The Unruh Effect

Marcell Howard

August 1, 2025

1 Introduction

Here we go through the steps to derive the thermal distribution seen from an accelerated observer in Minkowski space i.e. the Unruh Effect. We follow the derivation as laid out in Birrell and Davies *Quantum Fields in Curved Space*.

Conventions We use the mostly plus metric signature, i.e. $\eta_{\mu\nu} = (-,+,+,+)$ and units where $c = \hbar = 1$. The reduced four dimensional Planck mass is $M_{\rm Pl} = (8\pi G)^{-1/2} \approx 2.43 \times 10^{18}$ GeV. The d'Alembert and Laplace operators are defined to be $\Box \equiv g^{\mu\nu}\partial_\mu\partial_\nu$ and $\nabla^2 = \partial_i\partial^i$ respectively. We use boldface letters $\mathbf{r}$ to indicate 3-vectors and x and p to denote 4-vectors. Conventions for the curvature tensors, covariant and Lie derivatives are all taken from Carroll.

2 Inertial Particle Detector in Flat Space

We imagine there is some device (that we call a particle detector) that gives some response when a particle hits it. We imagine this particle detector moves along some worldline described by $x^\mu(\tau)$ where τ is the proper time. The interaction between the particle detector's monopole moment $m(\tau)$ and the (scalar) field for which produces the particle $\phi[x(\tau)]$, is given by $gm(\tau)\phi[x(\tau)]$, where g is the coupling constant. We generically expect for ϕ to be in its ground state which is the Minkowski vacuum that we denote $|0_M\rangle$. Upon registering the particle, the detector gets kicked up to a higher energy level which we call

$E > E_0$ with E_0 being the energy of the ground state. Thus, the matrix element of the transition is given by

$$S_{fi} = \langle E, \psi | \left(1 + ig \int_{-\infty}^{\infty} m(\tau)\phi[x(\tau)]\, \mathrm{d}\tau \right) |0_M, E_0\rangle = ig \langle E, \psi | \int_{-\infty}^{\infty} m(\tau)\phi[x(\tau)]\, \mathrm{d}\tau |0_M, E_0\rangle , \tag{1}$$

where $|\psi\rangle$ is some arbitrary excited state. Recall from first semester QFT, we can represent the time evolution of an operator by

$$\mathcal{O}(\tau) = e^{iH_0\tau}\mathcal{O}(0)e^{-iH_0\tau}, \tag{2}$$

where $H_0 |E\rangle = E |E\rangle$. This implies the transition amplitude turns into

$$ig \langle E, \psi | \int_{-\infty}^{\infty} e^{iH_0\tau} m(0) e^{-iH_0\tau} \phi[x(\tau)]\, \mathrm{d}\tau |0_M, E_0\rangle = ig \langle E, \psi | \int_{-\infty}^{\infty} e^{iE\tau} m(0) e^{-iE_0\tau} \phi[x(\tau)]\, \mathrm{d}\tau |0_M, E_0\rangle \tag{3}$$

$$= ig \langle E|m(0)|E_0\rangle \int_{\mathbb{R}} e^{i(E-E_0)\tau} \langle \psi|\phi[x(\tau)]|0_M\rangle . \tag{4}$$

Now we expand the scalar field in the basis of the solutions to the Klein-Gordon equation and quantize the Fourier components

$$\phi(\mathbf{r}, t) = \sum_{\mathbf{p}} \left(\hat{a}_{\mathbf{p}} u_{\mathbf{p}}(\mathbf{r}, t) + \hat{a}^\dagger_{\mathbf{p}} u^*_{\mathbf{p}}(\mathbf{r}, t) \right), \quad u_{\mathbf{p}}(\mathbf{r}, t) = \frac{e^{i(\mathbf{p}\cdot\mathbf{r} - \omega_{\mathbf{p}} t)}}{\sqrt{2\mathcal{V}\omega_{\mathbf{p}}}}, \tag{5}$$

