Lecture 1: Introduction - Basic Ideas and Terminology

a) Introduction

b) Performance Specifications

c) Open Loop and Closed-Loop Systems

d) Control System Terminology

e) Control and System Engineering Problems

a) Introduction

Control is an area dealing with physical systems or processes in terms of:

- **Performance Specifications**
- **Modelling** *(to build mathematical models)*
- **Analysis** *(to understand the behaviour)*
- **Control** *(to change the behaviour)*

Applications of control technology can be found in simple feedback amplifiers to complex industrial/engineering systems.

Application Examples:

- Auto-pilots of an aircraft
- Bio-electronic devices, e.g. prosthetics (artificial limbs, heart)
- Motor position and speed control, e.g. solar tracking systems
- Material handling control, e.g. cranes
- Robotics, e.g. manipulators etc…..

The objective of this course is to enable students to **model**, **analyze** and **control** engineering systems.

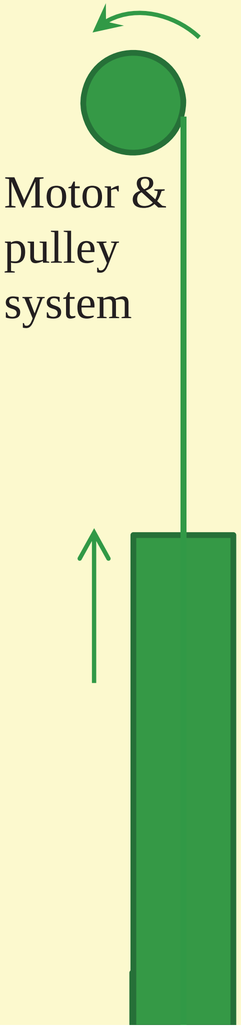
So that given a system (a robot, an elevator, a disk drive), you will be able to model the behaviour, perform an analysis, and then design controller so that the system behaves in a certain prescribed way.

Why are Control Systems important?

- Provide optimal performance of dynamic systems.
- Improve quality and lower production cost.
- Relieve the drudgery and monotony of routine, repetitive operations.
- Overcome limitations of manual operation because of hazardous conditions, power and/or speed limitations.

b) Performance Specifications

Let's take the example of an elevator. You want the following to happen:



1. Get to your destination (***desired floor***) as fast as possible. However, you want a smooth ride so that you don't get sick in the elevator.
2. You don't want the elevator to be bouncing up and down before it settles to the desired floor. For example, if your upward destination is the 4th floor, you don't want the elevator to go to the 5th floor first then comes back to the 4th floor (i.e. ***no overshoot***).
3. You want the elevator to stop at about the same level as the floor, otherwise you might trip and fall (i.e. you can only tolerate a very small ***steady state error***).

4. You want the speed of the elevator to be the about the same regardless of the load. That is, you want the elevator to take about the same time to travel from the 1^{st} floor to the 10^{th} floor, whether you have 1 or 10 passengers in the elevator.

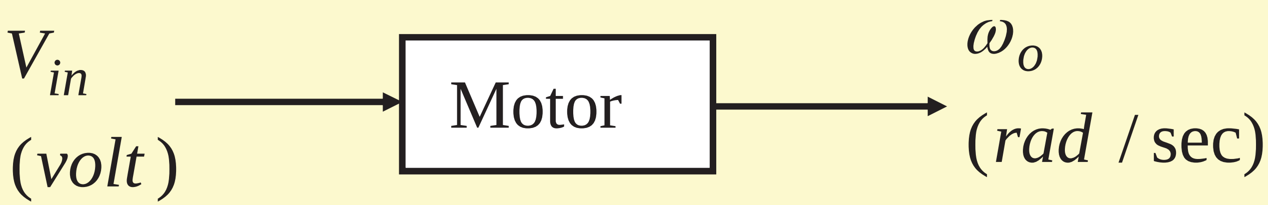
All these requirements are called **performance specifications**. A control engineer will have to ensure that their control systems meet the desired specifications, by implementing controllers to change the systems' behaviour, if necessary.

Q: Can you list similar specifications for a fighter aircraft or a disk drive control system?

c) Open Loop and Closed-loop Systems

Open loop systems are control systems in which the output has no control/effect on the control action (i.e. the output is not measured and fed back for comparison with input)

Consider the electrical motor:
(e.g. in the case of an electric fan)



The **output** speed ω_o is a function of the **input** voltage for a given load. An increase in V_{in} causes an increase in ω_o. However ω_o has no influence on V_{in}.

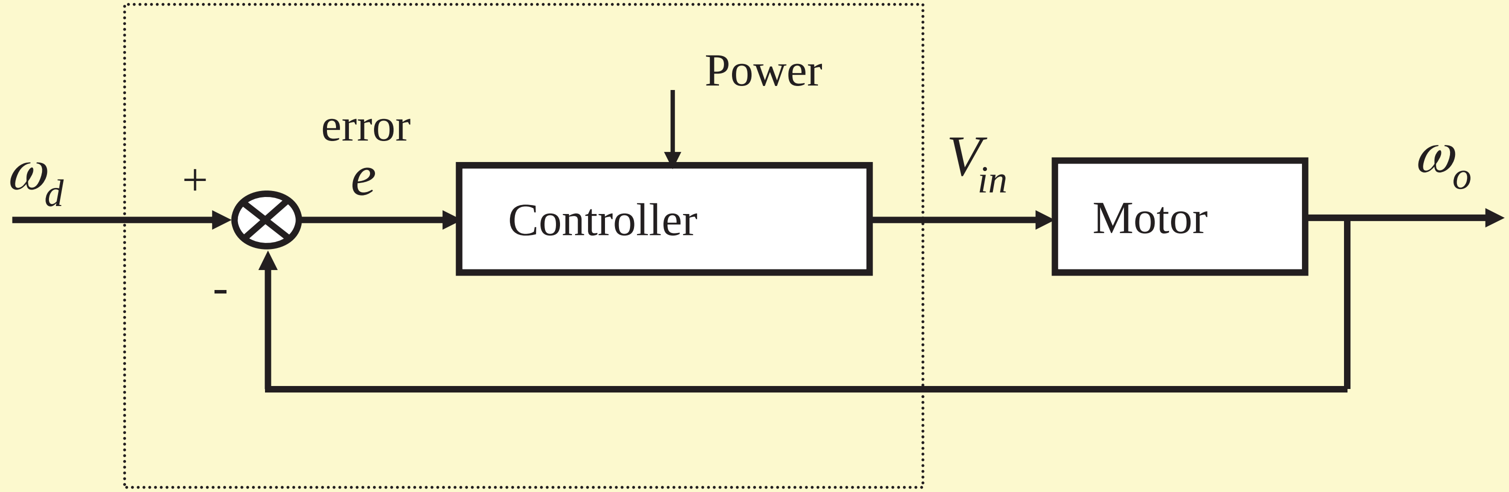
V_{in} can be calibrated in terms of the motor speed for any given load, and provided the load remains constant, a suitable V_{in} setting will give the predicted ω_o.

If the load varies, then ω_o will vary for the same setting of V_{in}. In this case an operator is required to alter V_{in} until ω_o achieves its desired value.

Two outstanding features of open loop control systems:

1. **The ability to perform accurately is determined by calibration.**
2. **Generally not troubled with problems of instability. (The system must be stable in the first place.) The concept of stability will be discussed in later section.**

In contrast, ***closed-loop*** systems make use of output information for feedback.



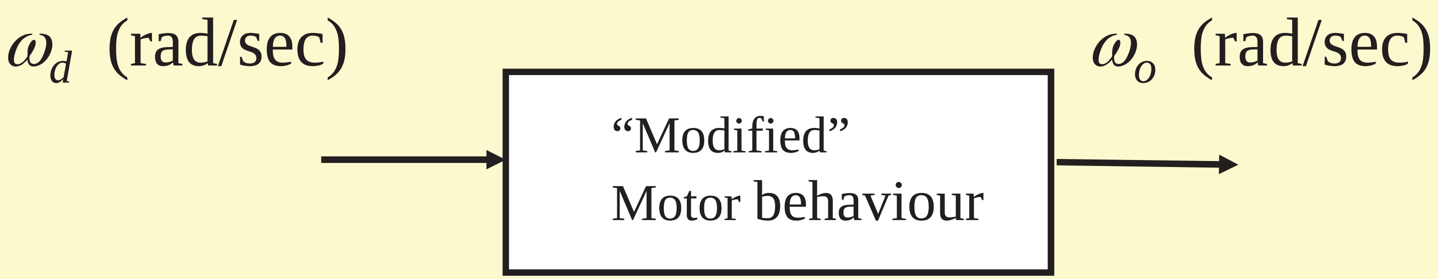
Clearly, the functioning of the system is directly dependent on the result of its action. A comparator is used to detect the error between the desired speed ω_d (the ***reference input***) and the actual speed ω_o (the ***output***) .

A controller is used to give an output that is equal to that required for the desired speed at initial conditions ($\omega_o = \omega_d$).

If $\omega_o = \omega_d$, the error e ($= \omega_d - \omega_o$) is zero and the voltage fed to the motor is that required to give $\omega_o = \omega_d$.

Assume that ω_d is constant. Suppose that ω_o changes due to changes in the load, an error will be generated. This error will drive the controller to produce a V_{in} such that ω_o is changed to reduce the error to zero. Similar arguments can also be made when ω_d is changing with time.

"Closing the loop" enables automatic control to be effected. Thus, we are able to change the input-output behaviour of the system. In the case of motor control, the "motor system" is now given by

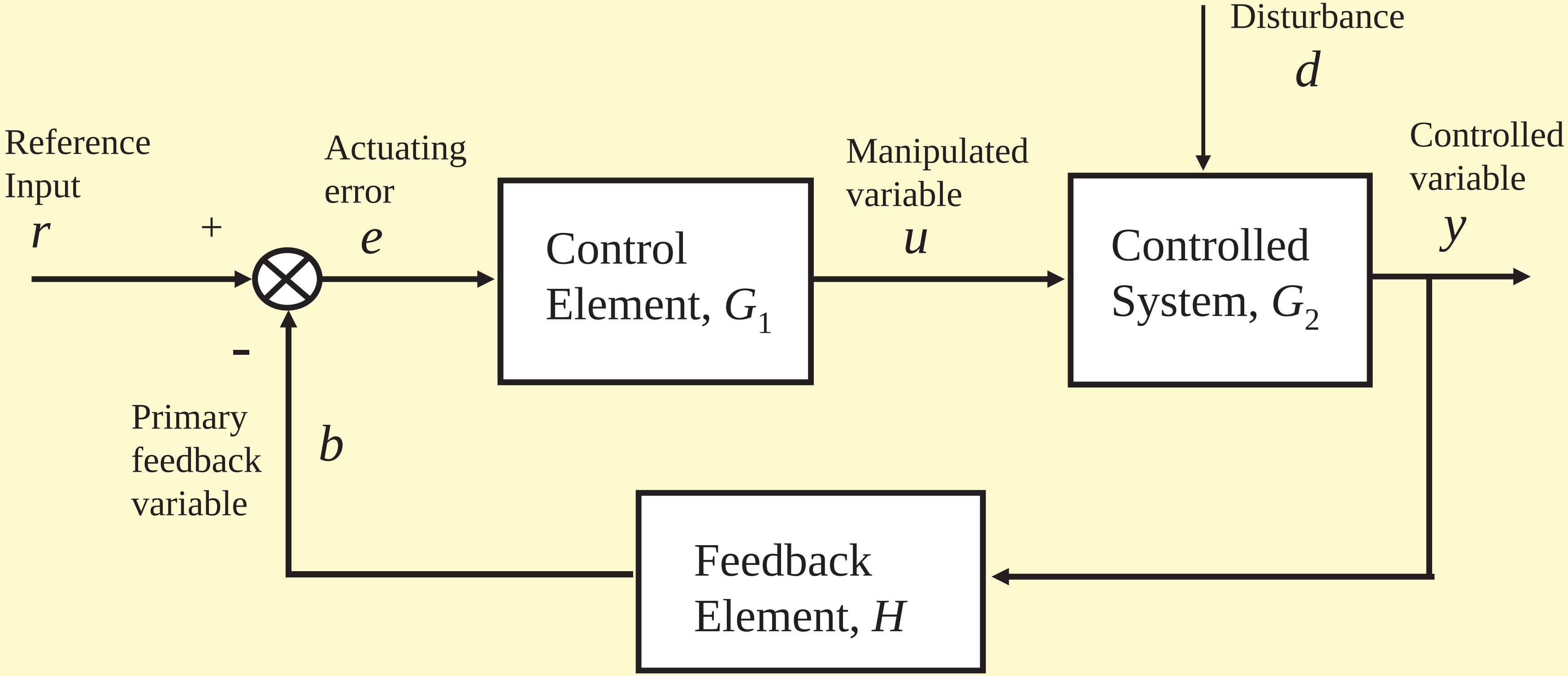


The important features of closed-loop control systems are:

1. **Increased accuracy.**
2. **Reduced sensitivity due to external disturbances and internal variations in system parameters.**
3. **Reduced effects of non-linearity and distortion.**
4. **Increased bandwidth. The bandwidth of a system is the range of frequencies of the input over which the system will respond satisfactorily.**
5. **Can lead to oscillation or instability if not done properly.**

d) Control System Terminology

A general (negative) feedback control system containing the basic elements is as shown in the figure below.



(In later sections, we will learn to represent a control system as in the above figure, which we called a block diagram representation)

Definitions: Variables in the System

***Reference Input*:**

r is the actual input to the system

***Controlled Variable*:**

y is the output of the controlled system

***Primary Feedback*:**

b is a function of y

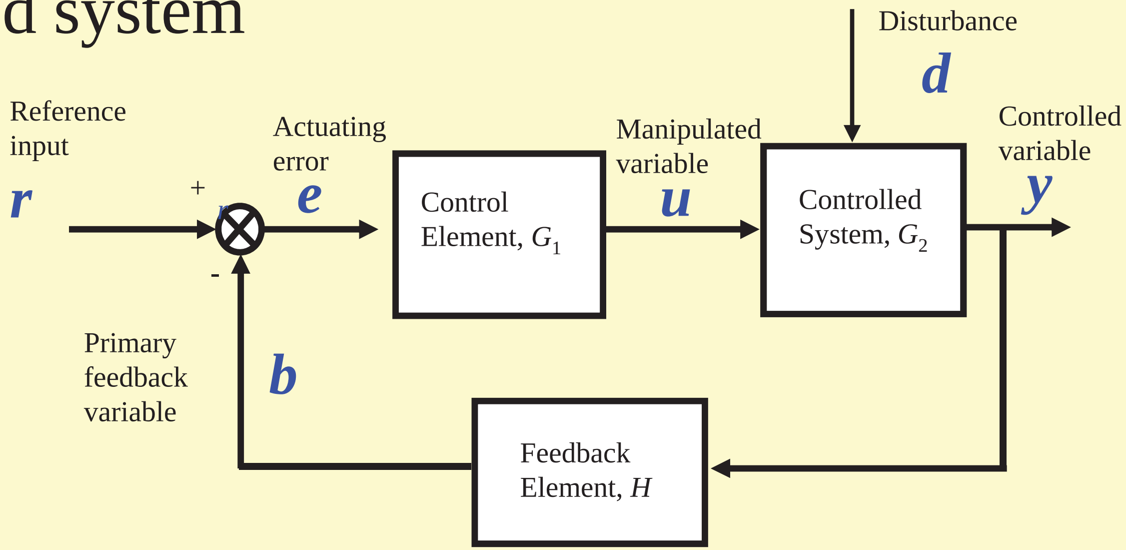
***Actuating Error*:**

e is $r - b$

***Manipulated Variable*:**

u is the output of G_1 and applied to G_2

- Generally of higher energy than e



***Disturbance*:**

d is the unwanted signal that will affect y. d may be introduced into the system at many places.

Definitions: System Components

Control Element:

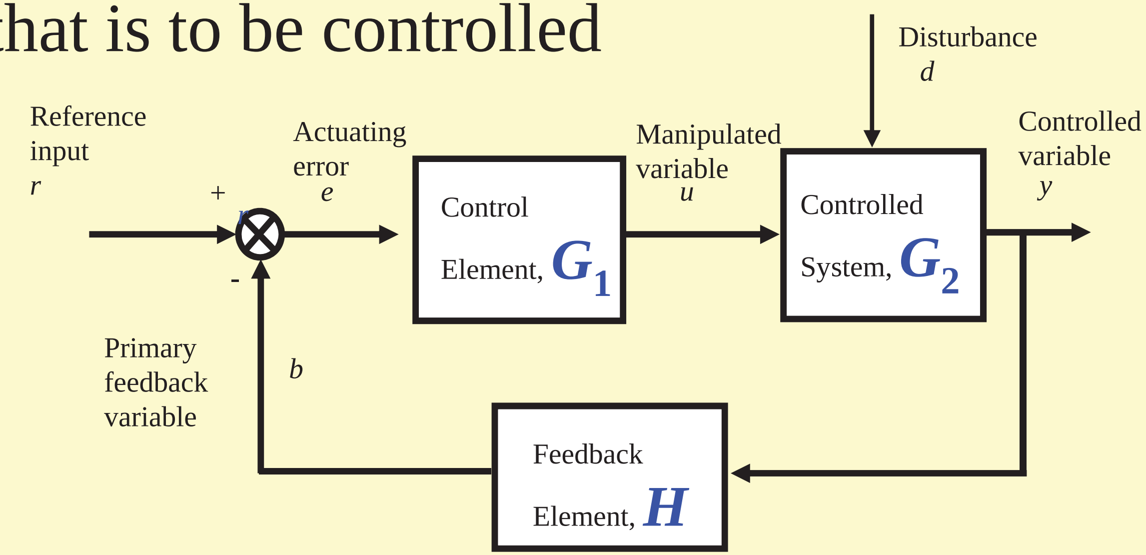
G_1 produces u from e

Controlled System:

G_2 is the process or plant that is to be controlled

Feedback Element:

H produces b from y



Supplementary Terminology

Forward Path:

The transmission path from the actuating error e to the controlled variable y

Feedback Path:

The transmission path from controlled variable y to the primary feedback b

e) Control and System Engineering Problems

In general, a control and system engineering problem can be divided into the following steps:

1. Established a set of performance specifications

2. Formulate/Derive a set of differential equations describing the behaviour of the physical system

3. Analyze the performance of the original system (many methods can be used)

4. Design and add control element to improve the response if the performance does not meet the specifications

5. System optimization may be required!!

Trade-offs

Any control system must be **stable**. In addition, it must have reasonable relative stability, i.e. the speed of response must be fast with reasonable damping. That is, the transient response will die out quickly.

A control system must also be capable of reducing the error to zero or to a small tolerable value (i.e. small ***steady-state error***).

The requirement of reasonable relative stability and that of steady-state accuracy tend to be incompatible. Thus, in designing a control system, it is necessary to make most ***effective compromise*** between these two requirements.

Non-linearity and Parameter Variation

Almost all physical systems are non-linear to a varied extent, and that the system parameter may change. However, in many practical control systems, the effects of non-linearity and parameter variation can be reduced by appropriate feedback. If non-linearity is small enough to be neglected, or that the operating limits are small enough, then a linear analysis is sufficient.

Disturbance and Noise Rejection

In practical situations, there is always some form of disturbances acting on the controlled system. These may be external or internal, and may be predictable or random. The control system must be able to attenuate the effects of the unwanted disturbances and/or noise.

Q: What happens when an aircraft passes through a thick cloud? (See Appendix 1.1.)

Appendix 1.1

When an aircraft passes through a thick cloud, there would be a significant pressure difference on the upper and the lower surfaces of the aircraft. This pressure difference results in a force acting on the aircraft, thus altering its altitude. This is a disturbance and the aircraft must be able to reject the disturbance, else its altitude may be affected.

A more common disturbance is the turbulent current and this results in a disturbance force acting on the aircraft. It will results in a slight bumpy ride if the aircraft has good controller to reject it, else the flight can be very bumpy.

Summary 1: Introduction

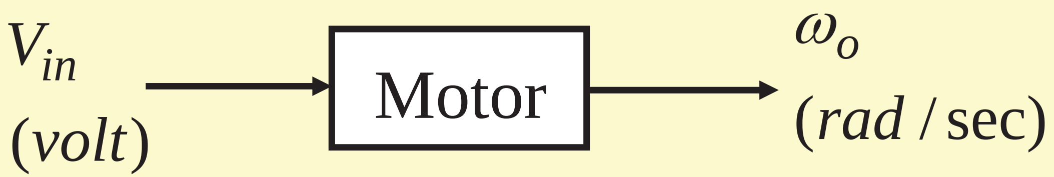
The objective of this course is to enable students to **model**, **analyze** and **control** real systems.

Performance Specifications

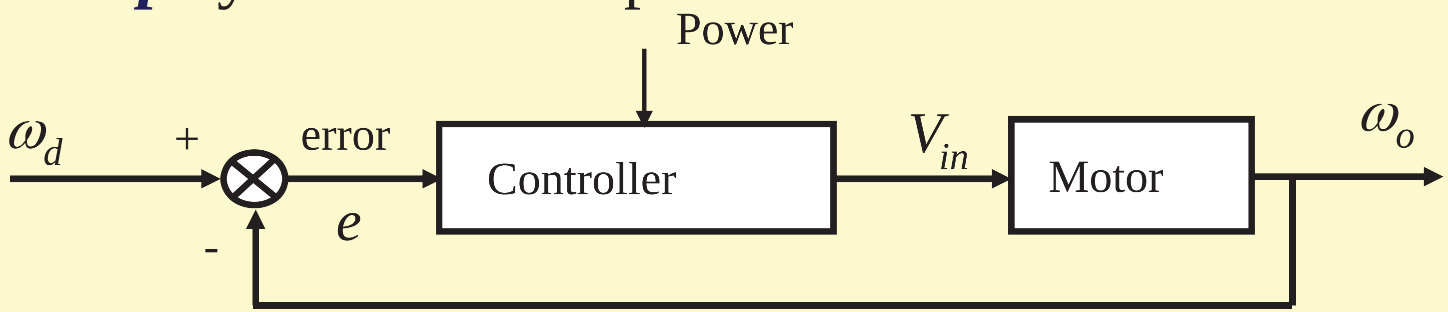
A set of specifications that a control system is required to perform/satisfy.

Open and Closed-loop Systems

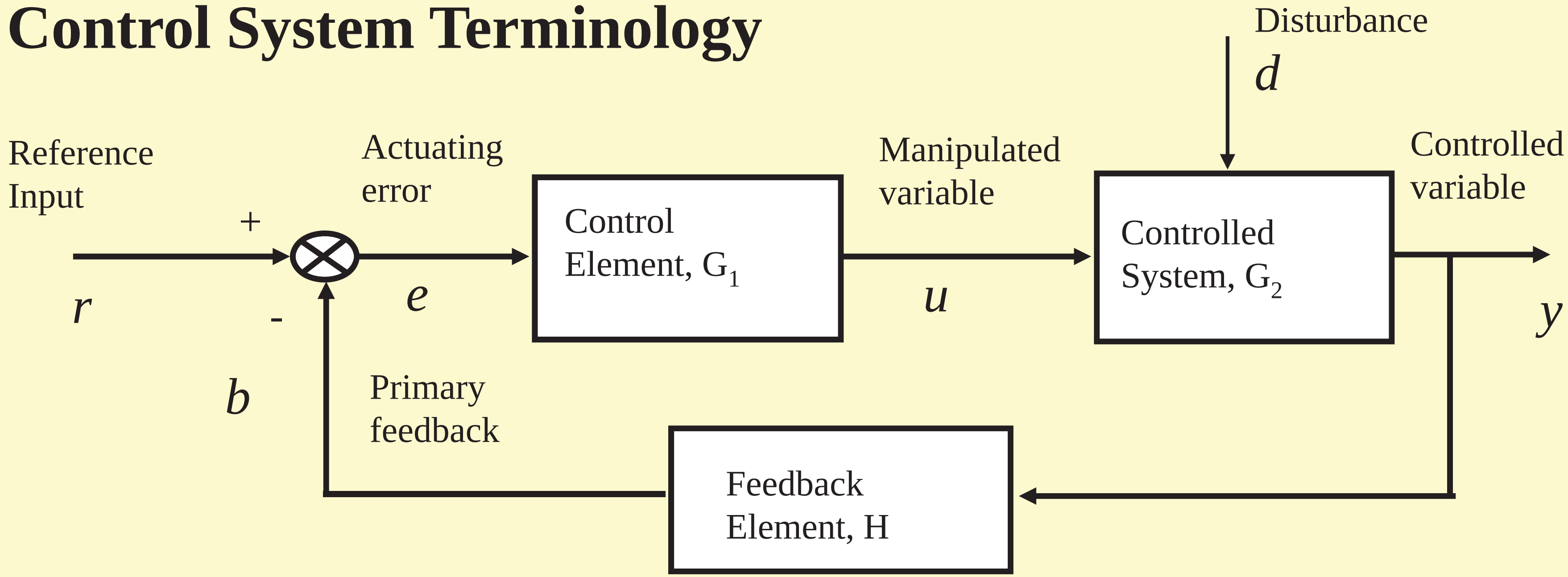
Open loop systems are control systems in which the output has no control/effect on the control action.



Closed-loop systems use output information for feedback.



Control System Terminology



Variables in the System:

$r, \; y, \; b, \; e, \; u, \; d$

System Components:

$G_1 , G_2 , \;\; H$

Forward Path:

The transmission path from e to y

Feedback Path:

The transmission path from y to b

Lecture 2: System Modelling - Block Diagrams

a) Linear Systems

b) Differential Equations

c) Transfer Functions

d) Block Diagrams

e) Multi-loop Block Diagrams

a) Linear Systems

In order to solve a control system problem, the descriptions of the system and its components must be put into a form suitable for analysis and evaluation.

The following methods can be used to model physical components and systems:

1. **Differential Equations**

2. ***Transfer Functions***

3. ***Block Diagrams***

4. *State-Space Model*

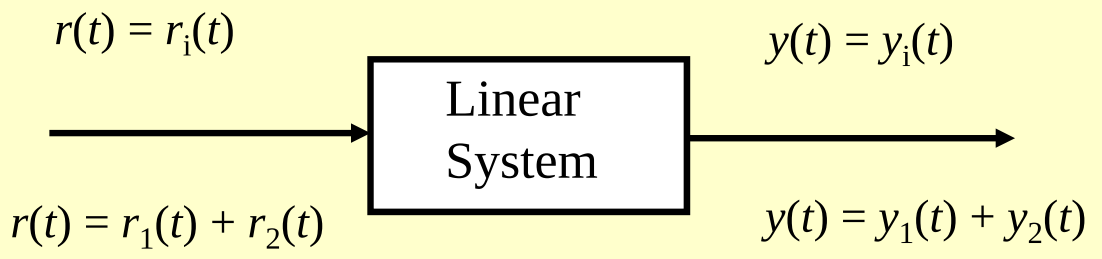
Our focus will be on linear systems. A system is **linear** if it obeys the principle of ***superposition*** and ***homogenity***, i.e.

If $r_1(t)$ produces $y_1(t)$

and $r_2(t)$ produces $y_2(t)$

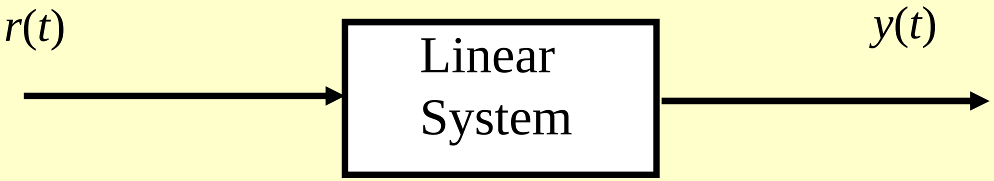
then, $r_1(t) + r_2(\mathrm{t})$ produces $y_1(t) + y_2(t)$

and $ar_1(t) + br_2(t)$ produces $ay_1(t) + by_2(t)$



b) Differential Equations

The input/output relationship for a linear system takes the form of a linear differential equation.



In general, the differential equation for the input/output relationship is:

$$\sum_{i=0}^{n} a_i \frac{d^i}{dt^i} y(t) = \sum_{i=0}^{m} b_i \frac{d^i}{dt^i} r(t) \qquad (1)$$

For physical systems, $n \geq m$. *(We say that these systems are* ***causal systems.****)*

We can apply ***Laplace Transformation*** to linear differential equations to obtain transfer functions that describe the behaviour of the systems. See Appendix 2.1 for a brief review of Laplace Transform.

c) Transfer Functions

Taking Laplace transform of (1), assuming zero initial conditions, we obtain

$$\left\{\sum_{i=0}^{n} a_i s^i\right\} Y(s) = \left\{\sum_{i=0}^{m} b_i s^i\right\} R(s)$$

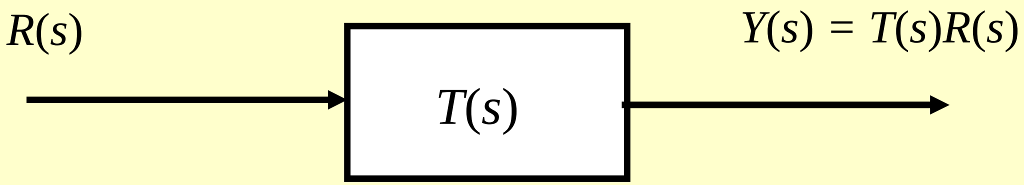
where $Y(s)$ and $R(s)$ are the Laplace transforms of $y(t)$ and $r(t)$.

The ***transfer function*** of the system is given by

$$T(s) = \frac{Y(s)}{R(s)} = \frac{\sum_{i=0}^{m} b_i s^i}{\sum_{i=0}^{n} a_i s^i} \qquad (2)$$

The transfer function describes the **input-output behaviour** of the system. In (2), the system is of order n.

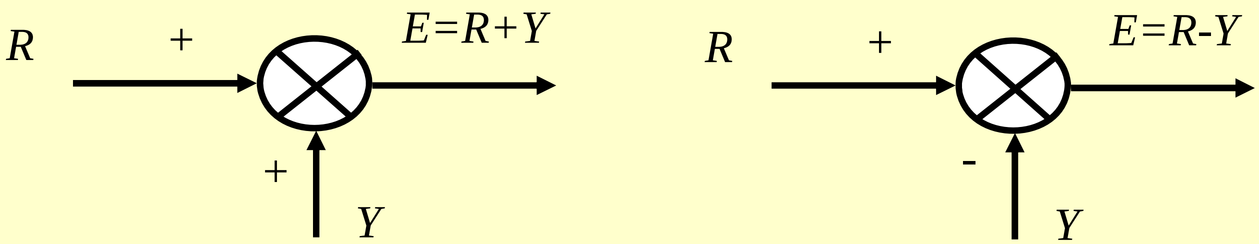
In **block diagram** representation, we have



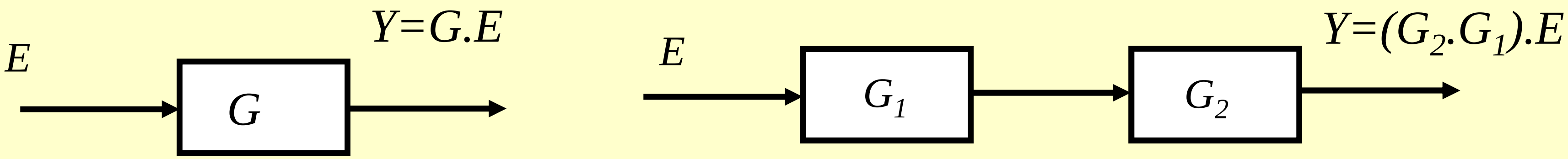
d) Block Diagrams

Block diagrams are used to give the functional representation of control systems. Three main symbols used are: summer/comparator, element block and take-off point.

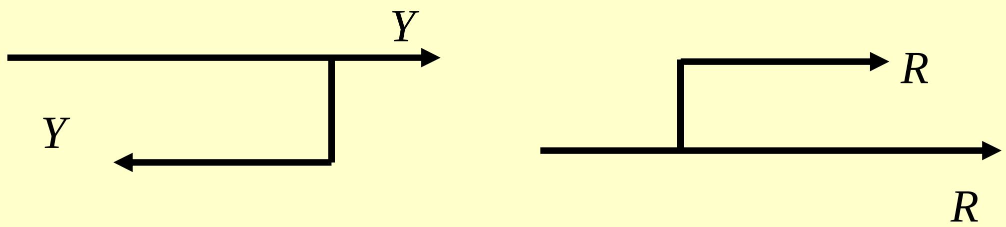
1. Summer or Comparator



2. Element Block

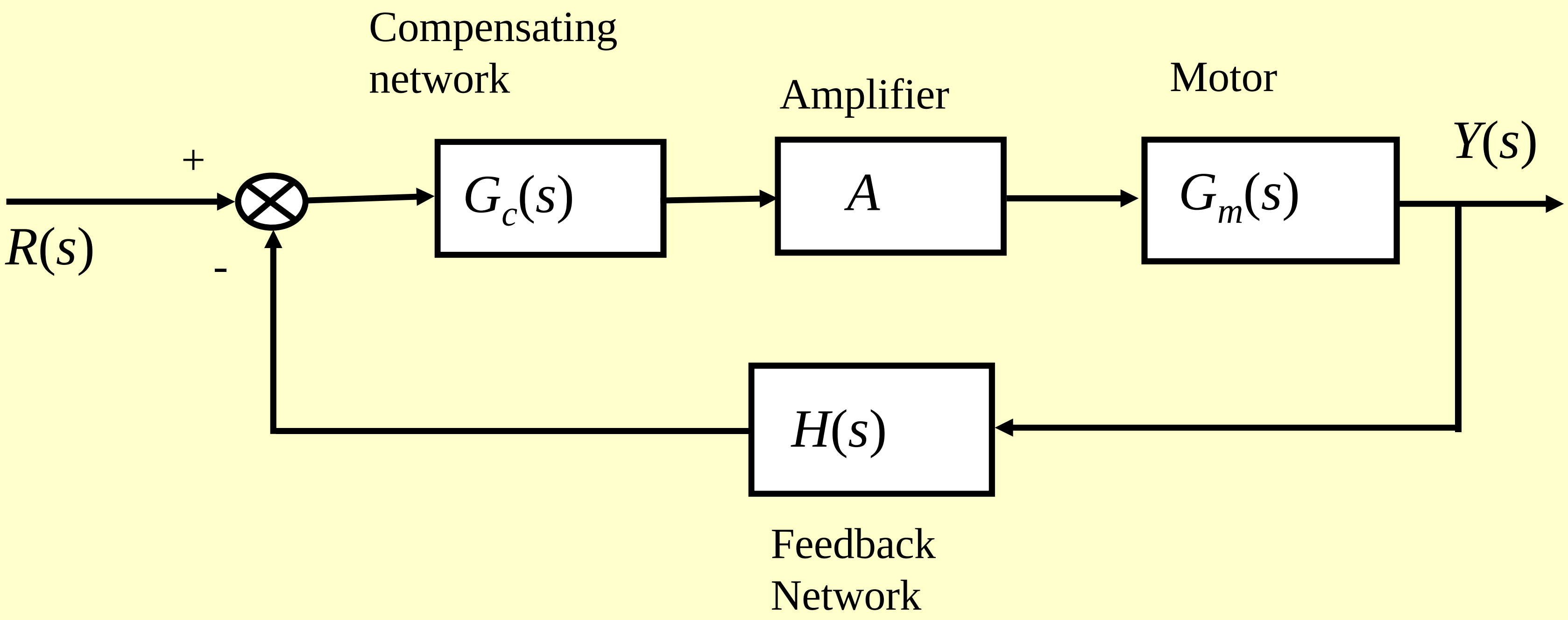


3. Take-off Point



Example:

A feedback servomechanism (e.g. in the case of the elevator system) with **negative feedback** is represented as

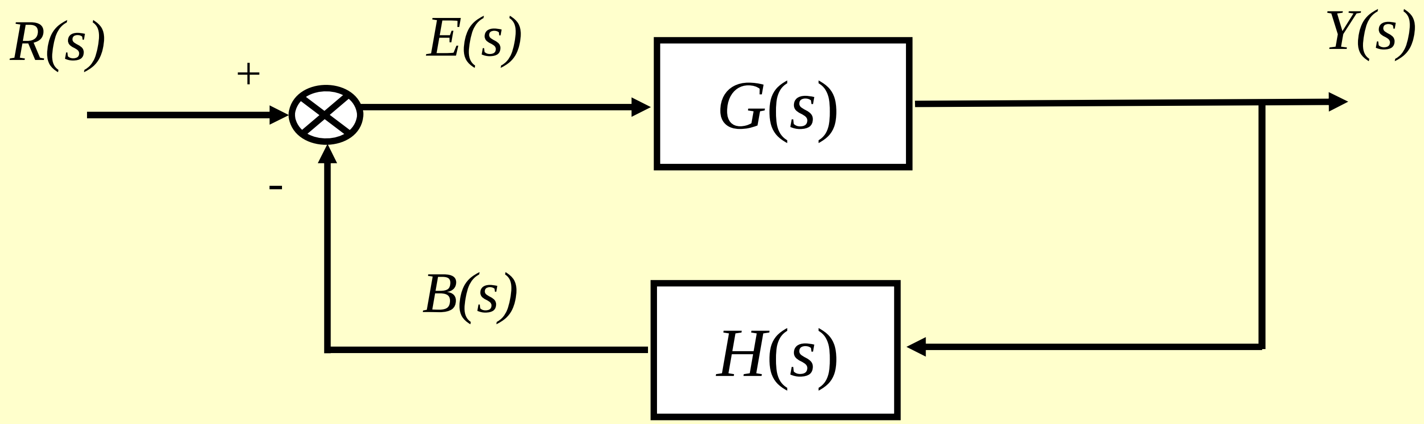


$Y(s)$ may be the displacement, velocity or acceleration.

To analyyze the behaviour of the system, we need to derive the relationship between $Y(s)$ and $R(s)$.

Generalized Block Diagram

- (single-input single-output systems)



$R(s)$ = reference input

$Y(s)$ = controlled output

$E(s)$ = actuating error

$B(s)$ = feedback variable

$G(s)$ = forward transfer function (FTF)

$H(s)$ = feedback transfer function

$G(s)H(s)$ = loop transfer function or open loop transfer function (OLTF)

$\dfrac{Y(s)}{R(s)}$ = overall transfer function or closed-loop transfer function (CLTF)

Clearly,

$$Y(s) = G(s)E(s)$$

$$E(s) = R(s) - B(s) = R(s) - H(s)Y(s)$$

$$\therefore \quad Y(s) = G(s)R(s) - G(s)H(s)Y(s)$$

$$\Rightarrow \quad (1 + G(s)H(s))Y(s) = G(s)R(s)$$

$$i.e. \quad \frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} = \frac{FTF}{1 + OLTF}$$

$1 + G(s)H(s) = 0$ is referred to as the ***characteristic equation*** of the system.

Exercise: Derive the closed-loop transfer function of a positive feedback system. (See Appendix 2.2)

e) Multi-loop Block Diagrams

The block diagrams of many practical systems contain several interacting loops. Such a complex block diagram can be simplified by block diagram algebra.

Two points to note when simplifying the block diagrams:

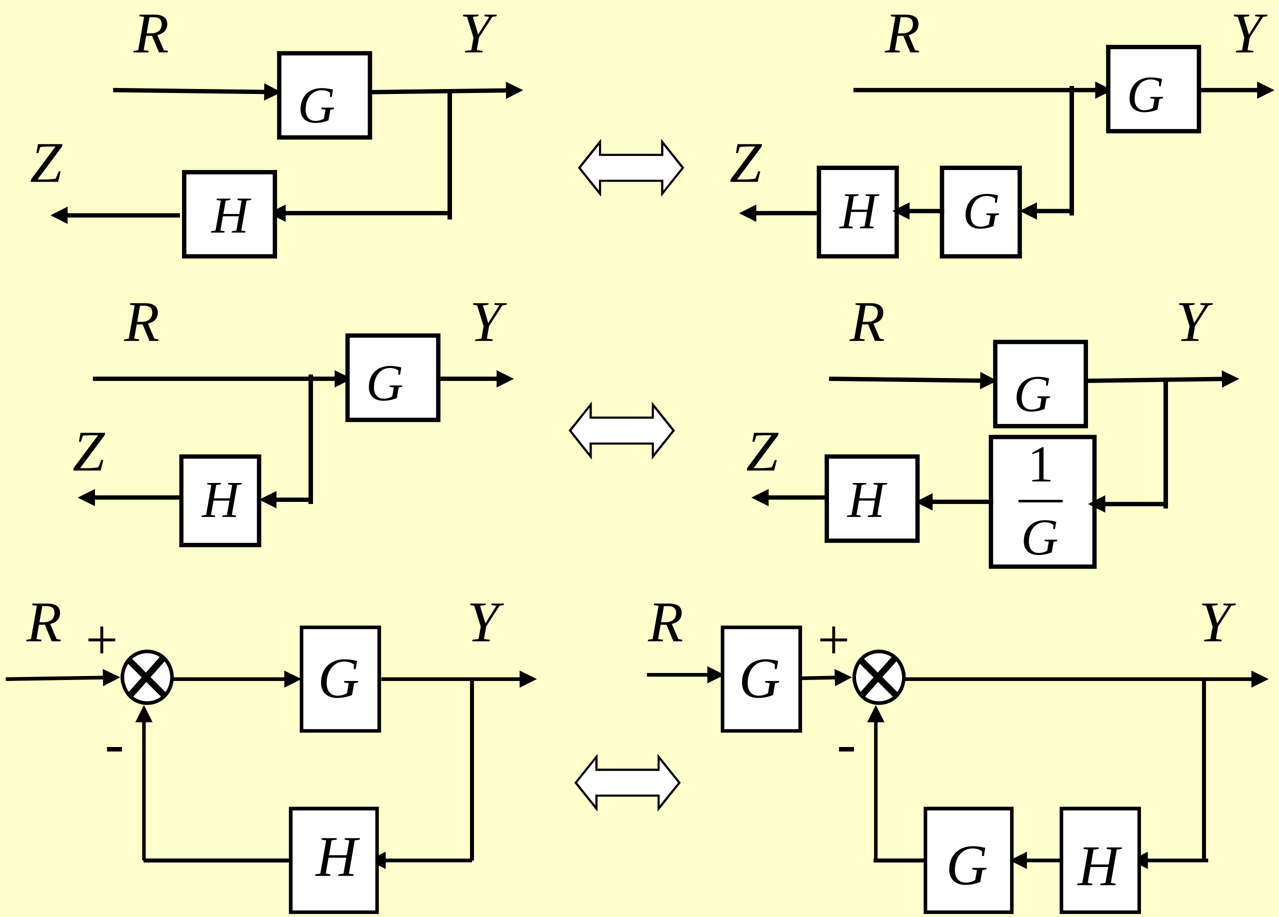
a) The product of the transfer functions in the loop forward path must remain the same.

b) The product of the transfer functions around the loop must remain the same.

See Appendix 2.3 for some standard manipulations.

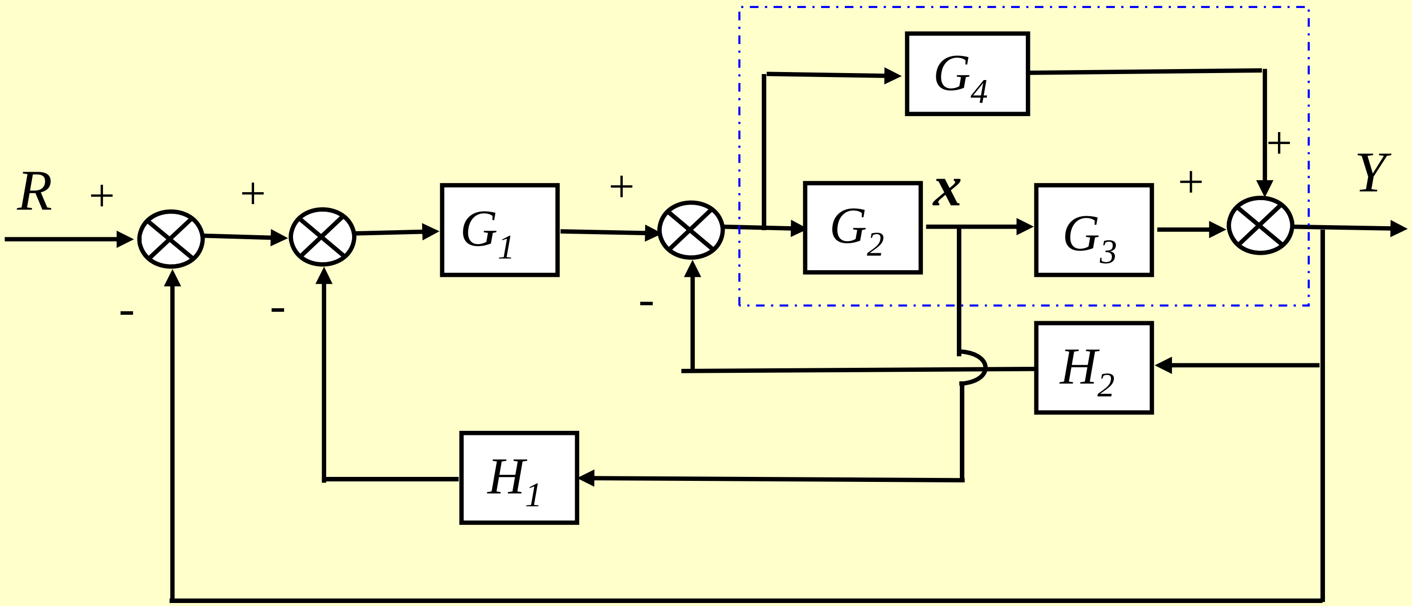
Mason's Gain Formula" can also be applied to determine the overall transfer function. Interested students can refer to the book by B. C. Kuo.

Three basic and useful block diagram manipulations are illustrated below. You verify the equivalence by checking the consistency of the signals.

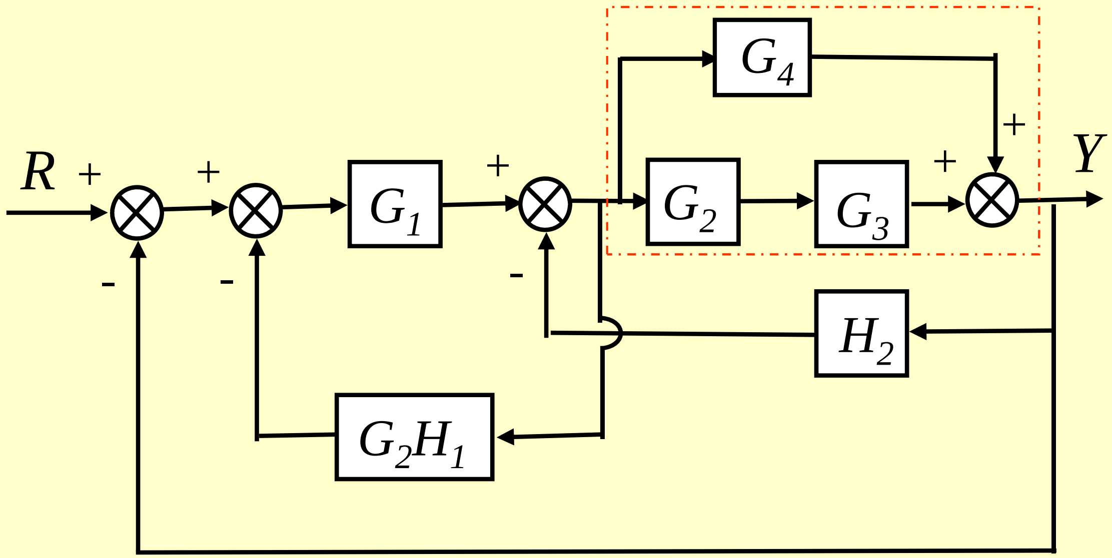


The main idea is that, after each manipulation, certain part of the system will yield a standard form (e.g. single-loop or parallel paths), which can then be simplified into a single block.

