Scalar Field Renormalization in Curved Spacetime

Marcell Howard

August 17, 2025

1 Introduction

We wish to calculate the contributions that a self-interacting scalar has on the amount of gravitational particle production in the early universe from the energy-momentum tensor. However, we quickly run into the presence of divergences that we must renormalize. Here we show the general procedure for renormalization in curved spacetime. We first perform this renormalization in flat spacetime and then generalize it to more dynamical spacetimes.

Conventions We use the mostly plus metric signature, i.e. $\eta_{\mu\nu} = (-,+,+,+)$ and units where $c = \hbar = 1$. The reduced four dimensional Planck mass is $M_{\rm Pl} = (8\pi G)^{-1/2} \approx 2.43 \times 10^{18}\,\mathrm{GeV}$. The d'Alembert and Laplace operators are defined to be $\Box \equiv g^{\mu\nu}\partial_\mu\partial_\nu$ and $\nabla^2 = \partial_i\partial^i$ respectively. We use boldface letters $\mathbf{r}$ to indicate 3-vectors and x and p to denote 4-vectors. Conventions for the curvature tensors, covariant and Lie derivatives are all taken from Carroll.

2 Minkowski Spacetime

Given the "bare" Lagrangian

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu\phi_0)^2 - \frac{1}{2}m_0^2\phi_0^2 - \frac{\lambda_0}{4!}\phi_0^4. \tag{2.1}$$

Here we use the subscript '0' to denote the bare/unrenormalized quantities. We wish to renormalize the wave function, mass, and coupling constant

$$\phi_0 = \sqrt{z}\phi, \quad m_0^2 = m^2 + \Delta m^2, \quad \Delta m^2 = z\,\delta m^2\,, \quad \lambda_0 = \frac{z_\lambda \lambda}{z^2} \tag{2.2}$$

by finding the renormalization parameters

$$z = 1 + z_1\lambda + z_2\lambda^2 + \ldots, \qquad \delta m^2 = \delta_1\lambda + \delta_2\lambda^2 + \ldots, \qquad z_\lambda = 1 + \zeta_1\lambda + \zeta_2\lambda^2 + \ldots, \tag{2.3}$$

up to a 1-loop correction. When introducing these new variables, the Lagrangian can be cast as

$$\begin{aligned}
\mathcal{L} &= -\frac{z}{2}(\partial_\mu\phi)^2 - \frac{1}{2}(m^2+\delta m^2)\phi^2 - \frac{z_\lambda\lambda}{4!}\phi^4 && (2.4)\\
&= -\frac{1}{2}(z-1+1)(\partial_\mu\phi)^2 - \frac{1}{2}(m^2+\delta m^2)\phi^2 - (z_\lambda - 1 + 1)\frac{\lambda}{4!}\phi^4 && (2.5)\\
&= -\frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 - \frac{1}{2}(z-1)(\partial_\mu\phi)^2 - \frac{1}{2}\,\delta m^2\,\phi^2 - (z_\lambda-1)\frac{\lambda}{4!}\phi^4. && (2.6)
\end{aligned}$$

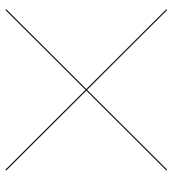
The Lagrangian when expressed in terms of these new renormalized variables is $\mathcal{L} = \mathcal{L}_R + \mathcal{L}_{\text{ct}}$ where

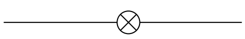
$$\begin{aligned}
\mathcal{L}_R &= -\frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 && (2.7)\\
\mathcal{L}_{\text{ct}} &= -\frac{1}{2}(z-1)(\partial_\mu\phi)^2 - \frac{1}{2}\,\delta m^2\,\phi^2 - (z_\lambda-1)\frac{\lambda}{4!}\phi^4. && (2.8)
\end{aligned}$$

Now we take the interaction Lagrangian to be

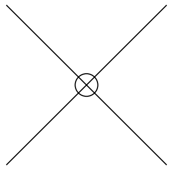
$$\mathcal{L}_{\text{int}} = -\frac{\lambda}{4!}\phi^4 + \mathcal{L}_{\text{ct}}. \tag{2.9}$$

We can treat the counter terms as if they are interactions with vertices (in Fourier space)

