Derivation of the One-Loop Effective Action

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1 Introduction

We are interested in deriving the general formalism of the one-loop effective action. Here we follow the derivations as laid out by Leonard Parker while filling in some of the gaps between the differing steps. All of this is in the service of eventually calculating the contributions that a self-interacting scalar has on the amount of gravitational particle production in the early universe. In the beginning we will deal with the most general case of N scalar fields and then we will specialize.

Conventions We use the mostly plus metric signature, i.e. $\eta_{\mu\nu} = (-,+,+,+)$ and units where $c = \hbar = 1$. The reduced four dimensional Planck mass is $M_{\rm Pl} = (8\pi G)^{-1/2} \approx 2.43 \times 10^{18}$ GeV. We use boldface letters $\mathbf{r}$ to indicate 3-vectors and x and p to denote 4-vectors. Conventions for the curvature tensors, covariant and Lie derivatives are all taken from Carroll.

2 Derivation

We start from the Schwinger action principle

$$\delta \langle \text{out}|\text{in}\rangle = \frac{i}{\hbar} \langle \text{out}|\delta S|\text{in}\rangle \,, \tag{2.1}$$

where $|\text{out}\rangle \,, |\text{in}\rangle$ can be any states and we have decided to keep powers of $\hbar$. We define the inner product between these two states as

$$\langle \text{out}|\text{in}\rangle \equiv \exp\left(\frac{iW}{\hbar}\right). \tag{2.2}$$

This leads to the variation of the inner product being

$$\delta \langle \text{out}|\text{in}\rangle = \delta \exp\left(\frac{iW}{\hbar}\right) = \exp\left(\frac{iW}{\hbar}\right)\frac{i}{\hbar}\,\delta W = \frac{i}{\hbar}\langle \text{out}|\delta S|\text{in}\rangle\,. \tag{2.3}$$

And thus, the variation in W becomes

$$\delta W = \frac{\langle \text{out}|\delta S|\text{in}\rangle}{\exp\left(\frac{iW}{\hbar}\right)} = \frac{\langle \text{out}|\delta S|\text{in}\rangle}{\langle \text{out}|\text{in}\rangle}. \tag{2.4}$$

Next we write the action for N scalar fields in d-dimensional curved spacetime

$$S = -\frac{1}{2}\int \sqrt{-g}\,\mathrm{d}^d x\,\left[\delta_{ij}\partial^\mu\varphi^i\partial_\mu\varphi^j + M^2(x)\varphi^i\varphi^j\right], \tag{2.5}$$

where $i \in [1, N]$ and $M^2_{ij}(x) = (m^2 + R(x))\delta_{ij}$. Now let's vary with respect to M^2_{ij}

$$\delta S = -\frac{1}{2}\int \sqrt{-g}\,\mathrm{d}^d x\,(\delta M^2_{ij}(x))\varphi^i(x)\varphi^j(x), \tag{2.6}$$

which when we plug it in for δW becomes

$$\delta W = -\frac{1}{2}\int \sqrt{-g}\,\mathrm{d}^d x\,(\delta M^2_{ij}(x))\frac{\langle \text{out}|\varphi^i(x)\varphi^j(x)|\text{in}\rangle}{\langle \text{out}|\text{in}\rangle}. \tag{2.7}$$

The correlation function in the above expression can be seen as the limit of

$$\langle \text{out}|\varphi^i(x)\varphi^j(x)|\text{in}\rangle \equiv \lim_{x'\to x}\,\langle \text{out}|\mathcal{T}(\varphi^i(x)\varphi^j(x'))|\text{in}\rangle\,, \tag{2.8}$$

where $\mathcal{T}(\varphi^i(x)\varphi^j(y))$ is the time-ordered product. Recall the Green's function for a set of N scalar fields in curved spacetime is

$$(-\delta_{ij}\Box_x + M^2_{ij}(x))G^{jk}(x, x') = \delta(x, x')\delta^k_i, \tag{2.9}$$

which implies

$$\frac{\langle \text{out}|\varphi^i(x)\varphi^j(x)|\text{in}\rangle}{\langle \text{out}|\text{in}\rangle} = -i\hbar G^{ij}(x, x). \tag{2.10}$$

This turns the expression of δW into

$$\delta W = \frac{i\hbar}{2}\int \sqrt{-g}\, \mathrm{d}^d x\, (\delta M^2_{ij}(x)) G^{ij}(x,x). \tag{2.11}$$