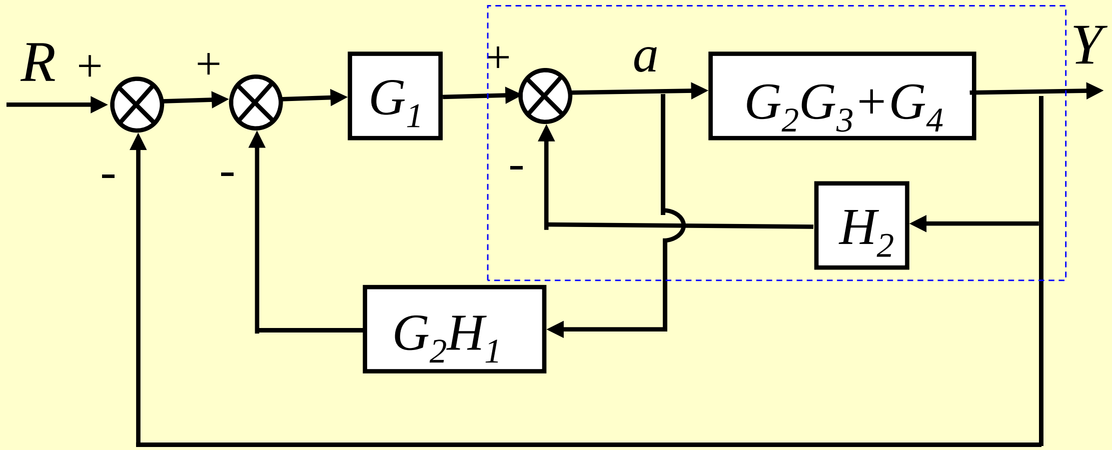
Example: Find the overall transfer function for the following block diagram:



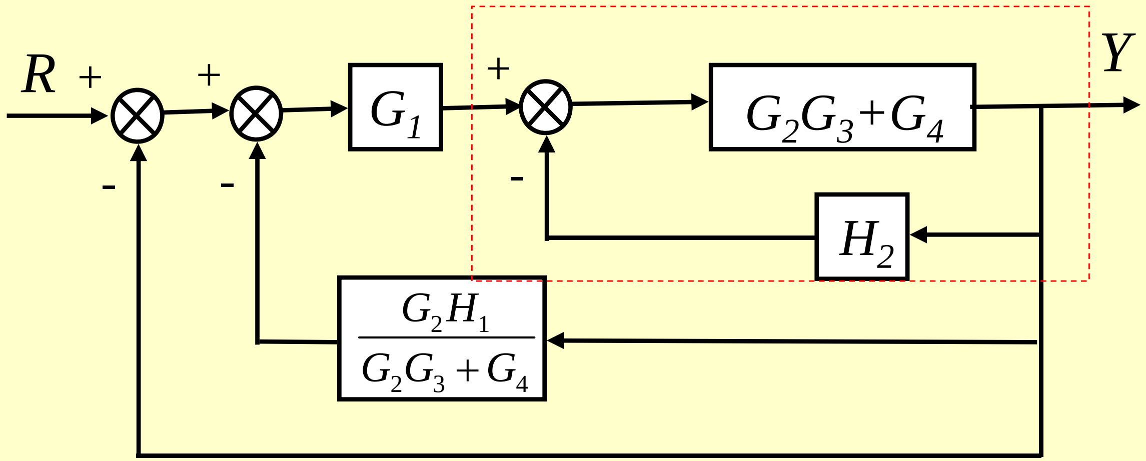
i) Move take-off point *x* backwards, we get



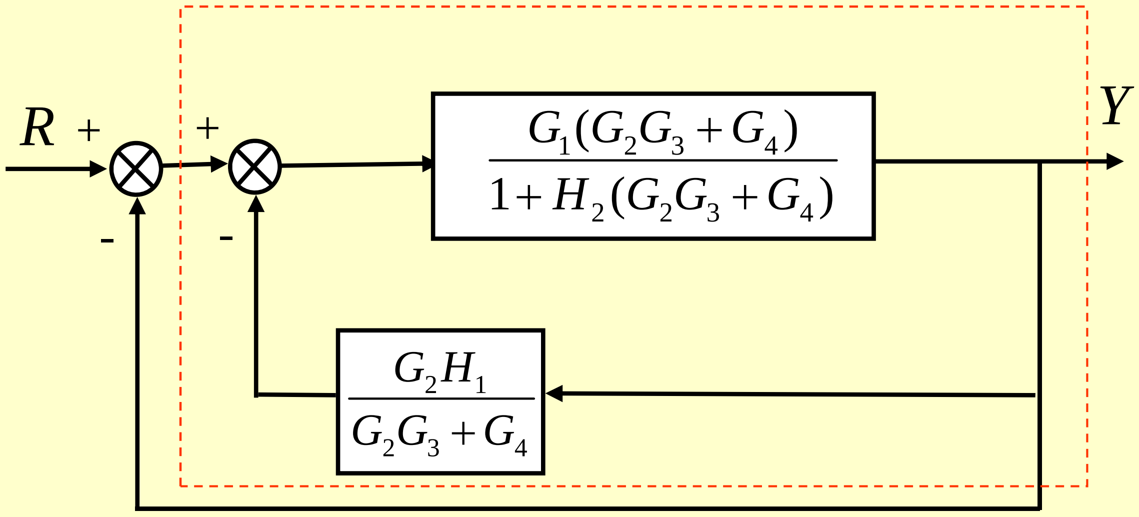
ii) Combining G_2, G_3 and G_4, we get



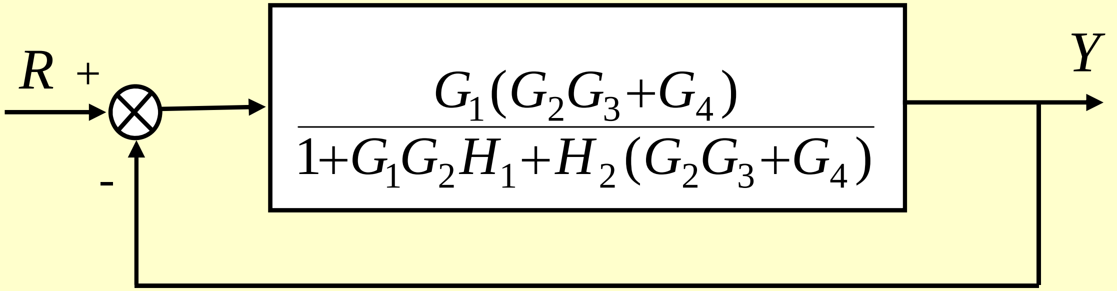
iii) Move take off point *a* forward,



iv) Simplify the inner block



v) Simplify further

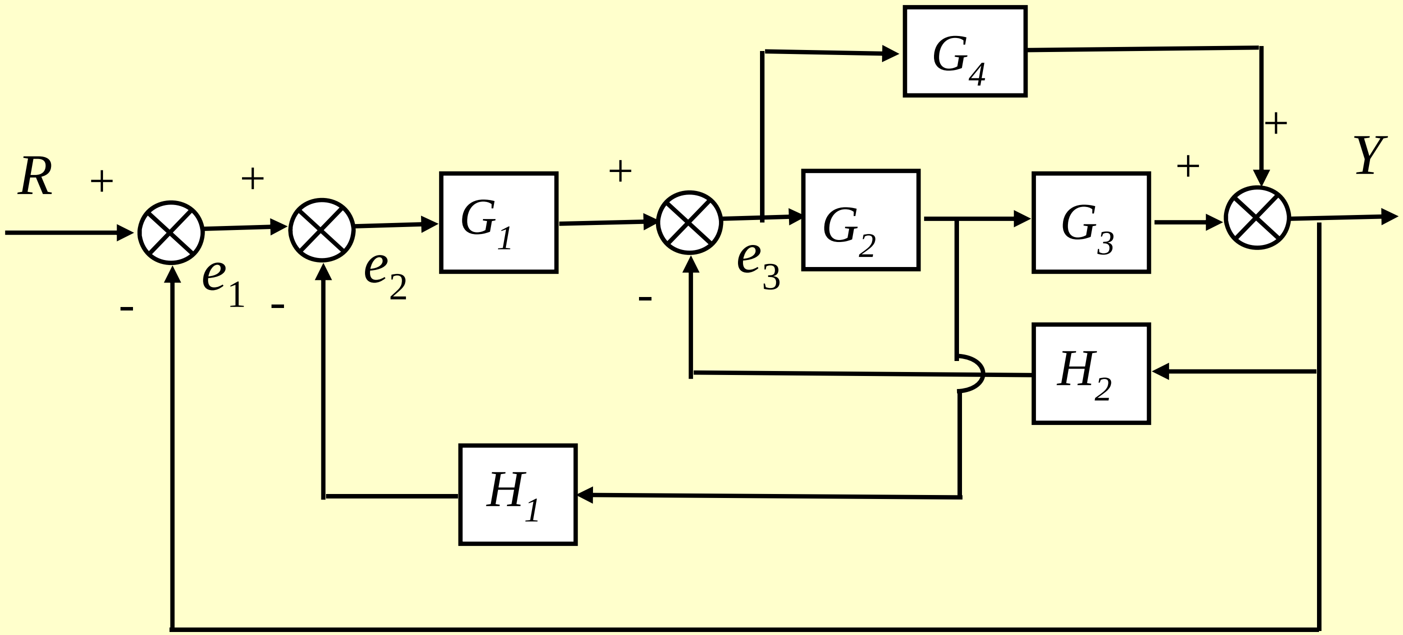


Finally,

$$\frac{Y}{R}=\frac{G_1(G_2G_3+G_4)}{1+G_1G_2H_1+H_2(G_2G_3+G_4)+G_1(G_2G_3+G_4)}$$

NB: For simplicity of expression, we'll drop the arguments where necessary.

Alternatively, we can define some auxiliary variables as follows:



Then, $$e_1 = R - Y$$

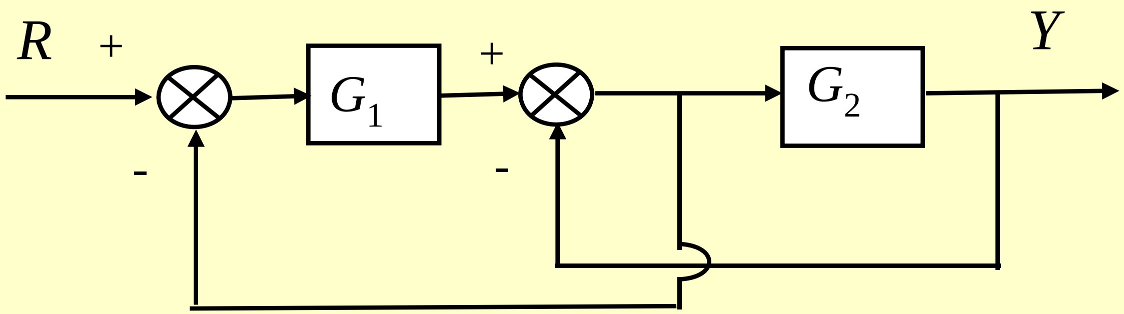
$$e_2 = e_1 - H_1 G_2 e_3$$

$$e_3 = G_1 e_2 - H_2 Y$$

$$Y = (G_4 + G_2 G_3) e_3$$

After eliminating $e_1 - e_3$, we will get the same transfer function. *Try it.* *(See Appendix 2.4)*

Exercise: Consider the block diagram given below.



Show that the closed-loop transfer function is given by

$$\frac{Y}{R} = \frac{G_1 G_2}{1 + G_1 + G_2}$$

(See Appendix 2.5)

Appendix 2.1: Laplace Transforms

Laplace transform of a time domain function *f*(*t*) is

$$\mathcal{L}\{f(t)\} = F(s) = \int_{-\infty}^{\infty} f(\tau)e^{-s\tau}d\tau$$

(Doubled-sided Laplace transform)

where $s = \sigma + j\omega$ is the complex variable.

In most engineering problems, *f*(*t*) is ***causal***, i.e.

$$f(t) = 0 \qquad \text{for} \qquad t < 0$$

Thus, single-sided Laplace transform is used, i.e.

$$\mathcal{L}\{f(t)\} = F(s) = \int_{0}^{\infty} f(\tau)e^{-s\tau}d\tau$$

Properties of the Laplace Transforms

Let $F(s) = \mathcal{L}\{f(t)\}$.

(1) *Time Differentiation*

$$\mathcal{L}\left\{\frac{d^n}{dt^n} f(t)\right\} = s^n F(s) - \sum_{i=0}^{n-1} s^i \frac{d^{n-1-i} f(0)}{dt^{n-1-i}}$$

(2) *Time Integration*

$$\mathcal{L}\left\{\int_0^{t_1}\int_0^{t_2}\cdots\int_0^{t_n} f(\tau_1)d\tau_1\cdots d\tau_n\right\} = \frac{F(s)}{s^n}$$

(3) *Complex Translation* (Shifting in the *s*-domain)

$$\mathcal{L}\left\{e^{\mp at} f(t)\right\} = F(s \pm a)$$

(4) ***Real Translation*** (Shifting in the time domain)

$$\mathcal{L}\left\{ f(t-T)u(t-T) \right\} = e^{-sT}F(s)$$

where $u(t-T)$ is a shifted unit-step function.

(5) ***Real Convolution*** (Complex Multiplication)

$$f_1(t) * f_2(t) = \int_0^t f_1(\tau) f_2(t-\tau) d\tau$$

$$= \int_0^t f_2(\tau) f_1(t-\tau) d\tau$$

$$\mathcal{L}\left\{ f_1(t) * f_2(t) \right\} = F_1(s) F_2(s)$$

(6) ***Initial Value Theorem***

$$\lim_{t \to 0} f(t) = f(0) = \lim_{s \to \infty} sF(s)$$

(7) *Final Value Theorem*

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

provided $sF(s)$ is <u>analytic</u> in the closed right-half of the s–plane (i.e. the denominator of $sF(s)$ has no roots on the $j\omega$–axis or in the right-half of the s–plane).

<u>Example</u>: Consider $\dfrac{Y(s)}{R(s)} = T(s) = \dfrac{2}{(s+1)(s+2)}$; $R(s) = \dfrac{1}{s}$

i.e. $Y(s) = \dfrac{2}{s(s+1)(s+2)} = \dfrac{1}{s} - \dfrac{2}{s+1} + \dfrac{1}{s+2}$

and hence $y(t) = 1 + e^{-2t} - 2e^{-t}$. So, $y_{ss} = 1$.

Alternatively, by f.v.t., we have

$$y_{ss} = \lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = 1$$

Example: $$F(s) = \frac{5}{s(s^2 + s + 2)}$$

Clearly, $sF(s)$ is analytic in the closed right-half of the s–plane, so the f.v.t. can be applied to give

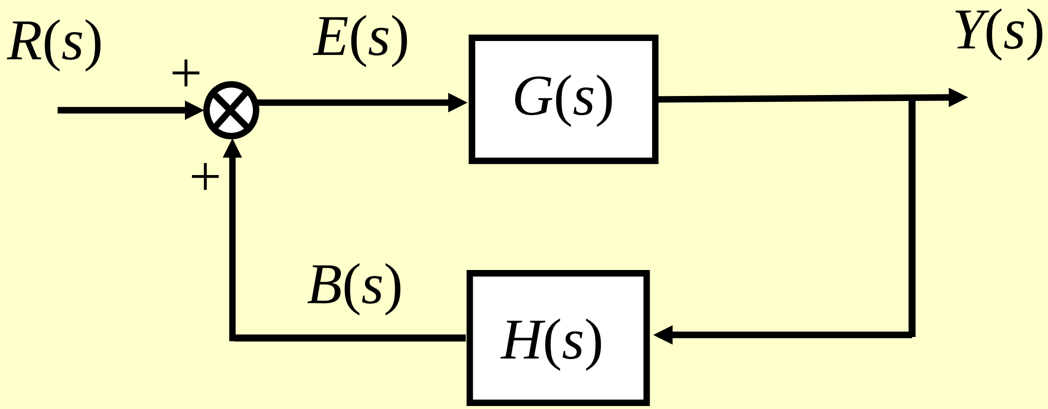
$$f_{ss} = \lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s) = \frac{5}{2}$$

Example: $$F(s) = \frac{\omega}{s^2 + \omega^2} \quad \text{i.e} \quad f(t) = \sin \omega t$$

In this case, denominator of $F(s)$ has 2 roots on the $j\omega$–axis, so the f.v.t. cannot be used. (If we applied the f.v.t., it gives a final value of zero which is wrong!)

Appendix 2.2

For positive feedback system, we have



Clearly, from the block diagram

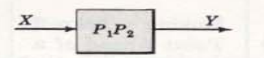
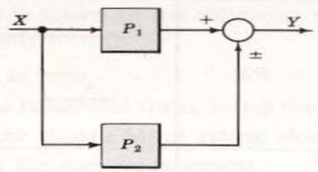
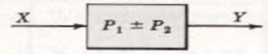
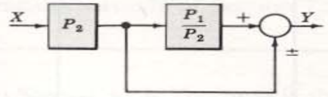
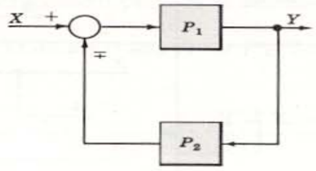
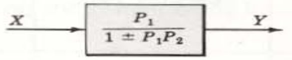
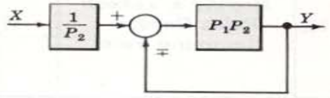
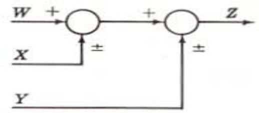
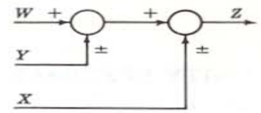
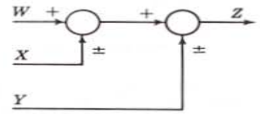
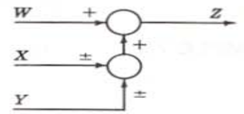
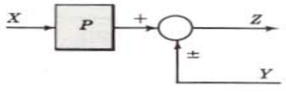
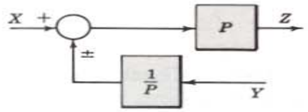
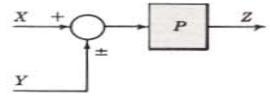
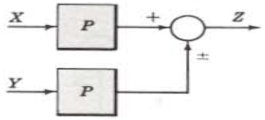
$$Y(s) = G(s)E(s)$$

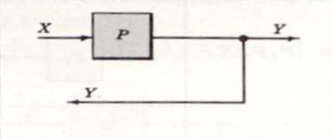
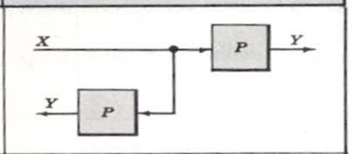
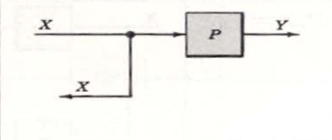
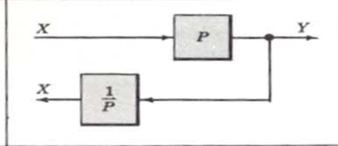
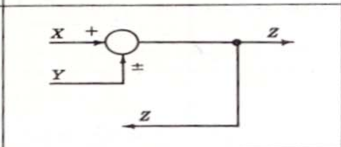
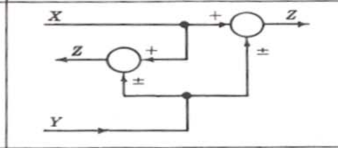
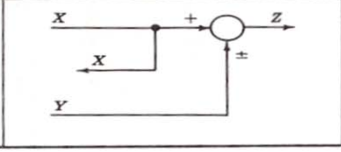
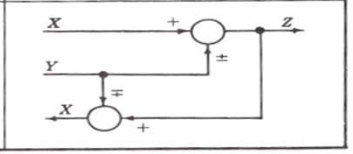
$$E(s) = R(s) + B(s) = R(s) + H(s)Y(s)$$

$$\therefore \quad Y(s) = G(s)R(s) + G(s)H(s)Y(s)$$

i.e. $$\frac{Y(s)}{R(s)} = \frac{G(s)}{1 - G(s)H(s)} = \frac{FTF}{1 - OLTF}$$

Appendix 2.3

	Transformation	Equation	Block Diagram	Equivalent Block Diagram
1	Combining Blocks in Cascade	$Y = (P_1 P_2)X$		
2	Combining Blocks in Parallel; or Eliminating a Forward Loop	$Y = P_1 X \pm P_2 X$		
3	Removing a Block from a Forward Path	$Y = P_1 X \pm P_2 X$		
4	Eliminating a Feedback Loop	$Y = P_1(X \mp P_2 Y)$		
5	Removing a Block from a Feedback Loop	$Y = P_1(X \mp P_2 Y)$		
6a	Rearranging Summing Points	$Z = W \pm X \pm Y$		
6b	Rearranging Summing Points	$Z = W \pm X \pm Y$		
7	Moving a Summing Point Ahead of a Block	$Z = PX \pm Y$		
8	Moving a Summing Point Beyond a Block	$Z = P[X \pm Y]$		

	Transformation	Equation	Block Diagram	Equivalent Block Diagram
9	Moving a Takeoff Point Ahead of a Block	$Y = PX$		
10	Moving a Takeoff Point Beyond a Block	$Y = PX$		
11	Moving a Takeoff Point Ahead of a Summing Point	$Z = X \pm Y$		
12	Moving a Takeoff Point Beyond a Summing Point	$Z = X \pm Y$		

Appendix 2.4

From the block diagram, we have

$$e_1 = R - Y \qquad \text{(A1)}$$

$$e_2 = e_1 - H_1 G_2 e_3 \qquad \text{(A2)}$$

$$e_3 = G_1 e_2 - H_2 Y \qquad \text{(A3)}$$

$$Y = (G_4 + G_2 G_3) e_3 \qquad \text{(A4)}$$

Sub. (A2) into (A3), we get

$$e_3 = G_1 e_1 - H_1 G_1 G_2 e_3 - H_2 Y$$

Eliminate e_1 by using (A1),

$$\begin{aligned}(1 + H_1 G_1 G_2) e_3 &= G_1 e_1 - H_2 Y \\ &= G_1 (R - Y) - H_2 Y\end{aligned}$$

i.e. $$e_3 = \frac{G_1 R - (G_1 + H_2)Y}{1 + H_1 G_1 G_2} \qquad \text{(A5)}$$

Sub. (A5) into (A4), we get

$$Y = (G_4 + G_2 G_3) \cdot \frac{G_1 R - (G_1 + H_2)Y}{1 + H_1 G_1 G_2}$$

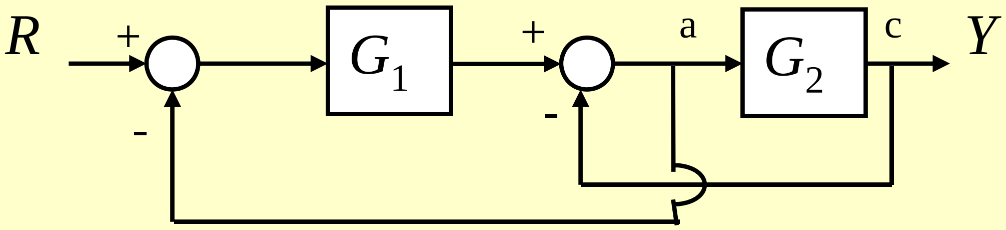
i.e. $$[1 + (G_4 + G_2 G_3)\frac{G_1 + H_2}{1 + H_1 G_1 G_2}]Y$$

$$= (G_4 + G_2 G_3) \cdot \frac{G_1}{1 + H_1 G_1 G_2} R$$

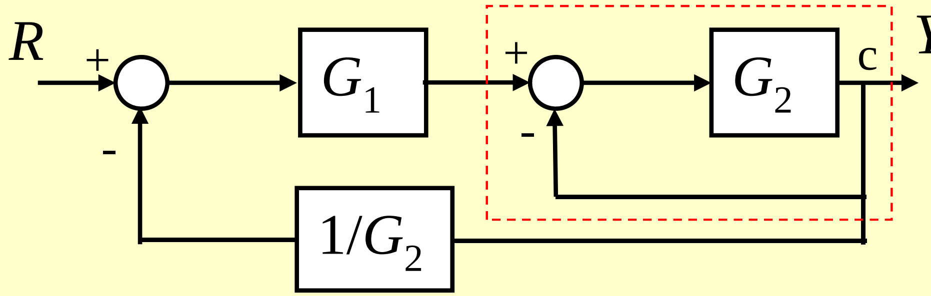
So, we have

$$\frac{Y}{R} = \frac{G_1 (G_4 + G_2 G_3)}{1 + H_1 G_1 G_2 + (G_1 + H_2)(G_4 + G_2 G_3)}$$

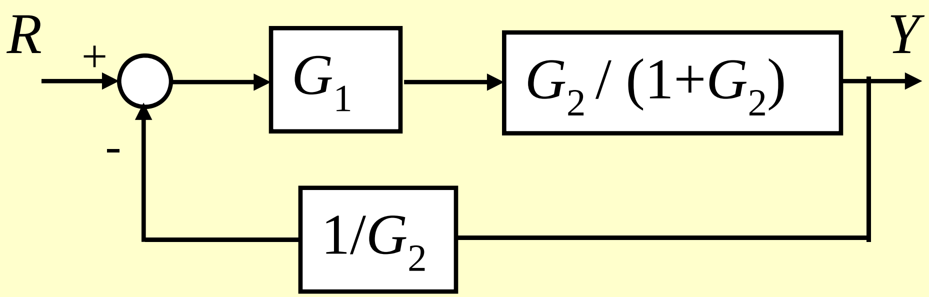
Appendix 2.5



Move take-off point a to c. We get



Simplifying the outlined feedback loop, we get:



Hence, $$\frac{Y}{R} = \frac{G_1 \cdot \dfrac{G_2}{1+G_2}}{1+G_1 \cdot \dfrac{G_2}{1+G_2} \cdot \dfrac{1}{G_2}} = \frac{G_1 G_2}{1+G_1+G_2}$$

Summary 2: Block Diagrams

The general input/output relationship of a linear system is expressed as

$$\sum_{i=0}^{n} a_i \frac{d^i}{dt^i} y(t) = \sum_{i=0}^{m} b_i \frac{d^i}{dt^i} r(t)$$

or in Laplace domain:

$$\left\{ \sum_{i=0}^{n} a_i s^i \right\} Y(s) = \left\{ \sum_{i=0}^{m} b_i s^i \right\} R(s)$$

The ***transfer function*** of the system defines the input-output relation and it is given by

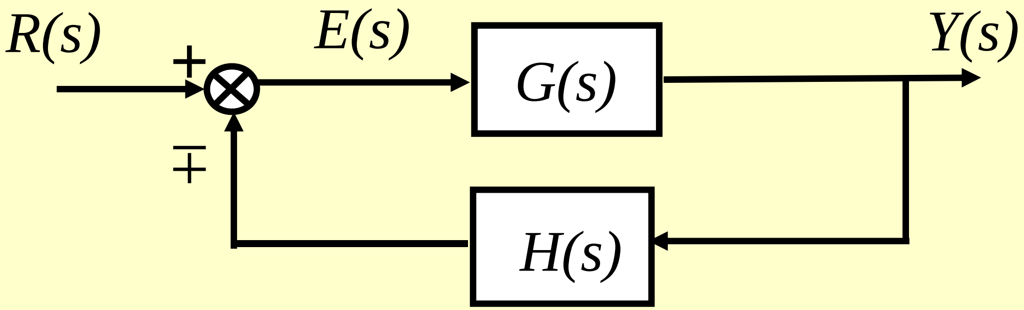
$$T(s) = \frac{Y(s)}{R(s)} = \frac{\sum_{i=0}^{m} b_i s^i}{\sum_{i=0}^{n} a_i s^i}$$

Block Diagrams

Block diagrams are used to simplify the representation of complex systems.

Generalized Block Diagram

- (single-input single-output systems)



We want to derive $Y(s)$ as a function of $R(s)$.

$$Y(s) = G(s)R(s) \mp G(s)H(s)Y(s)$$

i.e. $$\frac{Y(s)}{R(s)} = \frac{G(s)}{1 \pm G(s)H(s)} = \frac{FTF}{1 \pm OLTF}$$

$1 + G(s)H(s) = 0$ is the ***characteristic equation*** of the system.

Multi-loop Block Diagrams

If the block diagram contains several interacting loops, it can be systematically simplified by block diagram algebra.

Two points to note when simplifying the block diagrams:

a) The product of the transfer functions in the loop forward path must remain the same.

b) The product of the transfer functions around the loop must remain the same.

One can also define some auxiliary variables and then perform algebraic manipulations to obtain the transfer function.

Lecture 3: System Modelling - Modelling of Physical Systems

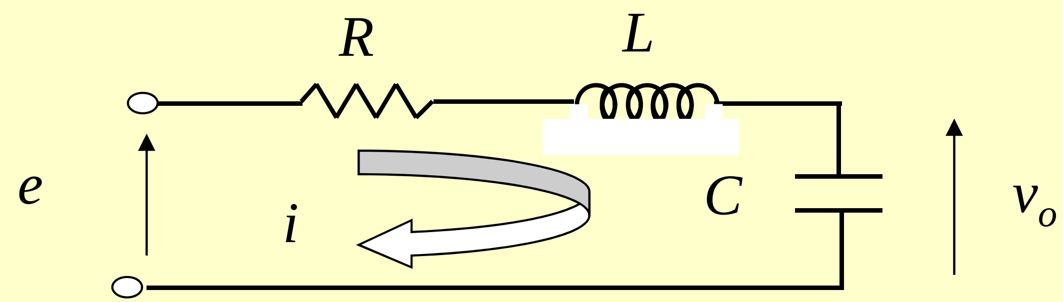
a) Electrical Systems
- Series RLC Circuit
- Parallel RLC Circuits

b) Mechanical and Electromechanical Systems
- Translational Systems

a) Electrical Systems

Resistors, inductors and capacitors are the 3 basic electrical elements. The electrical circuits are analyzed by the application of Kirchhoff's voltage and current laws.

Consider the RLC series circuit as shown below:



If we define the voltage e as the input and the charge q ($= \int idt$) across the capacitor as the output, then the system is described by

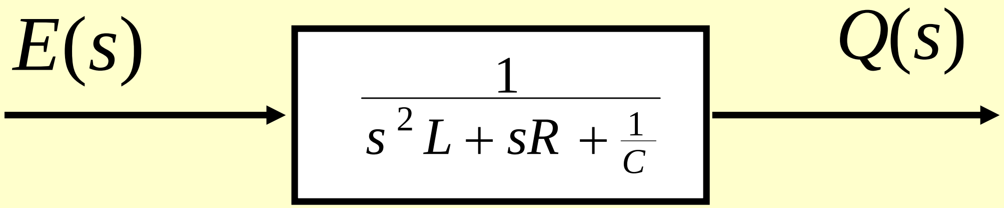
$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = e \qquad (1)$$

Eqn (1) is a 2nd-order ODE. Applying Laplace transform, with zero initial conditions, we get

$$(s^2 L + sR + \tfrac{1}{C})Q(s) = E(s)$$

or $$Q(s) = \frac{1}{s^2 L + sR + \frac{1}{C}} E(s)$$

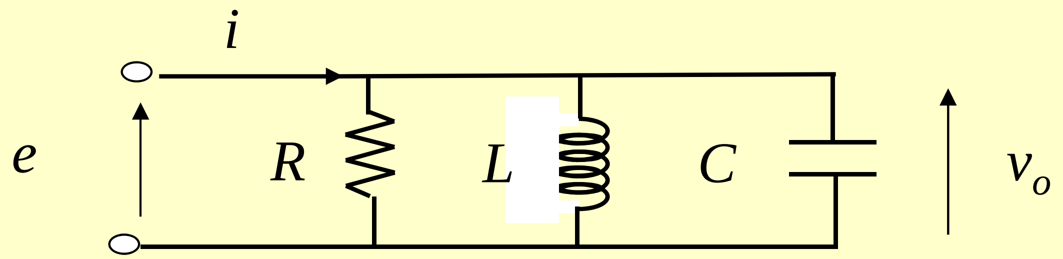
i.e. we have a 2nd-order dynamic system.



<u>Exercise</u>: If we define e as the input and i the output, then the 2nd order ODE is given by

$$L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = \frac{de}{dt}$$

Derive the transfer function. (See Appendix 3.1.)



For a parallel RLC circuit, let i be the input and the magnetic flux in the inductor ϕ $(=\int v dt)$ be the output, then the 2nd order ODE is

$$C\frac{d^2\phi}{dt^2}+\frac{1}{R}\frac{d\phi}{dt}+\frac{\phi}{L}=i \qquad (2)$$

In s-domain, we have $(s^2C+s\frac{1}{R}+\frac{1}{L})\Phi(s)=I(s)$

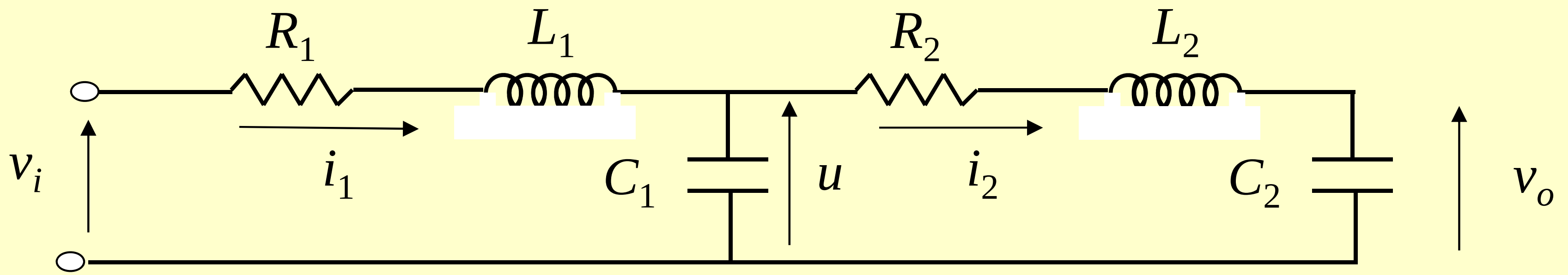
$$\Rightarrow \quad \Phi(s)=\frac{1}{(s^2C+s\frac{1}{R}+\frac{1}{L})}I(s)$$

$$I(s) \longrightarrow \boxed{\frac{1}{s^2C+s\frac{1}{R}+\frac{1}{L}}} \longrightarrow \Phi(s)$$

<u>Exercise</u>: Derive the transfer function if we define i as the input and v_o the output. The 2nd order ODE is given by

$$C\frac{d^2v_0}{dt^2}+\frac{1}{R}\frac{dv_o}{dt}+\frac{v_0}{L}=\frac{di}{dt}$$

<u>Example</u>: Derive a block diagram representation of the electric circuit given below and obtain the transfer function $V_o(s)/V_i(s)$.



In the s-domain, using KVL and KCL, we have:

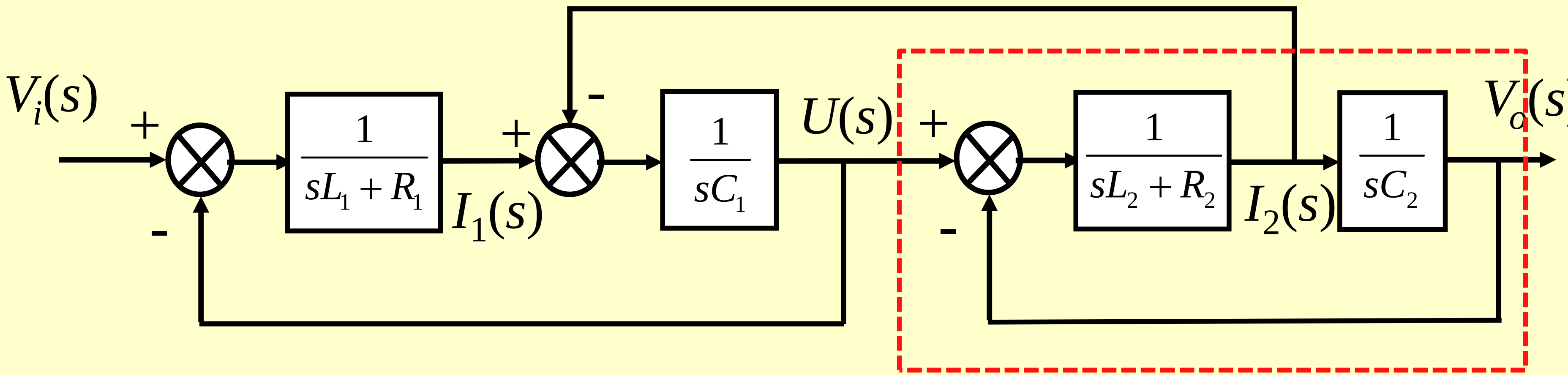
$$V_i(s) = I_1(s)(R_1 + sL_1) + U(s) \quad \Rightarrow \quad I_1(s) = \frac{1}{R_1 + sL_1}(V_i(s) - U(s))$$

$$sC_1U(s) = I_1(s) - I_2(s) \quad \Rightarrow \quad U(s) = \frac{1}{sC_1}(I_1(s) - I_2(s))$$

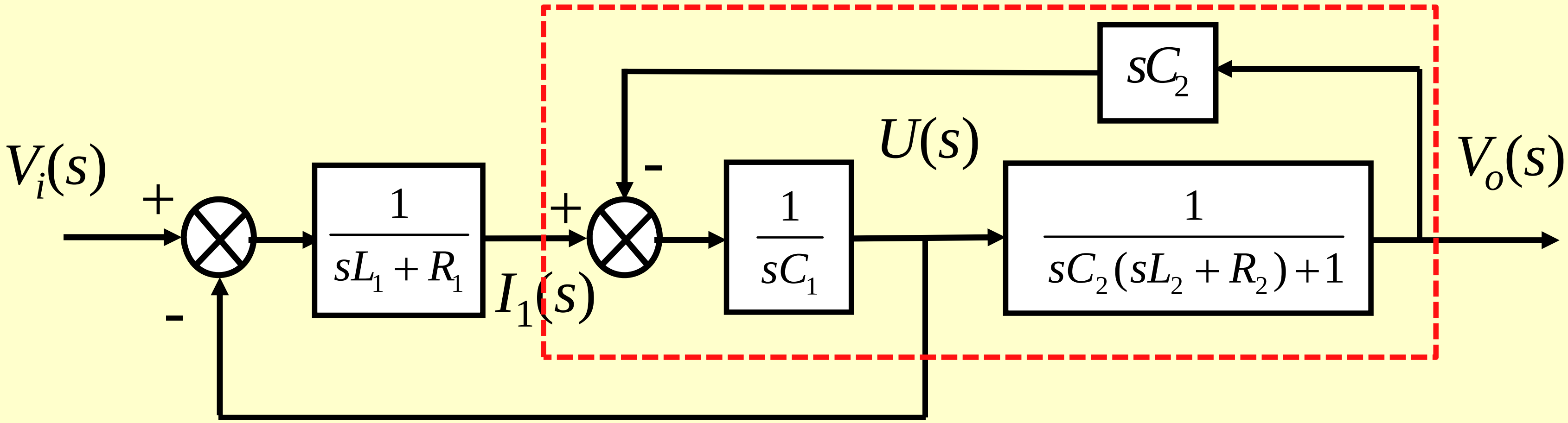
$$U(s) = I_2(s)(R_2 + sL_2) + V_o(s) \quad \Rightarrow \quad I_2(s) = \frac{1}{R_2 + sL_2}(U(s) - V_o(s))$$

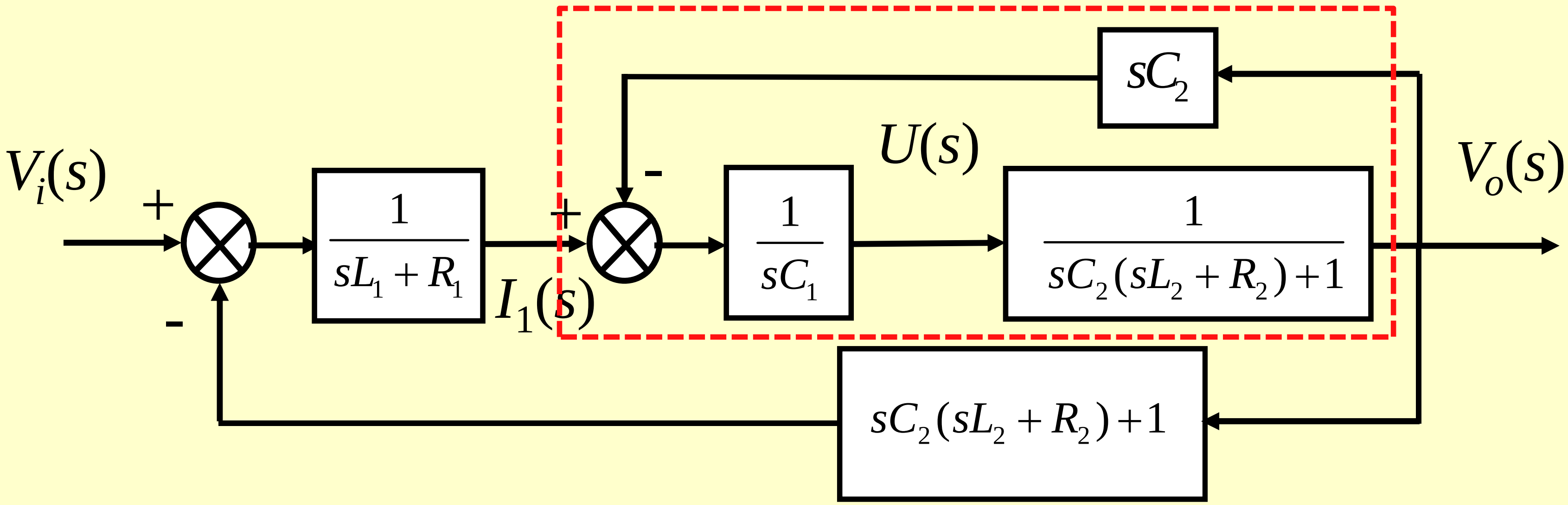
$$sC_2V_o(s) = I_2(s) \quad \Rightarrow \quad V_o(s) = \frac{1}{sC_2}I_2(s)$$

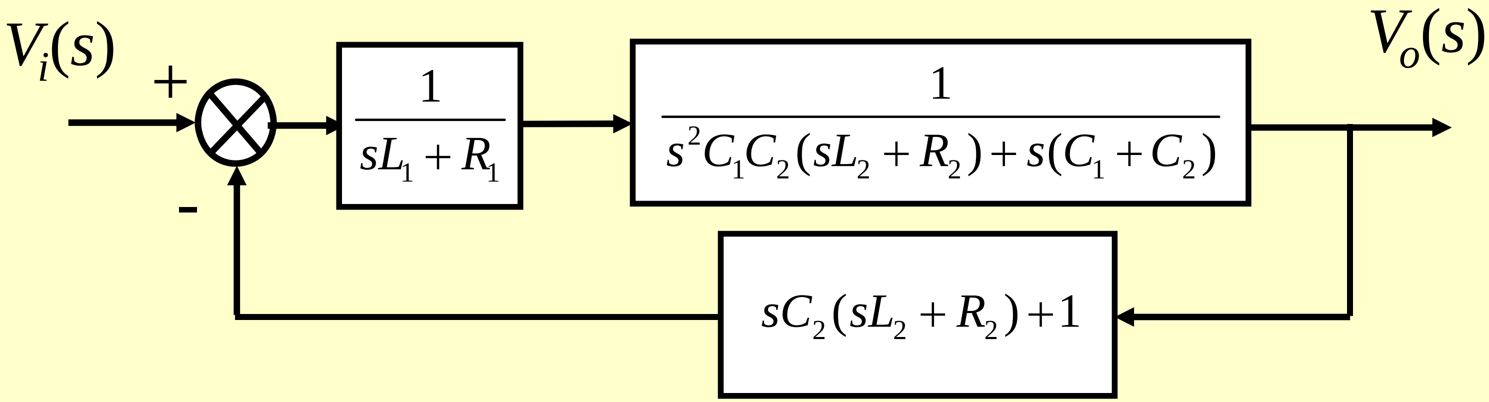
Thus, a block diagram representation is:



Simplifying (move take-off point and simplify):







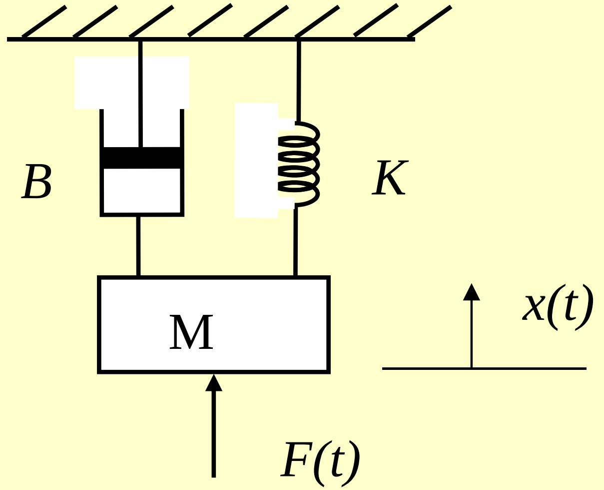
Thus,

$$\frac{V_0(s)}{V_i(s)} = \frac{1}{(sL_1+R_1)(s^2C_1C_2(sL_2+R_2)+s(C_1+C_2))+sC_2(sL_2+R_2)+1}$$

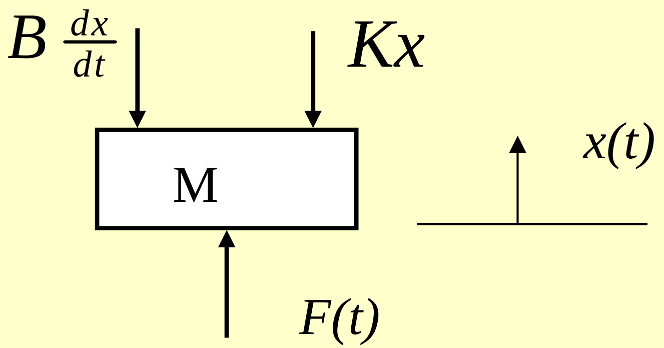
b) Mechanical and Electromechanical Systems

(i) *Translational Systems*

Example: Consider the mechanical system shown below. It models a car's suspension system. The output of interest is the displacement $x(t)$.



The *free-body diagram* of the system given below. (See Appendix 3.2) (*Q: What about the gravitational force*?)



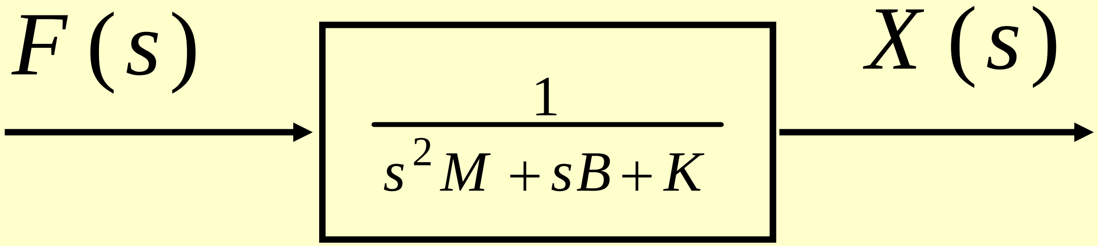
Applying Newton's law, the equation of motion is given by:

$$M\frac{d^2x}{dt^2} + B\frac{dx}{dt} + Kx = F$$

We have a 2nd-order ODE. In *s*-domain, we have

$$(s^2M + sB + K)X(s) = F(s)$$

i.e. $$X(s) = \frac{1}{s^2M + sB + K}F(s)$$



In an electromechanical system, electric current or voltage will be used to effect the actuation of the system to produce the motion.

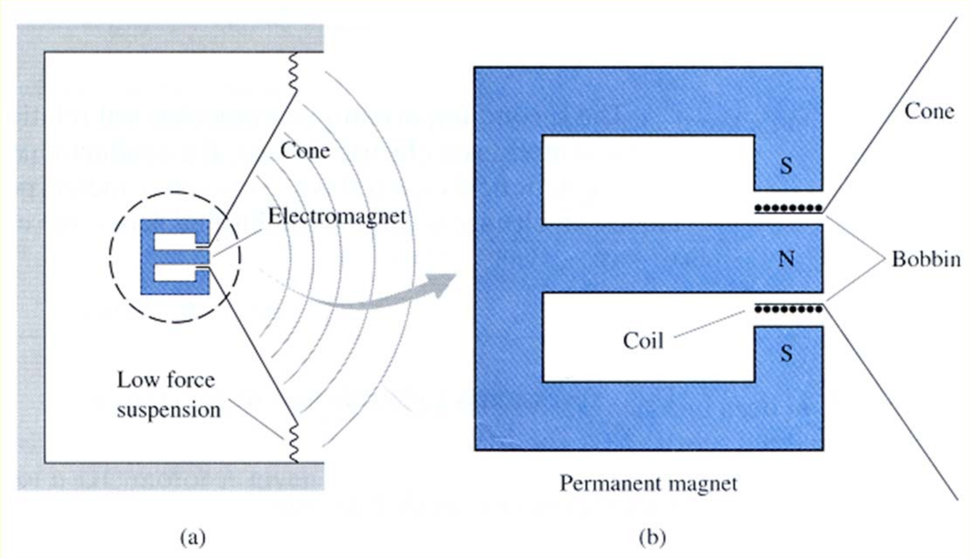
The first important electromechanical relationship is the effect of electric current on the mechanical motion:

If a current of i amp in a conductor of length l meters is arranged at right angles in a magnetic field of B_ϕ tesla, then there is a force on the conductor at right angles to the plane of i and B_ϕ, with magnitude

$$F = B_\phi l i \quad \text{newtons}$$

The force F will then produce a motion. The equation is the basis of conversion of electric energy to mechanical work and is called the **law of motors**.

Example (Modelling a Loudspeaker): In the following figure, the force on the conductor wound on the bobbin causes the voice coil to move, producing sound. The effects of the air can be modeled as if the cone had equivalent mass M and viscous friction coefficient B.



If the bobbin has N turns at a radius of r meters, then

$$l = 2N\pi r$$

The mechanical equation follows from Newton's laws (assuming spring effect is negligible):

$$M\ddot{x} + B\dot{x} = F$$

But $F = B_\phi l i.$

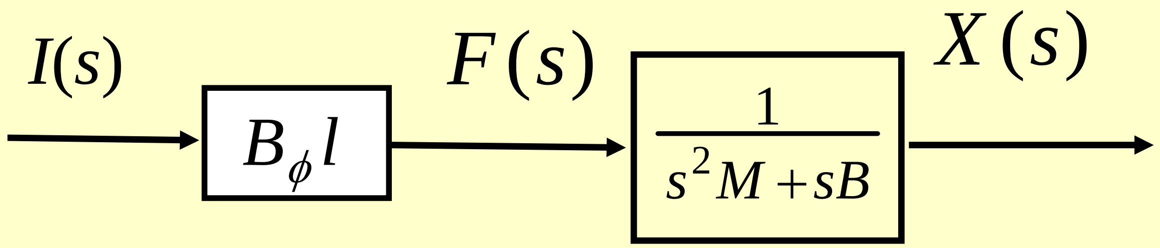
In s-domain, we have

$$(s^2 M + sB)X(s) = F(s) \;\Rightarrow X(s) = \frac{1}{s^2 M + sB} F(s)$$

and

$$F(s) = (B_\phi l) I(s)$$

Each of the above represents a subsystem (see Appendix 3.3). They are interconnected to give



$$\frac{X(s)}{I(s)} = \frac{B_\phi l}{s(sM + B)}$$

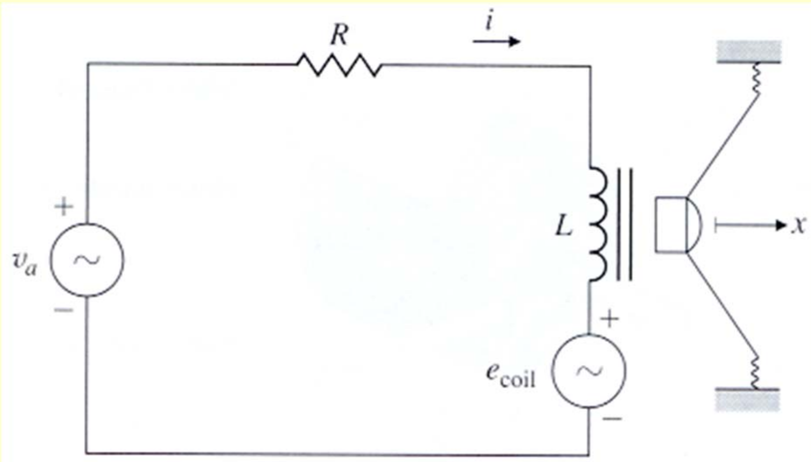
The second important electromechanical relationship is the effect of mechanical motion on electric voltage:

If a conductor of length l meters is moving in a magnetic field of B_ϕ tesla at a velocity of v m/s at mutually right angles, an electric voltage is established across the conductor:

$$e = B_\phi l v \quad \text{volts}$$

The above expression is called the **law of generators.**

Example: The loudspeaker considered earlier is now connected to a driving circuit as shown in the figure below.



