LECTURE NO 4

14 Observer Design

Motivations :

- Some of the system states may not be measurable and thus the state feedback controller may not be realisable.
- Even if some states are measurable, more sensors will be needed to implement the state feedback controller.

The system that estimates the states of another system is called an observer or a state estimator.

Problem formulation :
Design an observer which give estimates of the system states from the inputs and measured outputs. The estimated states will be used to replace the true states in the state feedback controller.
Two main types of observers for $\mathbf{x}(k)$:
(i) Prediction observers : The predicted estimate $\bar{\mathbf{x}}(k)$ is obtained based on measurements upto $y(k-1)$.
(ii) Current observers : The current estimate $\bar{\mathbf{x}}(k)$ is obtained based on the measurements up to $y(k)$.

15. Prediction Observer
(I) Open-Loop Observers

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}u(k)$$

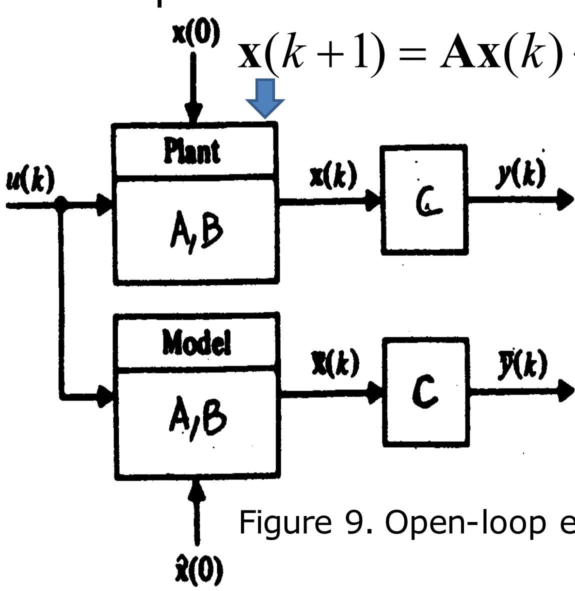

Figure 9. Open-loop estimator

$$\overline{\mathbf{x}}(k+1) = \mathbf{A}\overline{\mathbf{x}}(k) + \mathbf{B}u(k)$$

The observer is described above. The observer is implementable since $\mathbf{A}, \mathbf{B}$ and $u(k)$ are known. Let the estimation error be

$$\mathbf{x}_e(k) = \mathbf{x}(k) - \overline{\mathbf{x}}(k)$$

$$\Rightarrow \ \mathbf{x}_e(k+1) = \mathbf{x}(k+1) - \overline{\mathbf{x}}(k+1) = \mathbf{A}\mathbf{x}_e(k)$$

- The observer is only applicable to stable systems. If $\mathbf{A}$ is stable, then
  $$\mathbf{x}_e(k) \to 0 \text{ as } k \to \infty$$
- Any continuing measurements of the system's behaviour are not utilised.

(II) Closed-Loop Observers

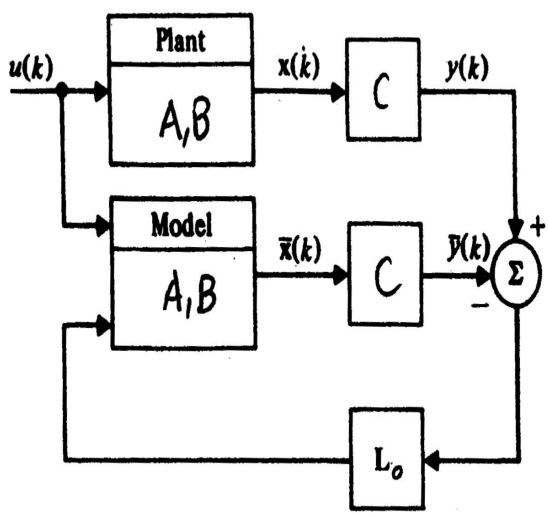

Figure 10. Closed-loop estimator

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$$\overline{\mathbf{x}}(k+1) = \mathbf{A}\overline{\mathbf{x}}(k) + \mathbf{B}u(k) + \mathbf{L}_O\left(y(k) - \mathbf{C}\overline{\mathbf{x}}(k)\right)$$
$$= \left[\mathbf{A} - \mathbf{L}_O\mathbf{C}\right]\overline{\mathbf{x}}(k) + \mathbf{B}u(k) + \mathbf{L}_O y(k) \quad ....(15.1)$$

$$\mathbf{L}_O = \left[l_1 \quad l_2 \quad \cdots \quad l_n\right]^T$$

The closed-loop observer is given in (15.1) where $\mathbf{L}_O$ is the observer gain to be determined. (15.1) is a prediction estimator. The measurements at time k results in an estimate of the states that are valid at time $k+1$. The estimation error is,

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$$\mathbf{x}_e(k) = \mathbf{x}(k) - \overline{\mathbf{x}}(k)$$
$$\Rightarrow \mathbf{x}_e(k+1) = \left[\mathbf{A} - \mathbf{L}_O\mathbf{C}\right]\mathbf{x}_e(k)$$

- The error dynamics are governed by the matrix $[\mathbf{A} - \mathbf{L}_O\mathbf{C}]$. If $\mathbf{L}_O$ is chosen such that $[\mathbf{A} - \mathbf{L}_O\mathbf{C}]$ is stable, then for any $\mathbf{x}(0)$,

