Scattering Amplitudes for (Non-) Lorentzian Quantum Field Theories

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1 Cubic Field

Here we are interested in computing the Feynman rules for a Lagrangian that is cubic in a massless, free scalar field. The Lagrangian is given by the following

$$\mathcal{L}_3 = -\frac{\lambda}{3!}\phi^3. \tag{1.1}$$

To find the factor that will be placed at each vertex of our Feynmann diagram, we need to keep taking derivatives until the scalar field has completely vanished from the Lagrangian. We proceed with the first derivative

$$\frac{\delta \mathcal{L}_3}{\delta \phi} = -\frac{1}{2}\lambda\phi^2. \tag{1.2}$$

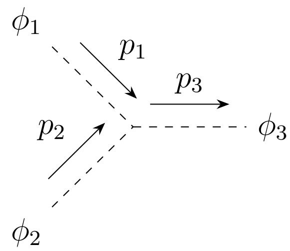
Since ϕ is still present within the Lagrangian we need to take another derivative. The second derivative is

$$\frac{\delta^2 \mathcal{L}_3}{\delta \phi^2} = -\lambda\phi. \tag{1.3}$$

Now that ϕ appears linearly within the functional, we need only take one more derivative. Hence:

$$\frac{\delta^3 \mathcal{L}_3}{\delta \phi^3} = -\lambda. \tag{1.4}$$

Next we can write down the amplitude for this Lagrangian.



Since this process has only one vertex, calculating the amplitude is fairly trivial. The amplitude is merely $\mathcal{A} = \lambda$. So that was easy, which is exactly what we expected since this is the simplest lowest order tree diagram that contributes to the overall scattering.

2 Quadratic Derivative

We are interested in calculating the scattering amplitude for a free and massless scalar field whose coupled to its own derivatives. The Lagrangian we wish to work with is

$$\mathcal{L}_3 = -\frac{\lambda}{3!}\phi(\partial\phi)^2, \tag{2.1}$$

where λ is the coupling constant of our toy theory, ϕ is the scalar field and $(\partial\phi)^2 = \partial_\mu\phi\partial^\mu\phi$ is a shorthand for the kinetic energy of the field. The first derivative is given by

$$\frac{\delta\mathcal{L}_3}{\delta\phi} = -\frac{\lambda}{3!}(\partial\phi)^2 - \frac{\lambda}{3}\phi(-ip_i^\mu)\partial_\mu\phi, \tag{2.2}$$

where we use the Feynman rule that

$$\frac{\delta\partial_\mu\phi\partial^\mu\phi}{\delta\phi} = 2(-ip^\mu)\partial_\mu\phi. \tag{2.3}$$

Taking the second derivative with respect to ϕ gives us

$$\frac{\delta^2\mathcal{L}_3}{\delta\phi^2} = -\frac{\lambda}{3}(ip_j^\mu)\partial_\mu\phi - \frac{\lambda}{3}(-ip_i^\mu)\partial_\mu\phi - \frac{\lambda}{3}\phi(-ip_{j,\mu})(-ip_i^\mu). \tag{2.4}$$

Differentiating one last time with respect to ϕ gets us,

$$\frac{\delta^3\mathcal{L}_3}{\delta\phi^3} = -\frac{1}{3}\lambda(ip_{k,\mu})(ip_j^\mu) - \frac{1}{3}\lambda(ip_{k,\mu})(ip_i^\mu) - \frac{1}{3}\lambda(ip_{j,\mu})(ip_i^\mu) \tag{2.5}$$

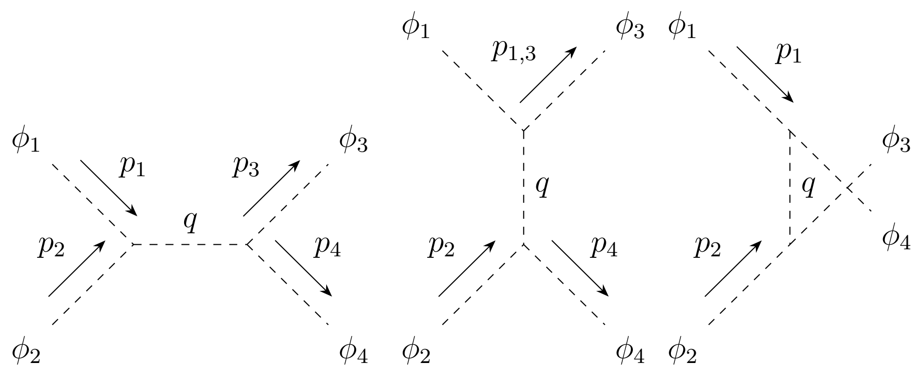
$$= \frac{\lambda}{3}\bigg(p_i\cdot p_j + p_j\cdot p_k + p_k\cdot p_i\bigg). \tag{2.6}$$

