

Coherent Scalar Field Oscillations in FRW Spacetime

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1 Introduction

We are interested in studying how scalar fields oscillate in an expanding universe. In particular, we are interested in the oscillation of ϕ about some local (or global minimum) of $V(\phi)$. If V has a min, then at some point in its evolution, ϕ will begin to oscillate around it. Furthermore, we assume the frequency of these oscillations¹ $\omega \simeq \dot{\phi}/\phi$ is always greater than the expansion rate. This is important for understanding the dynamics of the background/non-perturbative condensate scalar field that we expand around in order to study the quantum fluctuation.

Conventions We use the mostly plus metric signature, i.e. $\eta_{\mu\nu} = (-, +, +, +)$ and units where $c = \hbar = 1$. The reduced four dimensional Planck mass is $M_{\text{Pl}} = (8\pi G)^{-1/2} \approx 2.43 \times 10^{18} \text{ GeV}$. We use boldface letters \mathbf{r} to indicate 3-vectors and x and p to denote 4-vectors. Conventions for the curvature tensors, covariant and Lie derivatives are all taken from Carroll.

2 Coherent Oscillations

First we start off with the spacetime interval

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2(\theta) d\varphi^2 \right], \quad (1)$$

¹Need to investigate why the frequency of oscillations looks like the way it does. Brian Batell: That's just generically the behavior of oscillatory functions.

with the associated Friedmann equation given by

$$H^2 = \frac{8\pi G}{3}\rho_{\text{tot}} - \frac{k}{a^2}. \quad (2)$$

The Klein-Gordon equation for a scalar field with this metric is the usual

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0. \quad (3)$$

We can re-write this equation by multiplying the entire thing by $\dot{\phi}$ to get

$$\dot{\phi}\ddot{\phi} + 3H\dot{\phi}^2 + \dot{\phi}V'(\phi) = 0 \Leftrightarrow \frac{d}{dt}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi)\right] = -3H\dot{\phi}^2. \quad (4)$$

This equation is analogous to a harmonic oscillator with a driving term. That's why $3H\dot{\phi}$ is often referred to as the Hubble friction. The term on the left hand side is the total energy density

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi), \quad P = \frac{1}{2}\dot{\phi}^2 - V(\phi), \quad (5)$$

where we've introduced the pressure P . We can rewrite the kinetic energy of the scalar field by adding the energy density and pressure $\dot{\phi}^2 = \rho + P$. Therefore, the Klein-Gordon equation becomes

$$\frac{d\rho}{dt} = -3H(\rho + P) = -\frac{3\dot{a}}{a}(\rho + P). \quad (6)$$

This energy conservation equation is usually referred to as the continuity equation. We can then multiply both sides by a^3 and reverse the chain-rule to get

$$a^3\frac{d\rho}{dt} + 3a^2(\rho + P)\frac{da}{dt} = a^3\frac{d\rho}{dt} + 3a^2\rho\frac{da}{dt} + 3a^2P\frac{da}{dt} = \frac{d}{dt}(\rho a^3) + P\frac{da^3}{dt} \Leftrightarrow d(\rho a^3) = -P da^3. \quad (7)$$

Since ϕ oscillates, we expect ρ to change slowly over time as since $\dot{\rho} \sim H\rho$, it changes on a time scale of H^{-1} . However, $\dot{\phi}^2$ is expected to oscillate more quickly than ρ because $\dot{\phi}^2 \sim \omega^{-1}$ and since we're assuming $\omega \gg H \Leftrightarrow H^{-1} \gg \omega^{-1}$. Thus, we can write

$$\dot{\phi}^2 = \rho + P = (\gamma + \gamma_p)\rho, \quad (8)$$

where $\gamma = \langle \rho + P \rangle_{\text{one cycle}}$ and γ_p is the periodic part of $\dot{\phi}^2$. Since $\dot{\phi}^2$ varies on a time scale of ω^{-1} , it follows that

$$\int_0^t \gamma_p(t') dt' \leq \mathcal{O}(\omega^{-1}). \quad (9)$$

First we want to rewrite the continuity equation to include the terms with the oscillatory behavior of ϕ

$$\frac{d\rho}{dt} = -3H(\rho + P) = -3H(\gamma + \gamma_p)\rho \Leftrightarrow \frac{1}{\rho} \frac{d\rho}{dt} = \frac{d}{dt} \log \frac{\rho}{\rho_0} \frac{-3\dot{a}}{a} (\gamma + \gamma_p). \quad (10)$$

