Energy Momentum Tensor for Generic Scalar Field Coupling to Gravity

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1 Introduction

We want to derive the energy-momentum tensor for a scalar field, ϕ, from variations of the metric, $g_{\mu\nu}$. In a typical QFT course, one can derive the energy-momentum tensor by simply considering Noether's Theorem. The theorem states that every symmetry of a Lagrangian, $\mathcal{L}$, carries with it a conserved quantity i.e.

$$\mathcal{L} \to \mathcal{L} + \delta\mathcal{L} \Rightarrow \delta\mathcal{L} = 0. \tag{1.1}$$

In flat space, the Lagrangian for a (free) scalar field is given simply by

$$\mathcal{L} = -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi), \tag{1.2}$$

where $V(\phi)$ contains all the mass and self-interacting terms of the scalar field. The above expression is invariant under the transformation $x^\mu \to x'^\mu = x^\mu + \epsilon^\mu$ where ϵ^μ is a constant. Under this transformation, the Lagrangian varies by

$$\mathcal{L} \to -\frac{1}{2}\eta'^{\mu\nu}\frac{\partial\phi'}{\partial x'^\mu}\frac{\partial\phi'}{\partial x'^\nu} - V(\phi') = -\frac{1}{2}\eta^{\mu\nu}\frac{\partial\phi}{\partial x^\mu}\frac{\partial\phi}{\partial x^\nu} - V(\phi) = \mathcal{L}, \tag{1.3}$$

where we used the fact that $\phi'(x') = \phi(x)$, $\eta'_{\mu\nu} = \eta_{\mu\nu}$ and

$$\frac{\partial}{\partial x'^\mu} = \frac{\partial}{\partial x^\mu}. \tag{1.4}$$

So the Lagrangian is invariant under constant spacetime translation. Therefore we can identify the conserved quantity. Now we again let $x'^{\mu} = x^{\mu} + \epsilon^{\mu}$ and we get

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}\,\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu(\delta\phi) + \frac{\partial\mathcal{L}}{\partial x^\mu}\epsilon^\mu = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,\delta\phi\right) + \frac{\partial\mathcal{L}}{\partial x^\mu}\epsilon^\mu = 0, \tag{1.5}$$

where we used the Euler-Lagrange Equations with $\frac{\partial\mathcal{L}}{\partial\phi} = \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}$. Now we take $\delta\phi(x) = -\frac{\partial\phi}{\partial x^\nu}\epsilon^\nu = -\frac{\partial\phi}{\partial x_\nu}\epsilon_\nu$ and the total change in the Lagrangian is then

$$\delta\mathcal{L} = \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial^\nu\phi - g^{\mu\nu}\mathcal{L}\right]\epsilon_\nu \equiv \partial_\mu T^{\mu\nu}\epsilon_\nu = 0, \tag{1.6}$$

where we can define the energy-momentum tensor for the scalar field to be

$$T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial^\nu\phi - g^{\mu\nu}\mathcal{L}. \tag{1.7}$$

Unfortunately, the problem with this definition of the energy-momentum tensor is that (1) in general it's not symmetric and (2) its not gauge invariant. These both are exemplified in E&M where the energy momentum tensor is written as

$$T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu A^\lambda)}\partial^\nu A^\lambda - g^{\mu\nu}\mathcal{L}, \tag{1.8}$$

which is clearly not symmetric, let alone gauge invariant. Thus, we need a definition of the energy-momentum tensor that is symmetric (because the Einstein tensor is symmetric) as well as gauge invariant. The definition that does this is

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}, \tag{1.9}$$

where S is the action of whatever matter fields that exist in the Lagrangian.

Conventions We use the mostly plus metric signature, i.e. $\eta_{\mu\nu} = (-,+,+,+)$ and units where $c = \hbar = 1$. The reduced four dimensional Planck mass is $M_{\rm Pl} = (8\pi G)^{-1/2} \approx 2.43 \times 10^{18}$ GeV. The d'Alembert and Laplace operators are defined to be $\Box \equiv g^{\mu\nu}\partial_\mu\partial_\nu$ and $\nabla^2 = \partial_i\partial^i$ respectively. We use boldface letters $\mathbf{r}$ to indicate 3-vectors and x and p to denote 4-vectors. Conventions for the curvature tensors, covariant and Lie derivatives are all taken from Carroll.

2 The Derivation

Here we derive the proper form of the stress-energy tensor given arbitrary coupling to the curvature scalar. First we introduce the relevant expression

$$S_M = -\frac{1}{2}\int \sqrt{-g}\big[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + 2V(\phi) + \xi R\phi^2\big]\,\mathrm{d}^d x\,, \tag{2.1}$$

where $V(\phi)$ is the potential. Now we vary with respect to the metric. We write

$$\begin{aligned} \delta S_M = &-\frac{1}{2}\int \delta\sqrt{-g}\,\mathrm{d}^d x\,\big[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + 2V(\phi) + \xi R\phi^2\big] \\ &-\frac{1}{2}\int \sqrt{-g}\,\mathrm{d}^d x\,\big[\delta g^{\mu\nu}\,\partial_\mu\phi\partial_\nu\phi + \xi\,\delta R\,\phi^2\big]. \end{aligned} \tag{2.2}$$

