Canonical Transformations and Hamilton-Jacobi Theory

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Before we can start this paper, let us define a few concepts

Definition 0.1. *A canonical variable q_i is called* ***cyclic*** *if its canonical momentum is constant i.e. for some Hamiltonian, $H(q_1, ..., q_n, p_1, ..., p_n, t)$ we have*

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} = 0. \tag{1}$$

Cyclic variables are nice because they make solving dynamical systems easy. Whenever a variable, φ is cyclic then

$$\dot{\varphi} = \frac{\partial H}{\partial p_\varphi} \Rightarrow \varphi = \left(\frac{\partial H}{\partial p_\varphi}\right)t + \varphi_0, \tag{2}$$

since p_φ is a constant with respect to time which implies $\frac{\partial H}{\partial p_\varphi}$ will be independent of φ and t. Since cyclic variables makes solving the Hamiltonian equations much simpler, it would be convenient to find a set of variables where they are all cyclic. First we must define the concept of a canonical transformation

Definition 0.2. *A mapping $(q_i, p_i) \mapsto (Q_i, P_i)$ is called a* ***canonical transformation*** *if $\{Q_i, P_j\} = \delta_{ij}$.*

In general, these new canonical variables

$$Q_i = Q_i(\mathbf{q}, \mathbf{p}, t). \quad P_i = P_i(\mathbf{q}, \mathbf{p}, t), \quad H(\mathbf{q}, \mathbf{p}, t) \mapsto \tilde{H}(\mathbf{Q}, \mathbf{P}, t), \tag{3}$$

where $\mathbf{q} = (q_1, .., q_n)$. We demand that our new canonical coordinates satisfy their own Hamilton equations

$$\dot{Q}_i = \frac{\partial \tilde{H}}{\partial P_i}, \quad \dot{P}_i = -\frac{\partial \tilde{H}}{\partial Q_i}. \tag{4}$$

Now both sets of canonical variables and Hamiltonians need to satisfy the least action principle

$$\delta \int_{t_1}^{t_2} \mathrm{d}t \left[\sum_{i=1}^{n} \dot{q}_i p_i - H \right] = 0, \quad \delta \int_{t_1}^{t_2} \mathrm{d}t \left[\sum_{i=1}^{n} \dot{Q}_i P_i - \tilde{H} \right] = 0, \tag{5}$$

where the variations in each coordinate must vanish (as according to the least action principle). Since these two equations satisfy the least action principle on the same interval, that implies that they must be exactly the same up to a total derivative on a function that vanishes at both endpoints $\delta F(t_2) - \delta F(t_1) = 0$. Thus, we have

$$\sum_{i=1}^{n} \dot{Q}_i P_i - \tilde{H} = \sum_{i=1}^{n} \dot{q}_i p_i - H - \frac{\mathrm{d}F}{\mathrm{d}t}. \tag{6}$$

Suppose $F = F(\mathbf{q}, \mathbf{Q}, t)$. Then the total time derivative on F is

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\partial F}{\partial q_i}\dot{q}_i + \frac{\partial F}{\partial Q_i}\dot{Q}_i + \frac{\partial F}{\partial t}. \tag{7}$$

We could've done the same thing if we instead chose $F = F(\mathbf{q}, \mathbf{P}, t)$. It is simply the standard pick to go with what we chose. When plugging in the above condition, we can turn into

$$0 = \sum_{i=1}^{n} \dot{q}_i \left(\frac{\partial F}{\partial q_i} - p_i \right) + \sum_{i=1}^{n} \dot{Q}_i \left(\frac{\partial F}{\partial Q_i} + P_i \right) + H - \tilde{H} + \frac{\partial F}{\partial t}. \tag{8}$$

In order for the right-hand side of the equation to be zero, we require the following constraints:

$$p_i = \frac{\partial F}{\partial q_i}, \quad P_i = -\frac{\partial F}{\partial Q_i}, \quad \tilde{H} = H + \frac{\partial F}{\partial t}. \tag{9}$$

This motivates our third definition

Definition 0.3. *A function, $F(\mathbf{q}, \mathbf{Q}, t)$ is called a* ***generating function*** *if it satisfies*

$$p_i = \frac{\partial F}{\partial q_i}, \quad P_i = -\frac{\partial F}{\partial Q_i}, \quad \tilde{H} = H + \frac{\partial F}{\partial t} = 0. \tag{10}$$

The generating function guarantees that at least one of our new variables will be cyclic. That's great and all, but what if we can do better? What if we could find a set of coordinates where *all* of our (old) canonical variables are cyclic? To find such variables, we suppose there exists a new function $S(\mathbf{q}, \mathbf{P}, t)$ that is related to the original generating function via a Legendre transformation

$$F(\mathbf{q}, \mathbf{Q}, t) = S(\mathbf{q}, \mathbf{P}, t) - \sum_{i=1}^{n} Q_i P_i. \tag{11}$$

We can find how this new generating function relates to the old and new coordinates by considering the total derivative on F

$$\begin{aligned}
\mathrm{d}F &= \frac{\partial F}{\partial q_i}\,\mathrm{d}q_i + \frac{\partial F}{\partial Q_i}\,\mathrm{d}Q_i + \frac{\partial F}{\partial t}\,\mathrm{d}t \qquad (12)\\
&= p_i\,\mathrm{d}q_i - P_i\,\mathrm{d}Q_i + \frac{\partial F}{\partial t}\,\mathrm{d}t \qquad (13)\\
&= \mathrm{d}S - P_i\,\mathrm{d}Q_i - Q_i\,\mathrm{d}P_i\,. \qquad (14)
\end{aligned}$$

Solving for $\mathrm{d}S$ gives us

$$\mathrm{d}S = p_i\,\mathrm{d}q_i + Q_i\,\mathrm{d}P_i + \frac{\partial F}{\partial t}\,\mathrm{d}t\,. \tag{15}$$

Now we have something akin to a thermodynamic state function. Because of that, we have the following constraints on the new generating function

$$\frac{\partial S}{\partial q_i} = p_i, \quad \frac{\partial S}{\partial P_i} = Q_i, \quad \frac{\partial S}{\partial t} = \frac{\partial F}{\partial t}. \tag{16}$$

That relation yields the following equation

$$H(q_1, ..., q_n, \partial_{q_1} S, ..., \partial_{q_n} S, t) + \frac{\partial S}{\partial t} = 0. \tag{17}$$

This is the famous Hamilton-Jacobi Equation.