We will periodically make use of the following notation

$$G^{ij}(x,x') \equiv G^{ij'}, \tag{2.12}$$

and we define the differential operator

$$D_{ij} = \big(-\delta_{ij}\Box_x + M^2_{ij}(x)\big)\delta(x,x'). \tag{2.13}$$

We can check this operator leads to the correct form of the Green's function

$$\int D_{ij}(x,x')G^{jk}(x',x'')\sqrt{-g}\,\mathrm{d}^d x' = \int \big(-\delta_{ij}\Box_x + M^2_{ij}(x)\big)\delta(x,x')G^{jk}(x',x'')\sqrt{-g}\,\mathrm{d}^d x' \tag{2.14}$$

$$= \big(-\delta_{ij}\Box_x + M^2_{ij}(x)\big)G^{jk}(x,x'') = \delta_i^k\delta(x,x''). \tag{2.15}$$

We will also make use of the shorthand

$$D_{ij}(x,x')G^{jk}(x',x'') \equiv D_{ij'}G^{j'k''} = \delta_i^{k''}, \quad \delta M^2_{ij'} = \delta M^2_{ij}(x,x') \equiv \delta M^2_{ij}(x)\,\delta(x,x'). \tag{2.16}$$

Plugging this in to δW yields

$$\delta W = \frac{i\hbar}{2}\int \sqrt{-g}\,\mathrm{d}^d x\, \delta M^2_{ij}(x)\, G^{ij}(x,x) = \frac{i\hbar}{2}\int \sqrt{-g(x)}\,\mathrm{d}^d x \int \sqrt{-g(x')}\,\mathrm{d}^d x'\, \delta M^2_{ij}(x)\,\delta(x,x')G^{ij}(x,x') \tag{2.17}$$

$$= \frac{i\hbar}{2}\int \sqrt{-g(x)}\,\mathrm{d}^d x \int \sqrt{-g(x')}\,\mathrm{d}^d x'\, \delta M^2_{ij}(x,x')\, G^{ij}(x,x') \tag{2.18}$$

$$= \frac{i\hbar}{2}\int \sqrt{-g(x)}\,\mathrm{d}^d x \int \sqrt{-g(x')}\,\mathrm{d}^d x'\, \delta M^2_{ij'}\, G^{j'i} \tag{2.19}$$

$$\equiv \frac{i\hbar}{2}\,\mathrm{Tr}\Big[\delta M^2_{ij'}\, G^{j'i}\Big]. \tag{2.20}$$

We can also see that

$$\delta D_{ij} = \delta(-\delta_{ij}\Box_x + M_{ij}^2(x))\,\delta(x,x') = \delta M_{ij}^2(x,x')\,. \tag{2.21}$$

Recall that Green's functions can be seen as inverse differential operators i.e.

$$D_{ij'}G^{j'k''} = \delta_i^{k''} \Rightarrow G^{j'k''} = (D^{-1})^{j'k''}. \tag{2.22}$$

Therefore, δW can be expressed in the very compact form of

$$\delta W = \frac{i\hbar}{2}\,\mathrm{Tr}\Big[\delta M_{ij'}^2\,G^{j'i}\Big] = \frac{i\hbar}{2}\,\mathrm{Tr}\Big[\delta D_{ij'}\,(D^{-1})^{j'i}\Big] = \frac{i\hbar}{2}\,\mathrm{Tr}\big[\delta\ln\ell^2 D_{ij'}\big], \tag{2.23}$$

where ℓ is an integration constant that we put in to have dimensions of length. We therefore have an expression for W,

$$\delta W = \frac{i\hbar}{2}\,\delta\,\mathrm{Tr}\big[\ln\big(\ell^2 D_{ij'}\big)\big] \Rightarrow W = \frac{i\hbar}{2}\,\mathrm{Tr}\big[\ln\big(\ell^2 D_{ij'}\big)\big]. \tag{2.24}$$

We call W the *one-loop effective action*. Thus, the inner product we defined in the first few steps of this calculation is

$$\langle \mathrm{out}|\mathrm{in}\rangle = \exp\left(\frac{i}{\hbar}\cdot\frac{i\hbar}{2}\,\mathrm{Tr}\big[\ln\big(\ell^2 D_{ij'}\big)\big]\right) = \big(\det\big(\ell^2 D_{ij'}\big)\big)^{-1/2}, \tag{2.25}$$

where we used the fact that

$$\mathrm{Tr}\ln = \ln\det, \tag{2.26}$$

which can be easily seen/proved by expressing an arbitrary square, invertible matrix in terms of its eigenvalues. Now an interesting observation is that the term $(\det(\ell^2 D_{ij'}))^{-1/2}$ could've been arrived that had we integrated a Gaussian. We can see this from the following calculation

