The Caismir Effect

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1 Introduction

Here we derive the Caismir effect for a scalar field in Minkowski spacetime. The set up for the Caismir effect is two square neutral, conducting plates situated at $z = 0$ and $z = a$ whose area L^2 are situated in the x-y plane. We do it two different ways: solving the Klein-Gordon equation and utilizing the Heat Kernel method. First we solve the Klein-Gordon equation.

Conventions We use the mostly plus metric signature, i.e. $\eta_{\mu\nu} = (-, +, +, +)$ and units where $c = 1$. The reduced four dimensional Planck mass is $M_{\rm Pl} = (8\pi G)^{-1/2} \approx 2.43 \times 10^{18}\,\text{GeV}$. We use boldface letters $\mathbf{r}$ to indicate 3-vectors and x and p to denote 4-vectors. Conventions for the curvature tensors, covariant and Lie derivatives are all taken from Carroll.

2 The Klein-Gordon Equation

We start with a massless scalar field in Minkowski space

$$\Box\phi = (-\partial_t^2 + \nabla^2)\phi = 0. \tag{2.1}$$

We use the usual ansatz $\phi(x) = A_{\mathbf{p}} e^{i(\omega_{\mathbf{p}} t - \mathbf{p}\cdot\mathbf{r})}$. We have to impose periodic boundary conditions

$$\phi(x+L, y+L, z+a) = \phi(x,y,z) \Leftrightarrow \exp(i(\omega_{\mathbf{p}} t - (p_x x + p_x L + p_y y + p_y L + p_z z + p_z a)))$$
$$= \exp[i(\omega_{\mathbf{p}} t - \mathbf{p}\cdot\mathbf{r})], \tag{2.2}$$

which implies the different components of the momenta must satisfy

$$p_x = \frac{2\pi\ell}{L}, \quad p_y = \frac{2\pi m}{L}, \quad p_z = \frac{2\pi n}{a}, \tag{2.3}$$

where $(\ell, m, n) \in \mathbb{Z}^3$. Plugging this into the Klein-Gordon equation yields

$$\omega_{\ell,m,n}(a) = \sqrt{\left(\frac{2\pi}{L}\right)^2 (\ell^2 + m^2) + \left(\frac{2\pi}{a}\right)^2 n^2}. \tag{2.4}$$

Thus, our solution looks like

$$\phi(x) = \frac{1}{\sqrt{\mathcal{V}}} \sum_{\ell,m,n} \frac{1}{\sqrt{2\omega_{\ell,m,n}}} (\hat{a}_{\ell,m,n} e^{-ip\cdot x} + \hat{a}^\dagger_{\ell,m,n} e^{ip\cdot x}), \quad \mathcal{V} = aL^2. \tag{2.5}$$

Recall that the Hamiltonian (density) is given by

$$\frac{\hat{H}}{\mathcal{V}} = \frac{1}{\mathcal{V}} \sum_{\ell,m,n} \omega_{\ell,m,n}(a) \left(\hat{a}^\dagger_{\ell,m,n} \hat{a}_{\ell,m,n} + \frac{1}{2} \right). \tag{2.6}$$

Thus, the energy density as a function of the spacing of the plates is

$$\rho(a) \equiv \frac{1}{\mathcal{V}} \langle 0|\hat{H}|0\rangle = \frac{1}{2aL^2} \sum_{\ell,m,n} \omega_{\ell,m,n}(a). \tag{2.7}$$

Now this is a badly divergent quantity and we need to regulate this quantity. Therefore we will write

$$\rho(a) = -\frac{1}{2aL^2} \lim_{\alpha\to 0} \frac{\mathrm{d}}{\mathrm{d}\alpha} \sum_{\ell,m,n} e^{-\alpha\omega_{\ell,m,n}}, \tag{2.8}$$

and define the function

$$S(\alpha, a) \equiv \frac{1}{L^2} \sum_{\ell,m,n} e^{-\alpha\omega_{\ell,m,n}(a)}. \tag{2.9}$$

