# Cosmology for Calculus II Students - Solutions

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Here are the solutions to a problem set I made for calculus II students as part of a broader project to build up a repository of problems applying techniques from calc II to real problems in cosmology. The problem set I wrote is an amalgam of problems I've had to solve in my undergrad and grad cosmology courses over the years. The techniques involved are first order (separable) linear differential equations, integration by parts, l'Hoptial's rule, and power series solutions to integrals. Normally I would work in natural units and set $c = \hbar = k_B = 1$, but for the sake of clarity for undergraduate students, I've decided to keep those constant factors in.

**Problem 1.** We're told that the condition for energy conservation in an expanding universe is given by

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + 3H(t)\left(\rho(t) + \frac{P(t)}{c^2}\right) = 0, \tag{1}$$

where $\rho(t)$ is the energy density of a given species, $H(t)$ is the Hubble parameter and $P(t)$ is the pressure exerted on the energy density. The equation of state is given by $P(t) = w\rho(t)c^2$ which brings the continuity equation to the form

$$\dot{\rho} + 3(1+w)H(t)\rho(t) = 0. \tag{2}$$

This is a separable differential equation so we can solve via the following

$$\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a} \Rightarrow \int \frac{\mathrm{d}\rho}{\rho} = -3(1+w)\int \frac{\mathrm{d}a}{a}. \tag{3}$$

Thus, the energy density is

$$\log\left(\frac{\rho}{\rho_0}\right) = -3(1+w)\log\left(\frac{a(t)}{a_0}\right) \Rightarrow \rho(t) = \rho_0\left(\frac{a(t)}{a_0}\right)^{-3(1+w)}. \tag{4}$$

Now we find out what the density is given certain values of $w$. First we pick $w = \frac{1}{3}$:

$$\rho(t) = \rho_0\left(\frac{a(t)}{a_0}\right)^{-4} = \rho_0\left(\frac{a_0}{a(t)}\right)^{4}. \tag{5}$$

For $w = 0$:

$$\rho(t) = \rho_0\left(\frac{a(t)}{a_0}\right)^{-3} = \rho_0\left(\frac{a_0}{a(t)}\right)^{3}. \tag{6}$$

Lastly, for $w = -1$:

$$\rho(t) = \rho_0\left(\frac{a(t)}{a_0}\right)^{0} = \rho_0. \tag{7}$$

**Problem 2.** Now we want to solve Friedmann's equation for flat spatial geometry $k = 0$ for each energy density. The Friedmann equation is

$$H^2(t) = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho(t) - \frac{kc^2}{a^2(t)}. \tag{8}$$

In the case where $k = 0$ and $w = \frac{1}{3}$:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\frac{\rho_0 a_0^4}{a^4(t)} \Rightarrow \frac{\dot{a}}{a} = \sqrt{\frac{8\pi G\rho_0}{3}}\frac{a_0^2}{a^2} = \frac{Ca_0^2}{a^2}, \tag{9}$$

where $C \equiv \sqrt{\frac{8\pi G\rho_0}{3}}$. Our differential equation can be solved by multiplying both sides by $a^2$

$$a\dot{a} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}a^2 = Ca_0^2 \Rightarrow \frac{1}{2}a^2 = Ca_0^2 t. \tag{10}$$

Solving for $t_0$ from $a(t_0) \equiv a_0 = 1$ gives us $t_0 = (2C)^{-1}$. This brings the scale factor to the form

$$a(t) = \left(\frac{t}{t_0}\right)^{\frac{1}{2}}. \tag{11}$$

Next we want the situation $w = 0$:

$$\frac{\dot{a}(t)}{a} = C\frac{a_0^{\frac{3}{2}}}{a^{\frac{3}{2}}(t)} \Rightarrow a^{\frac{1}{2}}\dot{a} = Ca_0^3. \tag{12}$$

Integrating both sides to find the scale factor in terms of time gives us

$$\frac{2}{3}a^{\frac{3}{2}} = Ca_0^{\frac{3}{2}}t \Rightarrow a(t) = a_0\left(\frac{t}{t_0}\right)^{\frac{2}{3}}, \tag{13}$$

where $3t_0 = 2C^{-1}$. Lastly we write $w = -1$:

$$\frac{\dot{a}}{a} = C \Rightarrow a(t) = a_0 \exp\left(\sqrt{\frac{8\pi G\rho_0}{3}}t\right). \tag{14}$$

**Problem 3.** Lastly, we want to find the energy density for a massless species as a function of temperature. We start from

$$\rho = \frac{gc}{2\pi^2\hbar^3}\int_0^\infty \frac{p^3\,\mathrm{d}p}{e^{\beta pc} - 1}. \tag{15}$$

The integral in its current form is a bit unwieldy to evaluate. To get around that, we multiply the time and bottom by $e^{-\beta pc}$ to put it in a form that is more manageable.

