Homework #1 (Chapters 2-3)

1. Show that
$$\langle u^2 \rangle = \langle U^2 \rangle - \langle U \rangle^2$$

2. By considering the quantities $\left(u_i/\langle u_i^2\rangle^{1/2} \pm u_j/\langle u_j^2\rangle^{1/2}\right)^2$, show that $\langle u_i u_j\rangle^2 \leq \langle u_i^2\rangle\langle u_j^2\rangle$, $(i \neq j)$.

3. Like any other tensor, the Reynolds stress tensor, $\langle \vec{u}\vec{u}\rangle$, can be divided into isotropic and anisotropic parts
$$\underline{a} = \langle \vec{u}\vec{u}\rangle - \frac{2}{3}k\underline{I}$$
where, $\underline{a}$ is called the turbulence anisotropy (tensor) and defined by the above relation. Show that only the anisotropic part of the Reynolds stress and the symmetric part of the mean velocity gradient affect the rate of production of the turbulent kinetic energy, i.e.
$$\underline{P} = -\underline{a} \colon \langle \underline{S} \rangle$$

4. The turbulent dissipation, ε, is sometimes estimated by the quantity
$$\tilde{\varepsilon} = \nu \left\langle \vec{\nabla}\vec{u} \colon \left(\vec{\nabla}\vec{u}\right)^T \right\rangle = \nu \left\langle \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j} \right\rangle$$
which is called pseudo-dissipation. First verify the following relations
$$2\underline{s} \colon \underline{s} = \vec{\nabla}\vec{u} \colon \left(\vec{\nabla}\vec{u}\right)^T + \vec{\nabla}\vec{u} \colon \vec{\nabla}\vec{u}$$
$$2\underline{\Omega} \colon \underline{\Omega} = -\vec{\nabla}\vec{U} \colon \left(\vec{\nabla}\vec{U}\right)^T + \vec{\nabla}\vec{U} \colon \vec{\nabla}\vec{U}$$
then, with the aid of first relation, show that for an incompressible flow
$$\tilde{\varepsilon} = \varepsilon - \nu\vec{\nabla}.\left[\vec{\nabla}.\langle \vec{u}\vec{u}\rangle\right] = \varepsilon - \nu\frac{\partial^2 \langle u_i u_j\rangle}{\partial x_i \partial x_j}$$
Hence, the second term of the right-hand side should be much smaller than ε, which is usually the case, for the estimation to be true.