We want to derive the differential equations relating the input voltage v_a to the output displacement x.

As before, the motion equation is given by

$$M\ddot{x} + B\dot{x} = F \tag{E1}$$

with

$$F = B_\phi l i \tag{E2}$$

The motion with velocity $\dot{x}$ results in a voltage across the coil given by

$$e_{coil} = B_\phi l \dot{x} \tag{E3}$$

The equation for the electric circuit is

$$L\frac{di}{dt} + Ri = v_a - e_{coil} \tag{E4}$$

Applying Laplace transform to Eqns (E1)-(E4), we have

$$(s^2M + sB)X(s) = F(s) \Rightarrow X(s) = \frac{1}{s^2M + sB}F(s)$$

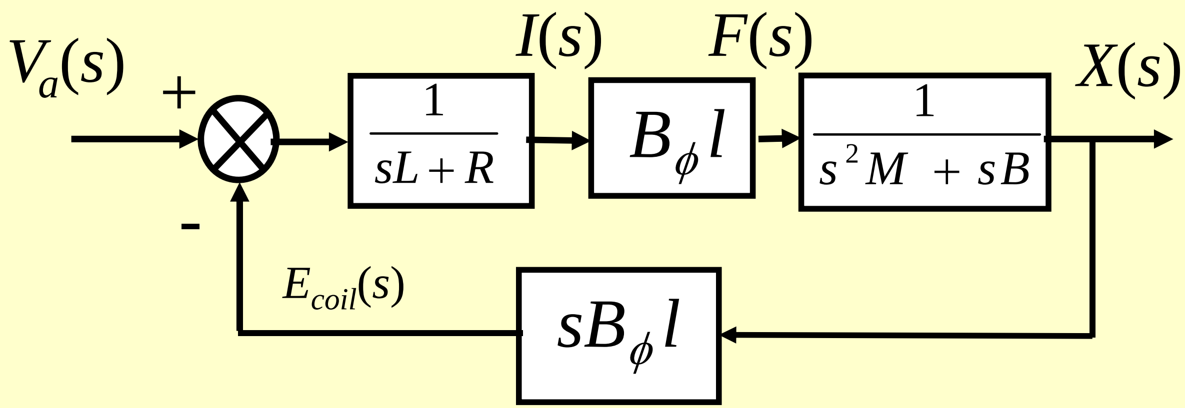
$$F(s) = (B_\phi l)\, I(s)$$

$$E_{coil}(s) = (sB_\phi l)X(s)$$

$$(sL + R)I(s) = V_a(s) - E_{coil}(s) \Rightarrow I(s) = \frac{1}{sL + R}(V_a(s) - E_{coil}(s))$$

Each of the above equation represents a sub-system. The required block diagram representation of the system can be obtained by interconnecting these subsystems appropriately. (See Appendix 3.4).

The block diagram representation of the system is given by:



So, the transfer function between the applied voltage and the loudspeaker displacement is

$$\frac{X(s)}{V_a(s)} = \frac{B_\phi l}{s[(Ms+B)(Ls+R)+(B_\phi l)^2]}$$

Appendix 3.1

Applying Laplace transform on the ODE,

$$L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = \frac{de}{dt}$$

Assuming zero initial conditions, we get

$$(s^2 L + sR + \tfrac{1}{C})I(s) = sE(s)$$

or

$$\frac{I(s)}{E(s)} = \frac{s}{s^2 L + sR + \frac{1}{C}}$$

i.e. we have

$$E(s) \longrightarrow \boxed{\frac{s}{s^2 L + sR + \frac{1}{C}}} \longrightarrow I(s)$$

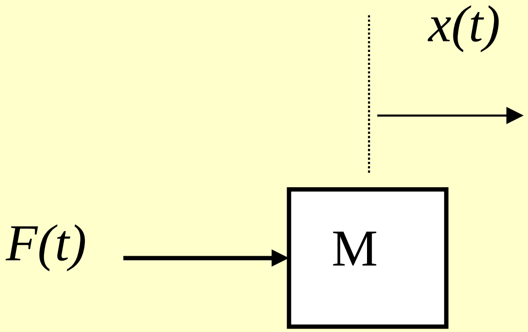
Appendix 3.2: Basic Force Components

Translational systems

Mass, friction and spring are 3 idealized parameters used in the analysis of mechanical systems.

The symbols and units used are : *x* [*m*], *M* [*Kg*], *F* [*N*], *t* [*sec*], *B* [*N/m/sec*], *K* [*N/m*]

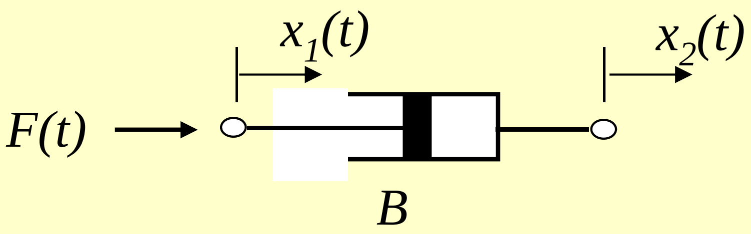
Mass



$$Force = mass \times acceleration$$

$$F = M \frac{d^2 x}{dt^2}$$

Damper (viscous friction)

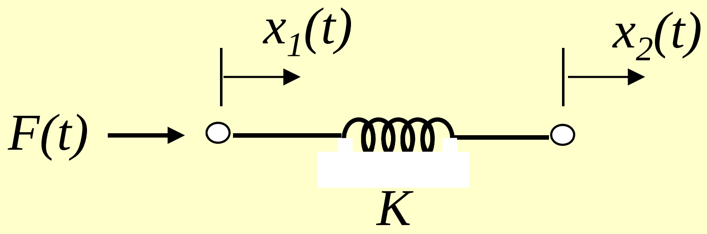


$$Force = B \times relative\_velocity$$

$$F = B\left(\frac{dx_1}{dt} - \frac{dx_2}{dt}\right)$$

where B = viscous friction coefficient.

Spring



$$Force = K \times elongation\,(compression)$$

$$F = K\left(x_1 - x_2\right)$$

where K = spring constant.

Appendix 3.3

From $X(s) = \frac{1}{s^2M+sB} F(s)$, we have the following sub-system

$$F(s) \longrightarrow \boxed{\frac{1}{s^2M+sB}} \longrightarrow X(s)$$

From $F(s) = B_\phi l I(s)$, we have

$$I(s) \longrightarrow \boxed{B_\phi l} \longrightarrow F(s)$$

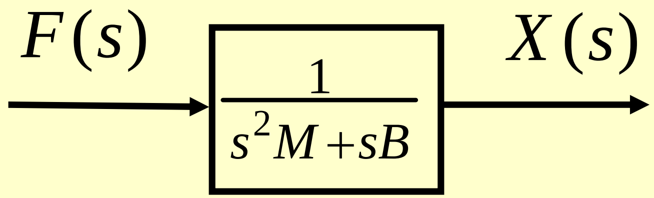
They are inter-connected to give

$$I(s) \longrightarrow \boxed{B_\phi l} \xrightarrow{F(s)} \boxed{\frac{1}{s^2M+sB}} \longrightarrow X(s)$$

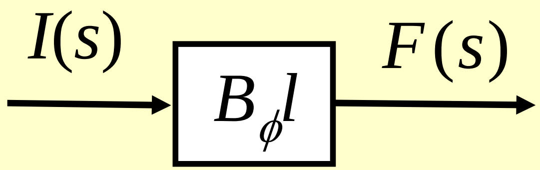
Hence, $$\frac{X(s)}{I(s)} = \frac{B_\phi l}{s(sM+B)}$$

Appendix 3.4

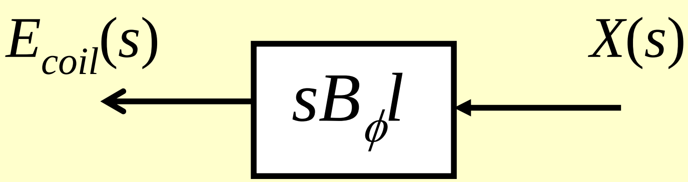
From $X(s) = \frac{1}{s^2 M + sB} F(s)$, we have the following sub-system



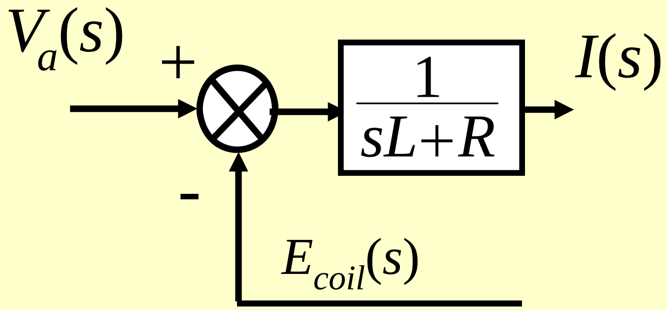
From $F(s) = B_\phi l I(s)$, we have



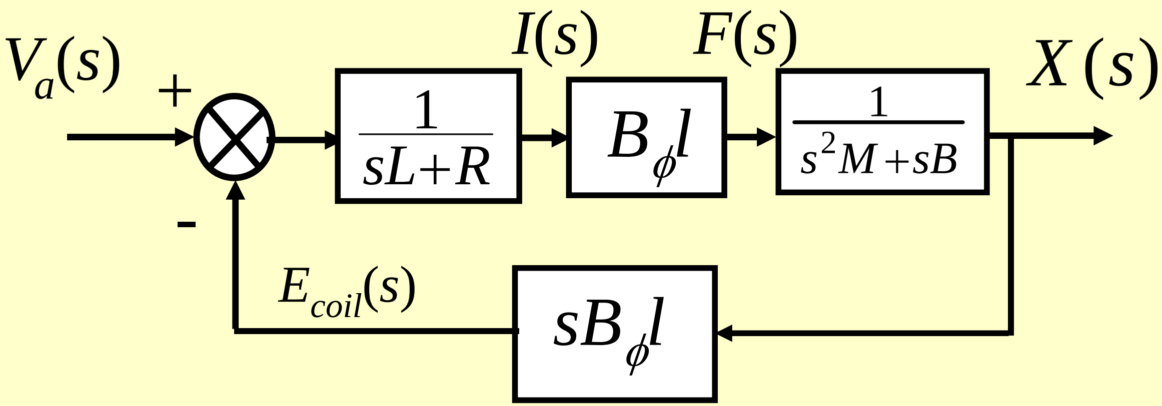
From $E_{coil}(s) = sB_\phi l X(s)$, we have



From $I(s) = \frac{V_a(s) - E_{coil}(s)}{sL+R}$, we have



They are inter-connected to give



Summary 3: Modelling of Physical Systems

RL electrical circuit: $L\frac{di}{dt} + Ri = e - e_b$

Translational mechanical system:

$$M\frac{d^2x}{dt^2} + B\frac{dx}{dt} + Kx = F$$

Electromechanical systems: Additional coupling between electrical and mechanical systems:

Law of motor: $F = a_1 i$

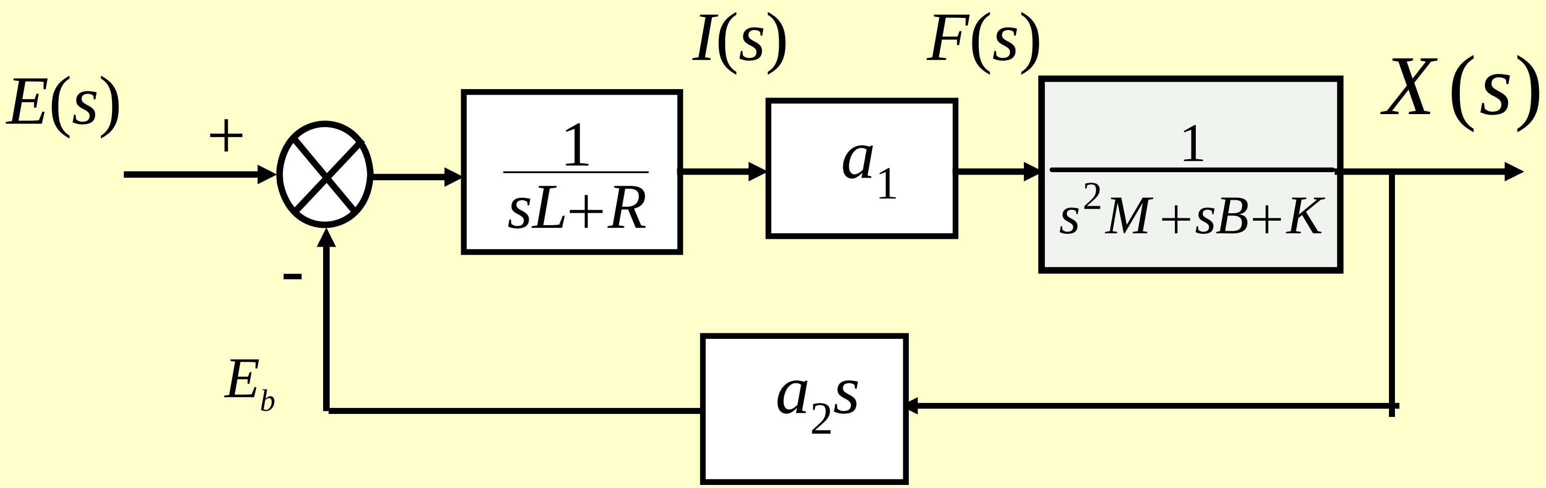
Law of generator: $e_b = a_2 \dot{x}$

where F, i, e, e_b, and $\dot{x}$ are force, current, voltage, induced voltage, and velocity, respectively.

Block Diagram Representation

We can use Laplace transform to transform the equations into *s*-domain. Each equation represents a subsystem.

These subsystems can be connected to form full block diagram representation of the system, e.g.



Lecture 4: System Modelling - Modelling of Physical Systems

a) Mechanical and Electromechanical Systems
   - Rotational Systems

b) Modelling of a DC Motor

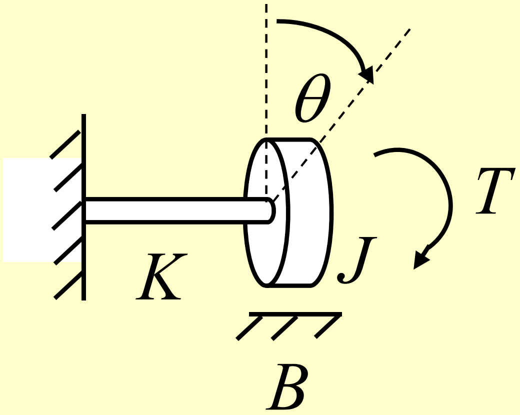
c) Servomechanism with Gear Trains

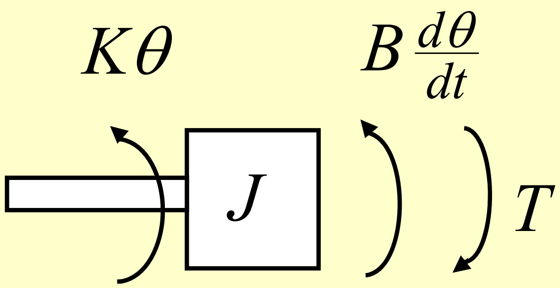
a) Mechanical and Electromechanical Systems

(ii) *Rotational Systems*

These involve fixed-axis rotation and are very important in many applications. (*Can you think of some?*)

<u>Example</u>: Consider the rotational mechanical system shown below. It models the wheel of a vehicle.





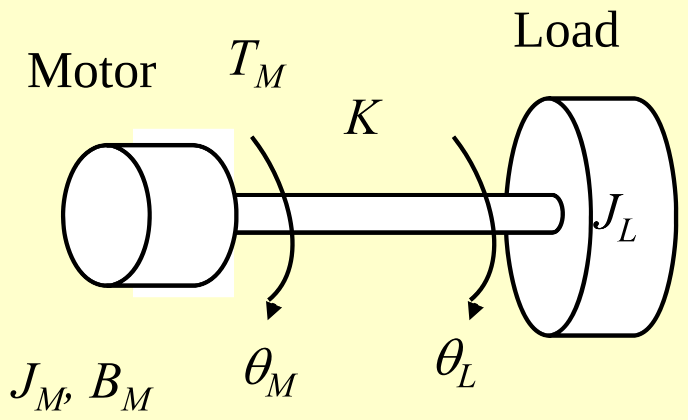
The torque equation: $J\dfrac{d^2\theta}{dt^2} + B\dfrac{d\theta}{dt} + K\theta = T$

That is, $\dfrac{\theta(s)}{T(s)} = \dfrac{1}{Js^2 + Bs + K}$

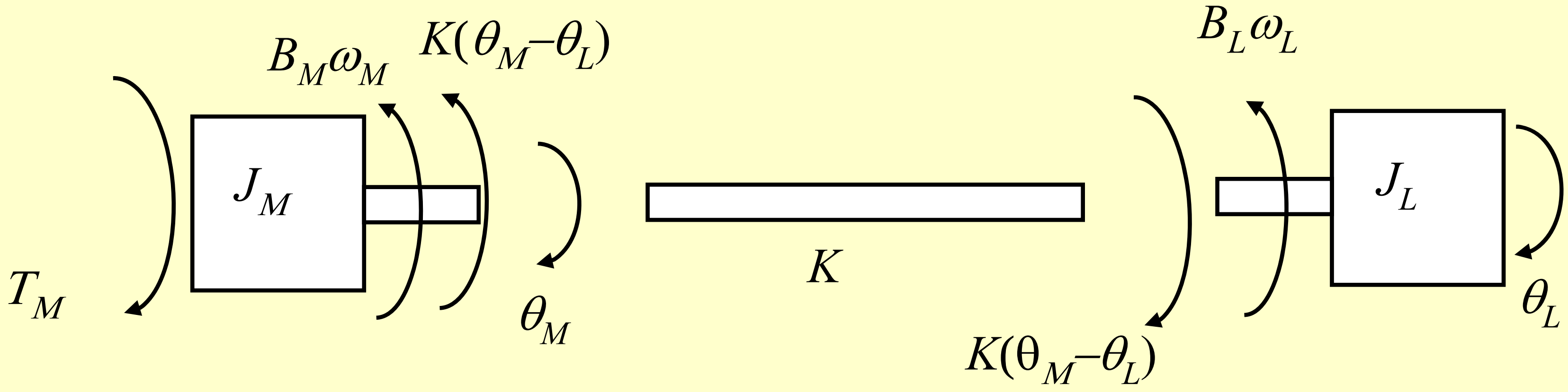
Example: Consider the situation where the load is connected to a motor through a non-rigid link as indicated below.

The system variables and parameters are defined below.

T_M = motor torque
J_M = motor inertia
B_M = motor viscous friction coefficient
J_L = load inertia
K = torsional shaft stiffness
θ_M = motor displacement
θ_L = load displacement
ω_M = motor velocity
ω_L = load velocity



The free-body diagrams of the system are shown below. (See Appendix 4.1 for review of basic torque components.)



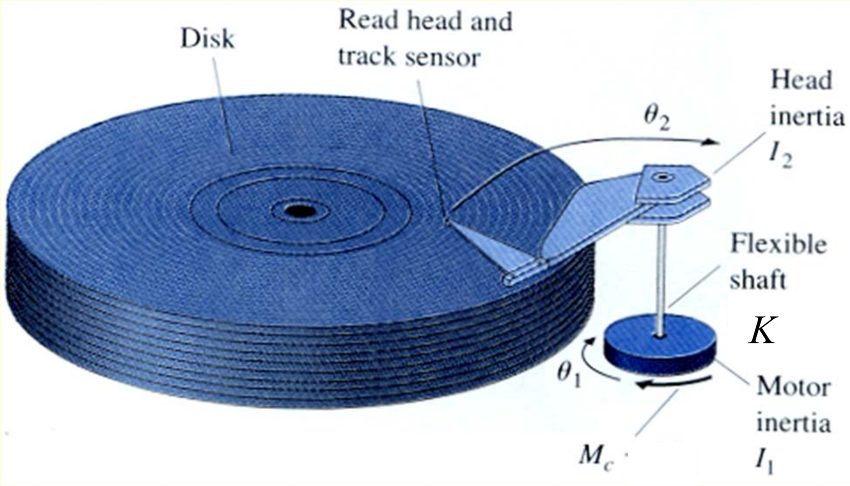
The torque equations of the system are:

$$T_M = J_M \frac{d^2\theta_M}{dt^2} + B_M \frac{d\theta_M}{dt} + K(\theta_M - \theta_L)$$

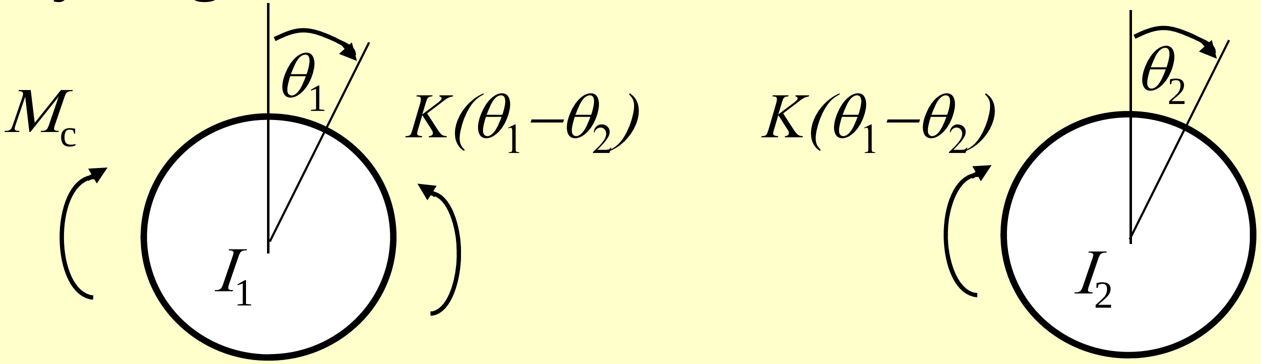
$$0 = J_L \frac{d^2\theta_L}{dt^2} + B_L \frac{d\theta_L}{dt} + K(\theta_L - \theta_M)$$

We have a coupled differential equation!

<u>Example:</u> The equations we derived above models the behaviour of a flexible Read/Write head of a Disk Drive. The viscous friction is assumed to be negligible.



The free-body diagram is



The torque equations are similar to the previous example:

$$M_c = I_1\ddot{\theta}_1 + K(\theta_1 - \theta_2) \quad \text{and} \quad 0 = I_2\ddot{\theta}_2 + K(\theta_2 - \theta_1)$$

Apply Laplace transform to the 2 equations, we get

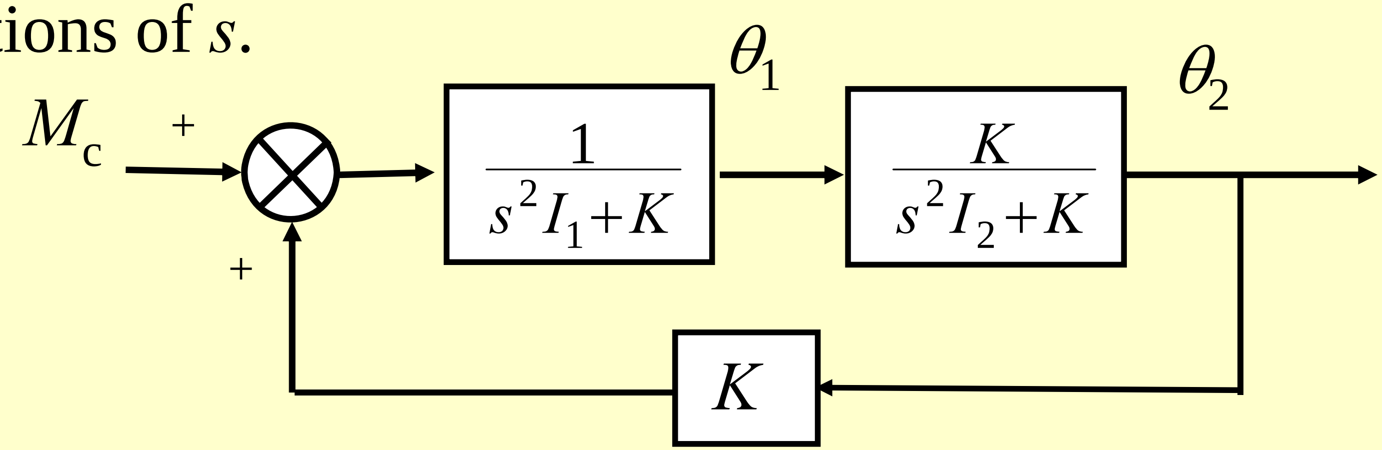
$$M_c(s) = s^2 I_1 \theta_1(s) + K(\theta_1(s) - \theta_2(s))$$

$$\Rightarrow \theta_1(s) = \frac{1}{s^2 I_1 + K}(M_c(s) + K\theta_2(s))$$

$$0 = s^2 I_2 \theta_2(s) + K(\theta_2(s) - \theta_1(s))$$

$$\Rightarrow \theta_2(s) = \frac{K}{s^2 I_2 + K}\theta_1(s)$$

The block diagram representation is given below. Note that M_c, θ_1, θ_2 are now functions of s.



Note that we have a positive feedback configuration!

The transfer function relating θ_2 to M_c is given by

$$\frac{\theta_2(s)}{M_c(s)} = \frac{K}{I_1 I_2 s^2 [s^2 + K(\frac{1}{I_1} + \frac{1}{I_2})]}$$

We have a 4th order system!

NB: A simplified and yet very effective model for the flexible disk drive is the double integrator model:

$$\frac{\theta(s)}{M_c(s)} = \frac{K_1}{s^2}$$

The double integrator is also a very effective model for the attitude control of communication satellites. It's an important class of control systems.

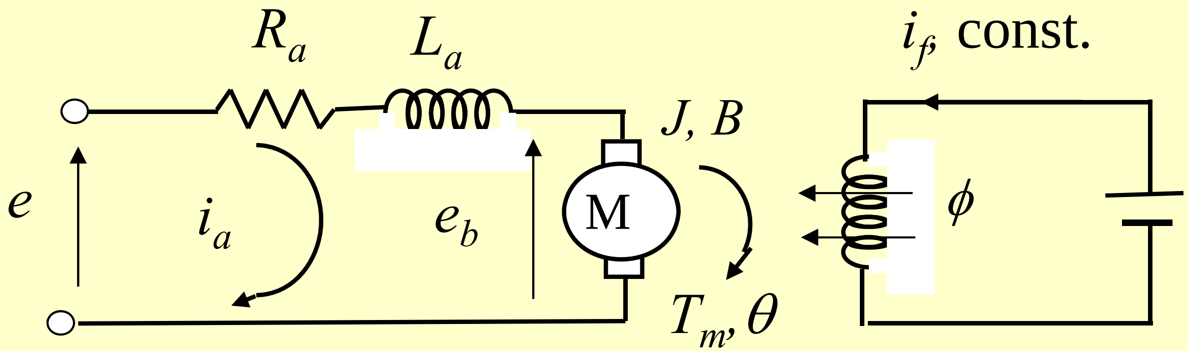
Exercise: For the disk drive system, derive the transfer function between θ_1 and M_c. *(See Appendix 4.2)*

b) Modelling of a DC Motor

DC motor is an electromechanical system with rotational motion. It has very wide applications, e.g. control the turning of a robot arm.

There are generally 2 control modes: ***Armature Control*** and *Field Control*.

Armature Control



When operating on the linear range of magnetization curve, the air-gap flux is $\phi = K_f i_f$, where K_f is a constant.

Motor torque developed is

$$T_m = K_1 \phi i_a \qquad \text{(law of motors)}$$

where K_1 is a constant. If

$$i_f \text{ constant} \quad \Rightarrow \quad \phi \text{ constant}$$

Hence $T_m = K_T i_a$ (law of motors) (e1)

where K_T is the motor torque constant.

Motor back emf is

$$e_b = K_b \frac{d\theta}{dt} \qquad \text{(law of generators)} \qquad \text{(e2)}$$

where K_b is a constant.

For the armature circuit,

$$L_a \frac{di_a}{dt} + Ri_a + e_b = e \qquad \text{(e3)}$$

When there is no load connected to the motor, the torque equation is

$$J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} = T_m \qquad \text{(e4)}$$

Taking Laplace transform of eqns (e1)-(e4), assuming zero initial conditions, we have

$$T_m(s) = K_T I_a(s) \qquad \text{(e5)}$$

$$E_b(s) = K_b s\theta(s) \qquad \text{(e6)}$$

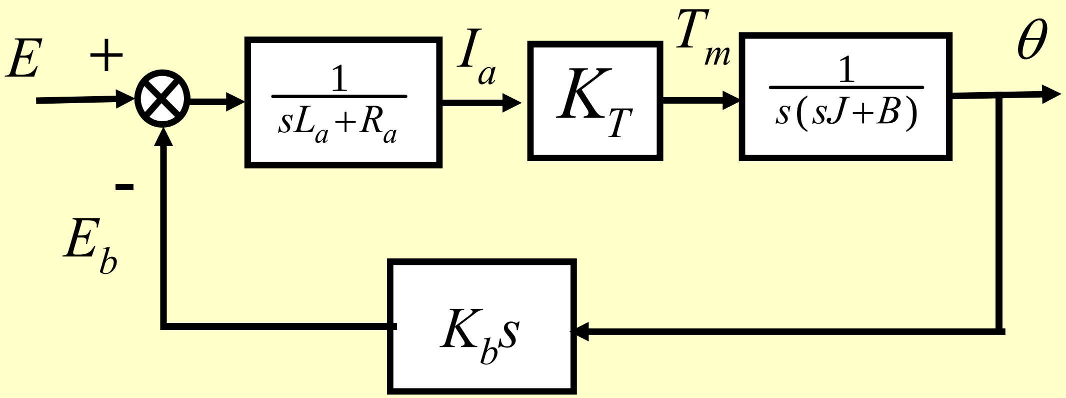
$$\left(L_a s + R_a\right) I_a(s) = E(s) - E_b(s)$$

$$\Rightarrow I_a(s) = \frac{1}{L_a s + R_a}\left(E(s) - E_b(s)\right) \qquad \text{(e7)}$$

$$\left(Js^2 + Bs\right)\theta(s) = T_m(s)$$

$$\Rightarrow \theta(s) = \frac{1}{Js^2 + Bs} T_m(s) \qquad \text{(e8)}$$

The block diagram representation of equations (e5)-(e8) is



So, the transfer function is given by

$$T(s) = \frac{\theta(s)}{E(s)} = \frac{K_T}{s\left[(L_a s + R_a)(Js + B) + K_T K_b\right]}$$

In many applications, L_a is usually very small, so the equation can be simplified to

$$T(s) = \frac{\theta(s)}{E(s)} = \frac{K_T / R_a}{s(Js + B + K_T K_b / R_a)} \qquad \text{(e9)}$$

The term $\left(B + \frac{K_T K_b}{R_a}\right)$ indicates that the back emf of the motor effectively increases the viscous friction of the system.

Let $B_1 = \left(B + \frac{K_T K_b}{R_a}\right)$,

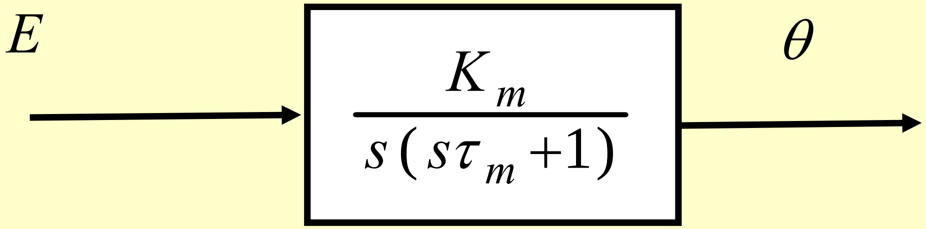
$\tau_m = \frac{J}{B_1}$, motor time constant,

$K_m = \frac{K_T}{R_a B_1}$, motor gain constant

Then the transfer function becomes

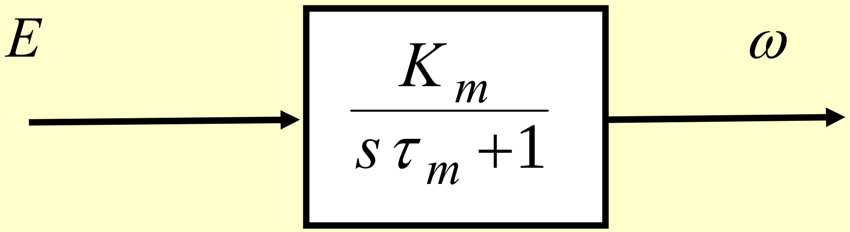
$$T(s) = \frac{\theta(s)}{E(s)} = \frac{K_m}{s(s\tau_m + 1)}$$

That is, we have a second order dynamic system. It describes the dynamic relationship between the output $\theta(s)$ and the input $E(s)$ of a DC motor. The block diagram is given by



In some applications, we are also interested in the transfer function between the motor input and the output speed $(\omega(s) = s\theta(s))$.
In that case, the transfer function would be

$$\frac{\omega(s)}{E(s)} = \frac{K_m}{s\tau_m + 1}$$

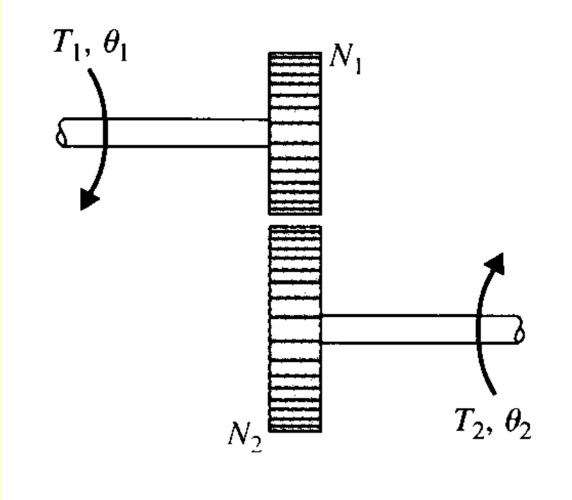


c) Servomechanism with Gear Trains

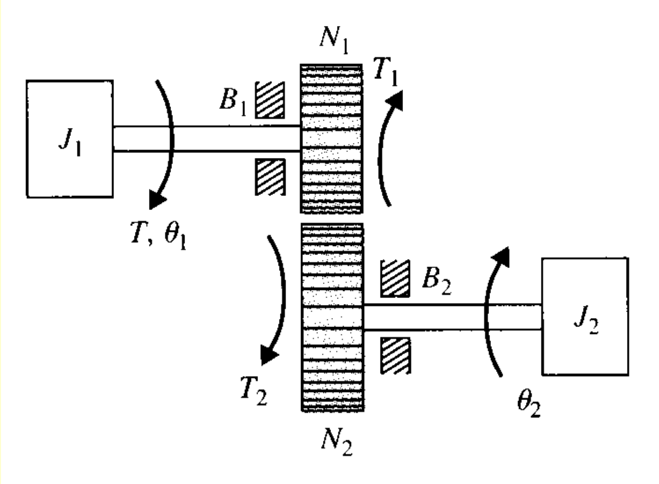
Gear train is a mechanical device that transmits energy from one part of the system to another in such a way that torque, speed and displacement may be altered. It is commonly found in many implementations.

The relationships between the torques, angular displacements, angular velocities, and the teeth numbers are:

$$\frac{T_1}{T_2} = \frac{\theta_2}{\theta_1} = \frac{N_1}{N_2} = \frac{\omega_2}{\omega_1}$$



In practice, gears do have inertia and friction between the coupled gear teeth. In the calculations, we need to reflect inertia, friction and torque from one side of gear train to the other. In the figure, T is the applied torque while T_1 and T_2 are transmitted torques.

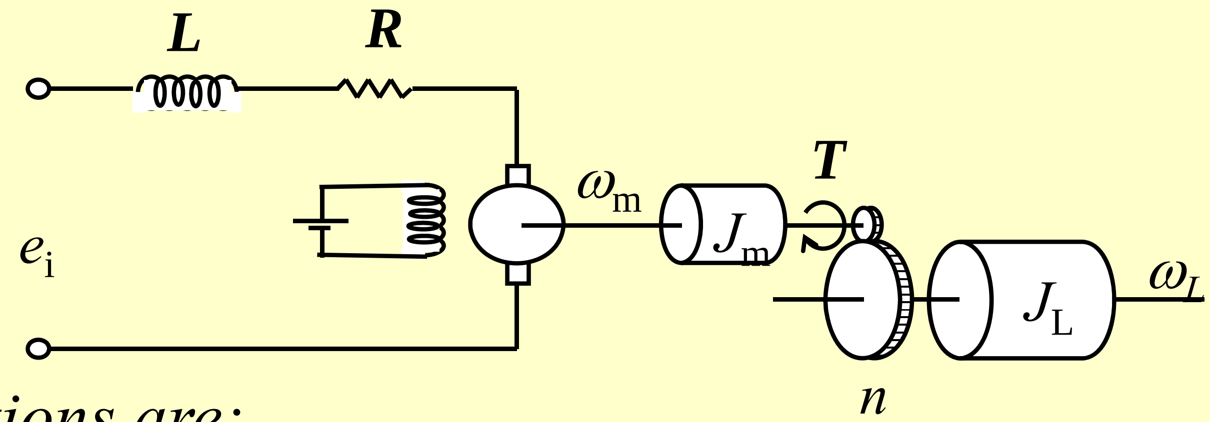


The following quantities are obtained when reflecting from gear 2 to gear 1:

Inertia: $\left(\frac{N_1}{N_2}\right)^2 J_2$ Viscous friction coefficient: $\left(\frac{N_1}{N_2}\right)^2 B_2$

Hence $J = J_1 + \left(\frac{N_1}{N_2}\right)^2 J_2$ and $B = B_1 + \left(\frac{N_1}{N_2}\right)^2 B_2$

<u>Exercise</u>: The figure below shows an armature-controlled dc servomotor driving a load with moment of inertia J_L. The load torque seen by the motor is T. The angular velocities of the motor rotor and the load elements are ω_m and ω_L respectively. The gear ratio is $n = \omega_m / \omega_L$. Let the coefficient of viscous friction, motor back emf constant and motor torque constant be B_m, K_b and K respectively. Assume zero friction at the load. Using the block diagram approach, obtain the overall transfer function of the system with ω_L as the output and e_i as the input. (See Appendix 4.3)



The relevant equations are:

$$L\frac{di_a}{dt} + Ri_a + e_b = e_i \; ; \quad T = Ki_a ; \quad J\frac{d\omega_m}{dt} + B\omega_m = T \; ; \quad e_b = K_b\omega_m$$

$$J = J_m + \tfrac{1}{n^2}J_L \; ; \qquad B = B_m \; ; \qquad \omega_L = \tfrac{1}{n}\omega_m$$

Appendix 4.1: *Basic Torque Components*

Rotational Systems

The variables of interest are torque and angular displacement, or velocity or acceleration.

The 3 basic components are: moment of inertia, torsional spring and viscous friction.

The symbols and units used in rotational motion are:

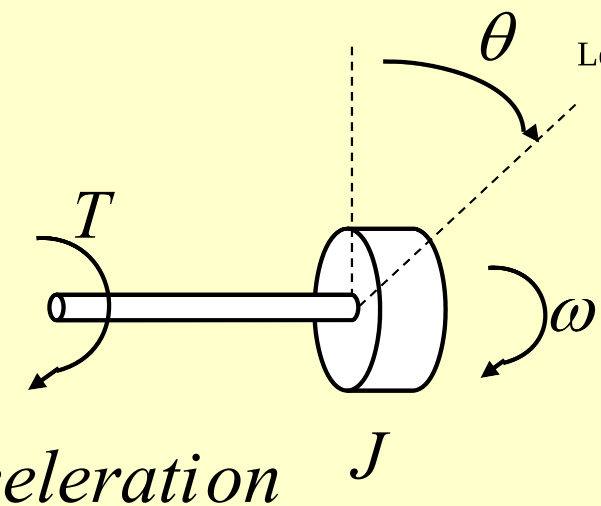
θ [*rad*] ; ω [*rad/sec*]

J [$Kg–m^2$] ; T [$N–m$]

t [*sec*] ; B [*N-m/rad/sec*]

K [*N–m/rad*]

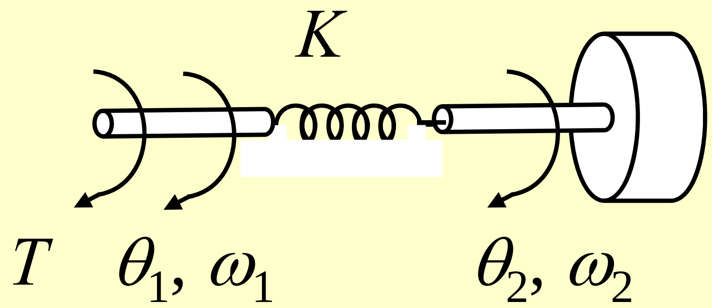
Moment of Inertia, J



$$Torque = J \times angular\_acceleration$$

$$T = J \times \frac{d\omega}{dt} = J \times \frac{d^2\theta}{dt^2}$$

Torsional Spring

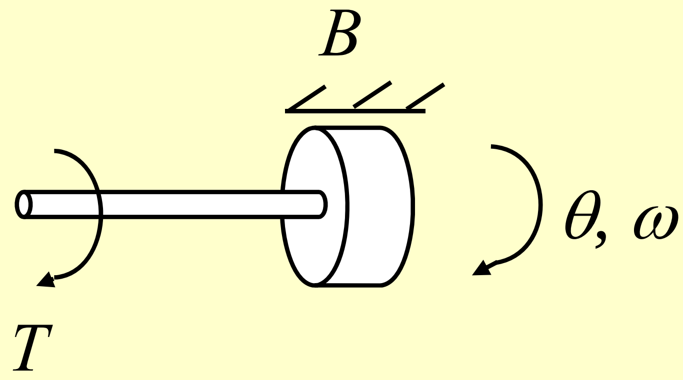


$$Torque = K \times angular\_displacement$$

$$T = K(\theta_1 - \theta_2) = K \int_{-\infty}^{t} (\omega_1 - \omega_2) d\tau$$

where K = torsional shaft stiffness.

Viscous Friction



$$Torque = B \times angular\_velocity$$

$$T = B\omega = B\frac{d\theta}{dt}$$

where $B =$ viscous friction coefficient.

Appendix 4.2

From the block diagram on page 4-5,

$$\theta_1 = \frac{1}{s^2 I_1 + K}(M_c + K\theta_2) \qquad \text{(A1)}$$

$$\theta_2 = \frac{K}{s^2 I_2 + K}\theta_1 \qquad \text{(A2)}$$

Sub. (A2) into (A1), and rearranging, we get

$$(1 - \frac{K^2}{(s^2 I_1 + K)(s^2 I_1 + K)})\theta_1 = \frac{1}{s^2 I_1 + K} M_c$$

So, we have $$\frac{I_1 I_2 s^2 [s^2 + K(\frac{1}{I_1} + \frac{1}{I_2})]}{(s^2 I_1 + K)(s^2 I_2 + K)}\theta_1 = \frac{1}{s^2 I_1 + K} M_c$$

i.e. $$\frac{\theta_1}{M_c} = \frac{s^2 I_2 + K}{I_1 I_2 s^2 [s^2 + K(\frac{1}{I_1} + \frac{1}{I_2})]}$$

Appendix 4.3

The relevant equations in s-domain are:

$$T(s) = KI_a(s) \qquad \text{(B1)}$$

$$E_b(s) = K_b\Omega_m(s) \qquad \text{(B2)}$$

$$(Ls+R)I_a(s) = E_i(s) - E_b(s)$$

$$\Rightarrow I_a(s) = \tfrac{1}{Ls+R}(E_i(s) - E_b(s)) \qquad \text{(B3)}$$

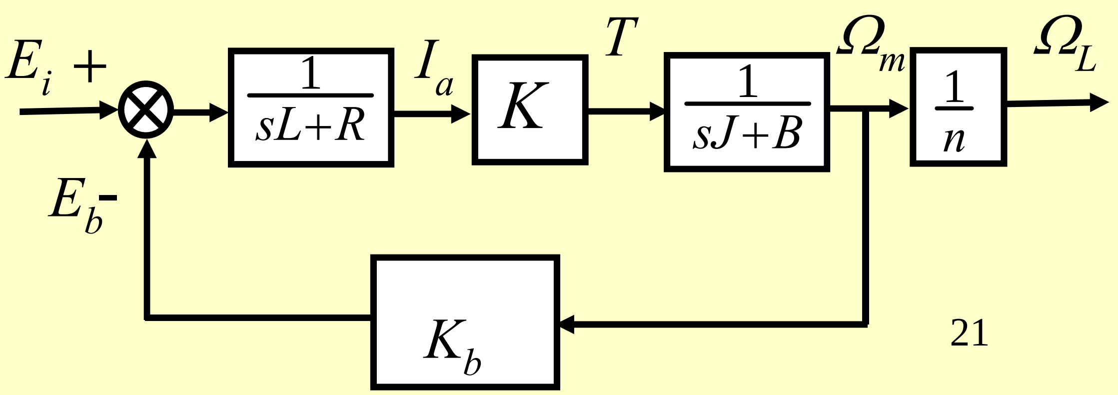
$$(Js+B)\Omega_m(s) = T(s)$$

$$\Rightarrow \Omega_m(s) = \tfrac{1}{Js+B}T(s) \qquad \text{(B4)}$$

$$\Omega_L(s) = \tfrac{1}{n}\Omega_m(s) \qquad \text{(B5)}$$

where $J = J_m + \frac{1}{n^2}J_L$ and $B = B_m$

Construct individual sub-systems and inter-connect them, we get the following speed control system:



Summary 4: Modelling of Physical Systems

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$$J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} + K\theta = T$$

Electromechanical systems: Additional coupling between electrical and mechanical systems:

<u>Law of motor</u>: $T = a_1 i$

<u>Law of generator</u>: $e_b = a_2\dot{\theta}$

where T, i, e, e_b and $\dot{\theta}$ are torque, current, voltage, induced voltage, and angular velocity, respectively.

Block Diagram Representation

We can use Laplace transform to transform the equations into *s*-domain. Each equation represents a subsystem.

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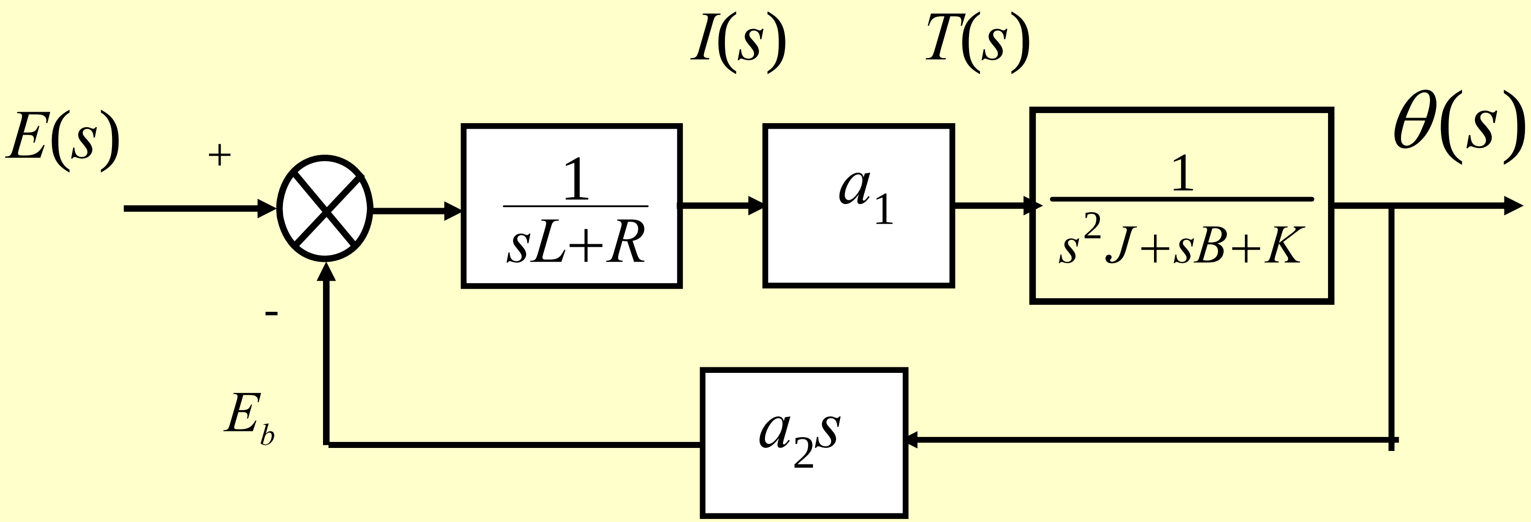


Lecture 5: Time Domain Analysis - Responses of 1st and 2nd Order Systems

a) Steady-state and Transient Responses

b) Typical Test Signals

c) Time Response of 1st Order Systems

- Time constant

d) Time Response of Standard 2nd Order Systems

- Damping ratio
- Un-damped natural frequency

a) Steady-state and Transient Responses

In the time domain analysis, a reference input signal is applied to a system, and the system performance is evaluated by studying the system's response in the time domain.

