Gravitational Waves

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1 Introduction

Gravitational waves are the newest frontier for gaining insight into the inner workings of the universe. It provides a window for us to understand the world around us in a way that hasn't been seen since the invention of the telescope. Particularly, the physics that we may discover through the Cosmic Gravitational Wave Background could possibly be paradigm shifting as the information that is carried by such an artifact could stretch to the first few seconds after the birth of our expanding universe. Understanding this background and the observables we may extract from it could answer questions about quantum gravity, inflation, and many more questions that are too numerous to name. As a result, it is crucial that we may study GWs and this potential background as it could lead to the next breakthrough in physics.

Conventions We use the mostly plus metric signature, i.e. $\eta_{\mu\nu} = (-,+,+,+)$ and units where $c = \hbar = 1$. The reduced four dimensional Planck mass is $M_P = \frac{1}{\sqrt{8\pi G}} \approx 2.43 \times 10^{18}$ GeV. The d'Alembert and Laplace operators are defined to be $\Box = \partial_\mu \partial^\mu$ and $\nabla^2 = \partial_i \partial^i$ respectively. We use boldface letters $\mathbf{r}$ to indicate 3-vectors and we use x and p to denote 4-vectors. Conventions for (anti-)symmetrization for tensors, the curvature tensors, covariant and Lie derivatives are all taken from Carroll. Greek indices $(\mu, \nu \ldots)$ and Latin indices $(a, b, c \ldots)$ denote spacetime indices.

2 Gauge Symmetry and Equations of Motion

We start off with the gauge symmetry inherent in GR

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = g_{\rho\lambda}\frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\lambda}{\partial x'^\nu}. \tag{2.1}$$

We can express this gauge transformation infinitesimally by $x^\mu \to x'^\mu = x^\mu + \xi^\mu(x)$. How does the metric perturbation transform under this infinitesimal generator? We can plug it into the transformation law for the metric above to get

$$\eta'_{\mu\nu} + h'_{\mu\nu} = \frac{\partial x^\lambda}{\partial x'^\mu}\frac{\partial x^\rho}{\partial x'^\nu} = \frac{\partial(x'^\lambda - \xi^\lambda(x'-\xi))}{\partial x'^\mu}\frac{\partial(x'^\rho - \xi^\rho(x'-\xi))}{\partial x'^\nu}. \tag{2.2}$$

We can expand the gauge parameter ξ^μ to first order to get

$$\eta'_{\mu\nu} + h'_{\mu\nu} = \left[\delta^\lambda_\mu\delta^\rho_\mu - \delta^\lambda_\mu\frac{\partial \xi^\rho}{\partial x'^\nu} - \delta^\rho_\nu\frac{\partial \xi^\lambda}{\partial x'^\mu}\right](\eta_{\lambda\rho} + h_{\lambda\rho}) \tag{2.3}$$

$$= \eta_{\mu\nu} + h_{\mu\nu} - \eta_{\lambda\rho}\delta^\lambda_\mu\frac{\partial \xi^\rho}{\partial x'^\nu} - \eta_{\lambda\rho}\delta^\rho_\nu\frac{\partial \xi^\lambda}{\partial x'^\mu}. \tag{2.4}$$

Canceling out the Minkowski metric on both sides leaves us with the following identity

$$h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu\xi_\nu - \partial_\nu\xi_\mu. \tag{2.5}$$

It is easy to see (using the same method we used for deriving the gauge symmetry) that for a global Lorentz transformation i.e. $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$ the linearized theory is also symmetric under this rotation i.e. given that $g_{\mu\nu} \to g'_{\mu\nu}(x') = \Lambda_\mu{}^\lambda\Lambda_\nu{}^\rho g_{\lambda\rho}(x)$ the metric perturbation transforms as

$$\eta'_{\mu\nu} + h'_{\mu\nu} = \Lambda_\mu{}^\lambda\Lambda_\nu{}^\rho(\eta_{\lambda\rho} + h_{\lambda\rho}) = \eta_{\mu\nu} + \Lambda_\mu{}^\lambda\Lambda_\nu{}^\rho h_{\lambda\rho} \Rightarrow h'_{\mu\nu}(x') = \Lambda_\mu{}^\lambda\Lambda_\nu{}^\rho h_{\lambda\rho}(x). \tag{2.6}$$

In addition to constant translations $x'^\mu = x^\mu + a^\mu$, we can say that the linearized theory is invariant under finite Poincare transformations (3 translations, 3 rotations, and 4 boosts). Since we're interested in the linearized theory, it would be nice to have what the curvature would be offhand, so we don't have to keep looking for that information. We want to know what the curvature will be up to first order in h. The Riemann, Ricci, and Einstein tensors along with the Ricci scalar takes the form

$$R^{(1)}_{\alpha\beta\mu\nu} = \frac{1}{2}(\partial_\alpha\partial_\nu h_{\beta\mu} - \partial_\alpha\partial_\mu h_{\beta\nu} - \partial_\beta\partial_\nu h_{\alpha\mu} + \partial_\beta\partial_\mu h_{\alpha\nu}), \tag{2.7}$$

$$R^{(1)}_{\mu\nu} = -\frac{1}{2}\partial_\mu\partial_\nu h - \frac{1}{2}\Box h_{\mu\nu} + \frac{1}{2}\partial^\alpha\partial_\mu h_{\nu\alpha} + \frac{1}{2}\partial^\alpha\partial_\nu h_{\mu\alpha}, \tag{2.8}$$

$$R^{(1)} = \partial_\mu\partial_\nu h^{\mu\nu} - \Box h, \tag{2.9}$$

$$G^{(1)}_{\mu\nu} = -\frac{1}{2}(\partial_\mu\partial_\nu h + \Box h_{\mu\nu} - \partial^\rho\partial_\mu h_{\nu\rho} - \partial^\lambda\partial_\nu h_{\mu\lambda} + \partial_\lambda\partial_\rho h^{\lambda\rho}\eta_{\mu\nu} - \Box h\eta_{\mu\nu}), \tag{2.10}$$

where $h = h^\mu_\mu = \eta^{\mu\nu}h_{\mu\nu}$. It is convenient for us to introduce the term $\overline{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu}$ which yields the following representation for the Einstein tensor

$$G^{(1)}_{\mu\nu} = -\frac{1}{2}(\Box\bar{h}_{\mu\nu} + \eta_{\mu\nu}\partial^\lambda\partial^\rho\bar{h}_{\lambda\rho} - 2\partial^\lambda\partial_{(\mu}\bar{h}_{\nu)\lambda}). \tag{2.11}$$

Recalling that Einstein's equations involve contributions from matter as a source for the curvature, we get

$$\Box\bar{h}_{\mu\nu} + \eta_{\mu\nu}\partial^\lambda\partial^\rho\bar{h}_{\lambda\rho} - 2\partial^\lambda\partial_{(\mu}\bar{h}_{\nu)\lambda} = -16\pi G T_{\mu\nu}. \tag{2.12}$$

Now we introduce the De Donder/Harmonic/Lorentz/Hilbert Gauge:

$$\partial^\nu h_{\mu\nu} - \frac{1}{2}\partial_\mu h = \partial^\nu\bar{h}_{\mu\nu} = 0. \tag{2.13}$$

This gauge is comes from the condition $\partial_\mu(\sqrt{-g}g^{\mu\nu}) = 0$ and is essentially the GR analog to the Lorenz gauge of E&M $\partial_\mu A^\mu = 0$. In order to go forward, it is convenient to understand how $\bar{h}_{\mu\nu}$ varies under a diffeomorphism. First we show how the trace of the perturbation transforms:

$$h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu\xi_\nu - \partial_\nu\xi_\mu \Rightarrow h' = \eta^{\mu\nu}h'_{\mu\nu} = h - 2\partial_\mu\xi^\mu. \tag{2.14}$$

Thus, $\bar{h}_{\mu\nu}$ goes as

$$\begin{aligned}\bar{h}_{\mu\nu} \to \bar{h}'_{\mu\nu} &= h'_{\mu\nu} - \frac{1}{2}h'\eta_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu} - \partial_\mu\xi_\nu - \partial_\nu\xi_\mu + \partial_\rho\xi^\rho\eta_{\mu\nu} \\ &= \bar{h}_{\mu\nu} - \partial_\mu\xi_\nu - \partial_\nu\xi_\mu + \partial_\rho\xi^\rho\eta_{\mu\nu}.\end{aligned} \tag{2.15}$$

The De Donder condition under a gauge transformation goes as

$$\partial^\nu \bar{h}_{\mu\nu} \to (\partial^\nu \bar{h}_{\mu\nu})' = \partial^\nu \bar{h}_{\mu\nu} - \Box \xi_\mu. \tag{2.16}$$

Thus, we can always move to a frame in which the De Donder gauge holds i.e. if

$$\partial^\nu \bar{h}_{\mu\nu} = f_\mu(x), \tag{2.17}$$

then we can pick a vector field ξ_μ such that

$$\Box \xi_\mu = f_\mu \Rightarrow \xi_\mu(x) = \int \mathrm{d}^4 y\, G(x-y) f_\mu(y), \quad \Box_x G(x-y) = -\delta^{(4)}(x-y). \tag{2.18}$$

The De Donder gauge is favored because Einstein's equations simplifies down to

$$G^{(1)}_{\mu\nu} = \Box \bar{h}_{\mu\nu} + \eta_{\mu\nu}\partial^\lambda \partial^\rho \bar{h}_{\lambda\rho} - 2\partial^\lambda \partial_{(\mu} \bar{h}_{\nu)\lambda} = \Box \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}. \tag{2.19}$$

Taking the divergence of the above equation yields

$$\partial^\nu T_{\mu\nu} = -\frac{1}{16\pi G}\Box \partial^\nu \bar{h}_{\mu\nu} = 0. \tag{2.20}$$

Because we are interested in investigating gravitational waves as well as the effect on test particles, it is necessary to look regions of spacetime for which we are outside the source i.e. $T_{\mu\nu} = 0$ which implies $\Box \bar{h}_{\mu\nu} = 0$. Now even though linearized GR carries only two degrees of freedom (and we will show this in the appendix), but $h_{\mu\nu}$ carries a total of 10 degrees of freedom (dof) (naively we would say $h_{\mu\nu}$ has 16 dofs but $h_{\mu\nu} = h_{\nu\mu}$ kills off 4 dofs) so we have extraneous fields we don't need. Working with $\bar{h}_{\mu\nu}$ kills off an additional 4 dofs so we are left with 6 in all. To get rid of the redundant information, we have to fix a gauge.

Given that $\partial^\nu \bar{h}_{\mu\nu} \to \partial^\nu \bar{h}_{\mu\nu} - \Box \xi_\mu$ then we can require $\Box \xi_\mu = 0$ to preserve the gauge condition $\partial^\nu \bar{h}_{\mu\nu} = 0$. Now we define the new gauge parameter

$$\xi_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \partial_\lambda \xi^\lambda \eta_{\mu\nu}, \tag{2.21}$$

which also carries $\Box \xi_{\mu\nu}$ if we restrict $\Box \xi_\mu = 0$. We can use $\xi_{\mu\nu}$ to eliminate the additional 6 dofs. The modified perturbation transforms as

$$\bar{h}_{\mu\nu} \to \bar{h}_{\mu\nu} - \xi_{\mu\nu} \Rightarrow \bar{h} \to \bar{h} - \xi^{\mu}_{\mu} = \bar{h} + 2\partial_{\lambda}\xi^{\lambda} = \bar{h} + 2\dot{\xi}^0 + \partial_i \xi^i. \tag{2.22}$$

We can pick $2\dot{\xi}^0 = -(\bar{h} + 2\partial_i\xi^i)$ to kill off the trace term in order to bring our count to 5 dofs left. Next we can see that

$$\bar{h}_{0i} \to \bar{h}_{0i} - \xi_{0i} = \bar{h}_{0i} - \dot{\xi}_i - \partial_i \xi_0. \tag{2.23}$$

And so we choose $\dot{\xi}_i = \bar{h}_{0i} - \partial_i\xi_0$ to reduce the dof count to 2 and so we are finally done. We have fixed/accounted for all the extraneous dofs from the theory and are now in a position to solve the equations of motion. Given that we are working in a gauge where the perturbation is traceless, there is now no distinction between $\bar{h}_{\mu\nu}$ and $h_{\mu\nu}$ and so we'll just be working with the original metric perturbation from now on. In light of the calculation we just did, the De Donder condition yields

$$\partial^{\nu} h_{0\nu} = \partial^0 h_{00} + \partial^i h_{0i} = 0 \Rightarrow h_{00} = 0, \tag{2.24}$$

so now both $h_{\mu\nu}$ and $\bar{h}_{\mu\nu}$ both only carry 2 degrees of freedom. Notice we also have

$$\partial^{\nu} h_{i\nu} = \partial^0 h_{i0} + \partial^j h_{ij} = 0 \Rightarrow \partial^j h_{ij} = 0. \tag{2.25}$$

