A Little Math Problem

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Let n be a positive integer. Given that

$$I_n = \int_0^{\pi/2} \cos^{2n} x \, \mathrm{d}x \, , \qquad J_n = \int_0^{\pi/2} x^2 \cos^{2n} x \, \mathrm{d}x \, ,$$

compute the following limit

$$\lim_{m \to \infty} 2 \sum_{n=1}^{m} \left(\frac{J_{n-1}}{I_{n-1}} - \frac{J_n}{I_n} \right).$$

For problems of these types, the strategy is to find some recursive relationship between the I_n's and J_n's and hope that simplifies the problem immensely. We start with the I_n's since that is the simpler expression.

$$I_n = \int_0^{\pi/2} \cos^{2n} x \, \mathrm{d}x \,. \tag{1}$$

Since we're looking for a recursive definition, let's try integration by parts. Let's take $u = \cos^{2n-1} x$ and $\mathrm{d}v = \cos x \, \mathrm{d}x$. These choices yield $\mathrm{d}u = (2n-1)\cos^{2n-2} x(-\sin x)\,\mathrm{d}x$ and $v = \sin x$. Thus, we have

$$I_n = \sin x \cos^{2n-1} x \Big|_0^{\pi/2} + (2n-1)\int_0^{\pi/2} \cos^{2n-2} x \sin^2 x \, \mathrm{d}x \,. \tag{2}$$

Since $\sin(0) = 0$ and $\cos\left(\frac{\pi}{2}\right) = 0$, the first term vanishes. Recalling that $\sin^2 x = 1 - \cos^2 x$, we're left with

$$I_n = (2n-1)\int_0^{\pi/2} \cos^{2n-2} x \, \mathrm{d}x - (2n-1)\int_0^{\pi/2} \cos^{2n} x \, \mathrm{d}x = (2n-1)I_{n-1} - (2n-1)I_n. \tag{3}$$

Rearranging to solve for I_n gives

$$I_n = \frac{2n-1}{2n} I_{n-1}. \tag{4}$$

Now let's focus on the J_n's. Let's use the same strategy of employing integration by parts to extract a recursive result

$$J_n = \int_0^{\pi/2} x^2 \cos^{2n} x \, \mathrm{d}x \,. \tag{5}$$

Let's take a page from the previous computation by letting $u = x^2 \cos^{2n-1} x$ with $\mathrm{d}v = \cos x \, \mathrm{d}x$. This yields $\mathrm{d}u = 2x \cos^{2n-1} x \, \mathrm{d}x$ and $v = \sin x$. This yields

$$J_n = x^2 \cos^{2n-1} x \sin x \Big|_0^{\pi/2} - \int_0^{\pi/2} 2x \cos^{2n-1} x \sin x - (2n-1)x^2 \cos^{2n-2} \sin^2 x \, \mathrm{d}x \,. \tag{6}$$

Since the first term vanishes at both boundaries, we have

$$J_n = (2n-1)\int_0^{\pi/2} x^2 \cos^{2n-2} x \, \mathrm{d}x - (2n-1)\int_0^{\pi/2} x^2 \cos^{2n} x \, \mathrm{d}x - \int_0^{\pi/2} 2x \cos^{2n-1} x \sin x \, \mathrm{d}x \tag{7}$$

$$= (2n-1)J_{n-1} - (2n-1)J_n - \int_0^{\pi/2} 2x \cos^{2n-1} x \sin x \, \mathrm{d}x \,. \tag{8}$$

Now let's take $u = 2x \cos^{2n-1} x \Rightarrow \mathrm{d}u = 2\cos^{2n-1} x + 2(2n-1)x \cos^{2n-2} x(-\sin x)$ and $\mathrm{d}v = \sin x \, \mathrm{d}x \Rightarrow v = -\cos x$, yielding

$$\begin{aligned} J_n &= (2n-1)J_{n-1} - (2n-1)J_n + 2x \cos^{2n} x \Big|_0^{\pi/2} - 2\int_0^{\pi/2} \cos x(\cos^{2n-1} x - (2n-1)x \cos^{2n-2} x \sin x) \, \mathrm{d}x \\ &= (2n-1)J_{n-1} - (2n-1)J_n - 2I_n + 2(2n-1)\int_0^{\pi/2} x \cos^{2n-1} x \sin x \, \mathrm{d}x \,. \end{aligned} \tag{9}$$

We're almost done. Now we will take $u = x \Rightarrow \mathrm{d}u = \mathrm{d}x$ and $\mathrm{d}v = \cos^{2n-1} x \sin x \,\mathrm{d}x \Rightarrow -\frac{1}{2n}\cos^{2n} x$. This brings J_n to the form

$$\begin{aligned}
J_n &= (2n-1)J_{n-1} - (2n-1)J_n - 2I_n + 2(2n-1)\left[\frac{-x\cos^{2n} x}{2n}\bigg|_0^{\pi/2} + \frac{1}{2n}\int_0^{\pi/2} \cos^{2n} x \,\mathrm{d}x\right] \\
&= (2n-1)J_{n-1} - (2n-1)J_n - 2I_n + \frac{2n-1}{n} I_n \\
&= (2n-1)J_{n-1} - (2n-1)J_n - \frac{1}{n} I_n.
\end{aligned} \tag{10}$$

Thus, we can solve for J_n to get

$$J_n = \frac{2n-1}{2n} J_{n-1} - \frac{1}{2n^2} I_n. \tag{11}$$

We can divide everything by I_n to get

$$\frac{J_n}{I_n} = \frac{2n-1}{2n}\frac{J_{n-1}}{I_n} - \frac{1}{2n^2}. \tag{12}$$

Recalling the recursive behavior we derived previously $I_n = \frac{2n-1}{2n} I_{n-1}$, we have

$$\frac{J_n}{I_n} = \frac{J_{n-1}}{I_{n-1}} - \frac{1}{2n^2} \Leftrightarrow \frac{J_{n-1}}{I_{n-1}} - \frac{J_n}{I_n} = \frac{1}{2n^2} \tag{13}$$

where we rearranged the terms to be similar to the original problem statement. Thus, plugging this expression into the limit yields

$$\lim_{m\to\infty} 2\sum_{n=1}^{m}\left(\frac{J_{n-1}}{I_{n-1}} - \frac{J_n}{I_n}\right) = \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}. \tag{14}$$

So the original problem is just a restatement of the Basel problem!