We've done previous derivations for what $\delta\sqrt{-g}$ and δR are, so we will just quote those results

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\,\delta g^{\mu\nu}\,, \qquad \delta R = R_{\mu\nu}\,\delta g^{\mu\nu} + g^{\mu\nu}\,\delta R_{\mu\nu}\,, \tag{2.3}$$

where the variation in the Ricci scalar can be further expanded into

$$\delta R = R_{\mu\nu}\,\delta g^{\mu\nu} + (g^{\mu\nu}g^{\lambda\rho} - g^{\mu\lambda}g^{\nu\rho})\nabla_\lambda[\nabla_\mu\,\delta g_{\nu\rho} + \nabla_\nu\,\delta g_{\mu\rho} - \nabla_\rho\,\delta g_{\mu\nu}]. \tag{2.4}$$

We also needed the following identities in order to compute the above

$$\delta g_{\mu\nu} = -g_{\mu\lambda}g_{\nu\rho}\,\delta g^{\lambda\rho}\,, \qquad g^{\mu\nu}\,\delta g_{\mu\nu} = -g_{\mu\nu}\,\delta g^{\mu\nu}\,. \tag{2.5}$$

Next we can throw these terms into the action to get

$$\begin{aligned} \delta S_M = &-\frac{1}{2}\int\left(-\frac{1}{2}\sqrt{-g}g_{\mu\nu}\,\delta g^{\mu\nu}\right)\mathrm{d}^d x\,\big[g^{\lambda\rho}\partial_\lambda\phi\partial_\rho\phi + 2V(\phi) + \xi R\phi^2\big] \\ &-\frac{1}{2}\int\sqrt{-g}\,\mathrm{d}^d x\,\big[\delta g^{\mu\nu}\,\nabla_\mu\phi\nabla_\nu\phi + \xi\big(\phi^2 R_{\mu\nu}\,\delta g^{\mu\nu} + \delta g^{\mu\nu}\,g_{\mu\nu}\Box\phi^2 - \delta g^{\mu\nu}\,\nabla_\mu\nabla_\nu\phi^2\big)\big]. \end{aligned} \tag{2.6}$$

We can write this under a single integral sign

$$\begin{aligned} \delta S_M = -\frac{1}{2}\int\sqrt{-g}\,\mathrm{d}^d x\,\delta g^{\mu\nu}\Bigg[&\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}g^{\lambda\rho}\nabla_\lambda\phi\nabla_\nu\phi - 2V(\phi)g_{\mu\nu} + \xi\big(g_{\mu\nu}\Box\phi^2 - \nabla_\mu\nabla_\nu\phi^2\big) \\ &+\xi\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right)\phi^2\Bigg]. \end{aligned} \tag{2.7}$$

Now by boldly defining the stress-energy tensor by the previous formula we gave, we're left with

$$T_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}g^{\lambda\rho}\nabla_\lambda\phi\nabla_\rho\phi - V(\phi)g_{\mu\nu} + \xi\left(g_{\mu\nu}\Box\phi^2 - \nabla_\mu\nabla_\nu\phi^2\right) + \xi\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right)\phi^2. \tag{2.8}$$

Specializing to the case $V(\phi) = \frac{1}{2}m^2\phi^2$, we can also compute the trace given by

$$T \equiv g^{\mu\nu}T_{\mu\nu} = \left(1 - \frac{d}{2}\right)\nabla^\mu\phi\nabla_\mu\phi + \xi(d-1)\Box\phi^2 + \xi\left(1 - \frac{d}{2}\right)R\phi^2 - \frac{d}{2}m^2\phi^2 \tag{2.9}$$

We can do partial integration on the first gradient-squared term to get

$$T = \left(1 - \frac{d}{2}\right)\nabla^\mu(\phi\nabla_\mu\phi) - \left(1 - \frac{d}{2}\right)\phi\Box\phi + \xi\left(1 - \frac{d}{2}\right)R\phi^2 - \frac{d}{2}m^2\phi^2 \tag{2.10}$$

$$= \left(1 - \frac{d}{2}\right)\nabla^\mu(\phi\nabla_\mu\phi) + \left(1 - \frac{d}{2}\right)\phi(-\Box + m^2 + \xi R)\phi + 2\xi(d-1)(\phi\nabla_\mu\phi) - m^2\phi^2 \tag{2.11}$$

$$= -\left[\left(\frac{d}{2} - 1\right) - 2\xi(d-1)\right]\nabla^\mu(\phi\nabla_\mu\phi) - m^2\phi^2, \tag{2.12}$$

where we made use of the Klein-Gordon equations of motion for the scalar field

$$-\Box\phi + m^2\phi + \xi R\phi = 0, \tag{2.13}$$

as well as the fact that

$$\Box\phi^2 = \nabla^\mu\nabla_\mu\phi^2 = 2\nabla^\mu(\phi\nabla_\mu\phi). \tag{2.14}$$

