Discrete Mathematics

Sets, Set Operations, &Functions

2nd Lecture

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Class: 1st stage.
Time:8:30AM-10:30AM

Sets & Set Operations

- A *set* is an unordered collection of objects, called *elements* or *members* of the set. A set is said to *contain* its elements. We write $a \in A$ to denote that a is an element of the set A. Then notation $a \notin A$ denotes that a is not an element of the set A.

EX_1 The set V of all vowels in the English alphabet can be written as $V = \{a, e, i, o, u\}$.

EX_2 The set O of odd positive integers less than 10 can be expressed by $O = \{1, 3, 5, 7, 9\}$.

EX_ 3 Although sets are usually used to group together elements with common properties, there is nothing that prevents a set from having seemingly unrelated elements. For instance, {*a*, 2, Fred, New Jersey} is the set containing the four elements *a*, 2, Fred, and New Jersey. Sometimes the roster method is used to describe a set without listing all its members. Some members of the set are listed, and then *ellipses* (. . .) are used when the general pattern of the elements is obvious.

EX_4 The set of positive integers less than 100 can be denoted by $\{1, 2, 3, \ldots, 99\}$.

Sets & Set Operations

- Another way to describe a set is to use **set builder** notation. We characterize all those elements in the set by stating the property or properties they must have to be members. For instance, the set O of all odd positive integers less than 10 can be written as:
  $O = \{x \mid x$ is an odd positive integer less than $10\}$

or, specifying the universe as the set of positive integers, as:

$O = \{x \in Z^+ \mid x$ is odd and $x < 10\}$.

- We often use this type of notation to describe sets when it is impossible to list all the elements of the set.
  the set Q^+ of all positive rational numbers can be written as:

$\mathbf{Q}^+ = \{x \in \mathbf{R} \mid x = \frac{p}{q}$, for some positive integers p and $q\}$.

These sets, each denoted using a boldface letter, play an important role in discrete mathematics:

$\mathbf{N} = \{0, 1, 2, 3, \ldots\}$, the set of **natural numbers**
$\mathbf{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, the set of **integers**
$\mathbf{Z}^+ = \{1, 2, 3, \ldots\}$, the set of **positive integers**
$\mathbf{Q} = \{p/q \mid p \in \mathbf{Z}, q \in \mathbf{Z}$, and $q \neq 0\}$, the set of **rational numbers**
$\mathbf{R}$, the set of **real numbers**
$\mathbf{R}^+$, the set of **positive real numbers**
$\mathbf{C}$, the set of **complex numbers**.

Sets & Set Operations

Recall the notation for **intervals** of real numbers. When a and b are real numbers with $a < b$, we write:

$$[a, b] = \{x \mid a \leq x \leq b\}$$
$$[a, b) = \{x \mid a \leq x < b\}$$
$$(a, b] = \{x \mid a < x \leq b\}$$
$$(a, b) = \{x \mid a < x < b\}$$

Note that $[a, b]$ is called the **closed interval** from a to b and (a, b) is called the **open interval** from a to b.

EX_5: The set $\{\mathbf{N},\mathbf{Z},\mathbf{Q},\mathbf{R}\}$ is a set containing four elements, each of which is a set. The four elements of this set are **N**, the set of natural numbers; **Z**, the set of integers; **Q**, the set of rational numbers; and **R**, the set of real numbers.

- Two sets are *equal* **if and only if they have the same elements**. Therefore, if A and B are sets, then A and B are equal if and only if $\forall x(x \in A \leftrightarrow x \in B)$.We write $A = B$ if A and B are equal sets.

EX_6: The sets {1, 3, 5} and {3, 5, 1} are equal, because they have the same elements.