We shall take the continuum limit in p_x, p_y

$$S(\alpha, a) = \frac{1}{(2\pi)^2} \sum_{n\in\mathbb{Z}} \int \mathrm{d}p_x \int \mathrm{d}p_y \exp\left[-\alpha\left(p_x^2 + p_y^2 + \left(\frac{2\pi}{a}\right)^2 n^2\right)^{1/2}\right]. \tag{2.10}$$

Next we move to polar coordinates with $p_x = p\cos\theta$ and $p_y = p\sin\theta$

$$S(\alpha, a) = \frac{1}{(2\pi)^2} \sum_{n\in\mathbb{Z}} \int_0^\infty \mathrm{d}p \int_0^{2\pi} \mathrm{d}\theta\, p \exp\left[-\alpha\left(p^2 + \left(\frac{2\pi}{a}\right)^2 n^2\right)^{1/2}\right] \tag{2.11}$$

$$= \frac{1}{2\pi} \sum_{n\in\mathbb{Z}} \int_0^\infty \mathrm{d}p\, p \exp\left[-\alpha\left(p^2 + \left(\frac{2\pi}{a}\right)^2 n^2\right)^{1/2}\right]. \tag{2.12}$$

Now we notice the integrand is an even function in n. Therefore we can write

$$\sum_{n\in\mathbb{Z}} = \sum_{n=1}^{\infty} + \sum_{n=-1}^{-\infty} + \sum_{n=0} = 2\sum_{n=1}^{\infty} + \sum_{n=0}, \tag{2.13}$$

and we get

$$S(\alpha, a) = \frac{1}{\pi} \sum_{n>0} \int_0^\infty \mathrm{d}p\, p \exp\left[-\alpha\left(p^2 + \left(\frac{2\pi}{a}\right)^2 n^2\right)^{1/2}\right] + \frac{1}{2\pi} \int_0^\infty p e^{-\alpha p}\, \mathrm{d}p\,. \tag{2.14}$$

We can do a u-sub with $u = \alpha p$ and end up with the Gamma function evaluated at 2 i.e. $\Gamma(2) = 1!$. The result is $1/\alpha^2$. We also define the function

$$F(n) = \int_0^\infty p \exp\left[-\alpha\left(p^2 + \left(\frac{2\pi}{a}\right)^2 n^2\right)^{1/2}\right] \mathrm{d}p\,, \tag{2.15}$$

which simplifies our expression for S with

$$\pi S(\alpha, a) = \sum_{n=1}^{\infty} F(n) + \frac{1}{2} F(0). \tag{2.16}$$

Now we will make use of the Euler-Maclaurin Formula i.e. when $F(n)$ where $b \le n < \infty$ is an analytic function and $\sum_{n=1}^{\infty} F(b+n)$ is a convergent sum, then

$$\frac{1}{2} F(b) + \sum_{n=1}^{\infty} F(b+n) = \int_b^\infty F(x)\, \mathrm{d}x - \sum_{m=1}^{\infty} \frac{B_{2m}}{(2m)!} F^{(2m-1)}(b), \tag{2.17}$$

where B_{2m} are the Bernoulli numbers with $B_2 = 1/6$, $B_4 = -1/30$, and so on with $F^{(2m-1)}$ being the $2m-1$ derivative of F. Thus, we're left with

$$\pi S(\alpha, a) = \int_0^\infty F(x)\,\mathrm{d}x - \sum_{m=1}^{\infty} \frac{B_{2m}}{(2m)!} F^{(2m-1)}(0). \tag{2.18}$$

Before proceeding, let's find the explicit form of the function $F(x)$:

$$F(x) = \int_0^\infty p \exp\left[-\alpha \left(p^2 + \left(\frac{2\pi}{a}\right)^2 x^2\right)^{1/2}\right] \mathrm{d}p\,. \tag{2.19}$$

This integral can be easily solved using a change of variables with $p^2 \to p^2 + \left(\frac{2\pi}{a}\right)^2 x^2$ which leads to

$$F(x) = \int_{\frac{2\pi x}{a}}^\infty p e^{-\alpha p}\,\mathrm{d}p\,. \tag{2.20}$$

