Coherent Scalar Field Oscillations in FRW Spacetime

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1 Introduction

We are interested in studying how scalar fields oscillate in an expanding universe. In particular, we are interested in the oscillation of ϕ about some local (or global minimum) of $V(\phi)$. If V has a min, then at some point in its evolution, ϕ will begin to oscillate around it. Furthermore, we assume the frequency of these oscillations $\omega \simeq \dot{\phi}/\phi$ is always greater than the expansion rate. This is important for understanding the dynamics of the background/non-perturbative condensate scalar field that we expand around in order to study the quantum fluctuation.

Conventions We use the mostly plus metric signature, i.e. $\eta_{\mu\nu} = (-,+,+,+)$ and units where $c = \hbar = 1$. The reduced four dimensional Planck mass is $M_{\mathrm{Pl}} = (8\pi G)^{-1/2} \approx 2.43 \times 10^{18}\,\mathrm{GeV}$. We use boldface letters $\mathbf{r}$ to indicate 3-vectors and x and p to denote 4-vectors. Conventions for the curvature tensors, covariant and Lie derivatives are all taken from Carroll.

2 Coherent Oscillations

First we start off with the spacetime interval

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\,\mathrm{d}\theta^2 + r^2\sin^2(\theta)\,\mathrm{d}\varphi^2\right], \tag{1}$$

with the associated Friedmann equation given by

$$H^2 = \frac{8\pi G}{3}\rho_{\text{tot}} - \frac{k}{a^2}. \tag{2}$$

The Klein-Gordon equation for a scalar field with this metric is the usual

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0. \tag{3}$$

We can re-write this equation by multiplying the entire thing by $\dot{\phi}$ to get

$$\dot{\phi}\ddot{\phi} + 3H\dot{\phi}^2 + \dot{\phi}V'(\phi) = 0 \Leftrightarrow \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi)\right] = -3H\dot{\phi}^2. \tag{4}$$

This equation is analogous to a harmonic oscillator with a driving term. That's why $3H\dot{\phi}$ is often refer to as the Hubble friction. The term on the left hand side is the total energy density

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi), \quad P = \frac{1}{2}\dot{\phi}^2 - V(\phi), \tag{5}$$

where we've introduced the pressure P. We can rewrite the kinetic energy of the scalar field by adding the energy density and pressure $\dot{\phi}^2 = \rho + P$. Therefore, the Klein-Gordon equation becomes

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -3H(\rho + P) = -\frac{3\dot{a}}{a}(\rho + P). \tag{6}$$

This energy conservation equation is usually referred to as the continuity equation. We can then multiply both sides by a^3 and reverse the chain-rule to get

$$a^3\frac{\mathrm{d}\rho}{\mathrm{d}t} + 3a^2(\rho+P)\frac{\mathrm{d}a}{\mathrm{d}t} = a^3\frac{\mathrm{d}\rho}{\mathrm{d}t} + 3a^2\rho\frac{\mathrm{d}a}{\mathrm{d}t} + 3a^2P\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\rho a^3\right) + P\frac{\mathrm{d}a^3}{\mathrm{d}t} \Leftrightarrow \mathrm{d}(\rho a^3) = -P\,\mathrm{d}(a^3)\,. \tag{7}$$

Since ϕ oscillates, we expect ρ to change slowly over time as since $\dot{\rho} \sim H\rho$, it changes on a time scale of H^{-1}. However, $\dot{\phi}^2$ is expected to oscillate more quickly than ρ because $\dot{\phi}^2 \sim \omega^{-1}$ and since we're assuming $\omega \gg H \Leftrightarrow H^{-1} \gg \omega^{-1}$. Thus, we can write

$$\dot{\phi}^2 = \rho + P = (\gamma + \gamma_p)\rho, \tag{8}$$

where $\gamma = \langle \rho + P \rangle_{\text{one cycle}}$ and γ_p is the periodic part of $\dot{\phi}^2$. Since $\dot{\phi}^2$ varies on a time scale of ω^{-1}, it follows that

$$\int_0^t \gamma_p(t')\,\mathrm{d}t' \leq \mathcal{O}\big(\omega^{-1}\big). \tag{9}$$

First we want to rewrite the continuity equation to include the terms with the oscillatory behavior of ϕ