  $\mathbf{x}_e(k) \rightarrow 0$ as $k \rightarrow \infty$
- The convergence speed is determined by the roots of (15.2) below, which are the observer poles.

$$\det\left[z\mathbf{I} - \mathbf{A} + \mathbf{L}_O\mathbf{C}\right] = 0 \quad ... (15.2)$$

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Suppose the desired observer poles are

$p_1^O, p_2^O, \ldots, p_n^O$

$$\Rightarrow \alpha_O(z) = \left(z - p_1^O\right)\left(z - p_2^O\right)\cdots\left(z - p_n^O\right)$$

$$= z^n + \beta_1^O z^{n-1} + \cdots + \beta_{n-1}^O z + \beta_n^O \qquad \ldots(15.3)$$

Equate

(15.2) = (15.3)

Then $\mathbf{L}_O$ can be determined.

Example 4.1

$$\mathbf{x}(k+1) = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} T^2/2 \\ T \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(k)$$

Consider the above plant in Example 3.12 where the position $x_1(k)$ is the measurement. Design an observer so that the observer poles are at

$$z_{1,2} = 0.4 \pm j0.4 \Rightarrow \alpha_O(z) = z^2 - 0.8z + 0.32$$

$$\left|z\mathbf{I}-\mathbf{A}+\mathbf{L}_O\mathbf{C}\right|$$

$$=\left|\left[z\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}-\begin{bmatrix}1 & T\\0 & 1\end{bmatrix}+\begin{bmatrix}l_1\\l_2\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix}\right]\right|$$

$$\Rightarrow z^2+\left(l_1-2\right)z+\left(Tl_2+1-l_1\right)$$

$$\equiv z^2-0.8z+0.32$$

$$\Rightarrow l_1=1.2; l_2=5.2 \quad \text{if} \quad T=0.1\ \text{sec}$$

Theorem :

$$\mathbf{x}(k+1)=\mathbf{A}\mathbf{x}(k)+\mathbf{B}u(k)$$

$$y(k)=\mathbf{C}\mathbf{x}(k)$$

Consider the above plant. Let $\alpha_O(z)$ be a polynomial of degree *n*, where *n* is the order of the system. If the system is observable, then there exists a matrix $\mathbf{L}_O$ in (15.1) such that

$$\det\left[z\mathbf{I}-\mathbf{A}+\mathbf{L}_O\mathbf{C}\right]=\alpha_O(z)$$

16 Computing $\mathbf{L}_O$ by Ackermann's Formula

The determination of $\mathbf{L}_O$ in the observer design problem is a similar mathematical problem as the determination of $\mathbf{K}$ in the pole placement problem.

In the controller design problem,

$$\mathbf{K} = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} \mathbf{W}_C^{-1} \alpha_C(\mathbf{A}) \quad ...(12.1)$$

$$\alpha_C(z) = \det \left[z\mathbf{I} - \mathbf{A} + \mathbf{BK} \right]$$

Transposing (15.2),

$$\Rightarrow \det \left[z\mathbf{I} - \mathbf{A}^T + \mathbf{C}^T \mathbf{L}_O^T \right]$$

Use the following substitutions in (12.1),

$$\mathbf{K} \to \mathbf{L}_O^T; \left\{ \mathbf{A} \to \mathbf{A}^T; \mathbf{B} \to \mathbf{C}^T \Rightarrow \mathbf{W}_C \to \mathbf{W}_O^T \right.$$

$$\Rightarrow \mathbf{L}_O^T = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} \left[\mathbf{W}_O^T \right]^{-1} \alpha_O(\mathbf{A}^T)$$

$$\Rightarrow \mathbf{L}_O = \alpha_O(\mathbf{A}) \left[\mathbf{W}_O \right]^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \qquad ...(16.1)$$

$$\alpha_O(\mathbf{A}) = \mathbf{A}^n + \beta_1^O \mathbf{A}^{n-1} + \cdots + \beta_{n-1}^O \mathbf{A} + \beta_n^O \mathbf{I} \quad ..(16.2)$$

(16.2) is the desired estimator characteristic polynomial.

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Example 4.2 Consider the plant in Example 1.4 and let the two observer poles be at 0.819.

$$\mathbf{x}(k+1) = \begin{bmatrix} 1 & 0.0952 \\ 0 & 0.905 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0.00484 \\ 0.0952 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(k)$$

$$\alpha_O(z) = \left(z - 0.819\right)^2 = z^2 - 1.638z + 0.671$$

$$\Rightarrow \alpha_O(\mathbf{A}) = \mathbf{A}^2 - 1.638\mathbf{A} + 0.671\mathbf{I}_2$$

$$= \begin{bmatrix} 0.033 & 0.0254 \\ 0 & 0.00763 \end{bmatrix}$$

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$$\mathbf{W}_O = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix} \Rightarrow \left[\mathbf{W}_O\right]^{-1} = \begin{bmatrix} 1 & 0 \\ -10.51 & 10.51 \end{bmatrix}$$

$$\Rightarrow \mathbf{L}_O = \alpha_O(\mathbf{A}) \left[\mathbf{W}_O\right]^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.267 \\ 0.0802 \end{bmatrix}$$

$$\Rightarrow \overline{\mathbf{x}}(k+1) = \left[\mathbf{A} - \mathbf{L}_O\mathbf{C}\right] \overline{\mathbf{x}}(k) + \mathbf{B}u(k) + \mathbf{L}_O y(k)$$

$$= \begin{bmatrix} 0.733 & 0.0952 \\ -0.0802 & 0.905 \end{bmatrix} \overline{\mathbf{x}}(k)$$

$$+ \begin{bmatrix} 0.00484 \\ 0.0952 \end{bmatrix} u(k) + \begin{bmatrix} 0.267 \\ 0.0802 \end{bmatrix} y(k)$$

