

Brans-Dicke Theory

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1 Equations of Motion

Here we are interested in deriving the Brans-Dicke equations of motion. First we start off with the Lagrangian for Brans-Dicke

$$\mathcal{L} = \sqrt{-g}(\phi R - \frac{\omega}{\phi} \nabla_\mu \phi \nabla^\mu \phi + \mathcal{L}_m). \quad (1)$$

Where ϕ is a scalar field, $R = R^\mu{}_\mu = g^{\mu\nu} R_{\mu\nu}$ is the Ricci scalar, ω is a dimensionless parameter and \mathcal{L}_m is the Lagrangian for matter. Next we place the Lagrangian in the action.

$$S = \int \sqrt{-g}(\phi R - \frac{\omega}{\phi} \nabla_\mu \phi \nabla^\mu \phi + \mathcal{L}_m) \, d^4x. \quad (2)$$

We will soon vary with respect to the inverse metric. But first we recognize that $R = g^{\mu\nu} R_{\mu\nu}$. So we then have

$$S = \int \sqrt{-g}(\phi g^{\mu\nu} R_{\mu\nu} - \frac{\omega}{\phi} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi + \mathcal{L}_m) \, d^4x. \quad (3)$$

Now we shall vary the action with respect to the inverse metric

$$\delta S = \delta S_{\phi R} + \delta S_\phi + \delta S_M \quad (4)$$

where

$$\begin{aligned}
\delta S_{\phi R} &= \int (\phi R_{\mu\nu} \delta g^{\mu\nu} + \phi g^{\mu\nu} \delta R_{\mu\nu}) \sqrt{-g} + \phi R \delta \sqrt{-g} \, d^4x \\
\delta S_\phi &= \int -\frac{\omega}{\phi} \nabla_\mu \phi \nabla_\nu \phi \delta g^{\mu\nu} \sqrt{-g} - \frac{\omega}{\phi} \nabla_\mu \phi \nabla^\mu \phi \delta \sqrt{-g} \, d^4x
\end{aligned} \tag{5}$$

and S_M is the action for matter. The second term in $S_{\phi R}$ can be found in Carroll's book.

Using the result from there we find $\delta S_{\phi R}$ takes the form

$$\begin{aligned}
\delta S_{\phi R} &= \int (\phi R_{\mu\nu} \delta g^{\mu\nu} + \nabla_\rho [g_{\mu\nu} \nabla^\rho \phi \delta g^{\mu\nu} - \nabla_\lambda \phi \delta g^{\rho\lambda}] \sqrt{-g} - \frac{1}{2} \phi R \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu}) \, d^4x \\
&= \int (\phi R_{\mu\nu} - \frac{1}{2} \phi R g_{\mu\nu} - [\nabla_\mu \nabla_\nu \phi - g_{\mu\nu} \nabla^2 \phi]) \sqrt{-g} \delta g^{\mu\nu} \, d^4x.
\end{aligned} \tag{6}$$

Looking now to δS_ϕ , the action becomes

$$\delta S_\phi = \int -\frac{\omega}{\phi} (\nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} (\nabla \phi)^2) \sqrt{-g} \delta g^{\mu\nu} \, d^4x, \tag{7}$$

with $\nabla_\mu \phi \nabla^\mu \phi = (\nabla \phi)^2$. Recall that the functional derivative of the action satisfies

$$\delta S = \int \sum_i \left(\frac{\delta S}{\delta \Psi^i} \delta \Psi^i \right) \, d^n x, \tag{8}$$

where Ψ^i is a complete set of fields being varied. This brings the total action δS to be

$$\begin{aligned}
\frac{1}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} &= \phi (R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}) - (\nabla_\mu \nabla_\nu \phi - g_{\mu\nu} \nabla^2 \phi) \\
-\frac{\omega}{\phi} (\nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} (\nabla \phi)^2) &+ \frac{1}{2\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}} = 0.
\end{aligned} \tag{9}$$

Defining the energy momentum tensor to be

$$T_{\mu\nu} = -\frac{1}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}} \tag{10}$$

Moving the last terms to the other side and dividing both sides by ϕ , we get

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{1}{2\phi} T_{\mu\nu} + \frac{1}{\phi} (\nabla_\mu \nabla_\nu \phi - g_{\mu\nu} \nabla^2 \phi) + \frac{\omega}{\phi^2} (\nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} (\nabla \phi)^2) \tag{11}$$

2 Degrees of Freedom

Brans-Dicke theory was one of the earliest competitors to GR. It is discussed today because it represent the prototypical example of a scalar tensor theory. The Lagrangian for the theory is given by

$$\mathcal{L} = \sqrt{-g}(\phi R - \frac{\omega}{\phi} \nabla_\mu \phi \nabla^\mu \phi). \quad (12)$$