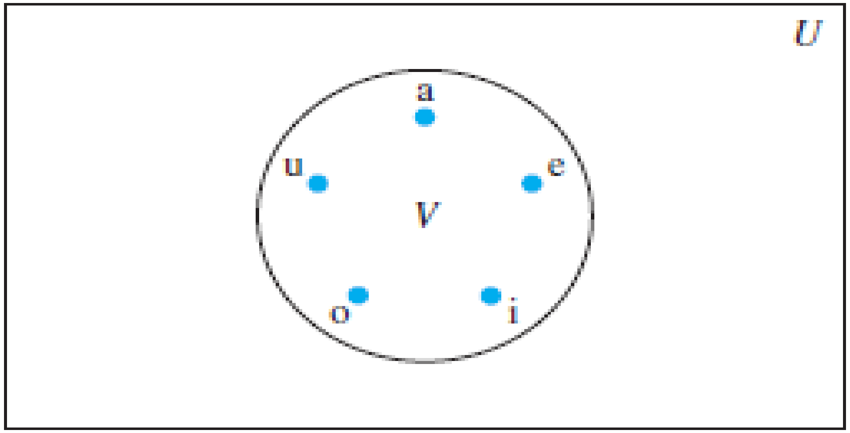
Note: the order in which the elements of a set are listed does not matter. Note also that it does not matter if an element of a set is listed more than once, so {1, 3, 3, 3, 5, 5, 5, 5} is the same as the set {1, 3, 5} because they have the same elements.

- There is a special set that has no elements. This set is called the **empty set**, or **null set**, and is denoted by $\emptyset$. The empty set can also be denoted by { }.
- A set with one element is called a **singleton set**.

Venn Diagrams:

Sets can be represented graphically using Venn diagrams. In Venn diagrams the **universal set** *U*, which contains all the objects under consideration, is represented by a rectangle. Inside this rectangle, circles or other geometrical figures are used to represent sets. Sometimes points are used to represent the particular elements of the set. Venn diagrams are often used to indicate the relationships between sets.

EX_7: Draw a Venn diagram that represents *V*, the set of vowels in the English alphabet.



Sol: We draw a rectangle to indicate the universal set *U*, which is the set of the 26 letters of the English alphabet. Inside this rectangle we draw a circle to represent *V* . Inside this circle we indicate the elements of *V* with points.

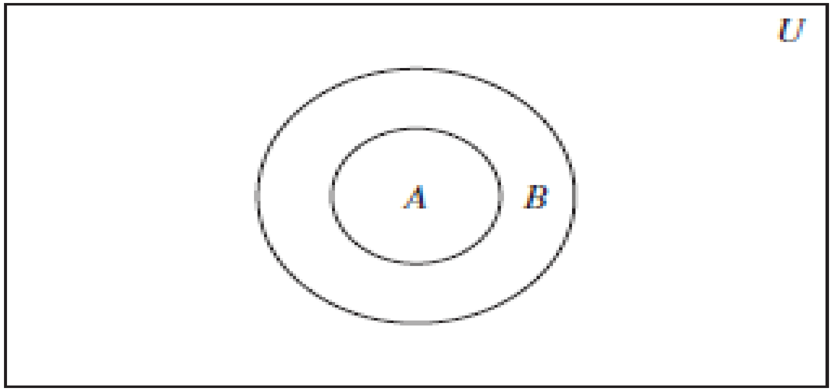
Subsets:

- It is common to encounter situations where the elements of one set are also the elements of a second set. We now introduce some terminology and notation to express such relationships between sets.
- The set A is a *subset* of B if and only if every element of A is also an element of B. We use the notation $A \subseteq B$ to indicate that A is a subset of the set B.

$A \subseteq B$ if and only if the quantification $\forall x(x \in A \rightarrow x \in B)$ *is true.*

Showing that A is a Subset of B To show that $A \subseteq B$, show that if x belongs to A then x also belongs to B.

Showing that A is Not a Subset of B To show that $A \not\subseteq B$, find a single $x \in A$ such that $x \notin B$.



Venn Diagram Showing that *A* is a Subset of *B*.

Sets & Set Operations

Theorem 1.: Let A, B, C be any sets. Then:
(i) $A \subseteq A$
(ii) If $A \subseteq B$ and $B \subseteq A$, then $A = B$
(iii) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$

The Size of a Set

- Sets are used extensively in counting problems, and for such applications we need to discuss the sizes of sets.
- Let S be a set. If there are exactly n distinct elements in S where n is a nonnegative integer, we say that S is a *finite set* and that n is the *cardinality* of S. The cardinality of S is denoted by $|S|$.