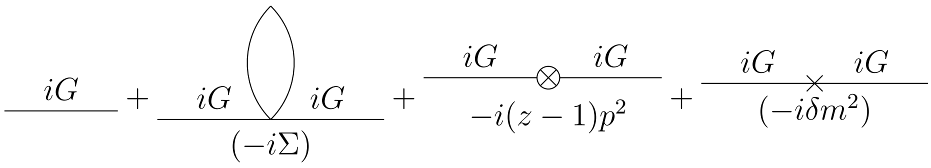
$\frac{\lambda}{4!} =$ 

$(z-1)p^2 =$ 

$\delta m^2 =$ 

$(z_\lambda - 1)\frac{\lambda}{4!} =$ 

We can fix the coefficients to cancel out the divergences. Now we express all quantities in terms of m_R, λ_R. Start with the propagator corrections



These all provide corrections to the self energy that go like

$$\tilde{\Sigma} = \Sigma_R(p) + (z-1)p^2 + \delta m^2\,, \tag{2.10}$$

where $\Sigma_R(p)$ is the renormalized self energy and the $(z-1)p^2$ comes from the kinetic part of the counter terms. Note how the self energy is independent of the momentum because the diagram prevents momentum transfer. The 1-loop self energy can be found by a simple application of the Feynman rules.

$$\Sigma_R(p) = \frac{-i\lambda}{2}\int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{k^2+m^2+i\epsilon} = \frac{i\lambda}{2}\int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{k^2-\Delta}, \tag{2.11}$$

where $\Delta = -m^2 - i\epsilon$. Using the integral formula

$$\int \frac{\mathrm{d}^d q}{(2\pi)^d}\frac{1}{(q^2-\Delta)^n} = \frac{(-1)^n}{(4\pi)^{d/2}}\frac{i\Gamma(n-\frac{d}{2})}{\Gamma(n)}\frac{1}{\Delta^{n-d/2}}, \tag{2.12}$$

the renormalized self energy becomes

$$\Sigma_R = \frac{\lambda}{2(4\pi)^{d/2}}\frac{\Gamma(1-\frac{d}{2})}{\Delta^{1-\frac{d}{2}}}. \tag{2.13}$$

Next, we set $d = 4 - \varepsilon$

$$\Sigma_R = \frac{-\lambda\Delta^{1-\frac{\varepsilon}{2}}(4\pi)^{\frac{\varepsilon}{2}}}{2(4\pi)^2}\frac{\Gamma(\frac{\varepsilon}{2})}{(1-\frac{\varepsilon}{2})}. \tag{2.14}$$

Now we expand to and keep up to $\mathcal{O}(\varepsilon)$

$$\Sigma_R = -\frac{\lambda\Delta^{1-\frac{\varepsilon}{2}}}{2(4\pi)^2}\Big(1+\frac{\varepsilon}{2}\ln 4\pi+\mathcal{O}(\varepsilon)\Big)\Big(1+\frac{\varepsilon}{2}+\mathcal{O}(\varepsilon^2)\Big)\Big(\frac{2}{\varepsilon}-\gamma+\mathcal{O}(\varepsilon)\Big) \tag{2.15}$$

$$= -\frac{\lambda m^2 m^{-\varepsilon}}{2(4\pi)^2}\left(\frac{2}{\varepsilon}-\gamma+1+\ln 4\pi\right). \tag{2.16}$$

Since the self energy is independent of the momentum at the 1-loop level, we can conclude that $z_1 \equiv 0$. This brings the self energy to the form

$$\tilde{\Sigma} = -\frac{\lambda m^2 m^{-\varepsilon}}{2(4\pi)^2}\left(\frac{2}{\varepsilon}-\gamma+1+\ln 4\pi\right)+\delta_1\lambda, \tag{2.17}$$

where, as we recall

$$\delta m^2 = \delta_1\lambda + \dots \tag{2.18}$$

That means we can use the expansion for δm^2 to eliminate the pole and/or constants in the renormalized self energy i.e.

