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Question 1: **derive the filtered equation and the required formulation.**

For deriving the filtered equation, first we should take filter from the equation:

$$\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}=\nu\frac{\partial^2 U}{\partial x^2}+\eta\left(x,t\right)\rightarrow\frac{\partial U}{\partial t}+\frac{1}{2}\frac{\partial UU}{\partial x}=\nu\frac{\partial^2 U}{\partial x^2}+\eta\left(x,t\right) \tag{1}$$

$$taking\ \ filter\rightarrow\frac{\partial \bar{U}}{\partial t}+\frac{1}{2}\frac{\partial \overline{UU}}{\partial x}=\nu\frac{\partial^2 \bar{U}}{\partial x^2}+\bar{\eta}\left(x,t\right) \tag{2}$$

Because we want to solve filtered velocity, we cannot compute $\overline{UU}$, so we can write it like below:

$$\overline{UU}=\bar{U}\bar{U}+\tau_{11}{}^{R} \tag{3}$$

And now we should use different models to simulate τ^R . By substituting eq. 3 into the eq.2, we can write:

$$\frac{\partial \bar{U}}{\partial t}+\frac{1}{2}\frac{\partial \bar{U}\bar{U}}{\partial x}=\nu\frac{\partial^2 \bar{U}}{\partial x^2}+\bar{\eta}\left(x,t\right)-\frac{1}{2}\frac{\partial \tau_{11}{}^{R}}{\partial x} \tag{4}$$

By using the continuity like eq. 1, we can rewrite eq. 4 like the paper:

$$\frac{\partial \bar{U}}{\partial t}+\bar{U}\frac{\partial \bar{U}}{\partial x}=\nu\frac{\partial^2 \bar{U}}{\partial x^2}+\bar{\eta}\left(x,t\right)-\frac{1}{2}\frac{\partial \tau_{11}{}^{R}}{\partial x} \tag{5}$$

The second order discretization for the DNS equation can be written as:

$$\frac{1}{2}\frac{\partial UU}{\partial x}=\frac{[U(x)+U(x+h)]^2-[U(x)+U(x-h)]^2}{8h} \tag{6}$$

$$\nu\frac{\partial^2 U}{\partial x^2}=\nu\frac{\mathrm{U}(x+h)-2U(x)+U(x-h)}{h^2} \tag{7}$$

Similarly, forth order discretization of DNS equation is:

$$\frac{1}{2}\frac{\partial UU}{\partial x}=\frac{4}{3}\times\frac{[U(x)+U(x+h)]^2-[U(x)+U(x-h)]^2}{8h}-\frac{1}{3}\times\frac{[U(x)+U(x+2h)]^2-[U(x)+U(x-2h)]^2}{16h} \tag{8}$$

$$\nu\frac{\partial^2 U}{\partial x^2}=\nu\frac{4}{3}\frac{U(x+h)-2U(x)+U(x-h)}{h^2}-\nu\frac{1}{3}\frac{U(x+2h)-2U(x)+U(x-2h)}{4h^2} \tag{9}$$

For simulating unresolved part of the LES equation, there are some dynamics models which we will discuss two of them, Smagornsky and Wong models:

Dynamic Smagorinsky Model

Smagorinsky proposed that we can use two different filter width to model the unresolved part. First, he used the below decomposition:

$$\tau^{R}{}_{ij} = \overline{U_i U_j} - \bar{U}_i \bar{U}_j \tag{10}$$

$$T^{d}{}_{ij} = \overline{U_i U_j} - \bar{U}_i \bar{U}_j \tag{11}$$

$$l_{ij} = T^{d}{}_{ij} - \tilde{\tau}^{R}{}_{ij} = \bar{U}_i \bar{U}_j - \tilde{\bar{U}}_i \tilde{\bar{U}}_j \tag{12}$$

As it can be seen, eq. 12 is the resolved part and we can compute it.