Before we carry on with the calculating, it is useful for us to work in Mandelstam variables (s,t,u) where

$$s = (p_1+p_2)^2 = (p_3+p_4)^2 = 2p_1\cdot p_2, \tag{2.7}$$

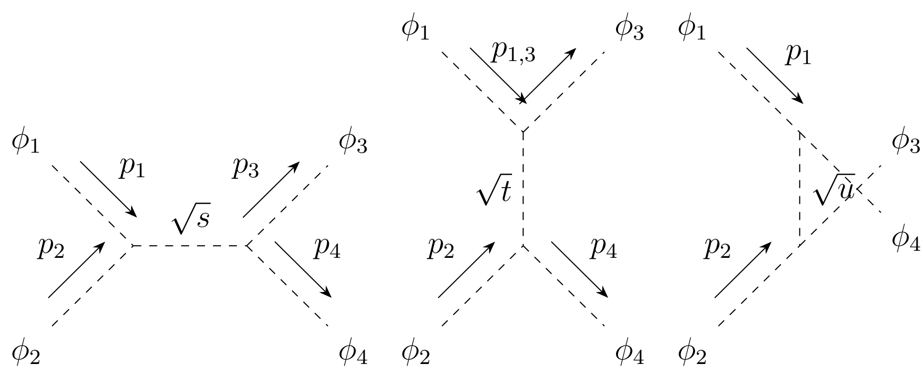
$$t = (p_1-p_3)^2 = (p_2-p_4)^2 = -2p_1\cdot p_3, \tag{2.8}$$

$$u = (p_1-p_4)^2 = (p_2-p_3)^2 = -2p_1\cdot p_4, \tag{2.9}$$



where we used the fact that $m_1 = m_2 = m_3 = m_4 = 0$. Next we need to compute the vertex rule for each diagram in Mandelstam variables. We introduce the parameter $\alpha = \frac{1}{3}\lambda(p_i \cdot p_j + p_j \cdot p_k + p_k \cdot p_i)$ for convenience.

Now we can compute the scattering amplitude, $\mathcal{A}$, for this Lagrangian. Since there are three possible diagrams that can be constructed from this theory, it is typical for one to break up each amplitude contribution into channels: s, t, *and* u. The most left diagram is called the s-channel, the center diagram is called the t-channel, and the right most diagram is the u-channel. Griffiths lays out a prescription on how to calculate the momenta of the internal line, but we can skip to the right answer.



First we calculate the vertex rule for the $s-$channel. From the diagram we can see

$$p_i = p_1, \quad p_j = p_2, \quad p_k = -(p_1 + p_2) \Rightarrow \alpha_s = -\frac{\lambda}{6}s. \tag{2.10}$$

The $t-$channel momenta and vertex rule is given by

$$p_i = p_1, \quad p_j = -(p_1 - p_3), \quad p_k = -p_3 \Rightarrow \alpha_t = -\frac{\lambda}{6}t. \tag{2.11}$$

And finally, the $u-$channel is shown to be

$$p_i = p_1, \quad p_j = -p_4, \quad p_k = -(p_1 - p_4) \Rightarrow \alpha_u = -\frac{\lambda}{6}u. \tag{2.12}$$

However, there is a constraint on the Mandelstam variables in that they satisfy the following equation

$$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2, \tag{2.13}$$

where m_i is the mass of the i-th particle. Since we're assuming all the scattered particles are massless, we can eliminate on of the variables in favor of the other two. The convention is to write $u = -(s+t)$. Now we can calculate the amplitude for each diagram

$$\mathcal{A}_s = \frac{\alpha_s^2}{s}, \quad \mathcal{A}_t = \frac{\alpha_t^2}{t}, \quad \mathcal{A}_u = \frac{\alpha_u^2}{u}. \tag{2.14}$$

