Deriving The Brans-Dicke Equations of Motion

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We start off with the Lagrangian for Brans-Dicke

$$\mathcal{L} = \sqrt{-g}(\phi R - \frac{\omega}{\phi}\nabla_\mu\phi\nabla^\mu\phi + \mathcal{L}_m). \tag{1}$$

Where ϕ is a scalar field, $R = R^\mu{}_\mu = g^{\mu\nu}R_{\mu\nu}$ is the Ricci scalar, ω is a dimensionless parameter and $\mathcal{L}_m$ is the Lagrangian for matter. Next we place the Lagrangian in the action.

$$S = \int \sqrt{-g}(\phi R - \frac{\omega}{\phi}\nabla_\mu\phi\nabla^\mu\phi + \mathcal{L}_m)\ \mathrm{d}^{n+1}x. \tag{2}$$

We will soon vary with respect to the inverse metric. But first we recognize that $R = g^{\mu\nu}R_{\mu\nu}$. So we then have

$$S = \int \sqrt{-g}(\phi g^{\mu\nu}R_{\mu\nu} - \frac{\omega}{\phi}\nabla_\mu\phi\nabla^\mu\phi + \mathcal{L}_m)\ \mathrm{d}^{n+1}x. \tag{3}$$

Now we shall vary the action with respect to the inverse metric

$$\delta S = \delta S_{\phi R} + \delta S_\phi + \delta S_M \tag{4}$$

where

$$\begin{aligned}\delta S_{\phi R} &= \int (\phi R_{\mu\nu}\delta g^{\mu\nu} + \phi g^{\mu\nu}\delta R_{\mu\nu})\sqrt{-g} + \phi R\delta\sqrt{-g}\ \mathrm{d}^{n+1}x \\ \delta S_\phi &= \int -\frac{\omega}{\phi}\nabla_\mu\phi\nabla_\nu\phi\delta g^{\mu\nu}\sqrt{-g} - \frac{\omega}{\phi}\nabla_\mu\phi\nabla^\mu\phi\delta\sqrt{-g}\ \mathrm{d}^{n+1}x\end{aligned} \tag{5}$$

and S_M is the action for matter. The second term in $S_{\phi R}$ can be found in Carroll's book. Using the result from there we find $\delta S_{\phi R}$ takes the form

$$\begin{aligned}\delta S_{\phi R} &= \int (\phi R_{\mu\nu}\delta g^{\mu\nu} + \nabla_\sigma[g_{\mu\nu}\nabla^\sigma\phi - \nabla_\lambda\phi\delta g^{\sigma\lambda}]\sqrt{-g} - \frac{1}{2}\phi R\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu})\ d^{n+1}x \\ &= \int (\phi R_{\mu\nu} - \frac{1}{2}\phi R g_{\mu\nu} - [\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\nabla^2\phi])\sqrt{-g}\delta g^{\mu\nu}\ \mathrm{d}^{n+1}x.\end{aligned} \tag{6}$$

Looking now to δS_ϕ, the action becomes

$$\delta S_\phi = \int -\frac{\omega}{\phi}(\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}(\nabla\phi)^2)\sqrt{-g}\delta g^{\mu\nu}\ \mathrm{d}^{n+1}x. \tag{7}$$

Recall that the functional derivative of the action satisfies

$$\delta S = \int \sum_i (\frac{\delta S}{\delta \Phi^i}\delta\Phi^i)\ d^n x, \tag{8}$$

where Φ^i is a complete set of fields being varied. This brings the total action δS to be

$$\begin{aligned}\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} &= \phi(R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu}) - (\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\nabla^2\phi) \\ &\quad - \frac{\omega}{\phi}(\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu_\nu}(\nabla\phi)^2) + \frac{1}{2\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}} = 0.\end{aligned} \tag{9}$$

Defining the energy momentum tensor to be

$$T_{\mu\nu} = -\frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}. \tag{10}$$

Moving the last terms to the other side and dividing both sides by ϕ, we get

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = \frac{1}{2\phi}T_{\mu\nu} + \frac{1}{\phi}(\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\nabla^2\phi) + \frac{\omega}{\phi^2}(\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}(\nabla\phi)^2) \tag{11}$$