We can finish the job using integration by parts with $u = p$ and $\mathrm{d}v = e^{-\alpha p}\,\mathrm{d}p$

$$F(x) = \frac{1}{\alpha}\left[\frac{2\pi x}{a} + \frac{1}{\alpha}\right] \exp\left[-\alpha\left(\frac{2\pi x}{a}\right)\right]. \tag{2.21}$$

Looking at the Euler-Maclaurin formula, we need to take some derivatives of $F(x)$. The first few derivatives are

$$F'(x) = \left(\frac{2\pi}{a}\right) x \exp\left[-\alpha\left(\frac{2\pi x}{a}\right)\right], \qquad F'(0) = 0, \tag{2.22}$$

$$F'''(x) = \frac{16\pi^3\alpha}{a^3}\left[1 - \frac{\pi\alpha x}{a}\right] \exp\left[-\alpha\left(\frac{2\pi x}{a}\right)\right], \qquad F^{(3)}(0) = 2\left(\frac{2\pi}{a}\right)^3 \alpha, \tag{2.23}$$

$$F^{(5)}(x) = \frac{64\pi^5\alpha^3}{a^5}\left[2 - \frac{\pi\alpha x}{a}\right] \exp\left[-\alpha\left(\frac{2\pi x}{a}\right)\right], \qquad F^{(5)}(0) = \mathcal{O}(\alpha^3). \tag{2.24}$$

At this point, it's clear that taking higher and higher derivatives of F, will lead terms proportional to higher powers in α. Therefore, in the limit where $\alpha \to 0$, they will vanish. Thus, we don't have to bother computing anymore derivatives of F since they will all be zero by the end. The only terms we need worry about are F' and $F^{(3)}$ as they don't vanish in the limit of small α. In fact, great care must be taken to deal with the divergence that's present in the F' term. We can compute it's integral now

$$\int_0^\infty F(x)\,\mathrm{d}x = \int_0^\infty \frac{1}{\alpha}\left[\frac{2\pi x}{a} + \frac{1}{\alpha}\right] \exp\left[-\alpha\left(\frac{2\pi x}{a}\right)\right] \mathrm{d}x\,. \tag{2.25}$$

A change of variables to make the exponential function dimensionless would be nice so $s = \frac{2\pi\alpha x}{a}$ yielding

$$\int_0^\infty F(x)\,\mathrm{d}x = \frac{a}{2\pi\alpha^3}\int_0^\infty (s+1)e^{-s}\,\mathrm{d}s = \frac{a}{2\pi\alpha^3}(\Gamma(2)+\Gamma(1)) = \frac{a}{\pi\alpha^3}, \tag{2.26}$$

where we made use of the Γ-function's properties when $\Gamma(n) = (n-1)!$. Getting back to our original S-function, we have

$$\begin{aligned}\pi S(\alpha, a) &= \int_0^\infty F(x)\,\mathrm{d}x - \sum_{m=1}^{\infty}\frac{B_{2m}}{(2m)!}F^{(2m-1)}(0) = \frac{a}{\pi\alpha^3} - \left(\frac{-1/30}{4!}\cdot\frac{16\pi^3\alpha}{a^3}\right) + \mathcal{O}\big(\alpha^3\big)\\ &= \frac{a}{\pi\alpha^3} + \frac{\pi^3\alpha}{45a^3} + \mathcal{O}\big(\alpha^3\big).\end{aligned} \tag{2.27}$$

Thus, we're left with

$$\frac{\partial S}{\partial\alpha} = -\frac{3a}{\pi^2\alpha^4} + \frac{\pi^2}{45a^3} + \mathcal{O}\big(\alpha^2\big). \tag{2.28}$$

Now we can go back to the energy density that we started off calculating

$$\rho(a) = -\frac{1}{2a}\lim_{\alpha\to 0}\frac{\partial S}{\partial\alpha} = -\frac{1}{2a}\lim_{\alpha\to 0}\left[-\frac{3a}{\pi^2\alpha^4} + \frac{\pi^2}{45a^3}\right]. \tag{2.29}$$