Thus the total scattering amplitude, the amplitude that is contributed by all three Feynman diagrams is

$$\mathcal{A} = \mathcal{A}_s + \mathcal{A}_t + \mathcal{A}_u. \tag{2.15}$$

Remembering that $\alpha_p = \alpha_p(s,t,u)$, we can simplify the total amplitude just in terms of the coupling constant and Mandelstam variables. Proceeding accordingly, we get

$$\mathcal{A} = \frac{1}{s}\left(\frac{\lambda}{6}\right)^2 s^2 + \frac{1}{t}\left(\frac{\lambda}{6}\right)^2 t^2 + \frac{1}{u}\left(\frac{\lambda}{6}\right)^2 u^2 \Leftrightarrow \left(\frac{\lambda}{6}\right)^2 (s+t+u) = 0. \tag{2.16}$$

And thus, the scattering amplitude for $\mathcal{A}_3$ is $\mathcal{A} = 0$. This was the expected result because in a Lagrangian with a term proportional to ϕ^2, we can do the following field redefinition to get rid of it

$$\phi^2 \to \frac{\lambda}{3!2}\left(\phi + (\partial\phi)^2\right)^2, \tag{2.17}$$

as a result, this term couldn't possibly contributed anything to the final amplitude because we can simply redefine it out of the Lagrangian.

3 Galileon Amplitude

Lastly, we would like to calculate the scattering amplitude for the following Lagrangian:

$$\mathcal{L}_3 = \frac{1}{\Lambda^3}(\partial\phi)^2\Box\phi, \tag{3.1}$$

where $\Box = \partial_\mu\partial^\mu$. Following our previous examples, the first derivative w.r.t ϕ gives us

$$\frac{\delta\mathcal{L}_3}{\delta\phi} = \frac{1}{\Lambda^3}\bigg(2(-ip_i^\mu)\partial_\mu\phi\Box\phi + (\partial\phi)^2(-ip_i^\nu)(-ip_{i,\nu})\bigg), \tag{3.2}$$

where we've employed Leibniz's rule. Next we take the second derivative which takes on the form

$$\frac{\delta^2\mathcal{L}_3}{\delta\phi^2} = \frac{1}{\Lambda^3}\bigg(2(-ip_i^\mu)(-ip_{j,\mu})\Box\phi + 2(-ip_i^\mu)\partial_\mu\phi(-ip_j^\nu)(-ip_{j,\nu}) + 2(-ip_j^\mu)\partial_\mu\phi(-ip_i^\nu)(-ip_{i,\nu})\bigg). \tag{3.3}$$

And finally, taking the third derivative gives us

$$\begin{aligned}\frac{\delta^3\mathcal{L}_3}{\delta\phi^3} = \frac{1}{\Lambda^3}(2(-ip_i^\mu)(-ip_{j,\mu})(-ip_k^\nu)(-ip_{k,\nu}) + 2(-ip_i^\mu)(-ip_{k,\mu})(-ip_j^\nu)(-ip_{j,\nu})\\ + 2(-ip_j^\mu)(-ip_{k,\mu})(-ip_i^\nu)(-ip_{i,\nu})).\end{aligned} \tag{3.4}$$

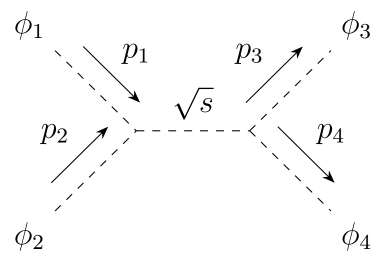
Collecting all the terms and simplifying a bit gets us

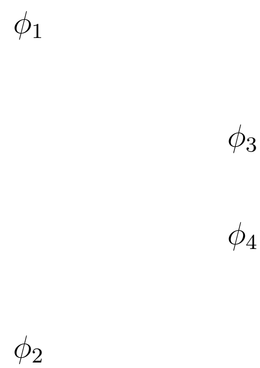
$$\frac{\delta^3\mathcal{L}_3}{\delta\phi^3} = \frac{2}{\Lambda^3}\bigg(p_i\cdot p_j\cdot p_k^2 + p_j\cdot p_k\cdot p_i^2 + p_k\cdot p_i\cdot p_j^2\bigg). \tag{3.5}$$

We shall rewrite the diagrams from the previous section in order for us to get a clearer picture as to what we wish to calculate.

