Derivation of the Euler-Heisenberg Lagrangian

Marcell Howard

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1 Introduction

We are interested in calculating the one-loop effective for a spin-0 field in a constant electromagnetic field. Here we follow the derivations as laid out by Leonard Parker while filling in some of the gaps between the differing steps. This is a famous result which has been used to calculate mixing of photons and axions.

Conventions We use the mostly plus metric signature, i.e. $\eta_{\mu\nu} = (-, +, +, +)$ and units where $c = \hbar = 1$. The reduced four dimensional Planck mass is $M_{\mathrm{Pl}} = (8\pi G)^{-1/2} \approx 2.43 \times 10^{18}\,\mathrm{GeV}$. The d'Alembert and Laplace operators are defined to be $\Box \equiv g^{\mu\nu}\partial_\mu\partial_\nu$ and $\nabla^2 = \partial_i\partial^i$ respectively. We use boldface letters $\mathbf{r}$ to indicate 3-vectors and x and p to denote 4-vectors. Conventions for the curvature tensors, covariant and Lie derivatives are all taken from Carroll.

2 The Heat Kernel for a Spin-less Charged Particle in a Constant Electromagnetic Field

We will calculate the one-loop effective action using the heat-kernel method

$$i\frac{\mathrm{d}}{\mathrm{d}\tau}K(\tau; x, x') = (-\partial_\mu\partial^\mu + m^2)K(\tau; x, x'). \tag{2.1}$$

We want to apply this formalism to a particular case: a scalar field in a constant electromagnetic field in Minkowski space. Since we're in a region with an electromagnetic field, the partial derivatives get replaced with covariant derivatives $\partial_\mu \to D_\mu = -i(\partial_\mu - ieA_\mu)$. Recall that the field strength tensor is defined by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2.2}$$

Since the electric and magnetic fields are uniform, $F_{\mu\nu}$ is a constant and therefore the 4-potential takes on the form

$$A_\mu = -\frac{1}{2} F_{\mu\nu} x^\nu + a_\mu, \tag{2.3}$$

with a_μ being an arbitrary constant. The kernel obeys the following differential equation with boundary condition

$$i\frac{\partial}{\partial\tau} K(\tau; x, x') = (-D_\mu D^\mu + m^2) K(\tau; x, x'), \quad K(0; x, x') = \delta(x, x'). \tag{2.4}$$

The above equation has the form of a time-dependent Schrodinger equation with $(-D^\mu D_\mu + m^2)$ playing the role of the Hamiltonian and the heat kernel playing the role of the Green's function/propagator for a scalar field in a constant electromagnetic field. As a result, τ often gets regarded as a sort of "proper time". Thus, our approach will be to deduce the form of the propagator/transition amplitude for this hypothetical particle. First, we start from the Hamiltonian for a charged particle

$$H = \eta^{\mu\nu}(p_\mu - eA_\mu)(p_\nu - eA_\nu) + m^2. \tag{2.5}$$

Hamilton's equations of motion dictate

$$\dot{x}^\mu = \frac{\partial H}{\partial p_\mu} = \eta^{\mu\nu}(2p_\nu - eA_\nu - eA_\lambda \delta_{\lambda\nu}) = 2p^\mu - 2eA^\mu \Leftrightarrow p^\mu = \frac{1}{2}\dot{x}^\mu + eA^\mu. \tag{2.6}$$

Taking the Legendre transformation to go back to the Lagrangian formalism yields

$$L = p_\mu \dot{x}^\mu - H = \left(\frac{1}{2}\dot{x}_\mu + eA_\mu\right)\dot{x}^\mu - \eta^{\mu\nu}\left[\left(\frac{1}{2}\dot{x}_\mu\right)\left(\frac{1}{2}\dot{x}_\nu\right)\right] - m^2 = \frac{1}{4}\dot{x}^\mu \dot{x}_\mu + eA_\mu \dot{x}^\mu - m^2, \tag{2.7}$$

and the associated action is defined to be

$$S[x] = \int_0^\tau L(\tau')\,\mathrm{d}\tau'\,. \tag{2.8}$$

Recall that we are regarding the kernel as a transition amplitude. Here we will think of it as a propagator from the points $x^\mu(0) = (x', 0)$ to $x^\mu(\tau) = (x, \tau)$. Thus it takes on the form (from the path-integral formulation of quantum mechanics)

$$K(\tau; x, x') = \int_{x^\mu(0)=x'^\mu}^{x^\mu(\tau)=x^\mu} \mathcal{D}x \exp(iS[x]), \tag{2.9}$$

where we've set $\hbar = 1$ again and the integral is taken to be over all paths between x'^μ and x^μ. To work out this transition amplitude, we first write

$$x^\mu(\tau') = x^\mu_{\mathrm{cl}}(\tau') + q^\mu(\tau'), \tag{2.10}$$

subject to the constraint

$$q(0) = q(\tau) = 0, \tag{2.11}$$

with x^μ_{cl} denoting the classical solution/path to the equations of motion. When we expand out the Lagrangian for a constant electromagnetic field

$$L = \frac{1}{4}\dot{x}^\mu \dot{x}_\mu + e\left(-\frac{1}{2}F_{\mu\nu}x^\nu + a_\mu\right)\dot{x}^\mu - m^2 = \frac{1}{4}\eta^{\mu\nu}\dot{x}_\mu \dot{x}_\nu - \frac{e}{2}F_{\mu\nu}\dot{x}^\mu x^\nu + e a_\mu \dot{x}^\mu - m^2, \tag{2.12}$$

we can see that it is at most quadratic in both the position and velocities. As a result, we can expand the initial action[1] to be

$$S[x] = S[x_{\mathrm{cl}}] + S_2[q], \tag{2.13}$$

This brings the path integral to the form

[1] When we go to expand out the action, we'll get cross terms that look like $x \cdot q$ and their derivatives. The electromagnetic field being constants means we can simply do integration by parts on the different terms and then the cross terms vanish when we plug the classical path into its equations of motion. Thereby making it so there are no cross terms and the total action can simply be written as the sum of the two actions.

$$K(\tau; x, x') = \int_{q^\mu(0)=0}^{q^\mu(\tau)=0} \mathcal{D}q \exp(iS[x_{\rm cl}] + iS_2[q]) = \exp(iS_{\rm cl}) \int_{q^\mu(0)=0}^{q^\mu(\tau)=0} \mathcal{D}q \exp(iS_2[q]), \tag{2.14}$$

where $S_{\rm cl} \equiv S[x_{\rm cl}]$. Since $q(0) = q(\tau) = 0$, it can't be a function of x, x' and therefore the heat kernel takes on the simple form

$$K(\tau; x, x') = f(\tau) \exp(iS_{\rm cl}). \tag{2.15}$$

2.1 The Classical Action for a Charged Particle in a Uniform Electromagnetic Field

Now we've reduced the problem to deducing what $f(\tau)$ is. This can be done by plugging the heat kernel into the differential equation and solving it for $f(\tau)$. To do so, we will plug the classical path into the equations of motion while also dropping the "cl" subscript. Proceeding accordingly yields

$$\frac{\partial L}{\partial \dot{x}_\mu} = \frac{1}{2}\dot{x}^\mu + eA^\mu \Rightarrow \frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial L}{\partial \dot{x}_\mu} = \frac{1}{2}\ddot{x}^\mu + e\dot{A}^\mu = \frac{1}{2}\ddot{x}^\mu - \frac{e}{2}F_{\mu\nu}\dot{x}^\nu, \tag{2.16}$$

$$\frac{\partial L}{\partial x_\mu} = e\dot{x}^\nu \partial_\mu A_\nu = -\frac{e}{2}F_{\nu\mu}\dot{x}^\nu, \tag{2.17}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial L}{\partial \dot{x}_\mu} - \frac{\partial L}{\partial x_\mu} = \frac{1}{2}\ddot{x}^\mu - \frac{e}{2}F^\mu{}_\nu\dot{x}^\nu - \frac{e}{2}F^\mu{}_\nu\dot{x}^\nu = 0, \tag{2.18}$$

which reduces to the slightly nicer form

$$\ddot{x}^\mu = 2eF^\mu{}_\nu\dot{x}^\nu. \tag{2.19}$$

From here on out, it'll be much easier to express everything in terms of matrices meaning:

$$\ddot{x}(\tau') = 2eF\dot{x}(\tau'), \qquad F^T = -F. \tag{2.20}$$

This is a pretty simple matrix differential equation. We can make the substitution $s = \dot{x}$ which turns the above equation into the form

$$\dot{s}(\tau') = 2eFs(\tau'). \tag{2.21}$$

The solutions to this are obvious and of second nature to us. They are simply exponentials of the form

$$s(\tau') = \dot{x}(\tau') = \exp(2e\tau' F)u, \tag{2.22}$$

where u is an integration constant (with respect to τ'). We can integrate this one more to get

$$x(\tau') = \frac{1}{2e}F^{-1}\exp(2e\tau' F)u + v, \tag{2.23}$$

where v is also an integration constant. These two constants can be fixed by utilizing the boundary conditions $x(\tau' = 0) = x'$ and $x(\tau' = \tau) = x$

$$x(\tau' = 0) = x' = \frac{1}{2e}F^{-1}u + v, \quad x(\tau' = \tau) = x = \frac{1}{2e}F^{-1}\exp(2e\tau F)u + v. \tag{2.24}$$

