Deriving the Force Law for Linearized General Relativity

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We start off with the Lagrangian for linearized General Relativity with a source term by

$$\mathcal{L} = \partial_\lambda h_{\mu\nu}\partial^\mu h^{\lambda\nu} + \frac{1}{2}\partial_\mu h\partial^\mu h - \frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} - \partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{M_P}h_{\mu\nu}T^{\mu\nu}, \tag{1}$$

where $M_P = (\sqrt{8\pi G_N})^{-1}$ is the Planck mass and the stress energy tensor is subject to the constraint that $\partial^\mu T_{\mu\nu} = 0$. Since we've already handled the hard part, lets skip straight to the gauge invariant action where we've taken the metric perturbation and expressed it into its irreducible representations via the following decompositions:

$$h_{00} = h^{00} = 2\Phi, \tag{2}$$

$$h_{0i} = -h^0_i = w_i, \tag{3}$$

$$h = h^\mu_\mu = \eta^{\mu\nu}h_{\mu\nu} = -2\Phi + \bar{h}, \tag{4}$$

$$h_{ij} = h^{TT}_{ij} + \partial_i v^T_j + \partial_j v^T_i + 2(\partial_i\partial_j\Psi - \frac{1}{n}\nabla^2\Psi\delta_{ij}) + \frac{1}{n}\bar{h}\delta_{ij}, \tag{5}$$

$$w_i = w^T_i + \partial_i\Omega, \tag{6}$$

$$\partial^i h^{TT}_{ij} = \delta^{ij}h^{TT}_{ij} = \partial^i v^T_i = \partial^i w^T_i = 0, \tag{7}$$

where $\bar{h} = Tr[h_{ij}]$ and δ_{ij} is the identity matrix. Defining the gauge-invariant fields $J \equiv \Phi - \dot{\Omega} + \ddot{\Psi}$, $L \equiv \frac{1}{3}(\bar{h} - 2\nabla^2\Psi)$, and $M_i = w_i^T - \dot{v}_i^T$, S_T, S_S and S_V take on the forms

$$S_T = \int \frac{1}{2} h_{TT}^{ij} \Box h_{ij}^{TT} \; d^4x, \tag{8}$$

$$S_V = \int \frac{1}{2} (\partial_i M_j)^2 \; d^4x, \tag{9}$$

$$S_S = \int [\Phi - \dot{\Omega} + \ddot{\Psi}] \nabla^2 L - L\nabla^2 L + 2L\ddot{L} \; d^4x + 2\Phi T_{00}(\mathbf{x}). \tag{10}$$

Taking the source of gravitational interaction to be a point particle, we can write T_{00} as

$$T_{00}(\mathbf{x}) = \rho(\mathbf{x}) = M\delta^{\mathbf{3}}(\mathbf{x}) + m\delta^{\mathbf{3}}(\mathbf{x} - \mathbf{r}), \tag{11}$$

where M and m are the masses of two point particles. Doing all the decompositions from above gives us the following actions

$$S_T = \int \frac{1}{2} h_{TT}^{ij} \Box h_{ij}^{TT} \; d^4x, \tag{12}$$

$$S_V = \int \frac{1}{2} (\partial_i M_j)^2 \; d^4x, \tag{13}$$

$$S_S = \int 4J\nabla^2 L - L\nabla^2 L + 2L\ddot{L} + \frac{2}{M_P}\Phi\rho \; d^4x. \tag{14}$$

Looking at the vector action we can see that no time derivatives of M_i are present in the action. Therefore it is an auxiliary field and we may use its equations of motion (EOM) to eliminate it. Proceeding accordingly we find

$$\frac{\delta \mathcal{L}_V}{\delta M^i} = \nabla^2 M_i = 0 \Rightarrow M_i = 0 \tag{15}$$

which implies that $S_V = 0$. Let's turn our attention to S_S. Recognizing that $\Phi = J + \dot{\Omega} - \ddot{\Psi}$, S_S becomes

$$S_S = \int 4J\nabla^2 L - L\nabla^2 L + 2L\ddot{L} + \frac{2}{M_p}J\rho + \frac{2}{M_p}\dot{\Omega}\rho - \frac{2}{M_p}\ddot{\Psi}\rho \; d^4x. \tag{16}$$

