Higher Derivative Gravity

Marcell Howard

August 5, 2025

1 Introduction

Here is a note repository on the action and variation of the higher curvature terms that could exist in a general gravitation theory. Theories of this form can be interesting in their own right as a sort of completion to Einstein's theory. From an effective field theory (EFTs) perspective, since the usual Einstein-Hilbert action only contains a [mass] dimension 2 operator in the form of the Ricci scalar R, it is not the most general action one can write down just from dimensional analysis grounds. EFTs tells us that we must include *all* possible terms in the theory that satisfies all the symmetries that are compatible with the theory. In this case, we wish to satisfy Lorentz and diffeomorphism invariance. People have also found use for these terms in a whole host of applications including models of inflation and models of quantum gravity.

In addition to these more general considerations, these sorts of terms are also necessary when considering semi-classical gravity (the motivation for this document). In semiclassical gravity, one is interested in $G_{\mu\nu} = 8\pi G \langle T_{\mu\nu} \rangle$ where $\langle \cdots \rangle$ is the vacuum expectation value. Because the stress energy tensor will have poles of the form $(d-4)^{-1}$ where d are the dimensions in spacetime, these poles can only be canceled out by the terms that are introduced by the higher curvature terms.

Conventions We use the mostly plus metric signature, i.e. $\eta_{\mu\nu} = (-,+,+,+)$ and units where $c = 1$. The reduced four dimensional Planck mass is $M_{\rm Pl} = (8\pi G)^{-1/2} \approx$

2.43×10^{18} GeV. We use boldface letters $\mathbf{r}$ to indicate 3-vectors and x and p to denote 4-vectors. Conventions for the curvature tensors, covariant and Lie derivatives are all taken from Carroll.

2 The Most General Gravitation Action

We start with the relevant action

$$S = \int \mathrm{d}^d x \sqrt{-g} \left[\frac{1}{16\pi G_0} (R - 2\Lambda_0) + \alpha_0 R^2 + \beta_0 R^{\mu\nu} R_{\mu\nu} + \gamma_0 R^{\lambda\rho\mu\nu} R_{\lambda\rho\mu\nu} \right], \tag{2.1}$$

where the subscripts 0 are to indicate these are the *bare* values of these constants in order highlight that we're interested in this action for renormalization purposes. We will derive the equations of motion for each term separately. First we'll start with the usual Einstein-Hilbert action.

3 Einstein-Hilbert Action

Going forward we'll write $2\kappa^2 = 16\pi G_0$ and we choose to vary with respect to the inverse metric. But first we recognize that $R = g^{\mu\nu} R_{\mu\nu}$. So we then have

$$S_{\mathrm{EH}} = \frac{1}{2\kappa^2} \int \mathrm{d}^d x \left[\sqrt{-g} (g^{\mu\nu} R_{\mu\nu} - 2\Lambda_0) \right]. \tag{3.1}$$

Now we shall vary the action with respect to the inverse metric

$$\delta S_{\mathrm{EH}} = \frac{1}{2\kappa^2} \int \mathrm{d}^d x \left[(R_{\mu\nu}\, \delta g^{\mu\nu} + g^{\mu\nu}\, \delta R_{\mu\nu}) \sqrt{-g} + (R - 2\Lambda_0)\, \delta \sqrt{-g} \right] \tag{3.2}$$

First we want to deal with the variation in the volume. Recall we can treat the variation operator as a differential operator. We get

$$\delta \sqrt{-g} = -\frac{1}{2} \frac{1}{\sqrt{-g}}\, \delta g\,, \tag{3.3}$$

where δg is the variation in the determinant. To figure out what this is, let's look at how the determinant is defined

$$g = \sum_{\mu\nu} (-1)^{\mu+\nu} M^{\mu\nu} g_{\mu\nu}, \tag{3.4}$$

where $M^{\mu\nu}$ is the determinant of the matrix whose μth-row and νth-column has been deleted. Thus, the variation in the determinant is simply

$$\delta g = (-1)^{\mu+\nu} M^{\mu\nu}\, \delta g_{\mu\nu}\,. \tag{3.5}$$

Next, we write the co-factor matrix in terms of the inverse metric and the determinant

$$g^{\mu\nu} = \frac{1}{g}(-1)^{\mu+\nu} M^{\mu\nu}, \tag{3.6}$$

and so the variant in the determinant becomes

$$\delta g = g g^{\mu\nu}\, \delta g_{\mu\nu}\,. \tag{3.7}$$

Recall we wish to vary the action with respect to the *inverse* metric. To get the above expression in those terms, we use the fact that

$$g^{\mu\lambda} g_{\lambda\nu} = \delta^\mu_\nu \Rightarrow g_{\lambda\nu}\, \delta g^{\mu\lambda} + g^{\mu\lambda}\, \delta g_{\lambda\nu} = 0. \tag{3.8}$$

The last expression implies that the variation in either the metric or inverse metric can be written as

$$\delta g_{\mu\nu} = -g_{\mu\lambda} g_{\nu\rho}\, \delta g^{\lambda\rho}\,, \qquad \delta g^{\lambda\rho} = -g^{\lambda\mu} g^{\rho\nu}\, \delta g_{\mu\nu}\,. \tag{3.9}$$

So the variation in the volume form is merely

$$\delta\sqrt{-g} = -\frac{1}{2\sqrt{-g}}(-g g_{\mu\nu}\, \delta g^{\mu\nu}) = -\frac{1}{2}\sqrt{-g} g_{\mu\nu}\, \delta g^{\mu\nu}\,. \tag{3.10}$$

Now we are interested in figuring out the variation in the curvature tensor. To figure out what $\delta R_{\mu\nu}$ is, we will first look at the Riemann tensor and its variation

$$R^\rho{}_{\lambda\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\lambda} - \partial_\nu \Gamma^\rho_{\mu\lambda} + \Gamma^\rho_{\mu\alpha}\Gamma^\alpha_{\nu\lambda} - \Gamma^\rho_{\nu\alpha}\Gamma^\alpha_{\mu\lambda}. \tag{3.11}$$

The variation in the Riemann tensor is then

$$\delta R^{\rho}{}_{\lambda\mu\nu} = \partial_\mu(\delta\Gamma^{\rho}_{\nu\lambda}) - \partial_\nu(\delta\Gamma^{\rho}_{\mu\lambda}) + (\delta\Gamma^{\rho}_{\mu\alpha})\Gamma^{\alpha}_{\nu\lambda} + \Gamma^{\rho}_{\mu\alpha}(\delta\Gamma^{\alpha}_{\nu\lambda}) - (\delta\Gamma^{\rho}_{\nu\alpha})\Gamma^{\alpha}_{\mu\lambda} - \Gamma^{\rho}_{\nu\alpha}(\delta\Gamma^{\alpha}_{\mu\lambda}). \quad (3.12)$$