The first term on the left still requires some massaging. We will do so my subtracting away this contribution by taking the $a\to\infty$ limit in the following way. First we write the energy density by

$$\rho(a) = \lim_{\alpha\to 0}\rho_0(\alpha, a) - \rho_0(\alpha, \infty), \qquad \rho_0(\alpha, a) = -\frac{1}{2a}\frac{\partial}{\partial\alpha}S(\alpha, a), \tag{2.30}$$

and the limit at infinity is

$$\rho_0(\alpha, \infty) = \lim_{a\to\infty} -\frac{1}{2a}\frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{a}{\pi^2\alpha^3} = \frac{3}{2\pi\alpha^4}. \tag{2.31}$$

Thus, the energy density as a function of the plate separation is just

$$\rho(a) = \lim_{\alpha\to 0}\left[\frac{3}{2\pi^2\alpha^4} - \frac{\pi^2}{90a^4} - \frac{3}{2\pi^2\alpha^4}\right] = -\frac{\pi^2}{90a^4}, \tag{2.32}$$

which is the desired result. The energy density of the Caismir vacuum is negative! This results in an attractive force between the plates.

3 The Heat Kernel Method

For the heat kernel method, we seek to quantize a (charged) scalar field, $\Phi(u, \mathbf{r})$, in a constant gauge field background

$$F_{\mu\nu} = 0 \Rightarrow A^\mu = a\delta^\mu_u, \tag{3.1}$$

in $\mathbb{R}^{d-1} \times \mathbb{S}^1$ where $\mathbb{S}^1$ is the circle. We decompose the manifold in this particular way because we will be imposing periodic boundary conditions on the spatial coordinate u with periodicity L. We require the Lagrangian to be single-valued i.e. $\mathcal{L}(u + L, \mathbf{r}) = \mathcal{L}(u, \mathbf{r})$. This single-valued-ness on the Lagrangian imposes a constraint on the charged scalar by enforcing that it can only change by a phase[1]

$$\Phi(u + L, \mathbf{r}) = \exp(2\pi i\delta)\Phi(u, \mathbf{r}), \tag{3.2}$$

where $\delta \in [0, 1)$. Thus, the heat kernel must satisfy

$$K(\tau; u+L, \mathbf{r}, v, \mathbf{r}') = \exp(2\pi i\delta)K(\tau; u, \mathbf{r}, v, \mathbf{r}'), \quad K(\tau; u, \mathbf{r}, v+L, \mathbf{r}') = \exp(-2\pi i\delta)K(\tau; u, \mathbf{r}, v, \mathbf{r}'). \tag{3.3}$$

We can expand the kernel in a Fourier series in the periodic coordinates u, v,

$$K(\tau; u, \mathbf{r}, v, \mathbf{r}') = \frac{1}{L}\sum_{n=-\infty}^{\infty} K_n(\tau; \mathbf{r}, \mathbf{r}') \exp\left[\frac{2\pi i}{L}(n + \delta)(u - v)\right], \tag{3.4}$$

where we regard $K_n(\tau; \mathbf{r}, \mathbf{r}')$ as the Fourier coefficients that satisfy the heat equation for the kernel

$$i\frac{\partial}{\partial\tau}K(\tau; \mathbf{r}, \mathbf{r}') = (-D^\mu D_\mu + m^2)K(\tau; \mathbf{r}, \mathbf{r}'), \quad D_\mu = \partial_\mu - ieA_\mu. \tag{3.5}$$

We can expand out the covariant derivative-squared term

$$\begin{aligned} D_\mu D^\mu &= (\partial_\mu - ieA_\mu)(\partial^\mu - ieA^\mu) = \partial^\mu\partial_\mu - ie\partial^\mu A_\mu - ieA^\mu\partial_\mu - e^2A^\mu A_\mu \\ &= \Box_{d-1} - \partial_u^2 i2iea\partial_u - e^2a^2 = \Box_{d-1} - (\partial_u + iea)^2, \end{aligned} \tag{3.6}$$

and the action this term has on the exponential is

[1] This is due to the fact that $\mathcal{L}$ contains terms like $\Phi^\dagger\Phi$.

