

# Cosmology for Calculus II Students

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Here is the problem set I made for calculus II students as part of a broader project to build up a repository of problems applying techniques from calc II to real problems in cosmology. This problem set is an amalgamation of problems I've had to solve in my undergrad and grad cosmology courses over the years. I have simply added additional context and provided hints to students where I thought was appropriate. The techniques involved are first order (separable) linear differential equations, integration by parts, l'Hoptial's rule, and power series solutions to integrals. Normally I would work in natural units and set  $c = \hbar = k_B = 1$ , but for the sake of clarity for undergraduate students, I've decided to keep those constant factors in.

## Introduction

In 1929, Edwin Hubble made the first discovery of the radial velocities of distant galaxies by measuring the spectral lines of molecular hydrogen. He found a roughly linear relationship: galaxies that were farther away were receding from our view faster than galaxies that were closer to us. Today, cosmologists recognize these observations as the first pieces of evidence for the expansion of the universe.

The expansion of the universe can be pictured as an inflating balloon with two observers standing on the surface. They maintain a constant angular separation  $\theta$ . As the balloon expands with the radial “expansion” determined by  $R(t)$  (which has units of length), the distance,  $s(t)$ , that separates the observers (stationary on the surface) is  $s(t) = R(t)\theta$  where  $\theta$  is a constant. The velocity for which they recede away from one another is given by

$$v(t) \equiv \frac{ds(t)}{dt} = \dot{R}\theta = \frac{\dot{R}}{R}R\theta = H(t)s(t) \quad (1)$$

where overhead dots

$$\dot{A} \equiv \frac{dA}{dt}, \quad (2)$$

are a common shorthand notation used to denote derivatives with respect to (physical) time and  $H(t)$  is called the Hubble parameter. This function tells cosmologists how quickly distant objects are moving away from us as a function of time. This equation is called Hubble's law. The expansion of space can be made manifest by writing  $R(t) = R_0 \frac{a(t)}{a(t_0)}$  where  $t_0$  is an (arbitrary) reference time. Thus we define  $H(t)$  to be

$$H(t) = \frac{\dot{R}}{R} = \frac{\dot{a}}{a}, \quad (3)$$

where  $a(t)$  is the scale factor (which is dimensionless) and is the object that cosmologists study and what we'll be focused on for the next two problems.

**Problem 1.** Often times in cosmology, we are more concerned with studying the evolution of the density of different particles in an expanding universe. One of our first lines of attack is considering conservation of energy. In an expanding universe, conservation of energy takes on the following form

$$\frac{d\rho}{dt} + 3H(t)\left(\rho(t) + \frac{P(t)}{c^2}\right) = 0, \quad (4)$$

here  $\rho(t)$  is the energy density of the relevant species,  $H(t)$  is the Hubble parameter defined above and  $P(t)$  is the pressure exerted on the energy density with  $c$  being the speed of light. Let's explore how different species become diluted in an expanding universe.

Solve the above linear differential equation in the case for which each constituent obeys<sup>1</sup>  $P(t) = w\rho(t)c^2$  where  $w$  is a constant. Show how each constituent dilutes as a function of the scale factor  $a(t)$  for radiation or massless particles ( $w = 1/3$ ), matter ( $w = 0$ ) and the cosmological constant ( $w = -1$ ).

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<sup>1</sup>We call this relation the equation of state. It's similar to the ideal gas law.

**Problem 2.** The *Friedmann Equation* which governs the rate of expansion due to the constituent species within the universe is given by

$$H^2(t) = \frac{8\pi G}{3}\rho_{\text{tot}}(t) - \frac{kc^2}{a^2(t)}, \quad (5)$$

where  $\rho_{\text{tot}}$  is the total energy density of all the particles that's driving the expansion,  $G$  is Newton's constant and  $k$  is a constant called the spatial curvature. It parameterizes the intrinsic shape of the universe and is normalized to take on values of  $k \in \{\pm 1, 0\}$ . When  $k = +1$ , we call this a closed geometry where the universe grows to a maximum length and contracts. The universe could be visualized as a sphere. A value of  $k = -1$  is an open geometry where the universe grows without bound and its shape is similar to a horse's saddle or a pringle. The case where  $k = 0$  is considered flat and also grows without bound.<sup>2</sup> We use the notation

$$A^2(t) \equiv (A(t))^2. \quad (6)$$

Now we'll assume the dominant component that's responsible for driving the expansion of the universe is radiation (this means to assume that  $\rho_{\text{tot}}$  is the energy density you found for massless particles). Using the relation between  $H(t)$  and  $a(t)$  in Eqn. 3 and relationship between  $\rho(a)$  and  $a$  you found in the previous problem, solve the Friedmann equation in the case with no spatial curvature. Plot the result. [HINT: Set the initial value of the scale factor  $a(t_0) = a_0 = 1$ . You are not explicitly told what  $t_0$  is. Find it and then plot the scale factor as a function of the ratio of  $t$  and this constant.] Do the same thing for a matter-dominated universe and a universe that's dominated by the cosmological constant. Call the energy density of the latter  $\rho_\Lambda = \Lambda c^2$ .

**Problem 3.** Very shortly after the initial expansion from the Big Bang, the universe was extremely hot. So hot that all the particles moved so fast that they were essentially massless. Therefore, we can say that the universe was dominated by radiation at this time. We shall compute the energy density of the universe during this time by considering

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<sup>2</sup>Current observations place the spatial curvature to be consistent with a flat geometry.

a gas of massless particles with integer internal angular momentum<sup>3</sup> at temperature  $T$ . The energy density can be written as

$$\rho = \frac{g}{2\pi^2\hbar^3} \int_0^\infty \frac{p^2 E(p) dp}{e^{\beta E(p)} - 1}, \quad (7)$$

where  $p$  is the momentum of the particles,  $E(p)$  is the kinetic energy of the (massless) particles as a function of momentum,  $\beta = (k_B T)^{-1}$  is the inverse temperature,  $k_B$  is the Boltzmann constant,  $\hbar$  is the reduced Planck's constant and  $g$  is essentially<sup>4</sup> the number of massless particles in the early universe. Given that the kinetic energy of massless particles is  $E(p) = pc$ , evaluate this integral. HINT: This integral cannot be solved indefinitely. In cases like these, it is common to use the power series to simplify the integrand. Use the fact that the geometric series

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad (8)$$

converges whenever  $|x| < 1$  as well as the following identity

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}. \quad (9)$$

Express the denominator of the integrand in a form where you can straightforwardly plug it in into the Maclaurin series of the geometric series. Make your  $u$ -sub to be the argument of the exponential. You may need to use integration by parts a few times in order to evaluate the integral. The temperature of the gas depends on the scale factor by

$$T(t) = \frac{T_0}{a(t)}, \quad (10)$$

where  $T_0 = 2.7$  K is the current temperature of the cosmic microwave background. The dependence on the temperature that you find from this integral should be consistent with the result you found in problem 1. Use this to check your answer.

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<sup>3</sup>We call these particles bosons.

<sup>4</sup>Technically,  $g$  is the effective number of degrees of freedom in the standard model of particle physics, but for our purposes it is sufficient to regard it as the number of massless particles at this epoch of the universe's history.