EX_10: Let A be the set of odd positive integers less than 10. Then $|A| = 5$.
EX_11: Let S be the set of letters in the English alphabet. Then $|S| = 26$.
EX_12: Because the null set has no elements, it follows that $|\emptyset| = 0$.

- A set is said to be *infinite* if it is not finite.

EX_13: The set of positive integers is infinite.

Sets & Set Operations

Power Sets:

- Many problems involve testing all combinations of elements of a set to see if they satisfy some property. To consider all such combinations of elements of a set *S*, we build a new set that has as its members all the subsets of *S*.
- Given a set *S*, the *power set* of *S* is the set of all subsets of the set *S*. The power set of *S* is denoted by *P(S)*.

EX_14: What is the power set of the set {0, 1, 2}?

Sol: The power set *P(*{0, 1, 2}*)* is the set of all subsets of {0, 1, 2}. Hence,
P({0, 1, 2}*)* = {∅, {0}, {1}, {2}, {0, 1}, {0, 2}, {1, 2}, {0, 1, 2}}.
Note that the empty set and the set itself are members of this set of subsets.

EX_15: What is the power set of the empty set? What is the power set of the set {∅}?

Sol: The empty set has exactly one subset, namely, itself. Consequently, *P(*∅*)* = {∅}.
The set {∅} has exactly two subsets, namely, ∅ and the set {∅} itself. Therefore,
P({∅}*)* = {∅, {∅}}.
If a set has *n* elements, then its power set has 2^n elements.

Cartesian Products:

- Let *A* and *B* be sets. The *Cartesian product* of *A* and *B*, denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. Hence,
  $A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$.

EX_16: Let A represent the set of all students at a university, and let B represent the set of all courses offered at the university. What is the Cartesian product $A \times B$ and how can it be used?

Sol: The Cartesian product $A \times B$ consists of all the ordered pairs of the form (a, b), where a is a student at the university and b is a course offered at the university. One way to use the set $A \times B$ is to represent all possible enrollments of students in courses at the university.

EX_17: What is the Cartesian product of $A = \{1, 2\}$ and $B = \{a, b, c\}$?

Sol: The Cartesian product $A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$.

EX_18: Show that the Cartesian product $B \times A$ is not equal to the Cartesian product $A \times B$, where A and B *sets* are as in Example 17.

Sol: The Cartesian product $B \times A = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}$.

This is not equal to $A \times B$, which was found in Example 17.

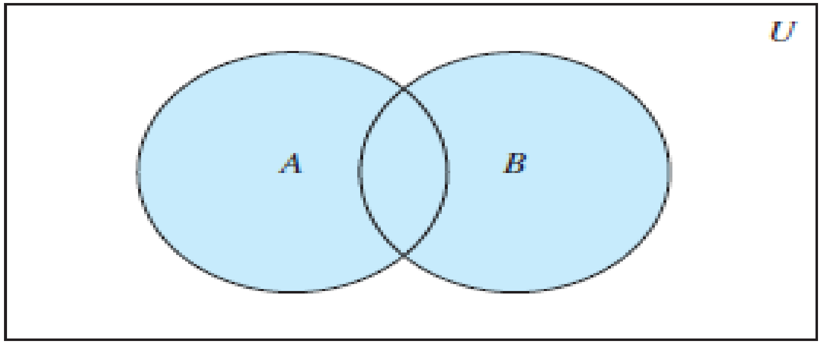
Set Operations:

- Let A and B be sets. The *union* of the sets A and B, denoted by $A \cup B$, is the set that contains those elements that are either in A or in B, or in both.
- $A \cup B = \{x \mid x \in A \vee x \in B\}$.

EX_19: The union of the sets $\{1, 3, 5\}$ and $\{1, 2, 3\}$ is the set $\{1, 2, 3, 5\}$; that is, $\{1, 3, 5\} \cup \{1, 2, 3\} = \{1, 2, 3, 5\}$.