So not only is the metric perturbation traceless, it is also transverse i.e. it's signals propagate transverse the direction of motion. We can conclude from this that the De Donder Gauge leads to the transverse traceless gauge, but only outside where the source of located. Thus, the full gauge conditions are

$$h_{0\mu} = 0, \quad h^i_i = 0, \quad \partial^j h_{ij} = 0. \tag{2.26}$$

From henceforth, we shall denote the metric perturbation (and the fact that it is both traceless and transverse) as h^{TT}_{ij}. Thus, the wave equation reads

$$\Box h^{TT}_{ij} = 0. \tag{2.27}$$

In analogy with E&M, we can make an ansatz for this differential equation to be

$h_{ij}^{TT}(x) = \epsilon_{ij}(p)e^{ip\cdot x}$ where ϵ_{ij} is a polarization tensor. Writing $p^\mu = (\omega, \mathbf{p})$ and plugging this all into the equation of motion, we get the dispersion relation

$$\left[-(-i\omega)^2 + (i\mathbf{p})^2\right]\epsilon_{ij}(p) = 0 \Rightarrow \omega = |\mathbf{p}|. \tag{2.28}$$

Defining the momentum unit vector in the usual way $\hat{\mathbf{n}} = \mathbf{p}/|\mathbf{p}|$, we can see that the perturbation tensor is (unsurprisingly) transverse to this direction

$$\partial^j h_{ij}^{TT} = \epsilon_{ij} i p^j e^{ip\cdot x} = n^j h_{ij} = 0. \tag{2.29}$$

Aligning our coordinate system so that the GW propagates in the z-direction (i.e. $\hat{\mathbf{n}} = \hat{\mathbf{z}}$) then the tensor becomes

$$h_{ij}^{TT} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij} \cos(\omega(t-z)), \tag{2.30}$$

where we used the fact that $\mathbf{p}\cdot\mathbf{r} = |\mathbf{p}||\mathbf{r}|\cos\theta = \omega z$, $\omega = |\mathbf{p}|$ and by convention we take the real part of the wave. The amplitudes h_+ and $h_\times$ denote the plus and cross polarizations of gravitational waves. We can write the above a bit more succinctly by recognizing we can restrict the matrix to be 2x2

$$h_{ab}^{TT} = \begin{pmatrix} h_+ & h_\times \\ h_\times & -h_+ \end{pmatrix}_{ab} \cos(\omega(t-z)). \tag{2.31}$$

Thus, the spacetime interval then becomes

$$\begin{aligned} \mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = &-\mathrm{d}t^2 + (1+h_+)\cos(\omega(t-z))\,\mathrm{d}x^2 + 2h_\times\cos(\omega(t-z))\,\mathrm{d}x\,\mathrm{d}y \\ &+ (1-h_+)\cos(\omega(t-z))\,\mathrm{d}y^2 + \mathrm{d}z^2\,. \end{aligned} \tag{2.32}$$

Next we define the projection tensor

$$P_{ij}(\hat{\mathbf{n}}) = \delta_{ij} - n_i n_j. \tag{2.33}$$

The projection tensor takes vectors and projects out the component that is transverse the direction of propagation. We can see that it has the following properties

$$n^j P_{ij} = n^j \delta_{ij} - n^j n_i n_j = 0, \tag{2.34}$$

$$P_{ik} P_{kj} = (\delta_{ik} - n_i n_k)(\delta_{kj} - n_k n_j) = \delta_{ik}\delta_{kj} - (n_k n_j \delta_{ik} + n_i n_k \delta_{kj}) = \delta_{ij} - n_i n_j = P_{ij}, \tag{2.35}$$

and $P_{ii} = 3 - 1 = 2$. From here we can construct a projection operator for tensors which maps tensors to their transverse traceless component i.e.

$$\Lambda_{ij,k\ell} = P_{ik} P_{j\ell} - \frac{1}{2} P_{ij} P_{k\ell}. \tag{2.36}$$

We can show that this map shares all the same features as the original projection map

$$\begin{aligned}
\Lambda_{ij,k\ell}\Lambda_{k\ell,mn} &= \left(P_{ik}P_{j\ell} - \frac{1}{2}P_{ij}P_{k\ell} \right)\left(P_{km}P_{\ell n} - \frac{1}{2}P_{k\ell}P_{mn} \right) \\
&= P_{ik}P_{j\ell}P_{km}P_{\ell n} - \frac{1}{2}(P_{ij}P_{k\ell}P_{km}P_{\ell n} + P_{ik}P_{j\ell}P_{k\ell}P_{mn}) + \frac{1}{4}P_{ij}P_{k\ell}P_{k\ell}P_{mn} \\
&= P_{im}P_{jn} - \frac{1}{2}P_{i\ell}P_{j\ell}P_{mn} - \frac{1}{2}P_{ij}P_{m\ell}P_{\ell n} + \frac{1}{4}P_{ij}P_{\ell\ell}P_{mn} \\
&= P_{im}P_{j\ell} - \frac{1}{2}P_{ij}P_{mn} = \Lambda_{ij,mn},
\end{aligned} \tag{2.37}$$

and that

$$\Lambda_{ii,k\ell} = P_{ik}P_{i\ell} - \frac{1}{2}P_{ii}P_{k\ell} = 0, \qquad \Lambda_{ij,kk} = P_{ik}P_{jk} - \frac{1}{2}P_{ij}P_{kk} = 0. \tag{2.38}$$

We can express the projection tensor in terms of the original unit vector $\hat{\mathbf{n}}$

$$\begin{aligned}
\Lambda_{ij,k\ell}(\hat{\mathbf{n}}) &= (\delta_{ik} - n_i n_k)(\delta_{j\ell} - n_j n_\ell) - \frac{1}{2}(\delta_{ij} - n_i n_j)(\delta_{k\ell} - n_k n_\ell) \\
&= \delta_{ik}\delta_{j\ell} - \frac{1}{2}\delta_{ij}\delta_{k\ell} - (n_i n_k \delta_{j\ell} + n_j n_\ell \delta_{ik}) + \frac{1}{2}(n_i n_j \delta_{k\ell} + n_k n_\ell \delta_{ij} + n_i n_j n_k n_\ell).
\end{aligned} \tag{2.39}$$

Thus for any symmetric 3-tensor T_{ij}, we define the new tensor T_{ij}^{TT} given by

$$T_{ij}^{TT} = \Lambda_{ij,k\ell} T_{k\ell}. \tag{2.40}$$

Notice that if we can write T_{ij} as a plane wave we get this relation for free. Going back to the equations of motion for the metric perturbation $\Box h_{ij}^{TT} = 0$ we can write down the solution in much the same way we do in any mass-less classical field theory

$$h_{ij}^{TT}(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} (\mathcal{A}_{ij}(p) e^{ip\cdot x} + \mathcal{A}_{ij}^*(p) e^{-ip\cdot x}). \tag{2.41}$$

Notice that the transverse and traceless conditions imply that the amplitudes must satisfy

$$h_{ii}^{TT} = \mathcal{A}_{ii} = 0, \quad \partial^j h_{ij}^{TT} = p^j \mathcal{A}_{ij} = 0. \tag{2.42}$$

Now recall: $p^\mu = (\omega, \mathbf{p})$, $|\mathbf{p}| = \omega = 2\pi f$, $\hat{\mathbf{n}} = \mathbf{p}/|\mathbf{p}|$. We can write $\mathbf{p} = |\mathbf{p}|\hat{\mathbf{n}} = 2\pi f \hat{\mathbf{n}}$ and $\mathrm{d}^3 p = |\mathbf{p}|^2 \,\mathrm{d}|\mathbf{p}| \,\mathrm{d}\Omega = (2\pi)^3 f^2 \,\mathrm{d}f \,\mathrm{d}^2\hat{\mathbf{n}}$ where we denote $\mathrm{d}^2\hat{\mathbf{n}} = \mathrm{d}\cos\theta \,\mathrm{d}\phi$. This brings the perturbation to the form

$$h_{ij}^{TT}(x) = \int \int_0^\infty f^2 (\mathcal{A}_{ij}(p) e^{-2\pi i f(t - \hat{\mathbf{n}}\cdot\mathbf{r})} + c.c.) \,\mathrm{d}f \,\mathrm{d}^2\hat{\mathbf{n}}\,, \tag{2.43}$$

where we used $p \cdot x = -\omega t + \mathbf{p}\cdot\mathbf{r} = -2\pi f t + 2\pi f \hat{\mathbf{n}}\cdot\mathbf{r} = -2\pi f(t - \hat{\mathbf{n}}\cdot\mathbf{r})$. If we assume that the GW was emitted from a single (astrophysical) source, we can write the amplitude as $\mathcal{A}_{ij}(p) = A_{ij}(f)\delta^{(2)}(\hat{\mathbf{n}} - \hat{\mathbf{n}}_0)$ where $\hat{\mathbf{n}}_0$ is the direction of the propagating wave. Plugging this into the integral gives us

$$\begin{aligned} h_{ab}(x) &= \int_0^\infty f^2 \int (A_{ab}(f)\delta^{(2)}(\hat{\mathbf{n}} - \hat{\mathbf{n}}_0) e^{-2\pi i f(t - \hat{\mathbf{n}}\cdot\mathbf{r})} + c.c.) \,\mathrm{d}^2\hat{\mathbf{n}} \,\mathrm{d}f \\ &= \int_0^\infty f^2 (A_{ab}(f) e^{-2\pi i f(t - \hat{\mathbf{n}}_0\cdot\mathbf{r})} + A_{ab}^*(f) e^{2\pi i f(t - \hat{\mathbf{n}}_0\cdot\mathbf{r})}) \,\mathrm{d}f\,, \end{aligned} \tag{2.44}$$

where we restrict ourselves to the a, b indices and drop the TT superscript since using the a, b indices already implies we're in the TT gauge. If we center our detector near the origin then $\exp(2\pi i f \hat{\mathbf{n}}\cdot\mathbf{r}) \approx 1$ since $\mathbf{r} \approx 0$. Writing

$$\tilde{h}_{ab}(f) = f^2 A_{ab}(f) = \begin{pmatrix} \tilde{h}_+(f) & \tilde{h}_\times(f) \\ \tilde{h}_\times(f) & -\tilde{h}_+(f) \end{pmatrix}_{ab} . \tag{2.45}$$

Lastly we can see that if we let $f \to -f$ in the second term and assume that h_{ab} is a real function (otherwise we just take the real part in the end anyway), we get

$$h_{ab}(t, \mathbf{r}=0) = \int_0^\infty \tilde{h}_{ab}(f) e^{-2\pi i f t}\, \mathrm{d}t - \int_0^{-\infty} \tilde{h}^*_{ab}(-f) e^{2\pi i f t}\, \mathrm{d}f$$
$$= \int_0^\infty \tilde{h}_{ab}(f) e^{-2\pi i f t}\, \mathrm{d}t + \int_{-\infty}^0 \tilde{h}_{ab}(f) e^{-2\pi i f t}\, \mathrm{d}t = \int_{-\infty}^\infty \tilde{h}_{ab}(f) e^{-2\pi i f t}\, \mathrm{d}f\,, \tag{2.46}$$

where we used the fact that the metric perturbation is real i.e. $\tilde{h}^*_{ab}(-f) = \tilde{h}_{ab}(f)$. This gives us the following Fourier pair

$$h_{ab}(t) = \int_{-\infty}^\infty \tilde{h}_{ab}(f) e^{-2\pi i f t}\, \mathrm{d}f\,, \quad \tilde{h}_{ab}(f) = \int_{-\infty}^\infty h_{ab}(t) e^{2\pi i f t}\, \mathrm{d}t\,. \tag{2.47}$$

Now suppose $\hat{\mathbf{u}}, \hat{\mathbf{v}}$ are unit vectors such that $\hat{\mathbf{u}} \cdot \hat{\mathbf{n}} = \hat{\mathbf{v}} \cdot \hat{\mathbf{n}} = 0$. Define the polarization tensors

$$\epsilon^+_{ij}(\hat{\mathbf{n}}) \equiv \hat{\mathbf{u}}_i \hat{\mathbf{u}}_j - \hat{\mathbf{v}}_i \hat{\mathbf{v}}_j, \quad \epsilon^\times_{ij}(\hat{\mathbf{n}}) \equiv \hat{\mathbf{u}}_i \hat{\mathbf{v}}_j + \hat{\mathbf{v}}_i \hat{\mathbf{u}}_j. \tag{2.48}$$

We can see that

$$\epsilon^+_{ij}\epsilon^{ij}_+ = (\hat{\mathbf{u}}_i \hat{\mathbf{u}}_j - \hat{\mathbf{v}}_i \hat{\mathbf{v}}_j)(\hat{\mathbf{u}}^i \hat{\mathbf{u}}^j - \hat{\mathbf{v}}^i \hat{\mathbf{v}}^j) = 1 - 0 - 0 + 1 = 2, \tag{2.49}$$

$$\epsilon^\times_{ij}\epsilon^{ij}_\times = (\hat{\mathbf{u}}_i \hat{\mathbf{v}}_j + \hat{\mathbf{v}}_i \hat{\mathbf{u}}_j)(\hat{\mathbf{u}}^i \hat{\mathbf{v}}^j + \hat{\mathbf{v}}^i \hat{\mathbf{u}}^j) = 1 - 0 - 0 + 1 = 2, \tag{2.50}$$

$$\epsilon^+_{ij}\epsilon^{ij}_\times = (\hat{\mathbf{u}}_i \hat{\mathbf{u}}_j - \hat{\mathbf{v}}_i \hat{\mathbf{v}}_j)(\hat{\mathbf{u}}^i \hat{\mathbf{v}}^j + \hat{\mathbf{v}}^i \hat{\mathbf{u}}^j) = 0 + 0 - 0 - 0 = 0. \tag{2.51}$$