$$\tau^{R}{}_{ij} - \frac{1}{3}\tau^{R}{}_{kk}\delta_{ij} = \tau^{r}{}_{ij} = -2Cs\bar{\Delta}^2 \,|\, \bar{S} \,|\, \bar{S}_{ij} \tag{13}$$

$$T^{d}{}_{ij} - \frac{1}{3}T^{d}{}_{kk}\delta_{ij} = T^{r}{}_{ij} = -2Cs\tilde{\bar{\Delta}}^2 \,|\, \tilde{\bar{S}} \,|\, \tilde{\bar{S}}_{ij} \tag{14}$$

From eq. 13 and 14:

$$l_{ij} - \frac{1}{3}l_{kk}\delta_{ij} = T^{r}{}_{ij} - \tilde{\tau}^{r}{}_{ij} = -2Cs\tilde{\bar{\Delta}}^2 \,|\, \tilde{\bar{S}} \,|\, \tilde{\bar{S}}_{ij} + 2Cs\bar{\Delta}^2 \,|\, \bar{S} \,|\, \bar{S}_{ij} \tag{15}$$

We can assume that Cs is constant and then we can write it out of the filter:

$$l_{ij} - \frac{1}{3}l_{kk}\delta_{ij} = Cs.M_{ij},\, M_{ij} = 2\bar{\Delta}^2 \,|\, \bar{S} \,|\, \bar{S}_{ij} - 2\tilde{\bar{\Delta}}^2 \,|\, \tilde{\bar{S}} \,|\, \tilde{\bar{S}}_{ij} \tag{16}$$

Because M_{ij} is a symmetric tensor, we have 5 independent equations and only 1 unknown parameter. From what we studied in advanced mathematics, we can compute the minimum error to find Cs. We can do it like below:

$$eQ = \sum error^2 = (l_{ij} - \frac{1}{3}l_{kk}\delta_{ij} - Cs.M_{ij})(l_{ij} - \frac{1}{3}l_{kk}\delta_{ij} - Cs.M_{ij}) \tag{17}$$

$$\begin{aligned}
&\frac{\partial eQ}{\partial Cs} = \sum error^2 = -M_{ij}(l_{ij} - \frac{1}{3}l_{kk}\delta_{ij} - Cs.M_{ij}) = 0 \\
&\rightarrow -M_{ij}l_{ij} + \frac{1}{3}M_{ij}l_{kk}\delta_{ij} + CsM_{ij}M_{ij} = 0,\ \frac{1}{3}M_{ij}l_{kk}\delta_{ij} = 0 \\
&\rightarrow M_{ij}l_{ij} = CsM_{ij}M_{ij} \rightarrow Cs = \frac{M_{ij}l_{ij}}{M_{ij}M_{ij}}
\end{aligned} \tag{18}$$

Because eq. 18 is unstable, we can take average in homogenous directions:

$$Cs = \frac{< M_{ij} l_{ij} >}{< M_{ij} M_{ij} >} \tag{19}$$

In current simulation, we want to solve burger equation which is one dimensional:

$$i = j = 1,\ \tilde{\bar{\Delta}} = 2\bar{\Delta} \tag{20}$$

So, we can simplify equations:

$$\tau_{11}{}^{R} = -2Cs\bar{\Delta}^2 \,|\,\bar{S}\,|\,\bar{S}_{11} = -2Cs\bar{\Delta}^2 \,|\,\frac{\partial \bar{U}_1}{\partial x}\,|\,\frac{\partial \bar{U}_1}{\partial x} \tag{21}$$

$$l_{11} = \bar{U}_1 \bar{U}_1 - \tilde{\bar{U}}_1 \tilde{\bar{U}}_1 \tag{22}$$

$$M_{11} = 2\bar{\Delta}^2 \,|\,\bar{S}_{11}\,|\,\bar{S}_{11} - 2\tilde{\bar{\Delta}}^2 \,|\,\tilde{\bar{S}}_{11}\,|\,\tilde{\bar{S}}_{11} = -2\bar{\Delta}^2 [4\,|\,\frac{\partial \bar{U}_1}{\partial x}\,|\,\frac{\partial \bar{U}_1}{\partial x} - |\,\frac{\partial \bar{U}_1}{\partial x}\,|\,\frac{\partial \bar{U}_1}{\partial x}] \tag{23}$$

$$Cs = \frac{< M_{11} l_{11} >}{< M_{11} M_{11} >} \tag{24}$$

Dynamic Wong Model

Wong proposed that we can use eq. 25 for turbulent viscosity:

$$\upsilon_t = C_w \bar{\Delta}^{4/3} \tag{25}$$

$$\tau_{ij}^{R} - \frac{1}{3}\tau_{kk}^{R}\delta_{ij} = -2\nu_t \bar{S}_{ij} \tag{26}$$