Going through the same procedure as we did in the quadratic derivative case, we find that the vertex rule for each diagram is

$$\alpha_s = -\frac{s^2}{\Lambda^3}, \quad \alpha_t = \frac{t^2}{\Lambda^3}, \quad \alpha_u = \frac{(s+t)^2}{\Lambda^3}. \tag{3.6}$$





And remembering that the scattering amplitude is nothing but the sum of the square of the vertex divided by the channel, we get

$$\mathcal{A} = \frac{\alpha_s^2}{s} + \frac{\alpha_t^2}{t} + \frac{\alpha_u^2}{u} \Leftrightarrow -\frac{3st}{\Lambda^6}(s+t) \tag{3.7}$$

And thus, we have the scattering amplitude for this Lagrangian.

4 Phonon-Phonon Scatter

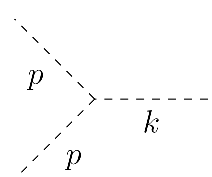
We are interested in (eventually) computing the scattering amplitude for Mimetic Gravity. To do so, we need a toy model in order to study non-relativistic scattering processes. To do so, we look to Condensed Matter Physics in which one maybe interested in the scattering process of two ”hard” phonons which yield a ”soft” phonon. Our toy model admits an effective field theory description

$$S = \int P(X)\ d^4x, \qquad X = \mu/m + \dot{\pi} - \frac{1}{2}(\nabla\pi)^2, \tag{4.1}$$

where P is the pressure, μ is the chemical potential, m is the mass of the superfluid, and $\pi(x)$ is the phonon field that transforms non-trivially under a Lorentz boost. We also need to define some additional terms

$$\rho = \frac{dP}{dX}, \qquad c_s^2 = \frac{d\rho}{dP}, \tag{4.2}$$

where ρ is the mass density of the fluid and c_s is the speed of sound. Since we don’t know a priori what the X dependence on the pressure is supposed to be, we do what physicists normally do when we don’t know what the function is and Taylor expand it around a certain point. In our case, we’re going to expand around $X = \mu/m$. We’re interested in the following diagram:



so we’ll need to keep only up to cubic order in $\pi(x)$. Now we Taylor expand P(X)

$$P(X) \approx P(\mu/m) + (X - \mu/m)\, \frac{dP}{dX}\bigg|_{X=\mu/m} + \frac{1}{2}(X-\mu/m)^2\, \frac{\mathrm{d}^2P}{\mathrm{d}X^2}\bigg|_{X=\mu/m} + \frac{1}{3!}(X-\mu/m)^3\, \frac{\mathrm{d}^3P}{\mathrm{d}X^3}\bigg|_{X=\mu/m}. \tag{4.3}$$

Since $P(\mu/m)$ is just a constant, we can ignore it under the action because we can simply redefine it in such a way that we can cancel the term out. Next we write

$$\frac{\mathrm{d}^2 P}{\mathrm{d}X^2} = \frac{\mathrm{d}\rho}{\mathrm{d}X} = \frac{\mathrm{d}\rho}{\mathrm{d}P}\frac{\mathrm{d}P}{\mathrm{d}X} = \frac{1}{c_s^2}\rho. \tag{4.4}$$

Similarly we can write

$$\frac{\mathrm{d}^3 P}{\mathrm{d}X^3} = \frac{\mathrm{d}}{\mathrm{d}X}\frac{\mathrm{d}^2 P}{\mathrm{d}X^2} = \frac{\mathrm{d}}{\mathrm{d}X}\left(\frac{1}{c_s^2}\rho\right) = \frac{1}{c_s^4}\rho. \tag{4.5}$$

So $P(X)$ is shown to be:

$$P(X) \simeq \rho\left(\dot{\pi} - \frac{1}{2}(\nabla\pi)^2\right) + \frac{1}{2}\frac{1}{c_s^2}\rho\left(\dot{\pi} - \frac{1}{2}(\nabla\pi)^2\right)^2 + \frac{1}{3!}\frac{\rho}{c_s^4}\left(\dot{\pi} - \frac{1}{2}(\nabla\pi)^2\right)^3. \tag{4.6}$$

