Counting the Degrees of Freedom in Massive Electromagnetism

Marcell Howard

March 13, 2021

We start with the Lagrangian for massive electromagnetism in flat space

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu \tag{1}$$

where $F_{\mu\nu}$ is the electromagnetic field strength tensor, $F^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}F_{\alpha\beta}$, m is the mass of the photon, and A_μ is the 4-potential. Next we write $F_{\mu\nu}$ in terms of the 4-potential

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2}$$

Next we write the 4-potential as

$$A_\mu = (A_0, A_i). \tag{3}$$

And now from the massless photon calculation, we can jump straight to the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_0 A_i - \partial_i A_0)^2 - \frac{1}{2}\partial_i A_j(\partial^i A^j - \partial^j A^i) - \frac{1}{2}m^2 A_0 A^0 - \frac{1}{2}m^2 A_i A^i. \tag{4}$$

Plugging (4) into the action S gives us

$$S = \int \frac{1}{2}(\partial_0 A_i - \partial_i A_0)^2 - \frac{1}{2}\partial_i A_j(\partial^i A^j - \partial^j A^i) - \frac{1}{2}m^2 A_0 A^0 - \frac{1}{2}m^2 A_i A^i \; d^4x. \tag{5}$$

Taking advantage of the result we calculated in the case of the massless photon while integrating certain terms out we get

$$S = \int \frac{1}{2}(\dot{A_i^T} + \partial_i \dot{\alpha} - \partial_i A_0)^2 + \frac{1}{2}A_T^i \nabla^2 A_i^T + \frac{1}{2}m^2 A_0^2 - \frac{1}{2}m^2(A_i^T + \partial_i \alpha)^2 \; d^4x \tag{6}$$

From (6) we observe that there are no $\dot{A}$ terms and thus we conclude A_0 is an auxiliary field. Meaning we can use it's equations of motions to eliminate it from the action. With this in mind, we're prepared to plug in the Lagrangian into the Euler-Lagrange equation

$$\frac{\delta \mathcal{L}}{\delta A_0} = \nabla^2(\dot{\alpha} - A_0) - m^2 A_0 = 0 \;\Rightarrow\; A_0 = D\dot{\alpha} \quad D \equiv \frac{\nabla^2}{\nabla^2 + m^2} \tag{7}$$

Plugging this equation into the action gives us

$$S = \int \frac{1}{2}[A_T^i(\Box - m^2)A_i^T - (\dot{\alpha}\nabla^2 + \nabla^2\dot{\alpha}\; D + \dot{\alpha}\nabla^2 D)\dot{\alpha} - m^2(D\dot{\alpha})^2 - m^2\partial_i\alpha\partial^i\alpha] \; d^4x \tag{8}$$

Where $\Box = \partial_\mu \partial^\mu$ is the d'Alembert operator. From here we can no longer identify anymore auxiliary fields and thus we conclude that for massive E+M, the 4-potential carries 3 degrees of freedom: two for A_i^T and one for α.