$$I[A] = \int \mathrm{d}^d x\,\exp\left(\frac{i}{2}(x, Ax)\right), \tag{2.27}$$

where $(x, Ax) \equiv x^T A x = A_{ij}x^i x^j$ and A is a symmetric matrix. Recall from linear algebra that any symmetric matrix can be decomposed into

$$A = U^T \Lambda U, \tag{2.28}$$

where $\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_d)$ is the diagonal matrix of the eigenvalues of A and $|\det U| = 1$. Then

$$I[A] = \int \mathrm{d}^d x \exp\biggl(\frac{i}{2} x^T U^T \Lambda U x\biggr), \tag{2.29}$$

doing a coordinate transformation/u-substitution with $u = Ux \Rightarrow \mathrm{d}^d u = |\det(U)|\, \mathrm{d}^d x = \mathrm{d}x$ puts the above integral in a simpler form

$$I[A] = \int \mathrm{d}^d u \exp\biggl(\frac{i}{2} u^T \Lambda u\biggr) = \int \mathrm{d}^d u \exp\biggl(\frac{i}{2}(\lambda_1 u_1^2 + \lambda_2 u_2^2 + \ldots \lambda_d u_d^2)\biggr) \tag{2.30}$$

$$= \prod_{n=1}^{d} \int \mathrm{d}u_n \exp\biggl(\frac{i}{2}\lambda_n u_n^2\biggr) = \prod_{n=1}^{d} \sqrt{\frac{2\pi i}{\lambda_n}} \tag{2.31}$$

$$= \frac{(2\pi i)^{d/2}}{\sqrt{\lambda_1 \lambda_2 \cdots \lambda_d}} = \frac{(2\pi i)^{d/2}}{\sqrt{\det \Lambda}} = \frac{(2\pi i)^{d/2}}{\sqrt{\det A}}. \tag{2.32}$$

By defining

$$(\det A)^{-1/2} = \int \mathrm{d}\mu(x) \exp\biggl(\frac{i}{2}(x, Ax)\biggr), \qquad \mathrm{d}\mu(x) = \prod_{n=1}^{d} \frac{\mathrm{d}x_n}{\sqrt{2\pi i}}. \tag{2.33}$$

Thus, the generalization to fields is easy to see

$$(\det \ell^2 D_{ij'})^{-1/2} = \int \mathrm{d}\mu[\varphi] \exp\biggl(\frac{i}{2\hbar}\varphi^i D_{ij'} \varphi^{j'}\biggr), \qquad \mathrm{d}\mu[\varphi] \equiv \prod_{n=1}^{d} \frac{\mathrm{d}\varphi^i}{\ell\sqrt{-2\pi i \hbar}}, \tag{2.34}$$

where

$$\frac{1}{2}\varphi^i D_{ij'} \varphi^{j'} = \frac{1}{2}\int \sqrt{-g(x)}\, \mathrm{d}^d x \int \sqrt{-g(x')}\, \mathrm{d}^d x'\, \varphi^i(x) D_{ij}(x, x') \varphi^j(x') \tag{2.35}$$

$$= \frac{1}{2}\int \sqrt{-g(x)}\, \mathrm{d}^d x \int \sqrt{-g(x')}\, \mathrm{d}^d x'\, \varphi^i(x) \bigl[-\delta_{ij}\Box_x + M_{ij}^2(x)\bigr] \delta(x, x') \varphi^j(x') \tag{2.36}$$

$$= \frac{1}{2}\int \sqrt{-g}\, \mathrm{d}^d x\, \varphi^i(x) \bigl[-\delta_{ij}\Box_x + M_{ij}^2(x)\bigr] \varphi^j(x) = -S[\varphi], \tag{2.37}$$

and we are thus left to conclude that

$$\langle \mathrm{out} | \mathrm{in} \rangle = \int \mathcal{D}\varphi \exp\biggl(\frac{i}{\hbar} S\biggr). \tag{2.38}$$