We can integrate the above equation straightforwardly to get

$$\log \frac{\rho}{\rho_0} = -3 \int_{t_0}^t \gamma \frac{d \log a}{dt'} dt' - 3 \int_{t_0}^t H(t') \gamma_p dt'. \quad (11)$$

We will now show the second term on the right hand side is negligible. We can do this via integration by parts by letting $u = H(t')$ and $dv = \gamma_p dt'$ to get

$$\int_{t_0}^t H(t') \gamma_p dt' = H(t') \int_{t_0}^{t'} \gamma_p dt'' \Big|_{t_0}^t - \int_{t_0}^t \dot{H}(t') \int_{t_0}^{t'} \gamma_p dt'' dt'. \quad (12)$$

Since we know the integral of the periodic part is $\mathcal{O}(\omega^{-1})$ and that $H\omega^{-1} \ll 1$, (and \dot{H} will similarly be slowly varying). This result tells us that in the limit that ϕ 's oscillations are much more rapid than the expansion of the universe, we can replace $\dot{\phi}^2$ by its average value over a cycle. Now we're going to assume that average value is constant. This leads to the following

$$\log \frac{\rho}{\rho_0} = -3\gamma \int_{t_0}^t d \log a = -3\gamma \log \left(\frac{a}{a_0} \right) \Leftrightarrow \frac{\rho}{\rho_0} = \left(\frac{a}{a_0} \right)^{-3\gamma}, \quad (13)$$

where $a_0 \equiv a(t_0)$. Now we'll assume that the dominant thing to the expansion is ρ i.e. $\rho_{\text{tot}} \simeq \rho$ and that the universe is flat $k = 0$. We can solve for the evolution of the scale factor

$$\left(\frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \left(\frac{a}{a_0} \right)^{-3\gamma} \Leftrightarrow \int a^{3\gamma/2-1} da = \frac{2}{3\gamma} a^{3\gamma/2} = \sqrt{\frac{8\pi G}{3}} a_0^{3\gamma/2} t \Leftrightarrow a(t) \propto t^{2/3\gamma}. \quad (14)$$

Now we need to compute γ . First, note that

$$\frac{\dot{\phi}^2}{\rho} = \gamma + \gamma_p \Leftrightarrow \left\langle \frac{\dot{\phi}^2}{\rho} \right\rangle_{\text{cycle}} = \gamma. \quad (15)$$

On time scales that are $\ll H^{-1}$, we can approximate the energy density $\rho = V(\phi_{\text{max}}) \equiv V_{\text{max}}$ as a constant. First we want to compute the one-cycle oscillation time

$$\frac{\dot{\phi}^2}{\rho} = \frac{2(\rho - V(\phi))}{V_{\text{max}}} \Rightarrow \frac{\dot{\phi}}{\sqrt{\rho}} = \sqrt{2\left(1 - \frac{V(\phi)}{V_{\text{max}}}\right)} \Leftrightarrow \frac{d\phi}{\sqrt{2\left(1 - \frac{V(\phi)}{V_{\text{max}}}\right)}} = \sqrt{\rho} dt. \quad (16)$$

Integrating both sides and recognizing $H^2 \sim \rho$

$$\int_{\phi_{\text{min}}}^{\phi_{\text{max}}} \frac{d\phi}{\sqrt{2\left(1 - \frac{V(\phi)}{V_{\text{max}}}\right)}} = \sqrt{\rho}(t - t_0) = H(t - t_0). \quad (17)$$