This is a system of linear equations which can be easily solved for u, v. We can find u by subtracting the expressions for x' and x

$$x' - x = \frac{1}{2e}F^{-1}u - \frac{1}{2e}F^{-1}\exp(2e\tau F)u \Rightarrow u = (1 - \exp(2e\tau F))^{-1}2eF(x' - x) \equiv \frac{2eF(x' - x)}{1 - \exp(2e\tau F)}. \tag{2.25}$$

Solving for v is similarly easy

$$v = x' + \frac{1}{2e}F^{-1}u = \frac{x' - \exp(-2e\tau F)x}{1 - \exp(-2e\tau F)}. \tag{2.26}$$

Remembering that

$$\dot{x}(\tau') = \exp(2e\tau' F)u, \tag{2.27}$$

the Lagrangian becomes

$$L(\tau') = \frac{1}{4}u^T u + e\left[\frac{-F}{2}\left(\frac{1}{2e}F^{-1}\exp(2e\tau' F)u + v\right) + a\right]^T \exp(2e\tau' F)u - m^2 \tag{2.28}$$

$$= \frac{1}{4}u^T u - m^2 + ea^T \exp(-2e\tau' F)u - \frac{1}{4}u^T \exp(-2e\tau' F)\exp(2e\tau' F)u - \frac{e}{2}v^T F \exp(2e\tau' F)u \tag{2.29}$$

$$= \frac{e}{2}v^T F \exp(2e\tau' F)u + ea^T \exp(2e\tau' F)u - m^2. \tag{2.30}$$

Plugging this Lagrangian into the action leaves us with

$$\int_0^\tau L(\tau')\,\mathrm{d}\tau' = -m^2\tau - ev^T F\left(\frac{F^{-1}}{2e}\right)(\exp(-2e\tau F) - 1)u + ea^T\left(\frac{-F^{-1}}{2e}(\exp(-2e\tau F) - 1)u\right) \tag{2.31}$$

$$= -m^2\tau + \frac{e}{2}v^T F(x' - x) + ea^T(x' - x) \tag{2.32}$$

$$= \frac{e}{2}(x' - x)^T F \frac{\exp(2e\tau F)x - x'}{\exp(2e\tau F) - 1} + ea^T(x - x') - m^2\tau. \tag{2.33}$$

Thus the classical action is simply

$$S_{\text{cl}}(x, x'; \tau) = \frac{e}{2}(x' - x)^T F\left[\frac{x}{1 - \exp(-2e\tau F)} - \frac{x'}{\exp(2e\tau F) - 1}\right] + ea^T(x - x') - m^2\tau. \tag{2.34}$$

2.2 The Heat Kernel for a Charged Particle in a Uniform Electromagnetic Field

Now we're ready to plug the heat kernel into the diffusion equation

$$i\frac{\partial}{\partial\tau}K(\tau; x, x') = (-D^\mu D_\mu + m^2)K(\tau; x, x'). \tag{2.35}$$

The right hand side of the equation can be written as

$$(-D^\mu D_\mu + m^2)K = (\partial_\mu - ieA_\mu)(\partial^\mu - ieA^\mu)K + m^2 K \tag{2.36}$$

$$= \Box_x K + m^2 K - ie\partial_\mu(KA^\mu) - ieA_\mu \partial^\mu K - e^2 A^\mu A_\mu K \tag{2.37}$$

$$= \Box_x K + m^2 K - 2ieA_\mu \partial^\mu K - e^2 A^\mu A_\mu K \tag{2.38}$$

$$= \Box_x K + m^2 K - 2ie\Big(-\frac{1}{2}F_{\mu\nu}x^\nu + a_\mu\Big)\partial^\mu K - e^2\Big(-\frac{1}{2}F_{\mu\lambda}x^\lambda + a_\mu\Big)\Big(-\frac{1}{2}F^\mu{}_\rho x^\rho + a^\mu\Big)K \tag{2.39}$$

$$= \Box_x K + m^2 K - \frac{e^2}{4}F^\mu{}_\lambda x^\lambda F_{\mu\rho}x^\rho K + e^2 K F^\mu{}_\nu x^\nu a_\mu - e^2 K a_\mu a^\mu + ieF^\mu{}_\nu x^\nu \partial_\mu K - 2iea^\mu \partial_\mu K. \tag{2.40}$$

We wish to express this equation in terms of matrices as to achieve parity to how we've written the classical action. We make the following identifications

$$A^\mu = -\frac{1}{2}F^\mu{}_\nu x^\nu + a^\mu \Leftrightarrow A = -\frac{1}{2}Fx + a \Rightarrow A^T = \frac{1}{2}x^T F + a^T, \tag{2.41}$$

$$-\frac{e^2}{4}F^\mu{}_\lambda x^\lambda F_{\mu\rho}x^\rho = \frac{e^2}{4}x_\rho F^\rho{}_\mu F^\mu{}_\lambda x^\lambda = \frac{e^2}{4}x^T FFx, \quad a_\mu a^\mu \Leftrightarrow a^T a, \tag{2.42}$$

$$-2iea^\mu \partial_\mu K \Leftrightarrow -2ie(\partial K)^T a, \quad ieF^\mu{}_\nu x^\nu \partial_\mu K \Leftrightarrow ie(\partial K)^T Fx. \tag{2.43}$$

Now we can act the partial derivatives of the heat kernel:

$$\partial_\mu K = if(\tau)\exp(iS_{\text{cl}})(\partial_\mu S_{\text{cl}}) = iK\partial_\mu S_{\text{cl}}. \tag{2.44}$$

We also have

$$\Box K = \partial^\mu(iK\partial_\mu S_{\text{cl}}) = iK\Box S_{\text{cl}} - K(\partial S_{\text{cl}})^2. \tag{2.45}$$

Now let's compute the derivatives of the classical action. Let's rewrite the action by distribution out the coordinates

$$\begin{aligned} S_{\text{cl}} = \frac{e}{2}\Bigg[&x'^T F\frac{1}{1-\exp(-2e\tau F)}x - x^T F\frac{1}{1-\exp(-2e\tau F)}x \\ &-x'^T F\frac{1}{\exp(2e\tau F)-1}x' + x^T F\frac{1}{\exp(2e\tau F)-1}x'\Bigg] + ea^T x - ea^T x' - m^2\tau. \end{aligned} \tag{2.46}$$

One thing to notice is that

$$[(\exp(2e\tau F)-1)^{-1}]^T = -(1-\exp(-2e\tau F))^{-1}, \quad [(1-\exp(-2e\tau F))^{-1}]^T = -(\exp(2e\tau F)-1)^{-1}. \tag{2.47}$$

This observation is important when paired with the basic fact that the inner product of vectors is commutative i.e. $u^T v = v^T u$. Thus, notice how

$$\left[x^T F \frac{1}{\exp(2e\tau F)-1} x'\right]^T = -x'^T \frac{1}{1-\exp(-2e\tau F)} F^T x = x'^T F \frac{1}{1-\exp(-2e\tau F)} x, \tag{2.48}$$

where we used the fact that F and $(1-\exp(-2e\tau F))^{-1}$ commute.[2] This simplifies the action to

$$\begin{aligned} S_{\text{cl}} =& \frac{e}{2}\left[2x'^T F \frac{1}{1-\exp(-2e\tau F)} x - x^T F \frac{1}{1-\exp(-2e\tau F)} x - x'^T F \frac{1}{\exp(2e\tau F)-1} x'\right] \\ & + e a^T x - a^T x' - m^2 \tau. \end{aligned} \tag{2.49}$$