By substituting eq. 25 into eq. 26, which was mentioned in eq. 6.40 in class, we can obtain Wong stress:

$$\tau_{ij}^{R} - \frac{1}{3}\tau_{kk}^{R}\delta_{ij} = -2(\mathrm{C}_w \bar{\Delta}^{4/3})\bar{S}_{ij} \tag{27}$$

Exactly like what we did in eq. 15 for Smagorinsky model we can write:

$$l_{ij} - \frac{1}{3} l_{kk}\delta_{ij} = T^{r}{}_{ij} - \tilde{\tau}^{r}{}_{ij} = -2C_w \tilde{\bar{\Delta}}^{4/3} \tilde{\bar{S}}_{ij} + 2C_w \bar{\Delta}^{4/3} \bar{S}_{ij} \tag{28}$$

Like eq. 16 we have:

$$l_{ij} - \frac{1}{3} l_{kk}\delta_{ij} = C_w . N_{ij}, N_{ij} = 2\bar{\Delta}^{4/3}\bar{S}_{ij} - 2\tilde{\bar{\Delta}}^{4/3}\tilde{\bar{S}}_{ij} \tag{29}$$

Similar to eq. 18 we can compute Wong coefficient:

$$\begin{aligned} &\frac{\partial eQ}{\partial Cs} = \sum error^2 = -N_{ij}(l_{ij} - \frac{1}{3} l_{kk}\delta_{ij} - C_w . N_{ij}) = 0 \\ &\rightarrow -N_{ij} l_{ij} + \frac{1}{3} N_{ij} l_{kk}\delta_{ij} + C_w N_{ij} N_{ij} = 0, \ \frac{1}{3} N_{ij} l_{kk}\delta_{ij} = 0 \\ &\rightarrow N_{ij} l_{ij} = C_w N_{ij} N_{ij} \rightarrow Cs = \frac{N_{ij} l_{ij}}{N_{ij} N_{ij}} \end{aligned} \tag{30}$$

We can simplify burgers equation for this model too:

$$\tau_{11}{}^R = -2(\mathrm{C}_w \bar{\Delta}^{4/3})\bar{S}_{11} = -2(\mathrm{C}_w \bar{\Delta}^{4/3}) \frac{\partial \bar{U}_1}{\partial x} \tag{31}$$

$$l_{11} = \bar{U}_1 \bar{U}_1 - \tilde{\bar{U}}_1 \tilde{\bar{U}}_1 \tag{32}$$

$$N_{11} = 2\bar{\Delta}^{4/3}\bar{S}_{11} - 2\tilde{\bar{\Delta}}^{4/3}\tilde{\bar{S}}_{11} = -2\bar{\Delta}^{4/3}[2^{4/3}\frac{\partial \bar{U}_1}{\partial x} - \frac{\partial \bar{U}_1}{\partial x}] \tag{33}$$

$$C_w = \frac{< N_{11} l_{11} >}{< N_{11} N_{11} >} \tag{34}$$

In order to write the code for unresolved part, we need to know the discretization for gradient of a parameter like U:

Second order discretization:

$$\frac{\partial U}{\partial x} = \frac{U(x+h) - U(x-h)}{2h} \tag{35}$$

And forth order discretization:

$$\frac{\partial U}{\partial x} = \frac{-U(x+2h) + 8U(x+h) - 8U(x-h) + U(x-2h)}{12h} \tag{36}$$