Thus, we have the relation

$$\epsilon^A_{ij}\epsilon^{ij}_B = 2\delta^A_B, \tag{2.52}$$

where $A = +, \times$. Now when we orient our coordinate system such that $\hat{\mathbf{n}} = \hat{\mathbf{z}} \Rightarrow \hat{\mathbf{u}} = \hat{\mathbf{x}}, \hat{\mathbf{v}} = \hat{\mathbf{y}}$ the polarization tensors in this basis becomes

$$\epsilon^+_{ab} \equiv \hat{\mathbf{x}}_a \hat{\mathbf{x}}_b - \hat{\mathbf{y}}_a \hat{\mathbf{y}}_b = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}_{ab}, \tag{2.53}$$

$$\epsilon^\times_{ab} \equiv \hat{\mathbf{x}}_a \hat{\mathbf{y}}_b + \hat{\mathbf{y}}_a \hat{\mathbf{x}}_b = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}_{ab}. \tag{2.54}$$

Thus we can write the amplitude as

$$f^2\mathcal{A}_{ij}(f,\hat{\mathbf{n}}) = \sum_{A=+,\times} \tilde{h}_A(f,\hat{\mathbf{n}})\epsilon^A_{ij}(\hat{\mathbf{n}}), \tag{2.55}$$

which brings the perturbed metric to the full form

$$h_{ab}(t,\mathbf{r}) = \sum_{A=+,\times} \int \mathrm{d}f \int \mathrm{d}^2\hat{\mathbf{n}}\, \tilde{h}_A(f,\hat{\mathbf{n}})\epsilon^A_{ab}(\hat{\mathbf{n}})e^{-2\pi i f(t-\hat{\mathbf{n}}\cdot\mathbf{r})}. \tag{2.56}$$

3 Dynamics of Free Non-Relativistic Particles in the Presence of Gravitational Waves

Consider a gravitational wave along the z-axis of a wave vector $\mathbf{k} = k\hat{\mathbf{z}}$. The gravitational wave has the form

$$\overline{h}_{\mu\nu}(\mathbf{r},t) = e^{-ik(ct-z)}(a(k)h^+_{\mu\nu}(k) + b(k)h^\times_{\mu\nu}(k) + C.C.). \tag{3.1}$$

Note in the Transverse Traceless Gauge $\bar{h}^\mu_\mu = 0$ which implies

$$\bar{h}^\mu_\mu = h^\mu_\mu - \frac{1}{2}\delta^\mu_\mu h = -h = 0 \Rightarrow \bar{h}_{\mu\nu} = h_{\mu\nu}. \tag{3.2}$$

Thus we have

$$h_{\mu\nu}(x) = \mathrm{Re}\big\{e^{-ik(ct-z)}(a(k)h^+_{\mu\nu}(k) + b(k)h^\times_{\mu\nu}(k))\big\}, \tag{3.3}$$

with $h^+_{xx} = -h^+_{yy} = 1, h^\times_{xy} = h^\times_{yx} = 1$ and all others are zero. In the TT gauge we have

$$R^x_{0x0} = R_{x0x0} = -\frac{1}{2}\ddot{h}_{xx},\quad R^y_{0y0} = R_{y0y0} = -\frac{1}{2}\ddot{h}_{yy} = -R_{x0x0},\quad R^y_{0x0} = R_{y0x0} = -\frac{1}{2}\ddot{h}_{xy} = R^x_{0y0} = -R_{x0y0}, \tag{3.4}$$

where overhead dots denote $\frac{\mathrm{d}}{\mathrm{d}t}$. All other components of the Riemann tensor vanish. Now we consider a particle initially at rest that is hit by the gravitational wave. Its geodesic equation

$$\frac{\mathrm{d}U^\mu}{\mathrm{d}\tau} + \Gamma^\mu_{\alpha\beta}U^\alpha U^\beta = 0, \tag{3.5}$$

and consider the acceleration shortly after being hit by the gravitational wave at $\tau = 0,\ U^\mu(0) = c(1,0,0,0)$ which gives us

$$\left.\frac{\mathrm{d}U^\mu}{\mathrm{d}\tau}\right|_{\tau=0} = -\Gamma^\mu_{00}c^2 \approx -\frac{c^2}{2}\eta^{\mu\nu}(\partial_t h_{\nu 0} + \partial_t h_{0\nu} - \partial_\nu h_{00}) = 0. \tag{3.6}$$

In TT-gauge, a particle originally at rest remains at rest. But we asked a coordinate dependent question since we asked about the acceleration with respect to a fixed coordinate. As the gravitational wave passes, both spacetime and the particle are perturbed in the same way, like a buoy on the surface of the ocean as the waves passes.

The buoy is at rest with respect to the fluid. But what about two particles (buoys) separated by a geodesic disturbance (proper distance) when the wave passes? Write

$$\mathrm{d}s^2 = -g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = -(\eta_{\mu\nu} + h_{\mu\nu})\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu\,. \tag{3.7}$$

Consider one particle at the origin and another at a coordinate distance ΔL along the x direction

$$\Delta S(t) = \int_0^{\Delta L} (1 + h_{xx}(\mathbf{r},t))^{1/2}\,\mathrm{d}x\,, \tag{3.8}$$

and consider $k\Delta L \ll 1$ ($\frac{\Delta L}{\lambda} \lll 1$) with wavelength λ. The proper distance is then

$$\Delta S(t) \sim \Delta L\left[1 + \frac{1}{2}h_{xx}(0,t)\right], \tag{3.9}$$

and the fractional displacement is

$$\delta(t) = \frac{\Delta S(t) - \Delta L}{\Delta L} \approx \frac{1}{2}h_{xx}(0,t), \tag{3.10}$$

which is called the fractional strain.

3.1 Generalizations

Consider an array of particles at $t = 0$ that form a ring in the $x - y$ plane with a test particle at the origin.

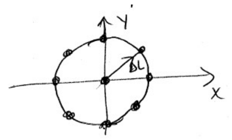

Figure 1: Coordinate plane for a ring of particles centered at the origin a distance ΔL away.

The coordinate of a particle on the ring is $\hat{\mathbf{n}}\Delta L$. Consider a gravitational wave propagating along the z axis (i.e. out of the page) with

$$h_{\mu\nu}(\mathbf{r}, t) = (ah^{+}_{\mu\nu} + bh^{\times}_{\mu\nu})\sin(k(ct - z)), \tag{3.11}$$

(monochromatic) with a long wavelength $k\Delta L \ll 1$. The proper distance becomes

$$\Delta S(t) \approx \Delta L\left[1 + \frac{1}{2}(ah^{+}_{ij} + bh^{\times}_{ij})\hat{\mathbf{n}}_i\hat{\mathbf{n}}_j \sin(kct)\right], \tag{3.12}$$

and the fractional strain is

$$\delta = \frac{\Delta S - \Delta L}{\Delta L} = \sin(kct)\left[\frac{a}{2}(\hat{\mathbf{n}}_x\hat{\mathbf{n}}_x - \hat{\mathbf{n}}_y\hat{\mathbf{n}}_y) + b\hat{\mathbf{n}}_x\hat{\mathbf{n}}_y\right], \tag{3.13}$$

where we used $h^{+}, h^{\times}$ obtained above for the wave with $\mathbf{k} = (0, 0, k)$. First suppose $a > 0, b = 0$, as t ranges from 0 ro $t = \frac{\pi}{2ck}$ the ring stretches along x and flattens along y. From $t = \frac{\pi}{2ck} \to t = \frac{2\pi}{2ck}$ we are back to the initial form. From $t = \frac{2\pi}{2ck} \to \frac{3\pi}{2ck}$ stretches along the y axis and flatters along x

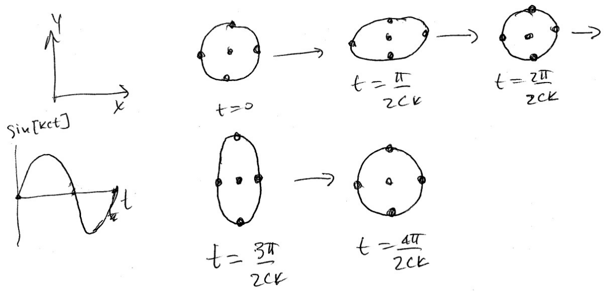

Figure 2: The first of two polarization modes of propagating gravitational waves. This is the plus-polarization where a ring of particles oscillate in the patter of a + sign.

This is the h^{+} polarization. Now suppose $a = 0,\ b > 0$. Now the ring of particles stretches along diagonals at 45 degrees ($\hat{\mathbf{n}}_x, \hat{\mathbf{n}}_y > 0$ for 45°) as $t = 0 \to \frac{\pi}{2ck}$ and we get back to the initial form for $t = \frac{\pi}{2ck} \to \frac{2\pi}{2ck}$ stretches along 135° between $\frac{2\pi}{2ck} \to \frac{3\pi}{2ck}$ and back to the original form between $\frac{3\pi}{2ck} \to \frac{4\pi}{2ck}$

This is the $h^{\times}$ polarization.

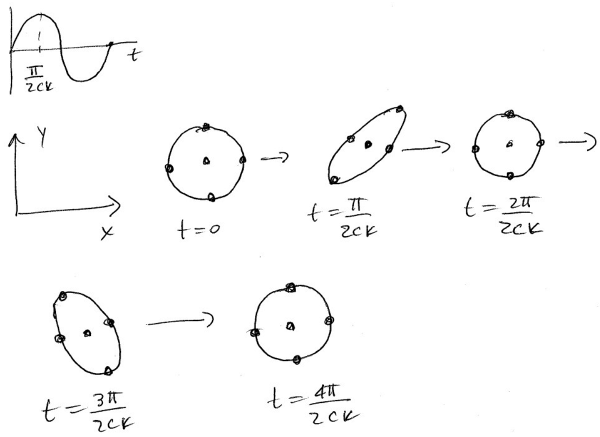

Figure 3: The second of the polarization modes for gravitational waves. This is the cross-polarization where a ring of particles oscillate in the patter of an $\times$ sign.

4 Energy and Momentum of Gravitational Waves

An important question we can ask is how to distinguish curvature due to the background metric and curvature as a result from propagating gravitational waves. Consider expanding the metric around some dynamical background spacetime

$$g_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x) + h_{\mu\nu}(x), \tag{4.1}$$

where $\bar{g}_{\mu\nu}$ is the background metric and $g_{\mu\nu}$ is the full metric where we raise and lower indices with respect to the background metric. One important concept for this is to introduce different scales for which these variations take place over. Let L_B be the length scale of the background and f_B be its frequency. Thinking about different scales is a natural way to try and distinguish between background curvature and curvature introduced by gravitational waves. First we start from Einstein's equations

$$R_{\mu\nu} = 8\pi G\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right). \tag{4.2}$$

We can expand out the curvature tensor and only keep up to quadratic powers of h

$$R_{\mu\nu} = \bar{R}_{\mu\nu} + R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu} + \dots, \tag{4.3}$$

where the overhead bars denote the curvature tensor constructed from $\bar{g}_{\mu\nu}$ that contains only the low frequency mode, $R^{(1)}_{\mu\nu}$ are the linear order h's that contains only the high frequency modes, and $R^{(2)}_{\mu\nu}$ is the quadratic h's composed of both high and low frequency modes. Thus we can write

$$\bar{R}_{\mu\nu} = -[R^{(2)}_{\mu\nu}]^{low} + 8\pi G\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)^{low}, \tag{4.4}$$

i.e. the low frequency modes

$$R^{(1)}_{\mu\nu} = -[R^{(2)}_{\mu\nu}]^{high} + 8\pi G\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)^{high}, \tag{4.5}$$

the high frequency modes. Because we are working with an arbitrary dynamical background spacetime, we must promote partial derivatives to covariant derivatives

$$R^{(1)}_{\mu\nu} = \frac{1}{2}\big[\bar{D}^{\lambda}(\bar{D}_{\mu}h_{\nu\lambda} + \bar{D}_{\nu}h_{\mu\lambda}) - \bar{D}^{\lambda}\bar{D}_{\lambda}h_{\mu\nu} - \bar{D}_{\mu}\bar{D}_{\nu}h\big], \tag{4.6}$$

where $\bar{D}_{\mu}$ is the covariant derivative with respect to $\bar{g}_{\mu\nu}$. The quadratic order curvature tensor is then

$$\begin{aligned} R^{(2)}_{\mu\nu} = \frac{1}{2}\bar{g}^{\lambda\rho}\bar{g}^{\alpha\beta}\Bigg[&\frac{1}{2}\bar{D}_{\mu}h_{\rho\alpha}\bar{D}_{\nu}h_{\lambda\beta} + (\bar{D}_{\rho}h_{\nu\alpha})(\bar{D}_{\lambda}h_{\mu\beta} - \bar{D}_{\beta}h_{\mu\lambda}) \\ &+h_{\rho\alpha}(\bar{D}_{\nu}\bar{D}_{\mu}h_{\lambda\beta} + \bar{D}_{\beta}\bar{D}_{\lambda}h_{\mu\nu} - \bar{D}_{\beta}\bar{D}_{\nu}h_{\mu\lambda} - \bar{D}_{\beta}\bar{D}_{\mu}h_{\nu\lambda}) \\ &+\left(\frac{1}{2}\bar{D}_{\alpha}h_{\rho\lambda} - \bar{D}_{\rho}h_{\alpha\lambda}\right)(\bar{D}_{\nu}h_{\mu\beta} + \bar{D}_{\mu}h_{\nu\beta} - \bar{D}_{\beta}h_{\mu\nu})\Bigg]. \end{aligned} \tag{4.7}$$