Next we notice that J is a Lagrange multiplier and therefore its equations of motion sets the other fields equal to zero. Varying w.r.t J gives us

$$\frac{\delta \mathcal{L}_S}{\delta J} = 4\nabla^2 L + \frac{2}{M_p}\rho = 0 \Rightarrow L = -\frac{1}{2M_P}\frac{1}{\nabla^2}\rho. \tag{17}$$

Plugging the equations of motion for J into the action gives us

$$S_S = \int -\frac{1}{4M_P^2}\rho\frac{1}{\nabla^2}\rho \; d^4x, \tag{18}$$

where we've integrated the last two terms out of the action. We notice that the term present in this Lagrangian is precisely the gravitational potential energy. To get the force law, we start with

$$U_G = \int -\frac{1}{4M_P^2}\rho\frac{1}{\nabla^2}\rho \; d^3\mathbf{x} \tag{19}$$

$$= -\frac{1}{(2\pi)^3}\frac{1}{4M_P^2}\int \rho(\mathbf{x})\frac{e^{i\mathbf{p}\cdot\mathbf{x}}}{p^2}\rho(\mathbf{p}) \; d^3\mathbf{p}d^3\mathbf{x} \tag{20}$$

Next we note that

$$\rho(\mathbf{p}) = \int e^{-i\mathbf{p}\cdot\mathbf{x}}\rho(\mathbf{x}) \; d^3\mathbf{x}, \tag{21}$$

which causes the expression in (12) to become

$$U_G = \frac{1}{(2\pi)^3}\int\int\int \frac{e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}}{p^2}\rho(\mathbf{x})\rho(\mathbf{x}') \; d^3\mathbf{p}d^3\mathbf{x}'d^3\mathbf{x}. \tag{22}$$

Using the formula

$$\frac{1}{(2\pi)^3}\int \frac{e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}}{p^2} \; d^3\mathbf{p} = \frac{1}{4\pi}\frac{1}{|\mathbf{x}-\mathbf{x}'|} \tag{23}$$

expression (14) becomes

$$U_G = -\frac{1}{4\pi}\frac{1}{4M_P^2}\int\int \frac{\rho(\mathbf{x})\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\ d^3\mathbf{x}' d\mathbf{x}. \tag{24}$$

Next we plug in the mass density for ρ into the integral

$$U_G = -\frac{1}{4\pi}\frac{1}{4M_P^2}\int\int \frac{(M\delta(\mathbf{x}) + m\delta(\mathbf{x}-\mathbf{r}))(M\delta(\mathbf{x}') + m\delta(\mathbf{x}'-\mathbf{r}))}{|\mathbf{x}-\mathbf{x}'|}\ d^3\mathbf{x}' d\mathbf{x}. \tag{25}$$

Since we wish to know what the force law is, we can ignore the terms that are for the gravitational energy that a particle gains from interacting with itself. With that in mind, the expression that we're working with reduces down to the following

$$U_G = -\frac{1}{2\pi}\frac{1}{4M_P^2}\int\int \frac{(Mm\delta(\mathbf{x})\delta(\mathbf{x}'-\mathbf{r}))}{|\mathbf{x}-\mathbf{x}'|}\ d^3\mathbf{x}' d\mathbf{x}. \tag{26}$$

Integrating over the delta functions and differentiating the gravitational potential energy gives us

$$\mathbf{F} = \frac{GMm\hat{\mathbf{r}}}{r^2} \tag{27}$$

where r is the distance between the two point masses. This is precisely the Newtonian force law! Gaze upon its beauty!!