$$\delta_1 = \frac{m^2 m^{-\varepsilon}}{2(4\pi)^2}\frac{2}{\varepsilon}, \qquad \bar{\delta}_1 = \frac{m^2 m^{-\varepsilon}}{2(4\pi)^2}\left(\frac{2}{\varepsilon}-\gamma+1+\ln 4\pi\right), \tag{2.19}$$

where the bar denotes whether we're using the MS or $\overline{\text{MS}}$ scheme. Next we want to estimate the renormalization parameter for the coupling constant, z_λ, by calculating the scattering amplitude. The matrix element to 1-st order in the renormalized coupling constant is simply

$$\mathcal{M}^{(1)}_{fi} = -\frac{i\lambda}{4!}\cdot 4\cdot 3\cdot 2 = -i\lambda, \tag{2.20}$$

The 1-loop correction is given by

$$\Sigma(p) = \frac{(-i\lambda)^2}{2}\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{-i}{k^2+m^2+i\epsilon}\frac{-i}{(k-p)^2+m^2+i\epsilon} \equiv \frac{(-i\lambda)^2}{2}I[p], \tag{2.21}$$

where $p^\mu = (p_1+p_2)^\mu$. Next we employ Feynman's trick

$$\frac{1}{[k^2+m^2+i\epsilon][(p-k)^2+m^2+i\epsilon]} = \int_0^1 \frac{\mathrm{d}x}{[(1-x)(k^2+m^2+i\epsilon)+x((p-k)^2+m^2+i\epsilon)]^2}. \tag{2.22}$$

We can simplify the denominator to be

$$(1-x)(k^2+m^2+i\epsilon)+x((p-k)^2+m^2+i\epsilon)=k^2-2xp\cdot k+m^2+xp^2+i\epsilon\pm x^2p^2 \tag{2.23}$$

$$=(k-xp)^2+M^2+x(1-x)p^2+i\epsilon \tag{2.24}$$

$$=(k-xp)^2-\Delta, \tag{2.25}$$

where $\Delta(x)\equiv -m^2-x(1-x)p^2-i\epsilon$. This brings our integral to the form

$$I[p]=\int\frac{\mathrm{d}^dk}{(2\pi)^d}\int_0^1\frac{\mathrm{d}x}{[(k-xp)^2-\Delta]^2}=\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{1}{(k^2-\Delta)^2}=\frac{-i}{(4\pi)^2}\frac{\Gamma(\frac{\varepsilon}{2})}{(4\pi)^{-\frac{\varepsilon}{2}}}\int_0^1\frac{\mathrm{d}x}{\Delta^{\frac{\varepsilon}{2}}}, \tag{2.26}$$

where we shifted the integral by $k\to k+xp$ in the second equality and set $d=4-\varepsilon$ after using the integral formula from the previous problem. When we expand and include powers of $\varepsilon<1$ we get

$$I[p]=\frac{-im^{-\varepsilon}}{(4\pi)^2}\left(\frac{2}{\varepsilon}-\gamma+\ln 4\pi+\int_0^1\mathrm{d}x\ln\left(1+\frac{p^2}{m^2}x(1-x)-i\epsilon\right)\right) \tag{2.27}$$

$$=\frac{-im^{-\varepsilon}}{(4\pi)^2}\left(\frac{2}{\varepsilon}-\gamma+\ln 4\pi+\tilde{I}[p^2]\right), \tag{2.28}$$

where $\tilde{I}[p^2]$ is the integral above. There are two more 1-loop diagrams whose momentum transfers are $k-(p_1-p_3)$ and $k-(p_1-p_4)$. However, the additional of these two other diagrams will result in the exact same mathematical steps. The only difference is now the integrals will be taken in the t- and u-channels as opposed to the s-channel. Thus the matrix element to 1-loop corrections is

$$\mathcal{M}_{fi}=-i\lambda-i(z_\lambda-1)\lambda-\frac{\lambda^2}{2}(I[s]+I[t]+I[u]) \tag{2.29}$$

$$=-i\lambda-i(z_\lambda-1)\lambda+\frac{i\lambda^2m^{-\varepsilon}}{2(4\pi)^2}\left[3\left(\frac{2}{\varepsilon}-\gamma+\ln 4\pi\right)+\tilde{I}[s]+\tilde{I}[t]+\tilde{I}[u]\right]. \tag{2.30}$$

Now we define

$$J[s,t,u]=\frac{1}{3}\left(\tilde{I}[s]+\tilde{I}[t]+\tilde{I}[u]\right), \tag{2.31}$$

which brings the scattering amplitude to the form

$$\mathcal{M}_{fi} = -i\lambda - i(z_\lambda - 1)\lambda + \frac{3i\lambda^2 m^{-\varepsilon}}{2(4\pi)^2}\left[\frac{2}{\varepsilon} - \gamma + \ln 4\pi + J[s,t,u]\right]. \tag{2.32}$$