Next we notice that

$$\nabla_\mu(\delta\Gamma^{\rho}_{\nu\lambda}) = \partial_\mu(\delta\Gamma^{\rho}_{\nu\lambda}) + \Gamma^{\rho}_{\mu\alpha}(\delta\Gamma^{\alpha}_{\nu\lambda}) - \Gamma^{\alpha}_{\nu\mu}(\delta\Gamma^{\rho}_{\alpha\lambda}) - \Gamma^{\alpha}_{\lambda\mu}(\delta\Gamma^{\rho}_{\alpha\nu}). \quad (3.13)$$

It is easy to see that $\nabla_\nu(\delta\Gamma^{\rho}_{\mu\lambda}) = (\mu \leftrightarrow \nu)$. Taking the difference between these two objects gives

$$\nabla_\mu(\delta\Gamma^{\rho}_{\nu\lambda}) - \nabla_\nu(\delta\Gamma^{\rho}_{\mu\lambda}) = \partial_\mu(\delta\Gamma^{\rho}_{\nu\lambda}) - \partial_\nu(\delta\Gamma^{\rho}_{\mu\lambda}) + (\delta\Gamma^{\rho}_{\mu\alpha})\Gamma^{\alpha}_{\nu\lambda} + \Gamma^{\rho}_{\mu\alpha}(\delta\Gamma^{\alpha}_{\nu\lambda}) - (\delta\Gamma^{\rho}_{\nu\alpha})\Gamma^{\alpha}_{\mu\lambda} - \Gamma^{\rho}_{\nu\alpha}(\delta\Gamma^{\alpha}_{\mu\lambda}) \quad (3.14)$$

$$= \delta R^{\rho}{}_{\lambda\mu\nu}\,. \quad (3.15)$$

Next we can find the variation in the Ricci tensor by taking the trace of the variation of the Riemann tensor

$$\delta R_{\mu\nu} = \delta R^{\lambda}{}_{\mu\lambda\nu} = \nabla_\lambda(\delta\Gamma^{\lambda}_{\nu\mu}) - \nabla_\nu(\delta\Gamma^{\lambda}_{\lambda\mu}). \quad (3.16)$$

Contracting this variation with the metric gives

$$g^{\mu\nu}\,\delta R_{\mu\nu} = \nabla_\lambda[g^{\mu\nu}\,\delta\Gamma^{\lambda}_{\mu\nu} - g^{\mu\nu}g^{\lambda}_{\nu}\,\delta\Gamma^{\rho}_{\rho\mu}] = \nabla_\lambda\big[g^{\mu\nu}\,\delta\Gamma^{\lambda}_{\mu\nu} - g^{\mu\lambda}\,\delta\Gamma^{\nu}_{\mu\nu}\big], \quad (3.17)$$

and is a total derivative term which we can normally throw out. The issue becomes when we have a term e.g. a scalar field that couples to the Ricci scalar. Now we find δS takes the form

$$\delta S = \frac{1}{\kappa^2}\int \mathrm{d}^d x\left(\sqrt{-g}R_{\mu\nu}\,\delta g^{\mu\nu} - \sqrt{-g}\frac{1}{2}(R - 2\Lambda_0)g_{\mu\nu}\,\delta g^{\mu\nu}\right). \quad (3.18)$$

This brings the total action δS to be

$$\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = \frac{1}{\kappa^2}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda_0 g_{\mu\nu}\right) = 0. \quad (3.19)$$

This leaves us with the famous Einstein vacuum equations with a cosmological constant

$$G_{\mu\nu} + \Lambda_0 g_{\mu\nu} = 0. \quad (3.20)$$

4 R^2-Gravity

We can find the equations of motion for R^2 gravity by computing the equations of motion for some general $f(R)$ gravity theory which we will reproduce below. In the interest of simplicity, we present the action as

$$S = \frac{1}{2\kappa^2}\int \mathrm{d}^d x\,\sqrt{-g}f(R), \tag{4.1}$$

where of course $f(R)$ is some general function of the curvature scalar. Again we wish to vary the action with respect to to inverse metric $g^{\mu\nu}$. Proceeding accordingly shows

$$\delta S = \frac{1}{2\kappa^2}\int \mathrm{d}^d x\,[\sqrt{-g}\,\delta f(R) + f(R)\,\delta\sqrt{-g}] = \frac{1}{2\kappa^2}\int \mathrm{d}^d x\left[\sqrt{-g}\frac{\delta f(R)}{\delta R}\,\delta R - \frac{1}{2}f(R)\sqrt{-g}g_{\mu\nu}\,\delta g^{\mu\nu}\right]. \tag{4.2}$$

f being some general scalar function of the curvature means the functional derivative should reduce down to just the normal derivative

$$\frac{\delta f}{\delta R} = \frac{\mathrm{d}f}{\mathrm{d}R} \equiv f'(R). \tag{4.3}$$

Next we need only to worry about varying the curvature scalar now

$$\delta S = \frac{1}{2\kappa^2}\int \mathrm{d}^d x\,[\sqrt{-g}f'(R)(R_{\mu\nu}\,\delta g^{\mu\nu} + g^{\mu\nu}\,\delta R_{\mu\nu}) - \frac{1}{2}\sqrt{-g}f(R)g_{\mu\nu}\,\delta g^{\mu\nu}]. \tag{4.4}$$

We've already dealt with the variation in the Ricci tensor, but only to a certain extent. Since it was already a total derivative term, we could safely ignore it. Now however, we have a coupling between our arbitrary function $f(R)$ and the curvature tensor which will yield non-trivial dynamics. Recall that the variation in the Ricci tensor is given by

$$g^{\mu\nu}\,\delta R_{\mu\nu} = \nabla_\lambda\big[g^{\mu\nu}\,\delta\Gamma^\lambda_{\mu\nu} - g^{\mu\lambda}\,\delta\Gamma^\nu_{\mu\nu}\big]. \tag{4.5}$$

Now we need to find out what the variation in the connection is. First we write

$$\Gamma_{\rho\mu\nu} = \frac{1}{2}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}), \tag{4.6}$$

where we can write

$$\Gamma^{\lambda}_{\mu\nu} = g^{\lambda\rho}\Gamma_{\rho\mu\nu}. \tag{4.7}$$

The variation in the connection can then be shown to be

$$\delta\Gamma^{\lambda}_{\mu\nu} = \delta g^{\lambda\rho}\,\Gamma_{\rho\mu\nu} + g^{\lambda\rho}\,\delta\Gamma_{\rho\mu\nu} \tag{4.8}$$

$$= -g^{\alpha\lambda}g^{\beta\rho}\,\delta g_{\alpha\beta}\,\Gamma_{\rho\mu\nu} + \frac{1}{2}g^{\lambda\rho}(\partial_\mu\,\delta g_{\nu\rho} + \partial_\nu\,\delta g_{\mu\rho} - \partial_\rho\,\delta g_{\mu\nu}) \tag{4.9}$$

$$= -g^{\lambda\rho}\Gamma^{\beta}_{\mu\nu}\,\delta g_{\beta\rho} + \frac{1}{2}g^{\lambda\rho}(\partial_\mu\,\delta g_{\nu\rho} + \partial_\nu\,\delta g_{\mu\rho} - \partial_\rho\,\delta g_{\mu\nu}) \tag{4.10}$$

$$= \frac{1}{2}g^{\lambda\rho}(\partial_\mu\,\delta g_{\nu\rho} + \partial_\nu\,\delta g_{\mu\rho} - \partial_\rho\,\delta g_{\mu\nu} - 2\Gamma^{\beta}_{\mu\nu}\,\delta g_{\beta\rho}). \tag{4.11}$$