Therefore the average over one cycle is

$$\gamma = \left\langle \frac{\dot{\phi}^2}{\rho} \right\rangle_{\text{cycle}} = \frac{\int_{\phi_{\text{min}}}^{\phi_{\text{max}}} \sqrt{2\left(1 - \frac{V(\phi)}{V_{\text{max}}}\right)} d\phi}{H(t - t_0)} = \frac{\int_{\phi_{\text{min}}}^{\phi_{\text{max}}} \sqrt{2\left(1 - \frac{V(\phi)}{V_{\text{max}}}\right)} d\phi}{\int_{\phi_{\text{min}}}^{\phi_{\text{max}}} \frac{d\phi}{\sqrt{2\left(1 - \frac{V(\phi)}{V_{\text{max}}}\right)}}} = 2 \frac{\int_0^{\phi_{\text{max}}} \sqrt{\left(1 - \frac{V(\phi)}{V_{\text{max}}}\right)} d\phi}{\int_0^{\phi_{\text{max}}} \frac{d\phi}{\sqrt{\left(1 - \frac{V(\phi)}{V_{\text{max}}}\right)}}}, \quad (18)$$

where we assume that the potential is an even function. Now we will consider potentials of the form $V(\phi) = \alpha\phi^n$, $V_{\text{max}} = \alpha\phi_{\text{max}}^n$. This brings γ to the form

$$\gamma = 2 \frac{\int_0^{\phi_{\text{max}}} \sqrt{1 - \frac{\phi^n}{\phi_{\text{max}}^n}} d\phi}{\int_0^{\phi_{\text{max}}} \frac{d\phi}{\sqrt{1 - \left(\frac{\phi}{\phi_{\text{max}}}\right)^n}}}. \quad (19)$$

These integrals are exactly solvable. Let $x = \frac{\phi}{\phi_{\text{max}}}$, the integrals become

$$\gamma = 2 \frac{\int_0^1 \sqrt{1 - x^n} dx}{\int_0^1 \frac{dx}{\sqrt{1 - x^n}}}. \quad (20)$$

First we'll do the integral in the numerator. Let's do an integral substitution $x = \sin^{2/n}(\theta)$ with $dx = \frac{2}{n} \sin^{2/n-1}(\theta) \cos \theta d\theta$

$$\int_0^1 \sqrt{1-x^n} dx = \frac{2}{n} \int_0^{\pi/2} \sin^{\frac{2}{n}-1}(\theta) \cos^2(\theta) d\theta = \frac{2}{n} \int_0^{\pi/2} \sin^{\frac{2}{n}-1}(\theta) \cos^{2\frac{3}{2}-1}(\theta) d\theta. \quad (21)$$

Recall the definition of the beta function

$$B(p, q) = 2 \int_0^{\pi/2} \sin^{2p-1}(\theta) \cos^{2q-1}(\theta) d\theta, \quad (22)$$

and the famous relationship between the gamma function and the beta function

$$B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \quad (23)$$

which brings the numerator to the form

$$\int_0^1 \sqrt{1-x^n} dx = \frac{1}{n} B\left(\frac{1}{n}, \frac{3}{2}\right) = \frac{\frac{1}{n} \Gamma\left(\frac{1}{n}\right) \Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{1}{n} + \frac{3}{2}\right)}. \quad (24)$$

Recalling the properties of the gamma function $\Gamma(z+1) = z\Gamma(z)$ and letting $z = 1/2$ in the numerator and $z = \frac{1}{n} + \frac{1}{2}$ in the denominator, we get

$$\int_0^1 \sqrt{1-x^n} dx = \frac{\frac{1}{n} \Gamma\left(\frac{1}{n}\right) \frac{1}{2} \Gamma\left(\frac{1}{2}\right)}{\left(\frac{1}{n} + \frac{1}{2}\right) \Gamma\left(\frac{1}{n} + \frac{1}{2}\right)} = \frac{\frac{1}{n} \Gamma\left(\frac{1}{n}\right) \frac{\sqrt{\pi}}{2}}{\left(\frac{1}{n} + \frac{1}{2}\right) \Gamma\left(\frac{1}{n} + \frac{1}{2}\right)}. \quad (25)$$