The typically input (test) signals are impulse, **step**, ramp and parabolic functions, and sinusoidal function (**frequency domain**)

For a **stable** system, the total time response is generally given by

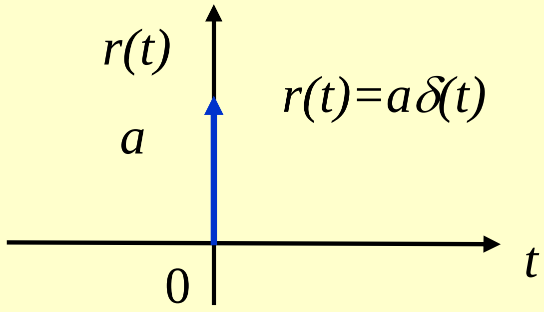
$$y(t) = y_t(t) + y_{ss}(t)$$

In many applications we want $y_{ss}(t)$, the steady-state response, to follow the test input signal (in a certain way) while $y_t(t)$, the transient response, to "disappear quickly".

b) Typical Test Signals

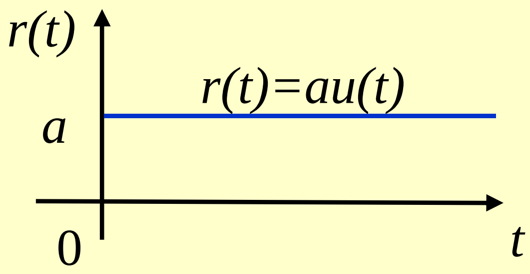
Impulse Function:

$r(t) = a\delta(t)$; a is a constant



Step Function:

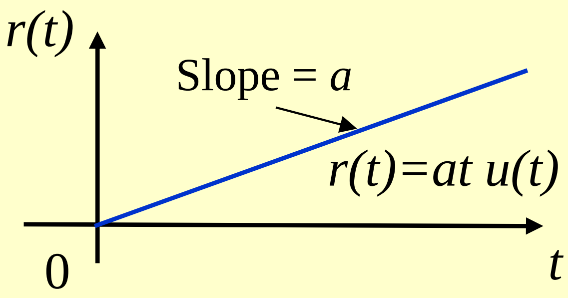
$r(t) = au(t)$; a is a constant



Ramp Function:

$$r(t) = \begin{cases} at & t \geq 0 \\ 0 & t < 0 \end{cases}$$

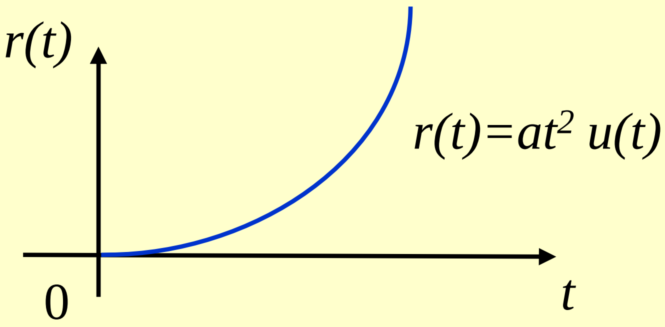
$$= \text{Integral of step function}$$



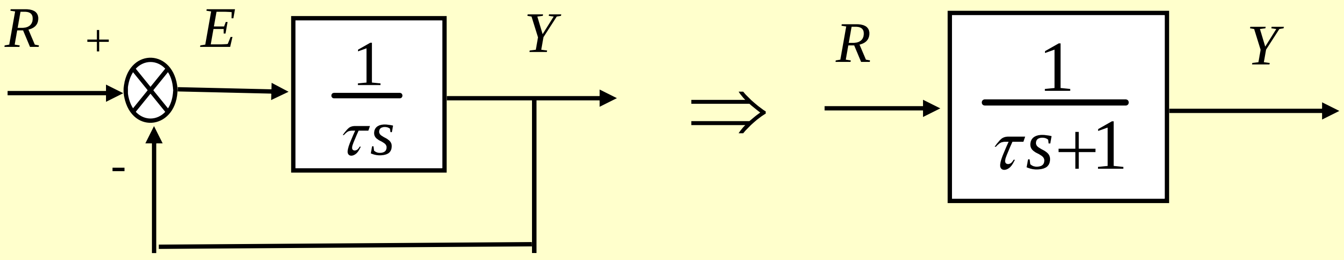
Parabolic Function:

$$r(t) = \begin{cases} at^2 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$= \text{Integral of ramp function}$$

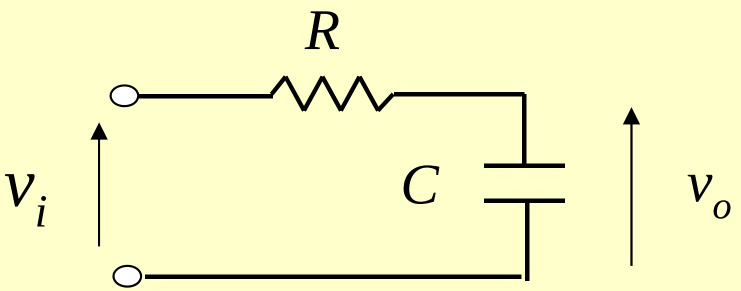


c) Time Response of First-Order Systems



Physically, a first-order system may represent an *RC* circuit, thermal system, pneumatic system, etc.

<u>Example</u>:

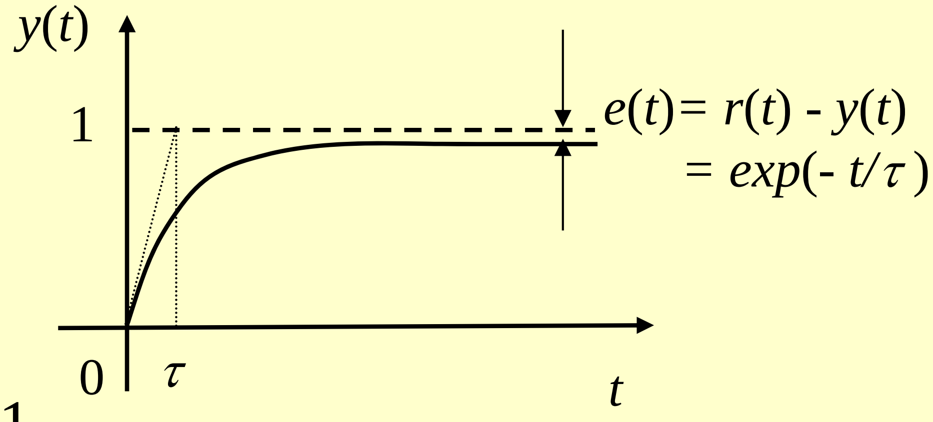


$$\frac{Y(s)}{R(s)} = \frac{V_o}{V_i} = \frac{1}{\tau s + 1} \quad ; \quad \tau = RC$$

Unit-step Response: $r(t) = u(t); \quad R(s) = \dfrac{1}{s}$

$$Y(s) = \frac{1}{(\tau s + 1)} \cdot \frac{1}{s} = \frac{1}{s} - \frac{\tau}{\tau s + 1}$$

$$\therefore \quad y(t) = 1 - e^{-\frac{t}{\tau}} \quad ; \quad t \geq 0$$



The slope at $t = 0$ is given by $\left.\dfrac{dy}{dt}\right|_{t=0} = \dfrac{1}{\tau}$.

τ is known as the **time constant**.

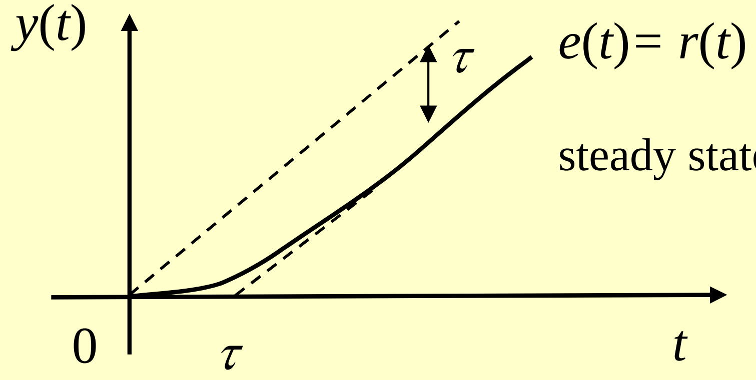
The error signal is $e(t) = r(t) - y(t)$

$$= e^{-\frac{t}{\tau}}$$

The steady-state error $e_{ss} = 0$.

Unit-ramp Response: $r(t) = t; \quad R(s) = \dfrac{1}{s^2}$

$$Y(s) = \frac{1}{(\tau s+1)} \cdot \frac{1}{s^2} = \frac{1}{s^2} - \frac{\tau}{s} + \frac{\tau^2}{\tau s+1}$$



So, $y(t) = t - \tau(1 - e^{-\frac{t}{\tau}}) \quad ; \quad t \geq 0$.

The eror signal is $e(t) = r(t) - y(t)$

$$= \tau(1 - e^{-\frac{t}{\tau}})$$

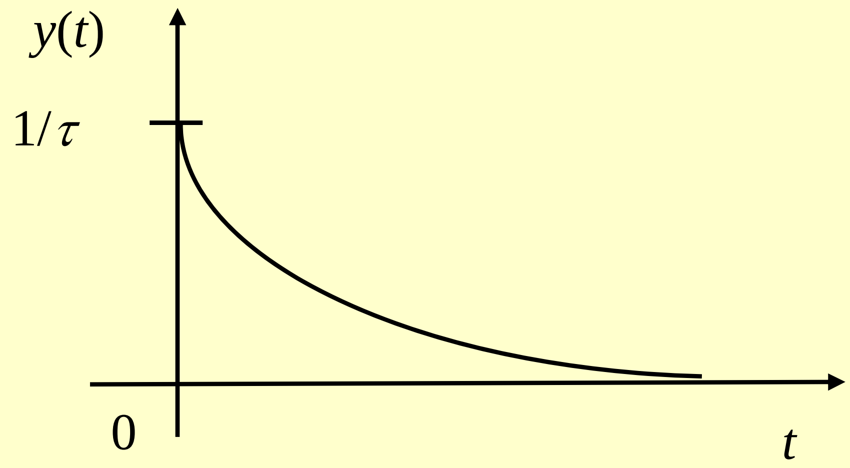
The steady-state error $e_{ss} = \tau$.

Reducing the time constant τ not only reduces the steady-state error to a ramp input but also speeds up the response.

Exercise: Show that $e_{ss} = a\tau$ *if* $R(s) = a/s^2$ *(See Appendix 5.1)* .

Unit-impulse Response: $r(t) = \delta(t)$; $R(s) = 1$

$$Y(s) = \frac{1}{(\tau s + 1)} \cdot 1$$



$$\therefore \quad y(t) = \frac{1}{\tau} e^{-\frac{t}{\tau}} \quad ; \quad t \geq 0$$

The error signal is

$$e(t) = r(t) - y(t)$$

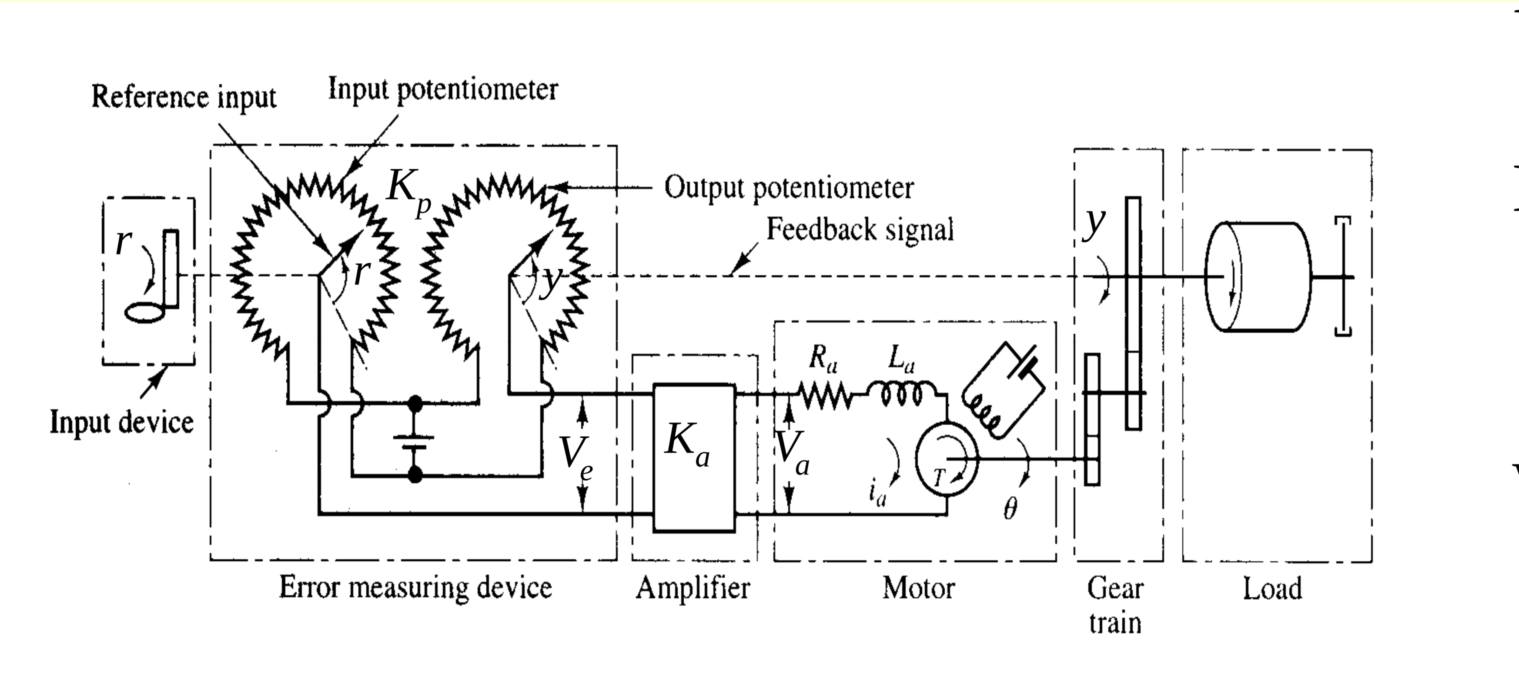
$$= \delta(t) - \frac{1}{\tau} e^{-\frac{t}{\tau}}$$

The steady-state error $e_{ss} = 0$.

d) Time Response of Standard Second-Order Systems

2nd-order systems are very important in the study of control engineering. Most control designs are based on 2nd-order system analysis (at least approximately). Even if the system is of higher order, its main response may be approximated by a 2nd-order system analysis.

Consider the dc motor servomechanism:



Potentiometer:

$$V_e = K_p(R - Y)$$

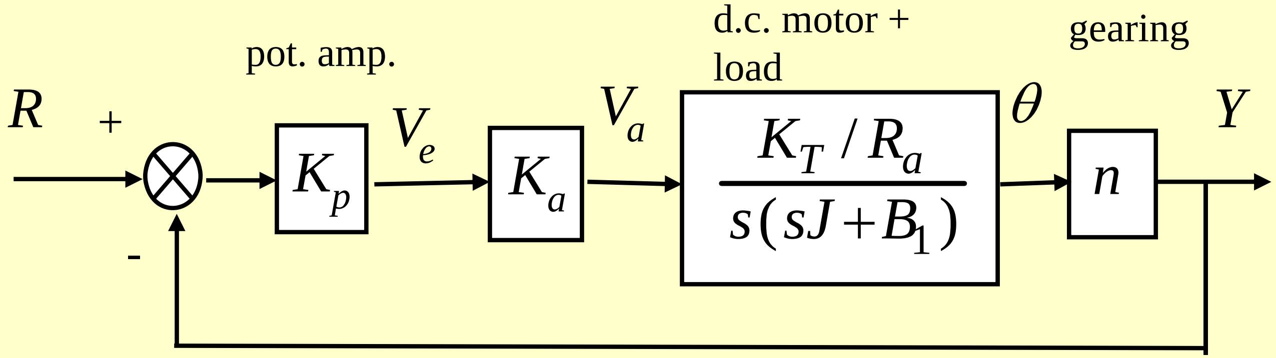
Motor and load:

$$\frac{\theta}{V_a} = \frac{K_T / R_a}{s(sJ + B_1)}$$

where

$$B_1 = B + \frac{K_T K_b}{R_a}$$

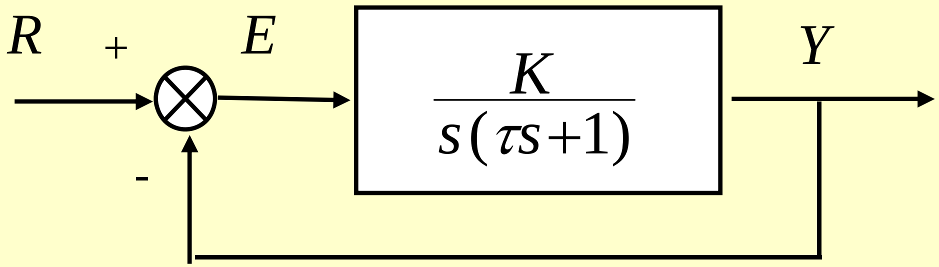
The block diagram is



Let $K' = K_p K_a \dfrac{K_T}{R_a} n$, then $G(s) = \dfrac{K'}{s(sJ+B_1)} = \dfrac{K}{s(s\tau+1)}$

where $K = \dfrac{K'}{B_1}$, $\quad \tau = \dfrac{J}{B_1}$.

The simplified block diagram is given by



The closed-loop transfer function is

$$T(s) = \frac{Y(s)}{R(s)} = \frac{K}{\tau s^2 + s + K} = \frac{\frac{K}{\tau}}{s^2 + \frac{1}{\tau}s + \frac{K}{\tau}} \qquad (1)$$

This is a (nice) **standard** 2nd-order system.

A standard 2nd order system is expressed as:

$$\frac{Y(s)}{R(s)} = T(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \qquad (2)$$

where ζ is called the **damping ratio** and ω_n the **un-damped natural frequency.**

For a given system, e.g. system (1), its ζ and ω_n can be obtained by comparing (1) with (2), i.e.

$$\zeta = \frac{1}{2\sqrt{K\tau}} \quad \left(= \frac{B_1}{2\sqrt{K'J}} \right) \quad \text{and} \quad \omega_n = \sqrt{\frac{K}{\tau}} \quad \left(= \sqrt{\frac{K'}{J}} \right)$$

The transfer function (2) defines ***a standard 2nd-order system.*** The denominator polynomial $q(s) = s^2 + 2\zeta\omega_n s + \omega_n^2$ is known as the **characteristic polynomial**, and $q(s) = 0$ is known as the ***characteristic equation***. It gives the poles of the system.

In this case,

$$q(s) = s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \qquad (3)$$

The roots are:

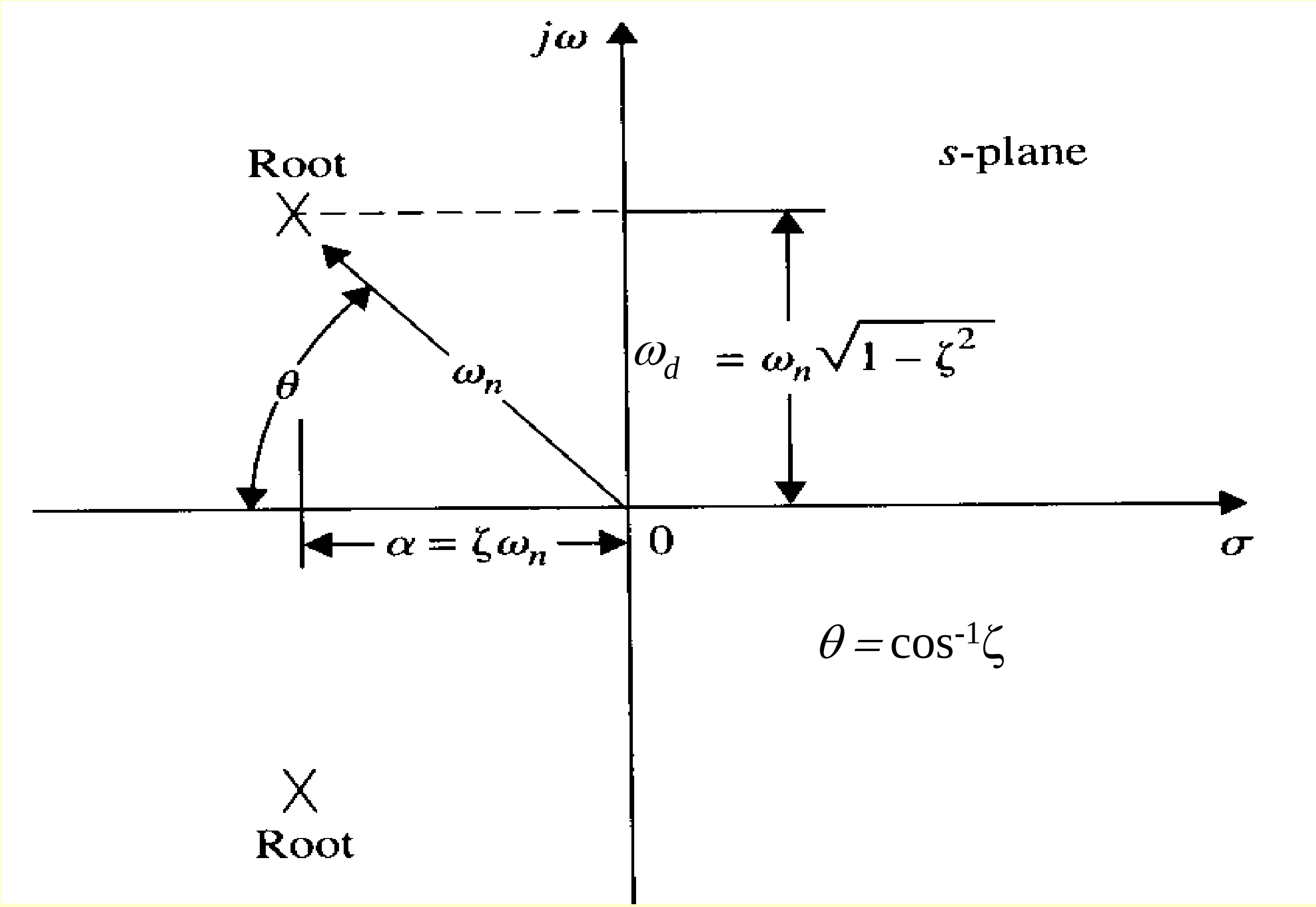
$$\begin{aligned} s_1, s_2 &= -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} \\ &= -\alpha \pm j\omega_d \end{aligned} \qquad (4)$$

where

α is the ***damping constant***
ω_d is the ***damped natural frequency***

The values of ζ and ω_n determine the nature of the 2nd order system's response.

The effect of ζ, ω_n and ω_d on the locations of roots is illustrated below.



Unit-step Response of the Standard 2nd-order System

The response of the standard 2nd-order system when subjected to a unit-step input will depend on the value of ζ. Its unit-impulse response is given in Appendix 5.2.

$$Y(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\times\frac{1}{s} \quad \text{as} \quad R(s)=\frac{1}{s}$$

(1) Undamped response, $\zeta=0$:

Poles at $s_1, s_2=\pm j\omega_n$

$$Y(s)=\frac{\omega_n^2}{(s^2+\omega_n^2)}\times\frac{1}{s}=\frac{1}{s}-\frac{s}{s^2+\omega_n^2}$$

$$\Leftrightarrow \quad y(t)=1-\cos\omega_n t$$

(2) Underdamped response, $\zeta < 1$:

Poles at $s_1, s_2 = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$

$$Y(s) = \frac{\omega_n^2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)} \times \frac{1}{s}$$

$$Y(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s+\zeta\omega_n)^2 + \left(\omega_n\sqrt{1-\zeta^2}\right)^2}$$

$$\Leftrightarrow \; y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_n\sqrt{1-\zeta^2}\,t + \theta\right)$$

where $\theta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta} = \cos^{-1}\zeta = \sin^{-1}\sqrt{1-\zeta^2}$

(3) Critically damped response, $\zeta = 1$:

Poles at $s_1, s_2 = -\omega_n, -\omega_n$

$$Y(s) = \frac{\omega_n^2}{(s^2 + 2\omega_n s + \omega_n^2)} \times \frac{1}{s} = \frac{\omega_n^2}{(s+\omega_n)^2} \times \frac{1}{s}$$

$$\Leftrightarrow \quad y(t) = 1 - e^{-\omega_n t}\left(1 + \omega_n t\right)$$

(4) Overdamped response, $\zeta > 1$:

Poles at $s_1, s_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} = -\beta_1\omega_n, -\beta_2\omega_n$

$$Y(s) = \frac{\omega_n^2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)} \times \frac{1}{s}$$

$$Y(s)=\frac{\omega_n^2}{\left[(s+\zeta\omega_n)^2-\left(\omega_n\sqrt{\zeta^2-1}\right)^2\right]}\times\frac{1}{s}$$

$$\Leftrightarrow y(t)=1 \quad -\frac{\zeta+\sqrt{\zeta^2-1}}{2\sqrt{\zeta^2-1}}e^{-\beta_1\omega_n t}+\frac{\zeta-\sqrt{\zeta^2-1}}{2\sqrt{\zeta^2-1}}e^{-\beta_2\omega_n t}$$

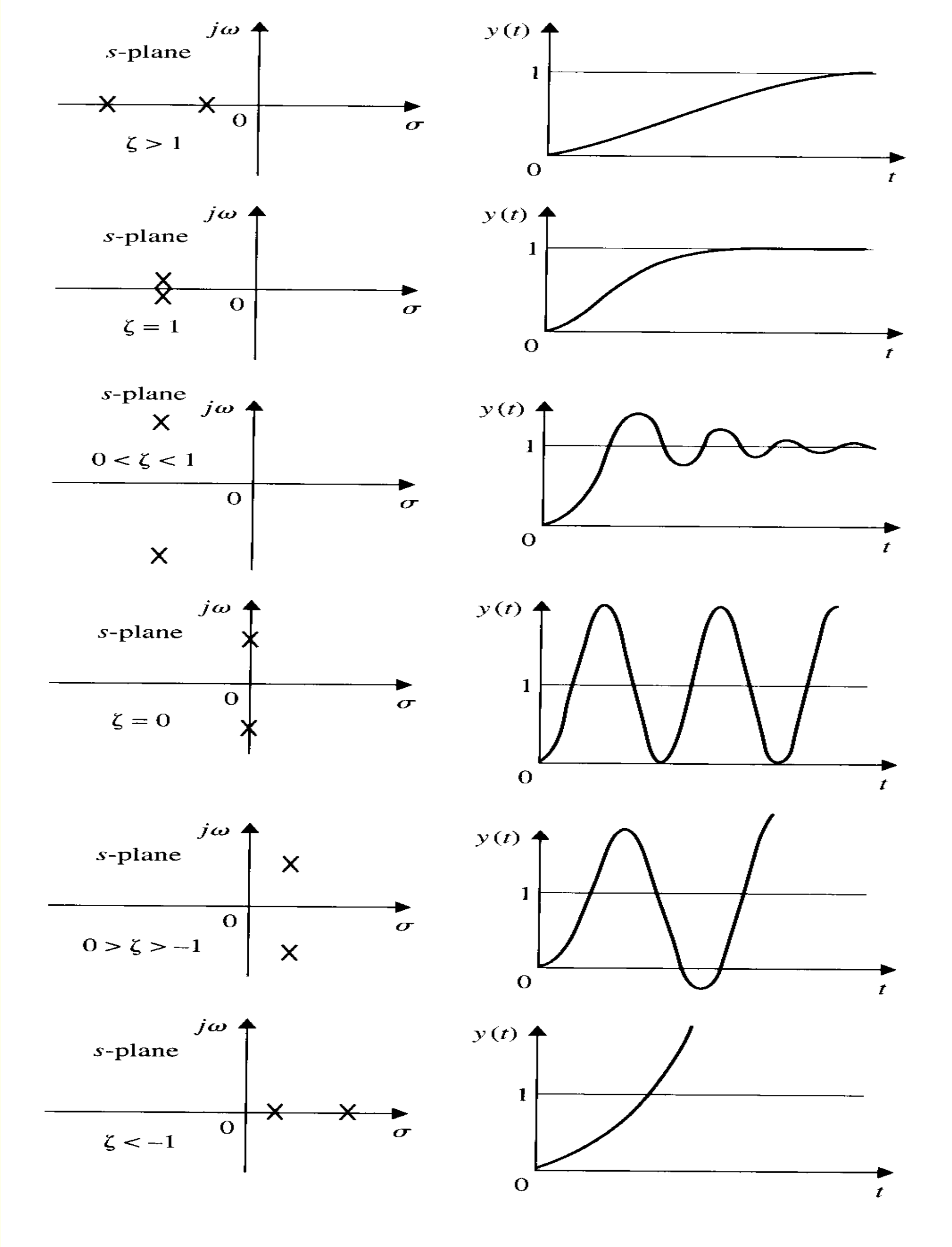
where

$$\beta_1=\zeta-\sqrt{\zeta^2-1}\ , \quad \beta_2=\zeta+\sqrt{\zeta^2-1}$$

If $\zeta >> 1,$ then

$$y(t)\approx 1-e^{-\beta_1\omega_n t}$$

That is, the response is very similar to that of a (slow) 1st-order system.

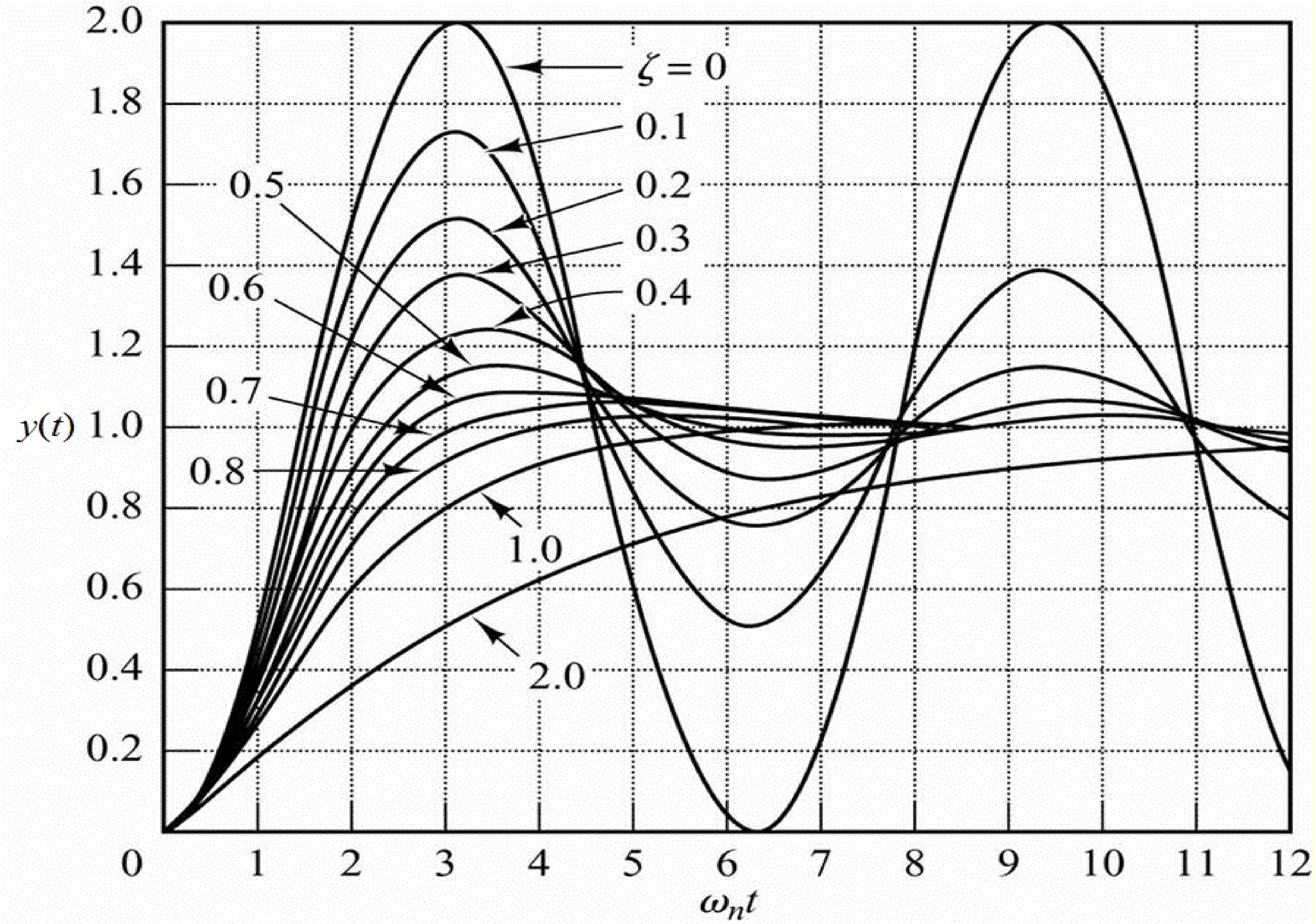


stable
responses

marginally
stable

unstable
responses

The unit-step responses for various values of ζ are shown below.



Appendix 5.1

If $r(t) = atu(t);\ R(s) = a/s^2$

$$Y(s) = \frac{1}{(\tau s+1)} \cdot \frac{a}{s^2} = a\left(\frac{1}{s^2} - \frac{\tau}{s} + \frac{\tau^2}{\tau s+1}\right)$$

So,

$$y(t) = a(t - \tau(1 - e^{-\frac{t}{\tau}}))u(t)$$

.

The error is $e(t) = r(t) - y(t) = a\tau(1 - e^{-\frac{t}{\tau}})u(t)$

The steady-state error is $e_{ss} = a\tau$

.

Appendix 5.2: *Unit-impulse response*

The unit-impulse response of the standard 2nd order system is given by

$$Y(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad \text{as} \quad R(s) = 1$$

Case 1: Un-damped oscillation $(\zeta = 0)$

$$s_1, s_2 = \pm j\omega_n$$

$$Y(s) = \frac{\omega_n^2}{(s^2 + \omega_n^2)} \iff y(t) = \omega_n \sin \omega_n t$$

Case 2: Under-damped (damped oscillatory) response $(\zeta < 1)$

$$s_1, s_2 = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

$$Y(s) = \frac{\omega_n^2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

$$\iff y(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin \omega_n \sqrt{1-\zeta^2}\, t$$

Case 3: Critically damped response $(\zeta = 1)$

$$s_1, s_2 = -\omega_n, -\omega_n$$

$$Y(s) = \frac{\omega_n^2}{(s+\omega_n)^2} \Leftrightarrow y(t) = \omega_n^2 e^{-\omega_n t} t$$

Case 4: Over-damped response $(\zeta > 1)$

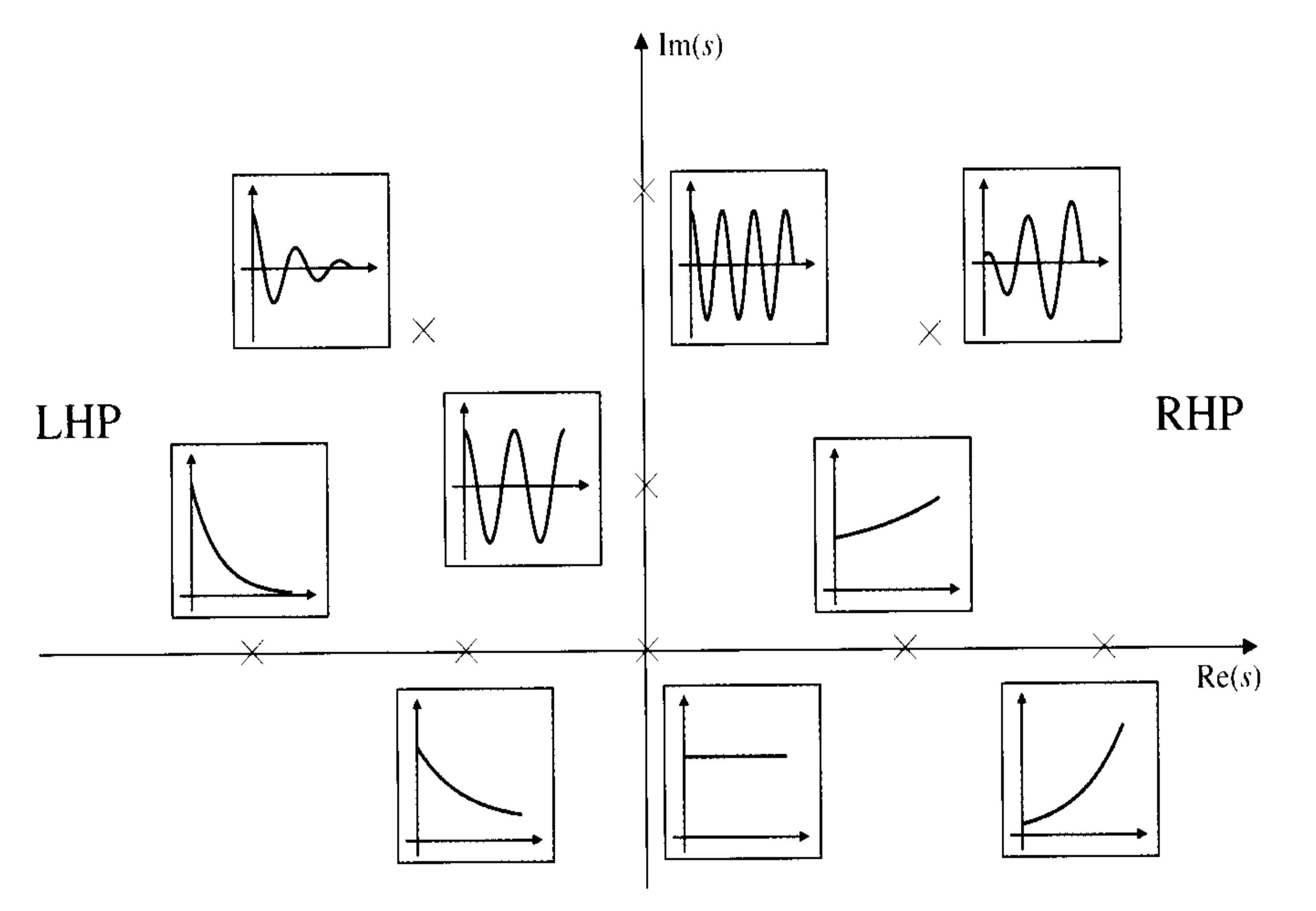
$$s_1, s_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} = -\beta_1\omega_n, -\beta_2\omega_n$$

$$(\ \beta_1, \beta_2 = \zeta \mp \sqrt{\zeta^2 - 1}\ )$$

$$Y(s) = \frac{\omega_n^2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

$$\Leftrightarrow y(t) = \frac{\omega_n}{2\sqrt{1-\zeta^2}}\left(e^{-\beta_1\omega_n t} - e^{-\beta_2\omega_n t}\right)$$

The impulse response for the various root locations for the standard 2nd-order system in the s–plane is illustrated below.



Summary 5. Responses of 1st and 2nd Order Systems

Study of time responses to impulse, **step**, ramp and parabolic functions
The total time response is generally given by

$$y(t) = y_t(t) + y_{ss}(t)$$

We normally want $y_t(t)$ to be small and to decay quickly.

Time Response of First-Order Systems

$$\frac{Y(s)}{R(s)} = \frac{1}{\tau s + 1}$$

The response depends on time constant τ.
A first order system gives a “good” response with respect to a step input but will yield a steady-state error of τ with respect to a unit-ramp input.

Time Response of Second-Order Systems

The standard 2nd-order system:

$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

The poles are: $s_1, s_2 = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -\alpha \pm j\omega_d$

The system's response is determined by the damping ratio ζ and undamped natural frequency ω_n. The decay rate of the transient term determined by damping constant $\zeta\omega_n$.

$\zeta = 0$, undamped response
$0 < \zeta < 1$, under-damped response
$\zeta = 1$, critical response
$\zeta > 1$, over-damped response
$\zeta < 0$, unstable response

Lecture 6: Tome Domain Analysis - Time-Response Specifications

a) Performance Indices of Standard 2nd Order Systems
   - Rise time
   - Peak time
   - Maximum overshoot
   - Settling time
   - Steady-state error

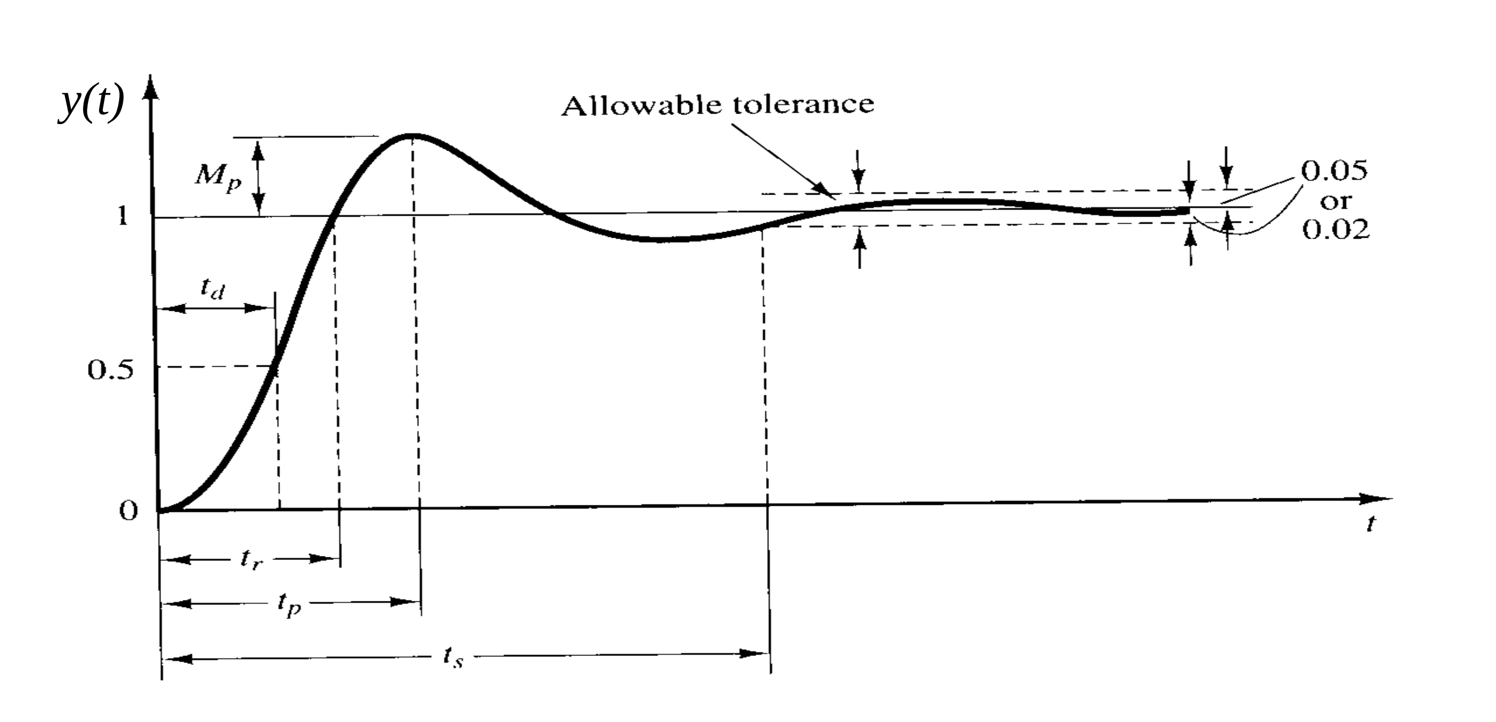
b) Other Second-order Systems

c) Higher-order Systems

a) Performance Indices of Standard 2nd Order Systems

For an under-damped standard 2nd -order control system, $0 < \zeta < 1$, the time response of the system to a unit-step input is given by

$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_n \sqrt{1-\zeta^2}\, t + \theta\right)$$



In the figure, t_r is defined for 0-100%. In practice where measurements are involved, t_r is usually defined for 10-90%.

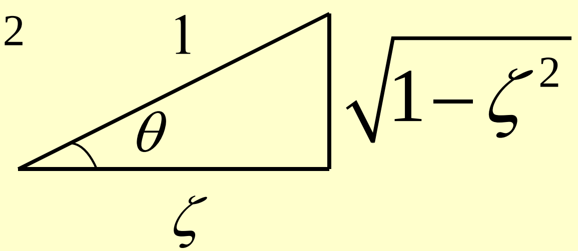
Time response indices are:

1. Rise time $\boldsymbol{t_r}$
2. Peak time $\boldsymbol{t_p}$
3. Maximum of overshoot $\boldsymbol{M_p}$
4. Settling time $\boldsymbol{t_s}$
5. Steady-state error $\boldsymbol{e_{ss}}$

These performance indices are to be derived from the unit-step response of the standard 2nd -order system

$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_n \sqrt{1-\zeta^2}\, t + \theta\right) \qquad (1)$$

where $\theta = \tan^{-1} \dfrac{\sqrt{1-\zeta^2}}{\zeta}$. $\Rightarrow$ $\sin\theta = \sqrt{1-\zeta^2}$, $\cos\theta = \zeta$



The derivations of the time-response indices are as follows:

(1) Rise Time t_r:

From (1), $y(t)$ reaches "1" for the first time at t_r (0-100% for under-damped system), i.e. $y(t_r) = 1$, so

$$y(t_r) = 1 - \frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}} \sin\left(\omega_n \sqrt{1-\zeta^2}\, t_r + \theta\right) = 1$$

Since $e^{-\zeta\omega_n t_r} \neq 0$, then

$$\sin\left(\omega_n \sqrt{1-\zeta^2}\, t_r + \theta\right) = 0$$

$$\Rightarrow \qquad \omega_n \sqrt{1-\zeta^2}\, t_r + \theta = n\pi \; [rad]; \;\; n = 0, 1, 2, \ldots$$

The valid value for t_r is given by $n = 1$.

Thus, $t_r = \dfrac{\pi - \theta}{\omega_n \sqrt{1-\zeta^2}} = \dfrac{\pi - \theta}{\omega_d}$

where $\omega_d = \omega_n \sqrt{1-\zeta^2}$.

Note that the derivation is valid when $0 < \zeta < 1$ $\left(\text{i.e. } 0 < \theta < \frac{\pi}{2}\right)$

(2) Peak Time t_p:

We differentiate $y(t)$ with respect to t, and set to zero, i.e.

$$\frac{dy(t)}{dt} = \zeta\omega_n \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_d t + \theta\right)$$

$$- \omega_n \sqrt{1-\zeta^2} \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \cos\left(\omega_d t + \theta\right) = 0$$

$$\therefore \quad \frac{dy(t)}{dt} = 0 \quad \text{if}$$

$$\left[\zeta \sin(\omega_d t + \theta) - \sqrt{1-\zeta^2}\cos(\omega_d t + \theta)\right] = 0$$

But $\sin\theta = \sqrt{1-\zeta^2}$ and $\cos\theta = \zeta$,

i.e. $\left[\sin(\omega_d t + \theta)\cos\theta - \cos(\omega_d t + \theta)\sin\theta\right] = 0$

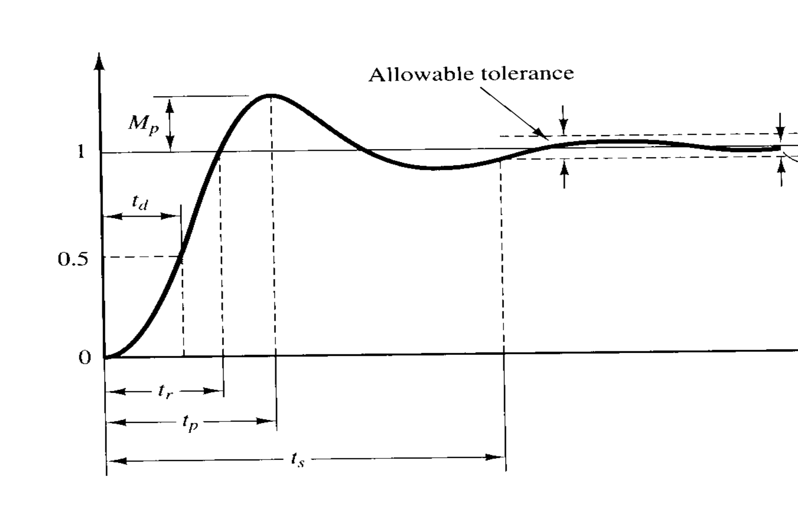
$$\Rightarrow \quad \sin(\omega_d t + \theta - \theta) = 0$$

$$\Rightarrow \quad \sin(\omega_d t) = 0 \quad \Rightarrow \quad \omega_d t = n\pi; \;\; n = 0,1,2,\ldots$$

Since t_p corresponds to the first peak, n=1, so

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d}$$

$$= \frac{1}{2}\times\left(\frac{2\pi}{\omega_d}\right) = \frac{1}{2}\times(Period - of - osc.)$$



(3) Maximum Overshoot M_p:

M_p occurs at t_p. So, $M_p = y(t_p) - 1 = 1 - \dfrac{e^{-\zeta\omega_n t_p}}{\sqrt{1-\zeta^2}}\sin\left(\omega_n\sqrt{1-\zeta^2}\,t_p + \theta\right) - 1$

Therefore, $M_p = -\dfrac{e^{-\zeta\omega_n t_p}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t_p + \theta\right)$; $\quad t_p = \dfrac{\pi}{\omega_d}$

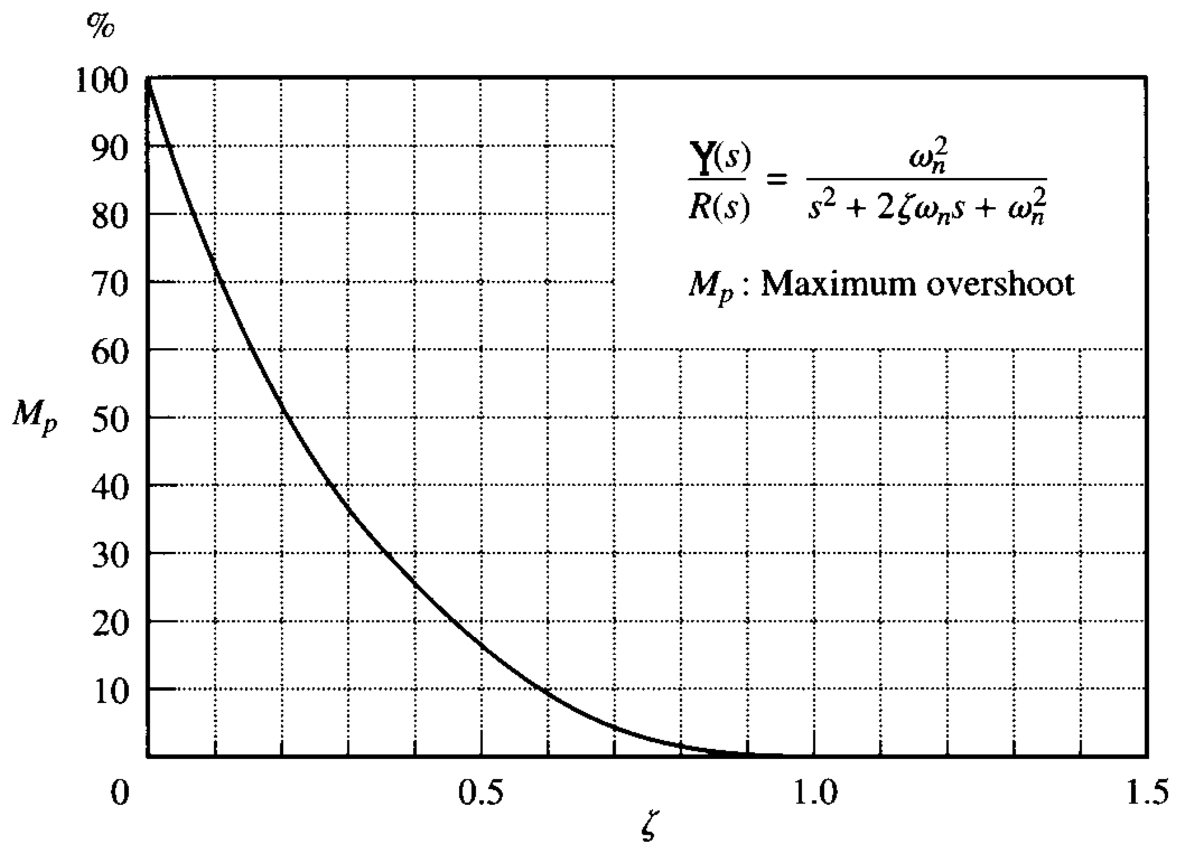
$$= -\frac{e^{-\pi\zeta/\sqrt{1-\zeta^2}}}{\sqrt{1-\zeta^2}}\sin(\pi + \theta)$$

$$= +\frac{e^{-\pi\zeta/\sqrt{1-\zeta^2}}}{\sqrt{1-\zeta^2}}\sin\theta = e^{-\pi\zeta/\sqrt{1-\zeta^2}}$$

That is, the percentage maximum overshoot = $100e^{-\pi\zeta/\sqrt{1-\zeta^2}}$ %

Note that M_p is a function of ζ and is independent of ω_n. M_p is therefore a good measure of system damping.