where $\omega_{\mathbf{p}}^2 = p^2 + m^2$. We can see that

$$\langle \psi|\phi[x(\tau)]|0_M\rangle = \sum_{\mathbf{q}} \langle 1_{\mathbf{p}}|\hat{a}^\dagger_{\mathbf{q}}|0_M\rangle \frac{e^{-i(\mathbf{q}\cdot\mathbf{r} - \omega_{\mathbf{q}} t)}}{\sqrt{2\mathcal{V}\omega_{\mathbf{q}}}}, \tag{6}$$

we had $|\psi\rangle \to |1_{\mathbf{p}}\rangle$ because the only allowed transition from the ground state to an excited state with a single creation operator is the first excited state. Now we move to the continuum limit

$$\langle \psi|\phi[x(\tau)]|0_M\rangle = \frac{1}{\sqrt{(2\pi)^3}} \int \mathrm{d}^3q\, \frac{e^{-i(\mathbf{q}\cdot\mathbf{r} - \omega_{\mathbf{q}} t)}}{\sqrt{2\omega_{\mathbf{q}}}} \langle 1_{\mathbf{p}}|\hat{a}^\dagger_{\mathbf{q}}|0_M\rangle . \tag{7}$$

Using the fact that $\langle 1_{\mathbf{p}}|\hat{a}^\dagger_{\mathbf{q}}|0_M\rangle = \delta^{(3)}(\mathbf{p} - \mathbf{q})$, we're left with

$$\langle\psi|\phi[x(\tau)]|0_M\rangle = \frac{e^{-i(\mathbf{p}\cdot\mathbf{r}-\omega_{\mathbf{p}}t)}}{\sqrt{16\pi^3\omega_{\mathbf{p}}}}. \tag{8}$$

Now recall that $\mathbf{r}$ is not in general a time independent object. In fact, in general it will be determined by the detector's trajectory through spacetime. We can study different trajectories to see the amplitude for each worldline. First we assume an inertial worldline with

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{v}t = \mathbf{r}_0 + \gamma\mathbf{v}\tau, \qquad \gamma = \frac{1}{\sqrt{1-\mathbf{v}^2}}. \tag{9}$$

This brings the transition amplitude to the form

$$\begin{aligned} S_{fi} &= \frac{ig\,\langle E|m(0)|E_0\rangle}{\sqrt{16\pi^3\omega_{\mathbf{p}}}}\int_{\mathbb{R}} e^{i(E-E_0)\tau}e^{-i[\mathbf{p}\cdot(\mathbf{r}_0+\gamma\mathbf{v}\tau)-\gamma\omega_{\mathbf{p}}\tau]}\,\mathrm{d}\tau && (10)\\ &= \frac{ig\,\langle E|m(0)|E_0\rangle\,e^{-i\mathbf{p}\cdot\mathbf{r}_0}}{\sqrt{16\pi^3\omega_{\mathbf{p}}}}\int_{\mathbb{R}} e^{i(E-E_0)\tau}e^{-i(\mathbf{p}\cdot\mathbf{v}-\omega)\gamma\tau}\,\mathrm{d}\tau && (11)\\ &= \frac{ig\,\langle E|m(0)|E_0\rangle\,e^{-i\mathbf{p}\cdot\mathbf{r}_0}}{\sqrt{16\pi^3\omega_{\mathbf{p}}}}2\pi\delta(E-E_0+\gamma(\omega_{\mathbf{p}}-\mathbf{p}\cdot\mathbf{v})). && (12)\end{aligned}$$

Because $E > E_0$ and $\mathbf{p}\cdot\mathbf{v} \leq |\mathbf{p}||\mathbf{v}| < \sqrt{\mathbf{p}^2+m^2} = \omega_{\mathbf{p}}$ this implies[1]

$$E - E_0 + \gamma(\omega - \mathbf{p}\cdot\mathbf{v}) > 0, \tag{13}$$

which means the delta function must vanish. This result implies that an observer/particle detector that is strictly moving in an inertial reference frame will not see *any* particles produced. Said in another way, a person standing still or moving at constant velocity will observe the universe around them and not notice anything strange happening. This is intuitively true.