The integral in the denominator can similarly be solved

$$\int_0^1 \frac{dx}{\sqrt{1-x^n}} = \frac{2}{n} \int_0^{\pi/2} \sin^{\frac{2}{n}-1}(\theta) \cos^0(\theta) d\theta = \frac{2}{n} \int_0^{\pi/2} \sin^{\frac{2}{n}-1}(\theta) \cos^{2\frac{1}{2}-1}(\theta) d\theta \quad (26)$$

$$= \frac{1}{n} B\left(\frac{1}{n}, \frac{1}{2}\right) = \frac{\frac{1}{n} \Gamma\left(\frac{1}{n}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{n} + \frac{1}{2}\right)}. \quad (27)$$

Therefore, γ becomes

$$\gamma = 2 \frac{\frac{1}{n} \Gamma\left(\frac{1}{n}\right) \frac{\sqrt{\pi}}{2}}{\left(\frac{1}{n} + \frac{1}{2}\right) \Gamma\left(\frac{1}{n} + \frac{1}{2}\right)} \cdot \frac{\Gamma\left(\frac{1}{n} + \frac{1}{2}\right)}{\frac{1}{n} \Gamma\left(\frac{1}{n}\right) \sqrt{\pi}} = \frac{2n}{n+2}. \quad (28)$$

Thus, the scale factor and energy density for $\gamma = \frac{2n}{n+2}$ is

$$a(t) \propto t^{\frac{2}{3\gamma}} = t^{\frac{n+2}{3n}}, \quad \rho = \rho_0 \left(\frac{a}{a_0} \right)^{-\frac{6n}{n+2}}, \quad (29)$$

and the maximum of field oscillations becomes

$$V_{\max} = \alpha \phi_{\max}^n = \rho_0 \left(\frac{a}{a_0} \right)^{-\frac{6n}{n+2}} \Rightarrow \phi_{\max} \propto a^{-\frac{6}{n+2}} \propto t^{-\frac{2}{n}}. \quad (30)$$

For $n = 2$, $\gamma = 1$, $a \propto t^{2/3}$ i.e. matter-dominated era and $\phi_{\max} \propto a^{-3/2}$ so ϕ redshifts like matter. This can further be seen by

$$\rho + P = (\gamma + \gamma_p)\rho \Leftrightarrow P = (\gamma - 1 + \gamma_p)\rho \Rightarrow \langle P \rangle = \langle (\gamma - 1 + \gamma_p)\rho \rangle = (\gamma - 1)\rho = 0, \quad (31)$$

for $\gamma = 1$. This is equivalent to saying ϕ acts as a pressureless gas which is a hallmark for (non-relativistic) matter. For $n = 4$,

$$\gamma = \frac{2(4)}{4+2} = \frac{4}{3} \Rightarrow \phi_{\max} \propto a^{-1} \propto t^{-1/2}, \quad (32)$$

and the average pressure and energy density becomes

$$\langle P \rangle = \left(\frac{4}{3} - 1 \right) \rho = \frac{1}{3} \rho, \quad \frac{\rho}{\rho_0} = \left(\frac{a}{a_0} \right)^{-3\frac{4}{3}} \propto a^{-4}, \quad a \propto t^{1/2}. \quad (33)$$

We can generalize these results to potentials of the form

$$V(\phi) = \alpha \phi^n (1 + \epsilon \phi^\ell), \quad V_{\max} = \alpha \phi_{\max}^n (1 + \epsilon \phi_{\max}^\ell) \Rightarrow \frac{V(\phi)}{V_{\max}} = \left(\frac{\phi}{\phi_{\max}} \right)^n \frac{1 + \epsilon \phi^\ell}{1 + \epsilon \phi_{\max}^\ell}. \quad (34)$$