Next we introduce the terms $\pm\Gamma^{\beta}_{\mu\rho}\,\delta g_{\beta\nu}$ and $\pm\Gamma^{\beta}_{\nu\rho}\,\delta g_{\beta\mu}$ to the top to get

$$\begin{aligned}\delta\Gamma^{\lambda}_{\mu\nu} &= \frac{1}{2}g^{\lambda\rho}(\partial_\mu\,\delta g_{\nu\rho} - \Gamma^{\beta}_{\mu\nu}\,\delta g_{\beta\rho} - \Gamma^{\beta}_{\mu\rho}\,\delta g_{\beta\nu} + \partial_\nu\,\delta g_{\mu\rho} - \Gamma^{\beta}_{\nu\mu}\,\delta g_{\beta\rho} - \Gamma^{\beta}_{\nu\rho}\,\delta g_{\beta\mu} \\ &- \partial_\rho\,\delta g_{\mu\nu} + \Gamma^{\beta}_{\rho\mu}\,\delta g_{\beta\nu} + \Gamma^{\beta}_{\rho\nu}\,\delta g_{\beta\mu}) \\ &= \frac{1}{2}g^{\lambda\rho}(\nabla_\mu\,\delta g_{\nu\rho} + \nabla_\nu\,\delta g_{\mu\rho} - \nabla_\rho\,\delta g_{\mu\nu}).\end{aligned} \tag{4.12}$$

This result implies we can write the variation in the Ricci tensor as

$$g^{\mu\nu}\,\delta R_{\mu\nu} = \frac{1}{2}(g^{\mu\nu}g^{\lambda\rho} - g^{\mu\lambda}g^{\nu\rho})\nabla_\lambda(\nabla_\mu\,\delta g_{\nu\rho} + \nabla_\nu\,\delta g_{\mu\rho} - \nabla_\rho\,\delta g_{\mu\nu}). \tag{4.13}$$

Plugging this into the action and integrating by parts gives us

$$\begin{aligned}\delta S = \frac{1}{2\kappa^2}\int \mathrm{d}^d x\,[\sqrt{-g}(f'(R)R_{\mu\nu}\,\delta g^{\mu\nu} - \frac{1}{2}(g^{\mu\nu}g^{\lambda\rho} - g^{\mu\lambda}g^{\nu\rho})[\nabla_\lambda f'(R)](\nabla_\mu\,\delta g_{\nu\rho} + \nabla_\nu\,\delta g_{\mu\rho} - \nabla_\rho\,\delta g_{\mu\nu})) \\ - \frac{1}{2}\sqrt{-g}f(R)g_{\mu\nu}\,\delta g^{\mu\nu}].\end{aligned} \tag{4.14}$$

We can integrate by parts again and manipulate the indices to get

$$\delta S = \frac{1}{2\kappa^2}\int \mathrm{d}^d x\,\sqrt{-g}\left[f'(R)R_{\mu\nu} - \nabla_\mu\nabla_\nu f'(R) + \Box f'(R)g_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu}\right]\delta g^{\mu\nu}\,. \tag{4.15}$$

Lastly, we can set the integral to zero and divide through by the variation and the volume element to get

$$\frac{1}{\sqrt{-g}}\frac{\delta S_{R^2}}{\delta g^{\mu\nu}} = f'(R)R_{\mu\nu} - (\nabla_\mu\nabla_\nu f'(R) - \Box f'(R)g_{\mu\nu}) - \frac{1}{2}f(R)g_{\mu\nu}. \tag{4.16}$$

And finally by taking $f(R) = R^2$, we get the desired equation

$$2RR_{\mu\nu} - 2(\nabla_\mu \nabla_\nu R - g_{\mu\nu}\Box R) - \frac{1}{2}R^2 g_{\mu\nu}. \tag{4.17}$$

5 $R^{\mu\nu}R_{\mu\nu}$-Gravity

Now we can work on the variation of $R_{\mu\nu}R^{\mu\nu}$. We start with the action

$$S = \int \mathrm{d}^d x \, \sqrt{-g} R^{\lambda\rho} R_{\lambda\rho}. \tag{5.1}$$

Now we take the variation

$$\delta S = \int \mathrm{d}^d x \, \sqrt{-g} \, \delta(R^{\lambda\rho} R_{\lambda\rho}) \, , \tag{5.2}$$

and notice that

$$\begin{aligned} \delta(R^{\lambda\rho}R_{\lambda\rho}) &= \delta g^{\alpha\lambda} g^{\beta\rho} R_{\alpha\beta} R_{\lambda\rho} = \delta g^{\alpha\lambda} \, g^{\beta\rho} R_{\alpha\beta} + g^{\alpha\lambda} \, \delta g^{\beta\rho} \, R_{\alpha\beta} R_{\lambda\rho} + g^{\alpha\lambda} g^{\beta\rho} (\delta R_{\lambda\rho} \, R_{\alpha\beta} + \delta R_{\alpha\beta} \, R_{\lambda\rho}) \\ &= \delta g^{\alpha\lambda} \, R^\rho_\alpha R_{\lambda\rho} + \delta g^{\beta\rho} \, R^\lambda_\beta R_{\lambda\rho} + 2R^{\lambda\rho} \, \delta R_{\lambda\rho} \\ &= 2R^\rho_\mu R_{\nu\rho} \, \delta g^{\mu\nu} + 2R^{\lambda\rho} \, \delta R_{\lambda\rho} \, . \end{aligned} \tag{5.3}$$