The above result, while intuitive, isn't very interesting and even suggests that looking at just the transition amplitude may not even be worth looking like. Much like we do in usual QFT fashion, lets look at the transition probability instead

$$|\mathcal{M}|^2 = g^2\sum_E |\langle E|m(0)|E_0\rangle|^2\mathcal{F}(E-E_0), \tag{14}$$

[1] The inequality $|\mathbf{p}||\mathbf{v}| < \sqrt{\mathbf{p}^2+m^2}$ must hold because $|\mathbf{v}| < 1$ and therefore $|\mathbf{p}||\mathbf{v}| < |\mathbf{p}|$. Because $\sqrt{\mathbf{p}^2+m^2} \geq \sqrt{\mathbf{p}^2} = |\mathbf{p}|$, the inequality is justified.

where

$$\mathcal{F}(E) = \int_{\mathbb{R}} \mathrm{d}\tau \int_{\mathbb{R}} \mathrm{d}\tau' \, e^{-iE(\tau-\tau')} G^+(x(\tau), x(\tau')), \tag{15}$$

with $G^+(x(\tau), x(\tau'))$ is the Green's function/propagator from $x(\tau)$ to $x(\tau')$. Because we're looking at Minkowski spacetime, we manifestly have time translation invariance. This implies that $G^+(x(\tau), x(\tau')) = G(\Delta\tau) = G(\tau - \tau')$. We shall also restrict our attention to the case of a massless scalar field, whose propagator has a well-known form

$$D^+(x, x') = -\frac{1}{4\pi^2[(t - t' - i\epsilon)^2 - |\mathbf{r} - \mathbf{r}'|^2]}. \tag{16}$$

Recall that

$$t - t' = \gamma\tau - \gamma\tau' = \gamma\Delta\tau, \qquad \mathbf{r} - \mathbf{r}' = \mathbf{r}_0 + \gamma\mathbf{v}\tau - (\mathbf{r}_0 + \gamma\mathbf{v}\tau') = \gamma\mathbf{v}\Delta\tau. \tag{17}$$

This leads to

$$|\mathbf{r} - \mathbf{r}'|^2 = |\gamma\mathbf{v}\Delta\tau|^2 = \frac{\mathbf{v}^2}{1 - \mathbf{v}^2}(\Delta\tau)^2. \tag{18}$$

Putting the results together yields

$$(t - t' - i\epsilon)^2 - |\mathbf{r} - \mathbf{r}'|^2 = (\gamma\Delta\tau - i\epsilon)^2 - \frac{\mathbf{v}^2}{1 - \mathbf{v}^2}(\Delta\tau)^2 \tag{19}$$

$$= \frac{\Delta\tau^2}{1 - \mathbf{v}^2} - 2i\gamma\epsilon\Delta\tau - \epsilon^2 - \frac{\mathbf{v}^2}{1 - \mathbf{v}^2}\Delta\tau^2 \tag{20}$$

$$= \Delta\tau^2 - 2i\gamma\epsilon\Delta\tau - \epsilon^2 = (\Delta\tau - i\epsilon)^2, \tag{21}$$

where we absorbed[2] the factor of γ into ϵ. Thus we have

$$\mathcal{F}(E - E_0) = \int_{\mathbb{R}} \mathrm{d}\Delta\tau \, e^{-i(E-E_0)\tau} \frac{1}{(\Delta\tau - i\epsilon)^2}. \tag{22}$$

We can evaluate this integral using the Residue theorem. Because $E - E_0 > 0$, we need to close the contour in the lower half-plane. However, the poles are located at $\Delta\tau = i\epsilon$ which because we take the $\epsilon \to 0^+$ limit, is located in the upper half plane. Therefore, we end up closing the contour where there are no residues and thus the integral vanishes.