Next we introduce a time scale $\bar{t}$ that is much longer than the period $1/f_B$ of the gravitational wave (i.e. $h_{\mu\nu}$ is a high frequency perturbation) which when we average over the low frequency Einstein Equations we get

$$\bar{R}_{\mu\nu} = -\langle R^{(2)}_{\mu\nu}\rangle + 8\pi G\langle T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\rangle, \tag{4.8}$$

where the angle brackets denote a spatial average. Because $T_{\mu\nu}$ will already be quite smooth, it'll be constant over the scale for which we are averaging so

$$\langle T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\rangle \approx \overline{T}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\overline{T}. \tag{4.9}$$

We can then define a stress energy tensor for the quadratic contributions

$$\tau_{\mu\nu} \equiv -\frac{1}{8\pi G}\langle R^{(2)}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}R^{(2)}\rangle, \tag{4.10}$$

where $R^{(2)} = \bar{g}^{\mu\nu}R^{(2)}_{\mu\nu}$. The trace is then

$$\tau = \bar{g}^{\mu\nu}\tau_{\mu\nu} = -\frac{1}{8\pi G}\bar{g}^{\mu\nu}\langle R^{(2)}_{\mu\nu}\rangle + \frac{1}{16\pi G}\bar{g}^{\mu\nu}\bar{g}_{\mu\nu}\langle R^{(2)}\rangle = \frac{1}{8\pi G}\langle R^{(2)}\rangle. \tag{4.11}$$

Therefore we have

$$\begin{aligned} \langle R^{(2)}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}R^{(2)}\rangle &= \langle R^{(2)}_{\mu\nu}\rangle - \frac{1}{2}\bar{g}_{\mu\nu}\langle R^{(2)}\rangle = -8\pi G\tau_{\mu\nu} \\ \Leftrightarrow \langle R^{(2)}_{\mu\nu}\rangle &= -8\pi G\tau_{\mu\nu} + \frac{1}{2}\bar{g}_{\mu\nu}(8\pi G\tau) = -8\pi G\left(\tau_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\tau\right). \end{aligned} \tag{4.12}$$

Plugging all of this into the expression for the background curvature tensor gives us

$$\begin{aligned} \bar{R}_{\mu\nu} &= -\langle R^{(2)}_{\mu\nu}\rangle + 8\pi G\langle T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\rangle = 8\pi G\left(\tau_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\tau\right) + 8\pi G\left(\overline{T}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\overline{T}\right) \\ &= 8\pi G\left[(\overline{T}_{\mu\nu} + \tau_{\mu\nu}) - \frac{1}{2}\bar{g}_{\mu\nu}(\overline{T} + \tau)\right], \end{aligned} \tag{4.13}$$

which we can rewrite this as

$$\bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{R} = 8\pi G(\overline{T}_{\mu\nu} + \tau_{\mu\nu}). \tag{4.14}$$

These are the coarse-grained Einstein Equations. This relates the curvature that is induced by a localized matter distribution *and* by the perturbation to the background to the background spacetime itself.

Now lets derive an explicit expression for the stress tensor for gravitational waves. Since we have a method to relate $\tau_{\mu\nu}$ to the quadratic order metric perturbations, it is simply a matter of plugging in the expression of 4.7 into 4.10. However we can do better. If we assume we're in the transverse-traceless gauge, the equation for $\tau_{\mu\nu}$ reduces immensely. When we also realize that multiple terms of equivalent up to a sign under an integration by parts, we get

$$\langle R^{(2)}_{\mu\nu}\rangle = \frac{1}{4}\langle\partial_\mu h_{\lambda\rho}\partial_\nu h^{\lambda\rho}\rangle \Rightarrow \tau_{\mu\nu} = \frac{1}{32\pi G}\langle\partial_\mu h_{\lambda\rho}\partial_\nu h^{\lambda\rho}\rangle. \tag{4.15}$$

An alternative way to derive the stress energy tensor in the TT gauge is to consider Noether's Theorem

$$\tau^{\mu\nu} = \langle -\frac{\partial \mathcal{L}}{\partial(\partial_\mu h_{\lambda\rho})}\partial^\nu h_{\lambda\rho} + \eta^{\mu\nu}\mathcal{L}\rangle. \tag{4.16}$$

The Lagrangian for linearized GR is

$$\mathcal{L} = -\frac{M_p^2}{4}\left(\partial_\lambda h_{\mu\nu}\partial^\mu h^{\lambda\nu} + \frac{1}{2}\partial_\mu h \partial^\mu h - \frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} - \partial_\mu h^{\mu\nu}\partial_\nu h\right), \tag{4.17}$$

i.e. this is the Lagrangian for which when you plug it into the Euler-Lagrange Equations, you recover Einstein's Equations for linearized GR. Imposing the TT gauge condition reduces the Lagrangian to

$$\mathcal{L} = -\frac{1}{8}M_p^2 \partial_\mu h_{\alpha\beta}\partial^\mu h^{\alpha\beta}. \tag{4.18}$$

Inserting this into the formula for the Noether current gives

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu h_{\lambda\rho})} = -\frac{M_p^2}{4}\left(\frac{\partial(\partial_\sigma h_{\alpha\beta})}{\partial(\partial_\mu h_{\lambda\rho})}\partial^\sigma h^{\alpha\beta}\right) = -\frac{M_p^2}{4}\partial^\sigma h^{\alpha\beta}\delta^\mu_\sigma \delta^\lambda_\alpha \delta^\rho_\beta = -\frac{M_p^2}{4}\partial^\mu h^{\lambda\rho}. \tag{4.19}$$

And it immediately follows

$$\langle\mathcal{L}\rangle = -\frac{M_p^2}{8}\langle \partial_\mu h_{\lambda\rho}\partial^\mu h^{\lambda\rho}\rangle. \tag{4.20}$$

Which leads to the following expression for the stress energy tensor

$$\tau^{\mu\nu} = \langle \frac{M_p^2}{4}\partial^\mu h^{\lambda\rho}\partial^\nu h_{\lambda\rho} + \eta^{\mu\nu}\left(-\frac{M^2}{8}\partial_\sigma h_{\lambda\rho}\partial^\sigma h^{\lambda\rho}\right) = \frac{M_p^2}{4}\langle \partial^\mu h^{\lambda\rho}\partial^\nu h_{\lambda\rho}\rangle, \tag{4.21}$$

wherein the last equality we used the fact that in the TT gauge, the equations of motion enforce $\Box h_{\mu\nu} = 0$ and we neglected the boundary term because that is of order $\mathcal{O}(1/L_B)$ and therefore is negligible.

5 The Gravitational Stochastic Background

The Stochastic Background of Gravitational Waves can emerge from the incoherent superposition of a large number of astrophysical sources that are too weak to be detected separately and such that the number of sources that contribute to each frequency bin is much larger than one. We make some assumptions about the background

Stationarity This means that all n-point correlation functions can only depend on time differences as opposed to absolute time i.e.

$$\langle h_A(t)h_B(t')\rangle \propto f(t-t')$$

but not on t, t' separate. This means we must have

$$\langle \tilde{h}_A^*(f)\tilde{h}_B(f')\rangle \propto \delta(f-f').$$

The typical time scale it can change substantially is of order the age of the universe.

Gaussianity All n-point correlators are or can be reduced to sums and products of the 2-point correlation function (and the vacuum expectation value but since we impose stationarity, the VEV has to be a constant that we set to zero for simplicity). This is a direct consequence of the central limit theorem.

Isotropy Because the early universe was highly isotropic (and we know this from the CMB), we expect the gravitational background should be isotropic as well. This implies

$$\langle \tilde{h}_A^*(f,\hat{\mathbf{n}})\tilde{h}_B(f',\hat{\mathbf{n}}')\rangle \propto \delta^2(\hat{\mathbf{n}},\hat{\mathbf{n}}')$$

where

$$\delta^2(\hat{\mathbf{n}},\hat{\mathbf{n}}') = \delta(\cos\theta - \cos\theta')\delta(\phi-\phi'). \tag{5.1}$$

This comes from the idea that waves that are coming from different directions should be uncorrelated.

Polarization : Lastly we expect the background to be unpolarized i.e.

$$\langle \tilde{h}_A^*(f,\hat{\mathbf{n}})\tilde{h}_B(f',\hat{\mathbf{n}}')\rangle \propto \delta_{AB}$$

All of these conditions taken together gives us

$$\langle \tilde{h}_A^*(f,\hat{\mathbf{n}})\tilde{h}_B(f',\hat{\mathbf{n}}')\rangle = \delta(f-f')\frac{\delta^2(\hat{\mathbf{n}},\hat{\mathbf{n}}')}{4\pi}\delta_{AB}\frac{S_h(f)}{2}, \tag{5.2}$$

where $S_h(f)$ is the spectral density of the stochastic background with dimensions Hz^{-1} and is an even function $S_h(-f) = S_h(f)$. The factor of 4π is there for normalization purposes

$$\int \mathrm{d}^2\hat{\mathbf{n}} \int \mathrm{d}^2\hat{\mathbf{n}}' \,\langle \tilde{h}_A^*(f,\hat{\mathbf{n}})\tilde{h}_B(f',\hat{\mathbf{n}}')\rangle = \delta(f-f')\delta_{AB}\frac{1}{2}S_h(f). \tag{5.3}$$

We can compute the mean-squared of the metric perturbation by first recalling our functional form of the perturbation

$$h_{ij}(t) = \sum_{A=+,\times} \int \mathrm{d}f \int \mathrm{d}^2\hat{\mathbf{n}}\, \tilde{h}_A(f,\hat{\mathbf{n}})\epsilon_{ij}^A(\hat{\mathbf{n}})e^{-2\pi i f t}, \tag{5.4}$$

where we've set $\mathbf{r} = 0$ because we've placed the origin at the detector. Next we can see that

$$h_{ij}(t)h^{ij}(t) = \left[\sum_{A=+,\times} \int \mathrm{d}f \int_{\mathbb{R}} \mathrm{d}^2\hat{\mathbf{n}}\, \tilde{h}_A(f,\hat{\mathbf{n}})\epsilon_{ij}^A(\hat{\mathbf{n}})e^{-2\pi i f t}\right]\left[\sum_{B=+,\times} \int_{\mathbb{R}} \mathrm{d}f' \int \mathrm{d}^2\hat{\mathbf{n}}'\, \tilde{h}_B(f',\hat{\mathbf{n}}')\epsilon_B^{ij}(\hat{\mathbf{n}}')e^{-2\pi i f' t}\right] \tag{5.5}$$

$$= \sum_{A,B=+,\times} \int_{\mathbb{R}} \mathrm{d}f \int_{\mathbb{R}} \mathrm{d}f' \int \mathrm{d}^2\hat{\mathbf{n}} \int \mathrm{d}^2\hat{\mathbf{n}}'\, \tilde{h}_A(f,\hat{\mathbf{n}})\tilde{h}_B(f',\hat{\mathbf{n}}')\epsilon_{ij}^A(\hat{\mathbf{n}})\epsilon_B^{ij}(\hat{\mathbf{n}}')e^{-2\pi i (f+f')t}. \tag{5.6}$$

Moving forward, we make the change of variables $f \to -f$ and use the fact that the amplitude of the gravitational wave must be real so $\tilde{h}_A(-f) = \tilde{h}_A^*(f)$ we can write

$$h_{ij}(t)h^{ij}(t) = \sum_{A,B=+,\times} \int_{\mathbb{R}} \mathrm{d}f \int_{\mathbb{R}} \mathrm{d}f' \int \mathrm{d}^2\hat{\mathbf{n}} \int \mathrm{d}^2\hat{\mathbf{n}}'\, \tilde{h}_A^*(f,\hat{\mathbf{n}})\tilde{h}_B(f',\hat{\mathbf{n}}')\epsilon_{ij}^A(\hat{\mathbf{n}})\epsilon_B^{ij}(\hat{\mathbf{n}}')e^{2\pi i (f-f')t}, \tag{5.7}$$

where the minus sign from the measure $\mathrm{d}f \to -\mathrm{d}f$, exactly cancels out from the minus sign in the new integration region $\int_{\infty}^{-\infty} = -\int_{-\infty}^{\infty}$. Now we can average over this product

$$\left\langle h_{ij}(t)h^{ij}(t)\right\rangle = \sum_{A,B=+,\times}\int_{\mathbb{R}}\mathrm{d}f\int_{\mathbb{R}}\mathrm{d}f'\int\mathrm{d}^2\hat{\mathbf{n}}\int\mathrm{d}^2\hat{\mathbf{n}}'\,\langle\tilde{h}^*_A(f,\hat{\mathbf{n}})\tilde{h}_B(f',\hat{\mathbf{n}}')\rangle\epsilon^A_{ij}(\hat{\mathbf{n}})\epsilon^{ij}_B(\hat{\mathbf{n}}')e^{2\pi i(f-f')t} \tag{5.8}$$

$$= \sum_{A,B=+,\times}\int_{\mathbb{R}}\mathrm{d}f\int_{\mathbb{R}}\mathrm{d}f'\int\mathrm{d}^2\hat{\mathbf{n}}\int\mathrm{d}^2\hat{\mathbf{n}}'\,\delta(f-f')\frac{\delta^2(\hat{\mathbf{n}},\hat{\mathbf{n}}')}{4\pi}\delta_{AB}\frac{S_h(f)}{2}\epsilon^A_{ij}(\hat{\mathbf{n}})\epsilon^{ij}_B(\hat{\mathbf{n}}')e^{2\pi it(f-f')}. \tag{5.9}$$