Now we can take its derivative

$$\partial S_{\text{cl}} = \frac{e}{2}\left[2x'^T F \frac{1}{1-\exp(-2e\tau F)} - x^T \left(\frac{F}{1-\exp(-2e\tau F)} + \frac{F}{\exp(2e\tau F)-1}\right) + 2a^T\right], \tag{2.50}$$

where we used the fact that

$$\frac{\partial \mathbf{A}\mathbf{x}}{\partial \mathbf{x}} = \mathbf{A}, \qquad \frac{\partial \mathbf{x}^T \mathbf{A}}{\partial \mathbf{x}} = \mathbf{A}^T. \tag{2.51}$$

The expression can be further simplified down to just

$$\partial S_{\text{cl}} = \frac{e}{2}\left[2x'^T F \frac{1}{1-\exp(-2e\tau F)} - x^T F \frac{\exp(2e\tau F)+1}{\exp(2e\tau F)-1} + 2a^T\right], \tag{2.52}$$

and whose transpose can be easily found to be

$$(\partial S_{\text{cl}})^T = \frac{e}{2}\left[\frac{1}{\exp(2e\tau F)-1} 2Fx' - \frac{1+\exp(-2e\tau F)}{1-\exp(-2e\tau F)} Fx + 2a\right]. \tag{2.53}$$

The product of the two is

[2] This is easy to see because they're diagonalized in the same basis given that they're both functions of F.

$$(\partial S_{\mathrm{cl}})(\partial S_{\mathrm{cl}})^T = e^2 \left[x'^T \frac{F\exp(2e\tau F)Fx'}{(\exp(2e\tau F)-1)^2} - x^T F \frac{\exp(2e\tau F)+1}{(\exp(2e\tau F)-1)^2} Fx' + 8a^T \frac{1}{\exp(2e\tau F)-1} Fx' \right.$$
$$\left. + \frac{1}{4} x^T F \frac{(\exp(2e\tau F)+1)^2}{(\exp(2e\tau F)-1)^2} Fx - x^T F \frac{\exp(2e\tau F)+1}{\exp(2e\tau F)-1} a + a^T a \right]. \tag{2.54}$$

We'll also need the d'Alembert operator

$$\Box S_{\mathrm{cl}} = \mathrm{Tr}[\partial_\mu \partial^\nu S_{\mathrm{cl}}] = -\frac{e}{2} \mathrm{Tr}\big[F(1-\exp(-2e\tau F))^{-1} + F(\exp(2e\tau F)-1)^{-1}\big] \tag{2.55}$$
$$= -\frac{e}{2} \mathrm{Tr}\big[F[(1-\exp(-2e\tau F))^{-1} + [(1-\exp(-2e\tau F))^{-1}]^T]\big] \tag{2.56}$$
$$= -e\, \mathrm{Tr}\big[F(1-\exp(-2e\tau F))^{-1}\big]. \tag{2.57}$$

We can also see that

$$ieF^\mu{}_\nu x^\nu \partial_\mu K = -eK\partial_\mu S_{\mathrm{cl}} F^\mu{}_\nu x^\nu \Leftrightarrow -eK(\partial S_{\mathrm{cl}})Fx, \tag{2.58}$$

which when we plug the expression for the gradient of the classical action, we get

$$-eK(\partial S_{\mathrm{cl}})Fx = -\frac{e^2}{2} K \left[2x'^T F \frac{1}{1-\exp(-2e\tau F)} - x^T F \frac{\exp(2e\tau F)+1}{\exp(2e\tau F)-1} + 2a^T \right] Fx \tag{2.59}$$
$$= -e^2 K x'^T F \frac{1}{1-\exp(-2e\tau F)} Fx + \frac{e^2}{2} K x^T F \frac{\exp(2e\tau F)+1}{\exp(2e\tau F)-1} Fx - e^2 K a^T Fx. \tag{2.60}$$

We also have the other terms

$$-2ie(iK\partial_\mu S_{\mathrm{cl}})a^\mu \Leftrightarrow 2eK\partial S_{\mathrm{cl}} a = e^2 K \left[2x'^T F \frac{1}{1-\exp(-2e\tau F)} - x^T F \frac{\exp(2e\tau F)+1}{\exp(2e\tau F)-1} + 2a^T \right] a \tag{2.61}$$
$$= 2e^2 K x'^T F \frac{1}{1-\exp(-2e\tau F)} a - e^2 K x^T F \frac{\exp(2e\tau F)+1}{\exp(2e\tau F)-1} a + 2e^2 K a^T a. \tag{2.62}$$

Now we can focus on the proper time derivatives

$$i\frac{\partial}{\partial \tau} K = i\frac{\partial}{\partial \tau}(f(\tau)\exp(iS_{\mathrm{cl}})) = if'(\tau)\exp(iS_{\mathrm{cl}}) - f(\tau)\exp(iS_{\mathrm{cl}})\frac{\partial S_{\mathrm{cl}}}{\partial \tau} = iK\frac{f'(\tau)}{f(\tau)} - K\frac{\partial S_{\mathrm{cl}}}{\partial \tau}. \tag{2.63}$$

The proper time derivative of the classical action is

$$\frac{\partial S_{\rm cl}}{\partial \tau} = \frac{e}{2}(x'-x)^T F\left[-\frac{(-\exp(-2e\tau F))(-2eF)}{(1-\exp(-2e\tau F))^2}x + \frac{\exp(2e\tau F)(2eF)}{(\exp(2e\tau F)-1)^2}x'\right] - m^2 \tag{2.64}$$

$$= e^2(x'-x)^T F\left[\frac{\exp(2e\tau F)Fx'}{(\exp(2e\tau F)-1)^2} - \frac{\exp(-2e\tau F)Fx}{(1-\exp(-2e\tau F))^2}\right] - m^2. \tag{2.65}$$

This can be simplified to be

$$\frac{\partial S_{\rm cl}}{\partial \tau} = e^2 x'^T \frac{F\exp(2e\tau F)F}{(\exp(2e\tau F)-1)^2}x' + e^2 x^T \frac{F\exp(2e\tau F)F}{(\exp(2e\tau F)-1)^2}x - 2e^2 x'^T \frac{F\exp(-2e\tau F)Fx}{(1-\exp(-2e\tau F))^2} - m^2, \tag{2.66}$$

where we used the fact that

$$x^T \frac{F\exp(2e\tau F)F}{(\exp(2e\tau F)-1)^2}x' = \left[x^T \frac{F\exp(2e\tau F)F}{(\exp(2e\tau F)-1)^2}x'\right]^T = x'^T \frac{F\exp(-2e\tau F)F}{(1-\exp(-2e\tau F))^2}x. \tag{2.67}$$

We're ready to plug all the terms into the differential equation

$$\begin{aligned}
&iK\frac{f'(\tau)}{f(\tau)} - e^2Kx'^T\frac{F\exp(2e\tau F)F}{(\exp(2e\tau F)-1)^2}x' - e^2Kx^T\frac{F\exp(2e\tau F)F}{(\exp(2e\tau F)-1)^2}x \\
&+ 2e^2Kx'^T\frac{F\exp(-2e\tau F)Fx}{(1-\exp(-2e\tau F))^2} + m^2K \\
&= iK\Box S_{\rm cl} - e^2Kx'^T\frac{F\exp(2e\tau F)Fx'}{(\exp(2e\tau F)-1)^2} - \frac{e^2K}{4}x^TF\left(\frac{\exp(2e\tau F)+1}{\exp(2e\tau F)-1}\right)^2 \\
&- e^2Ka^Ta + e^2Kx'^T\frac{F(1+\exp(-2e\tau F))Fx}{(1-\exp(-2e\tau F))^2} + e^2Kx^TF\frac{\exp(2e\tau F)+1}{\exp(2e\tau F)-1}a \\
&- 2e^2Kx'^TF\frac{1}{1-\exp(-2e\tau F)}a - e^2Kx'^T\frac{1-\exp(-2e\tau F)}{(1-\exp(-2e\tau F))^2}Fx \\
&+ \frac{e^2K}{2}x^TF\frac{\exp(2e\tau F)+1}{\exp(2e\tau F)-1}Fx - e^2Ka^TFx + 2e^2Kx'^TF\frac{1}{1-\exp(-2e\tau F)}a \\
&- e^2Kx^TF\frac{\exp(2e\tau F)+1}{\exp(2e\tau F)-1}a + 2e^2Ka^Ta + \frac{e^2}{4}Kx^TFFx + e^2Ka^TFx - e^2Ka^Ta + m^2K.
\end{aligned} \tag{2.68}$$