The relationship between the maximum percentage overshoot M_p and the damping ratio ζ is given in the graph below.



(4) Settling Time t_s:

The exact value of t_s is difficult to compute but good approximations for $0 < \zeta < 0.7$ are as follows:

For 2% tolerance criterion:

$$t_s \cong \frac{4}{\zeta\omega_n}$$

For 5% tolerance criterion:

$$t_s \cong \frac{3}{\zeta\omega_n}$$

For x% in general:

$$\frac{e^{-\zeta\omega_n t_s}}{\sqrt{1-\zeta^2}} \cong \frac{x}{100} \quad \text{or} \quad e^{+\zeta\omega_n t_s} \cong \frac{100}{x\sqrt{1-\zeta^2}}$$

$$\Rightarrow \quad t_s \approx \frac{\ln\dfrac{100}{x\sqrt{1-\zeta^2}}}{\zeta\omega_n}$$

Note that settling time t_s is inversely proportional to ω_n.

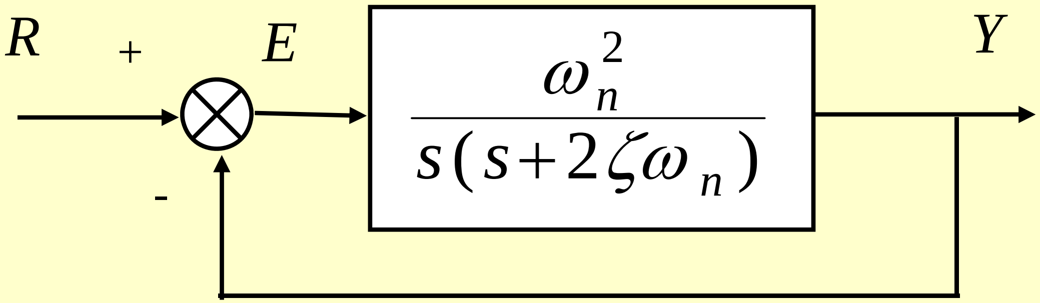
(5) Steady-State Error e_{ss}:

$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{t\to\infty} e(t)$$

$$= \lim_{t\to\infty}[r(t) - y(t)]$$

$$= 0 \quad \text{for standard 2nd order systems}$$

<u>Example</u>: Consider the system shown below where $\zeta = 0.6$ and $\omega_n = 5$ *rad*/sec. Find t_r, t_p, M_p and t_s when the system is subjected to a unit-step input.



The closed-loop transfer function is

$$\frac{Y}{R} = \frac{G}{1+G} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$= \frac{5^2}{s^2 + 2\times 0.6\times 5s + 5^2}$$

This is a standard second order system with an under-damped response as $\zeta < 1$.

Rise Time t_r:

$$t_r = \frac{\pi - \theta}{\omega_n \sqrt{1-\zeta^2}} = \frac{\pi - \theta}{\omega_d}$$

$$\theta = \cos^{-1}\zeta = \cos^{-1}0.6 = 0.93 \quad [rad]$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 5\sqrt{1-0.36} = 4\ [rad/\sec]$$

$$\therefore \qquad t_r = \frac{3.14-0.93}{4} = 0.55 \qquad [\sec]$$

Peak Time t_p:

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d} = \frac{3.14}{4} = 0.785 \quad [\sec]$$

Maximum Overshoot M_p:

$$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}} = e^{-\frac{0.6\pi}{\sqrt{1-0.36}}}$$

$$= 0.095 \quad \text{or} \quad 9.5\%$$

Settling Time t_s:

For 2% tolerance criterion:

$$t_s \cong \frac{4}{\zeta\omega_n} = \frac{4}{0.6\times 5} = 1.33 \quad \text{[sec]} \quad \text{or} \quad t_s = \frac{\ln \frac{100}{x\sqrt{1-\zeta^2}}}{\zeta\omega_n} = 1.38 \quad \text{[sec].}$$

For 5% tolerance criterion:

$$t_s \cong \frac{3}{\zeta\omega_n} = 1 \quad \text{[sec]} \quad \text{or} \quad t_s = \frac{\ln \frac{100}{x\sqrt{1-\zeta^2}}}{\zeta\omega_n} = 1.07 \quad \text{[sec].}$$

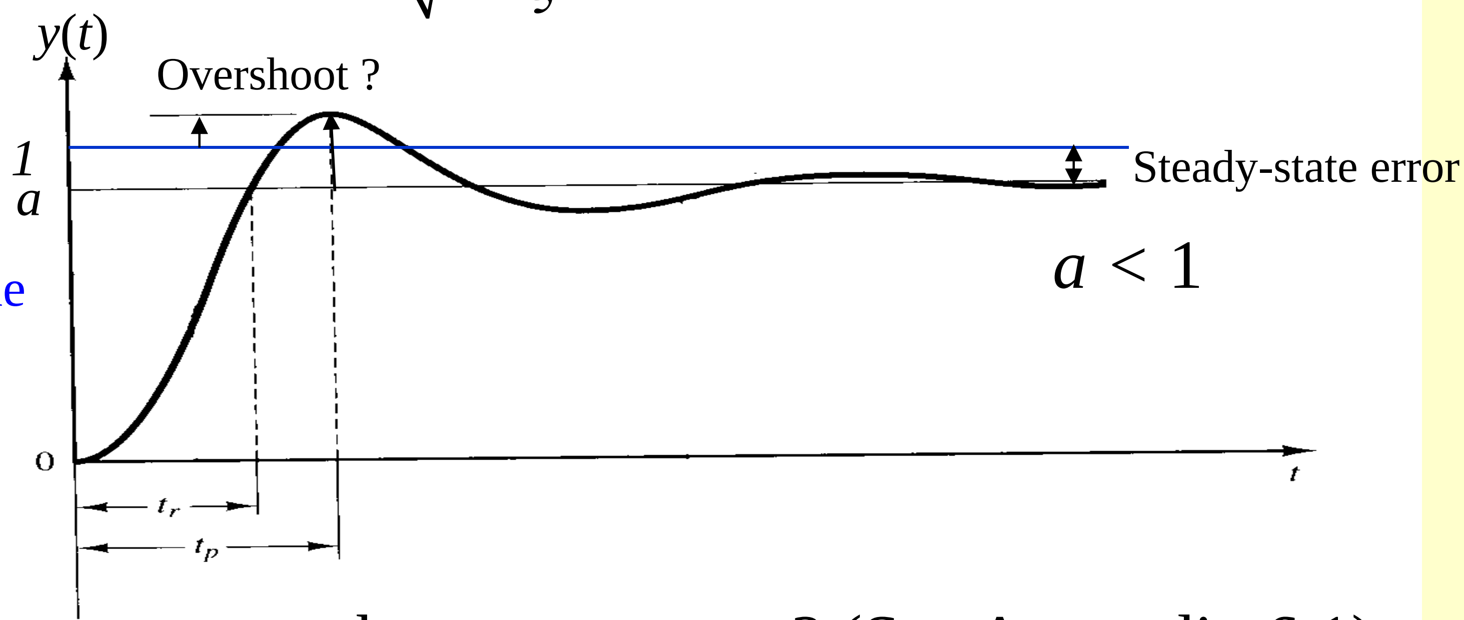
b) Other Second-order Systems

Consider the following 2nd order system:

$$\frac{Y}{R} = \frac{a\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

where a is the steady-state gain and it can be larger or smaller than 1.
The unit-step response is given by

$$y(t) = a[1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_n\sqrt{1-\zeta^2}\,t + \theta\right)]$$



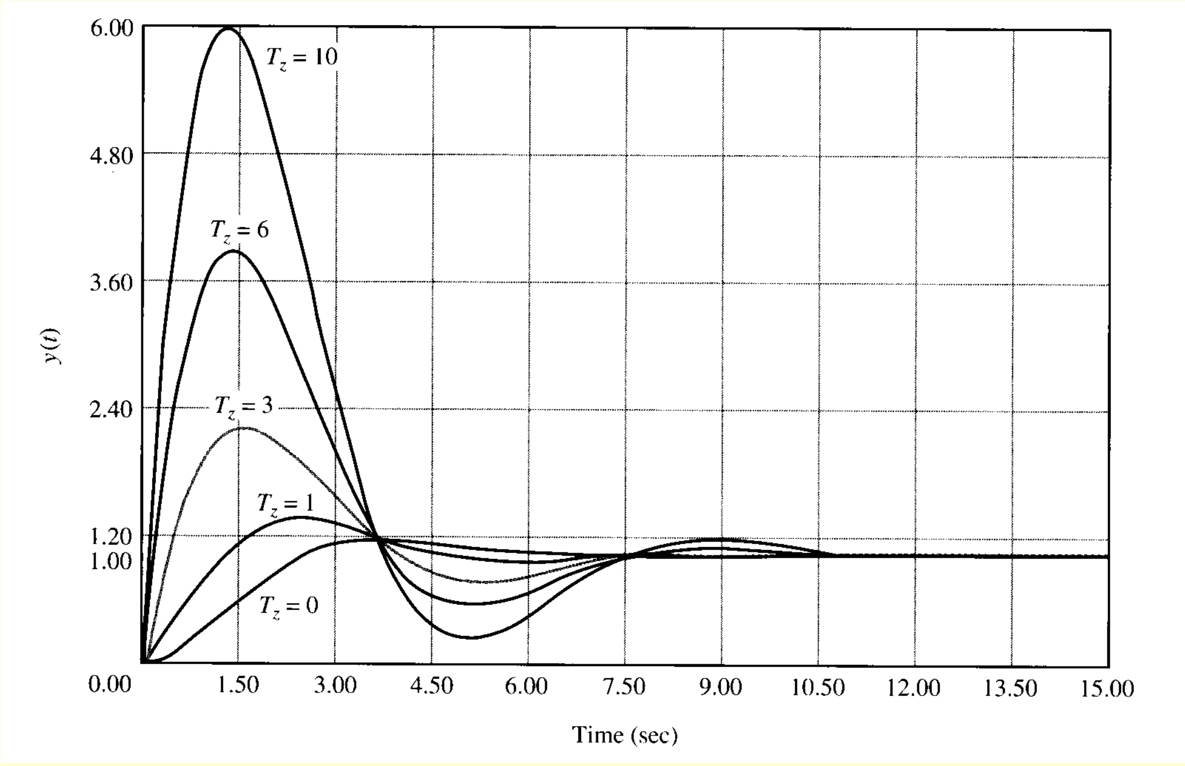
Q: How to define overshoot ?

How to compute overshoot, t_r, t_p, t_s etc? (See Appendix 6.1).

In some other cases, the non-standard 2nd order system has a zero such as

$$\frac{Y}{R} = \frac{\omega_n^2 (T_z s + 1)}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Note that this system's response will be affected by the presence of zero at $s = -1/T_z$. Some responses are illustrated below.



Can we derive the time domain output?

The effect of the zero on the closed-loop response can be explained as follows. Note that *Y/R* can be rewritten as

$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} + T_z \cdot s \cdot \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$i.e.\ \ Y(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} R(s) + T_z \cdot s \cdot \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} R(s)$$

$$i.e.\ \ Y(s) = Y_0(s) + T_z \cdot s \cdot Y_0(s)$$

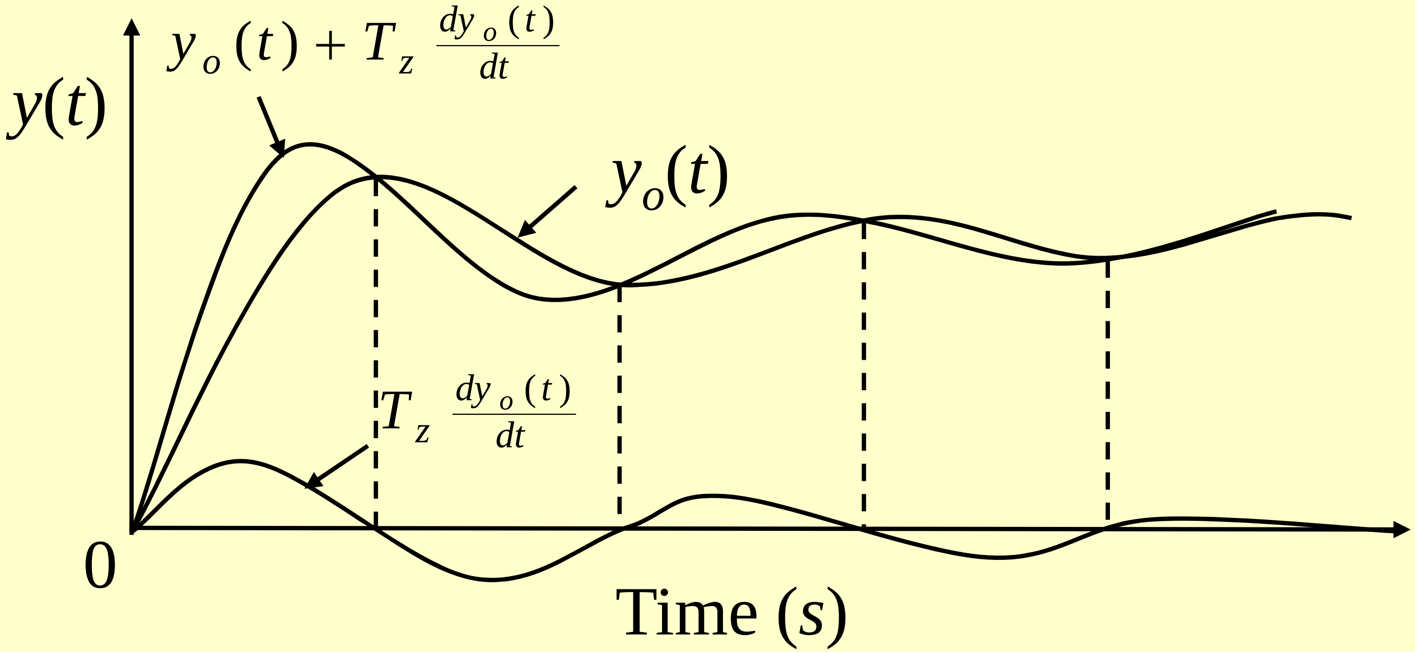
Therefore, with unit-step input, we get $y(t) = y_o(t) + T_z \dfrac{d}{dt} y_o(t)$

where $y_o(t)$ is the unit-step response of standard 2nd-order system.

For $\zeta < 1$,

$$y_o(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left[\omega_d t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right]$$

For $\zeta < 1$, we have



The computations of t_r, t_p, M_p from $y(t)$ can be done in the similar way as that for standard 2nd-order system, albeit more involved.

c) Higher-order Systems

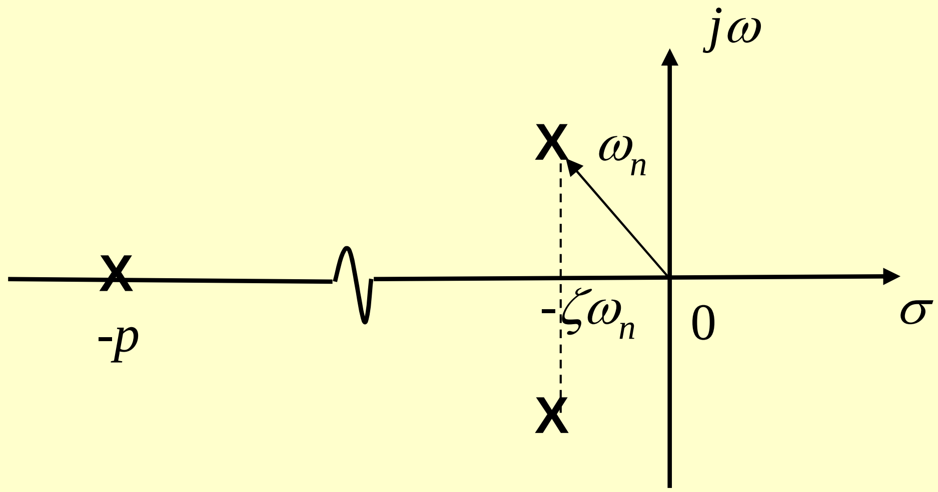
When a system has order higher than 2, its response can always be divided into responses of 1st and 2nd-order systems. *Do you know how to do that? Remember the partial fraction expansion?* The problem is that the sum of these simple response terms will yield a complicated expression.

In practice, the results of 2nd-order systems could be used to approximate higher order systems that have a pair of **dominant complex poles**. That is, the real parts of other poles are more negative than that of the dominant poles, generally more than 10 times. *Why?* In such cases, the high order system's response can be approximated with a 2nd-order system by neglecting all the non-dominant poles.

Example: Consider a 3rd-order system with transfer function

$$\frac{Y}{R} = \frac{p\omega_n^2}{\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)\left(s + p\right)}$$

$$= \frac{\omega_n^2}{\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)\left(1 + \frac{s}{p}\right)}$$



If $|p| \geq 10|\zeta\omega_n|$, the 3rd-order system's response as indicated by M_p and t_s can be well approximated by a 2nd-order system's response. That is,

$$\frac{Y}{R} \cong \frac{\omega_n^2}{\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)}$$

<u>Example</u>: Consider the following 2 systems:

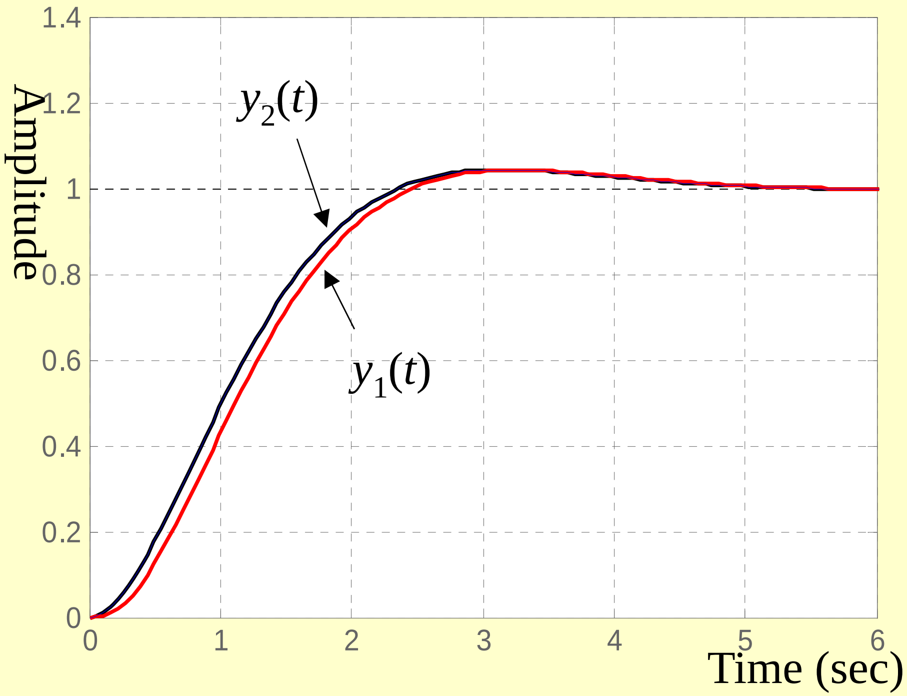
$$\frac{Y_1}{R} = \frac{10\times(\sqrt{2})^2}{\left(s^2+2\times\frac{1}{\sqrt{2}}\times\sqrt{2}s+(\sqrt{2})^2\right)(s+10)} \quad ; \quad \frac{Y_2}{R} = \frac{(\sqrt{2})^2}{\left(s^2+2\times\frac{1}{\sqrt{2}}\times\sqrt{2}s+(\sqrt{2})^2\right)}$$

i.e., $\zeta = \dfrac{1}{\sqrt{2}};\ \omega_n = \sqrt{2};$ and $p = 10 = 10\zeta\omega_n.$

Their unit-step responses are very similar:

$$y_1(t) = 1 - \tfrac{1}{41}e^{-10t} - \tfrac{40}{41}\sqrt{1+1.25^2}\,e^{-t}\sin\left(t+\tan^{-1}1.25\right)$$

$$y_2(t) = 1 - \sqrt{2}e^{-t}\sin\left(t+\tan^{-1}1\right)$$



Example: Consider the following 2 systems:

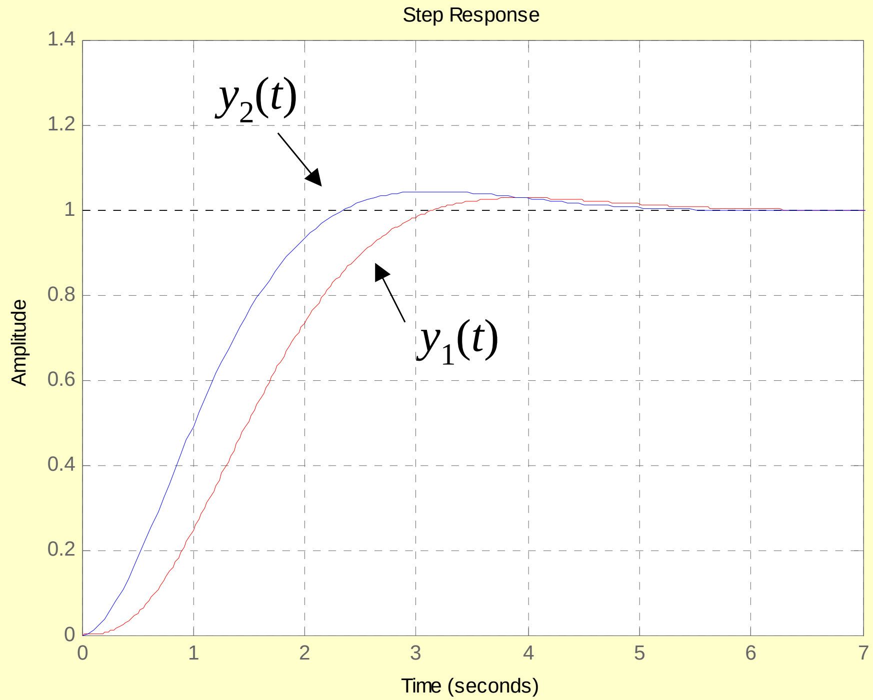
$$\frac{Y_1}{R} = \frac{2\times(\sqrt{2})^2}{\left(s^2 + 2\times\frac{1}{\sqrt{2}}\times\sqrt{2}s + (\sqrt{2})^2\right)(s+2)} \quad ; \quad \frac{Y_2}{R} = \frac{(\sqrt{2})^2}{\left(s^2 + 2\times\frac{1}{\sqrt{2}}\times\sqrt{2}s + (\sqrt{2})^2\right)}$$

i.e., $\zeta = \frac{1}{\sqrt{2}};\ \omega_n = \sqrt{2};$ and $p = 2 = 2\zeta\omega_n$.

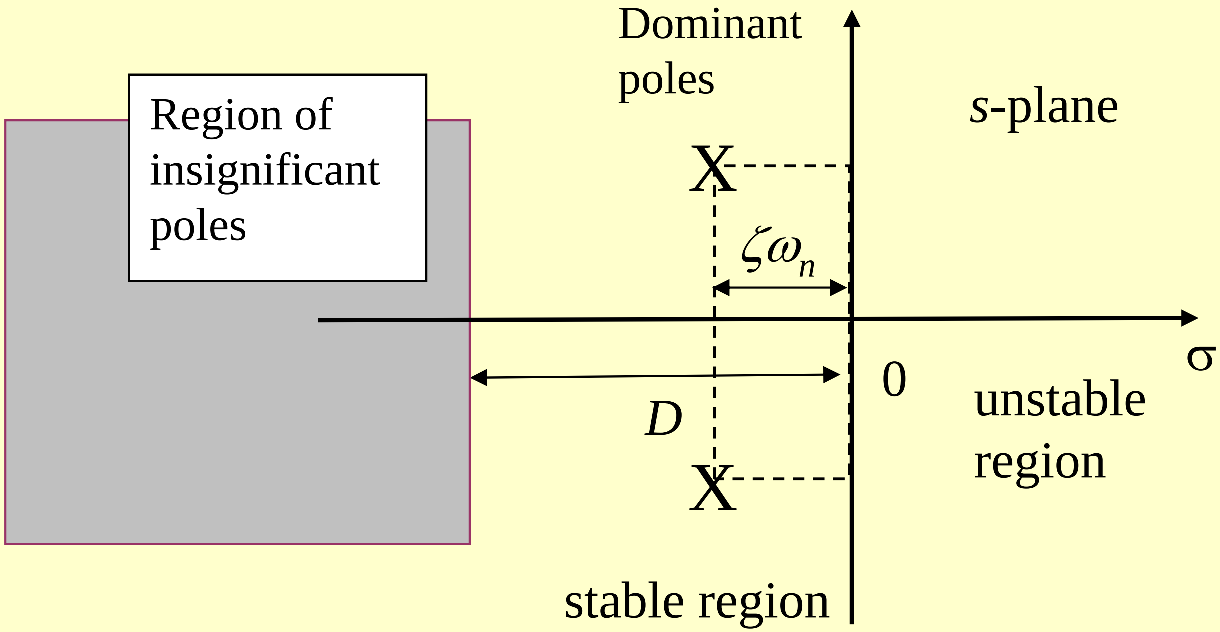
There is a significant difference in their unit-step responses:

$$y_1(t) = 1 - e^{-2t} - 2e^{-t}\sin t$$

$$y_2(t) = 1 - \sqrt{2}e^{-t}\sin\left(t + \tan^{-1} 1\right)$$



For higher order systems, the regions of dominant and insignificant poles are illustration below:



In practice, we normally want $D \geq 10\zeta\omega_n$

Appendix 6.1

If $\dfrac{Y(s)}{R(s)} = \dfrac{a\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$, then $Y(s) = aY_o(s)$

where $Y_o(s) = \dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} R(s)$

.

Hence the unit-step responses are:

$$y_o(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_n \sqrt{1-\zeta^2}\, t + \theta\right)$$

$$y(t) = ay_o(t) = a[1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_n \sqrt{1-\zeta^2}\, t + \theta\right)]$$

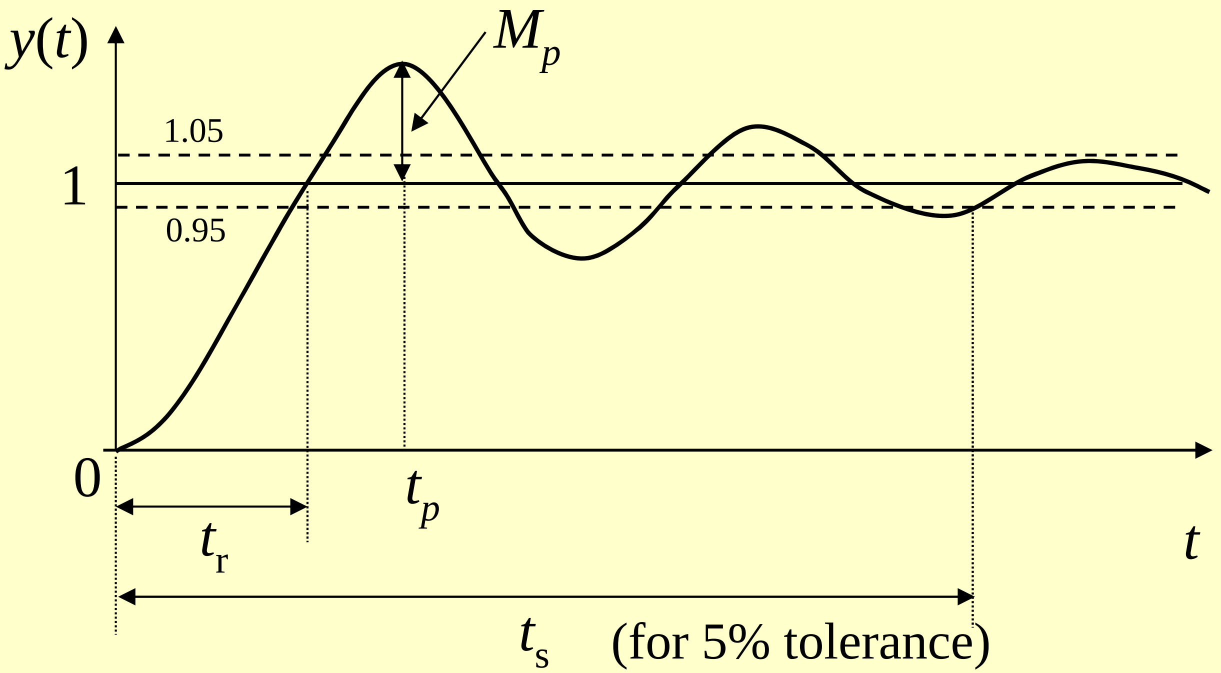
That is, $y(t)$ is a scaled version (in magnitude) of $y_o(t)$. So the formulae for t_r, t_p, and t_s will remain exactly the same. The magnitude of overshoot is given by $aM_p = ae^{-\pi\zeta / \sqrt{1-\zeta^2}}$. Note however that the percentage of overshoot remain unchanged.

Summary 6. Time Response Specifications

Unit-step Response of Standard 2nd-order Systems $(\zeta < 1)$

$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_n \sqrt{1-\zeta^2}\, t + \theta\right)$$

where $\theta = \cos^{-1}\zeta$



(1) Rise Time: $t_r = \dfrac{\pi - \theta}{\omega_n \sqrt{1-\zeta^2}} = \dfrac{\pi - \theta}{\omega_d}$

(2) Peak Time: $t_p = \dfrac{\pi}{\omega_n \sqrt{1-\zeta^2}} = \dfrac{\pi}{\omega_d}$

(3) Maximum Overshoot: $M_p = e^{-\pi\zeta / \sqrt{1-\zeta^2}}$

(4) Settling Time:

For 2% tolerance criterion: $t_s \cong \dfrac{4}{\zeta\omega_n}$

For 5% tolerance criterion: $t_s \cong \dfrac{3}{\zeta\omega_n}$

For *x*% in general: $t_s = \dfrac{\ln \frac{100}{x\sqrt{1-\zeta^2}}}{\zeta\omega_n}$

The 2nd order system's response can be greatly affected by the presence of zero!

Lecture 7: Stability Analysis - Routh-Hurwitz Stability

a) Steady-state and Transient Responses

b) Pole and Zeros

c) Routh-Hurwitz Criterion

d) Special Cases
   - Zero in the First Column of Array

a) Steady-state and Transient Responses

For a control system, the total time response is generally given by

$$y(t) = y_t(t) + y_{ss}(t)$$

where

$y_t(t)$ is the transient response

$y_{ss}(t)$ is the steady-state response

Further,

$$\lim_{t\to\infty} y_t(t) = 0 \quad ; \quad \text{if system is stable}$$

$$y_{ss}(t) = \lim_{t\to\infty} y(t)$$

Example: Consider

$$\frac{Y(s)}{R(s)} = T(s) = \frac{4}{(s+2)(s^2+2s+2)} \quad ; \quad R(s) = \frac{1}{s}$$

Then $$Y(s) = T(s)R(s) = \frac{4}{(s+2)(s^2+2s+2)} \times \frac{1}{s}$$

$$= \frac{1}{s} - \frac{1}{s+2} - \frac{2}{(s+1)^2+1^2}$$

$$y(t) = \underbrace{1}_{y_{ss}(t)} \underbrace{-e^{-2t} - 2e^{-t}\sin t}_{y_t(t)}$$

The transient terms determine the stability of the system. Notice that the transient responses are determined by the roots of the characteristic equation.

In physical systems, the transient terms are due to energy storage elements such as $C, L,$ inertia, spring, or equivalents.

b) Poles and Zeros

Let $T(s)$ be the (rational) transfer function of a control system, then

$$T(s) = \frac{p(s)}{q(s)}$$

where $p(s)$ and $q(s)$ are polynomial functions.

The ***poles*** are defined as the roots of the characteristic equation $q(s)=0$. The poles will determine the nature of the transient response and hence stability of the control systems.

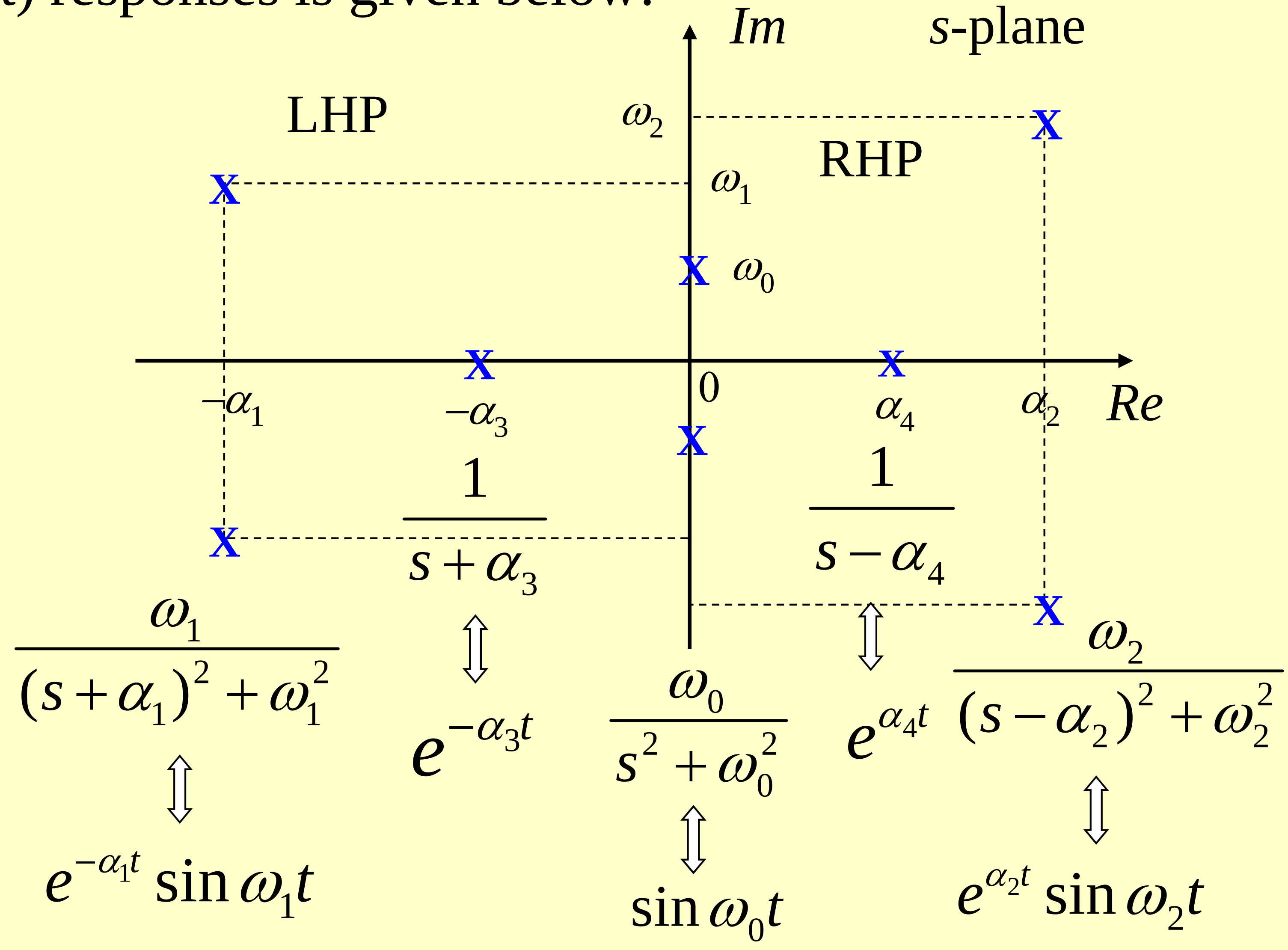
The ***zeros*** are defined as the roots of $p(s)=0$. The zero will affect the magnitude of transient response **but not** the stability of the control systems.

Example: $T(s) = \dfrac{10(s+2)}{s(s+1)(s+3)^2}$

Poles at $s=0, s=-1, s=-3$ and $s=-3$

Zeros at $s = -2$

An illustration of relationship between pole locations and impulse (transient) responses is given below:



Thus, <u>a control system $T(s)$ is stable if **ALL** its poles are in the open left-half plane</u>.

Example: Consider the following transfer functions and their corresponding unit-step responses:

$$H_1(s)=\frac{2(s+2)}{(s+1)(s+4)}\ ;\quad Y_1(s)=\frac{2(s+2)}{(s+1)(s+4)}\bullet\frac{1}{s}$$

$$\Leftrightarrow\ y_1(t)=1-\tfrac{2}{3}e^{-t}-\tfrac{1}{3}e^{-4t}$$

$$H_2(s)=\frac{-2(s+2)}{(s+1)(s-4)};\quad Y_2(s)=\frac{-2(s+2)}{(s+1)(s-4)}\bullet\frac{1}{s}$$

$$\Leftrightarrow\ y_2(t)=1-\tfrac{2}{5}e^{-t}-\tfrac{3}{5}e^{4t}$$

$$H_3(s)=\frac{-2(s-2)}{(s+1)(s+4)};\quad Y_3(s)=\frac{-2(s-2)}{(s+1)(s+4)}\bullet\frac{1}{s}$$

$$\Leftrightarrow\ y_3(t)=1-2.0e^{-t}+1.0e^{-4t}$$

From $y_1(t)$ and $y_2(t)$, we notice that the stability (i.e. the **characteristics** of the transient response) is influenced by the pole locations.

From $y_1(t)$ and $y_3(t)$, we note that the **magnitude** of transient response is influenced by the zero locations but stability remained.

c) Routh-Hurwitz Criterion

It determines the **absolute stability** of a linear time-invariant system. It provides information on the distribution of the roots of the characteristic equation with respect to the $j\omega$-axis.

Consider the (closed-loop) transfer function of a Single-Input Single-Output (SISO) system:

$$G(s) = \frac{Y(s)}{R(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_o}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_o}$$

where $a_n \neq 0, \quad m < n$. We assume without the loss of generality that $a_n > 0$. The characteristic equation is given by

$$q(s) = a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + a_{n-3} s^{n-3} + \cdots + a_o = 0$$

Necessary Conditions for Stability:

A necessary (but not sufficient) condition for stability of a linear system is that all the coefficients of its characteristic equation must have the **same sign**, and none of the coefficients should vanish. If any other coefficient is zero or negative, then the system is known immediately to be unstable. That is, it has at least a root with positive or zero real parts.

The converse is not true! That is, even if all the coefficients are of the same sign, it does not mean that the system is stable.

Routh-Hurwitz Stability Criterion:

If all coefficients are of the same sign, then the sufficient condition can be checked by setting up the **Routh array**. The method allows us to determine the distribution of the roots of the characteristic equation (C.E.) with respect to the $j\omega$–axis without actually solving the equation.

Consider the C.E. of an n[th]-order system:

$$q(s) = a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + a_{n-3} s^{n-3} + \cdots + a_o = 0 \quad (1)$$

Suppose that all the coefficients are of the same sign. Then we can proceed to determine the sufficient conditions for stability by setting up the Routh-Array as follows.

Step 1:

$$\begin{array}{l|llll} s^n & a_n & a_{n-2} & a_{n-4} & \cdots \\ s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \cdots \\ \hline \end{array}$$

Step 2:

$$\begin{array}{l|llll} s^{n-2} & b_1 & b_2 & b_3 & \cdots \\ s^{n-3} & c_1 & c_2 & c_3 & \\ \vdots & \vdots & \vdots & & \\ s^0 & h_1 & & & \end{array}$$

b_i etc is computed as the negative of the determinant of a 2x2 matrix divided by the lower-left element.

where the values of b_i etc are constructed as follows:

$$b_1 = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}} = -\frac{1}{a_{n-1}}\begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix}$$

$$b_2 = \frac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}} = -\frac{1}{a_{n-1}}\begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix} \qquad \text{etc.}$$

$$c_1 = \frac{a_{n-3}b_1 - a_{n-1}b_2}{b_1} = -\frac{1}{b_1}\begin{vmatrix} a_{n-1} & a_{n-3} \\ b_1 & b_2 \end{vmatrix}$$

$$c_2 = \frac{a_{n-5}b_1 - a_{n-1}b_3}{b_1} = -\frac{1}{b_1}\begin{vmatrix} a_{n-1} & a_{n-5} \\ b_1 & b_3 \end{vmatrix} \qquad \text{etc.}$$

Step 3:

If all the elements of the first column are of the same sign, then all the roots have negative real parts (hence yielding a stable system).

If there are ***changes*** of the signs in the elements of the first column, then the number of sign changes equals the number of roots with the positive real parts.

<u>Example</u>: $q(s) = s^3 - 4s^2 + s + 6 = 0$

Clearly, there is at least one root with positive real part. We can still set up the Routh array to check how roots are in the RHP.

Step 1:	s^3	1	1	+
	s^2	-4	6	-
Step 2:	s^1	$\frac{-4\times1-1\times6}{-4} = 2.5$		+
	s^0	$\frac{2.5\times6-(-4)\times0}{2.5} = 6$		+

There are 2 sign change, therefore there are 2 roots in the right-half plane. With a 3rd order polynomial, there are 3 and only 3 roots. So, the other one must be in the left-half plane. Actually the C.E. can be expressed as

$$q(s) = (s+1)(s-2)(s-3) = 0$$

Example: $q(s) = 2s^4 + s^3 + 3s^2 + 5s + 10 = 0$

The necessary conditions are met. Set up the Routh array.

Step 1:	s^4	2	3	10	+
	s^3	1	5		+
Step 2:	s^2	$\frac{1\times3-2\times5}{1} = -7$	$\frac{1\times10-2\times0}{1} = 10$		-
	s^1	$\frac{-7\times5-1\times10}{-7} = 6.4$			+
	s^0	10			+

There are 2 sign changes in the first column and hence C.E. has 2 roots in the right-half plane, and 2 in the left-half plane, because it is a 4th order polynomial.

Exercise: Given a 3rd-order system $q(s) = s^3 + a_2 s^2 + a_1 s + a_o = 0;\ a_i > 0$

Show that the system is stable iff $a_1 a_2 > a_0$.

d) Special Cases

(I) Special Case: Zero in the First Column

In the construction of the array, we can encounter a zero element in the first column, then the Routh's test breaks down. There are 2 ways to overcome the difficulty.

I. Replace the zero with an arbitrary small number ε and continue. Then examine the signs of the first column elements by letting $\varepsilon \to 0^+$.

II. Modify the original C.E. $q(s)$ by replacing s with z^{-1} to get $z^n q(z^{-1}) = q'(z)$. Then apply the test on the modified equation $q'(z)$. $q(s)$ and $q'(z)$ have the same distribution of the roots with respect to the $j\omega$–axis. *Why?* (See Appendix 7.1.) The method does not guarantee that you will not encounter zero in the 1st column again.

<u>Example</u>: Consider the C.E.

$$q(s) = 3s^4 + 5s^3 + 3s^2 + 5s + 8 = 0$$

The necessary conditions are met. Set up the Routh array.

Step 1:	s^4	3	3	8
	s^3	5	5	
Step 2:	s^2	0	8	
	s^1	∞		
	s^0	Premature termination		

To overcome the problem, replace '0' by ε. Then the modified Routh array becomes

Step 1:

s^4	3	3	8
s^3	5	5	

Step 2:

s^2	ε	8	
s^1	$\frac{5\varepsilon-40}{\varepsilon}$		
s^0	8		

Note that $\varepsilon > 0$.

$\therefore \quad \dfrac{5\varepsilon - 40}{\varepsilon}$ is negative.

So, there are 2 sign changes as $\varepsilon \to 0^+$. Thus there are 2 RHP roots.

With the second method, we replace s with z^{-1} in the C.E. to get

$$3z^{-4} + 5z^{-3} + 3z^{-2} + 5z^{-1} + 8 = 0$$

Multiply by z^4 and rearrange, we get

$$q'(z) = 8z^4 + 5z^3 + 3z^2 + 5z + 3 = 0$$

Set up the Routh array.

Step 1:	z^4	8	3	3
	z^3	5	5	
Step 2:	z^2	–5	3	
	z^1	8		
	z^0	3		

There are 2 sign changes, so $q'(z)$ has 2 roots in the RHP, so is $\boldsymbol{q(s)}$***!***

Appendix 7.1

Given $q(s) = 0$ and $q'(z) = z^n q(z^{-1}) = 0$, where $s = z^{-1}$

Let s_o be a root of $q(s) = 0$, i.e. $q(s_o) = 0$

Since $s_o = z_o^{-1}$, we have $q'(z_0) = z_o^n q(z_o^{-1}) = 0$

That is, z_o is a root of $q'(z) = 0$

Now, s_o is in general a complex number. Let $s_o = \alpha + j\beta$, then

$$z_o = s_o^{-1} = \frac{1}{\alpha + j\beta} = \frac{\alpha - j\beta}{\alpha^2 + \beta^2}$$

Notice that the sign of the real parts of s_o and z_o are the same! Hence if s_o is in the LHP (RHP), z_o is also in the LHP (RHP). That is,

$$q(s) = 0 \quad \text{and} \quad q'(z) = 0$$

have the same root distribution with respect to the $j\omega$-axis.

Summary 7. Routh-Hurwitz Stability

A linear time-invariant control system is stable if ALL its poles are in the left-half plane. Let its C.E. be given by

$$q(s) = a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + a_{n-3} s^{n-3} + \cdots + a_o = 0$$

Routh-Hurwitz Criterion

The root distribution of $q(s)$ with respect to the $j\omega$-axis can be checked by setting up the following **Routh array**:

$$\begin{array}{ll|llll}
\text{Step 1:} & s^n & a_n & a_{n-2} & a_{n-4} & \ldots \\
 & s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \ldots \\
\hline
\text{Step 2:} & s^{n-2} & b_1 & b_2 & b_3 & \ldots \\
 & s^{n-3} & c_1 & c_2 & c_3 & \\
 & \vdots & \vdots & \vdots & & \\
 & s^0 & h_1 & & &
\end{array}$$

b_i , c_i etc are constructed from determinants of 2x2 matrices.

The number of ***sign changes*** in the elements of the first column will give the number of roots with the positive real parts.

Special Case: Zero in the First Column

In this case, the Routh's test breaks down. There are 2 solutions.

I. Replace the zero with ε and continue. Then examine the signs of the first column elements by letting $\varepsilon \to 0^+$

II. Modify the original $q(s)$ by replacing s with z^{-1} to get $z^n q(z^{-1}) = q'(z)$ Then apply the test on the modified equation $q'(z)$. $q(s)$ and $q'(z)$ have the same distribution of the roots with respect to the $j\omega$ –axis.

