The Universal Wave Function

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August 23, 2023

Conventions We use the mostly plus metric signature, i.e. $\eta_{\mu\nu} = (-, +, +, +)$ and units where $c = \hbar = 1$. The reduced four dimensional Planck mass is $M_P = \frac{1}{\sqrt{8\pi G}} \approx 2.43 \times 10^{18}$ GeV. The d'Alembert and Laplace operators are defined to be $\Box = \partial_\mu \partial^\mu$ and $\nabla^2 = \partial_i \partial^i$ respectively. We use boldface letters $\mathbf{x}$ to indicate 3-vectors and we use x and p to denote 4-vectors. Conventions for the curvature tensors, covariant and Lie derivatives are all taken from Carroll [2].

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1 The Path Integral

Quantum mechanics (QM) represents one of our most experimentally tested and hence well-established theories in all of science. The breadth of phenomenon that it can describe covers large swaths of the world around us that to list of all them would likely be longer than the length of this document. When the theory was written down by Erwin Schrodinger, he wrote his famous Schrodinger Equation

$$i\frac{\partial}{\partial t}\left|\Psi\right\rangle = H\left|\Psi\right\rangle, \tag{1.1}$$

where $|\Psi\rangle$ is a state vector that we refer to as the wave function which lives in some Hilbert space and H is some operator that we call the Hamiltonian. The coordinate representation (the one that is taught to us when we first learn quantum mechanics) is simply

$$i\frac{\partial\Psi(\mathbf{x},t)}{\partial t} = -\frac{1}{2m}\nabla^2\Psi(\mathbf{x},t) + V(\mathbf{x})\Psi(\mathbf{x},t), \tag{1.2}$$

and $V(\mathbf{x})$ is some function of the local coordinates that we call the potential and we usually assume is time independent. From this equation, we can see that the dynamical evolution of the wave function is given by the action of the Hamiltonian so-called generator of time translations. Ever since Schrodinger first wrote down this formulation of quantum mechanics, there have been subsequent but equivalent ways to describe the theory. The approach that is favored by field theorists in the modern day was originally written down by Paul Dirac and eventually Richard Feynman called the Path Integral approach. Instead of understanding quantum mechanics through the dynamics of the evolving the wave function, we can study the evolution of a system by considering something called the propagator (sometimes called the time evolution operator)

$$\langle x,t|x',t'\rangle \equiv K(x,t;x',t') = \langle x|\exp(-iH(t-t'))\left|x'\right\rangle = \int \mathcal{D}x(t)\exp(iS[x(t)]), \tag{1.3}$$

where S is the classical action and the measure $\mathcal{D}x(t)$ can be regarded as a sum over all possible paths (sometimes called histories) $x(t)$ with probability amplitude $\exp(iS)$.

This formulation of the Path Integral is often favored because it highlights some of the aspects of the theory that we're working with. Additionally, we're given the beautiful picture that if we start at some point in spacetime (x', t') and then evolve the system to a new point in spacetime (x, t), we must take into account all possible ways that the system could've taken on its journey from one point to the other. The paths that are more likely to happen are given a greater weight and hence dominate over the other paths that aren't very likely to happen. The most probable path being the one which minimizes the classical action. The propagator also has the advantage of being unitary in this representation, implying that it preserves norms and hence probabilities. Once we have found the propagator, we are able to define the wave function

$$\Psi(x,t) = A \int_C \mathcal{D}x(t) \exp(iS[x(t)]), \tag{1.4}$$

where A is some normalization factor taken over some weighting of paths C. Let us express the propagator in terms of the eigenstates of the Hamiltonian

$$\begin{aligned} \langle x, t | x', t' \rangle &= \langle x | \exp(-iH(t-t')) | x' \rangle & (1.5) \\ &= \sum_{m,n} \langle x | n \rangle \langle n | \exp(-iH(t-t')) | m \rangle \langle m | x' \rangle & (1.6) \\ &= \sum_n \psi_n(x) \overline{\psi}_n(x') \exp(-iE_n(t-t')), & (1.7) \end{aligned}$$

where $|n\rangle$ is a normalized eigenstate of the Hamiltonian, $H|n\rangle = E_n |n\rangle$, and $\psi(x) \equiv \langle x|n\rangle$, and we used the completeness relation plus the orthonormality of the eigenstates of the Hamiltonian. As we can see, the propagator can also be expressed as a sum over all of the eigenmodes which makes picking out a particular state a matter of simply projecting the desired state onto the propagator. Unfortunately, the biggest knock against the time evolution operator is that it is not very well-defined, particularly when our integrand is an oscillatory function. To make this equation a bit more practical, we often preform what's called a Wick rotation, i.e. $t \to -i\tau$

$$K(\tau) = \exp(-H\tau). \tag{1.8}$$

The Wick rotated time is sometimes referred to as the Euclidean time because the signature of the metric is $(+,+,+,+)$ which is the signature of the metric on Euclidean (i.e. 3D) space and can be thought of as an analytical continuation into the complex plane. This will have the action of altering the formulae we derived for both the Path Integral as well as the sum over the eigenstates. For simplicity we shall set $x' = t = 0$. The Euclidean propagator is then

$$\langle x, 0|0, t'\rangle = \sum_n \psi_n(x)\overline{\psi}_n(0)\exp(iE_n t'). \tag{1.9}$$

Wick rotating this gives us simply

$$K(x, 0; 0, \tau) = \sum_n \psi_n(x)\overline{\psi}_n(0)\exp(-E_n\tau). \tag{1.10}$$

As a result of this analytical continuation we lose the unitarity of the propagator, but we get to keep its Hermiticity. The result of this is that our norms are no longer preserved as we evolve in time. One interesting property to note is what happens for when $\tau \to \infty$. For larger and larger time scales and assuming E_n is an increasing function of n, the $\exp(-E_n\tau)$ will get smaller and smaller. This implies that for long time scales, the ground-state energy will dominate over all of the others which means

$$K(x, 0; 0, \tau) \approx \psi_0(x)\overline{\psi}_0(0)\exp(-E_0\tau). \tag{1.11}$$

We can similarly define the ground-state wave function within the Euclidean time framework based off of this propagator

$$\Psi_0(x, 0) = A\int \mathcal{D}x(\tau)\exp(-I[x(\tau)]), \tag{1.12}$$

where $I[x(t)]$ is the Euclidean action, and we evaluate the ground-state at $t = 0$. As a result, when working in Euclidean time, we don't even need to worry about getting the excited states of the theory because *all states eventually evolve into the ground-state*. Which is to say, if you find your quantum system in an excited state, all you have to do is wait a sufficiently long while and your system will settle down into its ground-state.

2 The Hamiltonian of General Relativity

Our next order of business is to develop and Hamiltonian for General Relativity (GR)[1]. We shall answer the question of why we are interested in writing down the Hamiltonian in the first place. As we described in the previous section, the Hamiltonian of the system evolves the wave function in time. If we are to at least ostensibly write down a quantum theory of gravity, one of the most natural/obvious approach would be to apply the usual canonical quantization mechanism which we will describe in some detail a bit further on in this section. Unfortunately, there are some conceptual problems with trying to deal with GR in this way. The most important technical detail is the fact that there is no universal notion of time within GR. Time becomes simply another coordinate that labels your position within the manifold. It has no inherent value as a parameter and should thus be treated as such. This is in conflict with the Schrodinger equation. The Schrodinger Equation says to take time, t (idealized clock) and measure the changes of your system with respect to that clock. We shall see the result of this apparent contradiction in the principles of GR and QM firsthand once we've actually written down the Hamiltonian.

Now GR is a manifestly covariant theory. Some might even say it is *the* covariant theory of gravity. How do we write down a Hamiltonian for something like this? It is here we employ something called the 3+1 formalism[1] which is the idea that we can take the spacetime manifold and partition it into a family of three-dimensional surfaces (what mathematicians refer to as hypersurfaces Σ_t) called leaves, or foliated leaves. Treating the Hamiltonian in this way serves a number of functions. Firstly, it preserves the mathematical structure of GR. By taking the Lagrangian and splitting it up into a time and space part, we are able to maintain the integrity of the theory. Especially how its completely arbitrary as to which component you regard as the time. So to break up the Lagrangian in this way, we are always aware while in principle the 0-th component can stand for any of the other coordinates e.g. (x, y, z), we always assign it to be the time component.