3 Regularization

Now suppose $D_{ij'}$ satisfies

$$D_{ij'} {f_n}^{j'} = \lambda_n f_{ni}, \tag{3.1}$$

where the $\{f_n\}$ form a complete orthonormal basis. Specializing to Minkowski, the above equation reduces down to

$$(-\Box_x + m^2) f_n(x) = \lambda_n f_n(x). \tag{3.2}$$

The ansatz for this solution given the periodic boundary conditions are

$$f_{\vec{n}}(x) = \frac{1}{\sqrt{L_0 L_1 \cdots L_{d-1}}} \exp\left(2\pi i \sum_{\nu=0}^{d-1} \frac{n_\nu x^\nu}{L_\nu}\right), \tag{3.3}$$

with $\vec{n} = (n_0, n_1, \ldots n_{d-1})$ with $n_\nu \in \mathbb{Z}$. Plugging this into the eigenvalue equation yields

$$(\partial_t^2 - \nabla_{\mathbf{r}}^2 + m^2) f_{\vec{n}}(x) = \frac{1}{\sqrt{L_0 L_1 \cdots L_{d-1}}} \left[\left(\frac{2\pi i n_0}{L_0}\right)^2 - \sum_{j=1}^{d-1} \left(\frac{2\pi i n_j}{L_j}\right)^2 + m^2\right] \exp\left(2\pi i \sum_{\nu=0}^{d-1} \frac{n_\nu x^\nu}{L_\nu}\right), \tag{3.4}$$

which brings us with

$$\lambda_n = -\left(\frac{2\pi i n_0}{L_0}\right)^2 + \sum_{j=1}^{d-1} \left(\frac{2\pi i n_j}{L_j}\right)^2 + m^2. \tag{3.5}$$

We can define a new coordinate $p_\mu = \frac{2\pi}{L_\mu} n_\mu$ and take the continuum limit ($L_\mu \to \infty$) and the above sum becomes an integral. Thus we have

$$\det D_{ij'} = \prod_{n=1}^{d} \lambda_n \Rightarrow \ln \det\left(\ell^2 D_{ij'}\right) = \sum_{n=1}^{d} \ln\left(\ell^2 \lambda_n\right). \tag{3.6}$$

Taking the continuum limit of the eigenvalues yield

$$\lambda_n = -\left(\frac{2\pi i n_0}{L_0}\right)^2 + \sum_{j=1}^{d-1} \left(\frac{2\pi i n_j}{L_j}\right)^2 + m^2 \to -p_0^2 + p_1^2 + \ldots p_{d-1}^2 + m^2 \equiv p^2 + m^2. \tag{3.7}$$

This leaves the effective action to be

$$W = \frac{i\hbar}{2}\sum_n \ln\big(\ell^2\lambda_n\big) \xrightarrow[\text{limit}]{\text{continuum}} \frac{i\hbar}{2}(L_0 L_1 \cdots L_{d-1}) \int \frac{\mathrm{d}^d p}{(2\pi)^d} \ln\big(\ell^2[p^2+m^2]\big) \tag{3.8}$$

$$= \frac{i\hbar}{2}\int \mathrm{d}^d x \int \frac{\mathrm{d}^d p}{(2\pi)^d} \ln[\ell^2(p^2+m^2)]. \tag{3.9}$$

This integral very obviously and very badly diverges. We will need to regulate this somehow. First we define the effective potential density

$$V = -\frac{i\hbar}{2}\int \frac{\mathrm{d}^d p}{(2\pi)^d} \ln[\ell^2(p^2+m^2)], \tag{3.10}$$

so the effective potential is

$$W = -\int \mathrm{d}^d x\, V. \tag{3.11}$$

In order to regularize this quantity, we first let $D \to D + \delta D$:

$$W = \frac{i\hbar}{2}\,\mathrm{Tr}\big[\ln\big(\ell^2 D\big)\big] \Rightarrow \delta W = \frac{i\hbar}{2}\,\delta\,\mathrm{Tr}\ln\big(\ell^2 D\big) = \frac{i\hbar}{2}\,\mathrm{Tr}\big(D^{-1}\,\delta D\big). \tag{3.12}$$

We will now assume that $\mathrm{Im}\{D\} = \varepsilon \ll 1$. We also notice

$$D^{-1} = i\int_{\mathbb{R}_0^+} \exp(-iD\tau)\,\mathrm{d}\tau \Rightarrow -\frac{\delta D}{D^2} = i\int_0^\infty (-i\tau)\exp(-iD\tau)\,\mathrm{d}\tau \Leftrightarrow D^{-1}\,\delta D = -D\int_0^\infty \tau e^{-iD\tau}\,\mathrm{d}\tau \tag{3.13}$$