[2]We can do this because we're only interested in the $\epsilon \to 0^+$ limit.

3 Interlude: Accelerated Observers in Minkowski Space

Now we can consider the case of an accelerated observer. Before doing so, let us remove the basics of acceleration in Minkowski spacetime. We start with the formula for the relativistic addition of velocities

$$v(\tau + \mathrm{d}\tau) \simeq v(\tau) + \mathrm{d}v(\tau) = \frac{v(\tau) + \mathrm{d}v(\tau)}{1 + v(\tau)\,\mathrm{d}v(\tau)}. \tag{23}$$

We are justified in doing this because at any given instant proper time τ, the slope of the worldline corresponds to the instantaneous velocity of the worldline/observer. Therefore, at that instant, we can use the law of addition of velocities. Now recall that $\mathrm{d}v(\tau) = a(\tau)\,\mathrm{d}\tau$

$$v(\tau + \mathrm{d}\tau) - v(\tau) = \frac{v(\tau) + \mathrm{d}v}{1 + v(\tau)\,\mathrm{d}v(\tau)} - v(\tau) = \frac{\mathrm{d}v - v^2\,\mathrm{d}v}{1 + v\,\mathrm{d}v} \simeq a(\tau)(1 - v^2)\,\mathrm{d}\tau\,, \tag{24}$$

where we did a Taylor expansion in the last step and dropped the term that's proportional to $\mathcal{O}(dv^2)$. Since $v(\tau + \mathrm{d}\tau) - v(\tau) = \mathrm{d}v$, we have

$$a(\tau) = \frac{\mathrm{d}v}{\mathrm{d}\tau}\frac{1}{1 - v^2}. \tag{25}$$

This is an exactly solvable ordinary differential equation. Write

$$\frac{\mathrm{d}v}{1 - v^2} = a(\tau)\,\mathrm{d}\tau\,. \tag{26}$$

We have the integral

$$\int \frac{\mathrm{d}v}{1 - v^2}. \tag{27}$$

This is a simple integral to do with a hyperbolic trig substitution $v = \tanh(\theta)$. Thus we have

$$\mathrm{arctanh}(v) = \int_0^\tau a(\tau')\,\mathrm{d}\tau' \equiv \theta(\tau) \Rightarrow v(\tau) = \tanh\theta(\tau). \tag{28}$$

But $v(\tau) = \tanh\theta(\tau)$ further implies

$$\gamma(\tau) = \frac{1}{\sqrt{1-v^2}} = \frac{1}{\sqrt{1-\tanh^2(\theta)}} = \cosh\theta(\tau). \tag{29}$$

Next we recognize that

$$\frac{\mathrm{d}z}{\mathrm{d}\tau} = \frac{\mathrm{d}z}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau} = \gamma v, \tag{30}$$

where we used the fact that $u^\mu = \gamma(1, \mathbf{v})$. Now we are left with

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \gamma = \cosh\theta(\tau), \quad \frac{\mathrm{d}z}{\mathrm{d}\tau} = \cosh\theta \cdot \tanh\theta = \sinh\theta(\tau). \tag{31}$$

Therefore, the 4-velocity is

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = u^\mu = (\cosh\theta(\tau), 0, 0, \sinh\theta(\tau)), \quad a^\mu = \frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} = a(\tau)(\sinh\theta(\tau), 0, 0, \cosh\theta(\tau)). \tag{32}$$

We can see that the 4-velocity satisfies

$$u^\mu u_\mu = \eta_{\mu\nu}u^\mu u^\nu = -\cosh^2(\theta) + \sinh^2(\theta) = -1. \tag{33}$$