We can see that when we expand $(z_\lambda - 1)$ to $\mathcal{O}(\lambda)$ we can use ζ_1 to absorb the pole and/or constants i.e.

$$\zeta_1 = \frac{3m^{-\varepsilon}}{2(4\pi)^2}\frac{2}{\varepsilon}, \quad \bar{\zeta}_1 = \frac{3m^{-\varepsilon}}{2(4\pi)^2}\left(\frac{2}{\varepsilon} - \gamma + \ln 4\pi\right), \tag{2.33}$$

where the overhead bar again indicates $\overline{\text{MS}}$. Thus, the matrix element in both scenarios are

$$\mathcal{M}_{fi}^{\text{MS}} = -i\lambda + \frac{3i\lambda g_R}{2(4\pi)^2}[\ln 4\pi - \gamma + J[s,t,u]], \quad \mathcal{M}_{fi}^{\overline{\text{MS}}} = -i\lambda\left[1 - \frac{3g_R}{2(4\pi)^2}J[s,t,u]\right], \tag{2.34}$$

where $g_R = \lambda m^{-\varepsilon}$.

3 Curved Spacetime

Now that we've done the flat spacetime limit, we can generalize this to curved backgrounds. The first generalization comes at the level of the action. In the effective field theory framework, we are to include *all* possible renormalizable interactions. There's an additional term in the curved-space case that is not present in the flat space case

$$S_M[\phi] = \int \sqrt{-g}\,\mathrm{d}^d x \left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi_0\partial_\nu\phi_0 - \frac{1}{2}m_0^2\phi_0^2 - \frac{1}{2}\xi_0 R\phi_0^2 - \frac{\lambda_0}{4!}\phi_0^4 - \frac{\eta_0}{3!}\phi_0^3 - \sigma_0\phi_0 - \nu_0 R\phi_0\right]. \tag{3.1}$$

Our choice to include these non-renormalizable interactions will become much more apparent later. We also require the most general gravitational action

$$S_G = \int \sqrt{-g}\,\mathrm{d}^d x \left[\Lambda_0 + \kappa_0 R + \alpha_0 R^{\mu\nu\lambda\rho}R_{\mu\nu\lambda\rho} + \beta_0 R^{\mu\nu}R_{\mu\nu} + \gamma_0 R^2\right]. \tag{3.2}$$

The effective 1-loop action can be written as

$$\Gamma[\phi] = S[\phi] + \frac{i\hbar}{2}\ln\det\left(\ell^2\frac{\delta^2 S_M}{\delta\phi^2}\right), \tag{3.3}$$

where ℓ is an arbitrary length scale to keep the dimensions consistent. The term in the determinate is a (divergent) differential operator

$$\frac{\delta^2 S_M}{\delta\phi^2} = \left[\Box_x - m_0^2 - \xi_0 R - \eta_0\phi_0 - \frac{1}{2}\lambda_0\phi_0^2\right]\delta(x,y). \tag{3.4}$$

We've been working in $d-$dimensional spacetime as a way to suggest we plan on regularizing these quantities via dimensional regulation. First we can rewrite the bare quantities in terms of the renormalized quantities and the counter terms

$$\Lambda_0 = \Lambda + \delta\Lambda\,,\quad \kappa_0 = \kappa + \delta\kappa\,,\quad \alpha_0 = \alpha + \delta\alpha\,,\quad \beta_0 = \beta + \delta\beta\,,\quad \gamma_0 = \gamma + \delta\gamma\,, \tag{3.5}$$

with the wave function renormalization $\phi_0 = z\phi$

$$m_0^2 = m^2 + \delta m^2\,,\quad \xi_0 = \xi + \delta\xi\,,\quad \eta_0 = \eta + \delta\eta\,,\quad \lambda_0 = \lambda + \delta\lambda\,,\quad \sigma_0 = \sigma + \delta\sigma\,,\quad \nu_0 = \nu + \delta\nu\,. \tag{3.6}$$