Lecture 8: Stability Analysis - Roots on $j\omega$-axis, Relative Stability

a) Special Cases
   - All-zero Row
   - Roots on $j\omega$-axis

b) Characteristic Equation with Unknown Parameter

c) Relative Stability

a) Special Cases

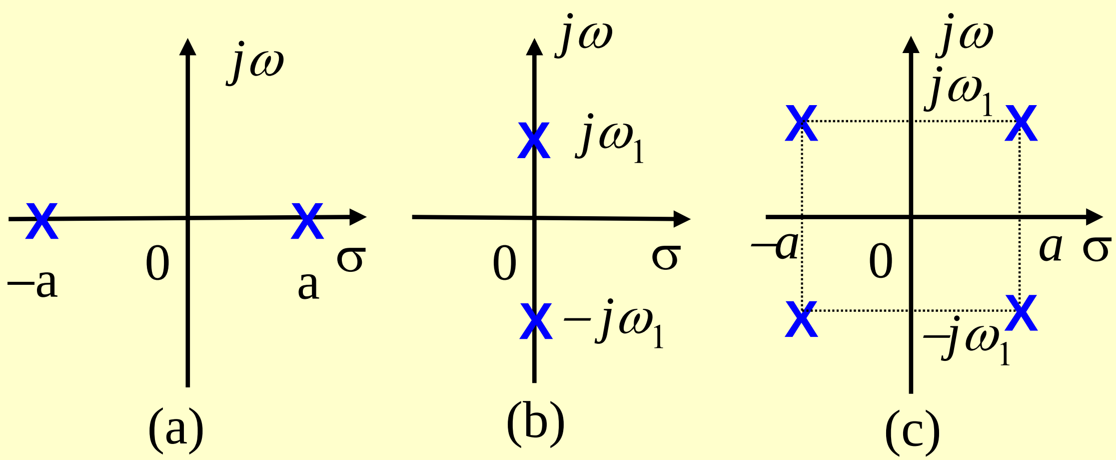
(II) Special Case: All-zero Row

The occurrence of an all-zero row indicates that there are symmetrically located roots in the *s*–plane. One or more of the following may happen:

i) Pair(s) of real roots with opposite signs.

ii) **Pair(s) of imaginary roots.**

iii) Pair(s) of complex-conjugate roots symmetrical about the origin of the *s*–plane.



Such roots can be found from the ***auxiliary equation*** formed from the elements in the row just above the all-zero row. The roots of the auxiliary equation are usually purely imaginary. In some special cases, you'll also get roots at the origin. *Do you know when? (See Appendix 8.1)*

An all-zero row terminates the Routh array prematurely. The difficulty can be overcome by replacing the row of zeros by the coefficients of the first derivative of the auxiliary equation.

Example: Consider the C.E.

$$s^4 + 2s^3 + 3s^2 + 2s + 2 = 0$$

The necessary conditions are met and the Routh array is

Step 1:	s^4	1	3	2
	s^3	2	2	
Step 2:	s^2	2	2	
	s^1	0		

The auxiliary equation is $A(s) = 2s^2 + 2$

and $\dfrac{dA(s)}{ds} = 4s$

The modified Routh array is:

Step 1:	s^4	1	3	2	
	s^3	2	2		
Step 2:	s^2	2	2		$\equiv A(s)$
	s^1	4			$\equiv \dfrac{dA(s)}{ds}$
	s^0	2			

There is no sign change in the 1st column. So the C.E. has no RHP roots *(?!)* The roots of the auxiliary equation is found from

$$A(s) = 2s^2 + 2 = 0 \quad \Rightarrow \quad s^2 + 1 = 0$$

$$\text{i.e.} \quad s_{1,2} = \pm j$$

Thus, there is a pair of imaginary roots. (There are 2 LHP roots, *why?*) So the system is <u>marginally stable</u>.

Example: Consider the C.E.

$$s^6 + 2s^5 + 8s^4 + 12s^3 + 20s^2 + 16s + 16 = 0$$

The necessary conditions are met and the Routh array is

Step 1:	s^6	1	8	20	16
	s^5	2	12	16	
Step 2:	s^4	2	12	16	
	s^3	0	0		
	s^2	Premature termination			

The auxiliary equation is $A(s) = 2s^4 + 12s^2 + 16$

and $\dfrac{dA(s)}{ds} = 8s^3 + 24s$

So the modified Routh array becomes

Step 1:	s^6	1	8	20	16	
	s^5	2	12	16		
Step 2:	s^4	2	12	16		$\equiv A(s)$
	s^3	8	24			$\equiv \frac{dA(s)}{ds}$
	s^2	6	16			
	s^1	$\frac{8}{3}$				
	s^0	16				

There is no sign change in the 1st column. So the C.E. has no RHP roots. *(?!)*

The roots of the auxiliary equation is found from

$$A(s) = 2s^4 + 12s^2 + 16 = 0$$

$$\Rightarrow \quad s^4 + 6s^2 + 8 = 0$$

$$\Rightarrow \quad (s^2)_{1,2} = \frac{-6 \pm \sqrt{36-32}}{2}$$

$$= -4, -2$$

i.e. $\quad s_{1,2} = \pm j2 \quad$ and $\quad s_{3,4} = \pm j\sqrt{2}$

Thus, they are 2 pairs of imaginary roots. (There are 2 LHP roots, *why?*) So the system is marginally stable.

<u>Exercise</u>: Consider the above example. Show that the roots of the auxiliary equation are also the roots of the original characteristic equation. (See Appendix 8.2.)

b) Characteristic Equation with Unknown Parameter

Example: Find the range of K for which the following C.E. is stable.

$$q(s) = s^3 + 34.5s^2 + 7500s + 7500K = 0$$

Necessary condition for stability is $K > 0$.

The Routh array is

Step 1:	s^3	1	7500	
	s^2	34.5	$7500K$	$\equiv A(s)$
Step 2:	s^1	$\dfrac{7500(34.5-K)}{34.5}$		
	s^0	$7500K$		

For stability, additional condition is

$$\frac{7500\,(34.5 - K)}{34.5} > 0 \quad \Rightarrow \quad K < 34.5$$

Hence, for stability,

$$0 < K < 34.5$$

If $K = 34.5$, the element of s^1 row are all zero. So the auxiliary equation is

$$A(s) = 34.5s^2 + 7500(34.5) = 0$$

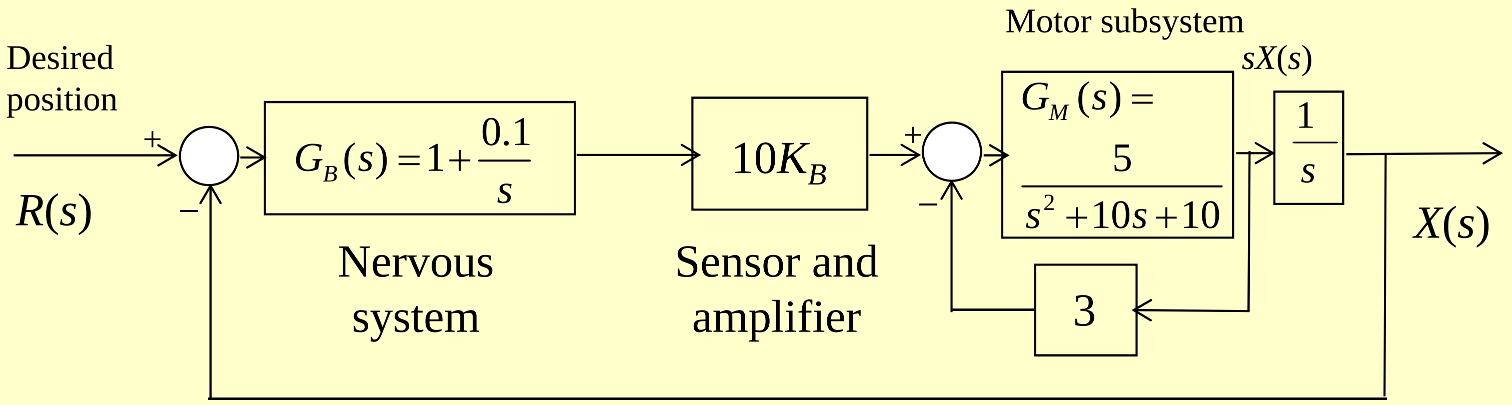
$$\Rightarrow \quad s^2 + 7500 = 0$$

$$\Rightarrow \quad s^2 = -7500$$

i.e. $s_{1,2} = \pm j\sqrt{7500} = \pm j86.6 \;\; rad/\sec$

Thus, with $K = 34.5$ the system will have an oscillation frequency of 86.6 *rad*/sec.

Example: A prosthetic and human artificial limb control system model is shown below.



Determine the range of K_B such that the closed-loop system is stable.

The motor arm subsystem is given by

$$G_T(s) = \frac{G_M(s)}{1+3G_M(s)} = \frac{5}{s^2+10s+25} = \frac{5}{(s+5)^2}$$

The closed-loop transfer function is

$$\frac{X(s)}{R(s)} = \frac{5K_B(10s+1)}{s^4+10s^3+25s^2+50K_Bs+5K_B}$$

Necessary condition is $K_B > 0$. The Routh array:

$$\begin{array}{llll}
s^4 & 1 & 25 & 5K_B \\
s^3 & 10 & 50K_B & \\
s^2 & 25\text{-}5K_B & 5K_B & \\
s^1 & \frac{240K_B - 50K_B^2}{5-K_B} & 0 & \\
s^0 & 5K_B & &
\end{array}$$

For stability, additional conditions are:

$$25 - 5K_B > 0 \quad \Rightarrow \quad K_B < 5.0$$

$$(240 - 50K_B)K_B > 0 \;\Rightarrow\; 0 < K_B < 4.8$$

That is, we need $0 < K_B < 4.8$

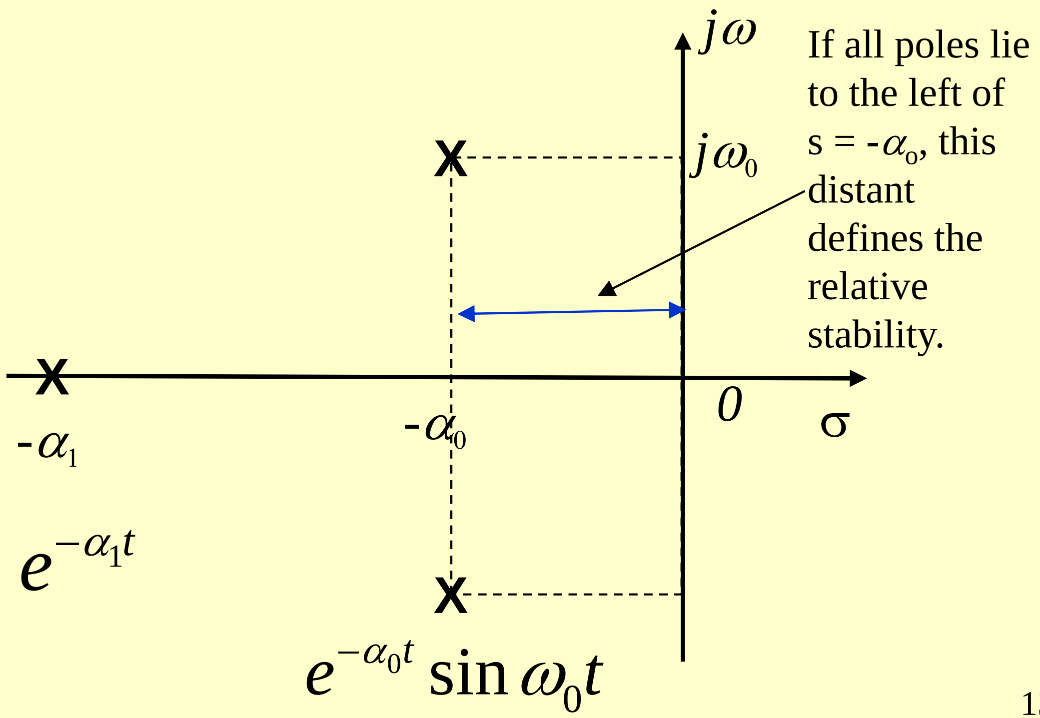
__Exercise__: For the above problem, determine the oscillation frequency when $K_B = 4.8$.

(*See Appendix 8.3 for an example with 2 unknown parameters.*)

c) Relative Stability

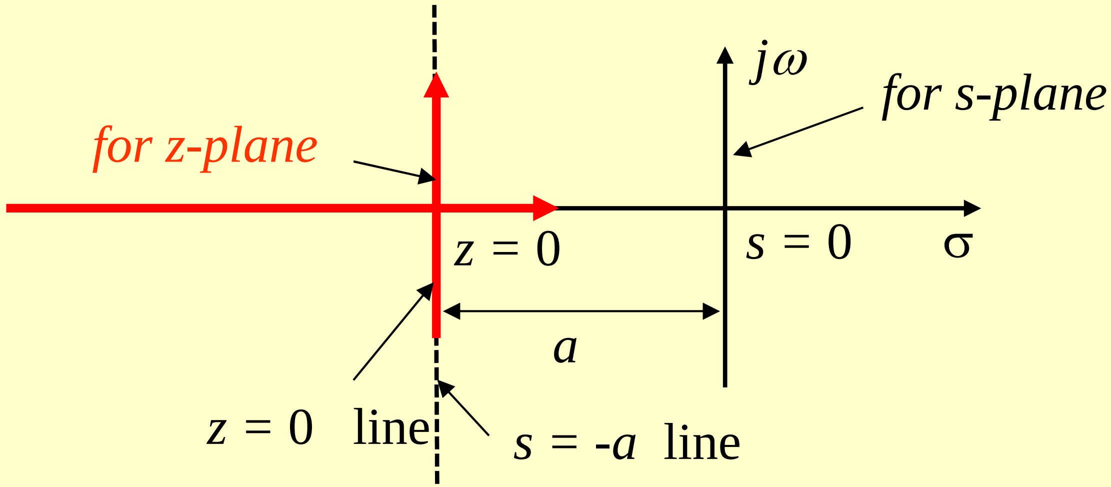
In addition to stability, we would also like to know "how stable" is the system, i.e. the ***relative stability*** of the system. The relative stability of a system is related to response of the dominant poles. We specify relative stability by requiring that the real parts of all the roots to be more negative than certain value. That is, all the roots must lie to the left of the line $s = -a;\ a > 0$.

Consider a 3rd-order system with 3 poles as shown below. The response term $e^{-\alpha_1 t}$ decays much faster than that of $e^{-\alpha_0 t}\sin\omega_0 t$. Thus, the distance between the dominant poles and the imaginary axis defines the relative stability of the system.



To determine whether all the roots of the C.E. are more negative than $-a$ we shift the origin of the s–plane to the left by a units. That is, we substitute $\boldsymbol{s = z - a}$ and consider stability w.r.t. the z–plane.

When $z = 0$, $s = -a$.



If the new C.E. in z satisfied the Routh-Hurwitz criterion, then all the roots in the z–domain lie to the left of $z = 0$ line and hence, to the left of the $s = -a$ line.

Example: Consider a system with C.E.

$$s^3 + 7s^2 + 25s + 39 = 0$$

By Routh-Hurwitz test, it can be shown that all roots are in the LHP.

But are all the roots having real parts more negative than –1.0? To answer this, we shift the origin to $s = -1$ by substituting $s = z - 1$ in the C.E.

The new C.E. in terms of z is

$$q'(z) = (z-1)^3 + 7(z-1)^2 + 25(z-1) + 39$$
$$= (z^3 - 3z^2 + 3z - 1) + 7(z^2 - 2z + 1)$$
$$+ 25(z-1) + 39 = 0$$

i.e. $$q'(z) = z^3 + 4z^2 + 14z + 20 = 0$$

The Routh array is

Step 1:	z^3	1	14
	z^2	4	20
Step 2:	z^1	9	
	z^0	20	

Since there is no sign change in the 1st column, all the roots $q'(z)$ lie in the LHP of the z–plane. Equivalently, all the roots of $q(s)$ lie to the left of the $s = -1$ line.

Exercise: For the same system, determine whether the roots have real parts more negative than –2.0. (See Appendix 8.4)

Appendix 8.1

Consider $q(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_o = 0$

where $a_o \neq 0$

Then $q(s) = 0$ will not have any root at the origin because $q(0) \neq 0$

Consider another polynomial,

$$q'(s) = s^m q(s) = a_n s^{n+m} + a_{n-1} s^{n+m-1} + \cdots + a_o s^m = 0$$

Clearly, $q'(s) = 0$ has m roots at the origin.

Appendix 8.2

The given characteristic equation can be factored as

$$s^4 + 2s^3 + 3s^2 + 2s + 2$$
$$= (s^2 + 1)(s^2 + 2s + 2) = 0$$

i.e. $A(s) = s^2 + 1$ is a factor of the C.E. and so the roots of $A(s) = 0$ are also the roots of the original C.E.

Similarly,

$$s^6 + 2s^5 + 8s^4 + 12s^3 + 20s^2 + 16s + 16$$
$$= (s^4 + 6s^2 + 8)(s^2 + 2s + 2) = 0$$

So, $A(s) = (s^4 + 6s^2 + 8)$ is a factor of the C.E. and so the roots of $A(s) = 0$ are also the roots of the original C.E.

The result is true in general but a formal proof is very difficult.

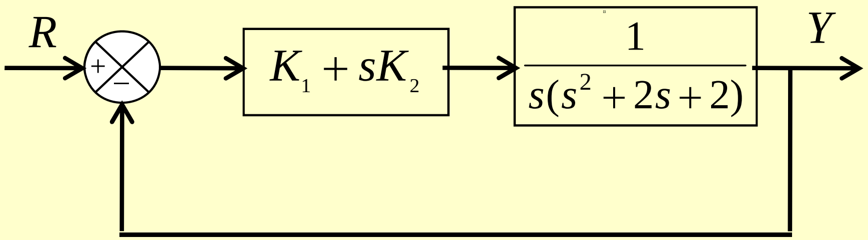
Appendix 8.3

Consider the following control system. Construct a parameter plane of K_1 versus K_2 and show the following regions in the plane.

(a) Stable and unstable regions.

(b) Trajectory on which the system will have sustained oscillation.

(c) The point in which the sustained oscillation frequency is 2 rad/s.



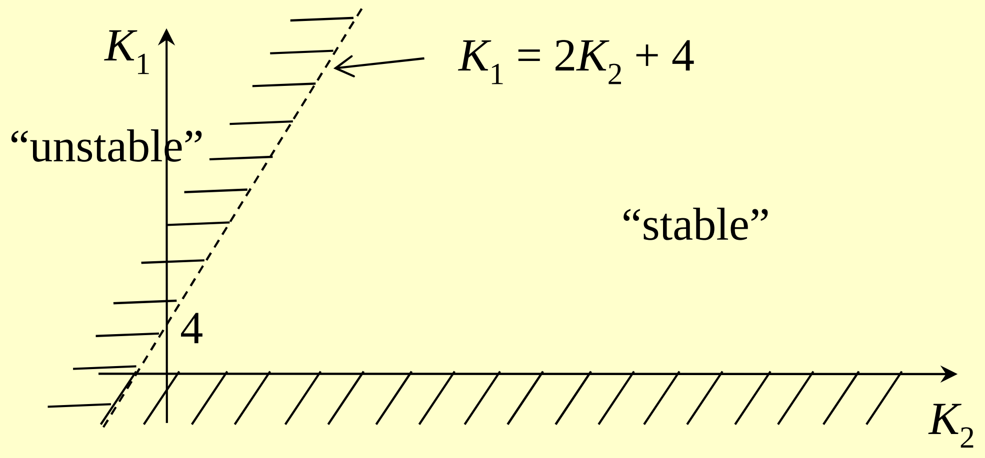
CLTF: $$\frac{Y}{R} = \frac{K_1 + sK_2}{s(s^2+2s+2) + sK_2 + K_1}$$

C.E.: $$q(s) = s^3 + 2s^2 + (K_2+2)s + K_1$$

s^3	1	$2+K_2$
s^2	2	K_1
s^1	$\frac{4+2K_2-K_1}{2}$	0
s^0	K_1	

For stability, we need $K_1 > 0$, and

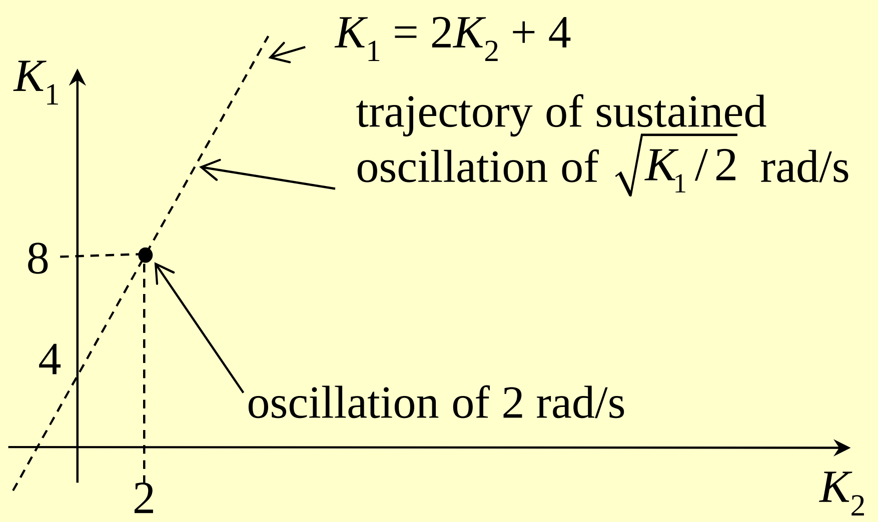
$$4 + 2K_2 - K_1 > 0 \;\Rightarrow\; K_1 < 2K_2 + 4$$



When $K_1 = 2K_2 + 4$, we get auxiliary equation:

$$A(s) = 2s^2 + K_1 = 0 \;\Rightarrow\; s = \pm j\sqrt{K_1/2}$$

If $\sqrt{K_1/2} = 2 \Rightarrow K_1 = 8$ (and $K_2 = 2$)



Appendix 8.4

$$q(s) = s^3 + 7s^2 + 25s + 39 = 0$$

Let $s = z - 2$ in the C.E. The new C.E. in terms of z is

$$q'(z) = (z-2)^3 + 7(z-2)^2 + 25(z-2) + 39$$

$$= z^3 + z^2 + 9z + 9 = 0$$

The Routh array and the modified array are

$$\begin{array}{c|ccc} z^3 & 1 & 9 & 0 \\ z^2 & 1 & 9 & 0 \\ \hline z^1 & 0 & 0 & \end{array} \qquad \begin{array}{c|ccc} z^3 & 1 & 9 & 0 \\ z^2 & 1 & 9 & 0 \\ \hline z^1 & 2 & 0 & \\ z^0 & 9 & & \end{array}$$

There is an auxiliary equation of $A(z) = z^2 + 9 = 0$. The roots are $z_{1,2} = \pm j3$

Thus $q'(z) = 0$ has 2 roots on the $j\omega$-axis of the z-plane. This means that $q(s) = 0$ has 2 roots on the line s = -2. Indeed, $s_{1,2} = -2 \pm j3$

The 3rd root of $q(s) = 0$ is to the left of s = -2.

Summary 8. Roots on $j\omega$-axis, Relative Stability

Special Case: All-zero Row

This yields roots symmetrically distributed in the s-plane, mainly roots on the $j\omega$-axis.

Such roots can be found from the ***auxiliary equation*** formed from the elements in the row just above the all-zero row.
An all-zero row terminates the Routh array prematurely. We replace the row of zeros by the coefficients of the first derivative of the auxiliary equation.

Relative Stability

Relative stability is achieved by requiring all the roots to lie to the left of the line $s = -a$; $a > 0$. In the original $q(s)$, we replace s by $z - a$ to get $q'(z) = q(z-a)$, and apply test to $q'(z)$.

Lecture 9: System Performance - Steady-state Errors

a) Steady-state Errors

b) Error Constants

c) “Types” of Control Systems

d) Steady-state Error in the Presence of Disturbance

a) Steady-state Errors

Steady-state error can be found by applying the final value theorem and is given by

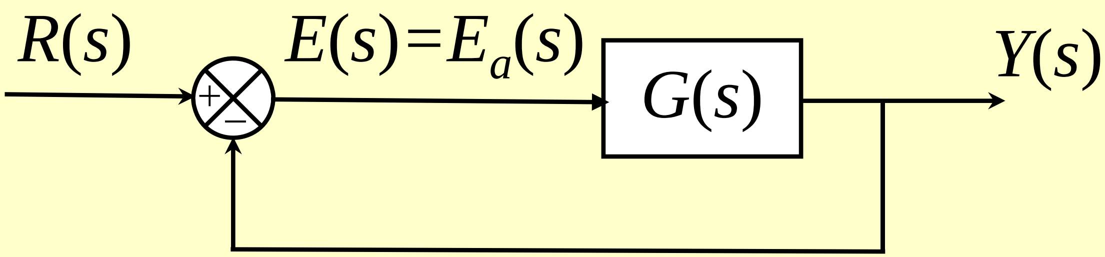
$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$$

provided that $sE(s)$ does not have poles on the $j\omega$-axis or in the right half of s-plane.

The **system error** is defined as the difference between the ***reference input*** and the ***output***:

$$e(t) = r(t) - y(t)\ ;\ \text{or}\ \ E(s) = R(s) - Y(s)\ \text{in the}\ s\text{-domain}$$

Consider the unity feedback system:



Since $$Y(s) = \frac{G(s)}{1+G(s)} \times R(s)$$

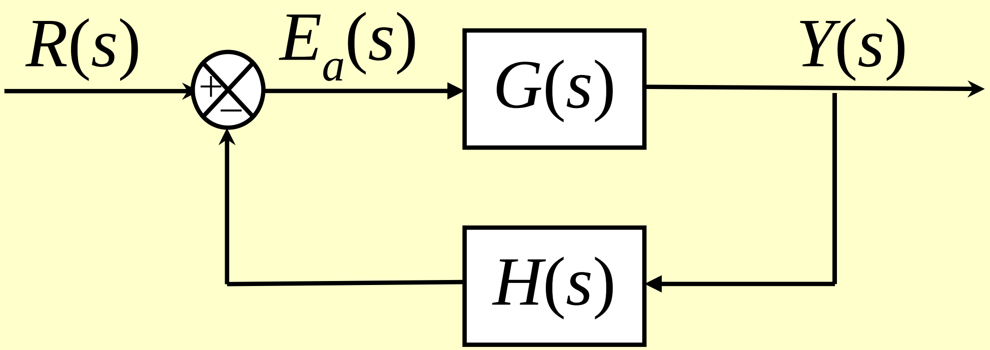
Then $$E(s) = R(s) - Y(s) = R(s)\left(1 - \frac{G(s)}{1+G(s)}\right)$$

That is, $$E(s) = \frac{R(s)}{1+G(s)}$$

Hence we have

$$e_{ss} = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{sR(s)}{1+G(s)}$$

For a general non-unity feedback control system shown below, we need to be careful about system error. Indeed, in some cases, it may not make sense to define $R(s)$ -$Y(s)$. *Why?*



b) Error Constants (for unity-feedback systems)

We examine the steady-state error w.r.t. the input function $R(s)$. Recall that

$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{sR(s)}{1+G(s)}$$

(1) Unit-Step Input: $R(s) = \dfrac{1}{s}$

$$e_{ss} = \lim_{s\to 0} \frac{s \times \frac{1}{s}}{\left[1+G(s)\right]}$$

$$= \frac{1}{1+\lim\limits_{s\to 0} G(s)} = \frac{1}{1+K_{pos}}$$

where $K_{pos} = \lim\limits_{s\to 0} G(s)$ is the position error constant.

(2) Unit-Ramp (velocity) Input: $R(s) = \dfrac{1}{s^2}$

$$e_{ss} = \lim_{s\to 0} \frac{s \times \frac{1}{s^2}}{[1 + G(s)]}$$

$$= \lim_{s\to 0} \frac{1}{s + sG(s)} = \frac{1}{\lim\limits_{s\to 0} sG(s)} = \frac{1}{K_{vel}}$$

where $K_{vel} = \lim\limits_{s\to 0} sG(s)$ is the velocity error constant.

(3) Unit Parabolic (acceleration) Input: $R(s) = \dfrac{1}{s^3}$

$$e_{ss} = \lim_{s\to 0} \frac{s \times \frac{1}{s^3}}{[1 + G(s)]}$$

$$= \lim_{s\to 0} \frac{1}{s^2 + s^2 G(s)} = \frac{1}{\lim\limits_{s\to 0} s^2 G(s)} = \frac{1}{K_{acc}}$$

where $K_{acc} = \lim\limits_{s\to 0} s^2 G(s)$ is the acceleration error constant.

Example: The closed-loop transfer function of a unity-feedback control system is given by

$$\frac{Y(s)}{R(s)} = \frac{Ks + b}{s^2 + as + b}$$

where $a, b > 0$ (*why?*). We want to find the steady-state error to a unit-ramp input.

We know that $\dfrac{Y(s)}{R(s)} = \dfrac{G(s)}{1 + G(s)} = \dfrac{Ks + b}{s^2 + as + b}$

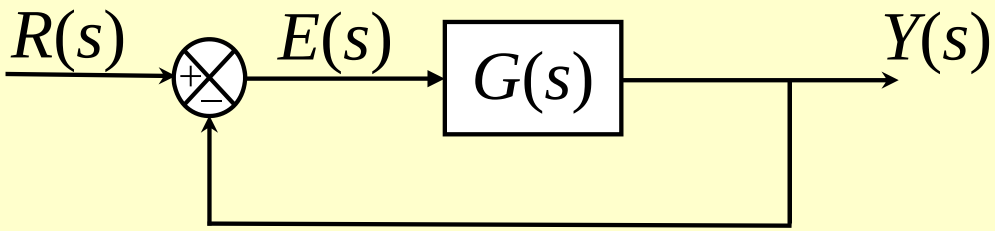
So, $(s^2 + as + b)G(s) = (Ks + b)[1 + G(s)]$, or

$$G(s) = \frac{Ks + b}{s(s + a - K)}$$

Thus the steady-state error to a unit-ramp input is

$$e_{ss} = \frac{1}{K_{vel}} = \frac{1}{\lim\limits_{s\to 0} sG(s)} = \lim_{s\to 0} \frac{s(s + a - K)}{s(Ks + b)} = \frac{a - K}{b}$$

Exercise: Given the unity feedback system



where $G(s) = \dfrac{10}{s^3 + 3s^2 + 3s + 1}$

A student calculated that the position error constant is

$$K_{pos} = \lim_{s \to 0} G(s) = 10$$

He then concluded that the steady-state error with respect to a step input is 1/11. Is the student's conclusion correct? (See Appendix 9.1.)

c) "Types" of Control Systems

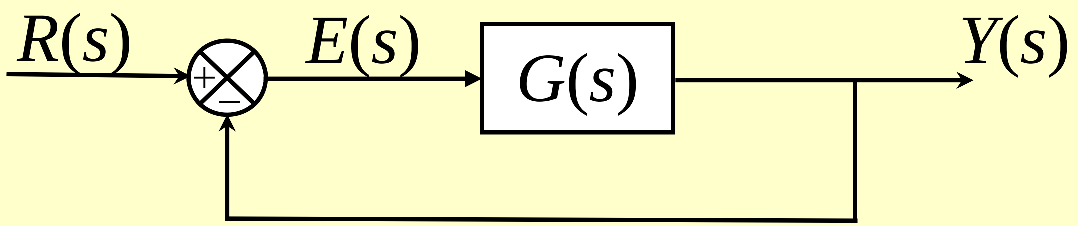
Let us now examine the steady-state error w.r.t the system OLTF $G(s)$. For a **unity-feedback system**, $G(s)$ can always be expressed as:

$$G(s) = \frac{G_1(s)}{s^j} \qquad (1)$$

where $G_1(s)$ has no pole nor zero at the origin. For example,

$$G(s) = \frac{10(s+3)}{s^2(s^2+3s+4)} = \frac{G_1(s)}{s^2}; \quad G_1(s) = \frac{10(s+3)}{s^2+3s+4}$$

$$G(s) = \frac{10}{s^3+3s^2+4s} = \frac{G_1(s)}{s}; \quad G_1(s) = \frac{10}{s^2+3s+4}$$



For a **unity feedback system**, the "**type**" of a control system w.r.t. $R(s)$ is defined by the order of the poles of $G(s)$ at the origin of the s-plane (i.e. the value of $j \in \{0, 1, 2, \ldots\}$). The system described by eqn (1) is of **type-*j***. It has a direct effect on the steady-state errors.

<u>Type –0 System</u>: (*j* = 0):

$$G(s) = \frac{G_1(s)}{s^0}; \quad G_1(0) = K \ (\neq \ \infty)$$

Then

$$K_{pos} \triangleq \lim_{s\to 0} G(s) = \lim_{s\to 0} G_1(s) = K$$

$$K_{vel} \triangleq \lim_{s\to 0} sG(s) = \lim_{s\to 0} sG_1(s) = 0$$

$$K_{acc} \triangleq \lim_{s\to 0} s^2 G(s) = 0$$

Thus,

$$e_{ss}(unit - step) = \frac{1}{1+K_{pos}} = \frac{1}{1+K}$$

$$e_{ss}(unit - ramp) = \frac{1}{K_{vel}} = \infty$$

$$e_{ss}(unit - parabolic) = \frac{1}{K_{acc}} = \infty$$

<u>Type –1 System</u>: (*j* = 1):

$$G(s) = \frac{G_1(s)}{s^1}; \quad G_1(0) = K \ (\neq 0 \text{ or } \infty)$$

Then

$$K_{pos} \triangleq \lim_{s\to 0} G(s) = \lim_{s\to 0} \frac{G_1(s)}{s} = \infty$$

$$K_{vel} \triangleq \lim_{s\to 0} sG(s) = \lim_{s\to 0} G_1(s) = K$$

$$K_{acc} \triangleq \lim_{s\to 0} s^2 G(s) = 0$$

Thus,

$$e_{ss}(unit-step) = \frac{1}{1+K_{pos}} = 0$$

$$e_{ss}(unit-ramp) = \frac{1}{K_{vel}} = \frac{1}{K}$$

$$e_{ss}(unit-parabolic) = \frac{1}{K_{acc}} = \infty$$

<u>Type –2 System</u>: (*j* = 2):

$$G(s) = \frac{G_1(s)}{s^2}; \quad G_1(0) = K \ (\neq 0 \text{ or } \infty)$$

Then

$$K_{pos} \triangleq \lim_{s \to 0} G(s) = \lim_{s \to 0} \frac{G_1(s)}{s^2} = \infty$$

$$K_{vel} \triangleq \lim_{s \to 0} sG(s) = \lim_{s \to 0} \frac{G_1(s)}{s} = \infty$$

$$K_{acc} \triangleq \lim_{s \to 0} s^2 G(s) = \lim_{s \to 0} G_1(s) = K$$

Thus,

$$e_{ss}(unit - step) = \frac{1}{1 + K_{pos}} = 0$$

$$e_{ss}(unit - ramp) = \frac{1}{K_{vel}} = 0$$

$$e_{ss}(unit - parabolic) = \frac{1}{K_{acc}} = \frac{1}{K}$$

The steady-state errors for various inputs and system types are summarized as follows:

System Types	Unit-step input, $u(t)$	Unit-ramp input, t	Parabolic input, $\frac{1}{2}t^2$
0	$\frac{1}{1+K_{pos}}$	∞	∞
1	0	$\frac{1}{K_{vel}}$	∞
2	0	0	$\frac{1}{K_{acc}}$
3	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Systems of type higher than 2 are not commonly used in practice because:

(i) They are more difficult to stabilize.

(ii) The dynamic errors tend to be larger.

d) Steady-state Error in the Presence of Disturbance

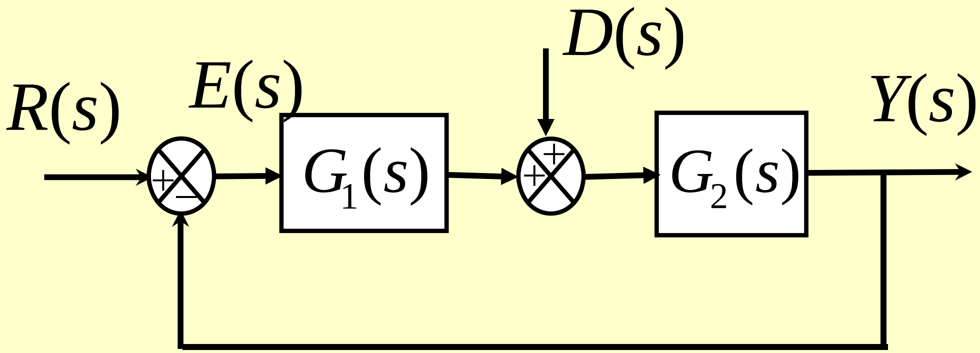
In the presence of disturbance, there are 2 components in the steady-state error:

- an error due to inability of the output to follow the reference input
- an unwanted contribution from the disturbance

It's important to note that the **output due to the disturbance is the unwanted output**!

It is desirable that each of these steady-state components are small, if not zero.

Consider the following feedback control system with disturbance:



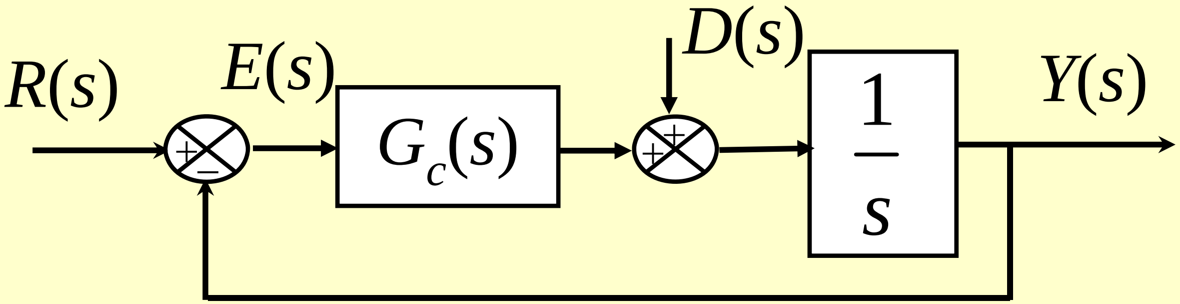
The output has 2 components, i.e. $Y(s) = Y_R(s) + Y_D(s)$, where $Y_R(s)$ is due to $R(s)$ alone and $Y_D(s)$ due to $D(s)$ alone.

The system error also has 2 components:

i.e. $E(s) = R(s) - (Y_R(s) + Y_D(s)) = (R(s) - Y_R(s)) - Y_D(s) = E_R(s) + E_D(s)$

where $E_R(s) = R(s) - Y_R(s);$ and $E_D(s) = -Y_D(s)$

Example: Consider the feedback control system as shown below:



If $G_c(s) = K$, calculate the steady-state error when $R(s)$ is a unit-ramp input and $D(s)$ is a unit-step disturbance.

Since $E(s) = E_R(s) + E_D(s)$, the total steady-state error is given by

$$e_{ss} = e_{Rss} + e_{Dss}$$

where $e_{Rss} \triangleq \lim_{s\to 0} sE_R(s) = \frac{1}{K_{vel}}$

and $e_{Dss} \triangleq \lim_{s\to 0} sE_D(s) = -\lim_{s\to 0} sY_D(s)$

Now, $e_{Rss} = \frac{1}{K_{vel}} = \frac{1}{\lim_{s\to 0} sG(s)} = \frac{1}{\lim_{s\to 0} s \times \frac{K}{s}} = \frac{1}{K}$

For a positive unit-step disturbance, we have

$$\frac{Y_D(s)}{D(s)} = \frac{\frac{1}{s}}{1+\frac{K}{s}} \quad ; \quad D(s) = \frac{1}{s}$$

Therefore,

$$y_{Dss} \triangleq \lim_{s\to 0} sY_D(s) = \lim_{s\to 0} \frac{s}{s+K}\frac{1}{s} = \frac{1}{K}$$

And

$$e_{Dss} = -y_{Dss} = -\frac{1}{K}.$$

So,

$$e_{ss} = e_{Rss} + e_{Dss} = \tfrac{1}{K} - \tfrac{1}{K} = 0 \quad !!$$

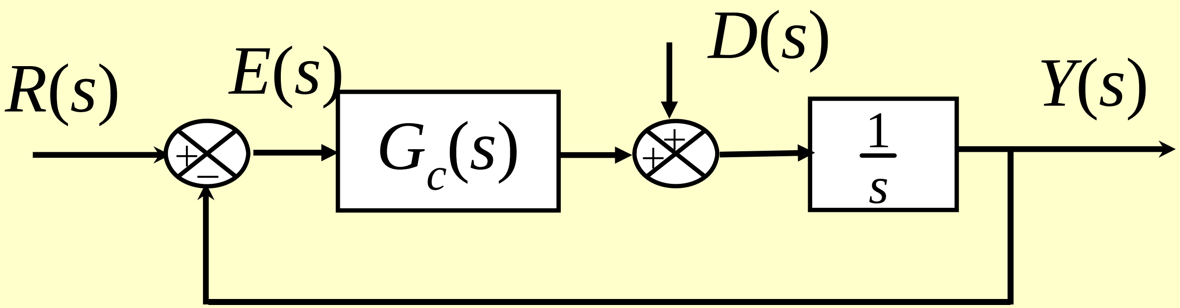
However, unlike the reference input, the disturbance can't be known exactly. In this example, it could be a negative step rather than a positive step, in which case

$$e_{ss} = e_{Rss} + e_{Dss} = \tfrac{1}{K} + \tfrac{1}{K} = \tfrac{2}{K} \quad !!$$

If K is small, then we would have a large error!

Hence, one should ensure that both e_{Rss} and e_{Dss} are zero or small!

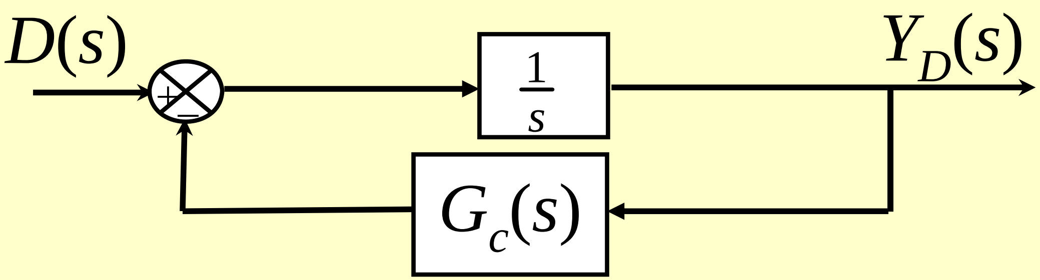
In feedback control system, we can eliminate (at steady-state) the effect of the step disturbance with a PI-control.



For the same problem, let's examine the design of

$$G_c(s) = \frac{sK_1 + K_2}{s}$$

to eliminate (at steady state) the error due to the disturbance $D(s)$, and at the same time ensure stability of the closed-loop control system. Setting $R(s)$ to zero and we have:



where $Y_D(s)$ denotes the output due to the disturbance.

For a unit-step disturbance input, $D(s) = \frac{1}{s}$, and

$$\frac{Y_D(s)}{D(s)} = \frac{\frac{1}{s}}{1 + \frac{1}{s}\left(\frac{sK_1 + K_2}{s}\right)} = \frac{s}{s^2 + sK_1 + K_2}$$

Hence

$$y_{Dss} = \lim_{s \to 0} sY_D(s) = \lim_{s \to 0} s \cdot \frac{s}{s^2 + sK_1 + K_2} \cdot \frac{1}{s}$$

$$= 0 \qquad \text{if} \quad K_2 \neq 0$$

The closed-loop transfer function w.r.t. $R(s)$ is given by

$$\frac{Y_R(s)}{R(s)} = \frac{\frac{G_c(s)}{s}}{1 + \frac{G_c(s)}{s}} = \frac{sK_1 + K_2}{s^2 + sK_1 + K_2}$$

For stability, we need $K_1 > 0$ and $K_2 > 0$.

The steady state error with respect to a step or ramp reference input is zero because the system with $G_c(s)$ is type 2. (Since $G_c(s) \bullet \frac{1}{s} = \frac{sK_1 + K_2}{s^2}$)

Appendix 9.1

Given $G(s) = \dfrac{10}{s^3 + 3s^2 + 3s + 1}$

then for the unity-feedback system,

$$E(s) = \frac{R(s)}{1 + G(s)} = \frac{s^3 + 3s^2 + 3s + 1}{s^3 + 3s^2 + 3s + 11} R(s)$$

Clearly, $sE(s)$ is not an analytic function in the closed RHP. This is because

$$1 + G(s) = 0 \quad (\Rightarrow s^3 + 3s^2 + 3s + 11 = 0)$$

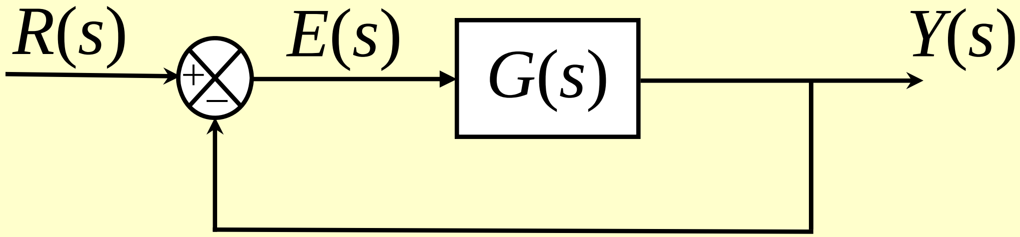
has roots in the RHP. That is, the closed-loop system is unstable and hence steady-state error is undefined.

Summary 9. Steady-State Errors

Steady-state error, $e_{ss} \triangleq \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$

For **stable** <u>unity feedback system</u> systems:

$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{sR(s)}{1+G(s)}$$



Error Constants (for unity-feedback systems)

Position Error Constant, $K_{pos} = \lim_{s\to 0} G(s)$

Velocity Error Constant, $K_{vel} = \lim_{s\to 0} sG(s)$

Acceleration Error Constant, $K_{acc} = \lim_{s\to 0} s^2 G(s)$

Hence,

$$e_{ss}(\text{unit-step}) = 1/\left(1+K_{pos}\right)$$

$$e_{ss}(\text{unit-ramp}) = 1/K_{vel}$$

$$e_{ss}(\text{unit-parabolic}) = 1/K_{acc}$$

"Types" of Control Systems

The type of a control system with respect to an input is specified by its steady-state error performance. Steady-state errors (for a unity feedback system) for various inputs and system types are summarized as follows:

System Types	Unit-step input, $u(t)$	Unit-ramp input, t	Parabolic input, $\frac{1}{2}t^2$
0	$\frac{1}{1+K_{pos}}$ (finite)	∞	∞
1	0	$\frac{1}{K_{vel}}$ (finite)	∞
2	0	0	$\frac{1}{K_{acc}}$ (finite)
3	0	0	0

Steady-state Error in the Presence of Disturbance:

When calculating the steady-state errors in the presence of disturbance, we want both the following components to be small:

- error due to inability of the output to follow the reference input
- unwanted contribution from the disturbance

Lecture 10: System Performance - Error and Sensitivity Analysis

a) Disturbance and Measurement Noise

b) Sensitivity Function and Complementary Sensitivity Function

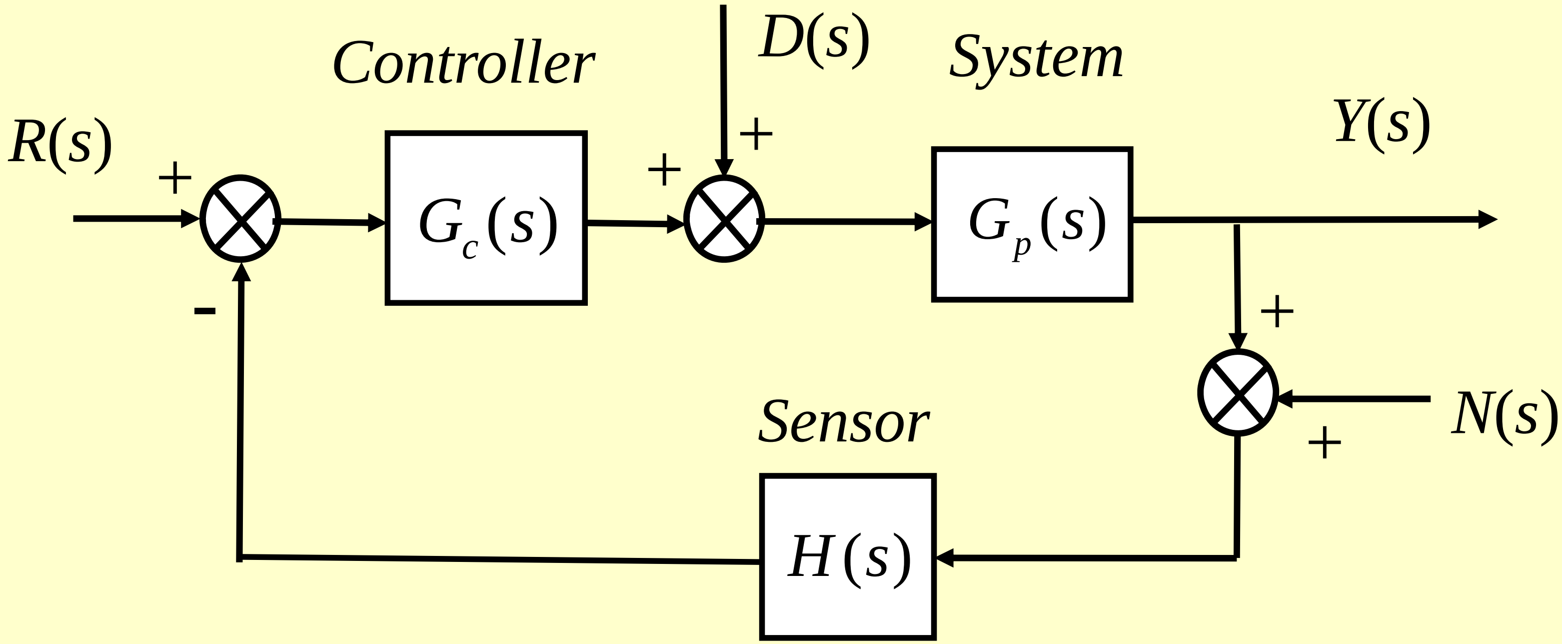
c) Disturbance Attenuation

d) Attenuation of Measurement Noise

e) Sensitivity to Parameter Variations

a) Disturbance and Measurement Noise

In a feedback control system, there is often the presence of both the disturbance $D(s)$ and the measurement noise $N(s)$.