Now we simplify to the case where an observer undergoes constant acceleration $a(\tau) = a$ which makes $\theta(\tau) = a\tau$. The trajectory through spacetime is

$$x^\mu = \frac{1}{a}(\sinh a\tau, 0, 0, \cosh a\tau) \Rightarrow t(\tau) = \frac{1}{a}\sinh a\tau, \quad z(\tau) = \frac{1}{a}\cosh a\tau. \tag{34}$$

Thus we have

$$-t^2 + z^2 = \frac{1}{a^2}. \tag{35}$$

4 Uniformly Accelerating Particle Detector in Minkowski Spacetime

Now we can consider the trajectory for a detector that accelerates uniformly with acceleration α^{-1} along the z-axis. The massless scalar propagator in these coordinates become

$$D^+(x,x') = -\frac{1}{(2\pi)^2}\frac{1}{(t-t'-i\epsilon)^2 - |\mathbf{r}-\mathbf{r}'|^2}. \tag{36}$$

This leads to

$$(t-t')^2 = \left(\alpha\sinh\frac{\tau}{\alpha} - \alpha\sinh\frac{\tau'}{\alpha}\right)^2 = \left[2\alpha\cosh\left(\frac{\tau+\tau'}{2\alpha}\right)\sinh\left(\frac{\tau-\tau'}{2\alpha}\right)\right]^2, \tag{37}$$

where we used and will use the hyperbolic trig identity

$$\sinh x - \sinh y = 2\sinh\left(\frac{x-y}{2}\right)\cosh\left(\frac{x+y}{2}\right), \quad \cosh x - \cosh y = 2\sinh\left(\frac{x+y}{2}\right)\sinh\left(\frac{x-y}{2}\right). \tag{38}$$

It then follows that

$$|\mathbf{r}-\mathbf{r}'|^2 = (z-z')^2 = \left(\alpha\cosh\frac{\tau}{\alpha} - \alpha\cosh\frac{\tau'}{\alpha}\right)^2 = \left[2\sinh\left(\frac{\tau+\tau'}{2\alpha}\right)\sinh\left(\frac{\tau-\tau'}{2\alpha}\right)\right]^2. \tag{39}$$

Letting $\epsilon \to 0$ for now, we get

$$(t-t')^2 - (z-z')^2 = 4\alpha^2\cosh^2\left(\frac{\tau+\tau'}{2\alpha}\right)\sinh^2\left(\frac{\tau-\tau'}{2\alpha}\right) - 4\alpha^2\sinh^2\left(\frac{\tau+\tau'}{2\alpha}\right)\sinh^2\left(\frac{\tau-\tau'}{2\alpha}\right) \tag{40}$$

$$= 4\alpha^2\sinh^2\left(\frac{\tau-\tau'}{2\alpha}\right)\left[\cosh^2\left(\frac{\tau+\tau'}{2\alpha}\right) - \sinh^2\left(\frac{\tau+\tau'}{2\alpha}\right)\right] \tag{41}$$

$$= 4\alpha^2\sinh^2\left(\frac{\tau-\tau'}{2\alpha}\right). \tag{42}$$

Thus, we can express the Green's function as

$$D^+(\Delta\tau) = -\frac{1}{(2\pi)^2}\frac{1}{\left(2\alpha\sinh\left(\frac{\tau-\tau'}{2\alpha}\right)\right)^2} = -\frac{1}{16\pi^2\sinh^2\left(\frac{\tau-\tau'}{2\alpha} - \frac{i\epsilon}{\alpha}\right)}, \tag{43}$$

where we point the ϵ back in. Now recall that we want to compute the transition probability $|\mathcal{M}|^2$ which involves integrating the above quantity. To do so, we will express the Green's function as a power series by recalling the Laurent series expansion for $\csc x$

$$\csc^2(\pi x) = \sum_{n=-\infty}^{\infty}\frac{1}{(\pi x - n\pi)^2} = \frac{1}{\pi^2}\sum_{n=-\infty}^{\infty}\frac{1}{(x-n)^2}. \tag{44}$$