We can clearly see that for $\xi = (d-2)/4(d-1)$ and $m = 0$ that the trace vanishes. This tells us that the stress-energy tensor and hence the action is conformally invariant when we set $\xi = (d-2)/4(d-1)$ and $m = 0$. Finally, we can express the energy-momentum tensor in an alternative way show to be

$$\begin{aligned} T_{\mu\nu} = (1 - 2\xi)\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}(1 - 4\xi)g_{\mu\nu}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2 g_{\mu\nu} + \xi G_{\mu\nu}\phi^2 \\ + 2\xi(\phi\Box\phi g_{\mu\nu} - \phi\partial_\mu\partial_\nu\phi + \phi\Gamma^\lambda_{\mu\nu}\partial_\lambda\phi), \end{aligned} \tag{2.15}$$

where $(\nabla\phi)^2 \equiv g^{\lambda\rho}\nabla_\lambda\phi\nabla_\rho\phi$.

3 Covariant Conservation

Next we can show that for an arbitrary interaction term in the form

$$\frac{\lambda}{n!}\phi^n, \tag{3.1}$$

where $n \in \mathbb{N} \geq 3$, the stress-energy tensor is still conserved. First, we recognize that

$$V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{n!}\phi^n, \tag{3.2}$$

which alters the equations of motion to be of the form

$$\big(-\Box + m^2 + \xi R\big)\phi + \frac{\lambda}{(n-1)!}\phi^{n-1} = 0. \tag{3.3}$$

Now take the divergence of the stress-energy tensor

$$\begin{aligned}\nabla^\nu T_{\mu\nu} = {} & \nabla^\nu(\nabla_\mu\phi)\nabla_\nu\phi + \nabla_\mu\phi\Box\phi - \frac{1}{2}\big(g^{\lambda\rho}\nabla_\mu(\nabla_\lambda\phi)\nabla_\rho\phi + g^{\lambda\rho}\nabla_\lambda\phi\nabla_\mu(\nabla_\rho\phi)\big) \\ & - m^2\phi\nabla_\mu\phi - \frac{\lambda}{(n-1)!}\phi^{n-1}\nabla_\mu\phi + \xi(\nabla_\mu\Box\phi^2 - \nabla^\nu\nabla_\mu\nabla_\nu\phi^2) + \xi\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right)\nabla^\nu\phi^2.\end{aligned} \tag{3.4}$$

Using the fact that for a scalar field

$$\nabla_\mu\nabla_\nu\phi = \nabla_\nu\nabla_\mu\phi, \tag{3.5}$$

we can see that

$$\begin{aligned}\nabla^\nu T_{\mu\nu} = {} & \nabla^\nu(\nabla_\mu\phi)\nabla_\nu\phi - \frac{1}{2}((\nabla_\nu\nabla_\mu\phi)\nabla^\nu\phi + \nabla^\nu\phi\nabla_\mu(\nabla_\nu\phi)) + \xi R_{\mu\nu}\nabla^\nu\phi^2 \\ & + \nabla_\mu\phi\left(\Box\phi - m^2\phi - \xi R\phi - \frac{\lambda}{(n-1)!}\phi^{n-1}\right) + \xi(\nabla_\mu\Box\phi^2 - \nabla^\nu\nabla_\mu\nabla_\nu\phi^2),\end{aligned} \tag{3.6}$$

and therefore the first two terms cancel each other. We can also see the first term on the second line vanishes due to the equations of motion. We are then left with

$$\nabla^\nu T_{\mu\nu} = \xi R_{\mu\nu}\nabla^\nu\phi^2 + \xi(\nabla_\mu\Box\phi^2 - \nabla^\nu\nabla_\mu\nabla_\nu\phi^2). \tag{3.7}$$

In order to show this vanishes, recall that

$$[\nabla_\mu, \nabla_\nu]V^\lambda = \nabla_\mu \nabla_\nu V^\lambda - \nabla_\nu \nabla_\mu V^\lambda = R^\lambda{}_{\rho\mu\nu} V^\rho. \tag{3.8}$$

Taking the trace of the Riemann tensor in the λ, μ indices gives us

$$R^\lambda{}_{\rho\lambda\nu} V^\rho = R_{\nu\rho} V^\rho = \nabla_\lambda \nabla_\nu V^\lambda - \nabla_\nu \nabla_\lambda V^\lambda. \tag{3.9}$$

Renaming our indices and letting $V^\nu = \nabla^\nu \phi^2$ we get

$$R_{\mu\nu} \nabla^\nu \phi^2 = \nabla_\nu \nabla_\mu \nabla^\nu \phi^2 - \nabla_\mu \Box \phi^2, \tag{3.10}$$

$$\nabla^\nu T_{\mu\nu} = \xi(\nabla_\nu \nabla_\mu \nabla^\nu \phi^2 - \nabla_\mu \Box \phi^2) + \xi(\nabla_\mu \Box \phi^2 - \nabla^\nu \nabla_\mu \nabla_\nu \phi^2) = 0. \tag{3.11}$$

Thus, polynomial self-interactions does not spoil the covariant conservation of the energy-momentum tensor.