In most cases, $D(s)$ is of low frequency content and $N(s)$ of high frequency content, i.e. $|D(j\omega)|$ is small for high frequency and $|N(j\omega)|$ is small for low frequency. This "separation" of frequency characteristics is important when designing the controller $G_c(s)$.

For ease of analysis, we assume that H(s) = 1.

From the diagram, we can derive that the output is given by

$$Y(s) = \frac{G_c(s)G_p(s)}{1+G_c(s)G_p(s)} \times R(s) + \frac{G_p(s)}{1+G_c(s)G_p(s)} \times D(s) - \frac{G_c(s)G_p(s)}{1+G_c(s)G_p(s)} \times N(s)$$

Let $L(s) = G_c(s)G_p(s)$, the loop gain (or open loop transfer function).

Then $Y(s) = \dfrac{L(s)}{1+L(s)} \times R(s) + \dfrac{G_p(s)}{1+L(s)} \times D(s) - \dfrac{L(s)}{1+L(s)} \times N(s)$

In many applications, we want $Y(s)$ to track $R(s)$, so we define the system error $E(s) = R(s) - Y(s)$:

$$E(s) = \underbrace{\frac{1}{1+L(s)} \times R(s)}_{E_R} - \underbrace{\frac{G_p(s)}{1+L(s)} \times D(s)}_{E_D} + \underbrace{\frac{L(s)}{1+L(s)} \times N(s)}_{E_N}$$

Clearly, to make errors w.r.t. R(s) and $D(s)$ small, we need a large $L(s)$, but to make error w.r.t. $N(s)$ small, we want a small $L(s)$. A contradiction?

The magnitude of a transfer function (i.e. "large" or "small") will be discussed under *Frequency Domain Analysis*. For the current purpose, it suffices to describe the magnitude of the loop gain $L(s)$ as the magnitude $\left|L(j\omega)\right|$ over the range of frequencies ω of interest.

To reduce the influence of the disturbance $D(s)$, we want a large $L(s)$ over the range of frequencies that characterize $D(s)$. Conversely, we need $L(s)$ small over the range of frequencies that characterize measurement noise $N(s)$.

Since the loop gain $L(s) = G_c(s)G_p(s)$, it can be made "big" and/or "small" with a proper design of the controller $G_c(s)$.

b) Sensitivity Function and Complementary Sensitivity Function

Instead of loop gain, we may also use Sensitivity Function and Complementary Sensitivity Function to quantify the 3 error terms.

Sensitivity Function is defined as $S(s) = \dfrac{1}{1+L(s)}$

Complementary Sensitivity Function is defined as $C(s) = \dfrac{L(s)}{1+L(s)}$

Then

$$E(s) = S(s) \times R(s) - S(s) G_p(s) \times D(s) + C(s) \times N(s)$$

Thus, to make errors w.r.t. *R*(s) and *D*(*s*) small, we need a small *S*(*s*) , and to make error w.r.t. *N*(*s*) small, we want a small *C*(*s*). However,

$$S(s) + C(s) = 1$$

So a trade-off is needed.

c) Disturbance Attenuation

To examine disturbance attenuation, we set $R(s) = N(s) = 0$, then

$$E_D(s) = -\frac{G_p(s)}{1+L(s)}D(s)$$

Clearly, to make error w.r.t. $D(s)$ small, we want a large $L(s)$. Since $L(s) = G_c(s)G_p(s)$, we can achieve this with proper design of the controller $G_c(s)$.

See Appendix 10.2 for an example comparing the merits of proportional closed-loop over open loop system, and an example of PI controller that eliminates steady state error due to disturbance.

d) Attenuation of Measurement Noise

To analyze noise attenuation, we set $R(s) = D(s) = 0$, then

$$E_N(s) = \frac{L(s)}{1 + L(s)} N(s)$$

Thus, to make error w.r.t. $N(s)$ small, we want a small $L(s)$.

If we design $G_c(s)$ such that $L(s) << 1$, then

$$E_N(s) \approx L(s) N(s)$$

If $L(s) >> 1$, then

$$E_N(s) \approx N(s)$$

In practice, measurement noise is often of high frequency and we only need to have low loop gain at high frequencies.

e) Sensitivity to Parameter Variations

One of the primary reasons for employing feedback control is to reduce possible changes to system characteristics with respect to parameter variations.

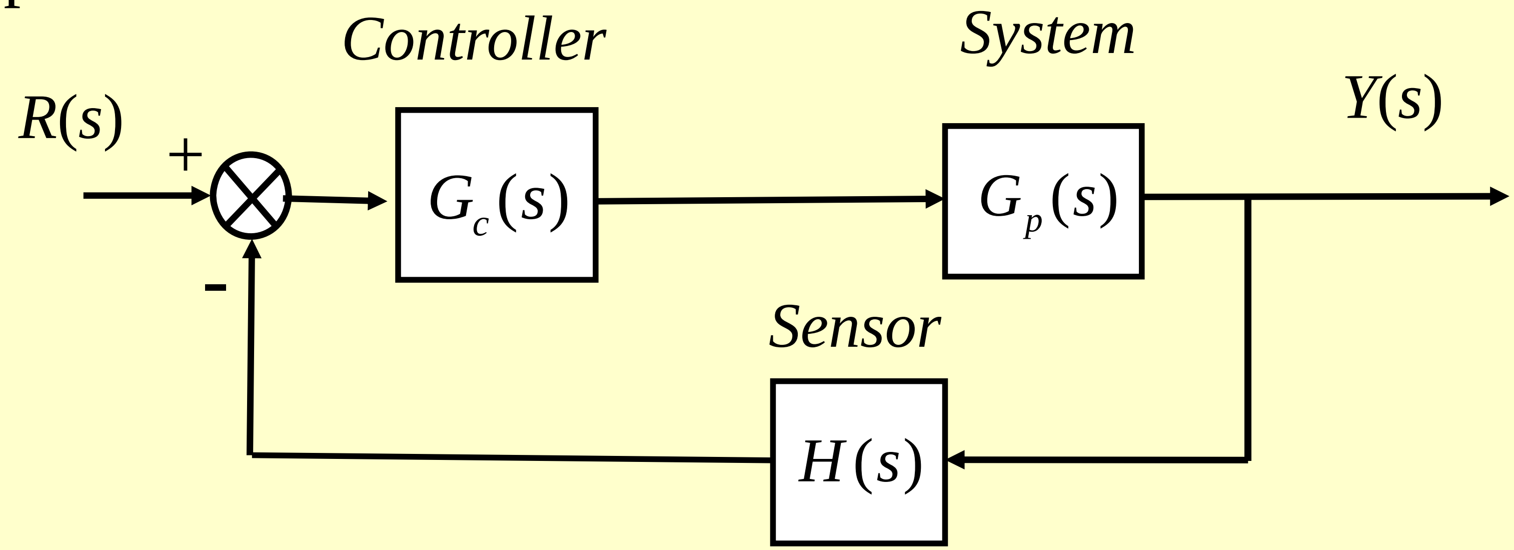
To quantify the sensitivity of change in system transfer function $T(s)$ with respect to a parameter b, we consider the following ratio:

$$S = \frac{\Delta T(s)/T(s)}{\Delta b/b} = \frac{\Delta T(s)}{\Delta b}\frac{b}{T(s)}$$

The above is evaluated in the limit as Δb approaches zero, thus we consider the following *sensitivity function*:

$$S_b^T = \lim_{\Delta b \to 0} \frac{\Delta T(s)}{\Delta b}\frac{b}{T(s)} = \frac{\partial T(s)}{\partial b}\frac{b}{T(s)}$$

Consider



Closed-loop transfer function: $T(s) = \dfrac{G_c(s)G_p(s)}{1 + G_c(s)G_p(s)H(s)}$

One of the key sensitivity functions is the sensitivity of the system transfer function $T(s)$ w.r.t. changes in plant transfer function $G_p(s)$:

$$S_{G_p}^{T} = \frac{\partial T(s)}{\partial G_p(s)} \frac{G_p(s)}{T(s)} = \frac{(1 + G_c G_p H)G_c - G_c G_p (G_c H)}{(1 + G_c G_p H)^2} \frac{G_p}{G_c G_p / (1 + G_c G_p H)}$$

$$= \frac{1}{1 + G_c G_p H}$$

Its frequency response:

$$S_{G_p}^{T}(j\omega) = \frac{1}{1 + G_c(j\omega)G_p(j\omega)H(j\omega)}$$

Hence, at frequencies within the system bandwidth, we want the loop gain G_cG_pH to be as large as possible to reduce the sensitivity of the system characteristics, by using controller G_c, but without destabilizing the system.

Another important sensitivity function is the sensitivity of the system transfer function $T(s)$ w.r.t. changes in the sensor $H(s)$:

$$S_H^T = \frac{\partial T(s)}{\partial H(s)}\frac{H(s)}{T(s)} = \frac{-G_cG_p(G_cG_p)}{(1+G_cG_pH)^2}\frac{H}{G_cG_p/(1+G_cG_pH)} = -\frac{G_cG_pH}{1+G_cG_pH}$$

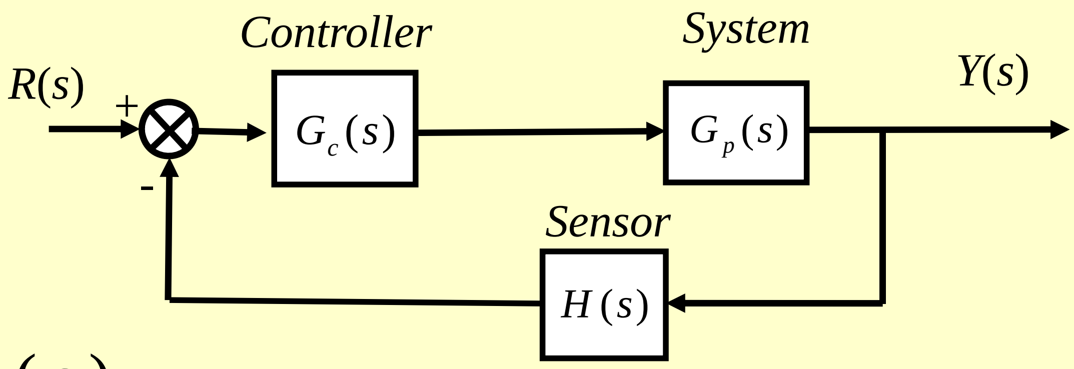
Q: Can you determine the sensitivity of $T(s)$ w.r.t. controller $G_c(s)$?

In general, $G_p(s)$ (or $H(s)$) may change due to variations of some system parameters within the system. If b is the variation parameter of interest, we then consider the sensitivity function:

$$S_b^T = \frac{\partial T(s)}{\partial b}\frac{b}{T(s)} = \frac{\partial T(s)}{\partial G_p(s)}\frac{\partial G_p(s)}{\partial b}\frac{b}{T(s)}$$

Example: A feedback system has proportional controller $G_c(s) = K_p$, and sensor transfer function $H(s) = H_k$, and the plant transfer function is

$$G_p(s) = \frac{K}{s+0.1}$$



The closed-loop transfer function is

$$T(s) = \frac{K_P G_p(s)}{1 + K_P G_p(s) H_k}$$

To examine the sensitivity of $T(s)$ w.r.t. K and H_k, we examine

$$S_K^T = \frac{\partial T(s)}{\partial G_p(s)} \frac{\partial G_p(s)}{\partial K} \frac{K}{T(s)}$$

$$= \frac{K_P}{(1+K_P G_p H_k)^2} \frac{G_p}{K} \frac{K}{K_P G_p/(1+K_P G_p H_k)} = \frac{1}{1+K_P G_p H_k}$$

$$S_H^T = \frac{\partial T(s)}{\partial H(s)} \frac{H(s)}{T(s)} = \frac{-(K_P G_p)^2}{(1+K_P G_p H_k)^2} \frac{H_k}{K_P G_p/(1+K_P G_p H_k)}$$

$$= -\frac{K_P G_p H_k}{1+K_P G_p H_k}$$

Suppose that the nominal values are $K = 5.0$ and $H_k = 0.05$, and we want to evaluate the sensitivity functions for $K_P = 1$ and $K_P = 10$.

From the sensitivity functions, we have

$$S_K^T(j\omega) = \frac{1}{1 + K_P\left[5/(j\omega + 0.1)\right](0.05)} = \frac{0.1 + j\omega}{0.1 + 0.25K_P + j\omega}$$

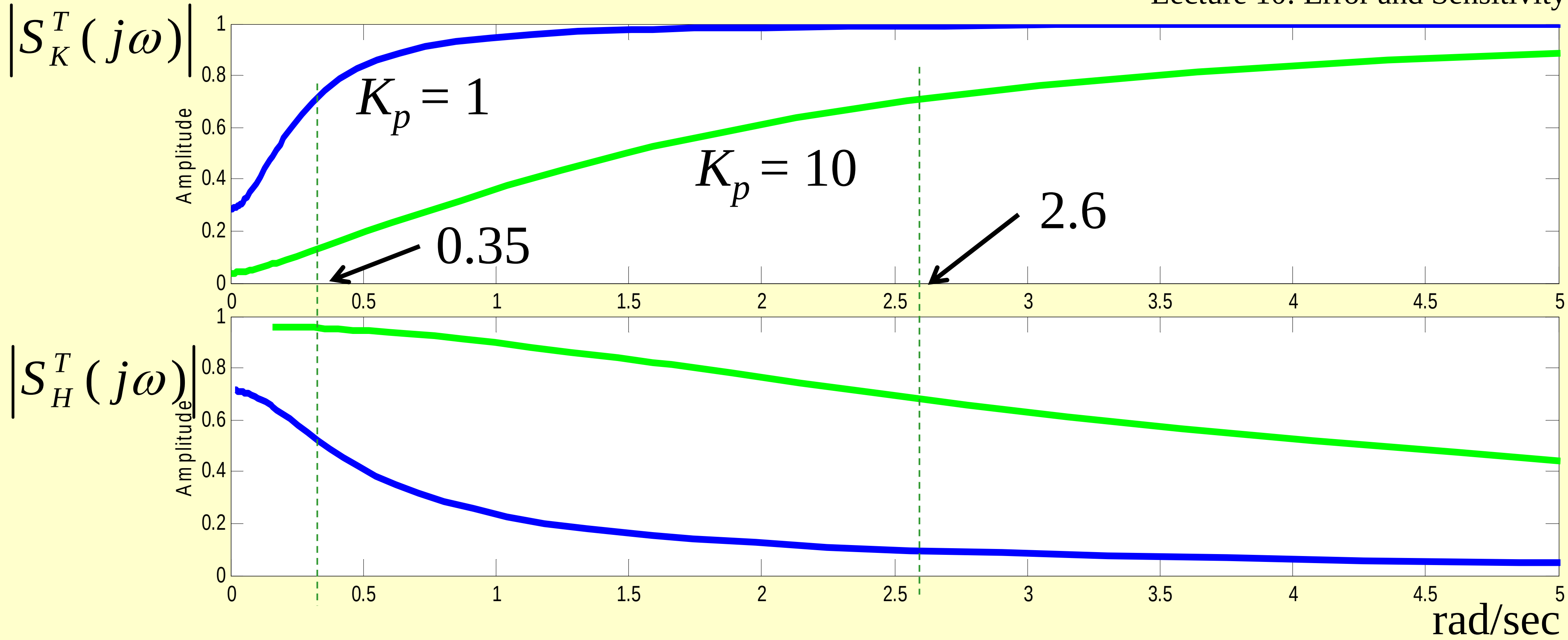
$$S_H^T(j\omega) = -\frac{K_P\left[5/(j\omega + 0.1)\right](0.05)}{1 + K_P\left[5/(j\omega + 0.1)\right](0.05)} = -\frac{0.25K_P}{0.1 + 0.25K_P + j\omega}$$

These are functions of K_P and frequencies. We need to examine these sensitivity functions within the bandwidth (BW) of the system.

From the closed-loop transfer function $T(s)$:

$$T(j\omega) = \frac{K_P G_p(j\omega)}{1 + K_P G_p(j\omega) H_k} = \frac{5K_P}{0.1 + 0.25K_P + j\omega}$$

it can be shown that (See Appendix 10.1) that for K_P = 1, BW = 0.35 rad/s, and for K_P = 10, BW = 2.60 rad/s.



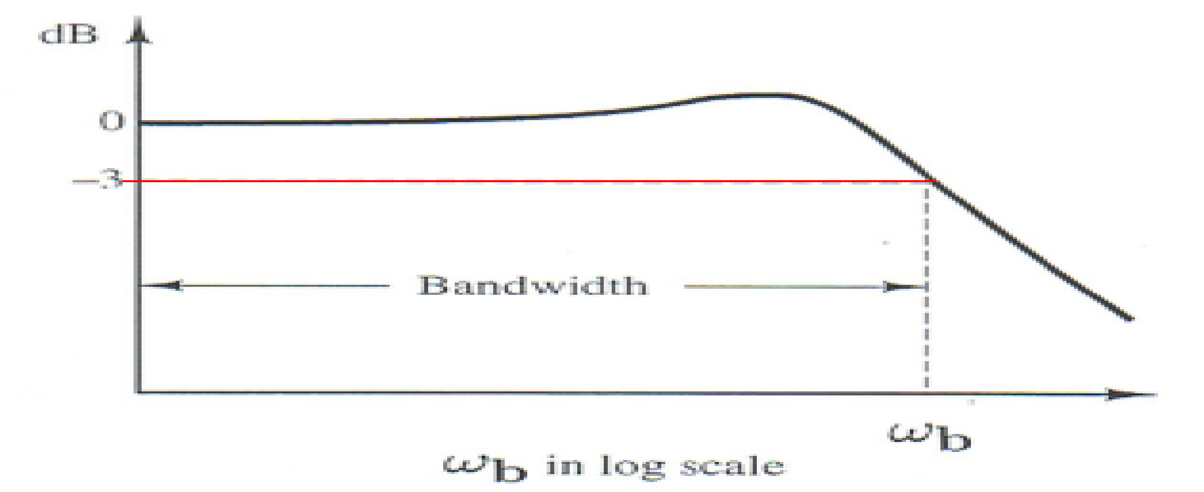
From the magnitude plots, we note that:

1. Sensitivity to K decreases with increasing loop gain (i.e. with larger K_p), while sensitivity to H increases with increasing loop gain;
2. System is very sensitive to K outside system bandwidth, which is not significant, and very sensitive to H inside system bandwidth, which is significant.

(See Appendix 10.3 for an example with PI controller)

Appendix 10.1: Bandwidth

- The bandwidth (BW) is the frequency ω_b at which the magnitude of transfer function $T(j\omega)$ drops 3 dB from 20log $|T(j0)|$. ω_b is also referred to as the cut off frequency.
- The closed-loop system filters out the signal components whose frequencies are greater than the cut off frequency.
- In general, the bandwidth of a system indicates its ability to reproduce the input signal. It gives a measure of the speed of response. The rise time and the bandwidth are inversely proportional to each other.
- A large bandwidth corresponds to a faster response. But it will not be good for noise (disturbance) attenuation (why?).



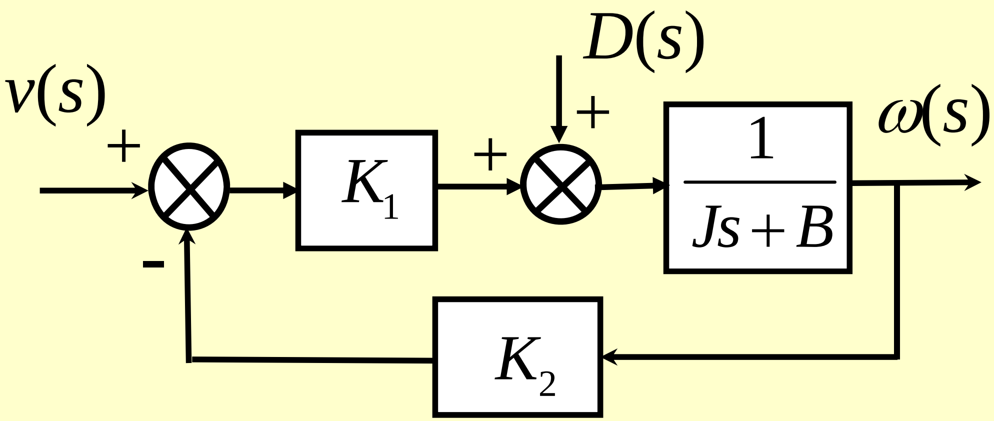
From $$T(j\omega) = \frac{K_P G_p(j\omega)}{1 + K_P G_p(j\omega) H_k} = \frac{5K_P}{0.1 + 0.25K_P + j\omega}$$

we note that the system gain is reduced to $0.707T(j0)$ (i.e. 3 dB down from $20\log|T(j0)|$) at the frequency $\omega = (0.1 + 0.25K_P)$, so the bandwidth of the system is BW = $(0.1 + 0.25K_P)$.

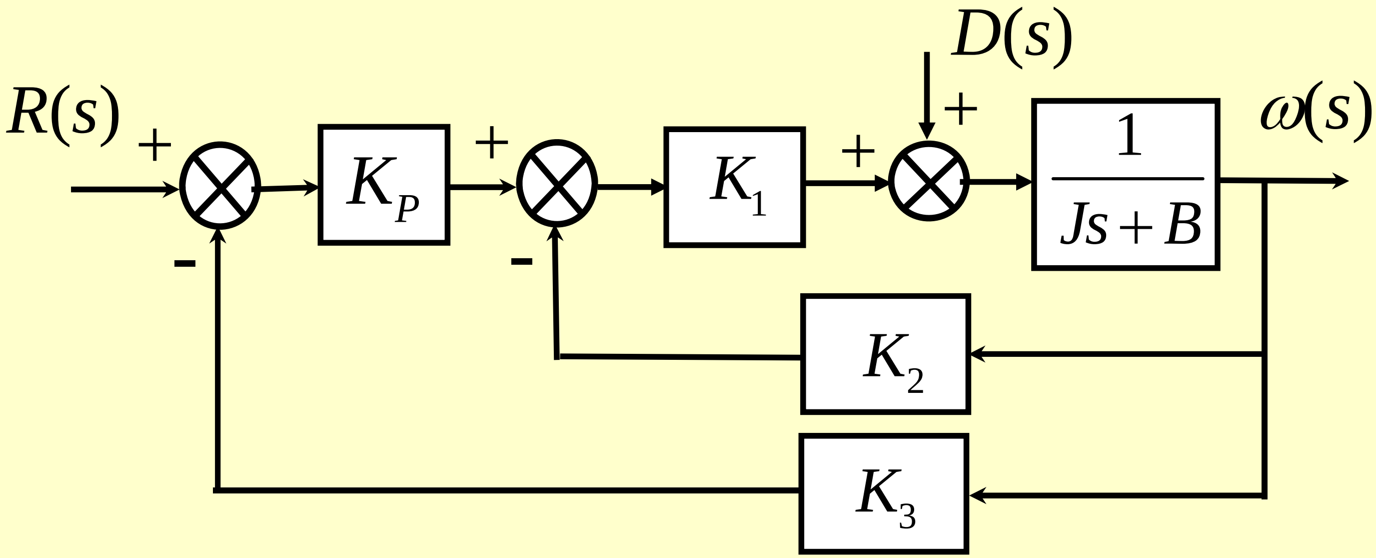
Thus, for $K_P = 1$, BW = 0.35, and for $K_P = 10$, BW = 2.60.

Appendix 10.2: Comparison of disturbance attenuation of open loop and closed-loop systems

<u>Example</u>: Consider the rolling mill speed control problem with disturbance torque $D(s)$, and where the open loop system is given below.



The closed-loop system with proportional control K_P is given below:



The respective speed errors due to step disturbance (obtained by setting $v(s) = 0$ and $R(s) = 0$ and computing the negative of the output) are:

$$E_{Dop}(s) = -\frac{1/(Js+B)}{1+K_1K_2/(Js+B)}D(s) = -\frac{1}{Js+B+K_1K_2}\frac{a}{s}; \qquad D(s) = \frac{a}{s}$$

$$E_{Dcl}(s) = -\frac{1/(Js+B)}{1+K_1(K_2+K_PK_3)/(Js+B)}D(s) = -\frac{1}{Js+B+K_1(K_2+K_PK_3)}\frac{a}{s}$$

By using a large enough proportional gain K_P , we can ensure that

$$\left|E_{Dcl}(j\omega)\right| \ll \left|E_{Dop}(j\omega)\right|$$

at least at low frequencies.

If we compare their steady-state values:

$$\left|e_{Dop}(\infty)\right| = \left|\lim_{s\to 0} s \frac{1}{Js + B + K_1 K_2} \frac{a}{s}\right| = \frac{a}{B + K_1 K_2}$$

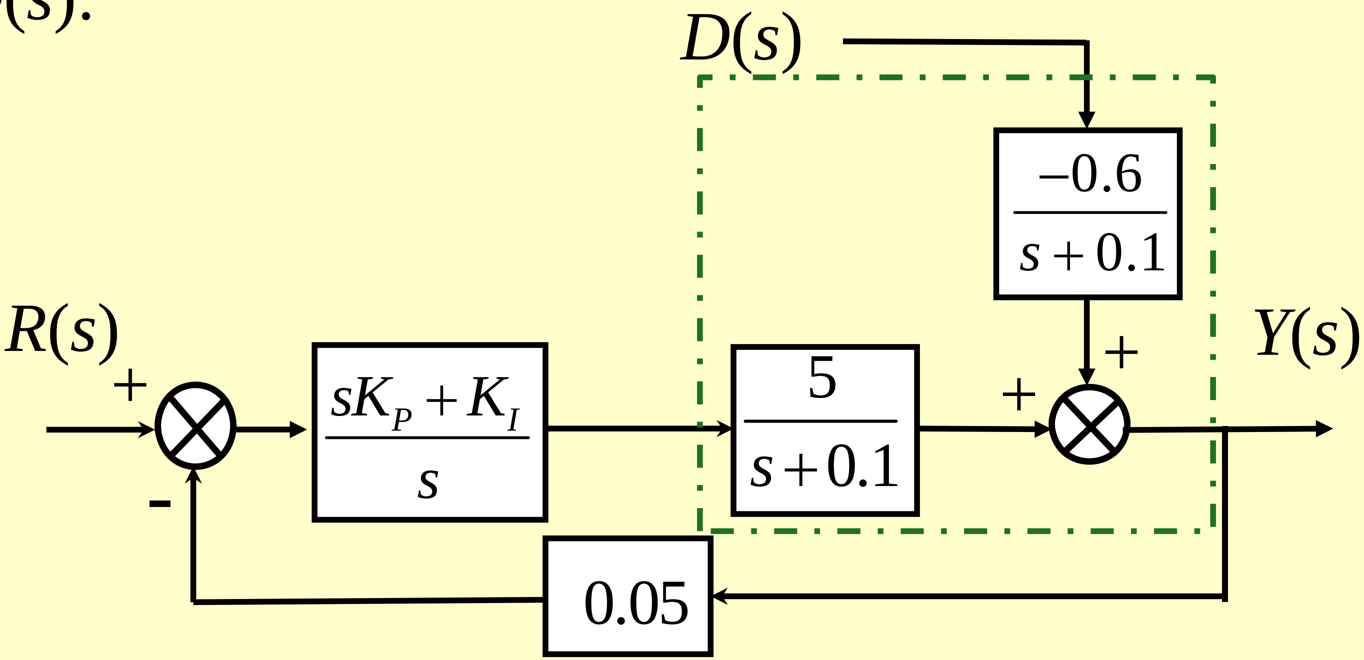
$$\left|e_{Dcl}(\infty)\right| = \left|\lim_{s\to 0} s \frac{1}{Js + B + K_1 (K_2 + K_P K_3)} \frac{a}{s}\right| = \frac{a}{B + K_1 (K_2 + K_P K_3)}$$

Thus, $$\frac{\left|e_{Dcl}(\infty)\right|}{\left|e_{Dop}(\infty)\right|} = \frac{B + K_1 K_2}{B + K_1 (K_2 + K_P K_3)}$$

$$\approx \frac{B + K_1 K_2}{K_1 K_P K_3} \quad \text{if} \quad K_P \quad \text{is sufficiently large}$$

Feedback control attenuates the effect of undesirable disturbance!

Example: Consider the following PI control problem with disturbance torque $D(s)$:



The transfer function from $D(s)$ to the output (with $R(s) = 0$) is given by

$$\frac{Y_D(s)}{D(s)} = \frac{\dfrac{-0.6}{(s+0.1)}}{1+\dfrac{sK_P+K_I}{s}\dfrac{5}{(s+0.1)}(0.05)} = \frac{-0.6s}{s(s+0.1)+0.25(sK_P+K_I)}$$

To determine the quality of disturbance rejection for a given PI controller, we can plot the function versus frequency:

$$\frac{Y_D(j\omega)}{D(j\omega)} = T_D(j\omega) = \frac{-0.6\,j\omega}{j\omega(j\omega + 0.1) + 0.25(j\omega K_P + K_I)}$$

Note however that if the disturbance is a step function, and the system is in steady-state, then the disturbance has no effect because $T_D(j0) = 0$!

This is the direct result of using the integrator which has infinite gain at dc.

However, there will be transient effect at the onset of disturbance, and the values of K_P and K_I will determine the characteristics of this transient effect.

Appendix 10.3:

Example: Similar to the previous example, but now a PI controller is

used, i.e. $G_c(s) = \dfrac{sK_P + K_I}{s}$. The rest remained the same i.e. $H(s) = H_k$, and

$G_p(s) = \dfrac{K}{s+0.1}$, and the nominal values are $K = 5.0$ and $H_k = 0.05$.

In this case, we have

$$S_K^T = \frac{G_c}{(1+G_cG_pH_k)^2}\frac{G_p}{K}\frac{K}{G_cG_p/(1+G_cG_pH_k)} = \frac{s(0.1+s)}{s(0.1+s)+0.25(sK_P+K_I)}$$

$$S_H^T = \frac{-(G_cG_p)^2}{(1+G_cG_pH_k)^2}\frac{H_k}{G_cG_p/(1+G_cG_pH_k)} = -\frac{0.25(sK_P+K_I)}{s(0.1+s)+0.25(sK_P+K_I)}$$

The values of these sensitivity functions are:

$$S_K^T(j\omega) = \frac{j\omega(0.1 + j\omega)}{j\omega(0.1 + j\omega) + 0.25(j\omega K_P + K_I)}$$

$$S_H^T(j\omega) = -\frac{0.25(j\omega K_P + K_I)}{j\omega(0.1 + j\omega) + 0.5(j\omega K_P + K_I)}$$

These are now more complicated, but you'll have a better appreciation on how to analyze them after you have learned Frequency Domain Analysis.

However, we note that $S_K^T(j0) = 0$. This means that if the input is constant and the system is in steady-state, then the system transfer function is not affected by changes in *K*, (provided not zero). This is one key reason why PI controller is popular.

Summary 10: Error and Sensitivity Analysis

Most control systems are subject to disturbance and measurement noise, thus the system error is given by

$$E(s) = \underbrace{\frac{1}{1+L(s)} \times R(s)}_{E_R} - \underbrace{\frac{G_p(s)}{1+L(s)} \times D(s)}_{E_D} + \underbrace{\frac{L(s)}{1+L(s)} \times N(s)}_{E_N}$$

where $L(s) = G_c(s)G_p(s)$ is the loop gain.

Sensitivity Function: $S(s) = \dfrac{1}{1+L(s)}$

Complementary Sensitivity Function: $C(s) = \dfrac{L(s)}{1+L(s)}$

$$\Rightarrow \quad S(s) + C(s) = 1$$

Disturbance Attenuation: Require small a $S(s)$ or a large $L(s)$

Measurement Noise Attenuation: Require small a $C(s)$ or a small $L(s)$

Sensitivity Function that characterizes the change in system transfer function $T(s)$ with respect to a parameter b:

$$S_b^T = \frac{\partial T(s)}{\partial b}\frac{b}{T(s)}$$

If b is the interested parameter in plant $G_p(s)$, then,

$$S_b^T = \frac{\partial T(s)}{\partial b}\frac{b}{T(s)} = \frac{\partial T(s)}{\partial G_p(s)}\frac{\partial G_p(s)}{\partial b}\frac{b}{T(s)}$$

Lecture 11: PID Controller Design - P and PD Controllers

a) The PID Controller Design

b) Proportional Controller

c) Proportional-plus Derivative Controller

a) The PID Controller Design

In order to meet performance specifications or to improve the system response, it may be necessary to modify the system by some types of controllers. One of the best-known controllers is the **Proportional-Integral-Derivative (PID)** controller of the form:

$$u(t) = K_P e(t) + K_P T_d \frac{de(t)}{dt} + \frac{K_P}{T_i}\int_0^t e(t)dt$$

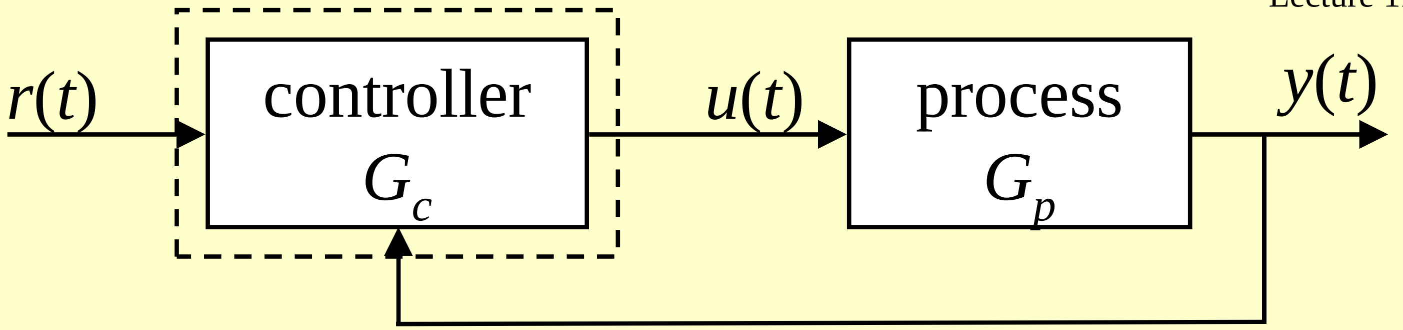
or $$\frac{U(s)}{E(s)} = G_c(s) = K_P + K_D s + \frac{K_I}{s}$$

where,

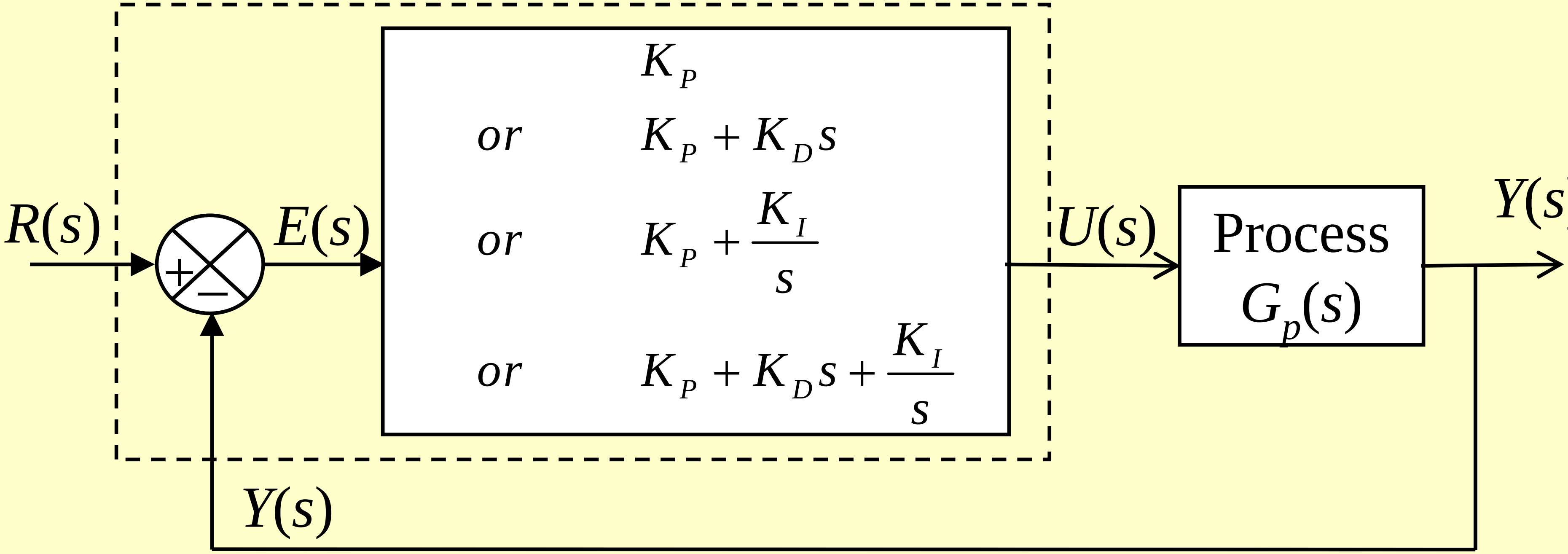
K_P = proportionality constant; $K_D = K_P T_d$ = derivative constant

$K_I = \dfrac{K_P}{T_i}$ = integral constant; T_d = derivative action time

T_i = integral action time



The controller configuration may be:



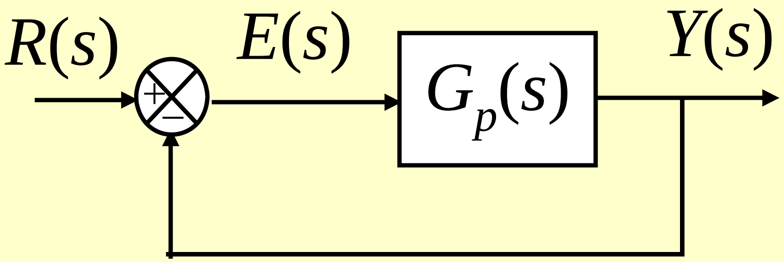
The design problem is to determine the values of K_P, K_D, K_I so that the performance of the system meets the specifications.

For the purpose of comparing the various control schemes, we let $G_p(s)$ be a 2nd-order system given by

$$G_p(s) = \frac{\omega_o^2}{s\left(s + 2\zeta_o\omega_o\right)}$$

This system models the position control problem of a DC motor. Clearly, with this system, the time domain response is not good. (*Show it!*)

If we simply close the loop, then we get

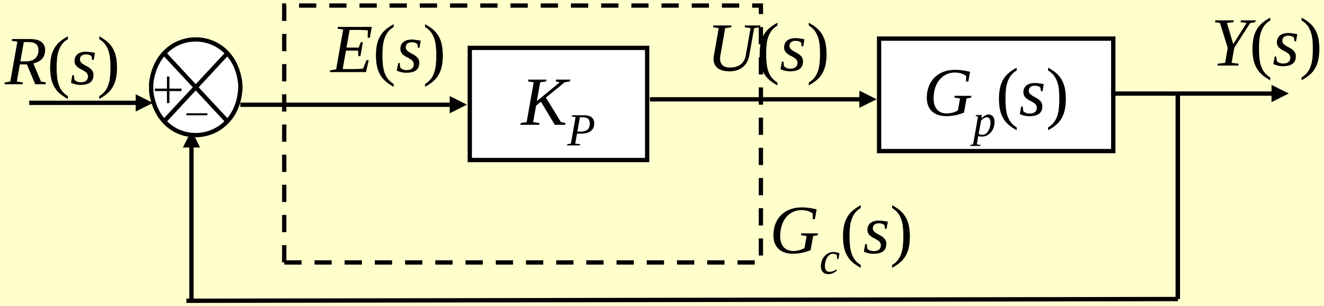


The closed-loop transfer function is a standard 2nd order system:

$$\frac{Y(s)}{R(s)} = \frac{\omega_o^2}{s^2 + 2\zeta_o\omega_o s + \omega_o^2}$$

The response will depend on ζ_o and ω_o!

Proportional Controller



$$\frac{U(s)}{E(s)} = G_c(s) = K_P; \qquad G_p(s) = \frac{\omega_o^2}{s\left(s + 2\zeta_o \omega_o\right)}$$

With the given $G_p(s)$, the above control system is a 2nd-order type-1 system. Thus we don't have to worry about the steady-state error for a step input.

The closed-loop transfer function is given by

$$\frac{Y(s)}{R(s)} = \frac{K_P \omega_o^2}{s^2 + 2\zeta_o \omega_o s + K_P \omega_o^2} \equiv \frac{\omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2}$$

Thus, with proportional control, $\omega_n = \omega_o\sqrt{K_P}$ is the un-damped natural frequency, while the damping ratio ζ is

$$\zeta = \frac{2\zeta_o\omega_o}{2\omega_n} = \frac{2\zeta_o\omega_o}{2\omega_o\sqrt{K_P}}$$

i.e. $$\zeta = \frac{\zeta_o}{\sqrt{K_P}} \qquad (1)$$

So, <u>with proportional control, the un-damped natural frequency is increased</u> while the <u>damping ratio is reduced</u>. With an increased natural frequency, the rise time will be reduced and hence a faster response but at the cost of higher overshoot.

Proportional control in this case also has the effect of reducing the steady-state error w.r.t. a ramp input:

$$e_{ss}(unit-ramp) = \lim_{s\to 0}\frac{1}{sK_pG_p(s)} = \frac{2\zeta_o}{K_p\omega_o}$$

<u>Example</u>: The OL transfer function of a simplified 2nd-order attitude control system with proportional control is given by

$$G(s) = G_p(s)G_c(s) = \frac{4500}{s(s+361.2)} \times K_P$$

Compare the performance of the CL system for $K_P = 10$ and $K_P = 50$.

The CLTF is $T(s) = \dfrac{G(s)}{1+G(s)} = \dfrac{4500K_P}{s^2 + 361.2s + 4500K_P} \equiv \dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

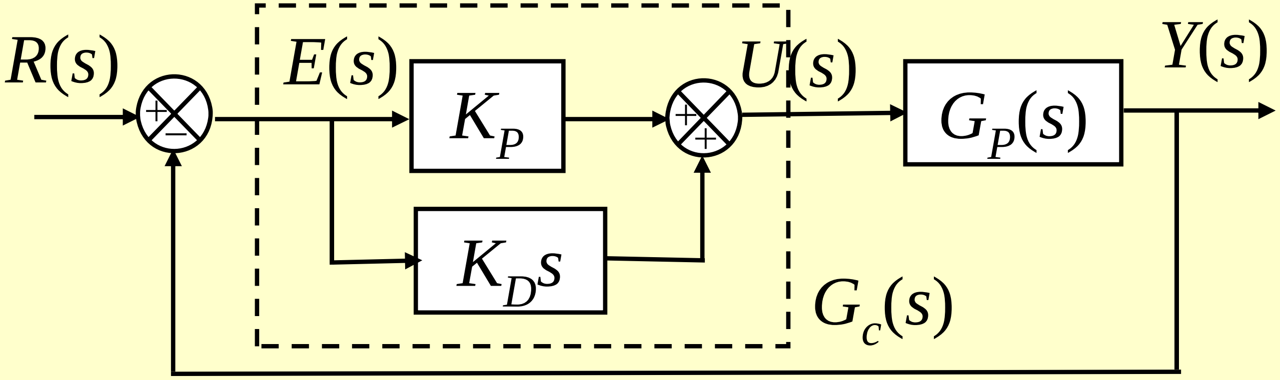
Equating coefficients, we have $\omega_n^2 = 4500K_P$ and $2\zeta\omega_n = 361.2$

i.e. $\omega_n = \sqrt{4500K_P}$ and $\zeta = \frac{361.2}{2\sqrt{4500K_P}}$

And $e_{ss}(unit-ramp) = \lim_{s\to 0} \dfrac{1}{sG(s)} = \dfrac{0.08}{K_P}$

K_P	ω_n	ζ	$e_{ss}(unit\text{-}ramp)$	t_r
10	212.14	0.851	0.008	0.0232
50	474.34	0.381	0.0016	0.0045

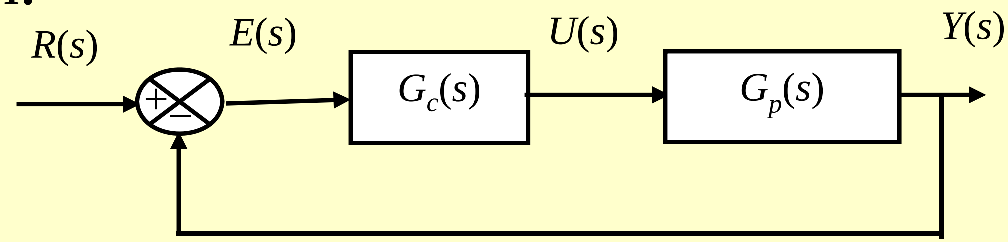
Proportional-plus-Derivative Controller

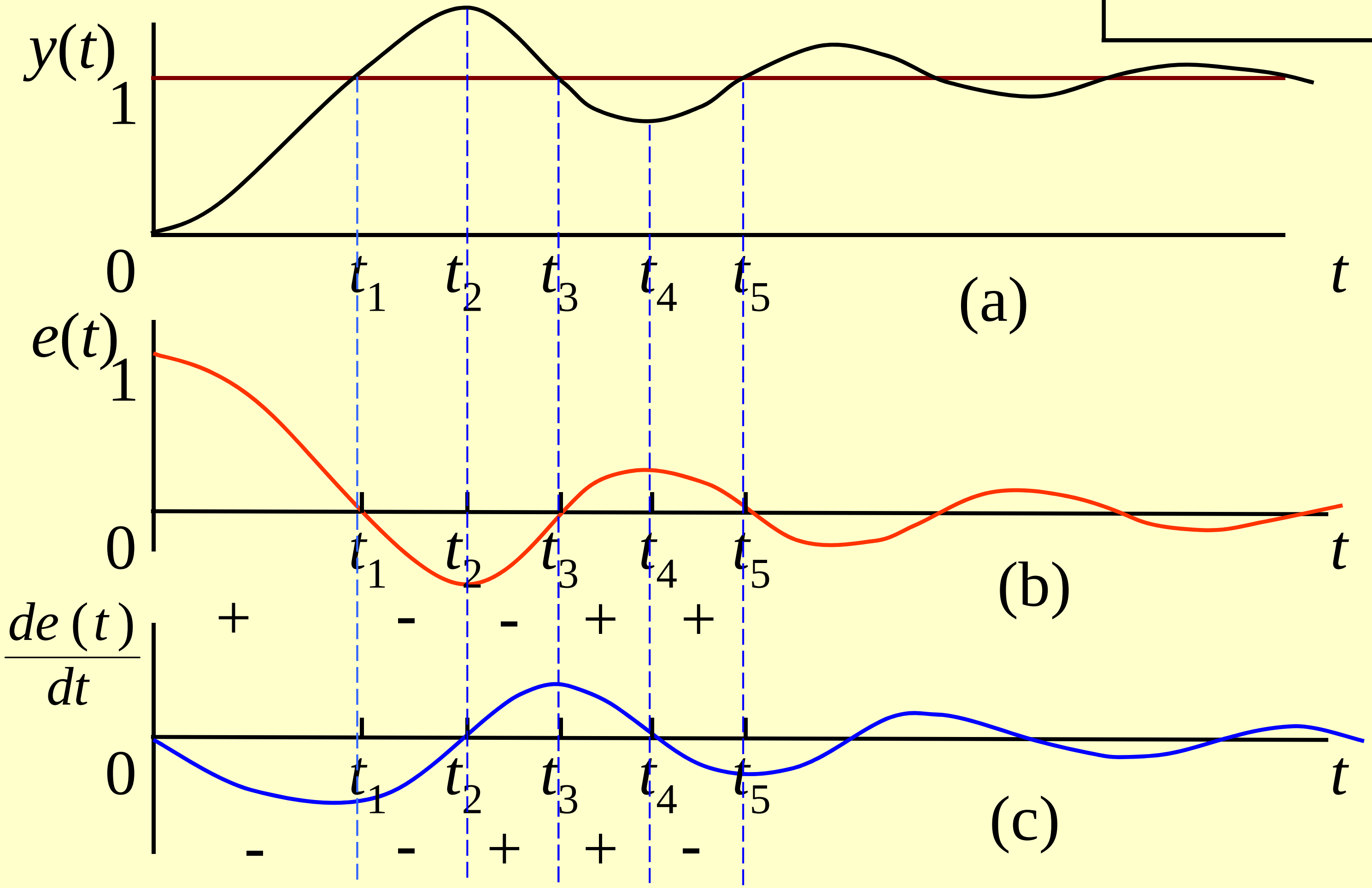


$$u(t) = K_P e(t) + K_D \frac{de(t)}{dt}; \qquad U(s) = (K_P + sK_D)E(s)$$

- This will result in small overshoots in the step response.
- Derivative control is anticipatory in that it predicts the large overshoot before it occurs and applies corrective effort.
- Since the overshoot can be reduced, the system is more stable. Derivative control is equivalent to increasing the damping ratio.
- However, it amplifies high frequency noise signals and may cause saturation. *Why?*

The following waveforms of the unit-step response $y(t)$, error signal $e(t)$, and derivative of the error signal $de(t)/dt$, can be used to explain the effect of derivative action.





$$u(t) = K_P e(t) + K_D \dot{e}(t)$$

For ease of illustration, we assume that the system is a servomechanism so that its torque is proportional the error $e(t)$ and that $y(t)$ is the unit-step response with only the P-control.

In interval $0 < t < t_1$, $e(t)$ is positive. The motor torque is positive and increasing. The overshoot and subsequent oscillations are due to excessive toque developed by the motor and the lack of damping.

In interval $t_1 < t < t_3$, $e(t)$ is negative. The motor torque is negative. This slows down the acceleration and eventually causes the direction of $y(t)$ to reverse and undershoot.

In interval $t_3 < t < t_5$, $e(t)$ is positive again. The motor torque is also positive and this reduces undershoot.