The other advantage of the 3+1 formalism is psychological. By conceptualizing our

[1]Not to be confused with the 1+3 formalism which is the idea that we slice up spacetime into 3 congruent one-dimensional worldlines.

spacetime as a family of hypersurfaces that foliate, or slice our manifold into leaves, we gain this beautiful and/or intuitive picture of an evolving universe. Imagine a set of 100 pictures that were taken about a millisecond apart. Now imagine scrolling through the images in rapid succession. The effect of which will be the appearance of the images that are captured by the pictures to be moving all on their own. We can think of each foliating hypersurface as one of the images that were captured. We can then think of the time coordinate as a sort of enumerator of the images. $t = 1$ corresponds to looking at the first image, or standing on the first leaf of the foliated spacetime. $t = 2$ is the second image and so on and so forth. So now we can think of this formalism as capturing the way in which we interact with the universe around us: as an ever successive passing of now. We start off with the Einstein-Hilbert action

$$S_{EH} = \frac{1}{2\kappa^2}\left[\int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}(\mathcal{R} - 2\Lambda) + 2\int_{\partial\mathcal{M}} \mathrm{d}^3x\sqrt{h}K\right] + S_{matter}, \tag{2.1}$$

$$S_{matter} = -\frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}[g^{\mu\nu}\partial_\mu\phi\partial_\mu\phi + V(\phi)] \tag{2.2}$$

where $\kappa^2 = M_P^{-2} = 8\pi G$, $\mathcal{R}$ is the scalar curvature on the full 4-d manifold, h is the determinant of the spatial metric , K is the trace of the extrinsic curvature and S_{matter} is the action for matter fields. As we have hinted at previously, writing down the Hamiltonian brings up technical difficulties with GR in particular that don't arise in other field theories. When Einstein first wrote down his general theory, he had the idea that there should be no prior geometry. Meaning, the choice of coordinates should not affect the form of the physical laws. As a result, the laws of gravitation must be *invariant* under an arbitrary (differentiable) coordinate transformations. This leaves GR with the title of being a theory that is already in parametrized form. This leads to an overall redundancy in the description of physical systems. To deal with this we need to take the Lagrangian and separate it into its truly dynamical variables and the content that characterizes the coordinate system. To that end, we introduce the so-called ADM variables

$$N \equiv (-g^{00})^{-\frac{1}{2}}, \quad g_{0i} \equiv N_i, \quad h_{ij} \equiv g_{ij}, \tag{2.3}$$

where N is the lapse function, N_i is the shift vector and h_{ij} is the spatial or 3-metric on the hypersurface. We can find the other components by using $\delta^\mu{}_\nu = g^{\mu\lambda} g_{\lambda\nu}$. Doing so gives us

$$g^0{}_i = -g^{00} N_i \quad g_{00} = -N^2 + N_i N^i. \tag{2.4}$$

A similar procedure to find the inverse metric yields

$$g^{ij} = h^{ij} - \frac{N^i N^j}{N^2}. \tag{2.5}$$

Lastly, the determinant of the spatial metric can be found by

$$g^{00} = \frac{C_{00}}{\det(g_{\mu\nu})} = \frac{C_{00}}{g} = \frac{h}{g} \Rightarrow g = \frac{h}{g^{00}} = -N^2 h, \tag{2.6}$$

where $C_{\mu\nu} = (-1)^{\mu+\nu} M_{\mu\nu}$ is the co factor matrix and $M_{\mu\nu}$ is the determinant of the metric with the 0-th column and row deleted. The spacetime interval is then

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = -(N^2 - N_i N^i)\,\mathrm{d}t^2 + 2N_i\,\mathrm{d}t\,\mathrm{d}x^i + h_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j\,. \tag{2.7}$$

Define a unit vector n^μ which satisfies $n^\mu n_\mu = -1$ and is normal to the hypersurface Σ_t. Because we want to retain the physical picture of this surface representing physical 3D space, we require that Σ_t be a spacelike surface i.e.

$$g_{ij} V^i V^j > 0, \tag{2.8}$$

$\forall V \in \Sigma_t$ so that our norms are positive definite (lengths are always positive). This justifies the constraint we place on our unit vector as being timelike. Since all of our calculations will be done on the hypersurface Σ_t, we want to make sure that we're only dealing with vectors that are not parallel to n^μ. Thus we can extend the 3-metric to the entire manifold

$$h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu. \tag{2.9}$$

We are ready to decompose the Einstein-Hilbert action in terms of the above variables. First we define the extrinsic curvature K_{ij} as

$$K_{ij} = -\frac{1}{2N}(\mathcal{L}_{\partial_t} h_{ij} - \mathcal{L}_N h_{ij}) = \frac{1}{2N}\Big(-\dot{h}_{ij} + D_i N_j + D_j N_i\Big), \tag{2.10}$$

where $\mathcal{L}_{\partial_t}$ is the Lie derivative in the direction of the vector field ∂_t and D_i is the covariant derivative on the hypersurface. The extrinsic curvature (sometimes called the second fundamental form) essentially tells us what the curvature of Σ_t is from the perspective of someone living on the 4-manifold. We finally have all the ingredients to write the Einstein-Hilbert action on 3-space:

$$S = \frac{1}{2\kappa^2}\left[\int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}(\mathcal{R} - 2\Lambda) + 2\int_{\partial\mathcal{M}} \mathrm{d}^3x\sqrt{h}K\right] + S_{matter} \tag{2.11}$$

$$= \frac{1}{2\kappa^2}\left[\int_{\mathcal{M}} \mathrm{d}^4x N\sqrt{h}[R - 2\Lambda + K^2 + K^{ij}K_{ij} - \frac{2}{N}\mathcal{L}_m K - \frac{2}{N}D_i D^i N] + 2\int_{\partial\mathcal{M}} \mathrm{d}^3x\sqrt{h}K\right] + S_{matter}. \tag{2.12}$$

The Lie derivative term is precisely $-1\cdot$ the Gibbons-Hawking-York and so that completely cancel out. We can also remove the covariant divergence term because it is a total derivative on the vector field. This simplies the action to the form

$$S = \frac{1}{2\kappa^2}\int_{\mathcal{M}} \mathrm{d}^4x N\sqrt{h}[R - 2\Lambda + K^{ij}K_{ij} - K^2] + S_{matter}. \tag{2.13}$$

We are finally in the position to write down the Hamiltonian for GR. In the full theory, the dynamical variable is the 4-metric $g_{\mu\nu}$. So it is reasonable to expect that the dynamical field will be the 3-metric. It is also the only term that enters into the Lagrangian with a time derivative (through the extrinsic curvature) which makes it a good candidate to construct a canonical momentum.

$$\pi^{ij} = \frac{\partial\mathcal{L}}{\partial\dot{h}_{ij}} = \sqrt{h}(h^{ij}K - K^{ij}). \tag{2.14}$$

We can invert this equation to solve for the extrinsic curvature

$$K_{ij} = \frac{1}{\sqrt{h}}\left(\frac{1}{2}h_{ij}h_{k\ell} - h_{ik}h_{j\ell}\right)\pi^{k\ell}. \tag{2.15}$$

And when we plug this into the action and define the Wheeler-DeWitt metric $G_{ijk\ell} = \frac{1}{2\sqrt{h}}(h_{ik}h_{j\ell} + h_{i\ell}h_{jk} - h_{ij}h_{k\ell})$, the quadratic extrinsic curvature term is simply

$$K^{ij}K_{ij} - K^2 = \frac{1}{\sqrt{h}} G_{ijk\ell}\pi^{ij}\pi^{k\ell}. \tag{2.16}$$

The Hamiltonian given by the Legendre transformation is then

$$\mathcal{H} = \dot{h}_{ij}\pi^{ij} - \mathcal{L} = \dot{h}_{ij}\pi^{ij} - N[G_{ijk\ell}\pi^{ij}\pi^{k\ell} + \sqrt{h}(R - 2\Lambda) + \mathcal{H}_{matter}] - N_i(2D_j\pi^{ij} + \mathcal{H}^i_{matter}). \tag{2.17}$$

Plugging this back into the action gives

$$S = \frac{1}{2\kappa^2}\int \mathrm{d}^3x\,\mathrm{d}t\left[\dot{h}_{ij}\pi^{ij} - N\mathcal{H}_{WDW} - N_i\mathcal{H}^i\right], \tag{2.18}$$

where $\mathcal{H}_{WDW} = G_{ijk\ell}\pi^{ij}\pi^{k\ell} - \sqrt{h}(R - 2\Lambda)$ and $\mathcal{H}^i = 2D_j\pi^{ij} + \mathcal{H}^i_{matter}$. We can see that the lapse and shift functions both enter into the action as Lagrange multipliers i.e. they only set things equal to zero. Meaning, we can eliminate them from the action using their equations of motion. Varying the action with respect to the Lagrange multipliers gives us

$$\frac{\delta S}{\delta N} = G_{ijk\ell}\pi^{ij}\pi^{k\ell} - \sqrt{h}(R - 2\Lambda) + \mathcal{H}_{matter} = 0, \qquad \frac{\delta S}{\delta N_i} = 2D_j\pi^{ij} + \mathcal{H}^i_{matter} = 0. \tag{2.19}$$

Now these constraints have enormous implications on the theory going forward. The fact that the Hamiltonian and momentum constraint vanishes is a hallmark of parametrized theories. In classical mechanics, a constraint such as this is the indication that we've picked a set of canonical variables/coordinates in phase space that satisfy the Hamilton-Jacobi equation. This cannot be the case here because we haven't picked a set of coordinates to work from. This vanishing of the Hamiltonian actually brings with it a whole new meaning. From the previous section, we know that the Hamiltonian is the generator of time translations. However, in a theory where time is just another coordinate with no special significance, it doesn't make sense to consider translations in a particular direction when it is a symmetry of the theory. Said in another way, the vanishing of the Hamiltonian reflects the diffeomorphism invariance of General Relativity.