We can do integration by parts with $u = \tau$ and $\mathrm{d}v = e^{-iD\tau}\,\mathrm{d}\tau$ which gives us

$$D^{-1}\,\delta D = D\left[\frac{\tau e^{-iD\tau}}{-iD}\bigg|_0^\infty + \int_0^\infty \frac{e^{-iD\tau}}{-iD}\,\mathrm{d}\tau\right] = i\int_0^\infty e^{-iD\tau}\,\mathrm{d}\tau = D^{-1}, \tag{3.14}$$

where we used the fact that $\mathrm{Im}\{D\} = \varepsilon \neq 0$ which means $e^{-\tau\varepsilon} \to 0$. Next we have

$$\delta W = \frac{i\hbar}{2}\left(i\int_0^\infty \mathrm{d}\tau\, e^{-iD\tau}\right) = \frac{i\hbar}{2}\,\delta\left(i\int_0^\infty \mathrm{d}\tau\,\frac{e^{-iD\tau}}{-i\tau}\right) \Rightarrow W = \frac{-i\hbar}{2}\int_0^\infty \frac{\mathrm{d}\tau}{\tau}\exp(-iD\tau). \tag{3.15}$$

Alternatively, in order to derive the top equation we could've used

$$\ln\left(\frac{\alpha}{\beta}\right) = -\int_0^\infty \frac{\mathrm{d}\tau}{\tau}\big(e^{-i\alpha\tau} - e^{-i\beta\tau}\big), \tag{3.16}$$

where $\mathrm{Im}(\alpha), \mathrm{Im}(\beta) < 0$ and $|\mathrm{Im}(\alpha)|, |\mathrm{Im}(\beta)| \ll 1$. Thus we can generalize this to be

$$\ln\left(\frac{D}{D_0}\right) = -\int_0^\infty \frac{\mathrm{d}\tau}{\tau}(e^{-i\tau D} - e^{-i\tau D_0}), \tag{3.17}$$

where $D_0 = \ell^{-2} I$ with I being the identity. Next we notice

$$\mathrm{Tr}\exp(-i\tau D) = \sum_n e^{-i\tau\lambda_n}, \tag{3.18}$$

and we define the following

$$K^i{}_j(\tau; x, x') = \sum_n e^{-i\tau\lambda_n} f_n{}^i(x) f^*_{nj}(x'). \tag{3.19}$$

Recall

$$\int f_n{}^i(x) f^*_{mi}(x)\sqrt{-g}\,\mathrm{d}^d x \equiv f_n{}^i f^*_{mi} = \delta_{nm}. \tag{3.20}$$

Thus we have

$$K^i{}_i(\tau; x, x) = \sum_n e^{-i\tau\lambda_n} f_n{}^i(x) f^*_{ni}(x), \tag{3.21}$$

which further implies

$$\int \sqrt{-g}\,\mathrm{d}^d x\, K^i{}_i(\tau; x, x) = \sum_n e^{-i\tau\lambda_n} \int \sqrt{-g}\,\mathrm{d}^d x\, f_n{}^i(x) f^*_{ni}(x) = \sum_n e^{-i\tau\lambda_n} = \mathrm{Tr}\exp(-i\tau D). \tag{3.22}$$

And thus the effective action can further be expressed as

$$W = \frac{-i\hbar}{2}\int_0^\infty \frac{\mathrm{d}\tau}{\tau}\,\mathrm{Tr}\exp(-i\tau D) = \frac{-i\hbar}{2}\int \frac{\mathrm{d}\tau}{\tau} K^i{}_i(\tau; x, x). \tag{3.23}$$

We can see that K satisfies

$$K^i{}_j(\tau; x, x') = \sum_n e^{-i\tau\lambda_n} f_n{}^i(x) f^*_{nj}(x') \xrightarrow{\tau\to 0} \sum_n f_n{}^i(x) f^*_{nj}(x') = \delta^i_j \delta(x, x'). \tag{3.24}$$

Differentiating K with respect to τ, leads to

$$i\frac{\partial}{\partial\tau} K^i{}_j = i\frac{\partial}{\partial\tau}\sum_n e^{-i\tau\lambda_n} f_n{}^i(x) f^*_{nj}(x') = \sum_n \lambda_n e^{-i\tau\lambda_n} f_n{}^i(x) f^*_{nj}(x'). \tag{3.25}$$

We can take advantage of the fact that this λ_n is an eigenvalue of D with f_{ni} being its eigenvector then

$$\sum_n \lambda_n e^{-i\tau\lambda_n} f_n{}^i(x) f^*_{nj}(x') = \sum_n e^{-i\tau\lambda_n}(-\delta^i_\ell \Box_x + M^{2i}_\ell) f_n{}^\ell(x) f^*_{nj}(x') \tag{3.26}$$

$$= (-\delta^i_\ell \Box_x + M^{2i}_\ell) \sum_n e^{-i\tau\lambda_n} f_n{}^\ell(x) f^*_{nj}(x') = D^i{}_\ell K^\ell{}_j. \tag{3.27}$$

