions, it follows that

$(f1 + f2)(x) = f1(x) + f2(x) = x^2 + (x - x^2) = x$

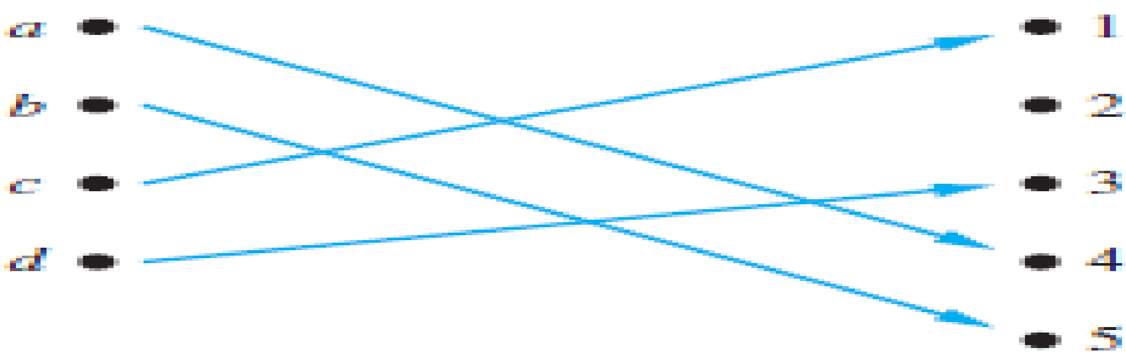
and

$(f1\, f2)(x) = x2(x - x2) = x^3 - x^4$.

One-to-One &Onto Functions

Some functions never assign the ***same value*** to two different domain elements. These functions are said to be **one-to-one**.

- A function f is said to be *one-to-one*, or an *injunction*, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f.
- A function is said to be *injective* if it is one-to-one.
- Note that a function f is one-to-one if and only if $f(a) = f(b)$ whenever $a = b$.



A One-to-One Function

EX_2 Determine whether the function f from $\{a, b, c, d\}$ to $\{1, 2, 3, 4, 5\}$ with $f(a) = 4$, $f(b) = 5$, $f(c) = 1$, and $f(d) = 3$ is one-to-one.

Sol: The function f is one-to-one because f takes on different values at the four elements of its domain. This is illustrated in the above figure.

Functions

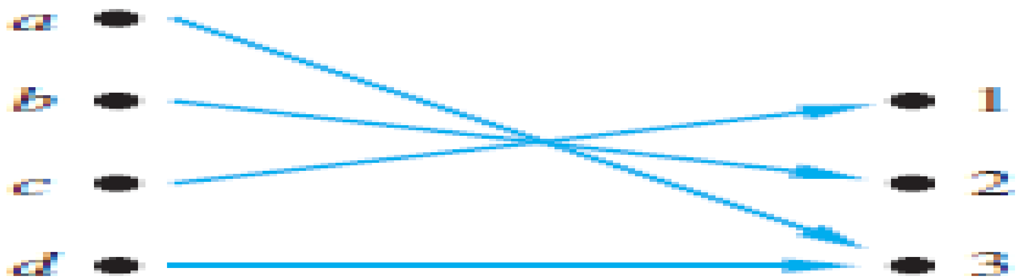
EX_3 Determine whether the function $f(x) = X^2$ from the set of integers to the set of integers is one-to-one.

Sol: The function $f(x) = X^2$ is not one-to-one because, for instance, $f(1) = f(-1) = 1$, but $1 \neq -1$.

- For some functions the ***range and the codomain are equal***. That is, every member of the codomain is the image of some element of the domain. Functions with this property are called **onto** functions.
- A function f from A to B is called ***onto***, or a *surjection*, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function f is called *surjective* if it is onto.

EX_4: Let f be the function from $\{a, b, c, d\}$ to $\{1, 2, 3\}$ defined by $f(a) = 3$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is f an onto function?