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Example 4.3

$$\mathbf{x}(k+1) = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} T^2/2 \\ T \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(k)$$

Consider the above plant in Example 3.12. Design an observer to give deadbeat response in the error vector.
The estimation error is given by

$$\mathbf{x}_e(k+1) = \left[\mathbf{A} - \mathbf{L}_O \mathbf{C}\right] \mathbf{x}_e(k)$$

$$\det\left[z\mathbf{I} - \mathbf{A} + \mathbf{L}_O \mathbf{C}\right]$$

$$= \det\left[z\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix} \right]$$

$$\Rightarrow z^2 + \left(l_1 - 2\right) z + \left(Tl_2 + 1 - l_1\right) \equiv z^2$$

$$\Rightarrow l_1 = 2; l_2 = 1/T \quad \Rightarrow \mathbf{L}_O = \begin{bmatrix} 2 \\ 1/T \end{bmatrix}$$

To verify deadbeat error response, let

$$\mathbf{x}(0) = \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}; \overline{\mathbf{x}}(0) = \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}$$

$$\Rightarrow \mathbf{x}_e(0) = \mathbf{x}(0) - \overline{\mathbf{x}}(0) = \begin{bmatrix} a_1 - a_2 \\ b_1 - b_2 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\mathbf{x}_e(k+1) = \left[\mathbf{A} - \mathbf{L}_O\mathbf{C}\right]\mathbf{x}_e(k)$$

$$\mathbf{x}_e(1) = \begin{bmatrix} -1 & T \\ -1/T & 1 \end{bmatrix}\mathbf{x}_e(0) = \begin{bmatrix} -a + bT \\ -a/T + b \end{bmatrix}$$

$$\mathbf{x}_e(2) = \begin{bmatrix} -1 & T \\ -1/T & 1 \end{bmatrix}\mathbf{x}_e(1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Hence, the error response is deadbeat.
The observer equation is

$$\overline{\mathbf{x}}(k+1) = \left[\mathbf{A} - \mathbf{L}_O\mathbf{C}\right]\overline{\mathbf{x}}(k) + \mathbf{B}u(k) + \mathbf{L}_O y(k)$$

$$= \begin{bmatrix} -1 & T \\ -1/T & 1 \end{bmatrix}\overline{\mathbf{x}}(k) + \begin{bmatrix} T^2/2 \\ T \end{bmatrix}u(k)$$

$$+ \begin{bmatrix} 2 \\ 1/T \end{bmatrix}y(k)$$

17 Reduced-order Observers

Motivations :

- Some states can be directly obtained from measurements and thus not necessary to be estimated.

- Therefore, a lower order observer can be designed to estimate those states which are not available for measurement.

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}u(k)$$

$$\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k)$$

Consider the above plant and partitioned the state vector as $\mathbf{x}(k) = \begin{bmatrix} \mathbf{x}_a(k) \\ \mathbf{x}_b(k) \end{bmatrix}$ where

$\mathbf{x}_a(k)$ - measurable part, i.e. $\mathbf{y}(k)$

$\mathbf{x}_b(k)$ - unmeasurable part, the remaining states.

Decomposed the state space model accordingly:

$$\begin{bmatrix} \mathbf{x}_a(k+1) \\ \mathbf{x}_b(k+1) \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{aa} & \mathbf{A}_{ab} \\ \mathbf{A}_{ba} & \mathbf{A}_{bb} \end{bmatrix} \begin{bmatrix} \mathbf{x}_a(k) \\ \mathbf{x}_b(k) \end{bmatrix} + \begin{bmatrix} \mathbf{B}_a \\ \mathbf{B}_b \end{bmatrix} \mathbf{u}(k) \qquad ...(17.1)$$

$$\mathbf{y}(k) = \begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x}_a(k) \\ \mathbf{x}_b(k) \end{bmatrix} \qquad ...(17.2)$$

The following equation described $\mathbf{x}_b(k)$. The term $[\mathbf{A}_{ba}\mathbf{x}_a(k) + \mathbf{B}_b\mathbf{u}(k)]$ can be considered as "known inputs" into the $\mathbf{x}_b(k)$ dynamics.

$$\mathbf{x}_b(k+1) = \mathbf{A}_{bb}\mathbf{x}_b(k) + \left[\mathbf{A}_{ba}\mathbf{x}_a(k) + \mathbf{B}_b\mathbf{u}(k)\right]$$

The following equation described $\mathbf{x}_a(k)$. The term $[\mathbf{x}_a(k+1) - \mathbf{A}_{aa}\mathbf{x}_a(k) - \mathbf{B}_a\mathbf{u}(k)]$ are "known measurements" and can be regarded as "outputs" from the $\mathbf{x}_b(k)$ dynamics.

$$\left[\mathbf{x}_a(k+1) - \mathbf{A}_{aa}\mathbf{x}_a(k) - \mathbf{B}_a\mathbf{u}(k)\right] = \mathbf{A}_{ab}\mathbf{x}_b(k)$$