Under the action, the above equation becomes

$$S = \int \frac{\rho}{c_s^2}\left\{\frac{1}{2}\left[\dot{\pi}^2 - c_s^2(\nabla\pi)^2\right] + \frac{g_3}{3! c_s^2}\dot{\pi}^3 - \frac{1}{2}\dot{\pi}(\nabla\pi)^2 + ...\right\} d^4x, \tag{4.7}$$

where the term $\rho\dot{\pi}$ was integrated out due to it being a total derivative, the ... are the terms that include powers of $\pi > 3$, and g_3 is defined to be

$$g_n = c_s^{2(n-2)}\ \frac{1}{P''}\frac{\mathrm{d}^n P}{\mathrm{d}X^n}\bigg|_{X=\mu/m}, \tag{4.8}$$

where P'' denotes the second derivative with respect to X. We are ready to apply the Feynman rules to the cubic terms. The rules are fairly similar to the Lorentz case, in fact they are exactly what we would naively expect. For every time derivative that exists, replace it with the energy of the field so $\frac{\mathrm{d}}{\mathrm{d}t} \to -iE$ and for every spatial derivative, replace it with the spatial momentum $\nabla \to -i\mathbf{p}$. Now we vary with respect to the field π. The first derivative is

$$\frac{c_s^2}{\rho}\frac{\delta\mathcal{L}}{\delta\pi} = \frac{1}{2}\frac{g_3}{c_s^2}(-iE)\dot{\pi}^2 - \frac{1}{2}(-iE)(\nabla\pi)^2 - \dot{\pi}(-i\mathbf{p})\cdot\nabla\pi, \tag{4.9}$$

where E and $\mathbf{p}$ are the energy and momentum of the incoming (i.e. hard) phonons. The second derivative brings us

$$\frac{c_s^2}{\rho}\frac{\delta}{\delta\pi}\frac{\delta\mathcal{L}}{\delta\pi} = \frac{g_3}{c_s^2}(-iE)(-iE)\dot{\pi} - (-iE)(-i\mathbf{p})\cdot\nabla\pi - (-iE)(-i\mathbf{p})\cdot\nabla\pi - \dot{\pi}(-i\mathbf{p})\cdot(-i\mathbf{p}). \tag{4.10}$$

The third (and final) derivative is given by

$$\begin{aligned}\frac{c_s^2}{\rho}\frac{\delta}{\delta\pi}\frac{\delta}{\delta\pi}\frac{\delta\mathcal{L}}{\delta\pi} = \frac{g_3}{c_s^2}&(-iE)(-iE)(iE_k) - (-iE)(-i\mathbf{p})\cdot(i\mathbf{k}) - (-iE)(-i\mathbf{p})\cdot(i\mathbf{k}) \\ &- (iE_k)(-i\mathbf{p})\cdot(-i\mathbf{p}),\end{aligned} \tag{4.11}$$

where E_k and $\mathbf{k}$ are the energy and momentum of the outgoing (i.e. soft) phonon. Next, we use the relation between the energy and momentum for a phonon $E \simeq c_s|\mathbf{p}|$. So our vertex rule becomes

$$\frac{\delta}{\delta\pi}\frac{\delta}{\delta\pi}\frac{\delta\mathcal{L}}{\delta\pi} \simeq \frac{2ip^2k\rho}{c_s}[\cos(\theta) + \frac{1}{2}(1-g_3)], \tag{4.12}$$

where $p = |\mathbf{p}|$, $k = |\mathbf{k}|$, and θ is the angle between $\mathbf{p}$ and $\mathbf{k}$. To compute the actual amplitude $\mathcal{M}$, we need to include the canonical normalization, $c_s/\sqrt{\rho}$, for each line and since there are no internal momenta for this process, $\mathcal{M}$ is simply the vertex rule multiplied by i. So the amplitude is just

$$i\mathcal{M} \simeq -\frac{2c_s^2}{\sqrt{\rho}}p^2k[\cos(\theta) + \frac{1}{2}(1-g_3)]. \tag{4.13}$$

And thus, we have computed the scattering amplitude for our effective field theory.