We wish to expand all the terms to at most second order in ϕ , R , and $\sqrt{-g}$. This brings us to

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad (13)$$

$$\phi = \phi_o + \varphi, \quad (14)$$

$$\sqrt{-g} = 1 + \frac{1}{2}h + \frac{1}{8}h^2 - \frac{1}{4}h_{\mu\nu}h^{\mu\nu}, \quad (15)$$

$$R = R^{(0)} + R^{(1)} + R^{(2)}, \quad (16)$$

where η and ϕ_o is some flat space solution, $h_{\mu\nu}$ and φ are perturbations and taken to be much less than 1, $h = h^\mu{}_\mu = \eta^{\mu\nu}h_{\mu\nu}$ is the trace, and $R^{(0)} = 0$, $R^{(1)}$, $R^{(2)}$ are the zeroth, first, and second order curvature respectively. Keeping only the terms quadratic in both $h_{\mu\nu}$ and φ , the Lagrangian becomes

$$\mathcal{L} = \phi_o(\sqrt{-g}R)^{(2)} - \frac{\omega}{\phi_o} \partial_\mu \varphi \partial^\mu \varphi. \quad (17)$$

Lets split these Lagrangians off into different sectors in order to deal with them easily.

$$\mathcal{L}_{R^{(2)}} = \phi_o(\sqrt{-g}R)^{(2)}, \quad (18)$$

$$\mathcal{L}_\varphi = -\frac{\omega}{\phi_o} \partial_\mu \varphi \partial^\mu \varphi, \quad (19)$$

Since we've done the degree of freedom count for $\mathcal{L}_{R^{(2)}}$ (being multiplied by a constant background won't change any of the calculations) we only need to focus our attention on \mathcal{L}_φ . We next perform the following decompositions:

$$h_{00} = h^{00} = 2\Phi, \quad (20)$$

$$h_{0i} = -h_i^0 = w_i, \quad (21)$$

$$h = h^\mu{}_\mu = \eta^{\mu\nu} h_{\mu\nu} = -2\Phi + \bar{h}, \quad (22)$$

$$w_i = w_i^T + \partial_i \Lambda, \quad (23)$$

$$h_{ij} = h_{ij}^{TT} + \partial_i v_j^T + \partial_j v_i^T + 2(\partial_i \partial_j \Psi - \frac{1}{3} \nabla^2 \Psi \delta_{ij}) + \frac{1}{3} \bar{h} \delta_{ij}, \quad (24)$$

$$\partial^i h_{ij}^{TT} = \delta^{ij} h_{ij}^{TT} = \partial^i v_i^T = \partial^i w_i^T = 0, \quad (25)$$

where $\bar{h} = \text{Tr}[h_{ij}]$ and δ_{ij} is the identity matrix. Defining the following gauge invariant fields

$$J \equiv \Phi - \dot{\Lambda} + \ddot{\Psi}, \quad L \equiv \frac{1}{3}(\bar{h} - 2\nabla^2 \Psi), \quad M_i \equiv w_i^T - \dot{v}_i^T. \quad (26)$$

We're ready to start our analysis on the degrees of freedom in these actions. We can turn our gaze to $\mathcal{L}_{R^{(2)}}$ and using the results from the linearized GR calculation while also scaling by the constant background ϕ_o we get the following actions:

$$S_T = \int \frac{1}{2} h_{TT}^{ij} \square h_{ij}^{TT} \, d^4x, \quad (27)$$

$$S_V = \int \frac{1}{2} (\partial_i M_j)^2 \, d^4x, \quad (28)$$

$$S_S = \int 4J \nabla^2 L - L \nabla^2 L - 2\dot{L}^2 \, d^4x, \quad (29)$$

$$S_\varphi = \int \frac{\omega}{\phi_o} \varphi \square \varphi \, d^4x. \quad (30)$$

We can now analyze the true degrees of freedom that are present in $h_{\mu\nu}$. First, looking at the vector action we can see that no time derivatives of M_i are present in the action. Therefore it is an auxiliary field and we may use its equations of motion (EOM) to eliminate it. Proceeding accordingly we find

$$\frac{\delta \mathcal{L}}{\delta M^i} = \nabla^2 M_i = 0 \Rightarrow M_i = 0, \quad (31)$$

which implies that $S_V = 0$. Next we turn our attention to the scalar action. Since J appears linearly with no time derivatives, we may interpret it as a Lagrange multiplier. From there we can see that the EOM of J enforces the following constraint:

$$\frac{\delta \mathcal{L}}{\delta J} = \nabla^2 L = 0 \Rightarrow L = 0, \quad (32)$$

and therefore, $S_S = 0$. The total action is now $S = S_T + S_\varphi$ and thus we can say that the Lagrangian for linearized Brans-Dicke has in total, three degrees of freedom. two from h_{ij}^{TT} and one from φ .