3 The Hartle-Hawking State

3.1 Quantization

We are finally ready to make the transition from classical GR to "quantum" GR. We will largely work in analogy to the non-relativistic QM (NRQM) procedure we laid out earlier in this document where we can. From the onset, there are already a few notable differences between NRQM and GR. For one, the wave function gets promoted from a function of the classical trajectories, to a **functional** of the classical hypersurfaces Σ_t i.e. $\Psi(x,t) \leftrightarrow \Psi[h_{ij}]$. The propagator also is quite different as well. Previously, the propagator told us the probability amplitude of starting from a point (x', t') and ending at the point (x, t). We then add up all possible paths between these two points while weighting classical trajectories more heavily than paths that are not classical paths. For GR, we are instead calculating the probability amplitude to go from an initial spacelike 3-geometry/surface h'_{ij} what a matter configuration ϕ' to a final spacelike 3-geometry h_{ij} and matter configuration ϕ

$$\langle h_{ij}, \phi | h'_{ij}, \phi' \rangle = \int \mathcal{D}g\mathcal{D}\phi \exp(iS_{EH}[g, \phi]), \tag{3.1}$$

where $\exp(iS_{EH}[g, \phi])$ serves as our analog to the amplitude we wrote down in the first section and the integral is taken over all 4-metrics and matter field configurations. Since time does not enter into the action explicitly, it won't be possible to specify a time on these states. It is possible, however, to specify the proper time since that enters through the lapse function N. Following our original procedure, we can write down the wave *functional* as

$$\Psi[h_{ij}, \phi] = \int_C \mathcal{D}g\mathcal{D}\phi \exp(iS_{EH}[g, \phi]), \tag{3.2}$$

where C is a class of spacetimes whose boundary is compact on which the induced metric is h_{ij} and ϕ is the matter field. We can now employ the canonical quantization procedure first introduced by Paul Dirac

$$\pi^{ij} \to -i\frac{\delta}{\delta h_{ij}}, \quad \pi_\phi \to -i\frac{\delta}{\delta \phi}, \tag{3.3}$$

which follow from the Poisson bracket relations

$$\{h_{ij}(x), \pi^{k\ell}(y)\} = \delta^k{}_i \delta^\ell{}_j \delta^4(x-y), \quad \{\phi(x), \pi_\phi(y)\} = \delta^4(x-y), \tag{3.4}$$

which when quantized becomes the usual commutation relations. The Hamiltonian and momentum constraints on the wave functional becomes

$$\mathcal{H}_{WDW}\Psi[h_{ij},\phi] = \left[-G_{ijk\ell}\frac{\delta}{\delta h_{ij}}\frac{\delta}{\delta h_{k\ell}} - \sqrt{h}(R-2\Lambda) + \mathcal{H}_{matter}\right]\Psi[h_{ij},\phi] = 0, \tag{3.5}$$

$$\mathcal{H}^i_{matter}\Psi[h_{ij},\phi] = 2iD_j\left(\frac{\delta\Psi}{\delta h_{ij}}\right) = 0. \tag{3.6}$$

The Hamiltonian constraint is given usually called the Wheeler De-Witt equation (WDWE). The momentum constraint implies that Ψ be invariant under arbitrary coordinate transformations on the hypersurface. Lastly, it is important to note that the ground-state wave function when constructed under the Euclidean functional integral **must** be a solution to the Wheeler-de Witt equation.

3.2 Boundary Conditions

Here we shall study in more detail the boundary conditions as well as HH's so-called "No Boundary Proposal". Our state vector $\Psi[h_{ij}]$ is a wave functional over the components of the 3-metric tensor h_{ij}. We can regard the 3-geometry as living on a space of spacelike, symmetric, 3-tensors (we shall denote by $H[\Sigma]$) for which $G_{ijk\ell}$ is a metric (technically $G^{ijk\ell}$ would be the metric for h_{ij} and $G_{ijk\ell}$ for its inverse). When we go and study the components of the WDW metric within the flat space basis, we find that it's signature is $(-,+,+,+,+,+)$. The fact that $H[\Sigma]$ is 6-dimensional should come as no surprise because it reflects the fact that there are 6 independent components of the 3-metric h_{ij}. The signature of the WDW metric implies that the WDWE is actually a hyperbolic partial differential equation on $H[\Sigma]$ (we're subtracting the partial derivatives from each other). Because the WDWE is a second order differential equation, we expect to impose boundary conditions on two hypersurfaces in order to ensure uniqueness of the solution.

Just like when we first learn QM and the usefulness of having a wave function in both the coordinate and momentum space representations, it will be useful to construct an additional wave function that is a functional over a different set of canonical variables. To that end, we'll need to pick a different set of coordinates for our wave functional. Since the DeWitt metric has a Lorentzian signature, we'll need a timelike coordinate as well as spacelike coordinates. Since we want to make the analogy to real-space, the timelike coordinate should be a scalar function $f(\mathbf{x})$ whose exterior derivative, when evaluated along the DeWitt metric, yields a negative number. Given we are working on the space of symmetric 3-metrics, a good candidate would be the determinant the metric. Since $h^{1/2}$ appears in the measure, we shall focus our attention on that. We can even check that its timelike!

$$\mathrm{d}h^{1/2} = \frac{\delta h^{1/2}}{\delta h_{ij}} h_{ij} = \frac{1}{2}\sqrt{h} h^{ij} h_{ij} \equiv J^{ij} h_{ij}. \tag{3.7}$$

Evaluating with along the direction of the super metric yields

$$G_{ijk\ell} J^{ij} J^{k\ell} = -\frac{1}{2}\frac{3}{4}\sqrt{h} < 0. \tag{3.8}$$

The above equation is true provided that $\sqrt{h} > 0$ holds. We can be reasonably confident it is the case because of the spacelike nature of the hypersurfaces. Spacelike surfaces imply $h_{ij}V^iV^j > 0$, $\forall V \in \Sigma_t$. The geometric interpretation of the determinant of a linear operator informs us that the positivity (or negativity) of the inner product of h_{ij} and the vectors $\Rightarrow$ the positivity of the determinant. This is also reasonable in light that $h^{1/2}$ is supposed to represent the volume of the hypersurface. For the spacelike coordinates, we choose the so-called conformal 3-metric $h^{1/3}\tilde{h}_{ij} = h_{ij}$. Since we're already using the determinant to play the role as our timelike coordinate, life is made easier if we render our spacelike coordinates to be independent of the timelike coordinates (again in analogy to real space). In order to enforce the condition that the 3-volume is non-negative, we impose the boundary condition that $\Psi[h^{1/2}, \tilde{h}_{ij}] \equiv 0$ whenever $h^{1/2} < 0$. The new wave functional Φ will also be a function of $h^{1/2}$'s (scaled) canonically conjugate variable: the trace of the extrinsic curvature $-\frac{4}{3(2\kappa^2)}K$. The wave functional is then

$$\Phi[\tilde{h}_{ij}, K] = \int_0^\infty \mathcal{D}h^{1/2} \exp\biggl(-\frac{4i}{3(2\kappa^2)}\int \mathrm{d}^3x\sqrt{h}K\biggr)\Psi[h_{ij}], \tag{3.9}$$

and we can get the original representation by doing the inverse transformation

$$\Psi[h_{ij}] = \int_{-\infty}^\infty \mathcal{D}K \exp\biggl(\frac{4i}{3(2\kappa^2)}\int \mathrm{d}^3x\sqrt{h}K\biggr)\Phi[\tilde{h}_{ij}, K]. \tag{3.10}$$

It'll also be helpful if we write down the Wick-rotated gravitational and matter actions for the K-representation of the wave functional. Under a $t \to -i\tau$, we get the following: $\mathrm{d}^4x \to -i\mathrm{d}^4x$, $\sqrt{-g} \to \sqrt{g}$, and $K \to -iK$. The Euclidean actions for the gravitational and matter sector are

$$2\kappa^2 I^k_{EH}[g] = -\frac{2}{3}\int_{\partial\mathcal{M}} \mathrm{d}^3x\sqrt{h}K - \int_{\mathcal{M}} \mathrm{d}^4x\sqrt{g}[R - 2\Lambda], \tag{3.11}$$

$$I^k_M[g, \phi] = \frac{1}{2}\int_{\mathcal{M}} \mathrm{d}^4x\sqrt{g}\left[(\nabla\phi)^2 + \frac{1}{6}R\phi^2\right], \tag{3.12}$$

where $(\nabla\phi)^2 \equiv g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$ is the kinetic energy of the scalar field and ∇_μ is the covariant derivative. We shall be using this form of the matter action for the rest of the paper.

3.3 The Ground-State and Minisuperspace

Now we are ready to discuss the ground-state wave functional. We will henceforth be working within a **closed** universe model. For more details as to why HH as well as quantum cosmologists at large work with closed universes, see A. When Wick rotated, the wave functional takes on the form

$$\Psi_0[h_{ij}, \phi] = A\int \mathcal{D}g\mathcal{D}\phi \exp(-I[g, \phi]). \tag{3.13}$$

HH informs us that the biggest advantage with using this functional integral definition of the ground-state is that it provides us with the semi-classical approximation (the WKB solution) of the wave functional directly. To actually preform the integral[2] we employ the

[2]HH don't actually do so in their paper, but they tell us that's they're doing.

method of steepest descents. Suppose $I_{cl}[h_{ij}]$ is the classical Euclidean action evaluated at the "smallest" stationary point h_{ij}. Then the ground-state wave functional becomes

$$\Psi_0[h_{ij}] = A\Delta^{-1/2}[h_{ij}]\exp(-I_{cl}[h_{ij}]), \tag{3.14}$$

where $\Delta^{-1/2}[h_{ij}]$ is the a combination of determinants of the wave operators defining the fluctuations about the classical 4-metric including those contributed by the ghosts. HH ignores this term for the rest of the paper and so will we. To fix the prefactor of the wave functional, we impose the normalization condition

$$\int \mathcal{D}h\overline{\Psi}_0[h_{ij}]\Psi_0[h_{ij}] = 1. \tag{3.15}$$

When we take the metric to be that of the 4-sphere and evaluate the action in this basis, the normalization is then

$$|A|^2\exp\left(\frac{2}{3H^2}\right) = 1 \Rightarrow A^2 = \exp\left(-\frac{2}{3H^2}\right). \tag{3.16}$$