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -3H(\rho + P) = -3H(\gamma + \gamma_p)\rho \Leftrightarrow \frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\log\frac{\rho}{\rho_0}\frac{-3\dot{a}}{a}(\gamma + \gamma_p). \tag{10}$$

We can integrate the above equation straightforwardly to get

$$\log\frac{\rho}{\rho_0} = -3\int_{t_0}^{t} \gamma\frac{\mathrm{d}\log a}{\mathrm{d}t'}\,\mathrm{d}t' - 3\int_{t_0}^{t} H(t')\gamma_p\,\mathrm{d}t'\,. \tag{11}$$

We will now show the second term on the right hand side is negligible. We can do this via integration by parts by letting $u = H(t')$ and $\mathrm{d}v = \gamma_p\,\mathrm{d}t'$ to get

$$\int_{t_0}^{t} H(t')\gamma_p\,\mathrm{d}t' = H(t')\int_{t_0}^{t'} \gamma_p\,\mathrm{d}t''\Bigg|_{t_0}^{t} - \int_{t_0}^{t} \dot{H}(t')\int_{t_0}^{t'} \gamma_p\,\mathrm{d}t''\,\mathrm{d}t'\,. \tag{12}$$

Since we know the integral of the periodic part is $\mathcal{O}(\omega^{-1})$ and that $H\omega^{-1} \ll 1$, (and $\dot{H}$ will similarly be slowly varying). This result tells us that in the limit that ϕ's oscillations are much more rapid than the expansion of the universe, we can replace $\dot{\phi}^2$ by its average value over a cycle. Now we're going to assume that average value is constant. This leads to the following

$$\log\frac{\rho}{\rho_0} = -3\gamma\int_{t_0}^{t} \mathrm{d}\log a = -3\gamma\log\left(\frac{a}{a_0}\right) \Leftrightarrow \frac{\rho}{\rho_0} = \left(\frac{a}{a_0}\right)^{-3\gamma}, \tag{13}$$

where $a_0 \equiv a(t_0)$. Now we'll assume that the dominant thing to the expansion is ρ i.e. $\rho_{\text{tot}} \simeq \rho$ and that the universe is flat $k = 0$. We can solve for the evolution of the scale factor

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\left(\frac{a}{a_0}\right)^{-3\gamma} \Leftrightarrow \int a^{3\gamma/2-1}\,\mathrm{d}a = \frac{2}{3\gamma}a^{3\gamma/2} = \sqrt{\frac{8\pi G}{3}}a_0^{3\gamma/2}t \Leftrightarrow a(t) \propto t^{2/3\gamma}. \tag{14}$$

Now we need to compute γ. First, note that

$$\frac{\dot{\phi}^2}{\rho} = \gamma + \gamma_p \Leftrightarrow \left\langle \frac{\dot{\phi}^2}{\rho} \right\rangle_{\text{cycle}} = \gamma. \tag{15}$$

On time scales that are $\ll H^{-1}$, we can approximate the energy density $\rho = V(\phi_{\max}) \equiv V_{\max}$ as a constant. First we want to compute the one-cycle oscillation time

$$\frac{\dot{\phi}^2}{\rho} = \frac{2(\rho - V(\phi))}{V_{\max}} \Rightarrow \frac{\dot{\phi}}{\sqrt{\rho}} = \sqrt{2\left(1 - \frac{V(\phi)}{V_{\max}}\right)} \Leftrightarrow \frac{\mathrm{d}\phi}{\sqrt{2\left(1 - \frac{V(\phi)}{V_{\max}}\right)}} = \sqrt{\rho}\,\mathrm{d}t\,. \tag{16}$$

Integrating both sides and recognizing $H^2 \sim \rho$

$$\int_{\phi_{\min}}^{\phi_{\max}} \frac{\mathrm{d}\phi}{\sqrt{2\left(1 - \frac{V(\phi)}{V_{\max}}\right)}} = \sqrt{\rho}(t - t_0) = H(t - t_0). \tag{17}$$