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Recall that the full-order observer is

$$\overline{\mathbf{x}}(k+1) = \mathbf{A}\overline{\mathbf{x}}(k) + \mathbf{B}\mathbf{u}(k) + \mathbf{L}_O\left(\mathbf{y}(k) - \mathbf{C}\overline{\mathbf{x}}(k)\right) \quad ...(15.1)$$

$$\begin{cases} \mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k) \\ \mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) \end{cases}$$

$$\begin{cases} \mathbf{x}_b(k+1) = \mathbf{A}_{bb}\mathbf{x}_b(k) + \left[\mathbf{A}_{ba}\mathbf{x}_a(k) + \mathbf{B}_b\mathbf{u}(k)\right] \\ \left[\mathbf{x}_a(k+1) - \mathbf{A}_{aa}\mathbf{x}_a(k) - \mathbf{B}_a\mathbf{u}(k)\right] = \mathbf{A}_{ab}\mathbf{x}_b(k) \end{cases}$$

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$$\begin{array}{ccc} \mathbf{x}_b(k) & \rightarrow & \mathbf{x}(k) \\ \mathbf{A}_{bb} & \rightarrow & \mathbf{A} \\ \left[\mathbf{A}_{ba}\mathbf{x}_a(k) + \mathbf{B}_b\mathbf{u}(k)\right] & \rightarrow & \mathbf{B}\mathbf{u}(k) \\ \left[\mathbf{x}_a(k+1) - \mathbf{A}_{aa}\mathbf{x}_a(k) - \mathbf{B}_a\mathbf{u}(k)\right] & \rightarrow & \mathbf{y}(k) \\ \mathbf{A}_{ab} & \rightarrow & \mathbf{C} \end{array}$$

The reduced-order observer to estimate $\mathbf{x}_b(k)$ can be obtained by making the above substitutions into (15.1).

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The reduced-order observer is then given by

$$\overline{\mathbf{x}}_b(k+1) = \mathbf{A}_{bb}\overline{\mathbf{x}}_b(k) + \left[\mathbf{A}_{ba}\mathbf{x}_a(k) + \mathbf{B}_b\mathbf{u}(k)\right] +$$

$$\mathbf{L}_r\left\{\left[\mathbf{x}_a(k+1) - \mathbf{A}_{aa}\mathbf{x}_a(k) - \mathbf{B}_a\mathbf{u}(k)\right] - \mathbf{A}_{ab}\overline{\mathbf{x}}_b(k)\right\} \quad ...(17.3)$$

$$\mathbf{y}(k) = \mathbf{x}_a(k)$$

$$\Rightarrow \overline{\mathbf{x}}_b(k+1) = \left[\mathbf{A}_{bb} - \mathbf{L}_r\mathbf{A}_{ab}\right]\overline{\mathbf{x}}_b(k) + \mathbf{L}_r\mathbf{y}(k+1) +$$

$$\left[\mathbf{A}_{ba} - \mathbf{L}_r\mathbf{A}_{aa}\right]\mathbf{y}(k) + \left[\mathbf{B}_b - \mathbf{L}_r\mathbf{B}_a\right]\mathbf{u}(k) \quad ...(17.4)$$

$$\mathbf{x}_{b_e}(k+1) = \left[\mathbf{A}_{bb} - \mathbf{L}_r\mathbf{A}_{ab}\right]\mathbf{x}_{b_e}(k) \quad ...(17.5)$$

$$\det\left[z\mathbf{I} - \mathbf{A}_{bb} + \mathbf{L}_r\mathbf{A}_{ab}\right] = \alpha_O(z) \quad ...(17.6)$$

The error equation is given by (17.5) where $\mathbf{L}_r$ is the reduced-order observer gain designed according to (17.6). If $y(k)$ is a scalar, Ackermann's formula from (16.2) :

$$\mathbf{L}_r = \alpha_O(\mathbf{A}_{bb})\begin{bmatrix} \mathbf{A}_{ab} \\ \mathbf{A}_{ab}\mathbf{A}_{bb} \\ \vdots \\ \mathbf{A}_{ab}\mathbf{A}_{bb}{}^{n-2} \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \quad ...(17.7)$$

Example 4.4 Consider the Example 3.12 for T = 0.1 s.

$$\mathbf{x}(k+1) = \begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0.05 \\ 0.1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(k)$$

Let $\mathbf{x}(k) = \begin{bmatrix} x_a(k) \\ x_b(k) \end{bmatrix}$ where

$x_a(k)$ - measured position of $y(k)$

$x_b(k)$ - velocity to be estimated

$$\Rightarrow \begin{bmatrix} A_{aa} & A_{ab} \\ A_{ba} & A_{bb} \end{bmatrix} = \begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} B_a \\ B_b \end{bmatrix} = \begin{bmatrix} 0.05 \\ 0.1 \end{bmatrix}$$

The observer gain l_r is a scalar.

Let the only estimator pole to be at $z = 0.5$.

$$(17.6) \Rightarrow z - 1 + 0.1 l_r \equiv z - 0.5 \Rightarrow l_r = 5$$

$$(17.4) \Rightarrow \overline{x}_b(k+1) = 0.5\overline{x}_b(k) + 5y(k+1) \\ - 5y(k) - 0.15u(k)$$

18 Current Observers

The full-order estimator in (15.1) is a prediction observer, since the estimate $\bar{\mathbf{x}}(k)$ is based on $y(k-1)$.