The case of the usual potential for a self-interacting scalar field with mass m can be achieved by letting $\alpha = \frac{1}{2}m^2$, $\epsilon = \frac{2\lambda}{m^2}$, and $n = \ell = 2$. We can expand out the second ratio so long as we can assure that $\epsilon \phi^\ell \leq \epsilon \phi_{\max}^\ell$ if $\epsilon \leq \phi_{\max}^{-\ell}$. This turns γ into a function of ϵ . We can compute $d\gamma/d\epsilon$ in order to calculate γ to first order in ϵ . First we have

$$\gamma(\epsilon) = 2 \frac{\int_0^{\phi_{\max}} \sqrt{1 - \left(\frac{\phi}{\phi_{\max}} \right)^n \frac{1 + \epsilon \phi^\ell}{1 + \epsilon \phi_{\max}^\ell}} d\phi}{\int_0^{\phi_{\max}} \frac{d\phi}{\sqrt{1 - \left(\frac{\phi}{\phi_{\max}} \right)^n \frac{1 + \epsilon \phi^\ell}{1 + \epsilon \phi_{\max}^\ell}}}}. \quad (35)$$

We want to compute $d\gamma/d\epsilon$. First we'll focus on the numerator

$$\frac{d}{d\epsilon} \int_0^{\phi_{\max}} \sqrt{1 - \left(\frac{\phi}{\phi_{\max}}\right)^n \frac{1 + \epsilon\phi^\ell}{1 + \epsilon\phi_{\max}^\ell}} d\phi = \int_0^{\phi_{\max}} \left(\frac{\phi}{\phi_{\max}}\right)^n \frac{(\phi_{\max}^\ell - \phi^\ell)}{\sqrt{1 - \left(\frac{\phi}{\phi_{\max}}\right)^n}} d\phi, \quad (36)$$

where we have set $\epsilon = 0$ in the last equality. The denominator then becomes

$$\frac{d}{d\epsilon} \int_0^{\phi_{\max}} \frac{d\phi}{\sqrt{1 - \left(\frac{\phi}{\phi_{\max}}\right)^n \frac{1 + \epsilon\phi^\ell}{1 + \epsilon\phi_{\max}^\ell}}} = \int_0^{\phi_{\max}} \left(\frac{\phi}{\phi_{\max}}\right)^n \frac{\phi_{\max}^\ell - \phi^\ell}{\left(1 - \left(\frac{\phi}{\phi_{\max}}\right)^n\right)^{3/2}} d\phi. \quad (37)$$

The full expression for $d\gamma/d\epsilon$ is then

$$\frac{d\gamma}{d\epsilon} = \frac{\int_0^{\phi_{\max}} \left(\frac{\phi}{\phi_{\max}}\right)^n \frac{(\phi_{\max}^\ell - \phi^\ell)}{\sqrt{1 - \left(\frac{\phi}{\phi_{\max}}\right)^n}} d\phi}{\int_0^{\phi_{\max}} \frac{d\phi}{\sqrt{1 - \left(\frac{\phi}{\phi_{\max}}\right)^n}}} + \frac{\int_0^{\phi_{\max}} \left(\frac{\phi}{\phi_{\max}}\right)^n \frac{\phi_{\max}^\ell - \phi^\ell}{\left(1 - \left(\frac{\phi}{\phi_{\max}}\right)^n\right)^{3/2}} d\phi \int_0^{\phi_{\max}} \sqrt{1 - \left(\frac{\phi}{\phi_{\max}}\right)^n} d\phi}{\left[\int_0^{\phi_{\max}} \frac{d\phi}{\sqrt{1 - \left(\frac{\phi}{\phi_{\max}}\right)^n}} \right]^2} \quad (38)$$

We can go on ahead and take $x = \phi/\phi_{\max}$ for each integral. Let's work out the numerator of the first time first

$$\phi_{\max}^{\ell+1} \int_0^1 \frac{x^n(1 - x^\ell)}{\sqrt{1 - x^n}} dx = \phi_{\max}^{\ell+1} \int_0^1 \frac{x^n dx}{\sqrt{1 - x^n}} - \phi_{\max}^{\ell+1} \int_0^1 \frac{x^{n+\ell}}{\sqrt{1 - x^n}} dx. \quad (39)$$