Now we need only to compute $R^{\lambda\rho}R_{\lambda\rho}$. Recall that the variation in the Ricci tensor is given by

$$\delta R_{\lambda\rho} = \nabla_\alpha \, \delta\Gamma^\lambda_{\lambda\rho} - \nabla_\rho \Gamma^\alpha_{\lambda\alpha} \tag{5.4}$$

$$= \nabla_\alpha \left[\frac{1}{2} g^{\alpha\sigma} (\nabla_\lambda \, \delta g_{\rho\sigma} + \nabla_\rho \, \delta g_{\lambda\sigma} - \nabla_\sigma \, \delta g_{\lambda\rho}) \right] - \nabla_\rho \left[\frac{1}{2} g^{\alpha\sigma} (\nabla_\lambda \, \delta g_{\alpha\sigma} + \nabla_\alpha \, \delta g_{\lambda\sigma} - \nabla_\sigma \, \delta g_{\lambda\alpha}) \right] \tag{5.5}$$

$$= \frac{1}{2} g^{\alpha\sigma} [\nabla_\alpha \nabla_\lambda \, \delta g_{\rho\sigma} + \nabla_\alpha \nabla_\rho \, \delta g_{\lambda\sigma} - \nabla_\alpha \nabla_\sigma \, \delta g_{\lambda\rho} - \nabla_\rho \nabla_\lambda \, \delta g_{\alpha\sigma} - \nabla_\rho \nabla_\alpha \, \delta g_{\lambda\sigma} + \nabla_\rho \nabla_\sigma \, \delta g_{\lambda\alpha} \, . \tag{5.6}$$

Now we can see that

$$R^{\lambda\rho}\,\delta R_{\lambda\rho} = \frac{1}{2}g^{\alpha\sigma}R^{\lambda\rho}[\nabla_\alpha\nabla_\lambda\,\delta g_{\rho\sigma} + \nabla_\alpha\nabla_\rho\,\delta g_{\lambda\sigma} - \nabla_\alpha\nabla_\sigma\,\delta g_{\lambda\rho} - \nabla_\rho\nabla_\lambda\,\delta g_{\alpha\sigma} - \nabla_\rho\nabla_\alpha\,\delta g_{\lambda\sigma} + \nabla_\rho\nabla_\sigma\,\delta g_{\lambda\alpha}] \tag{5.7}$$

$$= \frac{1}{2}g^{\alpha\sigma}[\delta g_{\rho\sigma}\,\nabla_\lambda\nabla_\alpha R^{\lambda\rho} + \delta g_{\lambda\sigma}\,\nabla_\rho\nabla_\alpha R^{\lambda\rho} - \delta g_{\lambda\rho}\,\nabla_\sigma\nabla_\alpha R^{\lambda\rho} \tag{5.8}$$

$$- \delta g_{\alpha\sigma}\,\nabla_\lambda\nabla_\rho R^{\lambda\rho} - \delta g_{\lambda\sigma}\,\nabla_\alpha\nabla_\rho R^{\lambda\rho} + \delta g_{\lambda\alpha}\,\nabla_\sigma\nabla_\rho R^{\lambda\rho}]$$

$$= \frac{1}{2}g^{\alpha\sigma}[\delta g_{\rho\sigma}\,\nabla_\lambda\nabla_\alpha R^{\lambda\rho} + \delta g_{\lambda\sigma}\,[\nabla_\rho,\nabla_\alpha]R^{\lambda\rho} - \delta g_{\lambda\rho}\,\nabla_\sigma\nabla_\alpha R^{\lambda\rho} - \delta g_{\alpha\sigma}\,\nabla_\lambda\nabla_\rho R^{\lambda\rho} + \delta g_{\lambda\alpha}\,\nabla_\sigma\nabla_\rho R^{\lambda\rho}], \tag{5.9}$$

where we integrated by parts and dropped the boundary terms. Next we choose to vary with respect to the inverse metric so we write

$$\begin{gathered}\delta g_{\rho\sigma} = -g_{\rho\mu}g_{\sigma\nu}\,\delta g^{\mu\nu}\,,\quad \delta g_{\lambda\sigma} = -g_{\lambda\mu}g_{\sigma\nu}\,\delta g^{\mu\nu}\,,\quad \delta g_{\lambda\rho} = -g_{\lambda\mu}g_{\rho\nu}\,\delta g^{\mu\nu}\,,\\ \delta g_{\alpha\sigma} = -g_{\alpha\mu}g_{\sigma\nu}\,\delta g^{\mu\nu}\,,\quad \delta g_{\lambda\alpha} = -g_{\lambda\mu}g_{\alpha\nu}\,\delta g^{\mu\nu}\,,\end{gathered} \tag{5.10}$$

which gives us the following

$$R^{\lambda\rho}\,\delta R_{\lambda\rho} = \frac{1}{2}\,\delta g^{\mu\nu}\,[\Box R_{\mu\nu} - \nabla_\lambda\nabla_\nu R^\lambda_\mu - [\nabla_\rho,\nabla_\nu]R^\rho_\mu - \nabla_\mu\nabla_\rho R^\rho_\nu + g_{\mu\nu}\nabla_\lambda\nabla_\rho R^{\lambda\rho}]. \tag{5.11}$$

Now using

$$[\nabla_\mu,\nabla_\nu]T^{\alpha\beta} = R^\alpha{}_{\lambda\mu\nu}T^{\lambda\beta} + R^\beta{}_{\lambda\mu\nu}T^{\lambda\alpha}, \tag{5.12}$$

we have

$$[\nabla_\alpha,\nabla_\nu]R^\rho_\mu = R^\rho{}_{\beta\alpha\nu}R^\beta_\mu - R^\beta{}_{\mu\alpha\nu}R^\rho_\beta, \tag{5.13}$$

which leads to

$$[\nabla_\rho,\nabla_\nu]R^\rho_\mu = R^\rho{}_{\beta\rho\nu}R^\beta_\mu - R^\beta{}_{\mu\rho\nu}R^\rho_\beta = R_{\beta\nu}R^\beta_\mu - R^\beta{}_{\mu\rho\nu}R^\rho_\beta. \tag{5.14}$$

From the fact that the Einstein tensor is divergence-free by construction, we have

$$\nabla_\lambda G^{\lambda\rho} = \nabla_\lambda\left(R^{\lambda\rho} - \frac{1}{2}Rg^{\lambda\rho}\right) = 0 \Rightarrow \nabla_\lambda R^{\lambda\rho} = \frac{1}{2}\nabla^\rho R, \tag{5.15}$$

and thus

$$\nabla_\lambda \nabla_\rho R^{\lambda\rho} = \frac{1}{2}\Box R. \tag{5.16}$$

This brings the contraction of the variation of the Ricci tensor with itself to be

$$R^{\lambda\rho}\,\delta R_{\lambda\rho} = \frac{1}{2}\,\delta g^{\mu\nu}\left[\Box R_{\mu\nu} + R_{\lambda\mu\rho\nu}R^{\lambda\rho} - R_{\mu\lambda}R^{\lambda}_{\nu} - \nabla_\lambda\nabla_\nu R^{\lambda}_{\mu} - \nabla_\mu\nabla_\rho R^{\rho}_{\nu} + \frac{1}{2}g_{\mu\nu}\Box R\right]. \tag{5.17}$$