A current estimator estimates the states at k^{th} instant using the measurements obtained at the k^{th} instant.

Consider the plant,

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}u(k)$$

$$y(k) = \mathbf{C}\mathbf{x}(k)$$

$$\hat{\mathbf{x}}(k+1) = \mathbf{A}\bar{\mathbf{x}}(k) + \mathbf{B}u(k) \qquad ...(18.1)$$

$$\bar{\mathbf{x}}(k+1) = \hat{\mathbf{x}}(k+1) + \mathbf{L}_C\left(y(k+1) - \mathbf{C}\hat{\mathbf{x}}(k+1)\right) \qquad ...(18.2)$$

One form of current observer is given by (18.1) and (18.2). In (18.1), $\hat{\mathbf{x}}(k+1)$ is the predicted estimate of the state at the $(k+1)^{th}$ instant, based on the dynamics of the system and on the signals at the k^{th} instant. In (18.2), the estimate $\hat{\mathbf{x}}(k+1)$ is corrected when the measurement $y(k+1)$ arrives at the $(k+1)^{th}$ instant.

$$\bar{\mathbf{x}}(k+1) = \left[\mathbf{A} - \mathbf{L}_C\mathbf{C}\mathbf{A}\right]\bar{\mathbf{x}}(k) + \left[\mathbf{B} - \mathbf{L}_C\mathbf{C}\mathbf{B}\right]u(k) + \mathbf{L}_C y(k+1) \qquad ...(18.3)$$

Substitute (18.1) into (18.2), gives (18.3). The estimator gain $\mathbf{L}_C$ determines the weight placed on the difference between the measurement at the $(k+1)^{th}$ instant and what we expect the measurement to be at that time. The final estimate is $\bar{\mathbf{x}}(k+1)$ and the estimation error is,

$$\mathbf{x}_e(k+1) = \mathbf{x}(k+1) - \bar{\mathbf{x}}(k+1)$$
$$= \left[\mathbf{A} - \mathbf{L}_C\mathbf{C}\mathbf{A}\right]\mathbf{x}_e(k)$$

The characteristic polynomial for the prediction observer is det $[z\mathbf{I} - \mathbf{A} + \mathbf{L}_O\mathbf{C}]$ while the characteristic polynomial for the current observer is det $[z\mathbf{I} - \mathbf{A} + \mathbf{L}_C\mathbf{C}\mathbf{A}]$.
In (16,1) replace $\mathbf{C}$ with $\mathbf{CA}$,

$$\mathbf{L}_C = \alpha_O(\mathbf{A})\begin{bmatrix} \mathbf{CA} \\ \mathbf{CA}^2 \\ \vdots \\ \mathbf{CA}^n \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \qquad ...(18.4)$$

Example 4.5 Consider the servo motor in Example 1.4. Using the same observer characteristic polynomial as in Example 4.2,

$$\mathbf{x}(k+1) = \begin{bmatrix} 1 & 0.0952 \\ 0 & 0.905 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0.00484 \\ 0.0952 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(k)$$

$$\alpha_O(z) = z^2 - 1.638z + 0.671$$

$$\Rightarrow \alpha_O(\mathbf{A}) = \begin{bmatrix} 0.033 & 0.0254 \\ 0 & 0.00763 \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{CA} \\ \mathbf{CA}^2 \end{bmatrix}^{-1} = \begin{bmatrix} 2.104 & -1.104 \\ -11.6 & 11.6 \end{bmatrix}$$

$$\Rightarrow \mathbf{L}_C = \alpha_O(\mathbf{A}) \begin{bmatrix} \mathbf{CA} \\ \mathbf{CA}^2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.258 \\ 0.0885 \end{bmatrix}$$

$$(18.3) \;\Rightarrow\; \overline{\mathbf{x}}(k+1) = \begin{bmatrix} 0.742 & 0.0706 \\ -0.0885 & 0.897 \end{bmatrix} \overline{\mathbf{x}}(k) + \begin{bmatrix} 0.00359 \\ 0.0948 \end{bmatrix} u(k) + \begin{bmatrix} 0.258 \\ 0.0885 \end{bmatrix} y(k+1)$$

19.1 Combined control law and observer. Using the observed states to replace the true states in the state feedback controller $u(k) = -\mathbf{K}\bar{\mathbf{x}}(k)$. See Figure 7.

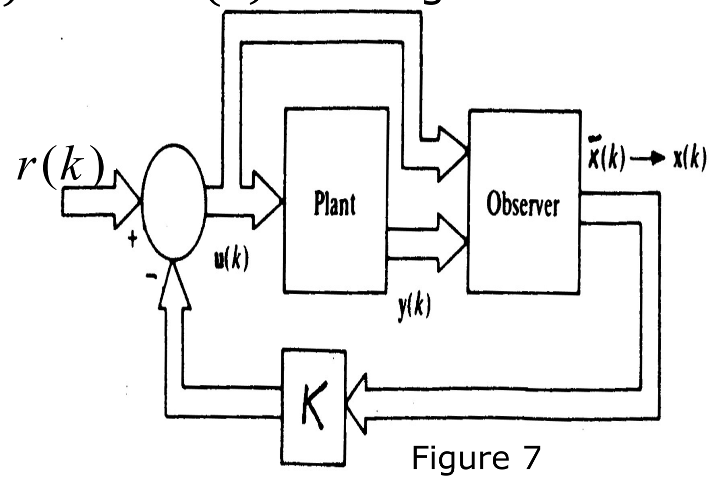


Figure 7

The poles of the closed-loop system are located at

$$p_1, p_2, \cdots, p_n \quad \cup \quad p_1^O, p_2^O, \cdots, p_n^O$$

i.e. the combination of the controller poles and estimator poles.