We cancel out the like terms on both sides including the overall factor of K. We can also notice that

$$\begin{aligned}
-\frac{e^2}{4}x^TF\left(\frac{\exp(2e\tau F)+1}{\exp(2e\tau F)-1}\right)^2 + \frac{e^2}{4}x^TFFx &= \frac{e^2}{4}x^TF\frac{(\exp(2e\tau F)-1)^2-(\exp(2e\tau F)+1)^2}{(\exp(2e\tau F)-1)^2}Fx \\
&= -e^2x^TF\frac{\exp(2e\tau F)}{(\exp(2e\tau F)-1)^2}Fx,
\end{aligned} \tag{2.69}$$

as well as

$$x^T F\left(\frac{\exp(2e\tau F)+1}{\exp(2e\tau F)-1}\right)Fx = \left[x^T F\left(\frac{\exp(2e\tau F)+1}{\exp(2e\tau F)-1}\right)Fx\right]^T \tag{2.70}$$

$$= -x^T F\left(\frac{1+\exp(-2e\tau F)}{1-\exp(-2e\tau F)}\right)Fx \tag{2.71}$$

$$= -x^T F\left(\frac{\exp(2e\tau F)+1}{\exp(2e\tau F)-1}\right)Fx, \tag{2.72}$$

and thus this term has to be zero. We also can see

$$x'^T F\frac{1+\exp(-2e\tau F)}{(1-\exp(-2e\tau F))^2}Fx - x'^T F\frac{1-\exp(-2e\tau F)}{(1-\exp(-2e\tau F))^2}Fx = 2x'^T F\frac{\exp(-2e\tau F)}{(1-\exp(-2e\tau F))^2}Fx. \tag{2.73}$$

After the dust clears, we're left with

$$i\frac{f'(\tau)}{f(\tau)} = i\Box S_{\text{cl}} \Leftrightarrow \frac{\partial}{\partial\tau}\ln f(\tau) = -e\,\mathrm{Tr}\big[F(1-\exp(-2e\tau F))^{-1}\big]. \tag{2.74}$$

To solve this differential equation, we first notice that

$$\frac{F}{1-\exp(-2e\tau F)} = \frac{\exp(2e\tau F)F}{\exp(2e\tau F)-1} = \frac{\partial}{\partial\tau}\ln(\exp(2e\tau F)-1). \tag{2.75}$$

We can go further than this. First we have

$$\ln(\exp(2e\tau F)-1) = \ln 2 + 2e\tau F + \ln\left(\frac{\exp(e\tau F)-\exp(-e\tau F)}{2}\right), \tag{2.76}$$

which when we take the derivative leaves

$$\frac{\partial}{\partial\tau}\ln(\exp(2e\tau F)-1) = 2eF + \frac{\partial}{\partial\tau}\ln\left(\frac{\exp(e\tau F)-\exp(-e\tau F)}{2}\right), \tag{2.77}$$

and since F is traceless, we can write

$$\frac{\partial}{\partial\tau}\ln f(\tau) = \mathrm{Tr}\left[-2eF - \frac{1}{2}\frac{\partial}{\partial\tau}\ln\left(\frac{\exp(e\tau F)-\exp(-e\tau F)}{2}\right)\right] = -\frac{1}{2}\frac{\partial}{\partial\tau}\mathrm{Tr}\ln\sinh(e\tau F). \tag{2.78}$$

This can be solved exactly to get

$$\ln f(\tau) = -\frac{1}{2}\mathrm{Tr}\ln\sinh(e\tau F) + C \Rightarrow f(\tau) = C\exp\left(-\frac{1}{2}\mathrm{Tr}\ln\sinh(e\tau F)\right). \tag{2.79}$$

Using the relation that for any square invertible matrix M

$$\operatorname{Tr}\ln M = \ln\det M, \tag{2.80}$$

which makes the form of $f(\tau)$ simplify to

$$f(\tau) = C(\det\sinh(e\tau F))^{-1/2}. \tag{2.81}$$

To determine the value of C, its necessary to go from the boundary conditions

$$\lim_{\tau\to 0}\int \mathrm{d}^d x'\, K(\tau; x, x') = \lim_{\tau\to 0} f(\tau)\int \mathrm{d}^d x' \exp(iS_{\mathrm{cl}}[x, x'; \tau]) = 1. \tag{2.82}$$

$$S_{\mathrm{cl}} = \frac{e}{2}(x'-x)^T F\left[\frac{1}{1-\exp(-2e\tau F)}x - \frac{1}{\exp(2e\tau F)-1}x'\right] + ea^T(x-x') - m^2\tau. \tag{2.83}$$

We can do integral substitution with $x'^\mu = x^\mu + \sqrt{4\tau}v^\mu$ which brings the action to the form

$$S_{\mathrm{cl}} = \frac{e}{2}\sqrt{4\tau}v^T F\left[\frac{1}{1-\exp(-2e\tau F)}x - \frac{1}{\exp(2e\tau F)-1}x - \frac{\sqrt{4\tau}}{\exp(2e\tau F)-1}v\right] - e\sqrt{4\tau}a^T v - m^2\tau \tag{2.84}$$

$$= \frac{e}{2}\sqrt{4\tau}v^T\left[x - \frac{\sqrt{4\tau}}{\exp(2e\tau F)-1}v\right] - e\sqrt{4\tau}a^T v - m^2\tau. \tag{2.85}$$

Now we can take $\tau \to 0$ limit

$$\lim_{\tau\to 0} S_{\mathrm{cl}} = \lim_{\tau\to 0}\frac{e}{2}\left[\sqrt{4\tau}v^T F x - v^T F\frac{4\tau}{\exp(2e\tau F)-1}v\right] - e\sqrt{4\tau}a^T v - m^2\tau \tag{2.86}$$

$$= -\lim_{\tau\to 0}\frac{e}{2}v^T F\frac{4}{\exp(2e\tau F)(2eF)}v \tag{2.87}$$

$$= -v^T v, \tag{2.88}$$

where we used l'Hopital's rule in the second to last equality sign. The integration measure becomes $\mathrm{d}^d x' = (4\tau)^{d/2}\,\mathrm{d}^d v$ so the integral is then

$$\lim_{\tau\to 0}\int \mathrm{d}^d x'\, f(\tau)\exp(iS_{\mathrm{cl}}) = \lim_{\tau\to 0} C(\det\sinh e\tau F)^{-1/2}(4\tau)^{d/2}\int \mathrm{d}^d v\exp(iS_{\mathrm{cl}}) \tag{2.89}$$

$$= \lim_{\tau\to 0} C\left(\det\left(\frac{\tau}{\sinh e\tau F}\frac{eF}{eF}\right)\right)^{1/2} 4^{d/2}\int \mathrm{d}^d v\exp(iS_{\mathrm{cl}}) \tag{2.90}$$

$$= \lim_{\tau\to 0} 2^d C\left[\det\left(\frac{e\tau F}{\sinh e\tau F}\right)\det(eF)^{-1}\right]^{1/2}\int \mathrm{d}^d v\exp(iS_{\mathrm{cl}}), \tag{2.91}$$

and using the fact that

$$\lim_{x\to 0}\frac{\sinh x}{x} = 1, \tag{2.92}$$

we are left with

$$\frac{2^d C}{\sqrt{\det eF}}\int \mathrm{d}^d v\exp(-i\eta_{\mu\nu}v^\mu v^\nu) = 1. \tag{2.93}$$