A completely equivalent way of describing the ground-state wave functional is through the K representation

$$\Phi_0[K, \tilde{h}_{ij}, \tilde{\phi}] = A\int \mathcal{D}g\mathcal{D}\phi\exp(-I_K[g,\phi]), \tag{3.17}$$

where $\tilde{\phi} = \phi h^{1/6}$ is a conformally invariant scalar field. Working with conformally invariant scalar fields will simplify our lives down the line. Now in order for us to continue onward, we must restrict the space of possible 3-metrics that we are considering. The reason is simply the fact that $H[\Sigma]$ is an infinite dimensional space when considering it as a configuration space. We don't know how to deal with infinite dimensional spaces, but we certainly know how to deal with finite dimensional spaces and when we restrict our view to the subspace of homogeneous and isotropic 3-metrics, $HI(\Sigma) \subset H(\Sigma)$. This space is often referred to as minisuperspace. We begin by taking the metric to be

$$\mathrm{d}s^2 = \sigma^2\left[-N^2(t) + a^2(t)\,\mathrm{d}\Omega_3^2\right], \tag{3.18}$$

where $\sigma^2 = \frac{2\kappa^2}{24\pi^2}$ and $\mathrm{d}\Omega_3^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\alpha^2 + \sin^2\theta\sin^2\alpha\,\mathrm{d}\beta^2$ is the metric on the unit 3-sphere. Again, we stress we are working in a **closed** universe with a positive cosmological constant. Because the universe is both homogeneous and isotropic (as far as we can tell), we usually take the matter fields to also be homogeneous i.e. $\phi = \phi(t)$. It is more convenient to work with our conformally invariant scalar field $\phi = \frac{\tilde{\phi}}{h^{1/6}} = \frac{\chi}{\sqrt{2\pi^2\sigma^2 a}}$. The Lorentzian (i.e. non-Wick rotated spacetime) action for the gravitational and matter section is

$$S = \frac{1}{2}\int \mathrm{d}t\,\frac{N}{a}\left[-\left(\frac{a}{N}\frac{\mathrm{d}a}{\mathrm{d}t}\right)^2 + a^2 - \lambda a^4 + \left(\frac{a}{N}\frac{\mathrm{d}\chi}{\mathrm{d}t}\right)^2 - \chi^2\right], \tag{3.19}$$

we show how this action is derived in Appendix C. Once we have the action, we can write down the Hamiltonian (constraint) by first computing the conjugate momenta of both a and χ. The Hamiltonian (and by extension, the constraint on the Hamiltonian) reads

$$NH[a, \pi_a, \chi, \pi_\chi] = \frac{1}{2}\frac{N}{a}\left[-\pi_a^2 - a^2 + \lambda a^4 + \pi_\chi^2 + \chi^2\right], \tag{3.20}$$

where

$$\pi_a = \frac{\partial L}{\partial \dot{a}} = -\frac{a\dot{a}}{N}, \quad \pi_\chi = \frac{\partial L}{\partial \dot{\chi}} = \frac{a\dot{\chi}}{N}. \tag{3.21}$$

Plugging in the usual canonical quantization conditions (while also keeping in mind operator ordering issues) $\pi_a^2 \to -\frac{1}{a^p}\frac{\partial}{\partial a}\left(a^p\frac{\partial}{\partial a}\right)$, $\pi_\chi \to -i\frac{\partial}{\partial\chi}$ we get the following Wheeler DeWitt equation

$$\frac{1}{2}\left[-\frac{1}{a^p}\frac{\partial}{\partial a}\left(a^p\frac{\partial}{\partial a}\right) - a^2 + \lambda a^4 - \frac{\partial^2}{\partial\chi^2} + \chi^2 - 2\epsilon_0\right]\Psi[a,\chi] = 0. \tag{3.22}$$

Because the matter part of the Wheeler DeWitt equation looks an awful lot like the Hamiltonian for a harmonic oscillator potential, we can guess that the matter dependence of the wave function can be expanded in terms of eigenstates of the harmonic oscillator i.e. the Hermite polynomials, so we write

$$\Psi[a,\chi] = \sum_n c_n(a)u_n(\chi), \quad \frac{1}{2}\left[-\frac{\mathrm{d}^2}{\mathrm{d}\chi^2} + \chi^2\right]u_n(\chi) = \left(n + \frac{1}{2}\right)u_n(\chi). \tag{3.23}$$

The conformal invariance of the matter field implies we can scale the matter fields by any function that preserves the angles between geodesics. In particular, it will be invariant under scaling by functions of the scale factor a since all the scale factor does is increase the total volume of the known universe thus justifying the separation of variables here. When plugging into WDWE, we get

$$\frac{1}{2}\left[\frac{u_n}{a^p}\frac{\mathrm{d}}{\mathrm{d}a}\left(a^p\frac{\mathrm{d}c_n}{\mathrm{d}a}\right)-(a^2-\lambda a^4)c_nu_n+c_n\left(-\frac{\mathrm{d}^2}{\mathrm{d}\chi^2}+\chi^2\right)u_n-2\epsilon_0c_nu_n\right]=0. \tag{3.24}$$

When using the recursion relation of the Hermite polynomials and the dividing through by u_n, the above equation becomes

$$\frac{1}{2}\left[-\frac{1}{a^p}\frac{\mathrm{d}}{\mathrm{d}a}\left(a^p\frac{\mathrm{d}c_n}{\mathrm{d}a}\right)+\left(a^2-\lambda a^4\right)c_n(a)\right]=\left(n+\frac{1}{2}-\epsilon_0\right)c_n(a) \tag{3.25}$$

$$\Leftrightarrow\ \frac{\mathrm{d}^2c_n}{\mathrm{d}a^2}+\frac{p}{a}\frac{\mathrm{d}c_n}{\mathrm{d}a}+(2n+1-2\epsilon_0)c_n(a)=0. \tag{3.26}$$

In order to get the solution for the scale factor function, we guess a power series of the form $c_n(a)=a^q\sum\limits_m z_{nm}a^m$. Define the coefficients of the first and zero order term $x(a)\equiv p/a$ and $y(a)\equiv 2n+1-2\epsilon_0$. Since the functions $s(a)\equiv ax(a)$ and $t(a)\equiv a^2y(a)$ are analytic at $a=0$, 0 is a regular singular point and hence, we can find power series solutions around this point. For $a\ll 1$ we get $q=\{1-p,0\}$ which corresponds to $c_n\approx a^{1-p}$, and $c_n\approx C$ where C is some arbitrary constant. For $a\gg 1$, we get $c_n\approx a^{-(1+p/2)}\exp\left(\pm\frac{iHa^3}{3}\right)$.

Now we want to take the Lorentzian action we derived and Wick rotate it $t\to -i\tau$. We will be working in the $N=1$ gauge from here on out. This is the necessary procedure for deriving the "ground-state" of this theory. The spacetime interval under this rotation becomes

$$\mathrm{d}s^2=\sigma^2\left[\mathrm{d}\tau^2+a^2(\tau)\,\mathrm{d}\Omega_3^2\right], \tag{3.27}$$

where $a_0\equiv a(0)$ on the boundary of the hypersurface of interest. We likewise define χ_0 to be the value of the matter field on the boundary also. It is common within the field of cosmology to work with the conformal time as opposed to the (Euclidean) time because it simplifies some of the math. The conformal time coordinate η is usually defined as $a(\eta)\,\mathrm{d}\eta = \mathrm{d}\tau$. The action when Wick-rotated (and setting $N = 1$) is

$$S[a,\chi] = \frac{-i}{2}\int \mathrm{d}\tau\,\frac{1}{a}\left[\left(a\frac{\mathrm{d}a}{\mathrm{d}\tau}\right)^2 + a^2 - \lambda a^4 - \left(a\frac{\mathrm{d}\chi}{\mathrm{d}\tau}\right)^2 - \chi^2\right]. \tag{3.28}$$

And when rotated along the conformal time axis, it becomes

$$I_{EH}[a,\chi] = iS[a,\chi] = \frac{1}{2}\int \mathrm{d}\eta\left[-\left(\frac{\mathrm{d}a}{\mathrm{d}\eta}\right)^2 - a^2 + \lambda a^4 + \left(\frac{\mathrm{d}\chi}{\mathrm{d}\eta}\right)^2 + \chi^2\right]. \tag{3.29}$$

The ground-state wave function is thus given by

$$\Psi_0(a_0,\chi_0) = \int \mathcal{D}a\mathcal{D}\chi\,\exp(-I_{EH}[a,\chi]). \tag{3.30}$$

We shall refer to the gravitational and matter sectors by

$$I_G = \frac{1}{2}\int \mathrm{d}\eta\left[-\left(\frac{\mathrm{d}a}{\mathrm{d}\eta}\right)^2 - a^2 + \lambda a^4\right], \quad I_M = \frac{1}{2}\int \mathrm{d}\eta\left[\left(\frac{\mathrm{d}\chi}{\mathrm{d}\eta}\right)^2 + \chi^2\right], \tag{3.31}$$

with $I_{EH} = I_G + I_M$. It'll actually be easier to calculate the ground-state wave function $\Psi_0(a_0,\chi_0)$ by calculating the K-representation of the wave function and then applying the semiclassical approximation to that. The ground-state wave function in the K representation is merely

$$\Phi_0(K,\chi_0) = \int \mathcal{D}a\mathcal{D}\chi\,\exp\bigl(-I^k[a,\chi]\bigr). \tag{3.32}$$

Let evaluate the Gibbons-Hawking-York term since the wave function is spherically symmetric:

$$-\frac{2}{3(2\kappa^2)}\int \mathrm{d}^3x\sqrt{h}K = -\frac{2}{3(2\kappa^2)}K\int \mathrm{d}^3x\sqrt{h}, \tag{3.33}$$

where $K \to -iK$ and $K = K(\tau)$ is also a homogeneous field. Recall that the extrinsic curvature is $K = -\frac{1}{2}h^{ij}\dot{h}_{ij} = \frac{3}{\sigma a}\frac{\mathrm{d}a}{\mathrm{d}\tau}$. Since the volume of a 3-sphere is simply $2\pi^2 a^3$ (a acts as a radius on this space), the Gibbons-Hawking-York term becomes

$$-\frac{2}{3(2\kappa^2)}K\int \mathrm{d}^3x\sqrt{h} = -\frac{(2\pi)^2\sigma^3 a_0^3 K}{3(2\kappa^2)} = \frac{-12\pi^2\sigma^2 a_0^3 k}{2\kappa^2} = -\frac{1}{2}k_0 a_0^3, \tag{3.34}$$

where we define the new variable $k \equiv \frac{\sigma}{9}K = \frac{1}{3a}\frac{\mathrm{d}a}{\mathrm{d}\tau}$ and utilize the fact that because we evaluated this integral at the boundary of the manifold, k and a both achieve the boundary values here. The k representation of the wave function is just

$$\Phi_0(k_0, \chi_0) = \int \mathcal{D}a\mathcal{D}\chi e^{-k_0 a_0^3} e^{-I_G}. \tag{3.35}$$

The original ground-state wave function is related via the familiar transformation

$$\Psi_0(a_0, \chi_0) = -\frac{1}{2\pi i}\int_C \mathrm{d}k\, e^{k a_0^3}\Phi_0(k, \chi_0), \tag{3.36}$$

where the contour integral is taken over the imaginary axis $(-i\infty, i\infty)$.

