The Feynman Rules

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0 Preliminary Stuff

Before we start writing down the Feynman rules, there are some other rules/definitions we need to know first. First, we define on-shell:

Definition 0.1 *We say that a quantity is* ***on shell*** *when it satisfies some classical equations of motion. Otherwise, we say it is off shell.*

For example, the Euler-Lagrange Equations gives on shell equations. While virtual particles are off shell because they don't satisfy the energy-momentum relation: $E^2 - |\mathbf{p}|^2 = m^2$. We call this the mass shell. Next, we should define what a scattering amplitude even is:

Definition 0.2 *A* ***scattering amplitude****, $\mathcal{A}$, for a Feynman diagram, $\mathcal{F}_n$, encodes the probability amplitude of one particle scattering off of another.*

We leave a more detailed explanation of finding a scattering amplitude below. Here we define renormalization.

Definition 0.3 *A field is* ***renormalizable*** *when through the process of introducing counter-terms into the Lagrangian, we prevent the path integral over all field configurations from diverging (i.e. we "tame" the infinities that is present within the theory). Otherwise we say* *that the field is non-renormalizable. By introducing counter-terms, we recognize that we do not measure infinities in nature, and thus, every infinity must be swallowed/absorbed into some parameter, while the other term must be what we find in experiment.*

The reason why we even talk about renormalization is because it helps in providing us a complete quantum mechanical description of the universe. Speaking of which:

Definition 0.4 *A (renormalizable) field theory is* ***complete*** *if its path integral is finite for any arbitrary length/energy scale.*

As an example, Quantum Chromodynamics (QCD) is considered complete for the reason articulated above. While we're at it, we should also define what an effective field theory is:

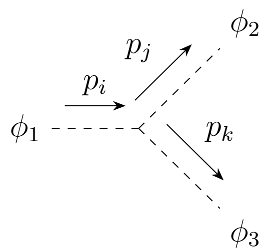
Definition 0.5 *A field theory is an* ***effective field theory*** *if its path integral is finite only up to some arbitrary cut-off length/energy scale. Therefore, it is an* ***effective*** *description of the underlying quantum phenomenon. Typically, for a theory to only be an effective description, it is non-renormalizable.*

For example, because General Relativity (GR) is non-renormalizable (requires an infinite number of counter-terms to tame the infinities), any quantization done on Einstein's equations would be an effective field theory description. Lastly, we have the rule that every delta function that gets written down, must also have a factor of 2π and every measure element (dq) gets a factor of $1/2\pi$. So for the 4-d delta function and 4-d volume element we write

$$(2\pi)^4\delta^4(x), \qquad \frac{1}{(2\pi)^4}d^4q. \tag{0.1}$$

1 Scalar Field Rules

Here we write down the Feynman rules for computing the scattering amplitude $\mathcal{A}$ from the Lagrangian. Say we have a generic scalar field ϕ, and we wish to compute its amplitude.



1. Diagram Order: Before we start computing amplitudes, we must decide to which order we may wish to look at. Since the above diagram has three lines, we'll focus on computing cubic order fields (since this is the lowest order Feynman diagram, $\mathcal{F}_3$, one can have). That being said, everything that we state for a three-point digram can be easily generalized to an n-point diagram.

$$\mathcal{L}_3 = -\frac{\lambda}{3!}\phi^3, \qquad \mathcal{L}_3 = -\frac{\lambda}{3!}\phi(\partial\phi)^2, \qquad \mathcal{L}_3 = \frac{1}{\Lambda^3}(\partial\phi)^2\Box\phi. \tag{1.1}$$

2. Differentiate: Once we decide what order we may wish to compute the diagrams, now we can actually get down to business. Our next rule is that we differentiate the Lagrangian $\mathcal{L}$ with respect to ϕ. The number of times we take differentiate w.r.t. ϕ corresponds to the order of the desired diagram. So if we want a three line diagram such as the one above, then we need to differentiate with respect to ϕ three times.