The wave function renormalization coupling constant can be expanded like

$$z = 1 + \delta z\,, \tag{3.7}$$

The expanded 1-loop action is

$$\frac{\delta^2 S_M}{\delta\phi^2} = \left[\Box_x - (m^2 + \delta m^2) - (\xi + \delta\xi)R - (\eta + \delta\eta)\phi - \frac{1}{2}(\lambda + \delta\lambda)\phi^2\right]\delta(x,y). \tag{3.8}$$

All counter terms can be expanded in a power series in Planck's constant

$$\delta q = \hbar\,\delta q^{(1)} + \hbar^2\,\delta q^{(2)} + \ldots, \tag{3.9}$$

which means the first order correction is on the order of $\hbar$. Since the 1-loop effective action is already of order $\hbar$ (its being multiplied by $\hbar$), we need not to include the counter terms inside the 1-loop action since those terms would correspond to the 2-loop correction. To regulate the action, we need to look at the divergent parts of both terms. We've previously worked out the divergent of $\ln\det \delta^2 S_M/\delta\phi^2$ in a previous document. Recall the divergent part is given by

$$\mathbf{divp}\left[\frac{i\hbar}{2}\ln\det\frac{\delta^2 S_M}{\delta\phi^2}\right]=\frac{-\hbar}{(4\pi)^2\epsilon}\int\sqrt{-g}\,\mathrm{d}^dx\,\mathrm{Tr}\,E_2(x), \tag{3.10}$$

where

$$E_2(x)=\left[-\frac{1}{30}\Box R(x)+\frac{1}{72}R^2-\frac{1}{180}\big(R^{\mu\nu}R_{\mu\nu}-R^{\mu\nu\lambda\rho}R_{\mu\nu\lambda\rho}\big)\right]I+\frac{1}{12}W^{\mu\nu}W_{\mu\nu}+\frac{1}{2}Q^2+\frac{1}{6}(\Box Q-RQ), \tag{3.11}$$

where $W_{\mu\nu}=[\nabla_\mu,\nabla_\nu]$ and in this instance

$$Q(x)=m^2+\xi R+\eta\phi+\frac{1}{2}\lambda\phi^2. \tag{3.12}$$

We can see that when we act $W_{\mu\nu}$ on a scalar field, we see that

$$W_{\mu\nu}\phi=\nabla_\mu(\partial_\nu\phi)-\nabla_\nu(\partial_\mu\phi)=\partial_\mu\partial_\nu\phi-\Gamma^\lambda_{\mu\nu}\partial_\lambda\phi-\partial_\nu\partial_\mu\phi+\Gamma^\lambda_{\nu\mu}\partial_\lambda\phi=0. \tag{3.13}$$

Now we compute $Q^2, \Box Q$

$$\Box Q=\xi\Box R+\eta\Box\phi+\frac{1}{2}\lambda\Box\phi^2, \tag{3.14}$$

$$Q^2=m^4+\xi^2R^2+\eta^2\phi^2+\frac{1}{4}\lambda^2\phi^2+2m^2\xi R+2m^2\eta\phi+m^2\lambda\phi^2+2\xi\eta R\phi+\xi\lambda R\phi^2+\eta\lambda\phi^3. \tag{3.15}$$

Putting these things together yields

$$\begin{aligned}\mathrm{Tr}\,E_2(x)=&\left(\frac{1}{72}+\frac{\xi^2}{2}-\frac{\xi}{6}\right)R^2+\frac{1}{180}\big(R^{\mu\nu\lambda\rho}R_{\mu\nu\lambda\rho}-R^{\mu\nu}R_{\mu\nu}\big)+\left(m^2\xi-\frac{1}{6}m^2\right)R+\frac{1}{2}m^4\\&+m^2\eta\phi+\left(\xi\eta-\frac{1}{6}\eta\right)R\phi+\frac{1}{2}\big(\eta^2+m^2\lambda\big)\phi^2+\frac{1}{2}(\xi\lambda-\lambda)R\phi^2+\frac{1}{2}\eta\lambda\phi^3+\frac{1}{8}\lambda^2\phi^4\\&+\frac{1}{6}\left(\xi-\frac{1}{5}\right)\Box R+\frac{1}{6}\eta\Box\phi+\frac{1}{12}\lambda\Box\phi^2.\end{aligned} \tag{3.16}$$