$$(\partial_u + iea)^2 \exp\left[\frac{2\pi i}{L}(n+\delta)(u-v)\right] = (\partial_u^2 + 2iea\partial_u - e^2a^2)\exp\left[\frac{2\pi i}{L}(n+\delta)(u-v)\right] \tag{3.7}$$

$$= -\left(\frac{2\pi}{L}\right)^2 \left[(n+\delta)^2 - 2(n+\delta)\frac{eaL}{2\pi} + \frac{e^2a^2L^2}{(2\pi)^2}\right] \exp\left[\frac{2\pi i}{L}(n+\delta)(u-v)\right] \tag{3.8}$$

$$= -\left(\frac{2\pi}{L}\right)^2 \left[n+\delta+\frac{eaL}{2\pi}\right]^2 \exp\left[\frac{2\pi i}{L}(n+\delta)(u-v)\right]. \tag{3.9}$$

Thus, acting the covariant derivative on the kernel yields

$$\begin{aligned}&(-D_\mu D^\mu + m^2)K_n(\tau;\mathbf{r},\mathbf{r}')\exp\left[\frac{2\pi i}{L}(n+\delta)(u-v)\right] \\ &= \left[-\Box_{d-1} + \left(\frac{2\pi}{L}\right)^2\left[n+\delta+\frac{eaL}{2\pi}\right]^2 + m^2\right]K_n(\tau;\mathbf{r},\mathbf{r}')\exp\left[\frac{2\pi i}{L}(n+\delta)(u-v)\right].\end{aligned} \tag{3.10}$$

The additional constants can be absorbed into a new definition of the mass

$$m_n^2 \equiv m^2 + \left(\frac{2\pi}{L}\right)^2\left[n+\delta+\frac{eaL}{2\pi}\right]^2, \tag{3.11}$$

which reduces the heat equation to the form

$$i\frac{\partial}{\partial\tau}K_n(\tau;\mathbf{r},\mathbf{r}') = \left[-\Box_{d-1} + m_n^2\right]K_n(\tau;\mathbf{r},\mathbf{r}'). \tag{3.12}$$

The right hand side is the equation for a free particle with mass m_n^2. Therefore, we know what the heat kernel should be

$$K_n(\tau;\mathbf{r},\mathbf{r}') = \frac{i}{(4\pi i\tau)^{\frac{d-1}{2}}}\exp\left[-\frac{i}{4\tau}|\mathbf{r}-\mathbf{r}'| - im_n^2\tau\right], \tag{3.13}$$

where

$$K_n(\tau;\mathbf{r},\mathbf{r}') \xrightarrow{\tau\to 0} \delta^{(d-1)}(\mathbf{r}-\mathbf{r}'). \tag{3.14}$$

Thus, the complete kernel is

$$K(\tau;u,\mathbf{r},v,\mathbf{r}') = \frac{i}{(4\pi i\tau)^{\frac{d-1}{2}}}\frac{1}{L}\sum_{m=-\infty}^{\infty}\exp\left[-\frac{i}{4\tau}|\mathbf{r}-\mathbf{r}'| - im_n^2\tau + \frac{2\pi i}{L}(n+\delta)(u-v)\right]. \tag{3.15}$$

This is a horribly complicated and unwieldy expression. Thankfully we're only interested in the coincident limit i.e. $v \to u$, $\mathbf{r}' \to \mathbf{r}$,

$$K(\tau; u, \mathbf{r}, u, \mathbf{r}) = \frac{i}{(4\pi i\tau)^{\frac{d-1}{2}}} \frac{1}{L} \sum_{m=-\infty}^{\infty} e^{-im_n^2\tau}, \tag{3.16}$$

which is the much simpler and reasonable expression to evaluate. Now we calculate the effective potential

$$V = i\hbar \int_0^\infty \frac{\mathrm{d}\tau}{\tau} K(\tau; u, \mathbf{r}, u, \mathbf{r}) = -\frac{\hbar}{L} \frac{1}{(4\pi i)^{(d-1)/2}} \sum_{n=-\infty}^{\infty} \int_0^\infty \tau^{-\frac{d-1}{2}-1} e^{-im_n^2\tau}\, \mathrm{d}\tau\,. \tag{3.17}$$