This can more succinctly be written as

$$i\frac{\partial}{\partial\tau} K^i{}_j = D^i{}_\ell K^\ell{}_j, \tag{3.28}$$

with $K^i{}_j(\tau = 0; x, x') = \delta^i_j \delta(x, x')$. This formula resembles the Schrodinger equation with Hamiltonian $D^i{}_\ell$. As a result, K is often referred to as the heat kernel in the literature. Now the heat kernel has the asymptotic form

$$K(\tau; x, x) \sim i(4\pi i\tau)^{-d/2} \sum_{n=0}^{\infty} (i\tau)^n E_n(x), \tag{3.29}$$

where $E_n(x)$ are local quantities constructed from the knowledge of the operator D_x which takes on the form

$$D_x = g^{\mu\nu}\nabla_\mu\nabla_\nu + Q(x). \tag{3.30}$$

The first few terms of the $E_n(x)$ series are

$$E_0(x) = I, \quad E_1(x) = \frac{1}{6}R(x)I - Q(x), \tag{3.31}$$

$$E_2(x) = \left[-\frac{1}{30}\Box R(x) + \frac{1}{72}R^2 - \frac{1}{180}\left(R^{\mu\nu}R_{\mu\nu} - R^{\mu\nu\lambda\rho}R_{\mu\nu\lambda\rho}\right)\right]I + \frac{1}{12}W^{\mu\nu}W_{\mu\nu} + \frac{1}{2}Q^2 + \frac{1}{6}(\Box Q - RQ), \tag{3.32}$$

where I is the identity matrix and $W_{\mu\nu} = [\nabla_\mu, \nabla_\nu]$. Now let $\mathbf{divp}\, W$ denote the divergent part of W. In order to regulate the effective action, it is crucial that we isolate the divergent part of the integral so

$$\mathbf{divp}\, W = -\frac{i\hbar}{2}\int \sqrt{-g}\, \mathrm{d}^d x\, \mathbf{divp} \int_0^{\tau_0} \frac{\mathrm{d}\tau}{\tau}\, \mathrm{Tr}[K(\tau; x, x)]. \tag{3.33}$$

We can expect that the divergence of W is at small values of τ since D has a small imaginary part that forces the integral to go to zero as $\tau \to \infty$. Assuming τ is small leads us to write

$$\mathbf{divp}\, W = -\frac{i\hbar}{2} \int \sqrt{-g}\, \mathrm{d}^d x\, \mathbf{divp} \left[\int_0^{\tau_0} \frac{\mathrm{d}\tau}{\tau} i(4\pi i\tau)^{-d/2} \sum_{n=0}^{\infty} (i\tau)^n \operatorname{Tr} E_n(x) \right] \tag{3.34}$$

$$= \frac{i\hbar}{2(4\pi)^{d/2}} \sum_{n=0}^{\infty} \int \sqrt{-g}\, \mathrm{d}^d x \operatorname{Tr} E_n(x) \int_0^{\tau_0} (i\tau)^{n-1-d/2}\, \mathrm{d}\tau\, . \tag{3.35}$$

Now we want to talk about the different ways we have of isolating the divergence in τ.

3.1 Cutoff

First we'll implement a cutoff parameter τ_c to the integral

$$\int_{\tau_c}^{\tau_0} (i\tau)^{n-1-d/2}\, \mathrm{d}\tau = i^{n-d/2-1} \frac{\tau_0^{n-d/2} - \tau_c^{n-d/2}}{n - d/2}, \tag{3.36}$$

for $n \neq d/2$. For $n = d/2$,

$$\int_{\tau_c}^{\tau_0} \frac{\mathrm{d}\tau}{i\tau} = \frac{1}{i} \ln\!\left(\frac{\tau_0}{\tau_c}\right). \tag{3.37}$$

We can see that when for the case when $n \neq d/2$, sending $\tau_c \to 0$ yields no issue and everything is finite. It's only when $n = d/2$ where the trouble arises. For even spacetime dimensions, we have

$$\mathbf{divp}\, W = \frac{i\hbar}{2(4\pi)^{d/2}} \int \sqrt{-g}\, \mathrm{d}^d x \sum_{n=0}^{d/2-1} \operatorname{Tr} E_n(x) \frac{i^{n-d/2}}{i} \left(\frac{-\tau_c^{n-d/2}}{n - d/2} \right) + \dots \tag{3.38}$$

$$= -\frac{\hbar}{2(4\pi)^{d/2}} \int \sqrt{-g}\, \mathrm{d}^d x \left[\sum_{n=0}^{d/2-1} \operatorname{Tr} E_n(x) \left(\frac{(i\ell^2\tau_c)^{n-d/2}}{n - d/2} \right) - \operatorname{Tr} E_{d/2}(x) \ln \tau_c \right]. \tag{3.39}$$