Now let $x = \sin^{\frac{2}{n}}(\theta)$ and we're left with

$$\frac{\phi_{\max}^{\ell+1}}{n} \left[2 \int_0^{\pi/2} \sin^{2\left(\frac{n+1}{n}\right)-1}(\theta) \cos^{2\frac{1}{2}-1}(\theta) d\theta - 2 \int_0^{\pi/2} \sin^{2\left(\frac{n+\ell+1}{n}\right)-1}(\theta) \cos^{2\frac{1}{2}-1}(\theta) d\theta \right] \quad (40)$$

$$= \frac{\phi_{\max}^{\ell+1}}{n} \left[B\left(\frac{n+1}{n}, \frac{1}{2}\right) - B\left(\frac{n+\ell+1}{n}, \frac{1}{2}\right) \right] = \frac{\phi_{\max}^{\ell+1}}{n} \left[\frac{\Gamma\left(\frac{n+1}{n}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n+1}{n} + \frac{1}{2}\right)} - \frac{\Gamma\left(\frac{n+\ell+1}{n}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n+\ell+1}{n} + \frac{1}{2}\right)} \right] \quad (41)$$

$$= \frac{\sqrt{\pi}\phi_{\max}^{\ell+1}}{n^2} \left[\frac{\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{3}{2} + \frac{1}{n}\right)} - \frac{n\Gamma\left(\frac{n+\ell+1}{n}\right)}{\Gamma\left(\frac{3}{2} + \frac{\ell+1}{n}\right)} \right]. \quad (42)$$

Now we can do the numerator of the second term

$$\int_0^{\phi_{\max}} \left(\frac{\phi}{\phi_{\max}} \right)^n \frac{\phi_{\max}^\ell - \phi^\ell}{\left(1 - \left(\frac{\phi}{\phi_{\max}} \right)^n \right)^{3/2}} d\phi = \phi_{\max}^{\ell+1} \int_0^1 \frac{x^n (1 - x^\ell)}{(1 - x^n)^{3/2}} dx. \quad (43)$$

Doing the same $x = \sin^{\frac{2}{n}}(\theta)$, we get

$$\frac{\phi_{\max}^{\ell+1}}{n} \left[2 \int_0^{\pi/2} \sin^{2(\frac{n+1}{n})-1}(\theta) \cos^{2(-\frac{1}{2})-1}(\theta) d\theta - 2 \int_0^{\pi/2} \sin^{2(\frac{n+\ell+1}{n})-1}(\theta) \cos^{2(-\frac{1}{2})-1}(\theta) d\theta \right] \quad (44)$$

$$= \frac{\phi_{\max}^{\ell+1}}{n} \left[B\left(\frac{n+1}{n}, -\frac{1}{2}\right) - B\left(\frac{n+\ell+1}{n}, -\frac{1}{2}\right) \right] = \frac{\phi_{\max}^{\ell+1}}{n} \left[\frac{\Gamma(\frac{n+1}{n})\Gamma(-\frac{1}{2})}{\Gamma(\frac{n+1}{n} - \frac{1}{2})} - \frac{\Gamma(\frac{n+\ell+1}{n})\Gamma(-\frac{1}{2})}{\Gamma(\frac{n+\ell+1}{n} - \frac{1}{2})} \right] \quad (45)$$

$$= \frac{2\sqrt{\pi}\phi_{\max}^{\ell+1}}{n^2} \left[\frac{n\Gamma\left(\frac{n+\ell+1}{n}\right)}{\Gamma\left(\frac{\ell+1}{n} + \frac{1}{2}\right)} - \frac{\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{1}{n} + \frac{1}{2}\right)} \right]. \quad (46)$$

where we used $\Gamma(-\frac{1}{2}) = -2\sqrt{\pi}$. Putting it all together gives