Now we use the commutation relation of the covariant derivatives

$$\nabla_\lambda\nabla_\nu R^{\lambda}_{\mu} = \nabla_\nu\nabla_\lambda R^{\lambda}_{\mu} + R_{\lambda\nu}R^{\lambda}_{\mu} - R_{\rho\mu\lambda\nu}R^{\lambda\rho} = \frac{1}{2}\nabla_\nu\nabla_\mu R + R_{\lambda\nu}R^{\lambda}_{\mu} - R_{\rho\mu\lambda\nu}R^{\lambda\rho}, \tag{5.18}$$

which brings us to

$$R^{\lambda\rho}\,\delta R_{\lambda\rho} = \frac{1}{2}\,\delta g^{\mu\nu}\left[\Box R_{\mu\nu} + 2R_{\lambda\mu\rho\nu}R^{\lambda\rho} - \nabla_\mu\nabla_\nu R - 2R_{\lambda\nu}R^{\lambda}_{\mu} + \frac{1}{2}g_{\mu\nu}\Box R\right]. \tag{5.19}$$

Thus, we're left with

$$\delta S = \int \mathrm{d}^d x\left[\sqrt{-g}\big(2R^{\rho}_{\mu}R_{\nu\rho}\,\delta g^{\mu\nu} + 2R^{\lambda\rho}\,\delta R_{\lambda\rho}\big) - \frac{1}{2}\sqrt{-g}R^{\lambda\rho}R_{\lambda\rho}g_{\mu\nu}\,\delta g^{\mu\nu}\right] \tag{5.20}$$

$$= \int \mathrm{d}^d x\,\sqrt{-g}\left[2R^{\rho}_{\mu}R_{\nu\rho} - \frac{1}{2}R^{\lambda\rho}R_{\lambda\rho}g_{\mu\nu} + \left(\Box R_{\mu\nu} + 2R_{\lambda\mu\rho\nu}R^{\lambda\rho} - \nabla_\mu\nabla_\nu R - 2R_{\lambda\nu}R^{\lambda}_{\mu} + \frac{1}{2}g_{\mu\nu}\Box R\right)\right]\delta g^{\mu\nu} \tag{5.21}$$

$$= \int \mathrm{d}^d x\,\sqrt{-g}\left[\Box R_{\mu\nu} + 2R_{\lambda\mu\rho\nu}R^{\lambda\rho} - \nabla_\mu\nabla_\nu R + \frac{1}{2}g_{\mu\nu}\Box R - \frac{1}{2}R^{\lambda\rho}R_{\lambda\rho}g_{\mu\nu}\right]\delta g^{\mu\nu}\,. \tag{5.22}$$

Thus, the equations of motion become

$$\frac{1}{\sqrt{-g}}\frac{\delta S_{R^{\mu\nu}R_{\mu\nu}}}{\delta g^{\mu\nu}} = \Box R_{\mu\nu} + 2R_{\lambda\mu\rho\nu}R^{\lambda\rho} - \nabla_\mu\nabla_\nu R + \frac{1}{2}g_{\mu\nu}\Box R - \frac{1}{2}R^{\lambda\rho}R_{\lambda\rho}g_{\mu\nu}. \tag{5.23}$$

6 $R^{\lambda\rho\mu\nu}R_{\lambda\rho\mu\nu}$-Gravity

Now we can finally focus on the action for Riemann-squared gravity

$$S = \int \mathrm{d}^d x\,\sqrt{-g}R^{\alpha\beta\lambda\rho}R_{\alpha\beta\lambda\rho}. \tag{6.1}$$

First we can see that

$$\delta(R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}) = \delta(g^{\alpha\mu}g^{\beta\nu}g^{\lambda\gamma}g^{\delta\rho}R_{\mu\nu\lambda\rho}R_{\alpha\beta\gamma\delta}) \tag{6.2}$$

$$= \delta g^{\alpha\mu}\, R_{\mu\nu\lambda\rho}R_{\alpha}{}^{\nu\lambda\rho} + \delta g^{\beta\nu}\, R_{\mu\nu\lambda\rho}R^{\mu}{}_{\beta}{}^{\lambda\rho} + \delta g^{\gamma\lambda}\, R_{\mu\nu\lambda\rho}R^{\mu\nu}{}_{\gamma}{}^{\rho} \tag{6.3}$$

$$+ \delta g^{\delta\rho}\, R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda}{}_{\delta} + 2R^{\alpha\beta\gamma\delta}\,\delta R_{\alpha\beta\gamma\lambda}\,.$$

We can use the symmetry and antisymmetry of both the variation in the metric and the Riemann tensor to get a slightly simpler form

$$\delta(R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}) = 4R_{\mu\nu\lambda\rho}R_{\alpha}{}^{\nu\lambda\rho}\,\delta g^{\alpha\mu} + 2R^{\alpha\beta\gamma\delta}\,\delta R_{\alpha\beta\gamma\delta}\,. \tag{6.4}$$

Now we can focus on the second term. First start with the variation in the Riemann tensor

$$\delta R^{\alpha}{}_{\beta\lambda\rho} = \nabla_\lambda\,\delta\Gamma^{\alpha}_{\rho\beta} - \nabla_\rho\,\delta\Gamma^{\alpha}_{\lambda\beta} \tag{6.5}$$

$$= \frac{1}{2}g^{\alpha\sigma}[\nabla_\lambda\nabla_\rho\,\delta g_{\beta\sigma} + \nabla_\lambda\nabla_\beta\,\delta g_{\rho\sigma} - \nabla_\lambda\nabla_\sigma\,\delta g_{\rho\beta} - \nabla_\rho\nabla_\lambda\,\delta g_{\rho\sigma} - \nabla_\rho\nabla_\beta\,\delta g_{\lambda\sigma} + \nabla_\rho\nabla_\sigma\,\delta g_{\lambda\beta}]. \tag{6.6}$$

Now taking the product $R^{\alpha\beta\gamma\delta}\,\delta R_{\alpha\beta\gamma\delta}$ and then integrating by parts twice much in the same way we did when dealing with the variation in the Ricci tensor gives

$$\begin{aligned} R_{\alpha}{}^{\beta\lambda\rho}\,\delta R^{\alpha}{}_{\beta\lambda\rho} = \frac{1}{2}g^{\alpha\sigma}[&[\nabla_\rho,\nabla_\lambda]R_{\alpha}{}^{\beta\lambda\rho}\,\delta g_{\beta\sigma} + \nabla_\beta\nabla_\lambda R_{\alpha}{}^{\beta\lambda\rho}\,\delta g_{\rho\sigma} \\ &+ \nabla_\sigma\nabla_\rho R_{\alpha}{}^{\beta\lambda\rho}\,\delta g_{\lambda\beta} - \nabla_\sigma\nabla_\lambda R_{\alpha}{}^{\beta\lambda\rho}\,\delta g_{\rho\beta} - \nabla_\beta\nabla_\rho R_{\alpha}{}^{\beta\lambda\rho}\,\delta g_{\lambda\sigma}], \end{aligned} \tag{6.7}$$

and reusing the formulas that relate the variation with the metric and the inverse metric yields

$$R_{\alpha}{}^{\beta\lambda\rho}\,\delta R^{\alpha}{}_{\beta\lambda\rho} = \frac{1}{2}\,\delta g^{\mu\nu}\,([\nabla_\rho,\nabla_\lambda]R_{\mu\nu}{}^{\lambda\rho} - \nabla_\rho\nabla_\lambda R_{\nu}{}^{\rho\lambda}{}_{\mu} - \nabla_\lambda\nabla_\rho R^{\lambda}{}_{\nu\mu}{}^{\rho} + \nabla_\rho\nabla_\lambda R^{\rho}{}_{\nu}{}^{\lambda}{}_{\mu} + \nabla_\lambda\nabla_\rho R_{\nu}{}^{\lambda}{}_{\mu}{}^{\rho}). \tag{6.8}$$