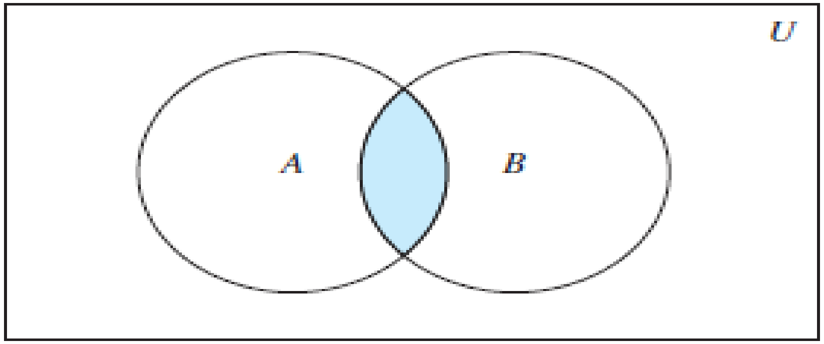
Sets & Set Operations

- Let A and B be sets. The *intersection* of the sets A and B, denoted by $A \cap B$, is the set containing those elements in both A and B.
- $A \cap B = \{x \mid x \in A \land x \in B\}$.
- Two sets are called *disjoint* if their intersection is the empty set.
- Let A and B be sets. The *difference* of A and B, denoted by $A - B$, is the set containing those elements that are in A but not in B. The difference of A and B is also called the *complement of B with respect to A.*
- $A - B = \{x \mid x \in A \land x \notin B\}$.



$A \cup B$ is shaded.

Venn Diagram of the Union of A and B.



$A \cap B$ is shaded.

Venn Diagram of the Intersection of A and B.

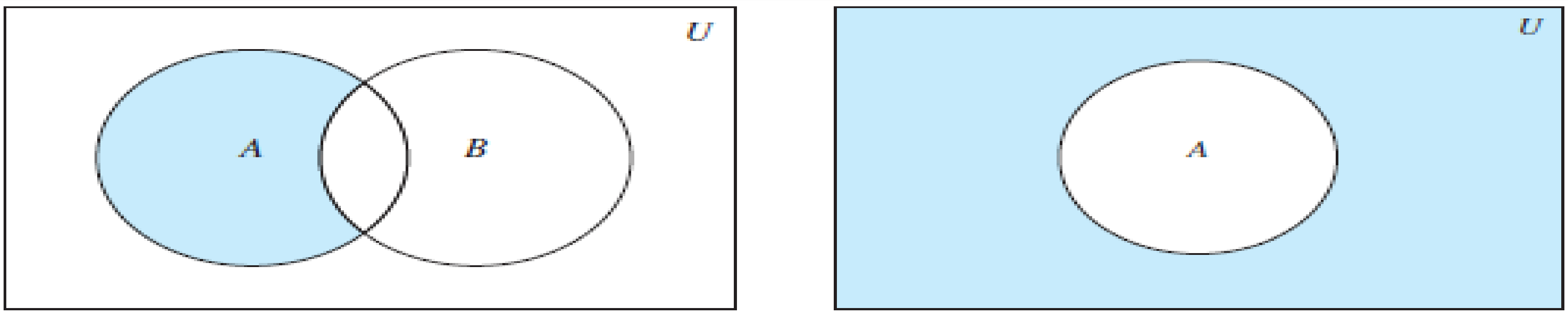
EX_20: The intersection of the sets $\{1, 3, 5\}$ and $\{1, 2, 3\}$ is the set $\{1, 3\}$; that is, $\{1, 3, 5\} \cap \{1, 2, 3\} = \{1, 3\}$.

EX_21: Let $A = \{1, 3, 5, 7, 9\}$ and $B = \{2, 4, 6, 8, 10\}$. Because $A \cap B = \emptyset$, A and B are disjoint.

Sets & Set Operations

EX_22: The difference of {1, 3, 5} and {1, 2, 3} is the set {5}; that is, {1, 3, 5} − {1, 2, 3} = {5}. This is different from the difference of {1, 2, 3} and {1, 3, 5}, which is the set {2}.

- ❑ Let U be the universal set. The *complement* of the set A, denoted by A, is the complement of A with respect to U. Therefore, the complement of the set A is $U - A$.
- ❑ $\bar{A} = \{x \in U \mid x \notin A\}$.



$A - B$ is shaded.

Venn Diagram for the Difference of A and B.

$\bar{A}$ is shaded.

Venn Diagram for the Complement of the Set A.