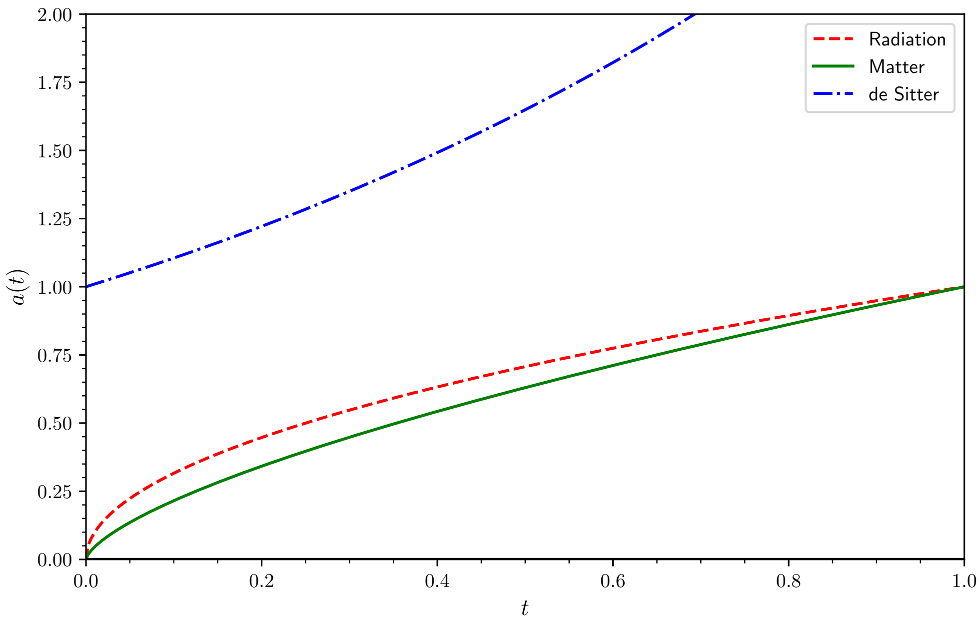


Figure 1: Plot of the three analytic solutions to the Friedman equations for a $k = 0$ flat spatial geometry.

$$\rho = \frac{gc}{2\pi^2\hbar^3} \int_0^\infty \frac{p^3 e^{-\beta pc}}{1 - e^{-\beta pc}} \, \mathrm{d}p \,. \tag{16}$$

On the interval for which we're doing integration, we have $|e^{-\beta pc}| \leq 1$. Thus, we can utilize the Taylor expansion for the geometric series

$$\rho = \frac{gc}{2\pi^2\hbar^3} \int_0^\infty p^3 e^{-\beta p} \sum_{n=0}^\infty e^{-n\beta pc} \, \mathrm{d}p = \frac{gc}{2\pi^2\hbar^3} \sum_{n=0}^\infty \int_0^\infty p^3 e^{-(n+1)\beta pc} \, \mathrm{d}p \,. \tag{17}$$

Lastly, we make the variable substitution $q = (n+1)\beta pc$

$$\rho = \frac{g}{2\pi^2(\hbar c)^3\beta^4} \sum_{n=0}^\infty \frac{1}{(n+1)^4} \int_0^\infty q^3 e^{-q} \, \mathrm{d}q \,. \tag{18}$$

We've reduced the integral to needing to do integration by parts, setting $u = q^3$ and $\mathrm{d}v = e^{-q}$

$$\rho = \frac{g}{2\pi^2(\hbar c)^3\beta^4} \sum_{n=0}^\infty \frac{1}{(n+1)^4} \left[ - q^3 e^{-q} \Big|_0^\infty + 3 \int_0^\infty q^2 e^{-q} \, \mathrm{d}q \right] . \tag{19}$$

We can do this two more times in order to reduce the power in the exponent of $q$ which leaves us with

$$\rho = \frac{g}{2\pi^2(\hbar c)^3\beta^4}\sum_{n=0}^{\infty}\frac{1}{(n+1)^4}\left[-q^3e^{-q}\Big|_0^{\infty} - 3q^2e^{-q}\Big|_0^{\infty} - 6qe^{-q}\Big|_0^{\infty} + 6\int_0^{\infty} e^{-q}\,\mathrm{d}q\right]. \tag{20}$$

Integrating the last exponetial yields

$$\rho = \frac{g}{2\pi^2(\hbar c)^3\beta^4}\sum_{n=0}^{\infty}\frac{1}{(n+1)^4}\left[-q^3e^{-q}\Big|_0^{\infty} - 3q^2e^{-q}\Big|_0^{\infty} - 6qe^{-q}\Big|_0^{\infty} - 6e^{-q}\Big|_0^{\infty}\right]. \tag{21}$$

Evaluating the each expression at the lower limit is easy. Each term except for the very last one vanishes. It is the upper limit that requires more care. We can express all of the terms in the sum as $q^n e^{-q}$ and we can use l'Hopital's rule to find the limit

$$\lim_{q\to\infty}\frac{q^n}{e^q}. \tag{22}$$

Since $n$ is an integer, we can differentiate the numerator $n$ times until it is reduced to a constant which leaves

$$\lim_{q\to\infty}\frac{q^n}{e^q} = \lim_{q\to\infty}\frac{n!}{e^q} = 0. \tag{23}$$

So all the terms vanish at the upper limit. Therefore, we are left with the following result

$$\rho = \frac{6g}{2\pi^2(\hbar c)^3\beta^4}\sum_{n=0}^{\infty}\frac{1}{(n+1)^4} = \frac{6g}{2\pi^2(\hbar c)^3\beta^4}\sum_{m=1}^{\infty}\frac{1}{m^4}, \tag{24}$$

where in the last step we replaced the only summation variable $n$ by setting $m = n+1$. Lastly we use the hint that the above series converges to the given result

$$\rho = \frac{6g}{2\pi^2(\hbar c)^3\beta^4}\cdot\frac{\pi^4}{90} = \frac{g\pi^2(k_BT)^4}{30(\hbar c)^3}. \tag{25}$$

We can see that the energy density scales like $T^4$. Given that the temperature in an expanding universe is inversely proportional to the scale factor i.e. $T \propto a^{-1}$, we can see that this is entirely consistent with the result we found for the energy density of a massless species as a function of the scale factor. This gives us greater confidence that we have found the correct expression.