3. Momentum: When differentiating, we might have a term that includes derivatives of ϕ, say $(\partial\phi)^2$. The way we go about dealing with terms like this is simply differentiating as we would normally do, but replace the derivative with a corresponding momentum i.e. $(\partial\phi)^2 \to 2p_\mu\partial^\mu\phi$. If one encounters a d'Alembert operator then the rule is $\Box \to p_\mu p^\mu$. However, this can only be done when our scalar field respect Lorentz symmetry. If it doesn't

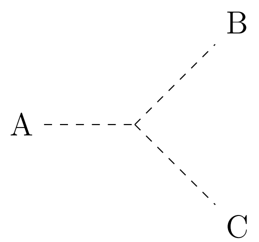
then the rules are slightly different. For every time derivative that exists, replace it with the energy of the field so $\frac{\mathrm{d}}{\mathrm{d}t} \to -iE$ and for every spatial derivative, replace it with the spatial momentum $\nabla \to -i\mathbf{p}$.

4. Notation: Clearly label the incoming and outgoing momenta after you've replaced the derivatives in favor of the aforementioned momenta. The first two derivatives correspond to the momenta of the incoming external lines that are connected at the vertex we're looking at. The next one or two correspond to the momenta of the outgoing external lines that are attached to the vertex of interest. It could be helpful to the audience to label them with Latin indices, so we don't confuse ourselves with what we call particle 1 (with momentum p_1) and with what we refer to as the momentum of the first external incoming line.

5. Vertex Rule: Once we've differentiated to the point where we eliminated all ϕ's of the order we wish to compute our diagrams to, whatever constant is leftover will be what we write down for every vertex. This is what we will call the coupling constant. Once we have this factor, its merely a matter of looking at the diagram and plugging in the appropriate momenta into the correct placement within the vertex rule.

2 Diagram Rules

Now, we shall follow the lead of Griffiths and write down the Feynman Rules for a toy model of spin 0 massive particles with the following interaction: $A \to B + C$. And of course, these are the rules for the lowest order (i.e. tree) $\mathcal{F}_3$ for a scattering amplitude $\mathcal{A}$, with the Lagrangian $\mathcal{L}$ and defining $\alpha \equiv \frac{\delta}{\delta\phi}\frac{\delta}{\delta\phi}\frac{\delta\mathcal{L}}{\delta\phi}$.



1. Notation: Label the incoming and outgoing 4-momenta as $p_1, p_2, ..., p_n$. Label the internal momenta as $q_1, q_2, ...$ Put an arrow beside each line to keep track of the 'positive' direction.

2. Vertex Factors: For each vertex, write down a factor

$$i\alpha, \tag{2.1}$$

where $i = \sqrt{-1}$ and α is the factor that was found from above; it specifies the strength of the interaction between A, B, and C. In this toy theory, g has dimensions of momentum however, typically coupling constants are dimensionless.

3. Propagators: Each internal line gets a factor of

$$\frac{i}{q_j^2 - m_j^2}, \tag{2.2}$$

where q_j and m_j is the 4-momentum and mass-squared of the j-th particle the internal line describes. (Note that $q_j \neq m_j$ because a virtual particle doesn't lie on its mass shell.)

4. Conservation of Energy/Momentum: Each vertex gets a delta function in the form of

$$(2\pi)^4\delta^4(k_1 + k_2 + k_3), \tag{2.3}$$

where the k's are the three four momenta coming into the vertex ($-k_i$ for outgoing momenta). This factor imposes conservation of energy/momentum at each vertex, since the delta function is zero unless the sum of the incoming momenta equals the sum of the outgoing momenta.

5. Integration Over Internal Momenta: For each internal line, write down a factor

$$\frac{1}{(2\pi)^4}d^4q_j. \tag{2.4}$$

and integrate over all internal momenta.

6. Cancel the Delta Function: The result will include a delta function

$$(2\pi)^4\delta^4(p_1 + p_2 + ... - p_n), \tag{2.5}$$

reflecting overall conservation of energy/momentum. Erase this factor and multiply by i. The result is $\mathcal{A}$ for that diagram. If there are more than one diagram to compute an amplitude for, then the total amplitude is the sum of all the little amplitudes, i.e. for n = 3 $\mathcal{F}_3$, the final amplitude is

$$\mathcal{A}_{tot} = \sum_{i=1}^{3}\mathcal{A}_i. \tag{2.6}$$