EX_23: Let $A = \{a, e, i, o, u\}$ (where the universal set is the set of letters of the English alphabet). Then $\bar{A} = \{b, c, d, f, g, h, j, k, l, m, n, p, q, r, s, t, v, w, x, y, z\}$.

EX_24: Let A be the set of positive integers greater than 10 (with universal set the set of all positive integers). Then $\bar{A} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

Sets & Set Operations

Set Identities

TABLE 1 Set Identities.

Identity	*Name*
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

EX_25: Let $A = \{0, 2, 4, 6, 8\}$, $B = \{0, 1, 2, 3, 4\}$, and $C = \{0, 3, 6, 9\}$. What are $A \cup B \cup C$ and $A \cap B \cap C$?

Sol: The set $A \cup B \cup C$ contains those elements in at least one of A, B, and C. Hence,
$A \cup B \cup C = \{0, 1, 2, 3, 4, 6, 8, 9\}$.
The set $A \cap B \cap C$ contains those elements in all three of A, B, and C. Thus,
$A \cap B \cap C = \{0\}$.

- The *intersection* of a collection of sets is the set that contains those elements that are members of all the sets in the collection.
- $A_1 \cap A_2 \cap \ldots \cap A_n = \bigcap_{i=1}^{n} A_i$
- The *union* of a collection of sets is the set that contains those elements that are members of at least one set in the collection.
- $A_1 \cup A_2 \cup \ldots \cup A_n = \bigcup_{i=1}^{n} A_i$

EX_26: For $i = 1, 2, \ldots$, let $A_i = \{i, i + 1, i + 2, \ldots\}$. Then,

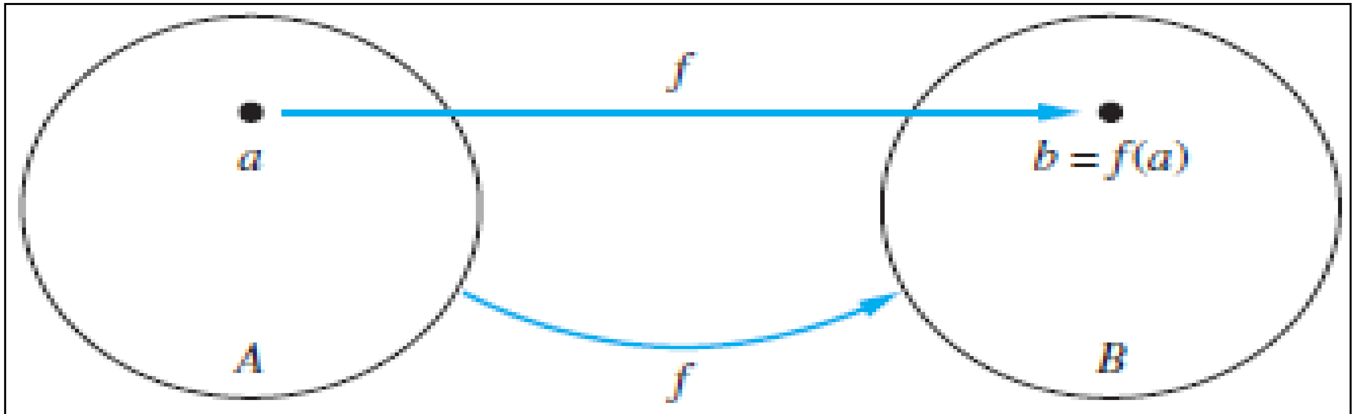
$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} \{i, i+1, i+2, \ldots\} = \{1, 2, 3, \ldots\},$$

and

$$\bigcap_{i=1}^{n} A_i = \bigcap_{i=1}^{n} \{i, i+1, i+2, \ldots\} = \{n, n+1, n+2, \ldots\} = A_n.$$

Functions

- Let A and B be nonempty sets. A *function* f from A to B is an assignment of exactly one element of B to each element of A. We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A. If f is a function from A to B, we write $f : A \to B$.