Validation

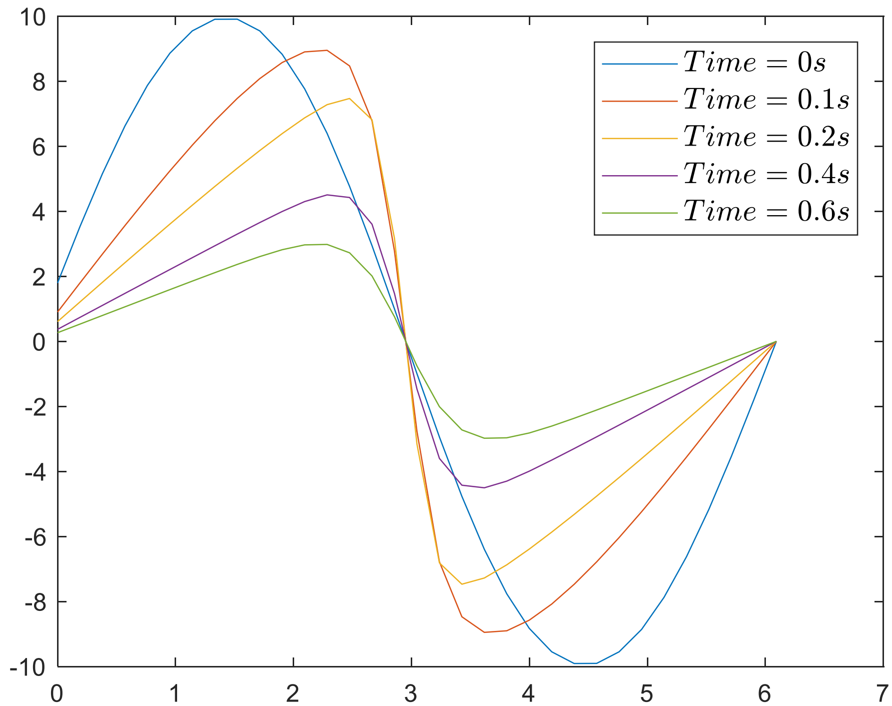


Fig 1. Example 6.8 from Parviz Moin.

Question 2: Use a fully conservative second-order discretization in space (and an explicit second-order scheme in time).

In this question we compared three different LES models including Dynamic Smagornisky, Dynamic Wong, and static Smagorinsky models by second order discretization in space and Adams–Bashforth discretization in time. You can see the results in Fig. 2. Wong model has more fluctuations near x=0, but for the rest of the domain, all the models are similar.

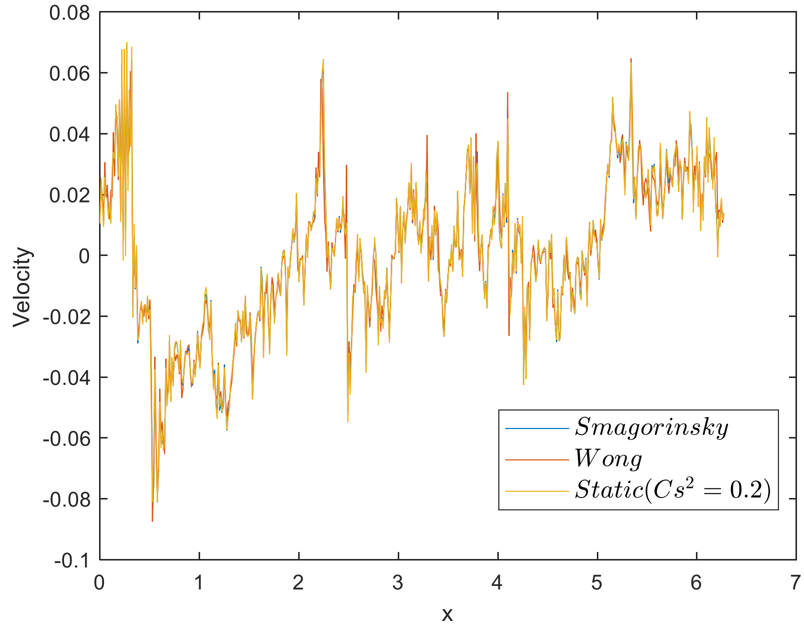


Fig 2. Different LES models by using second order discretization.

Question 3: Use 4th order fully conservative space discretization scheme and for time discretization, apply the 4th order Runge-Kutta scheme

As it can be seen in Fig. 3, all models are close to each other.