Next we use the fact

$$\int_{-\infty}^{\infty} e^{\pm ix^2} = \sqrt{\pi}e^{\pm\frac{i\pi}{4}}, \tag{2.94}$$

which when applied to our integral becomes

$$\int \mathrm{d}^d v\exp(-i\eta_{\mu\nu}v^\mu v^\nu) = \left[\sqrt{\pi}e^{\frac{i\pi}{4}}\right]^{d-1}\left[\sqrt{\pi}e^{-\frac{i\pi}{4}}\right] = \pi^{d/2}e^{\frac{i\pi d}{4}}e^{\frac{-i\pi}{4}}e^{\frac{-i\pi}{4}} = -i\pi^{d/2}e^{\frac{i\pi d}{4}}. \tag{2.95}$$

We now have enough to solve for C

$$\frac{2^d C(-i\pi^{d/2}e^{\frac{i\pi d}{4}})}{\sqrt{\det eF}} = -iC\sqrt{\frac{4^d\pi^d e^{i\pi d/2}}{\det eF}} = 1 \Rightarrow C = i\sqrt{\frac{\det eF}{(4\pi i)^d}} = i\sqrt{\det\left(\frac{eF}{4\pi i}\right)}. \tag{2.96}$$

Thus $f(\tau)$

$$f(\tau) = \frac{C}{\sqrt{\det\sinh e\tau F}} = i\sqrt{\det\left(\frac{eF}{4\pi i}\right)}\frac{1}{\sqrt{\det\sinh e\tau F}} = i\sqrt{\det\left(\frac{\frac{eF}{4\pi i}}{\sinh e\tau F}\right)}. \tag{2.97}$$

Before we proceed with this calculation, let us show that the heat kernel that we have here is correct by checking to see we recover the free particle case. This corresponds to the $F\to 0$ limit

$$\lim_{F\to 0} f(\tau) = \lim_{F\to 0} i\sqrt{\det\left(\frac{\frac{eF\tau}{4\pi i}}{\tau \sinh e\tau F}\right)} = i\sqrt{\det\left(\frac{1}{4\pi i\tau}\right)} = -\sqrt{(4\pi i\tau)^{-d}} = (4\pi i\tau)^{-d/2}. \tag{2.98}$$

The stationary action reduces down to

$$\lim_{F,a\to 0} S_{\text{cl}} = \lim_{F,a\to 0} \frac{e}{2}(x-x)^T F\left[\frac{1}{1-\exp(-2e\tau F)}x - \frac{1}{\exp(2e\tau F)-1}x'\right] - ea^T(x-x') - m^2\tau \tag{2.99}$$

$$= \lim_{F\to 0}(x'-x)^T\left[\frac{1}{\exp(-2e\tau F)(2e\tau)}x - \frac{1}{\exp(2e\tau F)(2e\tau)}x'\right] - m^2\tau \tag{2.100}$$

$$= \frac{(x'-x)^T}{2}\left(\frac{x-x'}{2\tau}\right) - m^2\tau = -\frac{1}{4\tau}\eta_{\mu\nu}(x-x')^\mu(x-x')^\nu - m^2\tau, \tag{2.101}$$

where in the second line, we used l'Hopital's rule. Therefore the heat kernel takes on the familiar form

$$K_{\text{free}}(\tau; x, x') = f(\tau)\exp(iS_{\text{cl}}) = \frac{i}{(4\pi i\tau)^{d/2}}\exp\left(\frac{i}{4\tau}(x-x')^2 + im^2\tau\right). \tag{2.102}$$

Let's show that this leads to the correct expression for the Feynman Green function. Recall that the Green's function is given by

$$G(x, x') = i\int_0^\infty K_{\text{free}}(\tau; x, x')\,\mathrm{d}\tau\,. \tag{2.103}$$

Next we write

$$\exp\left(\frac{i}{4\tau}(x-x')^2\right) = i(4\pi i\tau)^{d/2}\int\frac{\mathrm{d}^d p}{(2\pi)^d}\exp\big(-i\tau p^2 + ip\cdot(x-x')\big), \tag{2.104}$$

which we can then plug into the expression for the Green's function

$$G(x, x') = i\int_0^\infty \mathrm{d}\tau\,\frac{i}{(4\pi i\tau)^{d/2}}\left[i(4\pi i\tau)^{d/2}\int\frac{\mathrm{d}^d p}{(2\pi)^d}\exp\big(-i\tau p^2 + ip\cdot(x-x')\big)\right]\exp\big(-im^2\tau\big) \tag{2.105}$$

$$= -i\int\frac{\mathrm{d}^d p}{(2\pi)^d}\exp(ip\cdot(x-x'))\int_0^\infty \mathrm{d}\tau\exp\big(-i\tau(p^2+m^2)\big) \tag{2.106}$$

$$= \int\frac{\mathrm{d}^d p}{(2\pi)^d}\frac{\exp(ip\cdot(x-x'))}{p^2+m^2}. \tag{2.107}$$

3 The One-Loop Effective Action

Now we can calculate the one-loop effective action

$$W = -i\hbar \int \sqrt{-g}\,\mathrm{d}^d x \int_0^\infty \frac{\mathrm{d}\tau}{\tau} K(\tau; x, x), \tag{3.1}$$

with

$$K(\tau; x, x') = i\sqrt{\det\left(\frac{\frac{eF}{4\pi i}}{\sinh e\tau F}\right)} \exp(iS_{\mathrm{cl}}), \tag{3.2}$$

We can take the $x' \to x$ limit and the heat kernel simplifies to

$$\lim_{x'\to x} S_{\mathrm{cl}} = \lim_{x'\to x} \frac{e}{2}(x'-x)^T F\left[\frac{x}{1-\exp(-2e\tau F)} - \frac{x'}{\exp(2e\tau F)-1}\right] + ea^T(x-x') - m^2\tau = -m^2\tau, \tag{3.3}$$

which makes the heat kernel be the most pleasantly easy form

$$K(\tau; x, x) = i\sqrt{\det\left(\frac{\frac{eF}{4\pi i}}{\sinh e\tau F}\right)} \exp\bigl(-im^2\tau\bigr). \tag{3.4}$$

Now going forward, we're going to assume that the electromagnetic field is also weak. Weak enough where we can do a Taylor expansion

$$\frac{\theta}{\sinh\theta} = 1 - \frac{\theta^2}{6} + \frac{7}{360}\theta^4 + \dots, \tag{3.5}$$

as well as these matrix expansions

$$\ln(I+M) = M - \frac{1}{2}M^2 + \frac{1}{3}M^3 + \dots, \tag{3.6}$$

$$\det(I+M) = \exp\ln\det(I+M)) = \exp\operatorname{Tr}\ln(I+M) \tag{3.7}$$

$$= \exp\left(\operatorname{Tr} M - \frac{1}{2}\operatorname{Tr} M^2\right) = 1 + \operatorname{Tr} M + \frac{1}{2}(\operatorname{Tr} M)^2 - \frac{1}{2}\operatorname{Tr} M^2 + \dots, \tag{3.8}$$

and therefore we have

$$\det\left(\frac{\theta}{\sinh\theta}\right) = \det\left(I - \frac{1}{6}\theta^2 + \frac{7}{360}\theta^4 + \dots\right) \tag{3.9}$$

$$= 1 + \mathrm{Tr}\left[-\frac{1}{6}\theta^2 + \frac{7}{360}\theta^4 + \dots\right] + \frac{1}{2}\left[\mathrm{Tr}\left(-\frac{1}{6}\theta^2 + \frac{7}{360}\theta^4 + \dots\right)\right]^2 - \frac{1}{2}\,\mathrm{Tr}\left(-\frac{1}{6}\theta^2 + \frac{7}{360}\theta^4 + \dots\right)^2 \tag{3.10}$$

$$= 1 - \frac{1}{6}\,\mathrm{Tr}\,\theta^2 + \frac{7}{360}\,\mathrm{Tr}\,\theta^4 + \dots + \frac{1}{2}\left[-\frac{1}{6}\,\mathrm{Tr}\,\theta^2 + \frac{7}{360}\,\mathrm{Tr}\,\theta^4 + \dots\right]^2 \tag{3.11}$$

$$= 1 - \frac{1}{6}\,\mathrm{Tr}\,\theta^2 + \frac{1}{72}(\mathrm{Tr}\,\theta^2)^2 + \frac{1}{180}\,\mathrm{Tr}\,\theta^4 + \dots \tag{3.12}$$