3.4 The Ground-State Cosmological Wave Function

In this section, we will give some justification to call our model the "ground-state" of the wave function. Recall that a and χ both achieve their respective values $a(0) \equiv a_0$ and $\chi(0) \equiv \chi_0$ at the boundary which will henceforth be 0. Also recall that we require both a and χ to vanish at infinity. Next, we can take advantage of the fact that path integral factorizes into a gravitational and matter sector here. The matter sector is again

$$I_M = \frac{1}{2}\int \mathrm{d}\eta \left[\left(\frac{\mathrm{d}\chi}{\mathrm{d}\eta}\right)^2 + \chi^2\right], \tag{3.37}$$

which is exactly the path integral for the harmonic oscillator which hearkens back to the separation of variable expansion in terms of the Hermite polynomials that we did in the previous section. Because the ground-state wave function for that action is a simple Gaussian function in the integrated variable, the "ground-state" wave function becomes simply

$$\Psi_0(a_0, \chi_0) = e^{-\chi_0^2/2}\psi_0(a_0), \quad \psi(a_0) = \int \mathcal{D}a e^{-I_G[a]}, \tag{3.38}$$

this shows that at least when talking about the matter sector of the wave function, we are completely justified in referring to this state as a ground-state wave function. Similarly, the k-representation for the wave function is

$$\Phi_0(k_0,\chi_0)=e^{-\chi_0^2/2}\phi_0(k_0),\qquad \phi_0(k_0)=\int\mathcal{D}a e^{-I_G^k[a]}. \tag{3.39}$$

From here, all that is left is to perform a semiclassical approximation on the functional integral to get the ground-state wave function proper. To get k, we can compute the trace of the Lie derivative acting on the 3-metric. Because the compact geometry that extremizes the gravitational action is a part of the Euclidean 4-sphere, we take the metric to be $\mathrm{d}s^2=(\frac{\sigma}{H})^2(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\Omega_3^2)$. Thus the trace of the extrinsic curvature is simply

$$K=\frac{1}{2}h^{ij}\frac{H}{\sigma}\frac{\partial}{\partial\theta}h_{ij}=\frac{3H}{\sigma}\cot\theta\Rightarrow k=\frac{H}{3}\cot\theta. \tag{3.40}$$

By integrating the Euclidean action over the 4-sphere, making the substitution $k=\frac{1}{3}H\varkappa$, and extremizing the result, we get

$$I_E^k[k]=-\frac{1}{3H^2}\left(1-\frac{\varkappa}{\sqrt{1+\varkappa^2}}\right). \tag{3.41}$$

We extremized the result because we hope to employ the method of steepest descent. The critical points, in particular local and global minima of the exponent of the integrand will maximize the integrand and in turn dominates the most in the integral. The semiclassical approximation to the K-representation of the ground-state wave function is simply

$$\phi_0(k_0)\approx A\exp\bigl(-I_E^k[k_0]\bigr). \tag{3.42}$$

The wave function in the original representation is then

$$\psi_0(a)=-\frac{A}{2\pi i}\int_C\exp\bigl(ka_0^3-I_E^k[k]\bigr). \tag{3.43}$$

We evaluate this wave function via the method of steepest descent also. We wish to minimize the function

$$f(k)=\frac{1}{3}H\varkappa a_0^3+\frac{1}{3H^2}\left(1-\frac{\varkappa}{\sqrt{1+\varkappa^2}}\right). \tag{3.44}$$

The minima are then

$$\varkappa = \pm\frac{\sqrt{1-H^2a_0^2}}{Ha} \Rightarrow f_{\pm}(k) = -\frac{1}{3H^2}\big[1 \pm (1-H^2a_0^2)^{3/2}\big], \tag{3.45}$$

where $f_{-}(a_0)$ correspond to the $\varkappa > 0$ minimum and $f_{+}(a_0)$ corresponds to the $\varkappa < 0$ minimum. When $|Ha_0| < 1$ the minima are real. For $|Ha_0| > 1$ the extrema are complex and thus one must be more careful with their contour integration. The wave functions in both regimes are then

$$\psi_0(a_0) \approx \begin{cases} \exp\left(\frac{1}{3H^2}(1-(1-H^2a_0^2)^{3/2}) - \frac{1}{3}H^{-2}\right) & |Ha_0| < 1 \\ \cos\left(\frac{(H^2a_0^2-1)^{3/2}}{3H^2} - \frac{\pi}{4}\right) & |Ha_0| > 1 \end{cases}. \tag{3.46}$$

Alternatively, if one is interested in the $|Ha_0| \ll 1$ and $|Ha_0| \gg 1$ regimes, the wave function becomes

$$\psi_0(a_0) \approx \begin{cases} \exp\left(\frac{1}{2}a_0^2 - \frac{1}{3}H^{-2}\right) & |Ha_0| \ll 1 \\ \cos\left(\frac{1}{3}Ha_0^3\right) & |Ha_0| \gg 1 \end{cases}. \tag{3.47}$$

3.5 Excited States

Even though we could find some sense where the Hartle-Hawking state can be understood as a wave function for the ground-state of the entire universe, we are still very far away from describing the universe we live in. It could be the case that we are currently occupying the state of minimum excitation for the universe, or we could be living in an excited state. In the case of the latter, the path integral formalism will not be sufficient to calculate these excited states so we will have to turn to the Wheeler-DeWitt equation for that

$$-\frac{1}{2}\frac{1}{a^p}\frac{\mathrm{d}}{\mathrm{d}a}\left(a^p\frac{\mathrm{d}c_n}{\mathrm{d}a}\right) + V(a)c_n = \left(n+\frac{1}{2}-\epsilon_0\right)c_n, \quad V(a) = \frac{1}{2}(a^2-\lambda a^4), \tag{3.48}$$

which we can reasonably interpret to be a 1-d Schrödinger equation.

Appendices

A Why A Closed Universe?

We want to understand the motivation that lead to consider closed space time geometry in quantum gravity. Indeed in most of the fundamental papers on the subject, the work seems to focus on cases where $k = 1$. This, at first, may appear to be counter intuitive as the Planck data show that the universe is spatially flat to an accuracy of 0.2%. In this note we explore the consequence of enforcing $k = 0$ in the Wheeler-DeWitt equation and study the consequence of such an ansatz.

Toy Model

To start with, let us look at a specific example from the Halliwell lectures. We are considering a universe described by a homogeneous isotropic Robertson-Walker metric:

$$\mathrm{d}s^2 = \sigma^2 \left[-N^2(t)\mathrm{d}t^2 + e^{2\alpha(t)}\mathrm{d}\Omega_3^2(k) \right], \tag{A.1}$$

where $\sigma^2 = \frac{2}{3\pi m_p^2}$. The Einstein-scalar gravitational action for this system is:

$$S_{grav} = \frac{m_p^2}{16\pi} \int \mathrm{d}^4x \sqrt{-g} R. \tag{A.2}$$

With the metric we are working with, we have $\sqrt{-g} = \sigma^4 N e^{3\alpha(t)}$ and the Ricci scalar is:

$$\begin{aligned} R &= \frac{6}{\sigma^4 N^2 e^{2\alpha}} \left[\sigma \left(\ddot{\alpha} e^{\alpha} + \dot{\alpha} e^{\alpha} \right) \sigma e^{\alpha} + \left(\sigma \dot{\alpha} e^{\alpha} \right)^2 + k \sigma^2 N^2 \right] \\ &= \frac{6}{\sigma^2} \left[\frac{\ddot{\alpha} + \dot{\alpha}^2}{N^2} + \frac{\dot{\alpha}^2}{N^2} + k e^{-2\alpha} \right]. \end{aligned} \tag{A.3}$$

Hence, the gravitational action is:

$$
\begin{aligned}
S_{grav} &= \frac{m_p^2}{16\pi}\int \mathrm{d}^4x\sqrt{-g}R \\
&= \frac{m_p^2}{16\pi}\int \mathrm{d}^4x\sigma^4 N e^{3\alpha(t)}\frac{6}{\sigma^2}\left[\frac{\ddot{\alpha}+\dot{\alpha}^2}{N^2}+\frac{\dot{\alpha}^2}{N^2}+ke^{-2\alpha}\right] \\
&= \frac{6\sigma^2 m_p^2}{16\pi}\int \mathrm{d}^4x N e^{3\alpha(t)}\left[\frac{\ddot{\alpha}+\dot{\alpha}^2}{N^2}+\frac{\dot{\alpha}^2}{N^2}+ke^{-2\alpha}\right] \\
&= \frac{1}{2}\int \mathrm{d}t\, N e^{3\alpha}\left[-\frac{\dot{\alpha}^2}{N^2}+ke^{-2\alpha}\right].
\end{aligned}
\tag{A.4}
$$