5 Graviton-Graviton Scatter

In order to compute the scattering amplitude of Mimetic Gravity, we must first know what the amplitude $\mathcal{A}$ is within General Relativity. Since we're only interested the scattering at the vertex, we only need to go to cubic order in the metric perturbation. Fortunately, we have Mathematica that is able to do all the hard work in deducing what that Lagrangian is. We can also streamline the calculation quite a bit by taking into account that the polarization tensor will be traceless and transverse i.e.

$$p^{\mu}\epsilon_{\mu\nu} = \epsilon^{\mu}{}_{\mu} = 0. \tag{5.1}$$

With all of that in mind (and doing an additional reduction via the action) we can write down the cubic order Einstein-Hilbert Lagrangian

$$\mathcal{L}_{EH}^{(3)} = -\frac{3}{2}h^{\mu\nu}h^{\lambda\rho}\partial_{\lambda}\partial_{\rho}h_{\mu\nu} - 9h^{\mu\nu}\partial^{\rho}h^{\lambda}{}_{\nu}\partial_{\rho}h_{\mu\lambda} - 3\partial^{\lambda}h^{\mu\nu}\partial^{\rho}h_{\mu\lambda}h_{\nu\rho}. \tag{5.2}$$

For simplicity we scale the Lagrangian in such a way that we eliminate the common factor of three. Now we proceed with differentiating with respect to the metric perturbation. The first derivative gives us

$$\begin{aligned} \frac{\delta}{\delta\mathcal{L}_{EH}}^{(3)} h^{\mu\nu} = &-\frac{1}{2}[\epsilon^{1}_{\mu\nu}h^{\lambda\rho}\partial_{\lambda}\partial_{\rho}h^{\mu\nu} + h^{\mu\nu}\epsilon_{1}^{\lambda\rho}\partial_{\lambda}\partial_{\rho}h_{\mu\nu} + h_{\mu\nu}h^{\lambda\rho}(-ip^{1}_{\lambda})(-ip^{1}_{\rho})\epsilon_{1}^{\mu\nu}] \\ &- 3[\epsilon_{1}^{\mu\nu}\partial^{\rho}h^{\lambda}{}_{\nu}\partial_{\rho}h_{\mu\lambda} + h^{\mu\nu}(-ip_{1}^{\rho})\epsilon^{\lambda}{}_{1\nu}\partial_{\rho}h_{\mu\lambda} + h^{\mu\nu}\partial^{\rho}h^{\lambda}{}_{\nu}(-ip^{1}_{\rho})\epsilon^{1}_{\mu\lambda}] \\ &- [(-ip_{1}^{\lambda})\epsilon_{1}^{\mu\nu}h_{\nu\rho}\partial^{\rho}h_{\mu\lambda} + \partial^{\lambda}h^{\mu\nu}(-ip_{1}^{\rho})\epsilon^{1}_{\mu\lambda}h_{\nu\rho} + \partial^{\lambda}h^{\mu\nu}\partial^{\rho}h_{\mu\lambda}\epsilon^{1}_{\nu\rho}], \end{aligned} \tag{5.3}$$

where we have used the Feynman rules $\partial_{\mu} \to (-ip_{\mu})$ and $\frac{\delta}{\delta h^{\mu\nu}} \to \epsilon_{\mu\nu}$ and the numerical indices indicate which momenta on the Feynman diagram it represents. Since there are still h's left we must proceed with taking derivatives until we've eliminated all h's. The second derivative is