Let us recall some famous trig identities

$$\csc(iz) = \frac{1}{\sin(iz)} = \frac{1}{i\sinh z} \tag{45}$$

$$\csc^2 iz = -\frac{1}{\sinh^2 z} = \sum_{n=-\infty}^{\infty} \frac{1}{(iz - n\pi)^2}, \tag{46}$$

which implies the following series expansion for $D^+(\Delta\tau)$

$$D^+(\Delta\tau) = -\frac{1}{16\pi^2\alpha^2 \sinh^2\left(\frac{\Delta\tau}{2\alpha} - \frac{i\epsilon}{\alpha}\right)} = \frac{1}{16\pi^2\alpha^2} \sum_{n=-\infty}^{\infty} \left[\frac{i\Delta\tau}{2\alpha} + \frac{\epsilon}{\alpha} - n\pi\right]^{-2} \tag{47}$$

$$= -\frac{1}{4\pi^2} \sum_{n=-\infty}^{\infty} \frac{1}{(\Delta\tau - 2i\epsilon + 2\pi i n\alpha)^2}. \tag{48}$$

The transition probability then becomes

$$g^2 \sum_E |\langle E|m(0)|E_0\rangle|^2 \int_{\mathbb{R}} \mathrm{d}\Delta\tau \, e^{-i(E-E_0)\Delta\tau} D^+(\Delta\tau) \tag{49}$$

$$= -\frac{g^2}{(2\pi)^2} \sum_E |\langle E|m(0)|E_0\rangle|^2 \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} \frac{e^{i(E-E_0)\Delta\tau}}{(\Delta\tau - 2i\epsilon + 2\pi i n\alpha)^2} \mathrm{d}\Delta\tau \, . \tag{50}$$

We will make use of the Residue theorem once more by defining the function

$$f(z) = -\frac{e^{-i(E-E_0)z}}{(z - 2i\epsilon + 2\pi i n\alpha)^2}, \tag{51}$$

and recalling the definition of a residue of order m

$$\mathrm{Res}[f; z_0] = \lim_{z\to z_0} \frac{1}{(m-1)!} \frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}} [(z - z_0)^m f(z)]. \tag{52}$$

In our case $z_0 = 2i\epsilon - 2\pi i n\alpha$ and $m = 2$. Thus

$$\mathrm{Res}[f; 2i\epsilon - 2\pi i n\alpha] = \lim_{z\to z_0} \frac{-1}{(2-1)!} \frac{\mathrm{d}}{\mathrm{d}z} [e^{-i(E-E_0)z}] = i(E - E_0)e^{-i(E-E_0)z_0}, \tag{53}$$

and the sum of the resides is just

$$2\pi i \sum_{n\in\mathbb{Z}} i(E - E_0)e^{-i(E-E_0)(2i\epsilon - 2\pi i n\alpha)} = -2\pi(E - E_0) \sum_{n\in\mathbb{Z}} e^{-2\pi(E-E_0)\alpha n} = \frac{2\pi(E - E_0)}{e^{2\pi(E-E_0)\alpha} - 1}, \tag{54}$$

where we let $\epsilon \to 0$ in the middle step. Plugging this result back into the expression for the transition probability gives us

$$|\mathcal{M}|^2 = \frac{g^2}{(2\pi)^2} \sum_E |\langle E|m(0)|E_0\rangle|^2 \frac{2\pi(E-E_0)}{e^{2\pi(E-E_0)\alpha}-1} = \frac{g^2}{2\pi} \sum_E \frac{(E-E_0)|\langle E|m(0)|E_0\rangle|^2}{e^{2\pi(E-E_0)\alpha}-1}. \tag{55}$$

This final term is interesting because the factor of $(e^{2\pi(E-E_0)\alpha}-1)^{-1}$ greatly resembles a blackbody where we identify

$$k_B T = (2\pi\alpha)^{-1} \Rightarrow T = \frac{1}{2\pi k_B \alpha} = \frac{a}{2\pi k_B}. \tag{56}$$