Now we look at the action of the commutator between covariant derivatives on an arbitrary rank (0,4) or (2,2) tensor

$$[\nabla_\rho,\nabla_\lambda]T^{\alpha\beta}{}_{\gamma\delta} = R^{\alpha}{}_{\sigma\rho\lambda}T^{\sigma\beta}{}_{\gamma\delta} + R^{\beta}{}_{\sigma\rho\lambda}T^{\alpha\sigma}{}_{\gamma\delta} - R^{\sigma}{}_{\gamma\rho\lambda}T^{\alpha\beta}{}_{\sigma\delta} - R^{\sigma}{}_{\delta\rho\lambda}T^{\alpha\beta}{}_{\gamma\sigma}\,, \tag{6.9}$$

which upon setting $\alpha = \rho$ and $\beta = \lambda$ implies

$$
\begin{aligned}
[\nabla_\rho, \nabla_\lambda] T^{\lambda\rho}{}_{\gamma\delta} &= R^{\rho}{}_{\sigma\rho\lambda} T^{\sigma\lambda}{}_{\gamma\delta} + R^{\lambda}{}_{\sigma\rho\lambda} T^{\rho\sigma}{}_{\gamma\delta} - R^{\sigma}{}_{\gamma\rho\lambda} T^{\rho\lambda}{}_{\sigma\delta} - R^{\sigma}{}_{\delta\rho\lambda} T^{\rho\lambda}{}_{\gamma\sigma} \\
&= R_{\lambda\rho} T^{\lambda\rho}{}_{\gamma\delta} - R_{\lambda\rho} T^{\rho\lambda}{}_{\gamma\delta} - R^{\sigma}{}_{\gamma\rho\lambda} T^{\rho\lambda}{}_{\sigma\delta} - R^{\sigma}{}_{\delta\rho\lambda} T^{\rho\lambda}{}_{\gamma\sigma} \\
&= -R^{\sigma}{}_{\gamma\rho\lambda} T^{\rho\lambda}{}_{\sigma\delta} - R^{\sigma}{}_{\delta\rho\lambda} T^{\rho\lambda}{}_{\gamma\sigma} \,.
\end{aligned} \tag{6.10}
$$

Plugging in the Riemann tensor into the above relations gives

$$
[\nabla_\rho, \nabla_\lambda] R_{\mu\nu}{}^{\lambda\rho} = R^{\alpha}{}_{\mu\rho\lambda} R^{\rho\lambda}{}_{\nu\alpha} - R^{\alpha}{}_{\nu\rho\lambda} R^{\lambda\rho}{}_{\mu\alpha}, \tag{6.11}
$$

and therefore leaves us with

$$
\begin{aligned}
R_{\alpha}{}^{\beta\lambda\rho}\, \delta R^{\alpha}{}_{\beta\lambda\rho} &= \frac{1}{2}\, \delta g^{\mu\nu} \left[R^{\alpha}{}_{\mu\rho\lambda} R^{\rho\lambda}{}_{\nu\alpha} - R^{\alpha}{}_{\nu\rho\lambda} R^{\lambda\rho}{}_{\mu\alpha} + 2\nabla_\rho \nabla_\lambda R^{\rho}{}_{\nu}{}^{\lambda}{}_{\mu} - \nabla_\lambda \nabla_\rho R^{\lambda}{}_{\nu\mu}{}^{\rho} - \nabla_\rho \nabla_\lambda R_{\nu}{}^{\rho\lambda}{}_{\mu} \right] \\
&= \frac{1}{2}\, \delta g^{\mu\nu} \left(-2 R_{\mu\alpha\rho\lambda} R_{\nu}{}^{\alpha\rho\lambda} + 4 \nabla_\lambda \nabla_\rho R_{\mu}{}^{\lambda}{}_{\nu}{}^{\rho} \right),
\end{aligned} \tag{6.12}
$$

where we used the symmetric properties of the variation in the inverse metric and the antisymmetric properties of the Riemann tensor. This implies that the variation in the Riemann tensor squared is

$$
\delta(R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta}) = \delta g^{\mu\nu} \left(2 R_{\mu\alpha\lambda\rho} R_{\nu}{}^{\alpha\lambda\rho} + 4 \nabla^\lambda \nabla^\rho R_{\mu\lambda\nu\rho} \right). \tag{6.13}
$$

We're almost done! Now we need only take the double divergence of the Riemann tensor. To figure out what this quantity should be, we first look at the second Bianchi identity

$$
\nabla_\alpha R_{\mu\lambda\nu\rho} + \nabla_\mu R_{\lambda\alpha\nu\rho} + \nabla_\lambda R_{\alpha\mu\nu\rho} = 0. \tag{6.14}
$$

Now we take the trace by applying the above equation with $g^{\alpha\rho}$

$$
\nabla^\rho R_{\mu\lambda\nu\rho} + \nabla_\mu R_{\lambda}{}^{\rho}{}_{\nu\rho} + \nabla_\lambda R^{\rho}{}_{\mu\nu\rho} = 0. \tag{6.15}
$$

Rearranging the indices by using the antisymmetric properties of the Riemann tensor yields

$$
\nabla^\rho R_{\mu\lambda\nu\rho} + \nabla_\mu R^{\rho}{}_{\lambda\rho\nu} - \nabla_\lambda R^{\rho}{}_{\mu\rho\nu} = \nabla^\rho R_{\mu\lambda\nu\rho} + \nabla_\mu R_{\lambda\nu} - \nabla_\lambda R_{\mu\nu}, \tag{6.16}
$$

leaving us with

$$\nabla^\rho R_{\mu\lambda\nu\rho} = \nabla_\lambda R_{\mu\nu} - \nabla_\mu R_{\lambda\nu}. \tag{6.17}$$

Thus, taking the double divergence gives

$$\nabla^\lambda \nabla^\rho R_{\mu\lambda\nu\rho} = \Box R_{\mu\nu} - \nabla_\lambda \nabla_\mu R^\lambda_\nu. \tag{6.18}$$