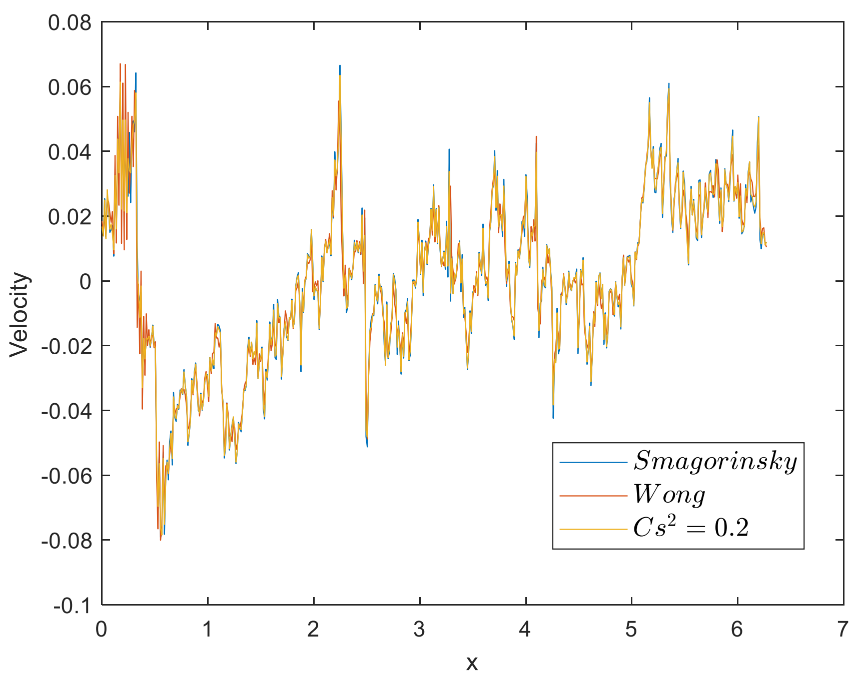
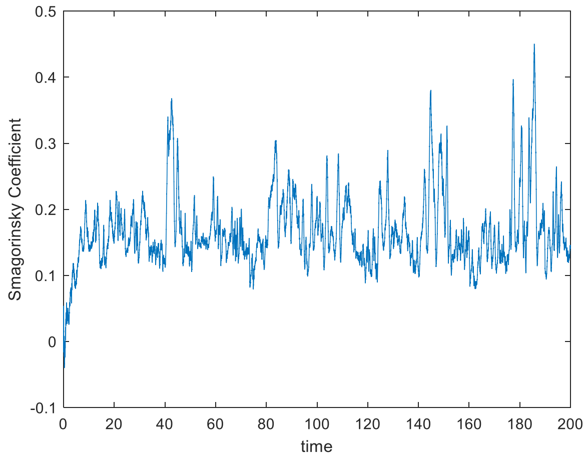


Fig 3. Different LES models by using 4rth order discretization.

Question 4: Plot the profile of the dynamic model coefficient versus time.



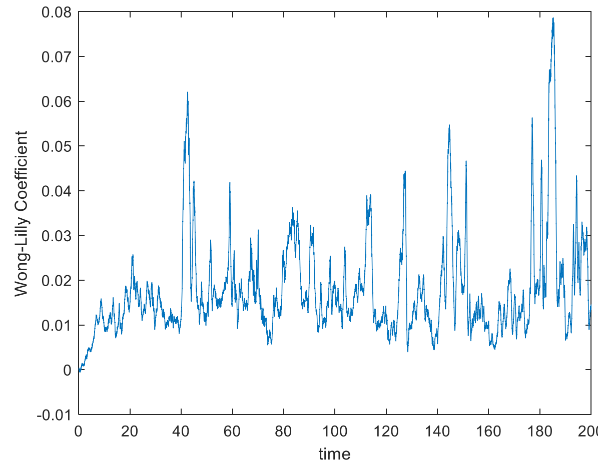


Fig 4. Dynamic Coefficient for different LES models

Question 5: Perform a DNS solution on a fine grid (Nx=8192). Forth order fully conservative space discretization and forth order Runge-Kutta scheme.

