FRW Metric in d-Dimensions

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1 Introduction

For the purposes of doing dimensional regularization for a scalar field in FRW coordinates, it is important to compute the d-dimensional FRW metric and then take the $d \to 4$ limit at the very end. Fortunately, the form of the FRW metric in 3+1 dimensions makes it very easy to generalize.

Conventions We use the mostly plus metric signature, i.e. $\eta_{\mu\nu} = (-,+,+,+)$ and units where $c = \hbar = k_B = 1$. The reduced four dimensional Planck mass is $M_{\text{Pl}} = (8\pi G)^{-1/2} \approx 2.43 \times 10^{18}\,\text{GeV}$. The d'Alembert and Laplace operators are defined to be $\Box \equiv \partial_\mu \partial^\mu = -\partial_t^2 + \nabla^2$ and $\nabla^2 = \partial_i \partial^i$ respectively. We use boldface letters $\mathbf{r}$ to indicate 3-vectors and x and p to denote 4-vectors. Conventions for the curvature tensors, covariant and Lie derivatives are all taken from Carroll.

2 Calculation

In 3+1 dimensions we have

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + a^2(t)\,\mathrm{d}\mathbf{r}^2\,, \tag{1}$$

where $\mathrm{d}\mathbf{r}^2 \equiv \mathrm{d}x^i\,\mathrm{d}x_i = \delta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$ where δ_{ij} is the Kroneckder delta symbol. The metric FRW metric in 3+1 dimensions can be easily read off

$$g_{\mu\nu} = \mathrm{diag}\left(-1, a^2, a^2, a^2\right). \tag{2}$$

Here we are implicitly working with a Cartesian coordinate system as it is a natural coordinate system to study a FRW spacetime. This can be written in a dimensionally-independent way

$$g_{00} = -1, \quad g_{0i} = 0, \quad g_{ij} = a^2\delta_{ij}. \tag{3}$$

Here it doesn't matter whether there are 3 spatial dimensions or d-1 spatial dimensions because the spatial components of the metric will retain the exact same form. Now we can compute the Christoffel symbols

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}) \tag{4}$$

Now we can calculate particular components

$$\Gamma^{t}_{\mu\nu} = \frac{1}{2}g^{t\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}) = -\frac{1}{2}(\partial_\mu g_{\nu t} + \partial_\nu g_{\mu t} - \partial_t g_{\mu\nu}) = \frac{1}{2}\dot{g}_{\mu\nu}. \tag{5}$$

From here, it is easy to see that

$$\Gamma^{t}_{tt} = 0, \quad \Gamma^{t}_{ti} = 0, \quad \Gamma^{t}_{ij} = a\dot{a}\delta_{ij}. \tag{6}$$

Next we can check

$$\Gamma^{i}_{\mu\nu} = \frac{1}{2}g^{i\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}) = \frac{1}{2a^2}\delta^{i\ell}(\partial_\mu g_{\nu\ell} + \partial_\nu g_{\mu\ell}). \tag{7}$$

It is similarly easy to see that

$$\Gamma^{i}_{tt} = 0, \quad \Gamma^{i}_{0j} = H\delta^{i}_{j}, \quad \Gamma^{k}_{ij} = 0. \tag{8}$$

Thus, the only non-zero components are

$$\Gamma^{t}_{ij} = a\dot{a}\delta_{ij}, \quad \Gamma^{i}_{tj} = H\delta^{i}_{j}. \tag{9}$$