Integrating $f', \hat{\mathbf{n}}'$ as well as using the Kronecker delta, the mean reduces down to

$$\left\langle h_{ij}(t)h^{ij}(t)\right\rangle = \frac{1}{2\cdot 4\pi}\sum_{A=+,\times}\int_{\mathbb{R}}\mathrm{d}f\int\mathrm{d}^2\hat{\mathbf{n}}\,\epsilon^A_{ij}(\hat{\mathbf{n}})\epsilon^{ij}_A(\hat{\mathbf{n}})S_h(f) \tag{5.10}$$

$$= \frac{1}{2\pi}\int\mathrm{d}f\int\mathrm{d}^2\hat{\mathbf{n}}\,S_h(f) = 2\int_{\mathbb{R}}S_h(f)\,\mathrm{d}f = 4\int_0^\infty S_h(f)\,\mathrm{d}f\,, \tag{5.11}$$

wherein we used the normalization of the polarization tensors

$$\sum_{A=+,\times}\epsilon^A_{ij}(\hat{\mathbf{n}})\epsilon^{ij}_A(\hat{\mathbf{n}}) = 4, \tag{5.12}$$

and in the last equality we used the evenness of $S_h(f)$. To get a physical understanding of what this spectral density means, we can look at the energy density of gravitational waves

$$\rho_{GW} = \frac{1}{32\pi G}\langle\dot{h}^{ij}\dot{h}_{ij}\rangle. \tag{5.13}$$

Recall the critical energy density required to close the universe is given by

$$\rho_c = \frac{3H_0^2}{8\pi G} \simeq 1.688\times 10^{-8}h_0^2\,\mathrm{erg/\,cm^3}, \tag{5.14}$$

where h_0 is the little Hubble parameter which is used to parametrize the uncertainty of the Hubble constant. Of which, we define to be $h_0 \simeq .73$. We also can define the fractional energy density to be

$$\Omega_{GW} \equiv \frac{\rho_{GW}}{\rho_c}. \tag{5.15}$$

Now to derive a physical interpretation for the mean squared perturbation, let us compute the spectral density for the energy density

$$\rho_{GW} = \int_{f=0}^{f=\infty} \mathrm{d}(\log f)\, \frac{\mathrm{d}\rho_{GW}}{\mathrm{d}(\log f)}, \tag{5.16}$$

and we choose to integrate over $\mathrm{d}\log f$ in order to have

$$\Omega_{GW}(f) \equiv \frac{1}{\rho_c}\frac{\mathrm{d}\rho_{GW}}{\mathrm{d}\log f}, \tag{5.17}$$

be dimensionless. We can also write down a spectral density for the frequency-dependent fractional energy density

$$\Omega_{GW} = \int_{f=0}^{f=\infty} \mathrm{d}(\log f)\, \Omega_{GW}(f). \tag{5.18}$$

Now we can compute the spectral density of the energy density. First we write

$$\dot{h}_{ij}(t) = (-2\pi i) \sum_{A=+,\times} \int \mathrm{d}f \int \mathrm{d}^2\hat{\mathbf{n}}\, f \tilde{h}_A(f,\hat{\mathbf{n}}) \epsilon^A_{ij}(\hat{\mathbf{n}}) e^{-2\pi i f t}. \tag{5.19}$$

Then we get

$$\dot{h}_{ij}\dot{h}^{ij} = \left[(-2\pi i)\sum_A \int_{\mathbb{R}} \mathrm{d}f \int \mathrm{d}^2\hat{\mathbf{n}}\, f\tilde{h}_A \epsilon^A_{ij} e^{-2\pi i f t}\right]\left[(-2\pi i)\sum_B \int_{\mathbb{R}} \mathrm{d}f' \int \mathrm{d}^2\hat{\mathbf{n}}'\, f'\tilde{h}_B \epsilon^{ij}_B e^{-2\pi i f' t}\right] \tag{5.20}$$

$$= (-2\pi i)^2 \sum_{A,B} \int_{\mathbb{R}} \mathrm{d}f \int_{\mathbb{R}} \mathrm{d}f' \int \mathrm{d}^2\hat{\mathbf{n}} \int \mathrm{d}^2\hat{\mathbf{n}}'\, ff' \tilde{h}_A(f,\hat{\mathbf{n}})\tilde{h}_B(f',\hat{\mathbf{n}}')\epsilon^A_{ij}(\hat{\mathbf{n}})\epsilon^{ij}_B(\hat{\mathbf{n}}')e^{-2\pi i(f+f')t}. \tag{5.21}$$

And again we let $f \to -f$ and use $\tilde{h}_A(-f) = \tilde{h}^*_A(f)$ to get

$$\left\langle \dot{h}_{ij}\dot{h}^{ij} \right\rangle = -(-2\pi i)^2 \sum_{A,B} \int_{\mathbb{R}} \mathrm{d}f \int_{\mathbb{R}} \mathrm{d}f' \int \mathrm{d}^2\hat{\mathbf{n}} \int \mathrm{d}^2\hat{\mathbf{n}}'\, ff' \langle \tilde{h}^*_A(f,\hat{\mathbf{n}})\tilde{h}_B(f',\hat{\mathbf{n}}')\rangle \epsilon^A_{ij}(\hat{\mathbf{n}})\epsilon^{ij}_B(\hat{\mathbf{n}}')e^{2\pi i(f-f')t} \tag{5.22}$$

$$= \sum_{A,B} \int_{\mathbb{R}} \mathrm{d}f \int_{\infty} \mathrm{d}f'\, ff' \int \mathrm{d}^2\hat{\mathbf{n}} \int \mathrm{d}^2\hat{\mathbf{n}}'\, \delta(f-f') \frac{\delta^2(\hat{\mathbf{n}},\hat{\mathbf{n}}')}{4\pi}\delta_{AB}\frac{S_h(f)}{2}\epsilon^A_{ij}(\hat{\mathbf{n}})\epsilon^{ij}_B(\hat{\mathbf{n}}')e^{-2\pi i f t(f-f')} \tag{5.23}$$

$$= \frac{\pi}{2} \sum_{A=+,\times} \int_{-\infty}^{\infty} \mathrm{d}f \int \mathrm{d}^2\hat{\mathbf{n}}\, f^2 S_h(f) \epsilon^A_{ij}(\hat{\mathbf{n}})\epsilon^{ij}_B(\hat{\mathbf{n}}) \tag{5.24}$$

$$= (4\pi)^2 \int_0^{\infty} f^2 S_h(f)\, \mathrm{d}f\,. \tag{5.25}$$

Thus the spectral density of the gravitational energy density is

$$\rho_{GW} = \frac{1}{32\pi G}\langle \dot{h}_{ij}\dot{h}^{ij}\rangle = \frac{\pi}{2G}\int_{f=0}^{f=\infty} \frac{\mathrm{d}f}{\mathrm{d}\log f} f^2 S_h(f)\,\mathrm{d}\log f = \frac{\pi}{2G}\int_{f=0}^{f=\infty} f^3 S_h(f)\,\mathrm{d}\log f\,, \tag{5.26}$$

which implies the following relation for the spectral density

$$\frac{\mathrm{d}\rho_{GW}}{\mathrm{d}\log f} = \frac{\pi}{2G} f^3 S_h(f). \tag{5.27}$$

The frequency-varying spectral density for the fractional energy density is then

$$\Omega_{GW}(f) = \frac{1}{\rho_c}\frac{\mathrm{d}\rho_{GW}}{\mathrm{d}\log f} = \frac{(2\pi)^2}{3H_0^2} f^3 S_h(f). \tag{5.28}$$

It will be prudent to work with $h_0^2\Omega_{GW}(f)$ as a way to circumvent any potential uncertainty. An alternative expression for the gravitational wave energy density is given in terms of the number of gravitons per cell of phase space $n(\mathbf{r},\mathbf{p}) = n_f$ which only depends on the frequency. The frequency is related to the momentum by $|\mathbf{p}| = \omega = 2\pi f$ and the fact that it only depends on the magnitude and not direction is a consequence of the isotropy condition we placed on the stochastic background. The energy density is then

$$\rho_{GW} = 2\int \frac{\mathrm{d}^3 p}{(2\pi)^3} E_{\mathbf{p}} n_f = 2\int \frac{p^2\,\mathrm{d}p\,\mathrm{d}\Omega}{(2\pi)^3} p n_f, \tag{5.29}$$

where the factor of 2 in front is for the two polarizations of the graviton and made use of the dispersion relation for a massless particle. Plugging in the relation $p = 2\pi f$ we get

$$\rho_{GW} = 2\cdot\frac{4\pi}{(2\pi)^3}\cdot(2\pi)^4\int_0^\infty f^2\cdot f n_f\,\mathrm{d}f = (4\pi)^2\int_{f=0}^{f=\infty}\mathrm{d}(\log f)\, f^4 n_f. \tag{5.30}$$

Thus the spectral density is then

$$\frac{\mathrm{d}\rho_{GW}}{\mathrm{d}\log f} = (4\pi f^2)^2 n_f \Rightarrow h_0^2\Omega_{GW}(f) = \frac{8\pi G}{3}\frac{h_0^2}{H_0^2}(4\pi f^2)^2 n_f. \tag{5.31}$$

It is common to express the covariance of the possible polarization states in terms of the *Stokes Parameters*[1]

$$\begin{bmatrix} \langle h_+^*(f,\hat{\mathbf{n}})h_+(f',\hat{\mathbf{n}}')\rangle & \langle h_+^*(f,\hat{\mathbf{n}})h_\times(f',\hat{\mathbf{n}}')\rangle \\ \langle h_\times^*(f,\hat{\mathbf{n}})h_+(f',\hat{\mathbf{n}}')\rangle & \langle h_\times^*(f,\hat{\mathbf{n}})h_\times(f',\hat{\mathbf{n}}')\rangle \end{bmatrix}$$
$$= \frac{1}{2}\delta(f-f')\frac{\delta^{(2)}(\hat{\mathbf{n}}-\hat{\mathbf{n}}')}{4\pi}\begin{bmatrix} I(f,\hat{\mathbf{n}})+Q(f,\hat{\mathbf{n}}) & U(f,\hat{\mathbf{n}})+iV(f,\hat{\mathbf{n}}) \\ U(f,\hat{\mathbf{n}})-iV(f,\hat{\mathbf{n}}) & I(f,\hat{\mathbf{n}})-Q(f,\hat{\mathbf{n}}) \end{bmatrix}, \tag{5.32}$$

where $I = I(f)$ is the overall intensity related to spectral density of the signal by $I(f) = S_h(f)$, V is the circular polarization, and U, Q are quantities that describe the horizontal/vertical (in the case of Q) and diagonal linear polarization states. We note that I and V transform as scalar (and pseudo-scalar) under rotations whereas parameters U, Q transform as spin-4 quantities under rotations. This presentation of the possible polarization states is a generalization of the condition for the Stochastic Background we originally wrote out. This can be seen by simply taking $Q = U = V = 0$ (which we originally did so due to the isotropy condition).

5.1 Signal-to-Noise Ratio

A gravitational wave background acts as an additional source of noise i.e. a background for a detector. One needs to model the noise inherent in the detector as well as all potential sources of noise in order to get a certain value for the spectral density of the noise $S_n(f)$.

Once the detector has been turned on, we measure $\langle s^2(t)\rangle$ where $s(t) = h(t) + n(t)$ with $h(t)$ being the response of the detector due to the gravitational wave signal and $n(t)$ is the noise.

We need to impose some signal-to-noise (SNR) cutoff to establish when a detection of the gravitational wave background is made. Given some S/N, we should compute the minimum value of $h_0^2\Omega_{GW}$ that can be measured (since this is a dimensionless observable). First, suppose there is no signal ($h(t) = 0$), the detector measures

[1]This is a concept that is borrowed from E&M where we use 4 parameters to describe all the possible polarization states.

$$\langle s^2(t)\rangle = \langle n^2(t)\rangle = \int_0^\infty S_n(f)\,\mathrm{d}f\,. \tag{5.33}$$

Now suppose there is some signal ($h(t) \neq 0$). Now for every propagation direction $\hat{\mathbf{n}}$, we can write $h(t) = h_+(t)F_+(\hat{\mathbf{n}}) + h_\times(t)F_\times(\hat{\mathbf{n}})$ where F_A are the detector pattern functions given by $F_A(\hat{\mathbf{n}}) = D^{ij}\epsilon^A_{ij}(\hat{\mathbf{n}})$ and D^{ij} is called the detector tensor. We break for a brief interlude.