$$
\begin{aligned}
\frac{\delta}{\delta h^{\mu\nu}}\frac{\delta\mathcal{L}_{EH}^{(3)}}{\delta h^{\mu\nu}} =& -\frac{1}{2}[\epsilon^1_{\mu\nu}(\epsilon_2^{\lambda\rho}\partial_\lambda\partial_\rho h^{\mu\nu} + h^{\lambda\rho}(-ip^2_\lambda)(-ip^2_\rho)\epsilon_2^{\mu\nu}) + \epsilon_1^{\lambda\nu}(\epsilon_2^{\mu\nu}\partial_\lambda\partial_\rho h_{\mu\nu} + h^{\mu\nu}(-ip^2_\lambda)(-ip^2_\rho)\epsilon^2_{\mu\nu}) \\
&+ (-ip^1_\lambda)(-ip^1_\rho)\epsilon_1^{\mu\nu}(\epsilon^2_{\mu\nu}h^{\lambda\rho} + h_{\mu\nu}\epsilon_2^{\lambda\rho})] \\
&- 3[\epsilon_1^{\mu\nu}((-ip_2^\rho)\epsilon^\lambda{}_{2\nu}\partial_\rho h_{\mu\lambda} + \partial^\rho h^\lambda{}_\nu(-ip^2_\rho)\epsilon^2_{\mu\lambda}) + (-ip_1^\rho)\epsilon^\lambda{}_{1\nu}(\epsilon_2^{\mu\nu}\partial_\rho h_{\mu\lambda} + h^{\mu\nu}(-ip^2_\rho)\epsilon^2_{\mu\lambda}) \\
&+ (-ip^1_\rho)\epsilon^1_{\mu\lambda}(\epsilon_2^{\mu\nu}\partial_\rho h_{\mu\lambda} + h^{\mu\nu}(-ip^2_\rho)\epsilon^2_{\mu\lambda})] \\
&- [(-ip_1^\lambda)\epsilon_1^{\mu\nu}(\epsilon^2_{\nu\rho}\partial^\rho h_{\mu\lambda} + h_{\nu\rho}(-ip^2_\rho)\epsilon^2_{\mu\lambda}) + (-ip_1^\rho)\epsilon^1_{\mu\lambda}((-ip_2^\lambda)\epsilon_2^{\mu\nu}h_{\nu\rho} + \partial^\lambda h^{\mu\nu}\epsilon^2_{\nu\rho}) \\
&+ \epsilon^1_{\nu\rho}((-ip_2^\lambda)\epsilon_2^{\mu\nu}\partial^\rho h_{\mu\lambda} + \partial^\lambda h^{\mu\nu}\epsilon^2_{\mu\lambda})].
\end{aligned} \tag{5.4}
$$

Whew, we're almost done. There's only one more h left which means we can stop after the third (functional) derivative. Proceeding accordingly, we have

$$
\begin{aligned}
\frac{\delta}{\delta h^{\mu\nu}}\frac{\delta}{\delta h^{\mu\nu}}\frac{\delta\mathcal{L}_{EH}^{(3)}}{\delta h^{\mu\nu}} =& -\frac{1}{2}[\epsilon^1_{\mu\nu}(\epsilon_2^{\lambda\rho}(-ip^3_\lambda)(-ip^3_\rho)\epsilon_3^{\mu\nu} + \epsilon_3^{\lambda\rho}(-ip^2_\lambda)(-ip^2_\rho)\epsilon_2^{\mu\nu}) \\
&+ \epsilon_1^{\lambda\nu}(\epsilon_2^{\mu\nu}(-ip^3_\lambda)(-ip^3_\rho)\epsilon^3_{\mu\nu} + \epsilon_3^{\mu\nu}(-ip^2_\lambda)(-ip^2_\rho)\epsilon^2_{\mu\nu}) \\
&+ (-ip^1_\lambda)(-ip^1_\rho)\epsilon_1^{\mu\nu}(\epsilon^2_{\mu\nu}\epsilon_3^{\lambda\rho} + \epsilon^3_{\mu\nu}\epsilon_2^{\lambda\rho})] \\
&- 3[\epsilon_1^{\mu\nu}((-ip_2^\rho)\epsilon^\lambda{}_{2\nu}(-ip^3_\rho)\epsilon^3_{\mu\lambda} + (-ip_3^\rho)\epsilon^\lambda{}_{3\nu}(-ip^2_\rho)\epsilon^2_{\mu\lambda}) \\
&+ (-ip_1^\rho)\epsilon^\lambda{}_{1\nu}(\epsilon_2^{\mu\nu}(-ip^3_\rho)\epsilon^3_{\mu\lambda} + \epsilon_3^{\mu\nu}(-ip^2_\rho)\epsilon^2_{\mu\lambda}) \\
&+ (-ip^1_\rho)\epsilon^1_{\mu\lambda}(\epsilon_2^{\mu\nu}(-ip^3_\rho)\epsilon^3_{\mu\lambda} + \epsilon_3^{\mu\nu}(-ip^2_\rho)\epsilon^2_{\mu\lambda})] \\
&- [(-ip_1^\lambda)\epsilon_1^{\mu\nu}(\epsilon^2_{\nu\rho}(-ip_3^\rho)\epsilon^3_{\mu\lambda} + \epsilon^3_{\nu\rho}(-ip^2_\rho)\epsilon^2_{\mu\lambda}) \\
&+ (-ip_1^\rho)\epsilon^1_{\mu\lambda}((-ip_2^\lambda)\epsilon_2^{\mu\nu}\epsilon^3_{\nu\rho} + (-ip_3^\lambda)\epsilon_3^{\mu\nu}\epsilon^2_{\nu\rho}) \\
&+ \epsilon^1_{\nu\rho}((-ip_2^\lambda)\epsilon_2^{\mu\nu}(-ip_3^\rho)\epsilon^3_{\mu\lambda} + (-ip_3^\lambda)\epsilon_3^{\mu\nu}\epsilon^2_{\mu\lambda})].
\end{aligned} \tag{5.5}
$$