Next we compute the Ricci Tensor

$$R_{\mu\nu} = \partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\mu\lambda} + \Gamma^\lambda_{\lambda\rho}\Gamma^\rho_{\mu\nu} - \Gamma^\lambda_{\nu\rho}\Gamma^\rho_{\mu\lambda} \tag{10}$$

$$= \partial_t \Gamma^t_{\mu\nu} - \partial_\nu \Gamma^t_{\mu t} - \partial_\nu \Gamma^k_{\mu k} + \Gamma^k_{kt}\Gamma^t_{\mu\nu} - \Gamma^t_{\nu t}\Gamma^t_{\mu t} - \Gamma^t_{\nu k}\Gamma^k_{\mu t} - \Gamma^k_{\nu t}\Gamma^k_{\mu t} - \Gamma^k_{\nu \ell}\Gamma^\ell_{\mu k}, \tag{11}$$

where we used the fact that $\partial_k \Gamma^k_{\mu\nu} = 0$. Now we can calculate particular components

$$R_{tt} = \partial_t \Gamma^t_{tt} - \partial_t \Gamma^t_{tt} - \partial_t \Gamma^i_{ti} + \Gamma^i_{it}\Gamma^t_{tt} - \Gamma^t_{tt}\Gamma^t_{tt} - \Gamma^t_{ti}\Gamma^i_{tt} - \Gamma^i_{tt}\Gamma^t_{it} - \Gamma^i_{tj}\Gamma^j_{it} \tag{12}$$

$$= -(d-1)\dot{H} - (H\delta^i_j)(H\delta^j_i) = -(d-1)(\dot{H} + H^2), \tag{13}$$

$$R_{ti} = \partial_t \Gamma^t_{ti} - \partial_i \Gamma^t_{tt} - \partial_i \Gamma^k_{tk} + \Gamma^k_{kt}\Gamma^t_{ti} - \Gamma^t_{it}\Gamma^t_{tt} - \Gamma^t_{ik}\Gamma^k_{tt} - \Gamma^k_{it}\Gamma^k_{tt} - \Gamma^k_{i\ell}\Gamma^\ell_{tk} \tag{14}$$

$$= 0, \tag{15}$$

$$R_{ij} = \partial_t \Gamma^t_{ij} - \partial_j \Gamma^t_{it} - \partial_j \Gamma^k_{ik} + \Gamma^k_{kt}\Gamma^t_{ij} - \Gamma^t_{jt}\Gamma^t_{it} - \Gamma^t_{jk}\Gamma^k_{it} - \Gamma^k_{jt}\Gamma^k_{it} - \Gamma^k_{j\ell}\Gamma^\ell_{ik} \tag{16}$$

$$= \dot{a}^2\delta_{ij} + a\ddot{a}\delta_{ij} + (H\delta^k_k)a\dot{a}\delta_{ij} - (a\dot{a}\delta_{jk})(H\delta^k_i) - (H\delta^k_j)(a\dot{a}\delta_{ik}) \tag{17}$$

$$= a\ddot{a}\delta_{ij} + (d-1)\dot{a}^2\delta_{ij} - \dot{a}^2\delta_{ij} = \big[a\ddot{a} + (d-2)\dot{a}^2\big]\delta_{ij}, \tag{18}$$

where we used the fact that taking the spatial trace is

$$\delta^{ij}\delta_{ij} = d - 1. \tag{19}$$

Lastly, we need only to compute the Ricci scalar

$$R = g^{\mu\nu}R_{\mu\nu} = -R_{tt} + \frac{1}{a^2}\delta^{ij}R_{ij} = (d-1)(\dot{H} + H^2) + \frac{1}{a^2}\big[a\dot{a} + (d-2)\dot{a}^2\big](d-1) \tag{20}$$

$$= 2(d-1)\frac{\ddot{a}}{a} + (d-1)(d-2)\frac{\dot{a}^2}{a^2}. \tag{21}$$

We can check that we have the correct form by taking the $d \to 4$ limit

$$\lim_{d\to 4} R(t;d) = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right). \tag{22}$$

This leaves the Einstein tensor to be

$$G_{tt} = R_{tt} + \frac{1}{2}R = \frac{(d-1)(d-2)}{2}\left(\frac{\dot{a}}{a}\right)^2, \quad G_{ij} = R_{ij} - \frac{a^2}{2}\delta_{ij}R = -(d-2)\left[\frac{(d-3)}{2}\dot{a}^2 + a\ddot{a}\right]\delta_{ij}. \tag{23}$$