Therefore the average over one cycle is

$$\gamma = \left\langle \frac{\dot{\phi}^2}{\rho} \right\rangle_{\text{cycle}} = \frac{\int_{\phi_{\min}}^{\phi_{\max}} \sqrt{2\left(1 - \frac{V(\phi)}{V_{\max}}\right)}\,\mathrm{d}\phi}{H(t - t_0)} = \frac{\int_{\phi_{\min}}^{\phi_{\max}} \sqrt{2\left(1 - \frac{V(\phi)}{V_{\max}}\right)}\,\mathrm{d}\phi}{\int_{\phi_{\min}}^{\phi_{\max}} \frac{\mathrm{d}\phi}{\sqrt{2\left(1 - \frac{V(\phi)}{V_{\max}}\right)}}} = 2\frac{\int_0^{\phi_{\max}} \sqrt{\left(1 - \frac{V(\phi)}{V_{\max}}\right)}\,\mathrm{d}\phi}{\int_0^{\phi_{\max}} \frac{\mathrm{d}\phi}{\sqrt{\left(1 - \frac{V(\phi)}{V_{\max}}\right)}}}, \tag{18}$$

where we assume that the potential is an even function. Now we will consider potentials of the form $V(\phi) = \alpha\phi^n$, $V_{\max} = \alpha\phi^n_{\max}$. This brings γ to the form

$$\gamma = 2\frac{\int_0^{\phi_{\max}} \sqrt{1 - \frac{\phi^n}{\phi^n_{\max}}}\,\mathrm{d}\phi}{\int_0^{\phi_{\max}} \frac{\mathrm{d}\phi}{\sqrt{1 - \left(\frac{\phi}{\phi_{\max}}\right)^n}}}. \tag{19}$$

These integrals are exactly solvable. Let $x = \frac{\phi}{\phi_{\max}}$, the integrals become

$$\gamma = 2\frac{\int_0^1 \sqrt{1 - x^n}\,\mathrm{d}x}{\int_0^1 \frac{\mathrm{d}x}{\sqrt{1 - x^n}}}. \tag{20}$$

First we'll do the integral in the numerator. Let's do an integral substitution $x = \sin^{2/n}(\theta)$ with $\mathrm{d}x = \dfrac{2}{n}\sin^{2/n-1}(\theta)\cos\theta\,\mathrm{d}\theta$

$$\int_0^1 \sqrt{1-x^n}\,\mathrm{d}x = \frac{2}{n}\int_0^{\pi/2} \sin^{\frac{2}{n}-1}(\theta)\cos^2(\theta)\,\mathrm{d}\theta = \frac{2}{n}\int_0^{\pi/2} \sin^{\frac{2}{n}-1}(\theta)\cos^{2\frac{3}{2}-1}(\theta)\,\mathrm{d}\theta\,. \quad (21)$$

Recall the definition of the beta function

$$B(p,q) = 2\int_0^{\pi/2} \sin^{2p-1}(\theta)\cos^{2q-1}(\theta)\,\mathrm{d}\theta\,, \quad (22)$$

and the famous relationship between the gamma function and the beta function

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \quad (23)$$

which brings the numerator to the form

$$\int_0^1 \sqrt{1-x^n}\,\mathrm{d}x = \frac{1}{n}B\left(\frac{1}{n},\frac{3}{2}\right) = \frac{\frac{1}{n}\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{1}{n}+\frac{3}{2}\right)}. \quad (24)$$

Recalling the properties of the gamma function $\Gamma(z+1) = z\Gamma(z)$ and letting $z = 1/2$ in the numerator and $z = \frac{1}{n}+\frac{1}{2}$ in the denominator, we get

$$\int_0^1 \sqrt{1-x^n}\,\mathrm{d}x = \frac{\frac{1}{n}\Gamma\left(\frac{1}{n}\right)\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{\left(\frac{1}{n}+\frac{1}{2}\right)\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)} = \frac{\frac{1}{n}\Gamma\left(\frac{1}{n}\right)\frac{\sqrt{\pi}}{2}}{\left(\frac{1}{n}+\frac{1}{2}\right)\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}. \quad (25)$$

The integral in the denominator can similarly be solved

$$\int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x^n}} = \frac{2}{n}\int_0^{\pi/2} \sin^{\frac{2}{n}-1}(\theta)\cos^0(\theta)\,\mathrm{d}\theta = \frac{2}{n}\int_0^{\pi/2} \sin^{\frac{2}{n}-1}(\theta)\cos^{2\frac{1}{2}-1}(\theta)\,\mathrm{d}\theta \quad (26)$$

$$= \frac{1}{n}B\left(\frac{1}{n},\frac{1}{2}\right) = \frac{\frac{1}{n}\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}. \quad (27)$$