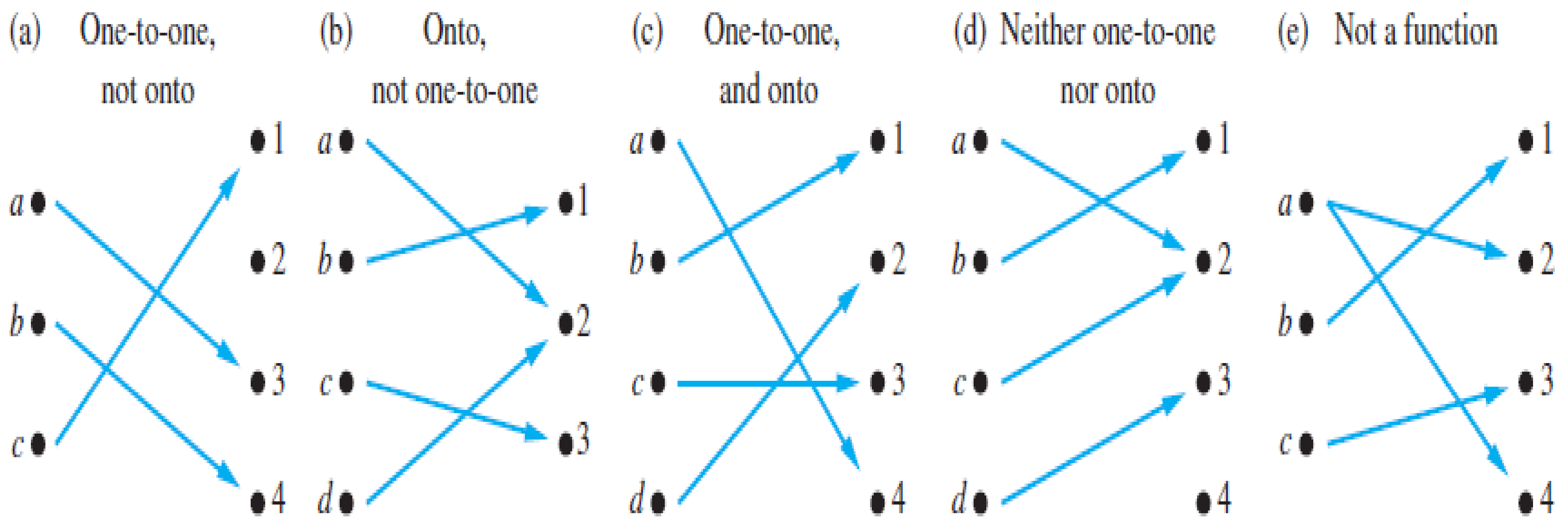
Sol: Because all three elements of the codomain are images of elements in the domain, we see that f is onto. This is illustrated in the figure below. Note that if the codomain were $\{1, 2, 3, 4\}$, then f would not be onto.



Functions

EX_5 : Is the function *f (x)* =X^2 from the set of integers to the set of integers onto?

Sol: The function *f* is not onto because there is no integer *x* with $X^2 = -1$, for instance