As it can be seen in Fig. 5, DNS velocity has more fluctuation which is obvious.

Note: for Nx=512, we could not compute the velocity without LES modeling.

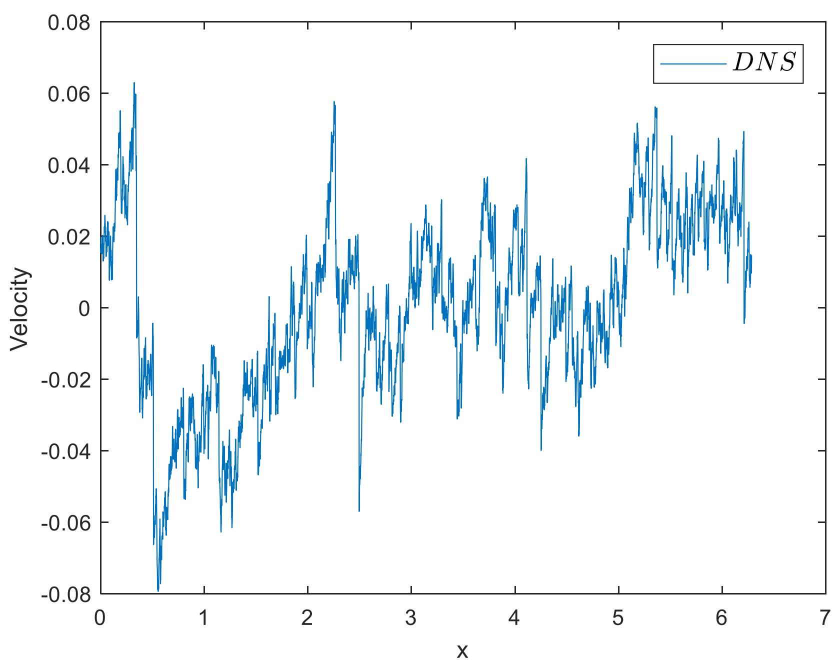


Fig 5. DNS velocity by using 4rth order discretization.

Question 6: Apply the proper filter on the reference DNS solution at t=200s, and compare the results with the LES models.

All LES models has a good accuracy in comparison with DNS and filtered DNS results.

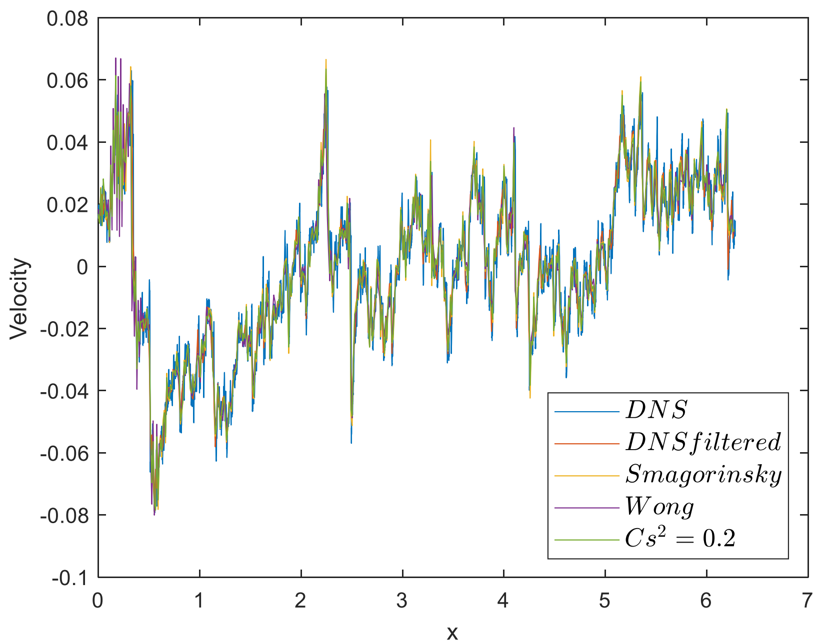


Fig 6. Comparison between filtered DNS velocity and different LES models.