Therefore, γ becomes

$$\gamma = 2\frac{\frac{1}{n}\Gamma\left(\frac{1}{n}\right)\frac{\sqrt{\pi}}{2}}{\left(\frac{1}{n}+\frac{1}{2}\right)\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}\cdot\frac{\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}{\frac{1}{n}\Gamma\left(\frac{1}{n}\right)\sqrt{\pi}} = \frac{2n}{n+2}. \quad (28)$$

Thus, the scale factor and energy density for $\gamma = \frac{2n}{n+2}$ is

$$a(t) \propto t^{\frac{2}{3\gamma}} = t^{\frac{n+2}{3n}}, \quad \rho = \rho_0 \left(\frac{a}{a_0} \right)^{-\frac{6n}{n+2}}, \tag{29}$$

and the maximum of field oscillations becomes

$$V_{\max} = \alpha \phi_{\max}^n = \rho_0 \left(\frac{a}{a_0} \right)^{-\frac{6n}{n+2}} \Rightarrow \phi_{\max} \propto a^{-\frac{6}{n+2}} \propto t^{-\frac{2}{n}}. \tag{30}$$

For $n = 2$, $\gamma = 1$, $a \propto t^{2/3}$ i.e. matter-dominated era and $\phi_{\max} \propto a^{-3/2}$ so ϕ redshifts like matter. This can further be seen by

$$\rho + P = (\gamma + \gamma_p)\rho \Leftrightarrow P = (\gamma - 1 + \gamma_p)\rho \Rightarrow \langle P \rangle = \langle (\gamma - 1 + \gamma_p)\rho \rangle = (\gamma - 1)\rho = 0, \tag{31}$$

for $\gamma = 1$. This is equivalent to saying ϕ acts as a pressureless gas which is a hallmark for (non-relativistic) matter. For $n = 4$,

$$\gamma = \frac{2(4)}{4+2} = \frac{4}{3} \Rightarrow \phi_{\max} \propto a^{-1} \propto t^{-1/2}, \tag{32}$$

and the average pressure and energy density becomes

$$\langle P \rangle = \left(\frac{4}{3} - 1 \right)\rho = \frac{1}{3}\rho, \quad \frac{\rho}{\rho_0} = \left(\frac{a}{a_0} \right)^{-3\frac{4}{3}} \propto a^{-4}, \quad a \propto t^{1/2}. \tag{33}$$

We can generalize these results to potentials of the form

$$V(\phi) = \alpha \phi^n (1 + \epsilon \phi^\ell), \quad V_{\max} = \alpha \phi_{\max}^n (1 + \epsilon \phi_{\max}^\ell) \Rightarrow \frac{V(\phi)}{V_{\max}} = \left(\frac{\phi}{\phi_{\max}} \right)^n \frac{1 + \epsilon \phi^\ell}{1 + \epsilon \phi_{\max}^\ell}. \tag{34}$$

The case of the usual potential for a self-interacting scalar field with mass m can be achieved by letting $\alpha = \frac{1}{2}m^2$, $\epsilon = \frac{2\lambda}{m^2}$, and $n = \ell = 2$. We can expand out the second ratio so long as we can assure that $\epsilon \phi^\ell \leq \epsilon \phi_{\max}^\ell$ if $\epsilon \leq \phi_{\max}^{-\ell}$. This turns γ into a function of ϵ. We can compute $\mathrm{d}\gamma/\mathrm{d}\epsilon$ in order to calculate γ to first order in ϵ. First we have

$$\gamma(\epsilon) = 2 \frac{\displaystyle\int_0^{\phi_{\max}} \sqrt{1 - \left(\frac{\phi}{\phi_{\max}} \right)^n \frac{1+\epsilon\phi^\ell}{1+\epsilon\phi_{\max}^\ell}}\, \mathrm{d}\phi}{\displaystyle\int_0^{\phi_{\max}} \frac{\mathrm{d}\phi}{\sqrt{1 - \left(\frac{\phi}{\phi_{\max}} \right)^n \frac{1+\epsilon\phi^\ell}{1+\epsilon\phi_{\max}^\ell}}}}. \tag{35}$$