The terms that are written with an overall box can be safely integrated out of the action and thus do not contribute to the regularization. The tree-level action can be written as

$$
\begin{aligned}
S[\phi] = &\int \sqrt{-g}\,\mathrm{d}^d x\,[\Lambda + \kappa R + \alpha R^{\mu\nu\lambda\rho}R_{\mu\nu\lambda\rho} + \beta R^{\mu\nu}R_{\mu\nu} + \gamma R^2] \\
&+ \int \sqrt{-g}\,\mathrm{d}^d x\,[\delta\Lambda + \delta\kappa\, R + \delta\alpha\, R^{\mu\nu\lambda\rho}R_{\mu\nu\lambda\rho} + \delta\beta\, R^{\mu\nu}R_{\mu\nu} + \delta\gamma\, R^2] \\
&- \int \sqrt{-g}\,\mathrm{d}^d x\,\bigg[\frac{1}{2}(1+\delta z)^2\partial^\mu\phi\partial_\mu\phi + \frac{1}{2}(m^2+\delta m^2)(1+\delta z)^2\phi^2 + \frac{1}{2}(1+\delta z)^2(\xi+\delta\xi)R\phi^2 \\
&+ (1+\delta z)(\sigma+\delta\sigma)\phi + (1+\delta z)(\nu+\delta\nu)\phi + \frac{1}{3!}\eta(1+\delta z)^3\phi^2 + \frac{1}{4!}\lambda(1+\delta z)^4\phi^4\bigg]\,.
\end{aligned}
\tag{3.17}
$$

The divergent part of this action of order $\mathcal{O}(\hbar)$ is then

$$
\begin{aligned}
\mathbf{divp}\, S = \hbar \int \sqrt{-g}\,\mathrm{d}^d x\,\big[&\delta\Lambda + \delta\kappa\, R + \delta\alpha\, R^{\mu\nu\lambda\rho}R_{\mu\nu\lambda\rho} + \delta\beta\, R^{\mu\nu}R_{\mu\nu} + \delta\gamma\, R^2 \\
&- \delta z\,\partial^\mu\phi\partial_\mu\phi - \left(\frac{1}{2}\,\delta m^2 + m^2\,\delta z\right)\phi^2 \\
&- \left(\frac{1}{2}\,\delta\xi + \xi\,\delta z\right)R\phi^2 - (\delta\sigma + \sigma\,\delta z)\phi - (\delta\nu + \nu\,\delta z)R\phi \\
&- \left(\frac{1}{3!}\,\delta\eta + \frac{1}{2}\eta\,\delta z\right)\phi^3 - \left(\frac{1}{4!}\,\delta\lambda + \frac{1}{3!}\lambda\,\delta z\right)\phi^4\bigg]\,.
\end{aligned}
\tag{3.18}
$$

Because the 1-loop action does not contain any terms with derivative divergences, we can take $\delta z = 0$. This simplifies things immensely by handing us the advantage to essentially read off the values of the counter terms

$$
\delta\Lambda = \frac{m^4}{2(4\pi)^2\epsilon}, \qquad \delta\kappa = \frac{m^2}{(4\pi)^2\epsilon}\left(\xi - \frac{1}{6}\right), \qquad \delta\alpha = \frac{1}{180(4\pi)^2\epsilon}, \tag{3.19}
$$

$$
\delta\beta = -\frac{1}{180(4\pi)^2\epsilon}, \qquad \delta\gamma = -\frac{1}{(4\pi)^2\epsilon}\left(\frac{1}{72} + \frac{\xi^2}{2} - \frac{\xi}{6}\right), \tag{3.20}
$$

$$
\delta m^2 = -\frac{1}{(4\pi)^2\epsilon}(m^2\lambda + \eta^2), \qquad \delta\xi = -\frac{\lambda}{(4\pi)^2\epsilon}\left(\xi - \frac{1}{6}\right), \qquad \delta\sigma = -\frac{m^2\eta}{(4\pi)^2\epsilon}, \tag{3.21}
$$

$$
\delta\nu = -\frac{\eta}{(4\pi)^2\epsilon}\left(\xi - \frac{1}{6}\right), \qquad \delta\eta = -\frac{3!}{2(4\pi)^2\epsilon}\eta\lambda, \qquad \delta\lambda = \frac{4!}{8(4\pi)^2\epsilon}\lambda^2. \tag{3.22}
$$