The characteristic polynomial of the complete control system is $\alpha_C(z)\alpha_O(z)$.

This is the ***separation principle*** by which a controller and an observer can be designed separately and yet used together.

The closed-loop system and the error equation are given by

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}u(k)$$

$$= \mathbf{A}\mathbf{x}(k) - \mathbf{B}\mathbf{K}\left[\mathbf{x}(k) - \mathbf{x}_e(k)\right]$$

$$\mathbf{x}_e(k+1) = \left[\mathbf{A} - \mathbf{L}_O\mathbf{C}\right]\mathbf{x}_e(k)$$

$$\Rightarrow \begin{bmatrix} \mathbf{x}(k+1) \\ \mathbf{x}_e(k+1 \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{B}\mathbf{K} & \mathbf{B}\mathbf{K} \\ \mathbf{0} & \mathbf{A} - \mathbf{L}_O\mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{x}(k) \\ \mathbf{x}_e(k) \end{bmatrix}$$

$$\Rightarrow \det\left[z\mathbf{I} - \begin{bmatrix} \mathbf{A} - \mathbf{B}\mathbf{K} & \mathbf{B}\mathbf{K} \\ \mathbf{0} & \mathbf{A} - \mathbf{L}_O\mathbf{C} \end{bmatrix} \right]$$

$$= \det\begin{bmatrix} z\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K} & -\mathbf{B}\mathbf{K} \\ \mathbf{0} & z\mathbf{I} - \mathbf{A} + \mathbf{L}_O\mathbf{C} \end{bmatrix}$$

$$= \left(\det\left[z\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K}\right]\right)\det\left[z\mathbf{I} - \mathbf{A} + \mathbf{L}_O\mathbf{C}\right]$$

$$= \left(\alpha_C(z)\right)\alpha_O(z)$$

Guidelines for choosing $\alpha_C(z)$ and $\alpha_O(z)$:

1. Choose the roots of $\alpha_C(z)$ to satisfy the performance specifications and actuator limitations.

2. Choose the roots of $\alpha_O(z)$ faster (by a factor of 2 to 4 times) so that the total response is dominated by the response of the poles in $\alpha_C(z)$.

Example 4.6 The regulator for the following plant was designed in Example 3.8 and the state observer was designed in Example 4.2. In the controller, the true states are replaced by the estimated states from the observer.

$$\mathbf{x}(k+1) = \begin{bmatrix} 1 & 0.0952 \\ 0 & 0.905 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0.00484 \\ 0.0952 \end{bmatrix} u(k) \qquad ...(19.1)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(k) \qquad ...(19.2)$$

$$\bar{\mathbf{x}}(k+1) = \begin{bmatrix} 0.733 & 0.0952 \\ -0.0802 & 0.905 \end{bmatrix} \bar{\mathbf{x}}(k) + \begin{bmatrix} 0.00484 \\ 0.0952 \end{bmatrix} u(k) + \begin{bmatrix} 0.267 \\ 0.0802 \end{bmatrix} y(k)$$

$$u(k) = -\begin{bmatrix} 4.52 & 1.12 \end{bmatrix} \bar{\mathbf{x}}(k) \qquad \text{...(19.3)}$$

$$\Rightarrow \bar{\mathbf{x}}(k+1) = \begin{bmatrix} 0.711 & 0.0898 \\ -0.510 & 0.798 \end{bmatrix} \bar{\mathbf{x}}(k) + \begin{bmatrix} 0.267 \\ 0.0802 \end{bmatrix} y(k) \qquad \text{...(19.4)}$$

(19.1), (19.2), (19.3) and (19.4) $\Rightarrow$

$$\mathbf{x}(k+1) = \begin{bmatrix} 1 & 0.0952 \\ 0 & 0.905 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} -0.02188 & -0.00542 \\ -0.4303 & -0.1066 \end{bmatrix} \bar{\mathbf{x}}(k)$$

$$\bar{\mathbf{x}}(k+1) = \begin{bmatrix} 0.267 & 0 \\ 0.0802 & 0 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0.711 & 0.0898 \\ -0.510 & 0.798 \end{bmatrix} \bar{\mathbf{x}}(k)$$

The closed-loop state equation and the closed-loop poles are :

$$\begin{bmatrix} \mathbf{x}(k+1) \\ \overline{\mathbf{x}}(k+1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0.0952 & -0.02188 & -0.00542 \\ 0 & 0.905 & -0.4303 & -0.1066 \\ 0.267 & 0 & 0.711 & 0.0898 \\ 0.0802 & 0 & -0.510 & 0.798 \end{bmatrix} \begin{bmatrix} \mathbf{x}(k) \\ \overline{\mathbf{x}}(k) \end{bmatrix}$$

$$\Rightarrow \lambda_i \left[\mathbf{A}_{cl} \right] = 0.888 \pm j0.173; 0.819; 0.819$$

Example 4.7

$$\mathbf{x}(k+1) = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} T^2/2 \\ T \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(k)$$

If T = 0.2 s, determine the state feedback gain $\mathbf{K}$ such that the closed-loop poles are at $z_{1,2} = 0.6 \pm j0.4$. If $y(k) = x_1(k)$ is the only state variable that can be measured, design a minimum-order observer to give deadbeat error response.

