Homework #3 (Chapter 6)

1. A property of the Navier-Stokes equations is the Galilean invariance. Therefore, it is desirable that turbulence models, e.g. SGS models, also possess this property. This property states that the Navier-Stokes equations have the same form in every inertial frame. In other words, equations for the transformed velocity field, $\vec{W}(\vec{x},t) = \vec{U}(\vec{x},t) + \vec{U}_0$, where $\vec{U}_0$ is a constant vector, have the same form as the equations for the original field, $\vec{U}$.
   a) Obtain the result
   $$\overline{W_i W_j} - \overline{W}_i \overline{W}_j = \overline{U_i U_j} - \overline{U}_i \overline{U}_j$$
   Showing that the residual stress, τ_{ij}^R, is Galilean invariant.
   b) Verify the validity of the Germano's decomposition, Eqs.(6.57)-(6.60).
   c) Show that $\mathcal{L}_{ij}^0$, C_{ij}^0, and $\mathcal{R}_{ij}^0$ are Galilean invariant.

2. In the Bardina model, Eq. (6.62) is used instead of the standard Smagorinsky model for the SGS stress, τ_{ij}^r. Using similar model for the SDS stress, T_{ij}^d, show that the dynamic value of C_s for the Bardina model, Eq. (6.62), is obtained by
   $$C_s = \left(M_{ij}\ell_{ij}^B\right)_{\text{ave}} / (M_{kl}M_{kl})_{\text{ave}}$$
   $$\ell_{ij}^B = \ell_{ij} - H_{ij}, \quad H_{ij} = \widetilde{\overline{\widetilde{\overline{U}}_i \widetilde{\overline{U}}_j}} - \widetilde{\overline{\widetilde{\overline{U}}}}_i \widetilde{\overline{\widetilde{\overline{U}}}}_j - \left(\widetilde{\overline{\overline{U}_i \overline{U}_j}} - \widetilde{\overline{\overline{U}}}_i \widetilde{\overline{\overline{U}}}_j\right)$$
   Hint: τ_{ij}^r is the SGS tensor of the filter $\overline{(.)}$, and T_{ij}^d is the SGS tensor of the filter $\widetilde{\overline{(.)}}$. Therefore, the model for T_{ij}^d is obtained by replacing every single filter $\overline{(.)}$ in the model for τ_{ij}^r with the double filter.