There are a few ways we can simplify the above mess. First we recognize that each term in the Lagrangian has a factor of $(-i)^2 = -1$ which calls all of the minus signs. Next we take advantage of the fact that in four dimensions we can write $\epsilon_{\mu\nu} = \epsilon_\mu\epsilon_\nu$. With these two observations in mind, we can simplify the above expression immensely as being

$$
\begin{aligned}
\frac{\delta}{\delta h^{\mu\nu}}\frac{\delta}{\delta h^{\mu\nu}}\frac{\delta \mathcal{L}_{EH}^{(3)}}{\delta h^{\mu\nu}} &= \frac{1}{2}(\epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot p_1 + \epsilon_2 \cdot \epsilon_3 \epsilon_1 \cdot p_2 + \epsilon_3 \cdot \epsilon_1 \epsilon_2 \cdot p_3)^2 \\
&+ \frac{1}{2}(\epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot p_2 + \epsilon_2 \cdot \epsilon_3 \epsilon_1 \cdot p_3 + \epsilon_3 \cdot \epsilon_1 \epsilon_2 \cdot p_1)^2 \\
&+ 6(\epsilon_1 \cdot \epsilon_2)(\epsilon_2 \cdot \epsilon_3)(\epsilon_3 \cdot \epsilon_1)(p_1 \cdot p_2 + p_2 \cdot p_3 + p_3 \cdot p_1).
\end{aligned}
\tag{5.6}
$$

Now that our expression is much cleaner, it is far easier to see there are a few more simplifications that we can do. For example, from energy/momentum conservation, we have

$$
p_1 + p_2 + p_3 = 0 \Leftrightarrow p_1^2 + p_2^2 + p_3^2 + 2p_1 \cdot p_2 + 2p_2 \cdot p_3 + 2p_3 \cdot p_1 = 0 \tag{5.7}
$$

$$
\Rightarrow p_1 \cdot p_2 + p_2 \cdot p_3 + p_3 \cdot p_1 = 0, \qquad p_i^2 = -m_i^2 = 0 \;\; \forall i \in \{1, 2, 3\}. \tag{5.8}
$$

So we can safely discard the last term in our expression. Lastly, from momentum conservation we can see that $p_3 = -(p_1 + p_2)$ (and likewise $p_2 = -(p_3 + p_1)$ and $p_1 = -(p_2 + p_3)$). Plugging this and the fact that the polarization tensor is transverse we find that the two expressions inside the parentheses are the same up to a minus sign. But taking the power of two into account, the terms are exactly the same. So we finally get

$$
\frac{\delta}{\delta h^{\mu\nu}}\frac{\delta}{\delta h^{\mu\nu}}\frac{\delta \mathcal{L}_{EH}^{(3)}}{\delta h^{\mu\nu}} = \Big[(\epsilon_1 \cdot \epsilon_2)(\epsilon_3 \cdot p_1) + (\epsilon_2 \cdot \epsilon_3)(\epsilon_1 \cdot p_2) + (\epsilon_3 \cdot \epsilon_1)(\epsilon_2 \cdot p_3)\Big]^2. \tag{5.9}
$$