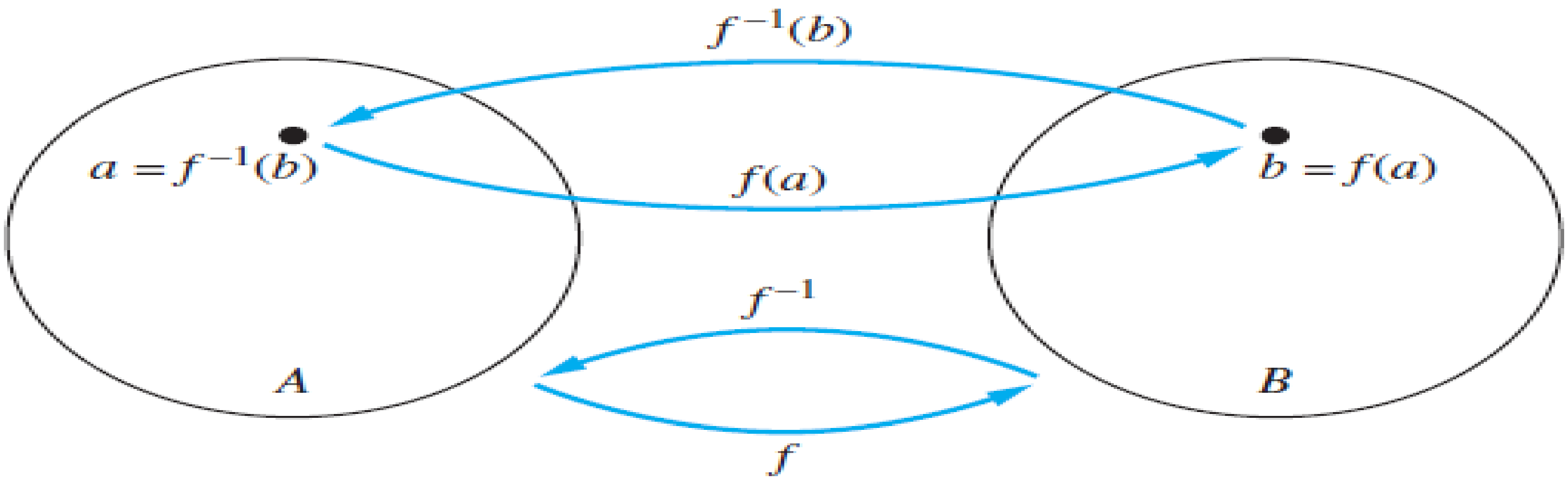
One-to-One &Onto Functions

H.W.: Describe each of the figures above how it is **One-to-One or Onto Functions**

Functions

Inverse Functions:

- Let f be a one-to-one correspondence from the set A to the set B.
- The *inverse function* of f is the function that assigns to an element b belonging to B the unique element a in A such that $f(a) = b$. The inverse function of f is denoted by f^{-1} Hence, $f^{-1}(b) = a$ when $f(a) = b$.



The Function f^{-1} Is the Inverse of Function f

EX_6: Let f be the function from $\{a, b, c\}$ to $\{1, 2, 3\}$ such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$. is f invertible, and if it is, what is its inverse?

Sol: The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence given by f, so $f^{-1}(1) = c$, $f^{-1}(2) = a$, and $f^{-1}(3) = b$.

EX_7: Let f be the function from **R** to **R** with $f(x) = X^2$. Is f invertible?

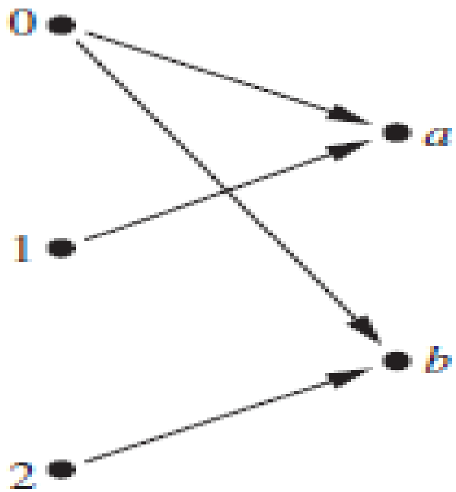
Sol: Because $f(-2) = f(2) = 4$, f is not one-to-one.

Note: If an inverse function were defined, it would have to assign two elements to 4. Hence, f is not invertible. (Note we can also show that f is not invertible because it is not onto.)

Relations

- Relationships between elements of sets are represented using the structure called a relation, which is just a <u>subset of the Cartesian product of the sets</u>.
- Relations can be used to solve problems such as determining which pairs of cities are linked by airline flights in a network, finding a viable order for the different phases of a complicated project, or producing a useful way to store information in computer databases.
- Let A and B be sets. A *binary relation from A to B* is a subset of $A \times B$.

EX_9: Let $A = \{0, 1, 2\}$ and $B = \{a, b\}$. Then $\{(0, a), (0, b), (1, a), (2, b)\}$ is a relation from A to B. This means, for instance, that $0\ R\ a$, but that $1 \not{R}\ b$.