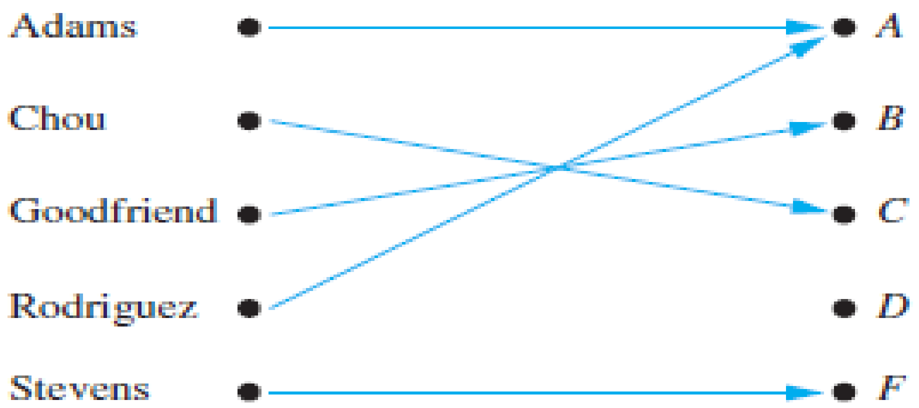


The Function f Maps A to B

- If f is a function from A to B, we say that A is the ***domain*** of f and B is the ***codomain*** of f. If $f(a) = b$, we say that b is the *image* of a and a is a *preimage* of b. The *range*, or *image*, of f is the set of all images of elements of A. Also, if f is a function from A to B, we say that f *maps* A to B.
- When we define a function we specify its *domain*, its *codomain*, and the *mapping of elements of the domain to elements in the codomain*.
- Two functions are **equal** when they have the same domain, have the same codomain, and map each element of their common domain to the same element in their common codomain.

Functions

EX_27: Suppose that each student in a discrete mathematics class is assigned a letter grade from the set $\{A,B,C,D,F\}$. And suppose that the grades are A for Adams, C for Chou, B for Goodfriend, A for Rodriguez, and F for Stevens. This assignment of grades is illustrated the Figure below:



Assignment of Grades in a Discrete Mathematics Class.

EX_28: What are the domain, codomain, and range of the function that assigns grades to students in EX_27?

Sol: Let G be the **function** that assigns a grade to a student in our discrete mathematics class. Note that $G(\text{Adams}) = A$, for instance. The domain of G is the set {Adams, Chou, Goodfriend, Rodriguez, Stevens}, and the codomain is the set $\{A,B,C,D,F\}$. The range of G is the set $\{A,B,C,F\}$, because each grade except D is assigned to some student.

Functions

EX_29 Let R be the relation with ordered pairs (Abdul, 22), (Brenda, 24), (Carla, 21), (Desire, 22), (Eddie, 24), and (Felicia, 22). Here each pair consists of a graduate student and this student's age. Specify a function determined by this relation.

Sol: If f is a function specified by R, then f (Abdul) = 22, f (Brenda) = 24, f (Carla) = 21, f (Desire) = 22, f (Eddie) = 24, and f (Felicia) = 22. (Here, $f(x)$ is the age of x, where x is a student.) For the domain, we take the set {Abdul, Brenda, Carla, Desire, Eddie, Felicia}. We also need to specify a codomain, which needs to contain all possible ages of students. Because it is highly likely that all students are less than 100 years old, we can take the set of positive integers less than 100 as the codomain. (Note that we could choose a different codomain, such as the set of all positive integers or the set of positive integers between 10 and 90, but that would change the function. Using this codomain will also allow us to extend the function by adding the names and ages of more students later.) The range of the function we have specified is the set of different ages of these students, which is the set {21, 22, 24}.

EX_30 Let $f : \mathbf{Z} \rightarrow \mathbf{Z}$ assign the square of an integer to this integer. Then, $f(x) = x^2$, where the domain of f is the set of all integers, the codomain of f is the set of all integers, and the range of f is the set of all integers that are perfect squares, namely, $\{0, 1, 4, 9, \ldots\}$.

Recommended Book

Discrete Mathematics and its Applications
by (Kenneth H. Rosen) **7th edition**

New Link:

https://www.mediafire.com/file/s668zj5rtro8iq0/Rosen_Discrete_Mathematics_and_Its_Applications_7th_Edition.pdf/file

THANK YOU