We are again in a position to utilize the commutator identity for the Ricci tensor

$$\begin{aligned}\nabla_\lambda \nabla_\mu R^\lambda_\nu &= \nabla_\mu \nabla_\lambda R^\lambda_\nu - R_{\mu\lambda} R^\lambda_\nu - R_{\rho\mu\lambda\nu} R^{\lambda\rho} &(6.19)\\ &= \frac{1}{2}\nabla_\mu \nabla_\nu R + R_{\mu\lambda} R^\lambda_\nu - R_{\mu\lambda\nu\rho} R^{\lambda\rho}, &(6.20)\end{aligned}$$

and therefore the double divergence is nothing but

$$\nabla^\lambda \nabla^\rho R_{\mu\lambda\nu\rho} = \Box R_{\mu\nu} - \frac{1}{2}\nabla_\mu \nabla_\nu R - R_{\mu\lambda} R^\lambda_\nu + R_{\mu\lambda\nu\rho} R^{\lambda\rho}, \tag{6.21}$$

and the variation in the Riemann product is

$$\delta(R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta}) = \delta g^{\mu\nu}\, [2R_{\mu\alpha\lambda\rho} R_\nu{}^{\alpha\lambda\rho} + 4\Box R_{\mu\nu} - 2\nabla_\mu \nabla_\nu R - 4R_{\mu\lambda} R^\lambda_\nu + 4R_{\mu\lambda\nu\rho} R^{\lambda\rho}]. \tag{6.22}$$

Finally, we have

$$\begin{aligned}\delta S &= \int \mathrm{d}^d x\, [\sqrt{-g}\, \delta(R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta}) + R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta}\, \delta\sqrt{-g}] &(6.23)\\ &= \int \mathrm{d}^d x\, \sqrt{-g}\, \delta g^{\mu\nu} \Big[-\frac{g_{\mu\nu}}{2} R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} + 2R_{\mu\alpha\lambda\rho} R_\nu{}^{\alpha\lambda\rho} + 4\Box R_{\mu\nu} - 2\nabla_\mu \nabla_\nu R - 4R_{\mu\lambda} R^\lambda_\nu + 4R_{\mu\lambda\nu\rho} R^{\lambda\rho}\Big]. &(6.24)\end{aligned}$$

Thus, the equations of motion for Riemann-squared gravity are simply

$$\frac{1}{\sqrt{-g}} \frac{\delta S_{R^{\lambda\rho\mu\nu} R_{\lambda\rho\mu\nu}}}{\delta g^{\mu\nu}} = -\frac{g_{\mu\nu}}{2} R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} + 2R_{\mu\alpha\lambda\rho} R_\nu{}^{\alpha\lambda\rho} + 4\Box R_{\mu\nu} - 2\nabla_\mu \nabla_\nu R - 4R_{\mu\lambda} R^\lambda_\nu + 4R_{\mu\lambda\nu\rho} R^{\lambda\rho}. \tag{6.25}$$

We can give these geometric tensors names

$$\begin{aligned}H_{\mu\nu} &\equiv \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} \int \mathrm{d}^d x\, \sqrt{-g} R^{\alpha\beta\lambda\rho} R_{\alpha\beta\lambda\rho}\\ &= -\frac{g_{\mu\nu}}{2} R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} + 2R_{\mu\alpha\lambda\rho} R_\nu{}^{\alpha\lambda\rho} + 4\Box R_{\mu\nu} - 2\nabla_\mu \nabla_\nu R - 4R_{\mu\lambda} R^\lambda_\nu + 4R_{\mu\lambda\nu\rho} R^{\lambda\rho},\end{aligned} \tag{6.26}$$

$$
\begin{aligned}
I_{\mu\nu} &\equiv \frac{1}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\int \mathrm{d}^d x\,\sqrt{-g}R^{\lambda\rho}R_{\lambda\rho} \\
&= \Box R_{\mu\nu} + 2R_{\lambda\mu\rho\nu}R^{\lambda\rho} - \nabla_\mu\nabla_\nu R + \frac{1}{2}g_{\mu\nu}\Box R - \frac{1}{2}R^{\lambda\rho}R_{\lambda\rho}g_{\mu\nu},
\end{aligned}
\tag{6.27}
$$

$$
\begin{aligned}
J_{\mu\nu} &\equiv \frac{1}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\int \mathrm{d}^d x\,\sqrt{-g}R^2 \\
&= 2RR_{\mu\nu} - 2(\nabla_\mu\nabla_\nu R - g_{\mu\nu}\Box R) - \frac{1}{2}R^2 g_{\mu\nu}.
\end{aligned}
\tag{6.28}
$$

Note: the literature names these geometric tensors to be slightly different

$$
{}^{(1)}H_{\mu\nu} \equiv J_{\mu\nu}, \quad {}^{(2)}H_{\mu\nu} \equiv I_{\mu\nu}. \tag{6.29}
$$

Another thing to note is the following quantity in $d = 4$ dimensions is a topological invariant

$$
\int \mathrm{d}^4 x\,\sqrt{-g}[R_{\lambda\rho\mu\nu}R^{\lambda\rho\mu\nu} + R^2 - 4R_{\mu\nu}R^{\mu\nu}]. \tag{6.30}
$$

Thus in 4-dimensions, the geometric tensors are not independent of one another

$$
H_{\mu\nu} = -J_{\mu\nu} + 4I_{\mu\nu}, \tag{6.31}
$$

which can be found by just varying the topological invariant. This term is referred to the Gauss-Bonnet term or Gauss-Bonnet gravity.

7 The FRW Metric

Since we're primarily interested in these geometric tensors within the context of cosmology (and possibly black holes in the future), let's write down the explicit form that these tensors will take on in d-dimensional FRW coordinates. First we write down the Christoffel symbols, Riemann and Ricci tensors and the Ricci scalar

$$
\Gamma^t_{ij} = a\dot{a}\delta_{ij}, \quad \Gamma^i_{tj} = H\delta^i_j \tag{7.1}
$$

$$
R^i{}_{tjt} = -(\dot{H} + H^2)\delta^i_j, \quad R^t{}_{itj} = a\ddot{a}\delta_{ij}, \quad R^\ell{}_{ikj} = \dot{a}^2(\delta_{ij}\delta^\ell_k - \delta_{ik}\delta^\ell_j), \tag{7.2}
$$

$$R_{tt} = -(d-1)(\dot{H} + H^2), \qquad R_{ij} = \left[a\ddot{a} + (d-2)\dot{a}^2\right]\delta_{ij}, \tag{7.3}$$

$$R = (d-1)\left[2\frac{\ddot{a}}{a} + (d-2)\frac{\dot{a}^2}{a^2}\right], \tag{7.4}$$

where we used the fact that taking the spatial trace is

$$\delta^{ij}\delta_{ij} = d-1, \tag{7.5}$$

and all unlisted components can be assumed to be zero[1]. To do so, it'll be simpler to be able to reference some of the necessary formulas. First we need the time derivatives of the Ricci scalar

