Counting the Degrees of Freedom in Linearized General Relativity

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We start off with the Lagrangian for linearized General Relativity given by

$$\mathcal{L} = \partial_\lambda h_{\mu\nu}\partial^\mu h^{\lambda\nu} + \frac{1}{2}\partial_\mu h\partial^\mu h - \frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} - \partial_\mu h^{\mu\nu}\partial_\nu h, \tag{1}$$

We know that since $h_{\mu\nu}$ is a symmetric (0,2) tensor, under spatial rotations the 00 component is a scalar, the 0i component forms a 3-vector, and the ij component forms a symmetric spatial tensor. This allows us to decompose the metric perturbation into it's constituent parts. Now we write $h_{\mu\nu}$ as

$$\begin{aligned} h_{00} &= h^{00} = -2\Phi, \\ h_{0i} &= -h^0_i = w_i, \\ h &= h^\mu_\mu = \eta^{\mu\nu}h_{\mu\nu} = 2\Phi + \bar{h}, \end{aligned} \tag{2}$$

where $\bar{h} = \mathrm{Tr}[h_{ij}]$. Plugging these expressions in while simplifying our Lagrangian immensely gives us

$$\begin{aligned} \mathcal{L} = &-2\partial_i w_j \dot{h}^{ij} - \partial_i w_j \partial^j w^i + \partial_i h_{jk}\partial^j h^{ik} - \frac{1}{2}\dot{\bar{h}}^2 + 2\partial_i\Phi\partial^i\bar{h} + \frac{1}{2}(\partial_i\bar{h})^2 \\ &+ \frac{1}{2}(\dot{h}_{ij})^2 + (\partial_i w_j)^2 - \frac{1}{2}(\partial_i h_{jk})^2 - 2w^i\partial_i\dot{\bar{h}} - 2\partial_i h^{ij}\partial_j\Phi - \partial_i h^{ij}\partial_j\bar{h}, \end{aligned} \tag{3}$$

Under the action, equation (3) takes on the form

$$\begin{aligned} S = \int & 2w^i\partial_j\dot{h}^{ij} + w_i(\partial^i\partial_k w^k - \nabla^2 w^i) + \partial^j h_{jk}\partial_i h^{ik} + \frac{1}{2}\bar{h}\Box\bar{h} - 2\Phi\nabla^2\bar{h} \\ &+ \frac{1}{2}h_{ij}\Box h^{ij} - 2w^i\partial_i\dot{\bar{h}} + 2\Phi\partial_i\partial_j h^{ij} + \bar{h}\partial_i\partial_j h^{ij}\,\mathrm{d}^4x\,, \end{aligned} \tag{4}$$

where $\Box = \partial_\mu \partial^\mu$ and $\nabla^2 = \partial_i \partial^i$. Next we perform the following decomposition for the spatial tensor h_{ij} and the vector field w_i:

$$
\begin{aligned}
h_{ij} &= h_{ij}^{TT} + \partial_i v_j^T + \partial_j v_i^T + 2\left(\partial_i \partial_j \Psi - \frac{1}{3}\nabla^2 \Psi \delta_{ij}\right) + \frac{1}{3}\bar{h}\delta_{ij}, \\
w_i &= w_i^T + \partial_i \Omega, \\
\partial^i \partial^i h_{ij}^{TT} &= \delta^{ij} h_{ij}^{TT} = \partial^i v_i^T = \partial^i w_i^T = 0,
\end{aligned}
\tag{5}
$$

where δ_{ij} is the identity matrix. We can streamline the calculation a bit by recognizing that we can treat the spin 0, 1, and 2 terms separately (i.e. we can assume there are no cross terms between differing spins). From this we can split the action into three different sectors:

$$
S = S_T + S_V + S_S, \tag{6}
$$

where

$$
S_T = \int -\frac{1}{2} h_{TT}^{ij} \ddot{h}_{ij}^{TT} + \frac{1}{2} h_{TT}^{ij} \nabla^2 h_{ij}^{TT} \, \mathrm{d}^4 x \,, \tag{7}
$$

$$
\begin{aligned}
S_V &= \int 2 w_i^T \nabla^2 \dot{v}_T^i - w_i^T \nabla^2 w_T^i + \nabla^2 v_T^i \left(\nabla^2 v_i^T - \left(-\frac{\partial^2}{\partial t^2} + \nabla^2\right) v_i^T\right) \mathrm{d}^4 x, \\
&= \int (\partial_i w_j^T - \partial_i \dot{v}_j^T)^2 \mathrm{d}^4 x,
\end{aligned}
\tag{8}
$$

$$
\begin{aligned}
S_S = \int & 2\Omega \nabla^2 \dot{\bar{h}} - \frac{8}{3}\Omega \nabla^4 \dot{\Psi} - \frac{2}{3}\Omega \nabla^2 \dot{\bar{h}} - \frac{16}{9}\nabla^4 \Psi \nabla^2 \Psi - \frac{8}{9}\nabla^2 \Psi \nabla^2 \bar{h} \\
& - \frac{1}{9}\bar{h}\nabla^2 \bar{h} - \frac{1}{2}\bar{h}\Box \bar{h} + \frac{4}{3}\nabla^2 \Psi \Box \nabla^2 \Psi + \frac{1}{6}\bar{h}\Box \bar{h} - 2\Phi \nabla^2 \bar{h} \\
& + \frac{2}{3}\Phi \nabla^2 \bar{h} + \frac{8}{3}\Phi \nabla^4 \Psi + \frac{4}{3}\bar{h}\nabla^4 \Psi + \frac{1}{3}\bar{h}\nabla^2 \bar{h} \mathrm{d}^4 x.
\end{aligned}
\tag{9}
$$

Defining the gauge-invariant fields $J \equiv -\Phi - \dot{\Omega} + \ddot{\Psi}$, $L \equiv \frac{2}{3}(\bar{h} - 2\nabla^2 \Psi)$, and $M_i = w_i^T - \dot{v}_i^T$, S_S and S_V take on the forms

$$
S_V = \int \frac{1}{2}(\partial_i M_j)^2 \mathrm{d}^4 x, \tag{10}
$$

$$
S_S = \int 2J\nabla^2 L - \frac{1}{4} L \nabla^2 L + \frac{1}{2} L \ddot{L} \mathrm{d}^4 x, \tag{11}
$$

We can now analyze the true degrees of freedom that are present in $h_{\mu\nu}$. First, looking at the vector action we can see that no time derivatives of M_i are present in the action. Therefore it is an auxiliary field and we may use its equations of motion (EOM) to eliminate it. Proceeding accordingly we find

$$\frac{\delta \mathcal{L}}{\delta M^i} = \nabla^2 M_i = 0 \Rightarrow M_i = 0, \tag{12}$$

which implies that $S_V = 0$. Next we turn our attention to the scalar action. Since J appears linearly with no time derivatives, we may interpret it as a Lagrange multiplier. From there we can see that the EOM of J enforces the following constraint:

$$\frac{\delta \mathcal{L}}{\delta J} = \nabla^2 L = 0 \Rightarrow L = 0, \tag{13}$$

and therefore, $S_S = 0$. The total action is now

$$S = S_T, \tag{14}$$

where

$$S_T = \int \frac{1}{2} h^{ij}_{TT} \Box h^{TT}_{ij} \mathrm{d}^4 x. \tag{15}$$

Since we've finally eliminated all of the purely gauge fields we're left with (15). Since h^{TT}_{ij} carries 2 independent modes, we can finally conclude our analysis that linearized General Relativity carries with it a maximum of two degrees of freedom.