$$\begin{aligned} \frac{1}{\phi_{\max}^\ell} \frac{d\gamma}{d\epsilon} &= \frac{1}{n^2} \frac{\Gamma\left(\frac{1}{n} + \frac{1}{2}\right)}{\Gamma\left(1 + \frac{1}{n}\right)} \left[\frac{\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{3}{2} + \frac{1}{n}\right)} - \frac{n\Gamma\left(\frac{n+\ell+1}{n}\right)}{\Gamma\left(\frac{3}{2} + \frac{\ell+1}{n}\right)} \right] \\ &\quad + \frac{1}{n^2} \frac{\Gamma\left(1 + \frac{1}{n}\right)}{\Gamma\left(\frac{1}{n} + \frac{3}{2}\right)} \left[\frac{\Gamma\left(\frac{1}{n} + \frac{1}{2}\right)}{\Gamma\left(1 + \frac{1}{n}\right)} \right]^2 \left[\frac{n\Gamma\left(\frac{n+\ell+1}{n}\right)}{\Gamma\left(\frac{\ell+1}{n} + \frac{1}{2}\right)} - \frac{\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{1}{n} + \frac{1}{2}\right)} \right]. \end{aligned} \quad (47)$$

Simplifying this mess while making use of the properties of the gamma function yields

$$\frac{1}{\phi_{\max}^\ell} \frac{d\gamma}{d\epsilon} \Big|_{\epsilon=0} = \frac{4\ell(\ell+1)}{(n+2)(n+2\ell+2)} \frac{\Gamma\left(\frac{\ell+1}{n}\right)\Gamma\left(\frac{n+2}{2n}\right)}{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{n+2\ell+2}{2n}\right)}. \quad (48)$$

Thus, we can approximate γ as being

$$\gamma(\epsilon) \simeq \gamma(0) + \epsilon \frac{d\gamma}{d\epsilon} \Big|_{\epsilon=0} = \frac{2n}{n+2} + \epsilon \phi_{\max}^\ell \frac{4\ell(\ell+1)}{(n+2)(n+2\ell+2)} \frac{\Gamma\left(\frac{\ell+1}{n}\right)\Gamma\left(\frac{n+2}{2n}\right)}{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{n+2\ell+2}{2n}\right)} + \mathcal{O}((\epsilon \phi_{\max}^\ell)^2). \quad (49)$$

For $n = \ell = 2$, γ then becomes

$$\gamma(\epsilon) = \frac{4}{2+2} + \epsilon \phi_{\max}^2 \cdot \frac{4 \cdot 3 \cdot 2}{(2+2)(2+4+2)} \frac{\Gamma(\frac{3}{2})\Gamma(\frac{2+2}{4})}{\Gamma(\frac{1}{2})\Gamma(\frac{2+4+2}{4})} = 1 + \frac{3}{8} \epsilon \phi_{\max}^2. \quad (50)$$

We eventually will have $\rho = V_{\max} \simeq \alpha \phi_{\max}^2$ because ϕ_{\max} decreases over time, γ becomes

$$\gamma(\epsilon) \simeq 1 + \frac{3}{8} \frac{\epsilon}{\alpha} \rho. \quad (51)$$

We can solve for the new dependence that the energy density will have on γ by plugging this above equation into the original equation that we found the dependence ρ had on the scale factor

$$\log \frac{\rho}{\rho_0} = -3 \int_{t_0}^t \gamma \, d \log a \simeq -3 \gamma \log \frac{a}{a_0} \Leftrightarrow -3 \log \frac{a}{a_0} = \frac{1}{\gamma} \log \frac{\rho}{\rho_0} = \frac{\log \frac{\rho}{\rho_0}}{1 + \frac{3\epsilon}{8\alpha} \rho} \simeq \log \frac{\rho}{\rho_0} \left(1 - \frac{3\epsilon}{8\alpha} \rho \right), \quad (52)$$

which yields a scale factor that looks like

$$\left(\frac{a}{a_0} \right)^{-3} \simeq \exp \left[\ln \frac{\rho}{\rho_0} \left(1 - \frac{3\epsilon}{8\alpha} \rho \right) \right] \simeq \left(\frac{\rho}{\rho_0} \right) \exp \left[-\frac{3\epsilon}{8\alpha} (\rho - \rho_0) \right], \quad (53)$$

where we inserted a factor of ρ_0 in the exponential to ensure consistency. If we wait for long times where $a_0 \gg a$ and $\rho_0 \gg \rho$, then

$$\frac{\rho}{\rho_0} \simeq \exp \left[-\frac{3\epsilon \rho_0}{8\alpha} \right] \left(\frac{a}{a_0} \right)^{-3}. \quad (54)$$