$$\dot{R} = 2(d-1)\left[\frac{a^{(3)}}{a} - (d-2)\frac{\dot{a}^3}{a^3} + (d-3)\frac{\dot{a}\ddot{a}}{a^2}\right], \tag{7.6}$$

$$\ddot{R} = 2(d-1)\left[3(d-2)\frac{\dot{a}^4}{a^4} + (12-5d)\frac{\dot{a}^2\ddot{a}}{a^3} + (d-3)\frac{\ddot{a}^2}{a^2} + (d-4)\frac{\dot{a}a^{(3)}}{a^2} + \frac{a^{(4)}}{a}\right], \tag{7.7}$$

and the box operator on the curvature scalar is then

$$\Box R = -2(d-1)\left[\frac{a^{(4)}}{a} + ((d-1)(d-8)+7)\frac{\dot{a}^2\ddot{a}}{a^3} + (2d-5)\frac{\dot{a}a^{(3)}}{a^2} + (d-3)\frac{\ddot{a}^2}{a^2} - (d-2)(d-4)\frac{\dot{a}^4}{a^4}\right]. \tag{7.8}$$

There's also the covariant derivatives which act on the Ricci scalar

$$\nabla_0\nabla_0 R = \ddot{R}, \qquad \nabla_i\nabla_j R = -\dot{R}\Gamma^0_{ij} = -a\dot{a}\dot{R}\delta_{ij}. \tag{7.9}$$

Now we can write down the different components of the geometric tensors

$$J_{tt} = \left[\frac{(d-1)^2(d-2)^2}{2} - 4(d-1)^2(d-2)\right]\frac{\dot{a}^4}{a^4} - 2(d-1)^2\left(\frac{\ddot{a}}{a}\right)^2 + 4(d-1)^2\frac{\dot{a}a^{(3)}}{a^2} + 4(d-1)^2(d-3)\frac{\dot{a}^2\ddot{a}}{a^3}, \tag{7.10}$$

where we employ the notation

$$a^{(n)} \equiv \frac{\mathrm{d}^n a}{\mathrm{d}t^n}. \tag{7.11}$$

Next we can do the same for the spatial components

[1]We checked!

$$J_{ij} = -a^2 \left[6(d-1)(d-3)\frac{\ddot{a}^2}{a^2} + 2(d-1)(3d^2-26d+44)\frac{\dot{a}^2\ddot{a}}{a^3} + \frac{1}{2}(d-1)(d-2)(d-5)(d-10)\frac{\dot{a}^4}{a^4} + 8(d-1)(d-3)\frac{\dot{a}a^{(3)}}{a^3} + 4(d-1)\frac{a^{(4)}}{a} \right] \delta_{ij}. \tag{7.12}$$

Now we want to compute I_{00} and I_{ij}. To that end, it'll be convenient to have some of the necessary quantities written out

$$R^{\lambda\rho}R_{\lambda\rho} = R^{00}R_{00} + R^{ij}R_{ij}, \tag{7.13}$$

and we note that

$$R^{\mu\nu} = g^{\mu\alpha}g^{\nu\beta}R_{\alpha\beta} \Rightarrow R^{00} = g^{0\alpha}g^{0\beta}R_{\alpha\beta} = R_{00}, \quad R^{ij} = g^{i\alpha}g^{j\beta}R_{\alpha\beta} = \frac{\delta^{ik}\delta^{j\ell}}{a^4}R_{k\ell}. \tag{7.14}$$

Thus we have

$$R^{00} = -(d-1)\frac{\ddot{a}}{a}, \quad R^{ij} = \frac{a\ddot{a} + (d-2)\dot{a}^2}{a^4}\delta^{ij}, \tag{7.15}$$

which yields

$$R^{\lambda\rho}R_{\lambda\rho} = (R_{00})^2 + R^{ij}R_{ij} = d(d-1)\frac{\ddot{a}^2}{a^2} + 2(d-1)(d-2)\frac{\dot{a}^2\ddot{a}}{a^3} + (d-1)(d-2)^2\frac{\dot{a}^4}{a^4}. \tag{7.16}$$

We also have the term $R_{\lambda\mu\rho\nu}R^{\lambda\rho} = R_{0\mu0\nu}R^{00} + R_{k\mu\ell\nu}R^{k\ell}$ leaving

$$R_{\lambda0\rho0}R^{\lambda\rho} = R_{k0\ell0}R^{k\ell} = -\frac{a\ddot{a} + (d-2)\dot{a}^2}{a^4}\delta^{k\ell}(a\ddot{a}\delta_{k\ell}) = -(d-1)\frac{\ddot{a}^2}{a^2} - (d-1)(d-2)\frac{\dot{a}^2\ddot{a}}{a^3}, \tag{7.17}$$

$$R_{\lambda i\rho j}R^{\lambda\rho} = R_{0i0j}R^{00} + R_{ki\ell j}R^{k\ell} = R_{0i0j}R_{00} + a^2\delta_{km}R^{m}{}_{i\ell j}R^{k\ell} \tag{7.18}$$

$$= (d-1)\ddot{a}^2\delta_{ij} + \frac{\dot{a}^2(a\ddot{a} + (d-2)\dot{a}^2)}{a^2}\delta^{\ell}_{m}(\delta_{ij}\delta^{m}_{\ell} - \delta_{i\ell}\delta^{m}_{j}) \tag{7.19}$$

$$= a^2\left[(d-1)\frac{\ddot{a}^2}{a^2} + (d-2)\frac{\dot{a}^2\ddot{a}}{a^3} + (d-2)^2\frac{\dot{a}^4}{a^4}\right]\delta_{ij}. \tag{7.20}$$

We also need to compute the d'Alembertian of the Ricci tensor

$$\Box R_{00} = -\ddot{R}_{00} - (d-1)\frac{\dot{a}}{a}\dot{R}_{00}, \tag{7.21}$$

giving

$$\dot{R}_{00} = (d-1)\left[\frac{\dot{a}\ddot{a}}{a^2} - \frac{a^{(3)}}{a}\right], \qquad \ddot{R}_{00} = (d-1)\left[\frac{\ddot{a}^2}{a^2} + 2\frac{\dot{a}a^{(3)}}{a^2} - 2\frac{\dot{a}^2\ddot{a}}{a^3} - \frac{a^{(4)}}{a}\right]. \tag{7.22}$$

Putting them together gives

$$\Box R_{00} = (d-1)\left[\frac{a^{(4)}}{a} - \frac{\ddot{a}^2}{a^2} - (d-3)\frac{\dot{a}^2\ddot{a}}{a^3} + (d-3)\frac{\dot{a}a^{(3)}}{a^2}\right]. \tag{7.23}$$

Now we can put all these pieces together to calculate I_{00}