For odd spacetime dimensions we have

$$\mathbf{divp}\, W = -\frac{\hbar}{2(4\pi)^{d/2}} \int \sqrt{-g}\, \mathrm{d}^d x \sum_{n=0}^{\lfloor d/2 \rfloor} \operatorname{Tr} E_n(x) (i\ell^2)^{n-d/2} \frac{\tau_c^{n-d/2}}{n - d/2}. \tag{3.40}$$

Here we've successfully managed to isolate the divergences to a finite number of terms.

3.2 Dimensional Regularization

Next we cover the case of dimensional regularization. We will take $\operatorname{Re}\{d\} < 0$. The divergent τ-integral is then

$$\int_0^{\tau_0} (i\tau)^{n-1-d/2} = i^{n-d/2-1} \frac{\tau_0^{n-d/2}}{n-d/2}. \tag{3.41}$$

We will then analytically continue d to $\mathrm{Re}\{d\} \geq 0$ since we only have simple poles for $d = 2n$ with $n \in \mathbb{N}$. Since all the poles appear for when d is even, there are no divergent terms for d being odd. This means there is only a single divergent term that we need to deal with. We will write $d = 2n + 2\epsilon$

$$\int_0^{\tau_0} (i\tau)^{n-1-d/2} = i^{n-d/2-1} \frac{\tau_0^{n-d/2}}{n-d/2} = \frac{\tau_0^{-\epsilon}}{-\epsilon} i^{-1-\epsilon}. \tag{3.42}$$

This leaves us with

$$\mathbf{divp}\, W = \frac{i\hbar}{2(4\pi)^{d/2}} \int \sqrt{-g}\, \mathrm{d}^d x\, \mathrm{Tr}\, E_{d/2}(x) \frac{(i\tau_0)^{-\epsilon}}{-i\epsilon} = -\frac{\hbar}{2(4\pi)^{d/2}} \int \sqrt{-g}\, \mathrm{d}^d x\, \frac{\mathrm{Tr}\, E_{d/2}(x)}{\epsilon}, \tag{3.43}$$

and therefore we need only to introduce a single counter term to cancel this out.

3.3 ζ-Function Regularization

The last regularization technique we're going to focus on is ζ-function regularization. Suppose we have the following action

$$S = \frac{1}{2} \int \sqrt{-g}\, \mathrm{d}^d x\, \varphi(x) D_x \varphi(x), \tag{3.44}$$

with $D_x = \Box_x + M^2(x)$. We want to consider this action because it leads to a Riemannian metric (as opposed to a pseudo-Riemannian metric). We can express the one-loop effective action as

$$\exp\left(-\frac{W}{\hbar}\right) = \int \mathcal{D}\varphi \exp\left(-\frac{S}{\hbar}\right) \Rightarrow W = \frac{\hbar}{2} \ln \det \ell^2 D_x = \frac{\hbar}{2} \sum_N \ln\bigl(\ell^2 \lambda_N\bigr), \tag{3.45}$$

where again, λ_N are the eigenvalues of D_x. We can define a new function

$$\zeta(s) = \sum_N \frac{1}{\lambda_N^2}, \tag{3.46}$$

where we note that when $\lambda_N = N$ then $\zeta(s)$ is merely the Riemann-zeta function. Now notice that

$$\zeta'(s) = -\sum_N \lambda_N^{-s} \ln \lambda_N \Rightarrow \zeta'(0) = -\sum_N \ln \lambda_N. \tag{3.47}$$

That means we can define the effective action to be

$$W = \frac{\hbar}{2} \sum_N \ln\big(\ell^2 \lambda_N\big) = \lim_{s \to 0} \frac{\hbar}{2} \sum_N \frac{\ln(\ell^2 \lambda_N)}{\lambda_N^s}. \tag{3.48}$$

Note:

$$-\zeta'(0) + \zeta(0) \ln \ell^2 = \sum_N \ln \lambda_N + \sum_N \ln \ell^2 = \sum_N \ln\big(\ell^2 \lambda_N\big). \tag{3.49}$$