5.1.1 Pattern Functions

First we introduce the detector tensor. The detector tensor D^{ij}, is the tensor that projects the relevant orientation of the gravitational wave according to the geometry of the detector. As an example, if the detector is purely driven by the xx component of the propagating wave then

$$D^{ij} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}^{ij}\,. \tag{5.34}$$

Because detectors can only receive scalar quantities as inputs and/or outputs, but gravitational waves are described by a rank 2 tensor, it is important that we express our mathematical tools in terms of things that can be measured. Now the pattern functions $F_A(\hat{\mathbf{n}}) = D^{ij}e^A_{ij}(\hat{\mathbf{n}})$, are a set of functions where we project out angular dependence of the polarization tensors. Given the propagation direction $\hat{\mathbf{n}}$, we can construct the plane orthogonal to this direction using the axes $\hat{\mathbf{u}}, \hat{\mathbf{v}}$. An interesting property that we can observe is what happens if we rotate these axes about the propagation direction. Because these are vectors under rotation they transform as

$$\hat{\mathbf{u}} \to \hat{\mathbf{u}}' = \hat{\mathbf{u}}\cos\psi - \hat{\mathbf{v}}\sin\psi, \quad \hat{\mathbf{v}} \to \hat{\mathbf{v}}' = \hat{\mathbf{u}}\sin\psi + \hat{\mathbf{v}}\cos\psi. \tag{5.35}$$

The amplitude for the polarizations $h_+, h_\times$ transform as

$$h'_+ = h_+\cos 2\psi - h_\times \sin 2\psi, \quad h'_\times = h_+\sin 2\psi + h_\times\cos 2\psi. \tag{5.36}$$

Now we recall the definition of the polarization tensors in terms of these vectors by

$$\epsilon^{+}_{ij} = \hat{\mathbf{u}}_i\hat{\mathbf{u}}_j - \hat{\mathbf{v}}_i\hat{\mathbf{v}}_j, \qquad \epsilon^{\times}_{ij} = \hat{\mathbf{u}}_i\hat{\mathbf{v}}_j + \hat{\mathbf{v}}_i\hat{\mathbf{u}}_j. \tag{5.37}$$

Under a rotation in the $\hat{\mathbf{u}}, \hat{\mathbf{v}}$ plane, these tensors transform as

$$(\epsilon^{+}_{ij}(\hat{\mathbf{n}}))' = \hat{\mathbf{u}}'_i\hat{\mathbf{u}}'_j - \hat{\mathbf{v}}'_i\hat{\mathbf{v}}'_j \tag{5.38}$$

$$= (\hat{\mathbf{u}}_i \cos\psi - \hat{\mathbf{v}}_i \sin\psi)(\hat{\mathbf{u}}_j \cos\psi - \hat{\mathbf{v}}_j \sin\psi) - (\hat{\mathbf{u}}_i \sin\psi + \hat{\mathbf{v}}_i \cos\psi)(\hat{\mathbf{u}}_j \sin\psi + \hat{\mathbf{v}}_j \cos\psi) \tag{5.39}$$

$$= \hat{\mathbf{u}}_i\hat{\mathbf{u}}_j \cos^2\psi - \hat{\mathbf{u}}_i\hat{\mathbf{v}}_j \sin\psi\cos\psi - \hat{\mathbf{v}}_i\hat{\mathbf{u}}_j \sin\psi\cos\psi + \hat{\mathbf{v}}_i\hat{\mathbf{v}}_j \sin^2\psi \tag{5.40}$$

$$- (\hat{\mathbf{u}}_i\hat{\mathbf{u}}_j \sin^2\psi + \hat{\mathbf{u}}_i\hat{\mathbf{v}}_j \sin\psi\cos\psi + \hat{\mathbf{v}}_i\hat{\mathbf{u}}_j \sin\psi\cos\psi + \hat{\mathbf{v}}_i\hat{\mathbf{v}}_j \cos^2\psi)$$

$$= \hat{\mathbf{u}}_i\hat{\mathbf{u}}_j(\cos^2\psi - \sin^2\psi) - 2\hat{\mathbf{u}}_i\hat{\mathbf{v}}_j \sin\psi\cos\psi - 2\hat{\mathbf{v}}_i\hat{\mathbf{u}}_j \sin\psi\cos\psi + \hat{\mathbf{v}}_i\hat{\mathbf{v}}_j(\sin^2\psi - \cos^2\psi) \tag{5.41}$$

$$= \hat{\mathbf{u}}_i\hat{\mathbf{u}}_j \cos 2\psi - \hat{\mathbf{u}}_i\hat{\mathbf{v}}_j \sin 2\psi - \hat{\mathbf{v}}_i\hat{\mathbf{u}}_j \sin 2\psi - \hat{\mathbf{v}}_i\hat{\mathbf{v}}_j \cos 2\psi \tag{5.42}$$

$$= (\hat{\mathbf{u}}_i\hat{\mathbf{u}}_j - \hat{\mathbf{v}}_i\hat{\mathbf{v}}_j) \cos 2\psi - (\hat{\mathbf{u}}_i\hat{\mathbf{v}}_j + \hat{\mathbf{v}}_i\hat{\mathbf{u}}_j) \sin 2\psi \tag{5.43}$$

$$= \epsilon^{+}_{ij} \cos 2\psi - \epsilon^{\times}_{ij} \sin 2\psi, \tag{5.44}$$

$$(\epsilon^{\times}_{ij}(\hat{\mathbf{n}}))' = \hat{\mathbf{u}}'_i\hat{\mathbf{v}}'_j + \hat{\mathbf{v}}'_i\hat{\mathbf{u}}'_j \tag{5.45}$$

$$= (\hat{\mathbf{u}}_i \cos\psi - \hat{\mathbf{v}}_i \sin\psi)(\hat{\mathbf{u}}_j \sin\psi + \hat{\mathbf{v}}_j \cos\psi) + (\hat{\mathbf{u}}_i \sin\psi + \hat{\mathbf{v}}_i \cos\psi)(\hat{\mathbf{u}}_j \cos\psi - \hat{\mathbf{v}}_j \sin\psi) \tag{5.46}$$

$$= \hat{\mathbf{u}}_i\hat{\mathbf{u}}_j \sin\psi\cos\psi + \hat{\mathbf{u}}_i\hat{\mathbf{v}}_j \cos^2\psi - \hat{\mathbf{v}}_i\hat{\mathbf{u}}_j \sin^2\psi - \hat{\mathbf{v}}_i\hat{\mathbf{v}}_j \sin\psi\cos\psi \tag{5.47}$$

$$+ \hat{\mathbf{u}}_i\hat{\mathbf{u}}_j \sin\psi\cos\psi - \hat{\mathbf{u}}_i\hat{\mathbf{v}}_j \sin^2\psi + \hat{\mathbf{v}}_i\hat{\mathbf{u}}_j \cos^2\psi - \hat{\mathbf{v}}_i\hat{\mathbf{v}}_j \sin\psi\cos\psi$$

$$= 2\hat{\mathbf{u}}_i\hat{\mathbf{u}}_j \sin\psi\cos\psi + \hat{\mathbf{u}}_i\hat{\mathbf{v}}_j(\cos^2\psi - \sin^2\psi) - 2\hat{\mathbf{v}}_i\hat{\mathbf{u}}_j \sin\psi\cos\psi + \hat{\mathbf{v}}_i\hat{\mathbf{v}}_j(\sin^2\psi - \cos^2\psi) \tag{5.48}$$

$$= \hat{\mathbf{u}}_i\hat{\mathbf{u}}_j \sin 2\psi - \hat{\mathbf{u}}_i\hat{\mathbf{v}}_j \cos 2\psi - \hat{\mathbf{v}}_i\hat{\mathbf{u}}_j \cos 2\psi - \hat{\mathbf{v}}_i\hat{\mathbf{v}}_j \sin 2\psi \tag{5.49}$$

$$= (\hat{\mathbf{u}}_i\hat{\mathbf{u}}_j - \hat{\mathbf{v}}_i\hat{\mathbf{v}}_j) \sin 2\psi - (\hat{\mathbf{u}}_i\hat{\mathbf{v}}_j + \hat{\mathbf{v}}_i\hat{\mathbf{u}}_j) \cos 2\psi \tag{5.50}$$

$$= \epsilon^{\times}_{ij} \cos 2\psi - \epsilon^{+}_{ij} \sin 2\psi, \tag{5.51}$$

thus we have the interesting fact that the polarization tensors

$$(\epsilon^{+}_{ij})' = \epsilon^{+}_{ij} \cos 2\psi - \epsilon^{\times}_{ij} \sin 2\psi, \qquad (\epsilon^{\times}_{ij})' = \epsilon^{\times}_{ij} \cos 2\psi + \epsilon^{+}_{ij} \sin 2\psi, \tag{5.52}$$

almost transform like vectors, the fact that they rotate twice as much likely being a consequence of the spin of the graviton. We also notice that the polarization tensors transform exactly the same as the polarization amplitudes. We next look at how the pattern functions transform under a rotation in the orthogonal plane of propagation $F'_A(\hat{\mathbf{n}}) = D^{ij}(\epsilon^A_{ij})'(\hat{\mathbf{n}})$

$$F'_+(\hat{\mathbf{n}}) = F_+(\hat{\mathbf{n}})\cos 2\psi - F_\times(\hat{\mathbf{n}})\sin 2\psi, \quad F'_\times = F_+ \sin 2\psi + F_\times \cos 2\psi, \tag{5.53}$$

which follows directly from the transformation properties of the polarization tensors and we used D_{ij} to project out the corresponding polarization amplitude. As a result of these definitions, the signal $h(t)$ should be independent of this rotation

$$h'(t) = h'_+F'_+ + h'_\times F'_\times \tag{5.54}$$

$$= (h_+\cos 2\psi - h_\times \sin 2\psi)(F_+(\hat{\mathbf{n}})\cos 2\psi - F_\times(\hat{\mathbf{n}})\sin 2\psi) \tag{5.55}$$

$$+ (h_+ \sin 2\psi + h_\times \cos 2\psi)(F_+ \sin 2\psi + F_\times \cos 2\psi)$$

$$= h_+F_+\cos^2 2\psi + h_\times F_\times \sin^2 \psi + h_\times F_\times \cos^2 2\psi + h_+F_+\sin^2 2\psi \tag{5.56}$$

$$= h_+F_+ + h_\times F_\times, \tag{5.57}$$

thus $h'(t) = h(t)$ as we anticipated. We then define the new pattern functions

$$F_+(\hat{\mathbf{n}};\psi) \equiv F_+(\hat{\mathbf{n}})\cos 2\psi - F_\times \sin 2\psi, \quad F_\times(\hat{\mathbf{n}};\psi) \equiv F_\times(\hat{\mathbf{n}})\sin 2\psi + F_+(\hat{\mathbf{n}})\cos 2\psi. \tag{5.58}$$

We next have the following useful identity:

$$\int \frac{\mathrm{d}^2\hat{\mathbf{n}}}{4\pi} F_+(\hat{\mathbf{n}})F_\times(\hat{\mathbf{n}}) = \int \frac{\mathrm{d}^2\hat{\mathbf{n}}}{4\pi}(D^{ij}\epsilon^+_{ij}(\hat{\mathbf{n}}))(D^{k\ell}\epsilon^\times_{k\ell}(\hat{\mathbf{n}})) = D^{ij}D^{k\ell}\int \frac{\mathrm{d}^2\hat{\mathbf{n}}}{4\pi}(\hat{\mathbf{u}}_i\hat{\mathbf{u}}_j - \hat{\mathbf{v}}_i\hat{\mathbf{v}}_j)(\hat{\mathbf{u}}_i\hat{\mathbf{v}}_j + \hat{\mathbf{v}}_i\hat{\mathbf{u}}_j) \tag{5.59}$$

$$= D^{ij}D^{k\ell}\int \frac{\mathrm{d}^2\hat{\mathbf{n}}}{4\pi}(\hat{\mathbf{u}}_i\hat{\mathbf{u}}_j\hat{\mathbf{u}}_k\hat{\mathbf{v}}_\ell + \hat{\mathbf{u}}_i\hat{\mathbf{u}}_j\hat{\mathbf{v}}_k\hat{\mathbf{u}}_\ell - \hat{\mathbf{v}}_i\hat{\mathbf{v}}_j\hat{\mathbf{u}}_k\hat{\mathbf{v}}_\ell - \hat{\mathbf{v}}_i\hat{\mathbf{v}}_j\hat{\mathbf{v}}_k\hat{\mathbf{u}}_\ell), \tag{5.60}$$

and when we integrate over all possible angles/directions $\hat{\mathbf{n}}$, we notice that for every vector $\hat{\mathbf{u}}$, there exist a vector with the same magnitude but opposite direction $-\hat{\mathbf{u}}$ which cancels both contributions out. We next can derive some other properties for these generalized pattern functions

$$\int \frac{\mathrm{d}\psi}{2\pi} F_+^2(\hat{\mathbf{n}};\psi) = \frac{F_+^2}{2\pi}\int_0^{2\pi} \cos^2 2\psi \,\mathrm{d}\psi + \frac{F_+F_\times}{2\pi}\int_0^{2\pi} \sin 4\psi \,\mathrm{d}\psi + \frac{F_\times^2}{2\pi}\int_0^{\infty} \sin^2 2\psi \,\mathrm{d}\psi \tag{5.61}$$

$$= \frac{F_+^2}{2\pi}\cdot\frac{\pi}{2} + \frac{F_\times^2}{2\pi}\cdot\frac{\pi}{2} = \frac{1}{4}\left(F_+^2 + F_\times^2\right) \tag{5.62}$$

$$= \frac{F_+^2}{2\pi}\int_0^{2\pi} \sin^2 2\psi \,\mathrm{d}\psi + \frac{F_+F_\times}{2\pi}\int_0^{2\pi} \sin 4\psi \,\mathrm{d}\psi + \frac{F_\times^2}{2\pi}\int_0^{\infty} \cos^2 2\psi \,\mathrm{d}\psi \tag{5.63}$$

$$= \int \frac{\mathrm{d}\psi}{2\pi} F_\times^2(\hat{\mathbf{n}};\psi). \tag{5.64}$$

