Shapiro Time Delay

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Introduction

Here we wish to provide a nice derivation of the Shapiro Time Delay. We will show here that we can solve the Einstein's equations exactly provided that the particle that experiences the time delay is massless. We follow the treatment as adopted in arXiv:1708.05716.

Conventions We use the mostly plus metric signature, i.e. $\eta_{\mu\nu} = (-,+,+,+)$ and units where $c = 1$. The reduced four dimensional Planck mass is $M_{\rm Pl} = (8\pi G)^{-1/2} \approx 2.43 \times 10^{18}\,\text{GeV}$. We use boldface letters $\mathbf{r}$ to indicate 3-vectors and x and p to denote 4-vectors. Conventions for the curvature tensors, covariant and Lie derivatives are all taken from Carroll.

First we start off with the flat space metric however, in order to do our analysis, it is convenient to work in light-cone coordinates so

$$v = \frac{t-x}{\sqrt{2}}, \quad u = \frac{t+x}{\sqrt{2}}. \tag{1}$$

So the line element in flat space becomes

$$\mathrm{d}s^2 = -2\,\mathrm{d}u\,\mathrm{d}v + \mathrm{d}\mathbf{r}^2, \tag{2}$$

and the Minkowski metric takes the form

$$\eta_{\mu\nu} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \delta_{ij} \end{pmatrix}. \tag{3}$$

Next we borrow the background metric from Kerr-Schild

$$g_{\mu\nu} = \eta_{\mu\nu} + F(u, \mathbf{r})\ell_\mu \ell_\nu, \tag{4}$$

where $\ell^\mu = (1, 0, \vec{0})$ is a covariantly constant null vector $\nabla^\mu \ell_\mu = \eta_{\mu\nu}\ell^\mu \ell^\nu = 0$ constructed to point in the v direction. Given the full metric $g_{\mu\nu}$, the line element becomes

$$\mathrm{d}s^2 = -2\,\mathrm{d}u\,\mathrm{d}v + F(u, \mathbf{r})\,\mathrm{d}u^2 + \mathrm{d}\mathbf{r}^2 \tag{5}$$

and the full metric can be written as

$$g_{\mu\nu} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & F(u, \mathbf{r}) & 0 \\ 0 & 0 & \delta_{ij} \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} F(u, \mathbf{r}) & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \delta_{ij} \end{pmatrix}. \tag{6}$$

Since we want to know how GR responds to a point particle moving at unit speed, at some momentum (say $p_v < 0$ and we use the subscripts u, v to denote which direction the vectors are parallel to), the stress energy tensor takes the form

$$T_{\mu\nu} = p_v \delta(u)\delta(\mathbf{r})\ell_\mu \ell_\nu, \tag{7}$$

where $\ell_\mu = (0, -1, \vec{0})$. This form of the stress tensor turns F into the 2 dimensional Green's function. Meaning, we can get an analytical expression for F in terms of elementary functions. Using the Fourier Transform approach, the solution to Einstein's equations becomes

$$F(u, \mathbf{r}) = -16\pi G_N \delta(u) \int \frac{e^{i\mathbf{p}\cdot\mathbf{r}}}{p^2} \frac{\mathrm{d}^2\mathbf{p}}{(2\pi)^2}. \tag{8}$$

First we'll give the integral that appears up top a name,

$$I(\mathbf{r}) = \int \frac{e^{i\mathbf{p}\cdot\mathbf{r}}}{p^2} \frac{\mathrm{d}^2\mathbf{p}}{(2\pi)^2}. \tag{9}$$

Switching to polar coordinates, the expression above becomes

$$I(r) = \int_0^\infty \int_0^{2\pi} \frac{e^{ipr\cos\theta}}{p} \frac{\mathrm{d}p\,\mathrm{d}\theta}{(2\pi)^2}, \tag{10}$$

where $r = |\mathbf{r}|$ and $p = |\mathbf{p}|$. Next we recognize that the theta integral is

$$J_0(pr) = \frac{1}{2\pi} \int_0^{2\pi} e^{ipr\cos\theta} \mathrm{d}\theta, \tag{11}$$

is merely the Bessel function of the first kind[1] . Now $I(r)$ becomes

$$I(r) = \frac{1}{2\pi} \int_0^\infty \frac{J_0(pr)}{p} \,\mathrm{d}p\,. \tag{12}$$

The integral at present is divergent, so we're going to have to be a little clever with how we carry out the next few steps. First we differentiate $I(r)$

$$I'(r) = \frac{1}{2\pi} \int_0^\infty J_0'(pr)\,\mathrm{d}p = \frac{1}{2\pi r} \int_0^\infty J_0'(s)\,\mathrm{d}s\,. \tag{13}$$

Next, taking advantage of the following properties of the Bessel function

$$J_n'(x) = \frac{1}{2}(J_{n-1}(x) - J_{n+1}(x)), \qquad J_{-n}(x) = (-1)^n J_n(x), \tag{14}$$

the integral can be written as

$$I'(r) = -\frac{1}{2\pi r} \int_0^\infty J_1(s)\,\mathrm{d}s\,. \tag{15}$$

The Bessel functions are normalized such that[2]

$$\int_0^\infty J_n(s)\,\mathrm{d}s = 1, \tag{16}$$

$I'(r)$ is shown to be

$$I'(r) = -\frac{1}{2\pi r} \Rightarrow I(r) = -\frac{1}{2\pi} \ln\left(\frac{r}{L}\right), \tag{17}$$

where L is some arbitrary length scale to make the argument of the logarithm dimensionless. Remembering we were looking for an elementary expression for F, we get

[1]See equation (71) here.

[2]See equation (74) here.

$$F(u,r) = \frac{p_v}{\pi M_{\rm Pl}^2}\delta(u)\ln\left(\frac{r}{L}\right). \tag{18}$$

Now we consider a 2nd particle that moves along the other light cone direction with momentum p_u that crosses the shock with impact parameter b. It's a little bit awkward that we have a delta function in our metric. No worries, we can remove this delta function at the point $r = b$ by making the following coordinate translation

$$v \to v + 4G_N p_v \ln\left(\frac{b}{L}\right)\Theta(u), \tag{19}$$

where $\Theta(u)$ is the Heaviside step function. Plugging this back into the line element of (2), we get

$$\mathrm{d}s^2 = -2\,\mathrm{d}u\,\mathrm{d}v + 8G_N p_v \ln\left(\frac{r}{b}\right)\delta(u)\,\mathrm{d}u^2 + \mathrm{d}\mathbf{r}^2\,. \tag{20}$$

This new line element has the benefit of the delta function disappearing at $r = b$ in this shifted coordinate system which keeps the geodesic continuous at this point. This means that the original coordinates suffers a shift Δv of

$$\Delta v = 4G_N p_v \ln\left(\frac{b}{L}\right), \tag{21}$$

and this is the Shapiro time delay.