Thus, by letting $\theta = e\tau F$, the heat kernel can be written as

$$K(\tau; x, x) = i(4\pi i\tau)^{-d/2}\sqrt{\det\left(\frac{e\tau F}{\sinh e\tau F}\right)}e^{-im^2\tau} \tag{3.13}$$

$$= i(4\pi i\tau)^{-d/2}e^{-im^2\tau}\sqrt{1 - \frac{1}{6}\,\mathrm{Tr}\,\theta^2 + \frac{1}{72}(\mathrm{Tr}\,\theta^2)^2 + \frac{1}{180}\,\mathrm{Tr}\,\theta^4 + \dots} \tag{3.14}$$

$$\simeq i(4\pi i\tau)^{-d/2}\left(1 - im^2\tau + \frac{(-im^2\tau)^2}{2} + \dots\right)\left(1 - \frac{1}{12}\,\mathrm{Tr}\big((eF\tau)^2\big) + \dots\right) \tag{3.15}$$

$$= i(4\pi i\tau)^{-d/2}\left[1 - im^2\tau - \frac{m^4\tau^2}{2} - \frac{e^2\tau^2}{12}\,\mathrm{Tr}\big(F^2\big) + \dots\right], \tag{3.16}$$

where

$$\mathrm{Tr}\,F^2 \equiv F^{\mu}{}_{\nu}F^{\nu}{}_{\mu} = -F_{\mu\nu}F^{\mu\nu}, \tag{3.17}$$

which finally leaves us with

$$K(\tau; x, x) \simeq i(4\pi i\tau)^{-d/2}\left[1 - im^2\tau + \tau^2\left(\frac{e^2}{12}F^{\mu\nu}F_{\mu\nu} - \frac{m^4}{2}\right) + \dots\right]. \tag{3.18}$$

We can define the one-loop effective Lagrangian as

$$\mathcal{L}^{(1)} = \hbar\int_0^\infty \frac{\mathrm{d}\tau}{\tau}\exp\big(-im^2\tau\big)i\sqrt{\det\left(\frac{\frac{eF}{4\pi i}}{\sinh e\tau F}\right)}. \tag{3.19}$$

This integral is going to be divergent and thus we'll need to regularize it. Leonard Parker in his textbook adopted to use dimensional regularization as we shall do here. We will also choose to work in $d = 4 + \epsilon$ dimensions. This allows us to write

$$\det\left[(4\pi i\tau)^{-1}\right] = (4\pi i\tau)^{-d} = (4\pi i\tau)^{-4}(4\pi i\tau)^{-\epsilon}, \tag{3.20}$$

which turns the integrand of the one-loop effect Lagrangian to be

$$\mathcal{L}^{(1)} = \hbar\ell^{\epsilon}\int_0^{\infty}\frac{\mathrm{d}\tau}{\tau}\exp\left(-im^2\tau\right)(4\pi i\tau)^{-\epsilon/2}\sqrt{\det\left(\frac{e\tau F}{4\pi i\sinh e\tau F}\right)}, \tag{3.21}$$

with $\mathrm{Im}(m^2) < 0$. We will perform a Wick rotation by $\tau \to -i\tau \Rightarrow \mathrm{d}\tau \to -i\,\mathrm{d}\tau$ and the Lagrangian becomes

$$\mathcal{L}^{(1)} = \hbar\ell^{\epsilon}\int_0^{\infty}\frac{-i\,\mathrm{d}\tau}{-i\tau}\exp\left(-m^2\tau\right)(4\pi\tau)^{-2-\epsilon/2}\sqrt{\det\left(\frac{-ie\tau F}{\sinh(-ie\tau F)}\right)} \tag{3.22}$$

$$= \hbar\ell^{\epsilon}\int_0^{\infty}\frac{\mathrm{d}\tau}{\tau}\exp\left(-m^2\tau\right)(4\pi\tau)^{-2-\epsilon/2}\sqrt{\det\left(\frac{e\tau F}{\sin e\tau F}\right)}. \tag{3.23}$$

Now in order to evaluate the determinant, we're going to need to calculate the F-matrix. The field strength tensor given by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ whose corresponds to

$$F_{0i} = \dot{A}_i - \partial_i A_0 = E_i, \quad F_{ij} = \partial_i A_j - \partial_j A_i = \epsilon_{ijk}B_k, \tag{3.24}$$

and matrix form is

$$F = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{bmatrix} \tag{3.25}$$

We also have the dual field strength tensor $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\lambda\rho}F^{\lambda\rho}$ which has the following matrix representation

$$\tilde{F} = \begin{bmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & -E_z & E_y \\ B_y & E_z & 0 & -E_x \\ B_z & -E_y & E_x & 0 \end{bmatrix} \tag{3.26}$$

In order to compute the determinant of F, it'll be easier to work with the diagonalization of the matrices $F^{\mu}{}_{\nu} = \eta^{\mu\lambda}F_{\lambda\nu}$ and $\tilde{F}^{\mu}{}_{\nu} = \eta^{\mu\lambda}\tilde{F}_{\lambda\nu}$

$$F^{\mu}{}_{\nu} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{bmatrix}, \quad \tilde{F}^{\mu}{}_{\nu} = \begin{bmatrix} 0 & -B_x & -B_y & -B_z \\ -B_x & 0 & E_z & E_y \\ -B_y & -E_z & 0 & E_x \\ -B_z & E_y & -E_x & 0 \end{bmatrix} \tag{3.27}$$

Defining the quantities

$$\mathcal{F} = \frac{1}{4}F^{\mu\nu}F_{\mu\nu} = \frac{1}{2}(\mathbf{B}^2 - \mathbf{E}^2), \quad \mathcal{G} = \frac{1}{4}\tilde{F}^{\mu\nu}F_{\mu\nu} = \mathbf{B}\cdot\mathbf{E}. \tag{3.28}$$

It's easy to prove the following identities from simple matrix multiplication: $\tilde{F}^{\mu}{}_{\lambda}F^{\lambda}{}_{\nu} = -\mathcal{G}\delta^{\mu}_{\nu}$ and $\tilde{F}^{\mu}{}_{\lambda}\tilde{F}^{\lambda}{}_{\nu} - F^{\mu}{}_{\lambda}F^{\lambda}{}_{\nu} = 2\mathcal{F}\delta^{\mu}_{\nu}$. These identities will make finding the eigenvalues and eigenvectors much easier. First we write down the standard eigenvalue-eigenvector equation

$$F^{\lambda}{}_{\nu}\psi^{\nu} = f\psi^{\lambda}. \tag{3.29}$$

We can act $\tilde{F}^{\mu}{}_{\lambda}$ on both sides to get

$$\tilde{F}^{\mu}{}_{\lambda}F^{\lambda}{}_{\nu}\psi^{\nu} = -\mathcal{G}\delta^{\mu}_{\nu}\psi^{\nu} = f\tilde{F}^{\mu}{}_{\lambda}\psi^{\lambda} \Rightarrow \tilde{F}^{\mu}{}_{\lambda}\psi^{\lambda} = -\frac{\mathcal{G}}{f}\psi^{\mu}. \tag{3.30}$$

This therefore leads to

$$(\tilde{F}^{\mu}{}_{\lambda}\tilde{F}^{\lambda}{}_{\nu} - F^{\mu}{}_{\lambda}F^{\lambda}{}_{\nu})\psi^{\nu} = 2\mathcal{F}\delta^{\mu}_{\nu}\psi^{\nu} = 2\mathcal{F}\psi^{\mu}, \tag{3.31}$$

and using the fact that ψ is both an eigenvector of F and $\tilde{F}$, with eigenvalue f and $-\mathcal{G}/f$ respectively gives us

$$\left(-\frac{\mathcal{G}}{f}\right)^2\psi^{\mu} - f^2\psi^{\mu} = 2\mathcal{F}\psi^{\mu} \Rightarrow \left[\left(\frac{\mathcal{G}}{f}\right)^2 - f^2 - 2\mathcal{F}\right] = 0. \tag{3.32}$$