In the last line we used the fact that:

$$
e^{3\alpha}\ddot{\alpha} = \frac{\partial}{\partial t}\left(\dot{\alpha}e^{3\alpha}\right) - 3\dot{\alpha}e^{3\alpha}. \tag{A.5}
$$

Now, we add a scalar field $\sqrt{2}\pi\sigma\phi(t)$ with the potential $2\pi^2\sigma^2 V(\phi)$. The complete action of this system is given by:

$$
S = \frac{1}{2}\int \mathrm{d}t\, N e^{3\alpha}\left[-\frac{\dot{\alpha}^2}{N^2}+\frac{\dot{\phi}^2}{N^2}-V(\phi)+ke^{-2\alpha}\right]. \tag{A.6}
$$

The momenta conjugate are defined as per usual:

$$
\pi_\alpha = \frac{\partial\mathcal{L}}{\partial\dot{\alpha}} = -e^{3\alpha}\frac{\dot{\alpha}}{N}, \qquad \pi_\phi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = e^{3\alpha}\frac{\dot{\phi}}{N}. \tag{A.7}
$$

The canonical Hamiltonian is given by:

$$
\begin{aligned}
H_c &= \pi_\alpha\dot{\alpha}+\pi_\phi\dot{\phi}-\mathcal{L} \\
&= -\frac{1}{2}e^{3\alpha}\frac{\dot{\alpha}^2}{N}+\frac{1}{2}e^{3\phi}\frac{\dot{\phi}^2}{N}-\frac{1}{2}Ne^{3\alpha}\left[-\frac{\dot{\alpha}^2}{N^2}+\frac{\dot{\phi}^2}{N^2}-V(\phi)+ke^{-2\alpha}\right].
\end{aligned}
\tag{A.8}
$$

Hence

$$
H_c = \frac{1}{2}Ne^{-3\alpha}\left[\pi_\alpha^2+\pi_\phi^2+e^{6\alpha}V(\phi)-ke^{4\alpha}\right]. \tag{A.9}
$$

We can obtain the Wheeler DeWitt equation for this system by substituting the canonical momenta by their corresponding operator:

$$H\Psi = \frac{1}{2}e^{-3\alpha}\left[\frac{\partial^2}{\partial\alpha^2} - \frac{\partial^2}{\partial\phi^2} + e^{6\alpha}V(\phi) - ke^{4\alpha}\right]\Psi = 0. \tag{A.10}$$

The flat universe ($k = 0$), Wheeler-DeWitt equation becomes:

$$H\Psi = \frac{1}{2}e^{-3\alpha}\left[\frac{\partial^2}{\partial\alpha^2} - \frac{\partial^2}{\partial\phi^2} + e^{6\alpha}V(\phi)\right]\Psi = 0. \tag{A.11}$$

For now, let's look for solutions that do not depend on ϕ very much. We will also use the WKB method i.e., we are looking for a solution of the form $\Psi(\alpha) = e^{\chi(\alpha)}$. We need to solve:

$$\chi''(\alpha) + \chi'(\alpha)^2 + e^{6\alpha}V(\phi) = 0. \tag{A.12}$$

Because we are considering very small dependence in ϕ, we can take $V(\phi) \simeq K$, with K being a constant. We can also assume that $\chi''(\alpha) \ll \chi'^2(\alpha)$. With those approximations, it is easy to see that the solution is:

$$\chi(\alpha) = \pm ie^{3\alpha}\frac{K^{1/2}}{3} = \pm ie^{3\alpha}K. \tag{A.13}$$

Hence $\Psi(\alpha) = e^{\pm ie^{3\alpha}K}$. There are only classically allowed solutions.

Solving the Wheeler DeWitt equation in flat FRLW space time.

We consider a FRW space-time:

$$\mathrm{d}s^2 = -c^2\,\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\mathrm{d}\Omega^2\right] = -c^2\,\mathrm{d}t^2 + a^2\mathrm{d}\Sigma^2. \tag{A.14}$$

In this metric, we have $\sqrt{-g} = ca^3(t)$ and $R = \frac{6}{c^2a^2}\left(\ddot{a}a + \dot{a}^2 + kc^2\right)$. The Einstein-Hilbert action is given by (ignoring the cosmological constant):

$$S_{grav} = \frac{1}{2k}\int \mathrm{d}^4x\sqrt{-g}R = \frac{1}{2k}\int \mathrm{d}^4xca^3\frac{6}{c^2a^2}\left(\ddot{a}a + \dot{a}^2 + kc^2\right). \tag{A.15}$$

We can use the following relation: $a^3\frac{\ddot{a}}{a} = -2a^3\left(\frac{\dot{a}}{a}\right)^2 + \frac{\partial}{\partial t}\left(a^2\dot{a}\right)$. The gravitational action becomes:

$$S_{grav} = \frac{1}{2k}\int \mathrm{d}^4x \frac{6}{c}a^3\left(-\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2}\right). \tag{A.16}$$

We consider a universe filled with a fluid with density ρ. The full action is:

$$\begin{aligned} S &= -\int \mathrm{d}t\, Na^3\left(\left(\frac{\dot{a}}{a}\right)^2 - \frac{kc^2}{a^2}\right) + S_{matter} \\ &= -\int \mathrm{d}t\, Na^3\left(\left(\frac{\dot{a}}{a}\right)^2 - \frac{kc^2}{a^2} + \frac{8\pi\mathcal{G}}{3c^2}\rho\right). \end{aligned} \tag{A.17}$$

The Lagrangian of our system is:

$$\mathcal{L} = Na^3\left(\left(\frac{\dot{a}}{a}\right)^2 - \frac{kc^2}{a^2} + \frac{8\pi\mathcal{G}}{3c^2}\rho\right), \tag{A.18}$$

with $N = \frac{3\pi c^2}{4\mathcal{G}}$. Hence the canonical Hamiltonian is: $H_c = \pi_a\dot{a} - \mathcal{L}$. We have $\pi_a = \frac{\partial\mathcal{L}}{\partial a} = -2Na\dot{a}$, hence:

$$\begin{aligned} H &= -2Na\dot{a} + Na^3\left(\left(\frac{\dot{a}}{a}\right)^2 - \frac{kc^2}{a^2} + \frac{8\pi\mathcal{G}}{3c^2}\rho\right) \\ &= -Na^3\left[\frac{p^2}{4N^2a^4} + \frac{kc^2}{a^2} - \frac{8\pi\mathcal{G}}{3c^2}\rho\right] = 0. \end{aligned} \tag{A.19}$$

We get the Wheeler De Witt equation by the usual substitution : $\hat{p} \to -i\hbar\partial_a$:

$$\left[\frac{\partial^2}{\partial a^2} - \frac{9\pi^2c^4a^2}{4\hbar^2\mathcal{G}}\left(kc^2 - \frac{8\pi\mathcal{G}}{3c^2}\rho a^2\right)\right]\Psi = 0. \tag{A.20}$$

Let's use the WKB approximation and look for solution of the form: $\Phi(a) = e^{\chi(a)}$. We are interested in solving this equation in flat space ($k = 0$). We need to solve:

$$\chi''(a) + \chi'^2(a) + B\rho a^4 = 0, \tag{A.21}$$

(where $B = \frac{6\pi^3c^2}{\hbar}$). As a first approximation, we can consider $\chi''(a) \ll 1$ and solve $\chi'^2(a) = -B\rho a^4$. If the universe is filled with a fluid with state parameter w, ρ satisfies $\rho = \rho_0 a^{-3(w+1)}$ and we need to solve:

$$\chi'^2(a) = -K\rho_0 a^{1-3w} \implies \chi'(a) = \pm i K^{1/2} a^{\frac{1-3w}{2}}. \tag{A.22}$$

Hence

$$\chi(a) = \pm i \frac{2K^{1/2}}{3(1-w)} a^{\frac{3(1-w)}{2}}, \tag{A.23}$$

and

$$\psi(a) = e^{\pm i \frac{2K^{1/2}}{3(1-w)} a^{\frac{3(1-w)}{2}}}. \tag{A.24}$$

And once again, the solution only has a classically allowed region.