Under a change of variables $s = im_n^2\tau$

$$V = \frac{-\hbar}{(4\pi i)^{(d-1)/2} L} \sum_{n=-\infty}^{\infty} (im_n^2)^{(d-1)/2} \int_0^\infty s^{\frac{1-d}{2}-1} e^{-s}\, \mathrm{d}s\,, \tag{3.18}$$

it's obvious that this integral is the Gamma function $\Gamma(z)$ evaluated at $z = (1-d)/2$ i.e.

$$\begin{aligned} V &= -\frac{\hbar\Gamma\left(\frac{1-d}{2}\right)}{(4\pi i)^{(d-1)/2}} \sum_{n=-\infty}^{\infty} \left[i\left(m^2 + \left(\frac{2\pi}{L}\right)^2 \left(n + \delta + \frac{eaL}{2\pi}\right)\right)^2\right]^{\frac{d-1}{2}} \\ &= -\frac{\hbar}{L} \frac{\Gamma\left(\frac{1-d}{2}\right)}{(4\pi)^{(d-1)/2}} \left(\frac{2\pi}{L}\right)^{d-1} \sum_{n=-\infty}^{\infty} \left[\left(\frac{mL}{2\pi}\right)^2 + \left(n + \delta + \frac{eaL}{2\pi}\right)^2\right]^{\frac{d-1}{2}}. \end{aligned} \tag{3.19}$$

Now we define the function

$$F(\lambda; \alpha, \beta) = \sum_{n\in\mathbb{Z}} [(n+\beta)^2 + \alpha^2]^{-\lambda}, \tag{3.20}$$

where

$$\lambda = -\frac{d-1}{2}, \qquad \alpha = \frac{mL}{2\pi}, \qquad \beta = \delta + \frac{eaL}{2\pi}. \tag{3.21}$$

This sum converges for $\mathrm{Re}\{\lambda\} > \frac{1}{2}$,

$$F(\lambda; \alpha, \beta) = \sqrt{\pi} \frac{\Gamma\left(\lambda - \frac{1}{2}\right)}{\Gamma(\lambda)} \alpha^{1-2\lambda} + 4\sin\pi\lambda f_\lambda(\alpha, \beta), \tag{3.22}$$

with

$$f_\lambda(\alpha,\beta) = \mathrm{Re}\int_\alpha^\infty \frac{(x^2-\alpha^2)^{-\lambda}\,\mathrm{d}x}{\exp(2\pi(x+i\beta))-1}. \tag{3.23}$$

Thus, the effective potential becomes

$$\begin{aligned} V &= -\frac{\hbar}{L}\frac{\Gamma\left(\frac{1-d}{2}\right)}{(4\pi)^{(d-1)/2}}\left(\frac{2\pi}{L}\right)^{d-1}\left[\sqrt{\pi}\frac{\Gamma(-d/2)}{\Gamma\left(\frac{1-d}{2}\right)} - 4\sin\frac{\pi(d-1)}{2}f_{(1-d)/2}(\alpha,\beta)\right] \\ &= -\hbar\Gamma\left(-\frac{d}{2}\right)\left(\frac{m^2}{4\pi}\right)^{d/2} - \frac{4\hbar}{L}\left(\frac{\pi}{L^2}\right)^{\frac{d-1}{2}}\Gamma\left(\frac{1-d}{2}\right)\cos\frac{\pi d}{2}f_{\frac{1-d}{2}}\left(\frac{mL}{2\pi},\delta+\frac{eaL}{2\pi}\right). \end{aligned} \tag{3.24}$$