We can rearrange the terms in this expression to get a quadratic equation for f^2 which looks like

$$f_{\pm}^2 = -\mathcal{F} \pm \sqrt{\mathcal{F}^2 \pm +\mathcal{G}^2} \Rightarrow \pm f_{\pm} = \pm\sqrt{-\mathcal{F} \pm \sqrt{\mathcal{F}^2 + \mathcal{G}^2}}. \tag{3.33}$$

We define the complex root of this eigenvalue as being

$$\psi = \sqrt{2(\mathcal{F} + i\mathcal{G})}, \qquad \psi^* = \sqrt{2(\mathcal{F} - i\mathcal{G})}, \tag{3.34}$$

with the properties

$$f_\pm = \frac{i}{2}(\psi^* \pm \psi) \Rightarrow f_+ + f_- = i\psi^*, \qquad f_+ - f_- = i\psi. \tag{3.35}$$

Now that we have our eigenvalues, we can write F as

$$F = U \operatorname{diag}(f_+, -f_+, f_-, -f_-)U^T, \tag{3.36}$$

where U is the matrix containing the eigenvectors of F. Thus

$$\frac{\sin(e\tau F)}{e\tau F} = U \operatorname{diag}\left(\frac{\sin e\tau f_+}{e\tau f_+}, \frac{-\sin e\tau f_+}{-e\tau f_+}, \frac{\sin e\tau f_-}{e\tau f_-}, \frac{-\sin e\tau f_-}{-e\tau f_-}\right)U^T, \tag{3.37}$$

and finally

$$\sqrt{\det\left(\frac{\sin e\tau F}{e\tau F}\right)} = \frac{\sin(e\tau f_+)\sin(e\tau f_-)}{(e\tau)^2 f_+ f_-}, \tag{3.38}$$

To simplify this expression, we have these identities:

$$f_+ f_- = -\frac{1}{4}(\psi^{*2} - \psi^*\psi + \psi\psi^* - \psi^2) = -\frac{1}{4}[2(\mathcal{F} + i\mathcal{G}) - 2(\mathcal{F} - i\mathcal{G})] = i\mathcal{G}, \tag{3.39}$$

$$\sin(e\tau f_+)\sin(e\tau f_-) = -\frac{1}{4}[e^{ie\tau(f_+ + f_-)} - e^{ie\tau(f_+ - f_-)} - e^{-ie\tau(f_+ - f_-)} + e^{-ie\tau(f_+ + f_-)}] \tag{3.40}$$

$$= \frac{1}{4}(e^{e\tau\psi} + e^{-e\tau\psi} - e^{e\tau\psi^*} - e^{-e\tau\psi^*}) \tag{3.41}$$

$$= \frac{1}{2}[\cosh e\tau\psi - \cosh e\tau\psi^*] = i\,\mathrm{Im}\{\cosh e\tau\psi\}, \tag{3.42}$$

thus we have

$$\sqrt{\det\left(\frac{\sin e\tau F}{e\tau F}\right)} = \frac{\mathcal{G}(e\tau)^2}{\mathrm{Im}\{\cosh e\tau\psi\}}. \tag{3.43}$$

The one-loop effective Lagrangian is then

$$\mathcal{L}^{(1)} = \hbar\ell^\epsilon \int_0^\infty \frac{\mathrm{d}\tau}{\tau} \exp\bigl(-m^2\tau\bigr)(4\pi\tau)^{-2-\epsilon/2} \frac{(e\tau)^2\mathcal{G}}{\mathrm{Im}\{\cosh e\tau\psi\}}. \tag{3.44}$$

We can Taylor expand the $1/\operatorname{Im}\{\cosh e\tau\psi\}$ in powers of τ to get

$$\begin{aligned}\frac{(e\tau)^2}{\operatorname{Im}\cosh e\tau\psi} &= \frac{4ie^2}{(e\psi)^2-(e\psi^*)^2} - \frac{2i(e\tau)^2((e\psi)^2+(e\psi^*)^2)}{6((e\psi)^2-(e\psi^*)^2)} \\ &\quad + 2i(e\tau)^2\tau^2\frac{3(e\psi)^4+8(e\psi)^2(e\psi^*)^2+3(e\psi^*)^4}{360((e\psi)^2-(e\psi^*)^2)} + \ldots,\end{aligned} \tag{3.45}$$

which can be simplified down to

$$\frac{(e\tau)^2}{\operatorname{Im}\cosh e\tau\psi} = \frac{4i}{4i\mathcal{G}} - \frac{4ie^2\mathcal{F}(e\tau)^2}{3(4i\mathcal{G})e^2} + \frac{3e^4(4(\mathcal{F}+i\mathcal{G})^2)+8e^4(\mathcal{F}^2+\mathcal{G}^2)+3e^4(4(\mathcal{F}-i\mathcal{G})^2)}{180(4i\mathcal{G})}i\tau^4 + \ldots \tag{3.46}$$

$$= \frac{1}{\mathcal{G}} - \frac{(e\tau)^2}{3}\frac{\mathcal{F}}{\mathcal{G}} + \frac{(e\tau)^4}{90}\frac{(7\mathcal{F}^2+\mathcal{G}^2)}{\mathcal{G}} + \ldots \tag{3.47}$$

The one-loop effective Lagrangian is now

$$\mathcal{L}^{(1)} = \hbar\ell^{\epsilon}\int_0^\infty \frac{\mathrm{d}\tau}{\tau}\frac{\exp(-m^2\tau)}{(4\pi)^{2+\epsilon/2}}\left[1-\frac{(e\tau)^2}{3}\mathcal{F}+\frac{(e\tau)^4}{90}(7\mathcal{F}^2+\mathcal{G}^2)+\ldots\right]. \tag{3.48}$$

We can do a u-substitution in order to evaluate this integral if we take $u = m^2\tau \Rightarrow \mathrm{d}u = m^2\,\mathrm{d}\tau$

$$\mathcal{L}^{(1)} = \frac{\hbar}{(4\pi)^2}\left(\frac{m^2\ell^2}{4\pi}\right)^{\epsilon/2}\int_0^\infty \mathrm{d}u\, e^{-u}u^{-3-\epsilon/2}m^4\left[1-\frac{(eu)^2}{3m^4}\mathcal{F}+\frac{(eu)^4}{90m^8}(7\mathcal{F}^2+\mathcal{G}^2)+\ldots\right] \tag{3.49}$$

$$= \frac{\hbar}{(4\pi)^2}\left(\frac{m^2\ell^2}{4\pi}\right)^{\epsilon/2}\left[m^4\Gamma(-2-\epsilon/2)-\frac{\Gamma(-\epsilon/2)e^2}{3}\mathcal{F}+\frac{e^4\Gamma(2-\epsilon/2)}{m^4}(7\mathcal{F}^2+\mathcal{G}^2)+\ldots\right]. \tag{3.50}$$

3.1 Renormalizing the Action

We can see that the first two terms clearly diverge when $\epsilon \to 0$. In order to cancel these terms out, we need a constant term to cancel out with $\Gamma(-2-\epsilon/2)$ (and it has to be a constant since its not attached to anything) and we need a term proportional to $\mathcal{F} = F^{\mu\nu}F_{\mu\nu}$ to cancel out with the $\Gamma(-\epsilon/2)$ term. Therefore we need to add $\mathcal{L}^{(1)}$ to the classical Lagrangian

$$\mathcal{L}^{(0)} = -\frac{1}{4}F_B^{\mu\nu}F^B_{\mu\nu} - c_B, \tag{3.51}$$

where the B subscript and superscript denotes the bare (which is to say unrenormalized) quantities. The renormalization procedure is the same as what we're taught in QFT which goes as follows

$$A_\mu^B = Z_A^{1/2} A_\mu \Rightarrow \mathcal{L}^{(0)} = -\frac{1}{4} Z_A F^{\mu\nu} F_{\mu\nu} - c_B = -Z_A \mathcal{F} - c_B, \tag{3.52}$$

and we expand the Z_A term like

$$Z_A = 1 + \delta Z_A + \ldots \tag{3.53}$$

Taking the total Lagrangian to be $\mathcal{L} = \mathcal{L}^{(0)} + \mathcal{L}^{(1)}$ gives