R	a	b
0	×	×
1	×	
2		×

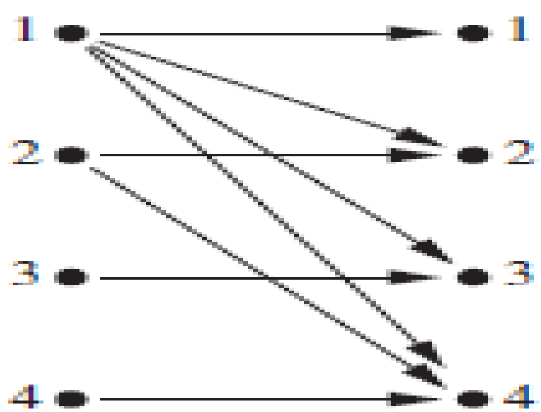
Relations

- A *relation on a set A* is a relation from A to A.
- In other words, a relation on a set A is a subset of $A \times A$.

EX_10: Let A be the set {1, 2, 3, 4}. Which ordered pairs are in the relation $R = \{(a, b) \mid a$ divides $b\}$?

Note: "a divides b" means "divides evenly". That is "a divides b“ if and only if b/a is itself an integer.

Sol: Because (a, b) is in R if and only if a and b are positive integers not exceeding 4 such that a divides b, we see that $R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$.



R	1	2	3	4
1	×	×	×	×
2		×		×
3			×	
4				×

Displaying the Ordered Pairs in the Relation *R* from Ex_10

EX_11: Consider these relations on the set of integers:
$R1 = \{(a, b) \mid a \le b\}$,
$R2 = \{(a, b) \mid a > b\}$,
$R3 = \{(a, b) \mid a = b \text{ or } a = -b\}$,
$R4 = \{(a, b) \mid a = b\}$,
$R5 = \{(a, b) \mid a = b + 1\}$,
$R6 = \{(a, b) \mid a + b \le 3\}$.
Which of these relations contain each of the pairs $(1, 1)$, $(1, 2)$, $(2, 1)$, $(1,-1)$, and $(2, 2)$?

Sol: The pair $(1, 1)$ is in $R1$, $R3$, $R4$, and $R6$; $(1, 2)$ is in $R1$ and $R6$; $(2, 1)$ is in $R2$, $R5$, and $R6$; $(1,-1)$ is in $R2$, $R3$, and $R6$; and finally, $(2, 2)$ is in $R1$, $R3$, and $R4$.

Relations

- A relation R on a set A is called *reflexive* if $(a, a) \in R$ for every element $a \in A$.

EX_12: Consider the following relations on {1, 2, 3, 4}:
$R1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$,
$R2 = \{(1, 1), (1, 2), (2, 1)\}$,
$R3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$,
$R4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$,
$R5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$,
$R6 = \{(3, 4)\}$.
Which of these relations are reflexive?

Sol: The relations $R3$ and $R5$ are reflexive because they both contain all pairs of the form (a, a), namely, $(1, 1)$, $(2, 2)$, $(3, 3)$, and $(4, 4)$. The other relations are not reflexive because they do not contain all of these ordered pairs. In particular, $R1$, $R2$, $R4$, and $R6$ are not reflexive because $(3, 3)$ is not in any of these relations.

Relations

EX_13: Which of the relations from Ex_11 are reflexive?

$R1 = \{(a, b) \mid a \leq b\}$,
$R2 = \{(a, b) \mid a > b\}$,
$R3 = \{(a, b) \mid a = b$ or $a = -b\}$,
$R4 = \{(a, b) \mid a = b\}$,
$R5 = \{(a, b) \mid a = b + 1\}$,
$R6 = \{(a, b) \mid a + b \leq 3\}$.

Sol: The reflexive relations from Ex_11 are $R1$ (because $a \leq a$ for every integer a), $R3$, and $R4$. For each of the other relations in this example it is easy to find a pair of the form (a, a) that is not in the relation.

Combining Relations

- Because relations from A to B are subsets of $A \times B$, two relations from A to B can be combined in any way two sets can be combined.

EX_14: Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$. The relations $R1 = \{(1, 1), (2, 2), (3, 3)\}$ and $R2 = \{(1, 1), (1, 2), (1, 3), (1, 4)\}$ can be combined to obtain:

$R1 \cup R2 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (3, 3)\}$,
$R1 \cap R2 = \{(1, 1)\}$,
$R1 - R2 = \{(2, 2), (3, 3)\}$,
$R2 - R1 = \{(1, 2), (1, 3), (1, 4)\}$.

To understand this , please refer the 2nd lecture in slids 9-12.

THANK YOU