We also have an identity involving the ensemble average over all possible angles

$$\langle F_+^2(\hat{\mathbf{n}};\psi)\rangle_{(\theta,\phi,\psi)} = \int_0^{2\pi}\frac{\mathrm{d}\psi}{2\pi}\int\frac{\mathrm{d}^2\hat{\mathbf{n}}}{4\pi}F_+^2(\hat{\mathbf{n}};\psi) = \int\frac{\mathrm{d}^2\hat{\mathbf{n}}}{16\pi}(F_+^2(\hat{\mathbf{n}})+F_\times^2(\hat{\mathbf{n}})) = \langle F_\times^2(\hat{\mathbf{n}};\psi)\rangle_{(\theta,\phi,\psi)}. \tag{5.65}$$

Lastly we define the *angular efficiency* F by

$$F \equiv \langle F_+^2\rangle + \langle F_\times^2\rangle = 2\langle F_+^2\rangle. \tag{5.66}$$

The angular efficiency relates the response of the detector with the angular dependence of the propagating wave.

5.2 SNR Analysis Continued

Now we can go back to the original discussion. The response of the detector due to a gravitational wave impacting it is

$$h(t) = h_+(t)F_+(\hat{\mathbf{n}})+h_\times(t)F_\times(\hat{\mathbf{n}}) \Rightarrow h^2(t) = h_+^2(t)F_+^2(\hat{\mathbf{n}})+2h_+(t)h_\times(t)F_+(\hat{\mathbf{n}})F_\times(\hat{\mathbf{n}})+h_\times^2(t)F_\times^2(\hat{\mathbf{n}}). \tag{5.67}$$

The time averaged ensemble is

$$\langle h^2(t)\rangle_t = \langle F_+^2h_+^2 + F_\times^2h_\times^2 + 2F_+F_\times h_+h_\times\rangle_t. \tag{5.68}$$

Next when we take the angular ensemble average we get

$$\int \frac{\mathrm{d}^2\hat{\mathbf{n}}}{4\pi} \int \frac{\mathrm{d}\psi}{2\pi} \langle h^2(t) \rangle_t = \int \frac{\mathrm{d}^2\hat{\mathbf{n}}}{4\pi} \int \frac{\mathrm{d}\psi}{2\pi} \left(F_+^2 \langle h_+^2 \rangle_t + F_\times^2 \langle h_\times^2 \rangle_t + 2F_+F_\times \langle h_+ h_\times \rangle_t \right) \tag{5.69}$$

$$= \int \frac{\mathrm{d}^2\hat{\mathbf{n}}}{4\pi} \int \frac{\mathrm{d}\psi}{2\pi} \left(F_+^2 \langle h_+^2 \rangle_t + F_\times^2 \langle h_\times^2 \rangle_t \right) \tag{5.70}$$

$$= \int \frac{\mathrm{d}^2\hat{\mathbf{n}}}{4\pi} \int \frac{\mathrm{d}\psi}{2\pi} F_+^2 \langle h_+^2 + h_\times^2 \rangle_t, \tag{5.71}$$

where we used the fact that the product of pattern functions belonging to different polarizations vanishes over this ensemble average as well as the fact that the average over all angles is the same for both pattern functions. Next we recognize that $\langle h^2 \rangle_t$ is isotropic and thus has no dependence on the angles which means

$$\int \frac{\mathrm{d}^2\hat{\mathbf{n}}}{4\pi} \int \frac{\mathrm{d}\psi}{2\pi} \langle h^2(t) \rangle_t = \langle h^2(t) \rangle_t \int \frac{\mathrm{d}^2\hat{\mathbf{n}}}{4\pi} \int \frac{\mathrm{d}\psi}{2\pi} = \langle h^2 \rangle_t. \tag{5.72}$$

We next can see that

$$h^{ij}h_{ij} = 2(h_+^2 + h_\times^2) \Rightarrow \langle h^{ij}h_{ij} \rangle_t = 2\langle h_+^2 + h_\times^2 \rangle_t = 4\int_0^\infty S_h(f)\,\mathrm{d}f \Rightarrow \langle h_+^2 + h_\times^2 \rangle_t = 2\int_0^\infty S_h(f)\,\mathrm{d}f\,. \tag{5.73}$$

The equal-time correlation function for the signal is thus

$$\langle h^2(t) \rangle_t = \int \frac{\mathrm{d}^2\hat{\mathbf{n}}}{4\pi} \int \frac{\mathrm{d}\psi}{2\pi} F_+^2 \left(2\int_0^\infty S_h(f)\,\mathrm{d}f \right) = 2\langle F_+^2 \rangle_{(\theta,\phi,\psi)} \int_0^\infty S_h(f)\,\mathrm{d}f = F\int_0^\infty S_h(f)\,\mathrm{d}f\,. \tag{5.74}$$

As a result, the correlation function for the strain amplitude $s(t) = h(t) + n(t)$ is thus

$$\langle s^2(t) \rangle_t = \langle h^2(t) \rangle_t + 2\langle h(t)n(t) \rangle_t + \langle n^2(t) \rangle_t. \tag{5.75}$$

The middle term can be ignored because given observation time T, the ensemble average

$$\frac{1}{T}\int_0^T h(t)n(t)\,\mathrm{d}t \sim \sqrt{\frac{\tau_0}{T}} h_0 n_0, \tag{5.76}$$

where τ_0 is a characteristic time, h_0 is a characteristic amplitude for the signal $h(t)$, and n_0 is a characteristic amplitude for the noise. It is evident that not only does this term vanish for long time-spans, but we only need $h_0 > \sqrt{\frac{\tau_0}{T}} n_0$ for our approximations. Now the strain amplitude is

$$\langle s^2(t)\rangle_t = F\int_0^\infty S_h(f)\,\mathrm{d}f + \int_0^\infty S_n(f)\,\mathrm{d}f = \int_0^\infty [FS_h(f) + S_n(f)]\,\mathrm{d}f\,. \tag{5.77}$$

Thus the correlation function for the strain amplitude in the presence of the stochastic background is slightly elevated than one would expect if the strain was due to pure noise. This is how we'll be able to detect the background. We can also make more direct comparisons of the signal to noise in a particular bin. Discretizing the integral, we can write

$$\int S_h(f)\,\mathrm{d}f \to \sum_i S_h(f_i)\Delta f, \qquad \int S_n(f)\,\mathrm{d}f \to \sum_i S_n(f_i)\Delta f. \tag{5.78}$$

Meaning the (square of the) signal-to-noise ratio in a particular frequency bin is

$$\left(\frac{S}{N}\right)^2 = \frac{FS_h(f_i)\Delta f}{S_n(f_i)\Delta f} = F\frac{S_h(f_i)}{S_n(f_i)}. \tag{5.79}$$

We can finally conclude that the minimum of $S_h(f)$ that is measurable by a single detector with a noise power spectrum $S_n(f)$ at a given S/N level is

$$[S_h(f)]_{min} = S_n(f)\frac{(S/N)^2}{F} \Rightarrow [\Omega_{GW}(f)]_{min} = \frac{4\pi^2}{3H_0^2}f^3 S_n(f)\frac{(S/N)^2}{F}. \tag{5.80}$$

We can make note of the following astounding fact: due to the form of the fractional energy density for gravitational waves, when provided a particular noise level for $S_n(f)$, at low frequencies we are able to squeeze out a much better sensitivity in Ω_{GW} relative to the comparably tiny variations in the power spectrum for the noise. This can be seen by the following: LISA will detect frequencies that go all the way down to $f \sim 10^{-3}\,\mathrm{Hz}$ for a strain amplitude of $S_n^{1/2}(f) \sim 4\times 10^{-21}\,\mathrm{Hz}^{-1/2}$. On the other hand a ground based detector such as LIGO or VIRGO can detect strain amplitudes of $4\times 10^{-23}\,\mathrm{Hz}^{-1/2}$ at much higher frequencies of $f \sim 10^2\,\mathrm{Hz}$. While this represents a drop off of

$$\frac{S_{n,1G}^{1/2}(f)}{S_{n,LISA}^{1/2}(f)} \sim 10^2 \Rightarrow \frac{S_{n,1G}(f)}{S_{n,LISA}(f)} \sim 10^4, \tag{5.81}$$

so a relative loss of amplitude of $10^4\,\mathrm{Hz}^{-1}$. Compare this to the cubic frequency in the definition of $\Omega_{GW}(f)$

$$\left(\frac{f_{high}}{f_{low}}\right)^3 \sim \left(\frac{10^2}{10^{-3}}\right)^3 = 10^{15}. \tag{5.82}$$

Meaning it is much easier to reach a lower level for $[\Omega_{GW}(f)]_{min}$ at low frequencies than it is for higher frequencies.

5.3 Anisotropies

Ultimately we are interested in studying anisotropies in the Stochastic Background. To this end we define the object

$$\Omega_{GW}(f, \hat{\mathbf{n}}) = \frac{1}{\rho_c}\frac{\partial^2 \rho_{GW}}{\partial(\log f)\partial\hat{\mathbf{n}}}, \tag{5.83}$$

such that we have

$$\overline{\Omega}_{GW}(f) = \int \frac{\mathrm{d}^2\hat{\mathbf{n}}}{4\pi}\Omega_{GW}(f, \hat{\mathbf{n}}). \tag{5.84}$$

From here, with a clear analogy to the CMB, we can define an overdensity field

$$\delta_{GW}(f, \hat{\mathbf{n}}) = \frac{\Omega_{GW}(f, \hat{\mathbf{n}}) - \overline{\Omega}_{GW}(f)}{\overline{\Omega}_{GW}(f)}. \tag{5.85}$$

Now the overdensity field can be decomposed into three distinct parts

$$\delta_{GW} \simeq \delta^s_{GW} + \delta^{los}_{GW} + \mathcal{D}\hat{\mathbf{n}} \cdot \hat{v}, \tag{5.86}$$

where δ^s_{GW} is the anisotropy that is induced from the various astrophysical backgrounds, δ^{los}_{GW} is an isotropy resulting from the accumulated line-of-sight effects, and $\mathcal{D}\hat{\mathbf{n}}\cdot\hat{v}$ is the dipole term with magnitude $\mathcal{D}$ induced by the peculiar velocity of the observer. Now δ^s_{GW} is the anisotropy that we shall constrain ourselves to because it accounts for $\sim 90\%$ of the anisotropic signal. Assuming the overdensity field can also be characterized by a Gaussian random field, then we only need to worry about the correlation function

$$C_{GW}(f, \cos\theta) = \langle \delta^s_{GW}(f, \hat{\mathbf{n}})\delta^s_{GW}(f, \hat{\mathbf{n}}')\rangle, \tag{5.87}$$

which can be expanded in terms of Legendre polynomials in the usual way

$$C_{GW}(f, \cos\theta) = \sum_{\ell=0}^{\infty} \frac{2\ell+1}{4\pi} C_\ell^\delta(f) P_\ell(\cos\theta). \tag{5.88}$$

Inverting this equation to get the angular power spectrum gives us

$$C_\ell^\delta(f) = 2\pi \int_{-1}^{1} C_{GW}(f, \cos\theta) P_\ell(\cos\theta)\, \mathrm{d}\cos\theta\,. \tag{5.89}$$

Much like in the case of the CMB, the quantity $\frac{\ell(\ell+1)C_\ell}{2\pi}$ is measured and subsequently plotted. This is essentially the contribution to the power spectrum per log bin to the variance of δ^s which quantifies C_ℓ^δ.

Appendices

A Degrees of Freedom Overview

We are interested in the degrees of freedom for a given Lagrangian because in field theory, degrees of freedom correspond to a particle i.e. the force carrier for the field. A degree of freedom, broadly speaking, is an independent function (in our case) of spacetime coordinates. First we consider a scalar field $f : \mathbb{R}^4 \to \mathbb{R}$ usually denoted as $f(t, \mathbf{r})$. We say that f only carries one degree of freedom because the only independent function it carries is itself. Next we have a 4-vector field $V : \mathbb{R}^4 \to \mathbb{R}^4$ denoted by

$$V_\mu = (V_0, V_i). \tag{.1}$$

Since V_0 is a scalar field, it carries a single degree of freedom like f. However the 3-vector field V_i is different because it is an array of three scalar fields and thus it carries three degrees of freedom. We can further decompose this vector field by using the following theorem from linear algebra:

Theorem A.1 *Let X, Y, Z be vector spaces, and T:X $\to$ Y, U:Y $\to$ Z be linear. If UT:X $\to$ Z is invertible, then* $Y = \ker(U) \oplus \mathrm{Im}(T)$.