$$\mathcal{L} = -Z_A \mathcal{F} - c_B + \frac{\hbar}{(4\pi)^2} \left(\frac{m^2 \ell^2}{4\pi} \right)^{\epsilon/2} \left[m^4 \Gamma(-2 - \epsilon/2) - \frac{\Gamma(-\epsilon/2) e^2}{3} \mathcal{F} + \ldots \right] \tag{3.54}$$

$$= -\mathcal{F} - \delta Z_A \, \mathcal{F} - c_B + \frac{\hbar}{(4\pi)^2} \left(\frac{m^2 \ell^2}{4\pi} \right)^{\epsilon/2} \left[m^4 \Gamma(-2 - \epsilon/2) - \frac{\Gamma(-\epsilon/2) e^2}{3} \mathcal{F} + \ldots \right] \tag{3.55}$$

$$= -\mathcal{F} - \left(\delta Z_A + \frac{e^2 \hbar}{3(4\pi)^2} \left(\frac{m^2 \ell^2}{4\pi} \right)^{\epsilon/2} \Gamma(-\epsilon/2) \right) \mathcal{F} + \left(\frac{\hbar m^4}{(4\pi)^2} \left(\frac{m^2 \ell^2}{4\pi} \right)^{\epsilon/2} \Gamma(-2 - \epsilon/2) - c_B \right) + \ldots \tag{3.56}$$

and thus we need only to make the choice

$$c_B = \frac{\hbar m^4}{(4\pi)^2} \left(\frac{m^2 \ell^2}{4\pi} \right)^{\epsilon/2} \Gamma(-2 - \epsilon/2), \quad \delta Z_A = -\frac{e^2 \hbar}{3(4\pi)^2} \left(\frac{m^2 \ell^2}{4\pi} \right)^{\epsilon/2} \Gamma(-\epsilon/2), \tag{3.57}$$

and those divergent terms are no longer a problem. Now we can freely take the $\epsilon \to 0$ limit. This is equivalent to writing

$$\mathcal{L} = -\mathcal{F} + \frac{\hbar}{(4\pi)^2} \int_0^\infty \frac{\mathrm{d}\tau}{\tau^3} \exp\bigl(-m^2 \tau\bigr) \left[\frac{(e\tau)^2 \mathcal{G}}{\operatorname{Im} \cosh e\tau\psi} - 1 + \frac{(e\tau)^2}{3} \mathcal{F} \right], \tag{3.58}$$

which can be evaluated to get

$$\mathcal{L} = -\mathcal{F} + \frac{\hbar e^4}{90(4\pi)^2 m^4} (7\mathcal{F}^2 + \mathcal{G}^2) = -\mathcal{F} + \frac{\hbar e^4}{90(4\pi)^2 m^4} \left[(\mathbf{B} \cdot \mathbf{E})^2 + \frac{7}{4} (\mathbf{B}^2 - \mathbf{E}^2)^2 \right]. \tag{3.59}$$

4 The Schwinger Effect

We can specialize to different cases: $\mathbf{E} \to 0$ with $\psi = |\mathbf{B}|$ and $\mathbf{B} \to 0$ with $\psi = i|\mathbf{E}|$. The $\mathbf{E} \to 0$ limit yields

$$(e\tau)^2 \lim_{\mathbf{E}\to 0} \frac{\mathbf{B}\cdot\mathbf{E}}{\operatorname{Im}\left\{\cosh e\tau\sqrt{2}\left(\frac{1}{2}(\mathbf{B}^2 - \mathbf{E}^2) + i\mathbf{E}\cdot\mathbf{B}\right)\right\}} \tag{4.1}$$

$$= (e\tau)^2 \lim_{\mathbf{E}\to 0} \frac{B\psi}{\operatorname{Im}\sinh e\tau\psi \cdot (B - iE)(e\tau)} \tag{4.2}$$

$$= \frac{e\tau B}{\sinh e\tau B}, \tag{4.3}$$

with the corresponding Lagrangian given by

$$\mathcal{L}(B) = -\frac{1}{2}B^2 + \frac{\hbar}{(4\pi)^2}\int_0^\infty \frac{\mathrm{d}\tau}{\tau^3}\exp\left(-m^2\tau\right)\left[\frac{e\tau B}{\sinh e\tau B} - 1 + \frac{(e\tau B)^2}{6}\right]. \tag{4.4}$$

We can do the same for the electric field

$$(e\tau)^2 \lim_{\mathbf{B}\to 0} \frac{\mathbf{B}\cdot\mathbf{E}}{\operatorname{Im}\left\{\cosh e\tau\sqrt{2}\left(\frac{1}{2}(\mathbf{B}^2 - \mathbf{E}^2) + i\mathbf{E}\cdot\mathbf{B}\right)\right\}} \tag{4.5}$$

$$= (e\tau)^2 \lim_{\mathbf{B}\to 0} \frac{E\psi}{\operatorname{Im}\sinh e\tau\psi \cdot (B - iE)(e\tau)} \tag{4.6}$$

$$= \frac{e\tau E}{\sin e\tau E}. \tag{4.7}$$

and the Lagrangian for this is

$$\mathcal{L}(E) = \frac{1}{2}E^2 + \frac{\hbar}{(4\pi)^2}\int_0^\infty \frac{\mathrm{d}\tau}{\tau^3}\exp\left(-m^2\tau\right)\left[\frac{e\tau E}{\sin e\tau E} - 1 - \frac{(e\tau E)^2}{6}\right]. \tag{4.8}$$

We can evaluate this last integral via the Residue theorem. The poles of the integrand are all on

$$e\tau_n E = n\pi \Rightarrow \tau_n = \frac{n\pi}{eE}, \tag{4.9}$$

for $n \in \mathbb{N}_{>0}$ (excluding 0). To apply the Residue theorem, define the function

$$f(z) = \frac{e^{-m^2 z}}{z^3}\frac{eEz}{\sin(eEz)} = \frac{e^{-m^2 z}}{z^2}\frac{eE}{\sin eEz}. \tag{4.10}$$

We can ignore the other two terms in the integrand since they are analytic functions of τ and thus the Residue theorem implies these terms are zero. Now we can find the residues for this function

$$\begin{aligned}\text{Res}\Big[f, \frac{n\pi}{eE}\Big] &= \lim_{z\to n\pi/eE} \frac{e^{-m^2 z}}{z^2}\frac{eE}{eE\cos(eEz)} &(4.11)\\ &= \frac{\exp(-m^2 n\pi/eE)}{(n\pi/eE)^2}\frac{1}{\cos n\pi} &(4.12)\\ &= (-1)^{n+1}\left(\frac{eE}{n\pi}\right)^2 \exp\left(-\frac{m^2 n\pi}{eE}\right), &(4.13)\end{aligned}$$

where we used the fact from Complex Analysis that for an analytic function that can be written as $f(z) = q(z)/p(z)$

$$\text{Res}[f, z_0] = \lim_{z\to z_0}(z-z_0)f(z) = \lim_{z\to z_0}\frac{zq(z)-z_0q(z)}{p(z)} = \lim_{z\to z_0}\frac{q(z)+zq'(z)-z_0q'(z)}{p'(z)} = \frac{q(z_0)}{p'(z_0)}. \tag{4.14}$$

Thus, the Lagrangian becomes

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}E^2 + \frac{2\pi i\hbar}{2(4\pi)^2}\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{eE}{n\pi}\right)^2 \exp\left(-\frac{\pi m^2}{eE}n\right) &(4.15)\\ &= \frac{1}{2}E^2 + \frac{i\hbar e^2E^2}{16\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\exp\left(-\frac{\pi m^2}{eE}n\right). &(4.16)\end{aligned}$$

The extra factor of 2 in the denominator is because we extended the integral bounds to go from $(-\infty, \infty)$ and cutting the integral in half by recognizing that since τ is positive, we can replace it with its absolute value, which upon doing so makes the integrand to be an even function. In order to interpret this result, recall the definition of the effective action

$$|\langle \text{out}|\text{in}\rangle|^2 = \exp\left(\frac{i}{\hbar}(W - W^*)\right) = \exp\left(-\frac{2}{\hbar}\,\text{Im}\,W\right). \tag{4.17}$$

Since $\text{Im}\,W \neq 0$, this implies the in-vacuum doesn't stay in the vacuum state. This is interpreted as the electric field being sufficiently strong enough that it physically creates particle-anti particle pairs. This is the famed Schwinger effect.