We want to compute $\mathrm{d}\gamma/\mathrm{d}\epsilon$. First we'll focus on the numerator

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\int_0^{\phi_{\max}}\sqrt{1-\left(\frac{\phi}{\phi_{\max}}\right)^n\frac{1+\epsilon\phi^\ell}{1+\epsilon\phi_{\max}^\ell}}\,\mathrm{d}\phi=\int_0^{\phi_{\max}}\left(\frac{\phi}{\phi_{\max}}\right)^n\frac{(\phi_{\max}^\ell-\phi^\ell)}{\sqrt{1-\left(\frac{\phi}{\phi_{\max}}\right)^n}}\,\mathrm{d}\phi\,, \tag{36}$$

where we have set $\epsilon=0$ in the last equality. The denominator then becomes

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\int_0^{\phi_{\max}}\frac{\mathrm{d}\phi}{\sqrt{1-\left(\frac{\phi}{\phi_{\max}}\right)^n\frac{1+\epsilon\phi^\ell}{1+\epsilon\phi_{\max}^\ell}}}=\int_0^{\phi_{\max}}\left(\frac{\phi}{\phi_{\max}}\right)^n\frac{\phi_{\max}^\ell-\phi^\ell}{\left(1-\left(\frac{\phi}{\phi_{\max}}\right)^n\right)^{3/2}}\,\mathrm{d}\phi\,. \tag{37}$$

The full expression for $\mathrm{d}\gamma/\mathrm{d}\epsilon$ is then

$$\frac{\mathrm{d}\gamma}{\mathrm{d}\epsilon}=\frac{\displaystyle\int_0^{\phi_{\max}}\left(\frac{\phi}{\phi_{\max}}\right)^n\frac{(\phi_{\max}^\ell-\phi^\ell)}{\sqrt{1-\left(\frac{\phi}{\phi_{\max}}\right)^n}}\,\mathrm{d}\phi}{\displaystyle\int_0^{\phi_{\max}}\frac{\mathrm{d}\phi}{\sqrt{1-\left(\frac{\phi}{\phi_{\max}}\right)^n}}}+\frac{\displaystyle\int_0^{\phi_{\max}}\left(\frac{\phi}{\phi_{\max}}\right)^n\frac{\phi_{\max}^\ell-\phi^\ell}{\left(1-\left(\frac{\phi}{\phi_{\max}}\right)^n\right)^{3/2}}\,\mathrm{d}\phi\int_0^{\phi_{\max}}\sqrt{1-\left(\frac{\phi}{\phi_{\max}}\right)^n}\,\mathrm{d}\phi}{\left[\displaystyle\int_0^{\phi_{\max}}\frac{\mathrm{d}\phi}{\sqrt{1-\left(\frac{\phi}{\phi_{\max}}\right)^n}}\right]^2} \tag{38}$$

We can go on ahead and take $x=\phi/\phi_{\max}$ for each integral. Let's work out the numerator of the first time first

$$\phi_{\max}^{\ell+1}\int_0^1\frac{x^n(1-x^\ell)}{\sqrt{1-x^n}}\,\mathrm{d}x=\phi_{\max}^{\ell+1}\int_0^1\frac{x^n\,\mathrm{d}x}{\sqrt{1-x^n}}-\phi_{\max}^{\ell+1}\int_0^1\frac{x^{n+\ell}}{\sqrt{1-x^n}}\,\mathrm{d}x\,. \tag{39}$$