The first term is divergent but gets renormalized away via the vacuum energy[2] just like in the flat space case. Thus the renormalized vacuum energy density is

$$V_{\mathrm{ren}}(\delta,d,m,a) = -\frac{4\hbar}{L}\left(\frac{\pi}{L^2}\right)^{\frac{d-1}{2}}\Gamma\left(\frac{1-d}{2}\right)\cos\frac{\pi d}{2}f_{\frac{1-d}{2}}\left(\frac{mL}{2\pi},\delta+\frac{eaL}{2\pi}\right). \tag{3.25}$$

The Casimir effect corresponds to the case where $a=m=0$ and $\delta=0$ for a (complex) scalar field and $\delta=1/2$ for a spin-1/2 field. The function f becomes

$$f_{\frac{1-d}{2}}(0,0) = \int_0^\infty \frac{(x^2-0)^{\frac{d-1}{2}}}{\exp(2\pi x)-1}\,\mathrm{d}x = \int_0^\infty \frac{x^{d-1}}{e^{2\pi x}-1}\,\mathrm{d}x\,. \tag{3.26}$$

We can do an integral substitution, $s=2\pi x$, to make the argument in the exponential to be dimensionless

$$f_{\frac{1-d}{2}}(0,0) = \frac{1}{(2\pi)^d}\int_0^\infty \frac{s^{d-1}}{e^s-1}\,\mathrm{d}s = \frac{\zeta(d)\Gamma(d)}{(2\pi)^d}, \tag{3.27}$$

where $\zeta(d)$ and $\Gamma(d)$ are given by

$$\zeta(d) = \sum_{n=1}^\infty \frac{1}{n^d}, \qquad \Gamma(d) = \int_0^\infty s^{d-1}e^{-s}\,\mathrm{d}s\,. \tag{3.28}$$

For $d=4$, $\zeta(4)=\dfrac{\pi^4}{90}$, $\Gamma(4)=3!$

$$f_{-\frac{3}{2}}(0,0) = \frac{1}{(2\pi)^4}\frac{\pi^4}{90}\cdot 3! = \frac{1}{2^4\cdot 3\cdot 5}. \tag{3.29}$$

Recalling one of the special formulas for the Gamma function

[2]It's proportional to an overall constant so the only thing we *could* renormalize it would be another divergent constant a la the vacuum energy.

$$\Gamma\left(\frac{1}{2}-n\right) = \frac{(-2)^n\sqrt{\pi}}{(2n-1)!!} = \frac{(-4)^n n!\sqrt{\pi}}{(2n)!}, \tag{3.30}$$

the renormalized vacuum energy density for the two cases are then

$$V_{\mathrm{ren}}(\delta=0) = -\frac{4\hbar}{L}\left(\frac{\pi}{L^2}\right)^{3/2}\frac{4}{3}\sqrt{\pi}\frac{1}{240} = -\frac{\pi^2\hbar}{45L^4}. \tag{3.31}$$

We can do the same calculation for spin-1/2 particles but with $\beta = 1/2$. The function f becomes

$$f_{\frac{1-d}{2}}(0,1/2) = -\int_0^\infty \frac{x^{d-1}\,\mathrm{d}x}{e^{2\pi x}+1} = -\frac{1}{(2\pi)^d}\eta(d)\Gamma(d) = \frac{(2^{1-d}-1)\zeta(d)\Gamma(d)}{(2\pi)^d}, \tag{3.32}$$

where we used the property of the $\eta(d)$ function

$$\eta(d) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^d} = (1-2^{1-d})\zeta(d). \tag{3.33}$$

The associated vacuum energy density is then

$$V_{\mathrm{ren}}(\delta=1/2) = -\frac{4\hbar}{L}\left(\frac{\pi}{L^2}\right)^{3/2}\frac{4}{3}\sqrt{\pi}\left(-\frac{1}{16}\cdot\frac{7}{8}\cdot\frac{1}{15}\right) = \frac{7\pi^2\hbar}{360L^4}. \tag{3.34}$$

The result for a real-valued scalar field is simply 1/2 of what is shown here.