Let's study the behavior of this zeta function. Recall that

$$\lambda_N^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty \tau^{s-1} \exp(-\lambda_N \tau) \, \mathrm{d}\tau \,, \tag{3.50}$$

where $\mathrm{Re}\{s\} > 0$. Now write

$$\zeta(s) = \sum_N \frac{1}{\lambda_N^s} = \frac{1}{\Gamma(s)} \int_0^\infty \tau^{s-1} \sum_N e^{-\lambda_N \tau}. \tag{3.51}$$

Let $\varphi_N(x)$ be an eigenfunction of D such that

$$\int \varphi_M(x) \varphi_N(x) \sqrt{-g} \, \mathrm{d}^d x = \delta_{MN}. \tag{3.52}$$

We also have

$$\sum_N e^{-\lambda_N \tau} = \mathrm{Tr} \exp(-\tau D) = \int \sqrt{-g} \, \mathrm{d}^d x \, K(\tau; x, x). \tag{3.53}$$

The heat kernel can be written as

$$K(\tau; x, x') = \sum_N e^{-\lambda_N \tau} \varphi_N(x) \varphi_N^*(x'), \tag{3.54}$$

and satisfies

$$-\frac{\partial}{\partial \tau} K(\tau; x, x') = D_x K(\tau; x, x'), \qquad K(\tau = 0; x, x') = \delta(x, x'). \tag{3.55}$$

We are now ready to incorporate these results to analyze the effective action. For large τ and $\lambda_N \leq \lambda_{N+1}$

$$\sum_N \exp(-\lambda_N \tau) \simeq e^{-\lambda_0 \tau}, \tag{3.56}$$

and the $\zeta(s)$-function becomes

$$\zeta(s) = \frac{1}{\Gamma(s)}\left[\int_0^{\tau_0} \tau^{s-1}\sum_N e^{-\lambda_N \tau} + \int_{\tau_0}^{\infty} \tau^{s-1}\sum_N e^{-\lambda_N \tau}\,\mathrm{d}\tau\right] \equiv \frac{1}{\Gamma(s)}(F_1(s)+F_2(s)). \tag{3.57}$$

$F_2(s)$ will be an analytic function of τ_0 and λ_0 so long as they're both positive. For τ_0 sufficiently small, we have the asymptotic relation

$$K(\tau; x, x') = (4\pi\tau)^{-d/2}\sum_N \tau^N E_N(x). \tag{3.58}$$

Now we assume that $\mathrm{Re}\{s\} > d/2$,

$$F_1(s) = \int_0^{\infty} \tau^{s-1}\sum_N e^{-\lambda_N \tau}\,\mathrm{d}\tau = \int_0^{\tau_0} \mathrm{d}\tau\, \tau^{s-1}\int \sqrt{-g}\,\mathrm{d}^d x\, K(\tau; x, x) \tag{3.59}$$

$$= \int \sqrt{-g}\,\mathrm{d}^d x \int_0^{\tau_0} \mathrm{d}\tau\, \tau^{s-1}\,\mathrm{Tr}\left[(4\pi\tau)^{-d/2}\sum_{n=0}^{\infty}\tau^n E_n(x)\right] \tag{3.60}$$

$$= \frac{1}{(4\pi)^{d/2}}\int \sqrt{-g}\,\mathrm{d}^d x \sum_{n=0}^{\infty}\mathrm{Tr}[E_n(x)]\int_0^{\tau_0} \tau^{s-1+n-d/2}\,\mathrm{d}\tau \tag{3.61}$$

$$= \sum_{n=0}^{\infty}\frac{\tau_0^{s+n-d/2}}{s+n-d/2}\int \sqrt{-g}\,\mathrm{d}^d x\,\mathrm{Tr}\,E_n(x). \tag{3.62}$$

Thus, $\zeta(s)$ is analytic in s except for when $s = d/2 - n$. Note: This is why we required $\mathrm{Re}\{s\} > d/2$. This prevents us from taking the $\tau_0 \to 0$ limit. For the case when $\mathrm{Re}\{s\} < d/2$, we can define $\zeta(s)$ via analytic continuation. This would force us to expand the Γ-function in the neighborhood of $s = -n$

$$\Gamma(s) = \frac{(-1)^n}{n!}(s+n)^{-n} + \ldots, \tag{3.63}$$

and then

$$\zeta(-n) = \frac{1}{\Gamma(s)}(F_1(s)+F_2(s)) \simeq \frac{(-1)^n n!}{(4\pi)^{d/2}}\int \sqrt{-g}\,\mathrm{d}^d x\,\mathrm{Tr}\,E_{n+d/2}(x). \tag{3.64}$$