Now let $x=\sin^{\frac{2}{n}}(\theta)$ and we're left with

$$\frac{\phi_{\max}^{\ell+1}}{n}\left[2\int_0^{\pi/2}\sin^{2\left(\frac{n+1}{n}\right)-1}(\theta)\cos^{2\frac{1}{2}-1}(\theta)\,\mathrm{d}\theta-2\int_0^{\pi/2}\sin^{2\left(\frac{n+\ell+1}{n}\right)-1}(\theta)\cos^{2\frac{1}{2}-1}(\theta)\,\mathrm{d}\theta\right] \tag{40}$$

$$=\frac{\phi_{\max}^{\ell+1}}{n}\left[B\left(\frac{n+1}{n},\frac{1}{2}\right)-B\left(\frac{n+\ell+1}{n},\frac{1}{2}\right)\right]=\frac{\phi_{\max}^{\ell+1}}{n}\left[\frac{\Gamma\left(\frac{n+1}{n}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n+1}{n}+\frac{1}{2}\right)}-\frac{\Gamma\left(\frac{n+\ell+1}{n}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n+\ell+1}{n}+\frac{1}{2}\right)}\right] \tag{41}$$

$$=\frac{\sqrt{\pi}\phi_{\max}^{\ell+1}}{n^2}\left[\frac{\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{3}{2}+\frac{1}{n}\right)}-\frac{n\Gamma\left(\frac{n+\ell+1}{n}\right)}{\Gamma\left(\frac{3}{2}+\frac{\ell+1}{n}\right)}\right]. \tag{42}$$

Now we can do the numerator of the second term

$$\int_0^{\phi_{\max}} \left(\frac{\phi}{\phi_{\max}}\right)^n \frac{\phi_{\max}^\ell - \phi^\ell}{\left(1-\left(\frac{\phi}{\phi_{\max}}\right)^n\right)^{3/2}}\,\mathrm{d}\phi = \phi_{\max}^{\ell+1}\int_0^1 \frac{x^n(1-x^\ell)}{(1-x^n)^{3/2}}\,\mathrm{d}x\,. \tag{43}$$

Doing the same $x = \sin^{\frac{2}{n}}(\theta)$, we get

$$\frac{\phi_{\max}^{\ell+1}}{n}\left[2\int_0^{\pi/2}\sin^{2\left(\frac{n+1}{n}\right)-1}(\theta)\cos^{2\left(-\frac{1}{2}\right)-1}(\theta)\,\mathrm{d}\theta - 2\int_0^{\pi/2}\sin^{2\left(\frac{n+\ell+1}{n}\right)-1}(\theta)\cos^{2\left(-\frac{1}{2}\right)-1}(\theta)\,\mathrm{d}\theta\right] \tag{44}$$

$$= \frac{\phi_{\max}^{\ell+1}}{n}\left[B\left(\frac{n+1}{n},-\frac{1}{2}\right) - B\left(\frac{n+\ell+1}{n},-\frac{1}{2}\right)\right] = \frac{\phi_{\max}^{\ell+1}}{n}\left[\frac{\Gamma\left(\frac{n+1}{n}\right)\Gamma\left(-\frac{1}{2}\right)}{\Gamma\left(\frac{n+1}{n}-\frac{1}{2}\right)} - \frac{\Gamma\left(\frac{n+\ell+1}{n}\right)\Gamma\left(-\frac{1}{2}\right)}{\Gamma\left(\frac{n+\ell+1}{n}-\frac{1}{2}\right)}\right] \tag{45}$$

$$= \frac{2\sqrt{\pi}\phi_{\max}^{\ell+1}}{n^2}\left[\frac{n\Gamma\left(\dfrac{n+\ell+1}{n}\right)}{\Gamma\left(\dfrac{\ell+1}{n}+\dfrac{1}{2}\right)} - \frac{\Gamma\left(\dfrac{1}{n}\right)}{\Gamma\left(\dfrac{1}{n}+\dfrac{1}{2}\right)}\right]. \tag{46}$$

where we used $\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}$. Putting it all together gives

$$\begin{aligned}\frac{1}{\phi_{\max}^\ell}\frac{\mathrm{d}\gamma}{\mathrm{d}\epsilon} =& \frac{1}{n^2}\frac{\Gamma\left(\dfrac{1}{n}+\dfrac{1}{2}\right)}{\Gamma\left(1+\dfrac{1}{n}\right)}\left[\frac{\Gamma\left(\dfrac{1}{n}\right)}{\Gamma\left(\dfrac{3}{2}+\dfrac{1}{n}\right)} - \frac{n\Gamma\left(\dfrac{n+\ell+1}{n}\right)}{\Gamma\left(\dfrac{3}{2}+\dfrac{\ell+1}{n}\right)}\right] \\ &+ \frac{1}{n^2}\frac{\Gamma\left(1+\dfrac{1}{n}\right)}{\Gamma\left(\dfrac{1}{n}+\dfrac{3}{2}\right)}\left[\frac{\Gamma\left(\dfrac{1}{n}+\dfrac{1}{2}\right)}{\Gamma\left(1+\dfrac{1}{n}\right)}\right]^2\left[\frac{n\Gamma\left(\dfrac{n+\ell+1}{n}\right)}{\Gamma\left(\dfrac{\ell+1}{n}+\dfrac{1}{2}\right)} - \frac{\Gamma\left(\dfrac{1}{n}\right)}{\Gamma\left(\dfrac{1}{n}+\dfrac{1}{2}\right)}\right].\end{aligned} \tag{47}$$