Determine the transfer function of the combined controller-estimator (compensator) transfer function $\frac{U(z)}{Y(z)}$.

For T = 0.2 s, $\mathbf{W}_C$ and $\mathbf{W}_O$ are non-singular.

$$\mathbf{A} = \begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix}; \mathbf{B} = \begin{bmatrix} 0.02 \\ 0.2 \end{bmatrix}; \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$\Rightarrow \det\left[z\mathbf{I} - \mathbf{A} + \mathbf{BK}\right] = \alpha(z)$$

$$= z^2 + \left(0.02k_1 + 0.2k_2 - 2\right)z + \left(0.02k_1 - 0.2k_2 + 1\right)$$

$$z_{1,2} = 0.6 \pm j0.4 \Rightarrow \alpha_C(z) = z^2 - 1.2z + 0.52$$

$$\alpha(z) \equiv \alpha_C(z) \Rightarrow \begin{cases} 0.02k_1 + 0.2k_2 - 2 = -1.2 \\ 0.02k_1 - 0.2k_2 + 1 = 0.52 \end{cases}$$

$$\Rightarrow k_1 = 8; \quad k_2 = 3.2$$

$$\Rightarrow u(k) = -\begin{bmatrix} 8 & 3.2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ \bar{x}_2(k) \end{bmatrix} = -\begin{bmatrix} 8 & 3.2 \end{bmatrix} \begin{bmatrix} y(k) \\ \bar{x}_2(k) \end{bmatrix} \quad ...(19.5)$$

$$\begin{bmatrix} A_{aa} & A_{ab} \\ A_{ba} & A_{bb} \end{bmatrix} = \begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} B_a \\ B_b \end{bmatrix} = \begin{bmatrix} 0.02 \\ 0.2 \end{bmatrix}$$

For deadbeat error response, $\alpha_O(z) = z$

$$(17.5) \Rightarrow \det\left[z - A_{bb} + l_r A_{ab}\right] = z$$

$$z - 1 + 0.2 l_r = z \Rightarrow l_r = 5$$

$$(17.4) \Rightarrow \overline{x}_b(k+1) = 5y(k+1) - 5y(k) + 0.1u(k) \qquad ...(19.6)$$

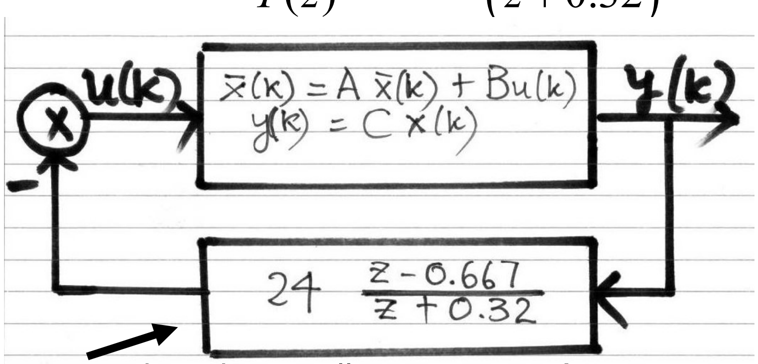
(19.5) and (19.6) and $\bar{x}_b(k) = \bar{x}_2(k)$

$$u(k+1) = -8y(k+1) - 3.2\overline{x}_2(k+1)$$
$$= -24y(k+1) + 16y(k) - 0.32u(k)$$

Take z-transform and zero IC,

$$zU(z) + 0.32U(z) = -24zY(z) + 16Y(z)$$

$$\Rightarrow \frac{U(z)}{Y(z)} = -24\frac{(z - 0.667))}{(z + 0.32)}$$



Combined controller-estimator (compensator) transfer function

19.2 Transfer function of the combined controller and current estimator.

Take z-transform of (18.3) with zero IC,

$$\bar{\mathbf{x}}(k+1) = \left[\mathbf{A} - \mathbf{L}_C\mathbf{CA}\right]\bar{\mathbf{x}}(k) + \left[\mathbf{B} - \mathbf{L}_C\mathbf{CB}\right]u(k) + \mathbf{L}_C y(k+1)$$

$$z\bar{\mathbf{X}}(z) = \left[\mathbf{A} - \mathbf{L}_C\mathbf{CA}\right]\bar{\mathbf{X}}(z) + \left[\mathbf{B} - \mathbf{L}_C\mathbf{CB}\right]U(z) + \mathbf{L}_C zY(z)$$

$$u(k) = -\mathbf{K}\bar{\mathbf{x}}(k) \Rightarrow U(z) = -\mathbf{K}\bar{\mathbf{X}}(z)$$

$$\Rightarrow z\bar{\mathbf{X}}(z) = \left[\mathbf{A} - \mathbf{L}_C\mathbf{CA}\right]\bar{\mathbf{X}}(z) + \left[\mathbf{B} - \mathbf{L}_C\mathbf{CB}\right]\left[-\mathbf{K}\bar{\mathbf{X}}(z)\right] + \mathbf{L}_C zY(z)$$

$$\Rightarrow \left[z\mathbf{I} - \mathbf{A} + \mathbf{L}_C\mathbf{CA} + \mathbf{BK} - \mathbf{L}_C\mathbf{CBK}\right]\bar{\mathbf{X}}(z) = \mathbf{L}_C zY(z)$$

$$\Rightarrow \frac{U(z)}{Y(z)} = -z\mathbf{K}\left[z\mathbf{I} - \mathbf{A} + \mathbf{L}_C\mathbf{CA} + \mathbf{BK} - \mathbf{L}_C\mathbf{CBK}\right]^{-1}\mathbf{L}_C$$