Question 7: Profiles of the resolved kinetic energy

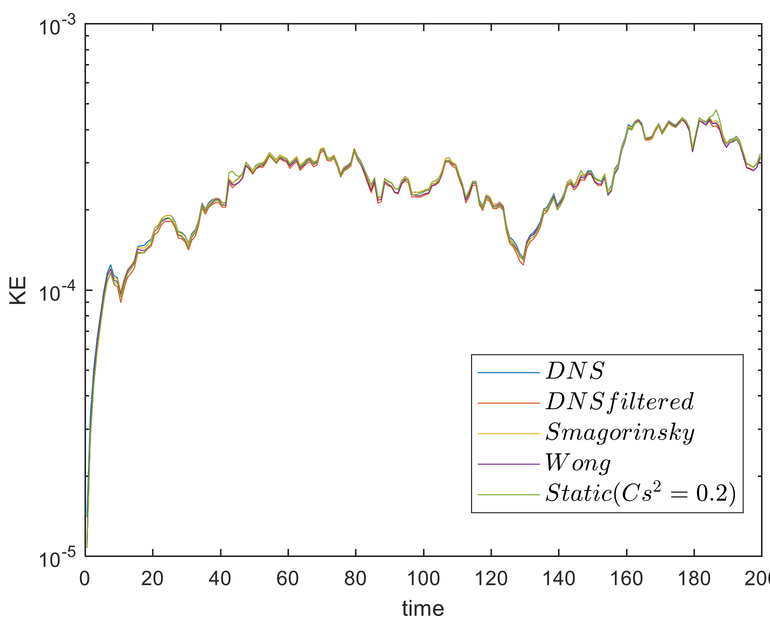


Fig 7. Comparison of kinetic energy between different solutions.

Question 8: Compare the energy spectrum

Fig. 8 shows the energy spectrum for different models. As it can be seen, LES models can only take smaller modes and larger eddies. That is what we do in LES modeling and we cannot take energy of small eddies. If we make our filter's width smaller, we can be closer to DNS results.

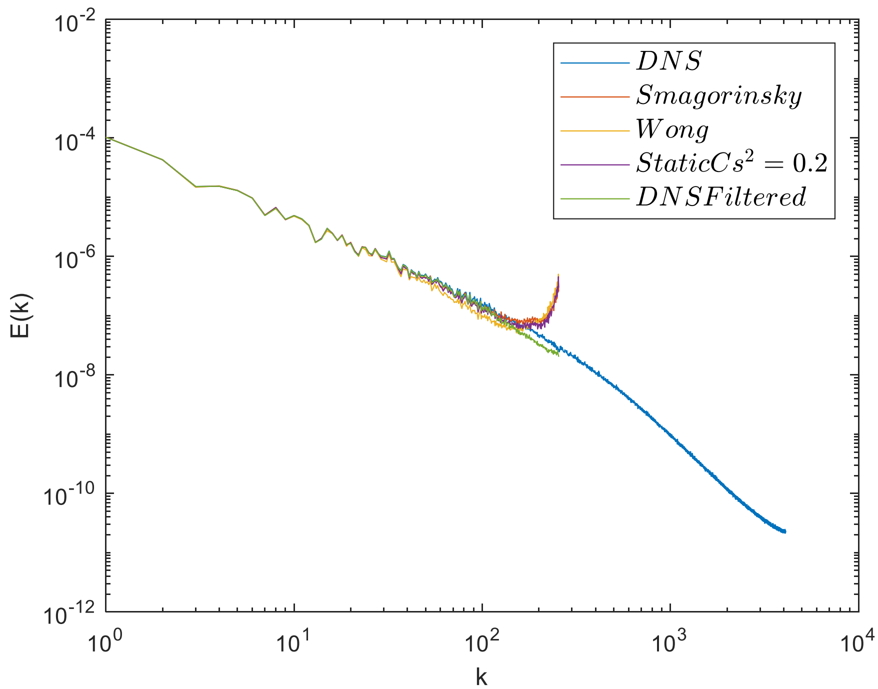


Fig 8. Comparison of energy spectrum between different solutions.