Taking Y to be the space of all vector fields $\mathcal{V}$, we can decompose it into two subspaces: the kernel of the divergence operator i.e. $\partial_i V^i = 0$ and the image of the gradient operator i.e. $\partial_i v$. Thus, any function[2] in $\mathcal{V}$ can be represented as

$$V_i = V_i^T + \partial_i v. \tag{.2}$$

Since v is a scalar function, it propagates only a single degree of freedom. Naively, since V_i^T is a 3 component object, we would say it has 3 degrees of freedom. However, since we can "solve" for one of the components (and provided the fields go to zero at infinity) we can see that

[2]Note that this theorem is merely a generalization of Helmholtz' theorem which states that any vector field (sufficiently smooth) can be written as the sum of a divergence-less part and a curl-less part.

$$\partial_x V_x^T + \partial_y V_y^T + \partial_z V_z^T = 0 \tag{.3}$$

$$\Rightarrow V_z^T(x,y,z) = -\int_{-\infty}^{z} \partial_x V_x^T(x,y,z') + \partial_y V_y^T(x,y,z')\mathrm{d}z'. \tag{.4}$$

Thus given some initial data, V_z^T is completely determined by the components V_x^T and V_y^T which implies that V_i^T has only two independent functions i.e. two degrees of freedom.

Lastly, we move on to discussing tensors. A tensor T_{ij} is an object that maps elements of a vector space to a basis. Generically, a 3x3 tensor has 9 components. However for this discussion, the tensors we will most be interested in are symmetric i.e. $T_{ij} = T_{ji}$.

The decomposition for a (spatial) tensor is slightly different than that of a vector. Symmetric tensors can be split into the image of the map taking functions to their traces, $T \to \frac{1}{3}T\delta_{ij}$, the space of transverse traceless tensors i.e. $\partial^i T_{ij} = 0, T^i_{\ i} = 0$, and the image of V under the map $V_i \to \partial_i V_j + \partial_j V_i - \frac{2}{3}(\partial^k V_k)\delta_{ij}$. This is another application of the above theorem where X is the space of vectors, Y is the space of traceless symmetric tensors, and Z is again the space of vectors, T maps $V_i \to \partial_i V_j + \partial_j V_i - \frac{2}{3}(\partial^k V_k)\delta_{ij}$ and U maps tensors to their divergence. Putting this all together, while also keeping plugging in the decomposition for vectors as well, any symmetric tensor can be written as

$$T_{ij} = T_{ij}^{TT} + \partial_i V_j^T + \partial_j V_i^T + 2\bigg(\partial_i\partial_j \upsilon - \frac{1}{3}\nabla^2 \upsilon \delta_{ij}\bigg) + \frac{1}{3}T\delta_{ij}, \tag{.5}$$

where we have the following constraints/conditions

$$\partial^i T_{ij}^{TT} = 0, \quad T^{TT\,i}_{\ \ \ \ i} = 0, \quad \partial_i V_T^i = 0, \quad T = \delta^{ij}T_{ij}. \tag{.6}$$

And now we ask how many degrees of freedom does T_{ij}^{TT} propagate? Since it's symmetric that means it has at most 6 independent components. Once we take into account its traceless-ness, that kills off an additional degree of freedom, so it can only have at most 5. Lastly, once we incorporate the fact that T_{ij}^{TT} is divergence-less, we find another three degrees of freedom are killed off and thus we can conclude T_{ij}^{TT} only propagates two degrees of freedom. For clarity's sake, the existence of a degree of freedom indicates a particle for that field, but the number of degrees of freedom for a particular field corresponds to the number of polarization modes.

B Gauge Transformations

Here we give an exhaustive list of all the gauge transformations for the components of the metric perturbation with gauge parameter A_μ. Firstly, how does the metric perturbation transform under the action of a gauge? Well we can see that $h_{\mu\nu}$ transforms as

$$h_{\mu\nu} \to h_{\mu\nu} - \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{.1}$$

Now that's all well and good, but how do the individual components themselves transform? First we should decompose the gauge parameter as outlined in the degree of freedom section of this document. We write

$$A_\mu = (A_0, A_i^T + \partial_i \alpha). \tag{.2}$$

Now we can plug into the transformation law from above and we get the following transformation rules for each component:

$$\Phi \to \Phi + \dot{A}_0, \tag{.3}$$

$$w_i^T \to w_i^T - \dot{A}_i^T, \tag{.4}$$

$$v_i^T \to v_i^T - A_i^T, \tag{.5}$$

$$\Omega \to \Omega - A_0 + \dot{\alpha}, \tag{.6}$$

$$\bar{h} \to \bar{h} - \nabla^2 \alpha, \tag{.7}$$

$$\Psi \to \Psi - \alpha, \tag{.8}$$

where $\bar{h} = \mathrm{Tr}[h_{ij}]$. Since there are 4 scalar fields and 2 scalar parameters, we expect 4 - 2 gauge invariant scalar fields. Likewise, since there are 2 vector fields and 1 vector gauge parameter than we expect 2 - 1 gauge invariant vector fields. From the above transformation laws we can define the following gauge invariant fields

$$J \equiv -\Phi - \dot{\Omega} + \ddot{\Psi}, \tag{.9}$$

$$L \equiv \frac{1}{3}(\bar{h} - 2\nabla^2\Psi), \tag{.10}$$

$$M_i \equiv w_i^T - \dot{v}_i^T. \tag{.11}$$

C Linearized General Relativity

We start off with the Lagrangian for quadratic General Relativity given by

$$\mathcal{L} = \partial_\lambda h_{\mu\nu}\partial^\mu h^{\lambda\nu} + \frac{1}{2}\partial_\mu h\partial^\mu h - \frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} - \partial_\mu h^{\mu\nu}\partial_\nu h, \tag{.1}$$

We use this Lagrangian because plugging this Lagrangian into the Euler-Lagrange Equations yields the equations of motion for linearized Generalized Relativity. Since $h_{\mu\nu}$ is a symmetric (0,2) tensor under spatial rotations, the 00 component is a scalar, the 0i component forms a 3-vector, and the ij component forms a symmetric spatial tensor. This allows us to decompose the metric perturbation into it's constituent parts. Now we write $h_{\mu\nu}$ as

$$\begin{aligned} &h_{00} = h^{00} = -2\Phi, \\ &h_{0i} = -h_i^0 = w_i, \\ &h = h_\mu^\mu = \eta^{\mu\nu}h_{\mu\nu} = 2\Phi + \bar{h}, \end{aligned} \tag{.2}$$

where $\bar{h} = \mathrm{Tr}[h_{ij}]$. Plugging these expressions in while simplifying our Lagrangian immensely gives us

$$\begin{aligned} \mathcal{L} = &-2\partial_i w_j \dot{h}^{ij} - \partial_i w_j \partial^j w^i + \partial_i h_{jk}\partial^j h^{ik} - \frac{1}{2}\dot{\bar{h}}^2 + 2\partial_i\Phi\partial^i\bar{h} + \frac{1}{2}(\partial_i\bar{h})^2 \\ &+ \frac{1}{2}(\dot{h}_{ij})^2 + (\partial_i w_j)^2 - \frac{1}{2}(\partial_i h_{jk})^2 - 2w^i\partial_i\dot{\bar{h}} - 2\partial_i h^{ij}\partial_j\Phi - \partial_i h^{ij}\partial_j\bar{h}, \end{aligned} \tag{.3}$$

Under the action, equation (3) takes on the form

$$\begin{aligned} S = \int &2w^i\partial_j\dot{h}^{ij} + w_i(\partial^i\partial_k w^k - \nabla^2 w^i) + \partial^j h_{jk}\partial_i h^{ik} + \frac{1}{2}\bar{h}\Box\bar{h} - 2\Phi\nabla^2\bar{h} \\ &+ \frac{1}{2}h_{ij}\Box h^{ij} - 2w^i\partial_i\dot{\bar{h}} + 2\Phi\partial_i\partial_j h^{ij} + \bar{h}\partial_i\partial_j h^{ij}\,\mathrm{d}^4x\,, \end{aligned} \tag{.4}$$

where $\Box = \partial_\mu \partial^\mu$ and $\nabla^2 = \partial_i \partial^i$. Next we perform the following decomposition for the spatial tensor h_{ij} and the vector field w_i:

$$
\begin{aligned}
h_{ij} &= h_{ij}^{TT} + \partial_i v_j^T + \partial_j v_i^T + 2\left(\partial_i \partial_j \Psi - \frac{1}{3}\nabla^2 \Psi \delta_{ij}\right) + \frac{1}{3}\bar{h}\delta_{ij}, \\
w_i &= w_i^T + \partial_i \Omega, \\
\partial^i \partial^i h_{ij}^{TT} = \delta^{ij} h_{ij}^{TT} &= \partial^i v_i^T = \partial^i w_i^T = 0,
\end{aligned}
\tag{.5}
$$

where δ_{ij} is the identity matrix. We can streamline the calculation a bit by recognizing that we can treat the spin 0, 1, and 2 terms separately (i.e. we can assume there are no cross terms between differing spins). From this we can split the action into three different sectors:

$$
S = S_T + S_V + S_S, \tag{.6}
$$

where

$$
S_T = \int -\frac{1}{2} h_{TT}^{ij} \ddot{h}_{ij}^{TT} + \frac{1}{2} h_{TT}^{ij} \nabla^2 h_{ij}^{TT} \, \mathrm{d}^4 x \, , \tag{.7}
$$

$$
\begin{aligned}
S_V &= \int 2 w_i^T \nabla^2 \dot{v}_T^i - w_i^T \nabla^2 w_T^i + \nabla^2 v_T^i \left(\nabla^2 v_i^T - \left(-\frac{\partial^2}{\partial t^2} + \nabla^2\right) v_i^T\right) \mathrm{d}^4 x, \\
&= \int (\partial_i w_j^T - \partial_i \dot{v}_j^T)^2 \mathrm{d}^4 x,
\end{aligned}
\tag{.8}
$$

$$
\begin{aligned}
S_S = \int & 2\Omega \nabla^2 \dot{\bar{h}} - \frac{8}{3}\Omega \nabla^4 \dot{\Psi} - \frac{2}{3}\Omega \nabla^2 \dot{\bar{h}} - \frac{16}{9}\nabla^4 \Psi \nabla^2 \Psi - \frac{8}{9}\nabla^2 \Psi \nabla^2 \bar{h} \\
& -\frac{1}{9}\bar{h}\nabla^2 \bar{h} - \frac{1}{2}\bar{h}\Box \bar{h} + \frac{4}{3}\nabla^2 \Psi \Box \nabla^2 \Psi + \frac{1}{6}\bar{h}\Box \bar{h} - 2\Phi \nabla^2 \bar{h} \\
& + \frac{2}{3}\Phi \nabla^2 \bar{h} + \frac{8}{3}\Phi \nabla^4 \Psi + \frac{4}{3}\bar{h}\nabla^4 \Psi + \frac{1}{3}\bar{h}\nabla^2 \bar{h} \mathrm{d}^4 x.
\end{aligned}
\tag{.9}
$$

Defining the gauge-invariant fields $J \equiv -\Phi - \dot{\Omega} + \ddot{\Psi}$, $L \equiv \frac{2}{3}(\bar{h} - 2\nabla^2 \Psi)$, and $M_i = w_i^T - \dot{v}_i^T$, S_S and S_V take on the forms

$$
S_V = \int \frac{1}{2} (\partial_i M_j)^2 \mathrm{d}^4 x, \tag{.10}
$$

$$
S_S = \int 2 J \nabla^2 L - \frac{1}{4} L \nabla^2 L + \frac{1}{2} L \ddot{L} \, \mathrm{d}^4 x \, , \tag{.11}
$$

We can now analyze the true degrees of freedom that are present in $h_{\mu\nu}$. First, looking at the vector action we can see that no time derivatives of M_i are present in the action. Therefore it is an auxiliary field and we may use its equations of motion (EOM) to eliminate it. Proceeding accordingly we find

$$\frac{\delta \mathcal{L}}{\delta M^i} = \nabla^2 M_i = 0 \Rightarrow M_i = 0, \tag{.12}$$

which implies that $S_V = 0$. Next we turn our attention to the scalar action. Since J appears linearly with no time derivatives, we may interpret it as a Lagrange multiplier. From there we can see that the EOM of J enforces the following constraint:

$$\frac{\delta \mathcal{L}}{\delta J} = \nabla^2 L = 0 \Rightarrow L = 0, \tag{.13}$$

and therefore, $S_S = 0$. The total action is now

$$S = S_T = \int \frac{1}{2}(\partial_\mu h_{ij}^{TT})^2 \, \mathrm{d}^4 x \tag{.14}$$

which yield to the equations of motion 2.27. Since we've finally eliminated all of the purely gauge fields we're left with (15). Since h_{ij}^{TT} carries 2 independent modes, we can finally conclude our analysis that linearized General Relativity carries with it a maximum of two degrees of freedom.