20 System with inputs

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}u(k)$$
$$y(k) = \mathbf{C}\mathbf{x}(k)$$

A control law of the following form is to be implemented. $r(k)$ is a scalar system input and k_r is a constant.

$$u(k) = -\mathbf{K}\mathbf{x}(k) + k_r r(k)$$

The closed-loop system transfer function,

$$\mathbf{x}(k+1) = \left[\mathbf{A} - \mathbf{B}\mathbf{K}\right]\mathbf{x}(k) + k_r\mathbf{B}r(k)$$
$$y(k) = \mathbf{C}\mathbf{x}(k)$$

$$\Rightarrow \frac{Y(z)}{R(z)} = k_r\mathbf{C}\left[z\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K}\right]^{-1}\mathbf{B}$$

k_r do not affect locations of the closed-loop poles.

The design of $\mathbf{K}$ proceeds as before.

Choose k_r, e.g., to satisfy DC gain requirements.

Example 4.8

$$\mathbf{x}(k+1) = \begin{bmatrix} 1 & 0.0952 \\ 0 & 0.905 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0.00484 \\ 0.0952 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(k)$$

Design a control law of the form

$$u(k) = -\mathbf{Kx}(k) + k_r r(k)$$

such that the closed-loop poles are at $0.888 \pm j0.173$ and the final value of $y(k)$ is unity for a unit-step $r(k)$.

From Example 3.8, we have

$$\mathbf{K} = \begin{bmatrix} 4.52 & 1.12 \end{bmatrix}$$

$$\frac{Y(z)}{R(z)} = k_r \mathbf{C} \left[z\mathbf{I} - \mathbf{A} + \mathbf{BK} \right]^{-1} \mathbf{B}$$

$$\Rightarrow Y(z) = k_r \left(\frac{0.00484z + 0.00468}{z^2 - 1.776z + 0.819} \right) R(z)$$

The final value theorem gives

$$\lim_{k \to \infty} y(k) = \lim_{z \to 1} \left(z - 1 \right) Y(z)$$

$$\Rightarrow \left(0.221 k_r = 1 \right) \Rightarrow k_r = 4.52$$

21 Integral Control

The idea of integral control by state augmentation is to add an integrator so as to obtain an integral of the error signal. Integral control is to eliminate steady-state error due to constant reference input inputs. One way to introduce an integrator in the mathematical model of a closed-loop system is to introduce a new state vector that integrates the difference between the command vector $r(k)$ and output vector $y(k)$. See Figure 11.

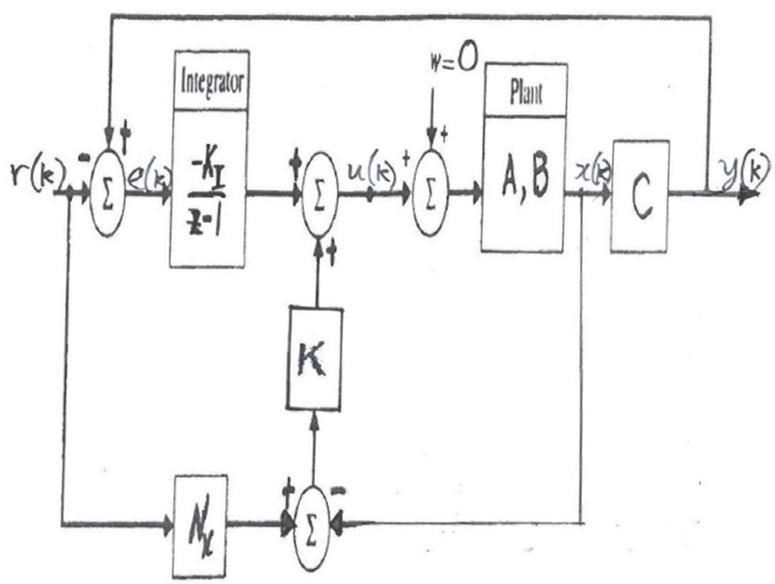


Figure 11. Block diagram for integral control with full state feedback

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$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}u(k)$$
$$y(k) = \mathbf{C}\mathbf{x}(k)$$

The plant is given as above. Augment the state with $x_I(k)$, the integral of the error

$$e(k) = y(k) - r(k)$$

From Figure 11,

$$x_I(k+1) = x_I(k) + e(k)$$
$$= x_I(k) + \mathbf{C}\mathbf{x}(k) - r(k)$$

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$$\Rightarrow \begin{bmatrix} \mathbf{x}(k+1) \\ x_I(k+1) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & 0 \\ \mathbf{C} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x}(k) \\ x_I(k) \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} u(k)$$
$$- \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} r(k)$$

$$u(k) = -\begin{bmatrix} \mathbf{K} & K_I \end{bmatrix} \begin{bmatrix} \mathbf{x}(k) \\ x_I(k) \end{bmatrix} + \mathbf{K}\mathbf{N}_x r(k)$$

Apply pole placement techniques for systems with inputs.

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