B Properties of the DeWitt Metric

Here we lay out some of the different properties of the DeWitt or supermetric $G_{ijk\ell}$. We note that $G_{ijk\ell}$ will be the *contravariant* metric tensor on superspace (because it acts on h^{ij}) and thus its inverse, $G^{ijk\ell}$ will be the the *covariant* metric. So what is the covariant metric expressed in terms of the 3-metric? The contravariant metric is $G_{ijk\ell} = \frac{1}{2}h^{-1/2}(h_{ik}h_{j\ell}+h_{i\ell}h_{jk}-h_{ij}h_{k\ell})$. Since we're just raising the indices, we can expect that the covariant metric will look a lot like the contravariant metric. We also want the contraction of both tensors to yield the identity operator on the space of symmetric 3-tensors since the 3-metric lies in the space of symmetric 3-tensors. To formalize all of these notions mathematically, we want

$$G_{ijmn}G^{mnk\ell} = \frac{1}{2}\left(\delta_i^k\delta_j^\ell + \delta_i^\ell\delta_j^k\right), \quad G^{ijk\ell} = \frac{1}{2}\sqrt{h}(h^{ik}h^{j\ell} + h^{i\ell}h^{jk} - ah^{ij}h^{k\ell}), \tag{B.1}$$

where we put the a in front of the $h^{ij}h^{k\ell}$ because we recognize when we expand out this product, we'll have multiple terms with a +'s so to cancel that out we're going to need an abundance of -'s. When we expand the product we get

$$\begin{aligned} G_{ijmn}G^{mnk\ell} &= \frac{1}{4}(h_{im}h_{jn} + h_{in}h_{jm} - h_{ij}h_{mn})(h^{mk}h_{j\ell} + h^{m\ell}h^{nk} - ah^{mn}h^{k\ell}) && \text{(B.2)} \\ &= \frac{1}{2}(\delta_i^k\delta_j^\ell + \delta_i^\ell\delta_j^k) + \frac{1}{4}(a-2)h_{ij}h^{k\ell}. && \text{(B.3)} \end{aligned}$$

As we can see, if we want the inhomogeneous term to vanish, we require $a = 2$ so $G^{ijk\ell} = \frac{1}{2}\sqrt{h}(h^{ik}h^{j\ell} + h^{i\ell}h^{jk} - 2h^{ij}h^{k\ell})$. Next we can show what happens when we contract the variation in the supermetric with itself

$$G_{ijk\ell}\delta G^{ijk\ell} = \frac{1}{2}G_{ijk\ell}[\delta h^{1/2}(h^{ik}h^{j\ell} + h^{i\ell}h^{jk} - 2h^{ij}h^{k\ell}) + \sqrt{h}\delta((h^{ik}h^{j\ell} + h^{i\ell}h^{jk} - 2h^{ij}h^{k\ell}))]. \tag{B.4}$$

Next we used the following facts

$$\delta h^{1/2} = \frac{1}{2}\sqrt{h}h^{ij}\delta h_{ij}, \quad \delta h^{ij} = -h^{ik}\delta h_{k\ell}h^{\ell j}, \tag{B.5}$$

and plug this into the contracted variation equation while also plugging in the expression for $G_{ijk\ell}$ to get

$$G_{ijk\ell}\delta G^{ijk\ell} = -h^{ij}\delta h_{ij}. \tag{B.6}$$

We can use the variation in the 3-determinant, $\delta h = hh^{ij}\delta h_{ij}$, to rewrite the above

$$G_{ijk\ell}\delta G^{ijk\ell} = -\frac{\delta h}{h}. \tag{B.7}$$

Now we see that the variation in the determinant of the supermetric can be related to the determinant of the 3-metric

$$\delta G = \delta(\det\big(G^{ijk\ell}\big)) = GG_{ijk\ell}\delta G^{ijk\ell} = -\frac{G}{h}\delta h. \tag{B.8}$$

This equation implies that

$$\frac{\delta G}{G} = -\frac{\delta h}{h} \Rightarrow \delta \ln G = -\delta \ln h \Leftrightarrow \delta(\ln Gh) = 0. \tag{B.9}$$

Which we can further deduce that

$$G = -\frac{a}{h}, \tag{B.10}$$

where a is some arbitrary constant. To figure out what a is, we can pick a particular basis of the h_{ij} and figure out what the components of $G^{ijk\ell}$ come out to be (we can do this because the determinant of a matrix/linear operator is basis independent). So pick the basis where $h_{ij} = \delta_{ij}$ and pick particular values of i, j, k, ℓ to find $G^{ijk\ell}$. Doing so gives us $-\frac{1}{2}, 1, 1, 1, 1, 1$ as the only independent components of the metric in this basis. It follows that $a = -\frac{1}{2}$ and that the signature for this metric is $(-, +, +, +, +, +)$.

C Deriving the Lorentzian Action

Here we derive the Lorentzian action for the minisuperspace model. We start off with the spacetime interval for our model

$$\mathrm{d}s^2 = \sigma^2\big[-N^2(t)\,\mathrm{d}t^2 + a^2(t)\,\mathrm{d}\Omega_3^2\big], \tag{C.1}$$

where $\mathrm{d}\Omega_3^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\alpha^2 + \sin^2\theta\sin^2\alpha\,\mathrm{d}\beta^2$ is the metric on the unit 3-sphere with $\alpha, \theta \in [0, \pi]$ and $\beta \in [0, 2\pi]$. The total action is $S = S_{EH} + S_M$ with the Einstein-Hilbert action and the action for a conformally invariant scalar field being

$$S_{EH} = \frac{1}{2\kappa^2}\int \mathrm{d}^4x\sqrt{-g}(R - 2\Lambda), \qquad S_M = -\frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\left[(\nabla\phi)^2 + \frac{1}{6}R\phi^2\right], \tag{C.2}$$

We shall deal with these two actions separately and then we'll put all the pieces back together at the very end. First we'll focus on the gravitational sector. The 4-volume and Ricci scalar in terms of the above variables are

$$R = 6\left[\frac{N^3 - a\dot{a}\dot{N} + N(\dot{a}^2 + a\ddot{a})}{\sigma^2 a^2 N^3}\right], \qquad \sqrt{-g} = \sigma^4 a^3 N \sin^2\theta\sin\alpha. \tag{C.3}$$

Next when we plug these quantities back into the action, we get

$$S_{EH} = \frac{6\sigma^2}{2\kappa^2}\int \mathrm{d}t\,\mathrm{d}\theta\,\mathrm{d}\alpha\,\mathrm{d}\beta\sin^2\theta\sin\alpha\left[\frac{N^3 - a\dot{a}\dot{N} + N(\dot{a}^2 + a\ddot{a})}{N^3} - \lambda N a^3\right], \tag{C.4}$$

where we used the substitution $\Lambda = \frac{3\lambda}{\sigma^2}$. Since the angular integrals are independent of one another, we can use Frobini's theorem to evaluate them

$$\int_0^\pi \sin^2\theta\,\mathrm{d}\theta\int_0^\pi \sin\alpha\,\mathrm{d}\alpha\int_0^{2\pi}\mathrm{d}\beta = 2\pi^2, \tag{C.5}$$

where we used the double angle formula for sine to reduce the power we were raising sine to. The Einstein-Hilbert action becomes

$$S_{EH} = \frac{12\pi^2\sigma^2}{2\kappa^2} \int \mathrm{d}t \left[aN - \frac{a^2\dot{a}\dot{N}}{N^2} + \frac{a\dot{a}^2}{N} + \frac{a^2\ddot{a}}{N} - \lambda N a^3 \right]. \tag{C.6}$$

Next we can use integration by parts on the second term where $u = a^2\dot{a}$ and $\mathrm{d}v = -\frac{\dot{N}}{N^2}$. Doing so gives us $-\int \frac{a^2\ddot{a}+2a\dot{a}^2}{N}\,\mathrm{d}t$ which reduces the above action to

$$\begin{aligned} S_{EH} &= \frac{1}{2} \int \mathrm{d}t \left[aN - \frac{a\dot{a}^2}{N} - \lambda N a^3 \right] && \text{(C.7)} \\ &= \frac{1}{2} \int \mathrm{d}t \, \frac{N}{a} \left[-\left(\frac{a}{N}\frac{\mathrm{d}a}{\mathrm{d}t} \right)^2 + a^2 - \lambda a^4 \right]. && \text{(C.8)} \end{aligned}$$

Now we focus on the matter action. Since the field is homogeneous and covariant derivatives act as partial derivatives when operating on scalar fields, the action becomes

$$S_M = -\frac{1}{2} \int \mathrm{d}^4x [\sigma^4 a^3 N \sin^2\theta \sin\alpha] \left[-\frac{1}{\sigma^2 N^2}\dot{\phi}^2 + \left[\frac{N^3 - a\dot{a}\dot{N} + N(\dot{a}^2 + a\ddot{a})}{\sigma^2 a^2 N^3} \right] \phi^2 \right]. \tag{C.9}$$

We introduce our conformally invariant scalar field $\phi = \frac{\tilde{\phi}}{h^{1/6}} = \frac{\chi}{a\sqrt{2\pi^2\sigma^2}}$. This gives us

$$\sqrt{2\pi^2\sigma^2}\dot{\phi} = \frac{\dot{\chi}}{a} - \frac{\dot{a}\chi}{a^2} \Rightarrow 2\pi^2\sigma^2\dot{\phi}^2 = \left(\frac{\dot{\chi}}{a} \right)^2 - \frac{2\dot{a}\chi\dot{\chi}}{a^3} + \frac{\dot{a}^2\chi^2}{a^4}. \tag{C.10}$$

Plugging all of these relations into the action yields

$$S_M = \frac{1}{2} \int \mathrm{d}t \left[\frac{a\dot{\chi}^2}{N} - \frac{2\dot{a}\chi\dot{\chi}}{N} + \frac{\dot{a}^2\chi^2}{aN} - \frac{N}{a}\chi^2 + \frac{\dot{a}\dot{N}}{N^2}\chi^2 - \frac{\dot{a}^2\chi^2}{aN} - \frac{\ddot{a}}{N}\chi^2 \right]. \tag{C.11}$$

Preforming another integration by parts with: $u = \dot{a}\chi^2$ and $\mathrm{d}v = -\frac{\dot{N}}{N^2}$ gives us

$$\begin{aligned} S_M &= \frac{1}{2} \int \mathrm{d}t \left[\left(\frac{a\dot{\chi}^2}{N} - \frac{2\dot{a}\chi\dot{\chi}}{N} + \frac{\dot{a}^2\chi^2}{aN} \right) - \left(\frac{N}{a}\chi^2 - \frac{2\dot{a}\chi\dot{\chi}}{N} + \frac{\dot{a}^2\chi^2}{aN} \right) \right] && \text{(C.12)} \\ &= \frac{1}{2} \int \mathrm{d}t \, \frac{N}{a} \left[\left(\frac{a}{N}\frac{\mathrm{d}\chi}{\mathrm{d}t} \right)^2 - \chi^2 \right]. && \text{(C.13)} \end{aligned}$$

And when we put all of the pieces back together again, we get

$$S = S_{EH} + S_M = \frac{1}{2} \int \mathrm{d}t \, \frac{N}{a} \left[-\left(\frac{a}{N}\frac{\mathrm{d}a}{\mathrm{d}t} \right)^2 + a^2 - \lambda a^4 + \left(\frac{a}{N}\frac{\mathrm{d}\chi}{\mathrm{d}t} \right)^2 - \chi^2 \right]. \tag{C.14}$$

D The Question of Boundary Conditions in the Hartle-Hawking Wave Function of the Universe

Introduction

We have recently been studying and thinking about the role of boundary and initial conditions in the Hartle-Hawking wave function of the universe. The question of boundary conditions is a complicated issue, and so far a satisfying answer to the exact role of boundary conditions has not been found.