Simplifying this mess while making use of the properties of the gamma function yields

$$\frac{1}{\phi_{\max}^\ell}\left.\frac{\mathrm{d}\gamma}{\mathrm{d}\epsilon}\right|_{\epsilon=0} = \frac{4\ell(\ell+1)}{(n+2)(n+2\ell+2)}\frac{\Gamma\left(\dfrac{\ell+1}{n}\right)\Gamma\left(\dfrac{n+2}{2n}\right)}{\Gamma\left(\dfrac{1}{n}\right)\Gamma\left(\dfrac{n+2\ell+2}{2n}\right)}. \tag{48}$$

Thus, we can approximate γ as being

$$\gamma(\epsilon) \simeq \gamma(0) + \epsilon\left.\frac{\mathrm{d}\gamma}{\mathrm{d}\epsilon}\right|_{\epsilon=0} = \frac{2n}{n+2} + \epsilon\phi_{\max}^\ell\frac{4\ell(\ell+1)}{(n+2)(n+2\ell+2)}\frac{\Gamma\left(\frac{\ell+1}{n}\right)\Gamma\left(\frac{n+2}{2n}\right)}{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{n+2\ell+2}{2n}\right)} + \mathcal{O}\left((\epsilon\phi_{\max}^\ell)^2\right). \tag{49}$$

For $n = \ell = 2$, γ then becomes

$$\gamma(\epsilon) = \frac{4}{2+2} + \epsilon\phi_{\max}^2 \cdot \frac{4\cdot 3\cdot 2}{(2+2)(2+4+2)} \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{2+2}{4}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{2+4+2}{4}\right)} = 1 + \frac{3}{8}\epsilon\phi_{\max}^2. \tag{50}$$

We eventually will have $\rho = V_{\max} \simeq \alpha\phi_{\max}^2$ because $\phi_{\max}$ decreases over time, γ becomes

$$\gamma(\epsilon) \simeq 1 + \frac{3}{8}\frac{\epsilon}{\alpha}\rho. \tag{51}$$

We can solve for the new dependence that the energy density will have on γ by plugging this above equation into the original equation that we found the dependence ρ had on the scale factor

$$\log\frac{\rho}{\rho_0} = -3\int_{t_0}^{t} \gamma \,\mathrm{d}\log a \simeq -3\gamma\log\frac{a}{a_0} \Leftrightarrow -3\log\frac{a}{a_0} = \frac{1}{\gamma}\log\frac{\rho}{\rho_0} = \frac{\log\frac{\rho}{\rho_0}}{1+\frac{3\epsilon}{8\alpha}\rho} \simeq \log\frac{\rho}{\rho_0}\left(1 - \frac{3\epsilon}{8\alpha}\rho\right), \tag{52}$$

which yields a scale factor that looks like

$$\left(\frac{a}{a_0}\right)^{-3} \simeq \exp\left[\ln\frac{\rho}{\rho_0}\left(1-\frac{3\epsilon}{8\alpha}\rho\right)\right] \simeq \left(\frac{\rho}{\rho_0}\right)\exp\left[-\frac{3\epsilon}{8\alpha}(\rho-\rho_0)\right], \tag{53}$$

where we inserted a factor of ρ_0 in the exponential to ensure consistency. If we wait for long times where $a_0 \gg a$ and $\rho_0 \gg \rho$, then

$$\frac{\rho}{\rho_0} \simeq \exp\left[-\frac{3\epsilon\rho_0}{8\alpha}\right]\left(\frac{a}{a_0}\right)^{-3}. \tag{54}$$