The outline of this paper is as follows. In Section 2, we introduce the issue of initial conditions and present Hartle and Hawking's "No-Boundary Proposal." Section 3 discusses the path integral formalism used for the Hartle-Hawking wave function. After touching on criticisms of the no-boundary proposal in Section 4, we discuss the idea of a universality of inflation in the minisuperspace model in Section 5 before turning to the topic of the generalized Hartle-Hawking state in Section 6. We summarize and conclude in Section 7.

No-Boundary Proposal

Consider the following quote from Halliwell's *Introductory Lectures on Quantum Cosmology*, where he argues that inflation depends on the initial conditions of the universe [5]:

"Whilst it is certainly true that, as a result of inflation, the observed universe could have arisen from a much larger class of initial conditions than in the hot big bang model, it is certainly not true that it could have arisen from any initial state - one could choose initial conditions which did not lead to the correct density perturbation spectrum, and indeed, one could choose initial conditions for which inflation does not occur."

In quantum cosmology, one is interested in the wave function of a closed universe,

$$\Psi\left[h_{ij}(\mathbf{x}), \Phi(\mathbf{x}), B\right], \tag{D.1}$$

which is the amplitude that the universe contains a three-surface B on which the three-metric is $h_{ij}(\mathbf{x})$ and the matter field configuration is $\Phi(\mathbf{x})$. From this, one can derive the *Wheeler-DeWitt equation*, which the wave function must satisfy and is analogous to the Schrödinger equation. Because the Wheeler-DeWitt equation will in general have many solutions, it is necessary to use initial conditions to pick out a particular solution.

In regions where spacetime is essentially classical, one finds the wave function to be peaked about a particular set of solutions to the classical Einstein equations. This set of solutions is a subset of the general solution, by virtue of the boundary conditions. Additionally, the boundary conditions pick out a particular choice of vacuum state for the matter fields [5].

In light of the seeming importance of boundary conditions, Hartle and Hawking's "no-boundary proposal" from their original 1983 paper is intriguing. They argue that the initial boundary conditions of the universe is that there is no boundary. According to this proposal, the universe smoothly expanded from a point of zero size (similar to a shuttlecock). In this way, their proposed "wave function of the universe" encompasses all time at once - the past, present and future.

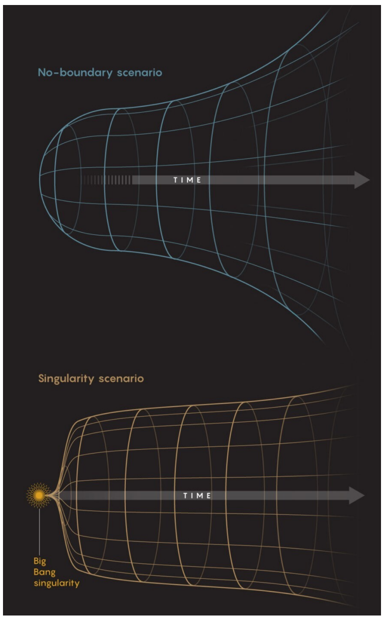

Figure 1: A pictorial representation of the no-boundary proposal, in which the universe expands smoothly from its beginning, versus the case in which there was a singularity when the universe originated. Taken from [8].

Each moment in time in the universe corresponds to a cross-section, or slice, of this so-called shuttlecock; one evolves from one moment in time to the next via correlations between the size of the cross-section and its other properties, such as entropy. As the shuttlecock rounds off, these correlations become less reliable, and time ceases to exist [8].

Path Integral Formalism

In their 1983 paper, Hartle and Hawking express the wave function of the universe as the sum of all possible ways it could have smoothly expanded from a single point, *i.e.* a sum over all possible expansion histories. More concretely, Hartle and Hawking write for the ground state wave function

$$\Psi_0[h_{ij}] = N \int \delta g e^{-I_E[g]} \tag{D.2}$$

where N is a normalization factor and $I_E[g]$ is the Euclidean action for gravity which includes the cosmological constant Λ. They propose that this integral is over compact 4-geometries, since it includes 1) the Euclidean four-geometries that have a boundary on which the metric is h_{ij}, and 2) the remaining class of geometries, which determine the ground state.

Here is the confusing statement: an integral over all compact 4-geometries indicates that the universe has no boundaries in space or time. However, Hartle and Hawking say that this integral over all compact four-geometries is *bounded by a given three-geometry*, and that this three-geometry is its only boundary [6].

An interesting fact to point out is that Hartle and Hawking only actually use the path integral formalism in the minisuperspace model and semiclassical approximations, and not for the general case when one is truly integrating over all four-geometries. When using the path integral semiclassically and/or in minisuperspace, then one would restrict the integral to be over a certain class of four-geometries.

How does one go about choosing the right contour for the path integral? The contour "picks out" the expansion history that physically makes sense. There are two classical solutions that dominate in the minisuperspace model; one that corresponds to our universe (radiation density fluctuations from a normal distribution around zero) and one that doesn't (radiation density fluctuations go to infinity) [6] [8].

The problem becomes when one tries to generalize the Hartle-Hawking wave function beyond the minisuperspace model. In theory, extension of Eq. (D.2) would just be

$$\Psi_0[\partial M^{(1)}, h_{ij}^{(1)}, ..., \partial M^{(N)}, h_{ij}^{(N)}] = \int \delta g e^{-I_E[g]}, \tag{D.3}$$

where the $\partial M^{(i)}$ are the compact boundaries and $h_{ij}^{(i)}$ their three-metrics [6]. There are a couple of problems with Eq. (D.3). The first is that the integral is over N disconnected compact boundaries; however, integrating this for all superspace would mean $N = \infty$, since superspace is infinite-dimensional. Another way of saying this is that there are infinitely many possible shapes and sizes of the universe that one would be summing over. So already there is an issue with the definition of Eq. (D.3). The second problem is that the Wheeler-Dewitt equation is no longer sufficient to calculate the wave function Ψ_0 because the correlations between the three-geometries are no longer trivial [6]. However, there may be a solution around this by looking at what is known as the *Kodama state*. This is discussed further in Section 6.

Criticisms of the No-Boundary Proposal

One of the main objections to the no-boundary proposal is that is does not corresponds to our physical observations of the universe. If it were true, the Hartle-Hawking path integral would have an extremum resulting from the cosmological constant that would therefore lead to rapid early inflation. The result is a very nearly empty de Sitter spacetime for the early universe, which does not match what we observe. There are several possible resolutions to this challenge, but it's unclear whether any of these are satisfactory [7].

Universality of Inflation?

Consider general relativity coupled to a scalar field ϕ. The action of the Hamiltonian constraint is gauge fixed so that ϕ is constant on surfaces of constant time:

$$\partial_a \phi = 0, \tag{D.4}$$

where a is a spatial index. Eq. (D.4) is not only a choice of gauge, but also a restriction on the space of solutions.

In the spatially homogeneous case, one finds an attractor that would seem to indicate that all initial conditions preserving the gauge condition (D.4) merge to the same solution, which suggests a universality of inflation [4].

It appears that the reason for restricting first to the spatially homogeneous case and demanding certain gauge conditions and diffeomorphism constraints is to select conditions that will yield physical conditions, *i.e.* to match what we actually observe in the universe. Then the results from [4] may indicate that all of the solutions in this subset of the general solution correspond to inflation. It seems to be a bit of circular reasoning in the sense that one gets back inflation for all conditions that would correspond to inflation in the first place. Nevertheless, the fact that the no-boundary proposal does *not* exclude non-physical solutions is one of its main criticisms.

Generalizing the Hartle-Hawking Wave Function

The question now is how to generalize the Hartle-Hawking wave function. One possible way may be through the Kodama (or Chern-Simons) state. The Hartle-Hawking wave function is known to be the Fourier dual to the Kodama or Chern-Simons state in the minisuperspace model; therefore, it is theorized that the generalization of the Kodama state would correspond to generalizing the Hartle-Hawking wave function beyond minisuperspace. Indeed, the generalized Hartle-Hawking state has already been computed for a number of models, but these results still needs to be interpreted [3].

Conclusion

The issue of boundary conditions of the universe and in the Hartle-Hawking wave function remains an unresolved problem in quantum cosmology. Hartle and Hawking's "no-boundary proposal" is certainly not flawless, and it has been met with criticism. The issue of boundary conditions seems to be a reason why generalizing the Hartle-Hawking wave function beyond a spatially homogeneous and isotropic model, *i.e.* minisuperspace, is non-trivial. While there are promising next steps, a lot of work remains to be done in this area of quantum cosmology.

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