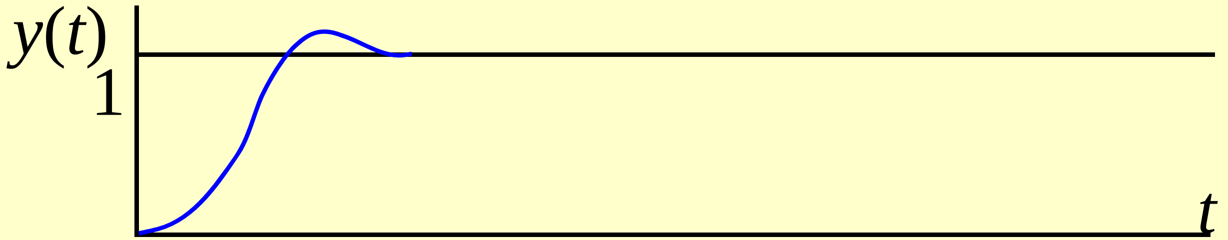
With PD control, we also make use of the information of $\frac{de(t)}{dt}$ for feedback.

For $0 < t < t_1$, $e(t)$ is positive but $\frac{de(t)}{dt}$ is negative. This will reduce the original torque developed due to $e(t)$ alone.

For $t_1 < t < t_2$, both $e(t)$ and $\frac{de(t)}{dt}$ are negative. This means that the negative retarding torque developed will be large than that due to *P*-control alone.

For $t_2 < t < t_3$, $e(t)$ and $\frac{de(t)}{dt}$ have opposite signs. Thus the negative torque that originally contributes to the undershoot is reduced.

With good PD control, we expect to get:



The transfer function of the PD-controller is

$$\frac{U(s)}{E(s)} = G_c(s) = K_P + sK_D$$

Consider the example where the process $G_p(s)$ is given below:

$$G_p(s) = \frac{\omega_o^2}{s\left(s+2\zeta_o\omega_o\right)}$$

With PD-control, the open-loop transfer function of the overall system becomes

$$\frac{Y(s)}{E(s)} = G(s) = G_p(s)G_c(s) = \frac{\omega_o^2}{s\left(s+2\zeta_o\omega_o\right)} \times \left(K_P + K_D s\right)$$

The CLTF is $$\frac{Y(s)}{R(s)}=\frac{G(s)}{1+G(s)}=\frac{\left(sK_D+K_P\right)\omega_o^2}{s^2+\left(2\zeta_o\omega_o+K_D\omega_o^2\right)s+K_P\omega_o^2}$$

$$\equiv\frac{(\mathrm{s}K_D+K_P)\omega_o^2}{s^2+2\zeta\,\omega_n s+\omega_n^2} \qquad (2)$$

Thus, equating coefficients, $\omega_n=\omega_o\sqrt{K_p}$, and

$$\zeta=\frac{2\zeta_o\omega_o+K_D\omega_o^2}{2\omega_n}=\frac{2\zeta_o\omega_o+K_D\omega_o^2}{2\omega_o\sqrt{K_P}}$$

i.e. $$\zeta=\frac{\zeta_o}{\sqrt{K_P}}+\frac{K_D\omega_o}{2\sqrt{K_P}} \qquad (3)$$

The damping ratio with derivative control has been increased by an amount proportional to the derivative constant K_D as compared with just the proportional control.

Example: The OL transfer function of a simplified 2nd-order attitude control system with PD control is given by

$$G(s) = G_p(s)G_c(s) = \frac{815265}{s(s+361.2)} \times (K_P + sK_D)$$

Design a PD-controller such that the following specifications are met:

$$e_{ss}(unit - ramp) \le 0.00045$$
$$\text{Maximum overshoot} \le 5\%$$

Solution: The system is type-1 and $K_{vel} = \lim_{s \to 0} sG(s) = \dfrac{815265K_P}{361.2}$, so the steady-state error due to unit-ramp is

$$e_{ss}(unit - ramp) = \frac{1}{K_{vel}} = \frac{361.2}{815265K_P} = \frac{0.000443}{K_P}$$

Hence we can set $K_P = 1$.

The closed-loop transfer function (assuming unity-feedback) is

$$\frac{\theta_y(s)}{\theta_r(s)} = \frac{815265(K_P + sK_D)}{s^2 + (361.2 + 815265K_D)s + 815265K_P} \equiv \frac{815265(K_P + sK_D)}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

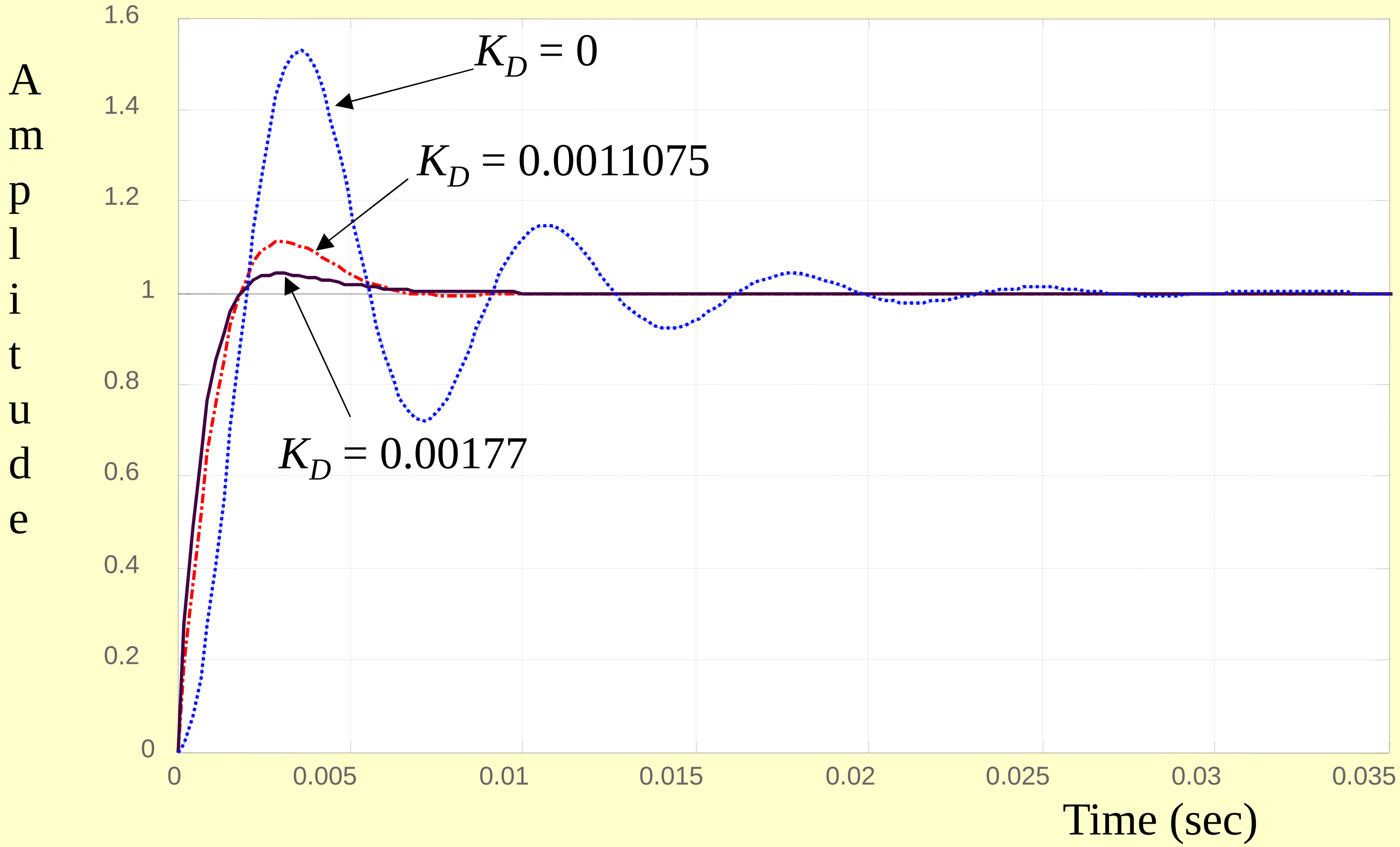
A non-standard 2nd-order system!

Then (by equating coefficients with K_P = 1), ω_n = 902.92, and the damping ratio of the system is

$$\zeta = \frac{361.2 + 815265K_D}{1805.84} = 0.2 + 451.46K_D$$

Note that we have a non-standard 2nd-order system due to the presence of a zero added by the PD control. This will give rise to additional overshoot as compared to a standard 2nd-order system. So, on the conservative side and because of tight requirement on overshoot, we set ζ = 1. Then K_D = 0.00177.

If one uses the result of standard 2nd-order system and set $\zeta = 0.7$ (which would yield a 5% overshoot for a standard 2nd-order system), then K_D = 0.0011075, and the actual overshoot will be higher.



Example: The OL transfer function of a PD-controlled aircraft attitude control system is given by

$$G(s) = \frac{2.718 \times 10^{9}}{s(s^{2} + 3408.3s + 1{,}204{,}000)} \times (K_P + K_D s)$$

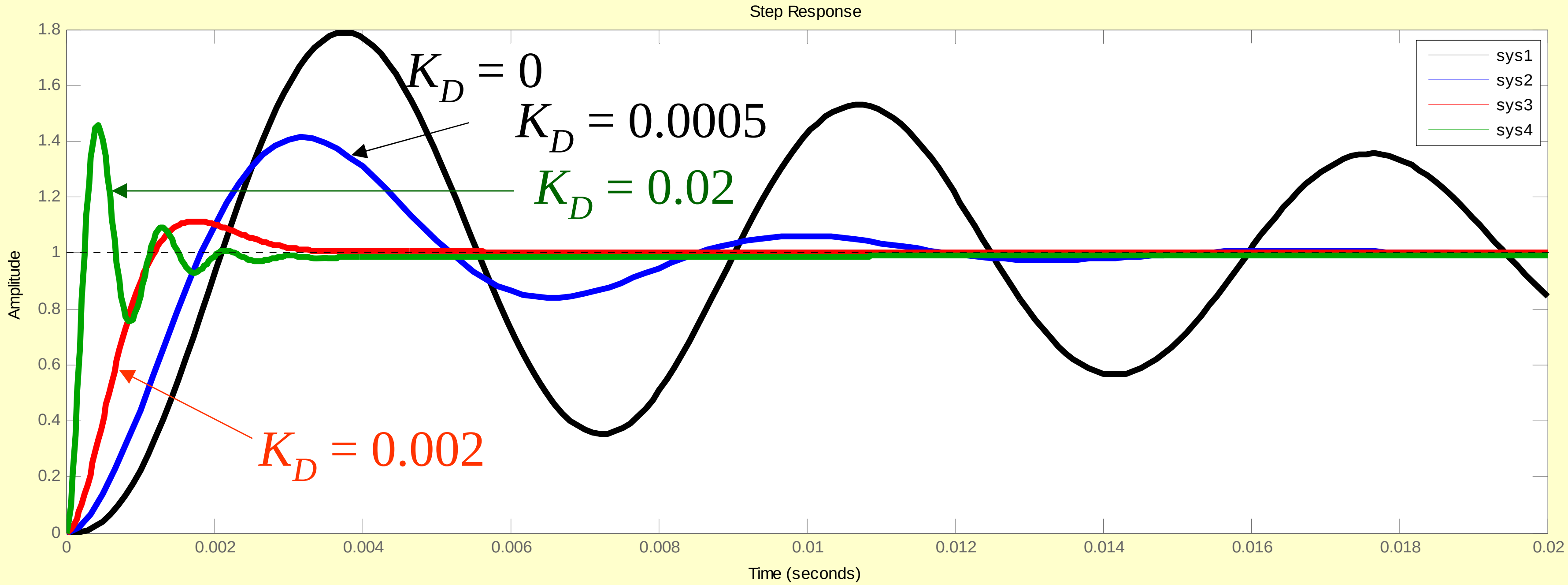
Design K_P such that $K_{vel} > 2.25\text{x}10^3$, and examine the step response for different values of K_D.

Solution: Since $K_{vel} = \lim_{s \to 0} sG(s) = 2.257 \times 10^{3} K_P$, we can set $K_P = 1$.

The closed-loop transfer function (assuming unity-feedback) is

$$\frac{\theta_y(s)}{\theta_r(s)} = \frac{G(s)}{1 + G(s)} = \frac{2.718 \times 10^{9}(sK_D + K_P)}{s(s^{2} + 3408.3s + 1{,}204{,}000) + 2.718 \times 10^{9}(sK_D + K_P)}$$

With $K_P = 1$, the unit-step response of the system with various K_D values is shown below. If K_D is set higher than 0.002, at $K_D = 0.02$ say, the system will have reduced damping. *Why?*



One may examine the poles and zero of the CL system to gain some understanding of the difference in the responses. See Appendix 11.1

For this 3rd-order system, the design procedure is to equate the closed-loop C.E. to a **desired C.E.**, i.e.

$$s^3 + 3408.3s^2 + 1{,}204{,}000s + 2.718\times10^9(sK_D + K_P)$$

$$\equiv (s^2 + 2\zeta\omega_n s + \omega_n^2)(s+p) = s^3 + (p + 2\zeta\omega_n)s^2 + (2p\zeta\omega_n + \omega_n^2)s + p\omega_n^2$$

The desired 3rd C.E. should satisfy certain specifications (in terms of some "specified" ζ, ω_n and/or p).

Then equating the coefficients:

$$3408.3 = p + 2\zeta\omega_n$$

$$1.204\times10^6 + 2.718\times10^9 K_D = 2p\zeta\omega_n + \omega_n^2$$

$$2.718\times10^9 K_P = p\omega_n^2$$

However in this case, we cannot specify ζ, ω_n and p independently because we only have 2 parameters of K_P and K_D for our design. Hence, this is a difficult constrained design problem. See Appendix 11.2 for a design example.

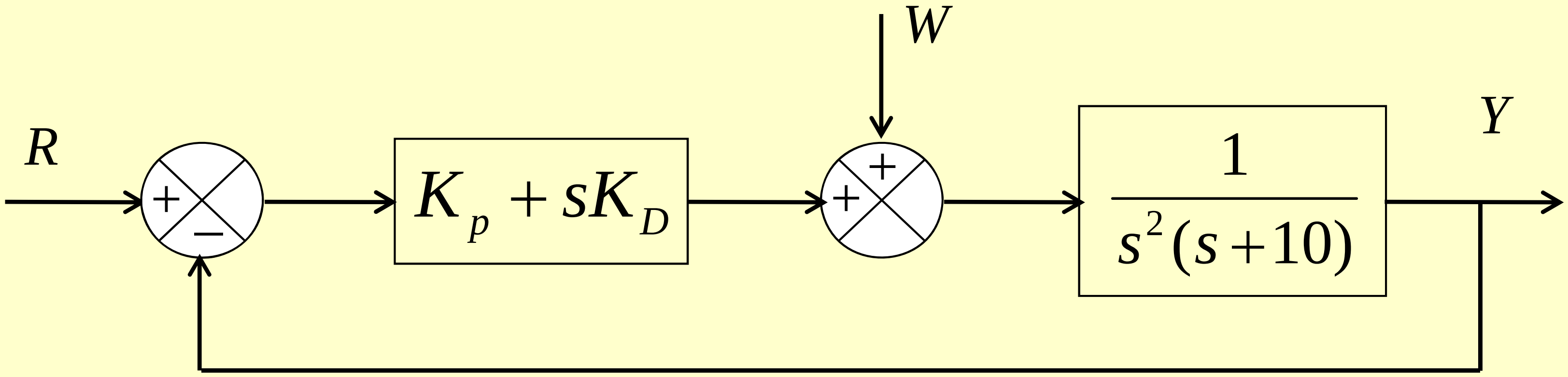
Appendix 11.1:

If one computes the poles and zero of the CL system, one will have a better idea about the relative influence of the poles and zero, and hence the type of responses:

K_D	0	0.0005	0.002	0.02
Closed-loop poles	$-57.5 \pm j906.6$; -3293.3	$-282.6 \pm j936.0$; -2843.1	$-1438.2 \pm j1744.0$; -531.9	$-1679.6 \pm j7251.1$; -49.1
Zero	-	-2000	-500	-50

Clearly, with $K_D = 0.02$, the zero at $s = -50$ almost cancels the pole at $s = -49.1$, so the response is essentially due to the 2 complex poles.

Appendix 11.2: Consider the system shown below.



Design the PD controller such that the closed-loop system satisfies the following specifications:

(a) The steady-state error with respect to a step disturbance $W(s)$ is no more than 10%.

(b) The third order system gives a dominant 2nd order response such that the third pole $s = -p$ satisfies $p = 10\zeta\omega_n$, where $\zeta\omega_n$ is the damping constant.

Note that we are dealing with a 3rd order system, but we have only 2 degrees of freedom from the PD-controller.

Consider the spec on disturbance attenuation:

$$\frac{Y_W}{W} = \frac{\dfrac{1}{s^2(s+10)}}{1+\dfrac{K_P + sK_D}{s^2(s+10)}} = \frac{1}{s^2(s+10)+K_P+sK_D}$$

$$Y_W = \frac{1}{s^2(s+10)+K_P+sK_D}\cdot\frac{1}{s} \quad ; \quad W(s)=\frac{1}{s}$$

We use W(s) = 1/s for simplicity, as the error specification is given in percentage.

$$y_w(\infty) = \lim_{s\to 0} sY_W = \frac{1}{K_P} \le 0.1$$

Set $K_P = 10$.

CLTF: $$\frac{Y_R}{R} = \frac{K_P + sK_D}{s^2(s+10) + K_P + sK_D} = \frac{sK_D + K_P}{s^3 + 10s^2 + sK_D + K_P}$$

We want

$$s^3 + 10s^2 + sK_D + K_P \equiv \left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)(s + p)$$

$$= s^3 + (p + 2\zeta\omega_n)s^2 + \left(\omega_n^2 + 2p\zeta\omega_n\right)s + p\omega_n^2$$

Equating coefficients and using $K_P = 10$:

$$10 = p + 2\zeta\,\omega_n \qquad \text{------ (1)}$$

$$K_D = \omega_n^2 + 2p\zeta\omega_n \qquad \text{------ (2)}$$

$$K_P = p\omega_n^2 = 10 \qquad \text{------ (3)}$$

With $p = 10\zeta\omega_n$, from (1) we get $\zeta\omega_n = \frac{10}{12}$

Then from (3), $\omega_n^2 = 1.2$

Substitute into (2), we get $K_D = 1.2 + 2 \times 10 \times \left(\frac{10}{12}\right)^2 = 15.09$

If specification (b) is replaced by wanting a critically damped response, i.e. $\zeta = 1.0$, we cannot fix p and we'll need to solve a more difficult problem:

With $\zeta = 1.0$:

From (1), $10 = p + 2\omega_n \quad \Rightarrow p = 10 - 2\omega_n$ ------ (4)

From (3) and (4), we have $\left(10 - 2\omega_n\right)\omega_n^2 = 10$, i.e.

$$\omega_n^3 - 5\omega_n^2 + 5 = 0 \qquad \text{------ (5)}$$

Solving (5) for a valid ω_n (must be positive). Then substitute into (2) to get K_D.

The roots of (5) are 4.7813, 1.1378 and ~~-0.9191~~ (why?).

If ω_n = 1.1378, then

$$K_D = (1.1378)^2 + 2(10 - 2 \times 1.1378)(1.1378) = 18.87$$

If ω_n = 4.7813, then

$$K_D = (4.7813)^2 + 2(10 - 2 \times 4.7813)(4.7813) = 27.04$$

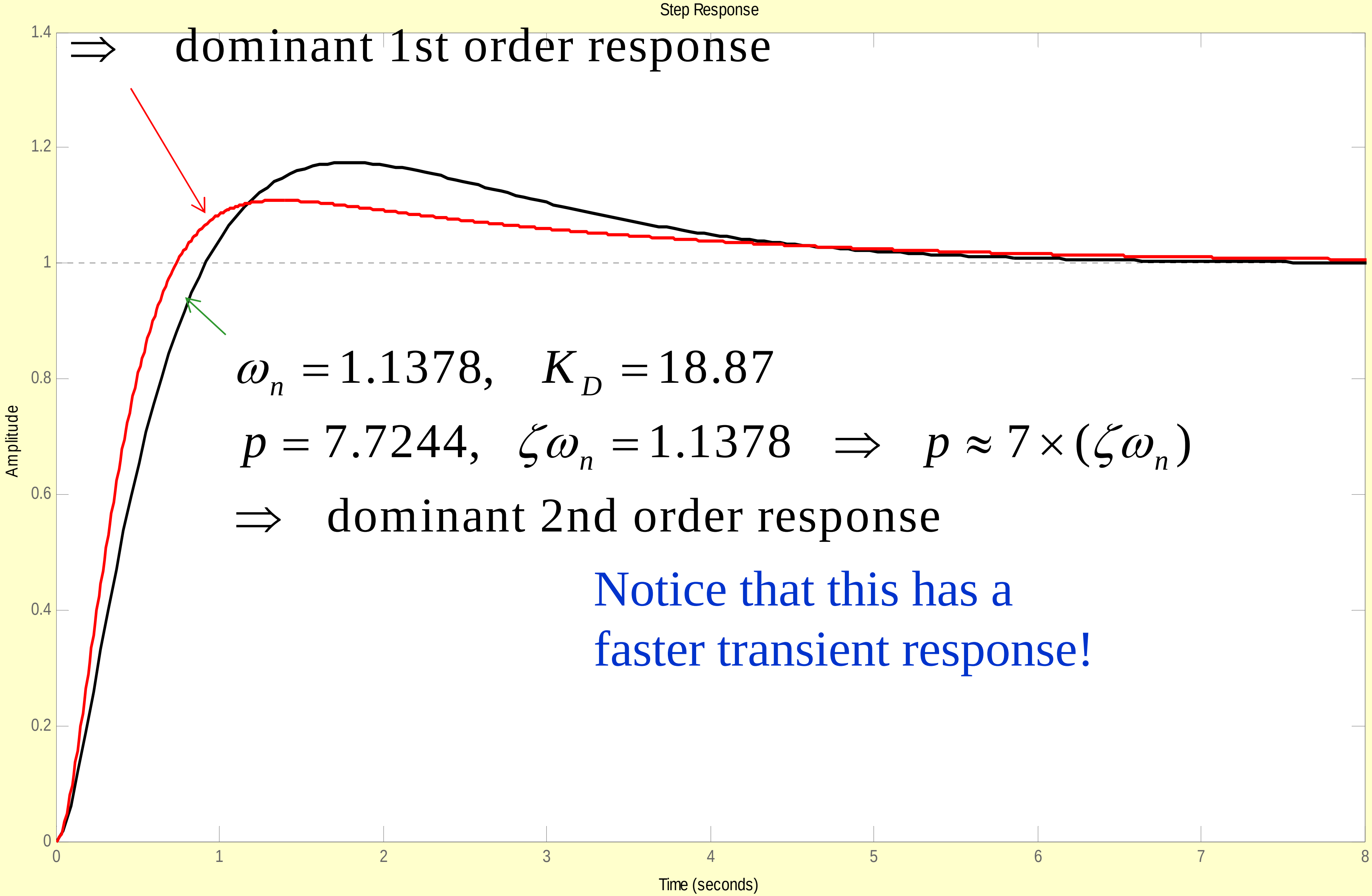
(Note that $K_P = 10$)

Discuss the merits of the 2 solutions!

$$\omega_n = 4.7813, \quad K_D = 27.04$$

$$p = 0.4374, \quad \zeta\omega_n = 4.7813 \quad \Rightarrow \quad \zeta\omega_n \approx 9 \times p$$

$$\Rightarrow \quad \text{dominant 1st order response}$$



$$\omega_n = 1.1378, \quad K_D = 18.87$$

$$p = 7.7244, \quad \zeta\omega_n = 1.1378 \quad \Rightarrow \quad p \approx 7 \times (\zeta\omega_n)$$

$$\Rightarrow \quad \text{dominant 2nd order response}$$

Notice that this has a
faster transient response!

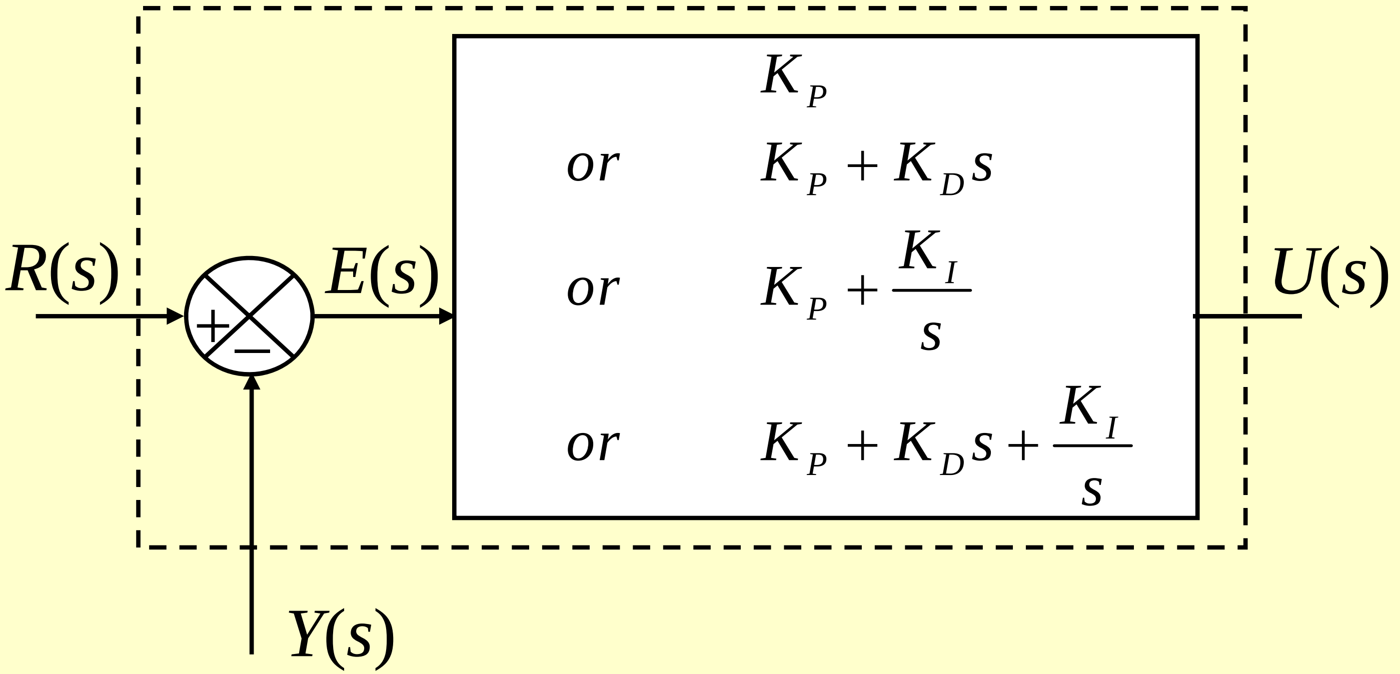
Summary 11: P and PD Controllers

The Proportional-Integral-Derivative (PID) controller of the form:

$$\frac{U(s)}{E(s)} = G_c(s) = K_P + K_D s + \frac{K_I}{s}$$

where K_P = proportionality constant
K_D = derivative constant
K_I = integral constant

The controller configuration may be:



P Controller (for $G_p(s) = \dfrac{\omega_o^2}{s(s+2\xi_o\omega_o)}$):

$G_c(s) = K_P$

- Increases the undamped natural frequency and hence reduces the rise time.
- Reduction in damping ratio, giving higher overshoot.
- Can reduce steady-state error.

PD Controller (for $G_p(s) = \dfrac{\omega_o^2}{s(s+2\xi_o\omega_o)}$):

$G_c(s) = K_P + sK_D$

- Derivative control increases the damping ratio and hence reduces overshoot.
- The presence of zero can add additional overshoot!
- It can amplify high frequency noise!

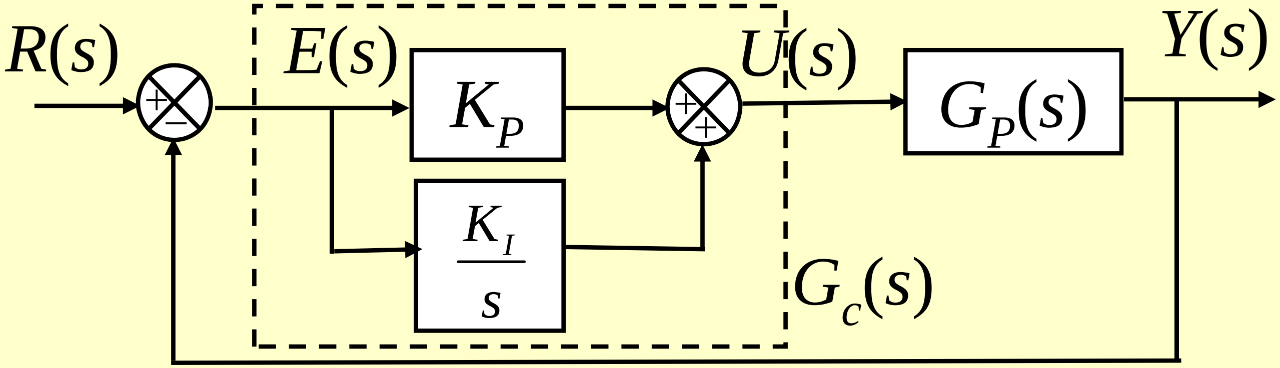
Lecture 12: PID Controller Deign - PI and PID Controllers

a) Proportional-plus-Integral Controller

b) PID Controller Design

c) Rate Feedback Control

a) Proportional-plus-Integral Controller



Integral control is often used to eliminate the constant steady-state error to meet high accuracy requirements. The controller is

$$u(t) = K_P e(t) + K_I \int_0^t e(\tau)\,d\tau; \quad U(s) = K_P E(s) + \frac{K_I}{s} E(s)$$

The transfer function of the PI-controller is

$$G_c(s) = \frac{U(s)}{E(s)} = K_P + \frac{K_I}{s} = \frac{sK_P + K_I}{s}$$

Given again that $G_P(s)$ is a 2nd-order system given by

$$G_P(s) = \frac{\omega_o^2}{s(s+2\zeta_o\omega_o)}$$

With PI control, the open-loop transfer function becomes

$$G(s) = G_p(s)G_c(s) = \frac{\omega_o^2}{s\left(s+2\zeta_o\omega_o\right)} \times \frac{\left(sK_P + K_I\right)}{s}$$

The CLTF is

$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

$$= \frac{\left(sK_P + K_I\right)\omega_o^2}{s^3 + 2\zeta_o\omega_o s^2 + K_P\omega_o^2 s + K_I\omega_0^2} \qquad (1)$$

- PI controller increases the system type by one!

- PI controller **increases the system order by one**. Since the original 2nd-order system now becomes a 3rd-order system, it may be less stable or becomes unstable if K_P and K_I are not properly chosen.

From (1), it can be shown (e.g. by Routh-Hurwitz array) that C.E. of eqn (1) will have 2 roots in the right-half s-plane if

$$K_I > 2\zeta_o \omega_o K_P$$

i.e. we would have an unstable system.

In general, integral control tends to de-stabilize the system. But we may need the integral action to eliminate steady-state error. Thus a proper design is necessary.

With PI-controller: $G(s) = \dfrac{\omega_o^2}{s\left(s+2\zeta_o\omega_o\right)} \times \dfrac{\left(K_P s + K_I\right)}{s}$

Then the velocity error constant is $K_{vel} = \lim_{s \to 0} sG(s) \to \infty$

Hence for a unit-ramp input, $e_{ss}(unit-ramp) = \lim_{s \to 0} sE(s) = \dfrac{1}{K_{vel}} = 0$

(Or we can just state that it's a type-2 system, so $e_{ss}(unit-ramp) = 0\,!$)

With just the P-controller (i.e. $K_I = 0$): $G'(s) = G_p(s)G_c(s) = \dfrac{K_P\omega_o^2}{s\left(s+2\zeta_o\omega_o\right)}$

Then $K_{vel} = \lim_{s \to 0} sG'(s) = \dfrac{K_P\omega_o}{2\zeta_o}$

Hence, $e_{ss}(unit-ramp) = \dfrac{1}{K_{vel}} = \dfrac{2\zeta_o}{K_P\omega_o}$

(We have a type-1 system, so steady-state error w.r.t. ramp input is not zero)

<u>Example</u>: The OLTF of a PI-controlled attitude control system is given by

$$G(s) = G_p(s)G_c(s) = \frac{815,265}{s(s+361.2)} \times \frac{sK_P + K_I}{s}$$

Clearly, $K_{vel} = \lim_{s \to 0} sG(s) = \infty$ if $K_I \neq 0$. Thus, the closed-loop system will have a zero steady-state error w.r.t ramp input. (Or one can simply state that the system is now type-2 and hence the steady state error w.r.t ramp input is zero.)

The closed-loop transfer function is

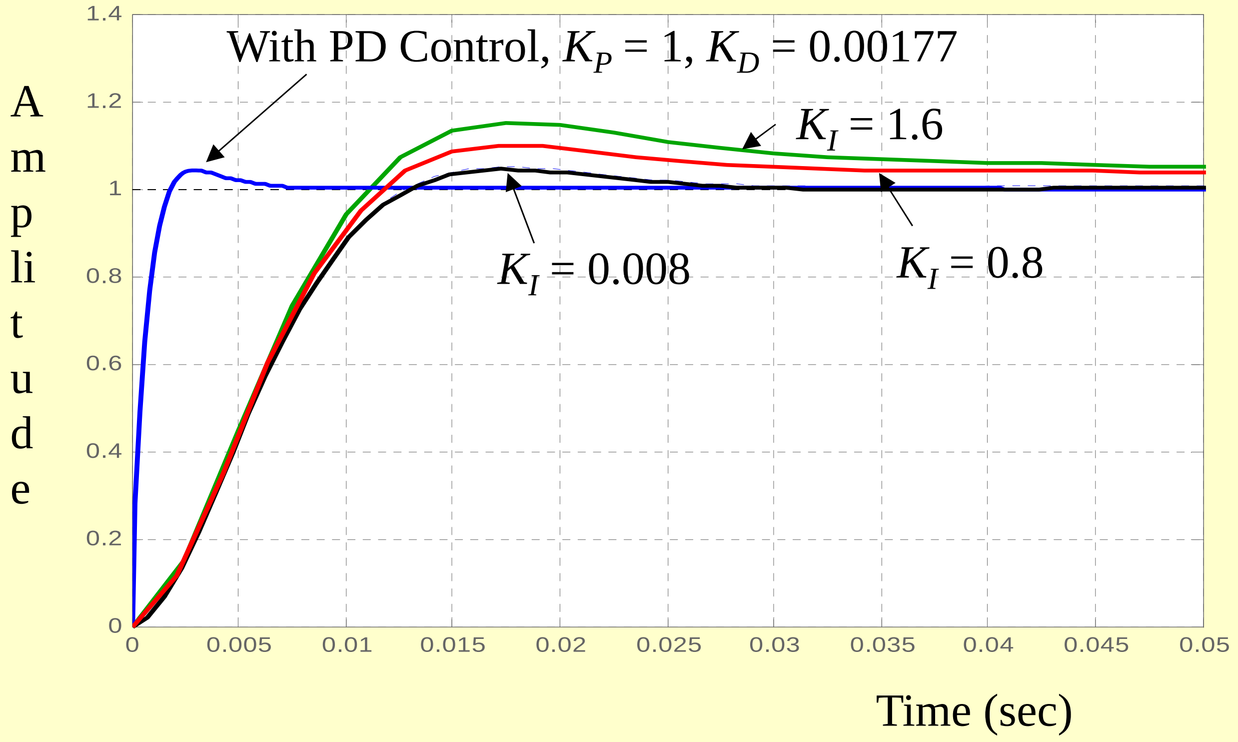
$$T(s) = \frac{G(s)}{1+G(s)}$$

$$= \frac{815265(K_P s + K_I)}{s^2(s+361.2) + 815265(K_P s + K_I)}$$

With K_P = 0.08, the unit-step response of the system for various value of K_I is shown in the figure below.

The same system with PD control has OL and CL transfer functions:

$$G'(s) = \frac{815,265(K_P + sK_D)}{s(s+361.2)}; \quad T(s) = \frac{815,265(K_P + sK_D)}{s(s+361.2) + 815,265(K_P + sK_D)}$$



PD has better step response. If the input is unit-ramp, then e_{ss} = 0 with PI control, but it is not zero with PD control. *Can you show it ?*

b) PID Controller Design

If $G_P(s)=\dfrac{\omega_o^2}{s(s+2\zeta_o\omega_o)}$, with PID-controller, the OLTF is

$$G(s)=\frac{\omega_o^2}{s\left(s+2\zeta_o\omega_o\right)}\times\frac{K_D s^2+K_P s+K_I}{s}$$

The closed-loop C.E. is $q(s)=s^2\left(s+2_o\zeta_o\omega_o\right)+\omega_o^2(K_D s^2+K_P s+K_I)$

We equate it to a "desired" C.E., i.e.

$$s^3+\left(2_o\zeta_o\omega_o+\omega_o^2K_D\right)s^2+\omega_o^2K_P s+\omega_o^2K_I$$

$$\equiv(s^2+2\zeta\omega_n s+\omega_\mathrm{n}^2)(s+p)$$

$$=s^3+(p+2\zeta\omega_n)s^2+(2p\zeta\omega_n+\omega_\mathrm{n}^2)s+p\omega_\mathrm{n}^2$$

The values of ζ,ω_n and p are "specified" to represent a desired 3rd order system.

Equating coefficients: $\omega_o^2K_D+2\zeta_o\omega_o=p+2\zeta\omega_n$

$$\omega_o^2K_P=2p\zeta\omega_n+\omega_n^2$$

$$\omega_o^2K_I=p\omega_n^2$$

Clearly, with PID control, we now have 3 parameters of K_P, K_I and K_D (i.e. 3 degrees of freedom) for the design, so we can use them to meet 3 independent specifications of ζ, ω_n and p, which may be specified directly or indirectly, i.e. in terms of overshoot, rise time, specific pole locations (typically dominant pole locations), etc.

If there is no other additional specifications, the PID controller design for the 2^{nd} order system seems easy. This is also true for PID controller design for type-0 and type-2 2^{nd} order systems. See Appendix 12.1.

One should note that the exact effects of the P- and D-actions on ζ and ω_n will depend on the system concerned. However, the effect of I-action is more specific. (*What's it?*)

On the choice of p, we normally want to have a dominant 2nd order response, even if not explicitly stated, so that the 3rd order system's response can be approximated by that of a 2nd order system. This is achieved by setting $p = a \times \zeta\omega_n$, with a = 10 or a suitably large value (*why?*), if we have enough degrees of freedom to impose this specification.

In most applications, in addition to specifications on ζ, ω_n and/or p, we do need to meet steady-state error and/or disturbance attenuation specifications, and these may impose specific values on K_P and/or K_I. Then, we can no longer have 3 independent specifications of ζ, ω_n and p.

Example: A control system with PID controller is shown in the figure below. Design the parameters of the PID controller so that the following specifications are satisfied:

- e_{ss} **(unit-step) = 0**
- e_{ss} **(unit-ramp)** $\leq$ **0.1**
- **Max overshoot,** $M_P \leq$ **1.5% (for step input)**

$$R(s) \xrightarrow{+} \bigcirc \xrightarrow{E(s)} \boxed{\frac{s^2K_D + sK_P + K_I}{s}} \longrightarrow \boxed{\frac{10}{s^2+10s+10}} \longrightarrow Y(s) \quad (\text{unity negative feedback, } -)$$

Solution: First, we take care of the steady-state error w.r.t. unit-step. Note that we have a unity-feedback system and a type-1 system if $K_I \neq 0$.
So, e_{ss} (unit-step) = 0!

To determine e_{ss}(unit-ramp), we have $K_{vel} = \lim_{s\to 0} sG(s) = K_I$, so

$$e_{ss}(\text{unit-ramp}) = \frac{1}{K_{vel}} = \frac{1}{K_I} \leq 0.1$$

We set K_I = 10. (So, K_I is fixed!) (E1)

The closed-loop transfer function is

$$\frac{Y}{R} = \frac{10(s^2 K_D + sK_P + K_I)}{s(s^2 + 10s + 10) + 10(s^2 K_D + sK_P + K_I)}$$

It's a 3rd-order system and the response is more complicated than standard 2nd-order system. Since the overshoot requirement is very tight, we opt to be more conservative and set the damping ratio ζ to 1.

(You can also equate 1.5% to a damping ratio $\zeta = 0.8$, for standard 2nd order system, and proceed from there.)

We equate the closed-loop C.E. to a "desired" 3rd order system, i.e.

$$s(s^2 + 10s + 10) + 10(s^2 K_D + sK_P + K_I)$$

$$\equiv (s^2 + 2\zeta\omega_n s + \omega_n^2)(s + p)$$

Expanding the characteristic equations,

$$s^3 + (10 + 10K_D)s^2 + (10 + 10K_P)s + 10K_I$$

$$\equiv s^3 + (2\zeta\omega_n + p)s^2 + (\omega_n^2 + 2p\zeta\omega_n)s + p\omega_n^2$$

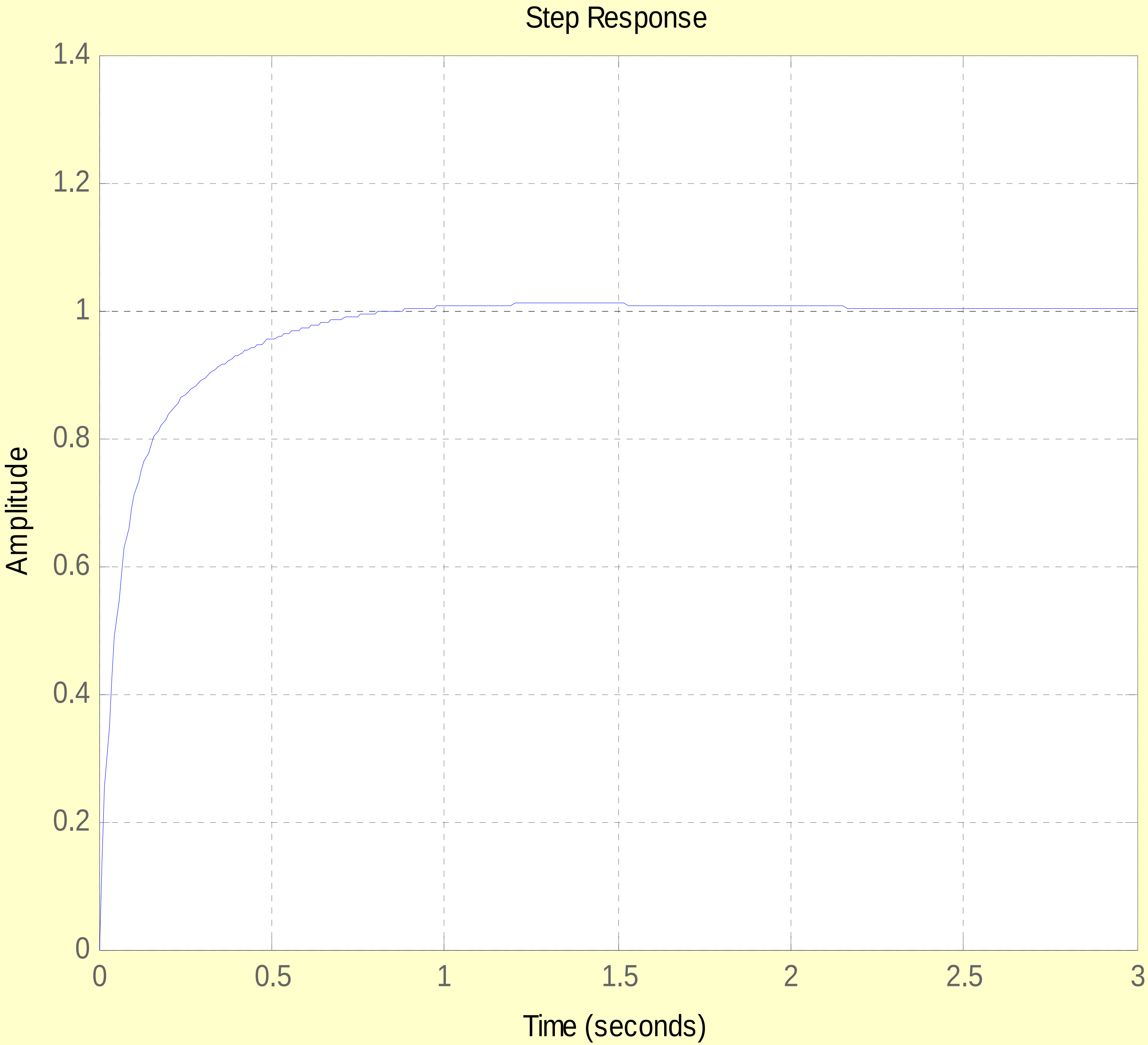
We usually want our system to behave like a dominant 2nd-order system, i.e. $p \geq 10\zeta\omega_n$ so we set $p = 10\zeta\omega_n$. We equate the coefficients, with $\zeta = 1.0$ and $p = 10\omega_n$, we get

$$10 + 10K_D = 2\zeta\omega_n + p = 12\omega_n \qquad \text{(E2)}$$

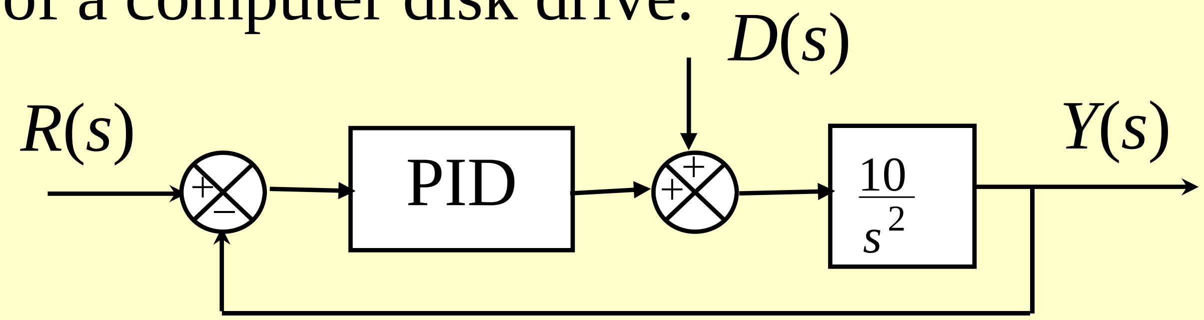
$$10 + 10K_P = \omega_n^2 + 2p\zeta\omega_n = 21\omega_n^2 \qquad \text{(E3)}$$

$$10K_I = p\omega_n^2 = 10\omega_n^3 \qquad \text{(E4)}$$

With K_I = 10 (from (E1) to meet the steady state-error requirement), ω_n is obtained from (E4) as ω_n = 2.154, (i.e. ω_n cannot be set independently to other value) and we solve (E2)-(E3) to get the required values for K_P = 8.743 and K_D = 1.585 (and K_I = 10).

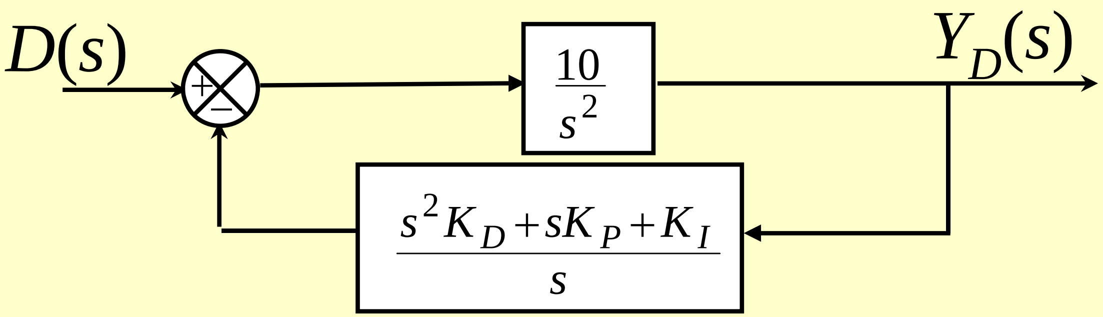

Step Response
1.4
1.2
1
0.8
0.6
0.4
0.2
0
Amplitude
0
0.5
1
1.5
2
2.5
3
Time (seconds)

Example: The following block diagram represents simplified tracking control of a computer disk drive.



Design a controller that is able to provide a **damping ratio of no less than 0.8** for the closed-loop system and at the same time **able to reject step disturbance**.

Solution: First, we take care of disturbance rejection. Let $Y_D(s)$ denote the output due to the disturbance. Thus we have



where $\text{PID} \equiv \dfrac{s^2K_D + sK_P + K_I}{s}$.

The output due to disturbance is given by

$$Y_D(s) = \frac{10s}{s^3 + 10(s^2 K_D + sK_P + K_I)} \bullet \frac{a}{s}$$

$$\therefore \quad y_{Dss} = \lim_{s \to 0} sY_D(s) = 0 \quad \text{if} \quad K_I \neq 0$$

That is, if integral action is used, we will reject the disturbance (at steady-state).

Next, we take care of the damping. The closed-loop transfer function is

$$\frac{Y}{R} = \frac{10(s^2 K_D + sK_P + K_I)}{s^3 + 10(s^2 K_D + sK_P + K_I)}$$

We want our system to behave like a desired 3rd order system with a dominant 2nd-order response, i.e.

$$s^3 + 10(s^2 K_D + sK_P + K_I)$$

$$\equiv (s^2 + 2\zeta\omega_n s + \omega_n^2)(s + p)$$

with $p \geq 10\zeta\omega_n$.

Set $p = 10\zeta\omega_n$, then

$$s^3 + 10K_D s^2 + 10K_P s + 10K_I$$

$$\equiv (s^2 + 2\zeta\omega_n s + \omega_n^2)(s + 10\zeta\omega_n)$$

$$= s^3 + 12\zeta\omega_n s^2 + (\omega_n^2 + 20\zeta^2\omega_n^2)s + 10\zeta\omega_n^3$$

Equating coefficients with $\zeta = 0.8$:

$$10K_D = 12 \times 0.8 \times \omega_n$$

$$10K_P = \omega_n^2 + 20 \times 0.8^2 \times \omega_n^2$$

$$10K_I = 10 \times 0.8 \times \omega_n^3$$

With no additional condition imposed on ω_n, we have many possible solutions! That is, we have one more degree-of-freedom and it can be used to specify the rise time (which would then "specify" a certain ω_n).

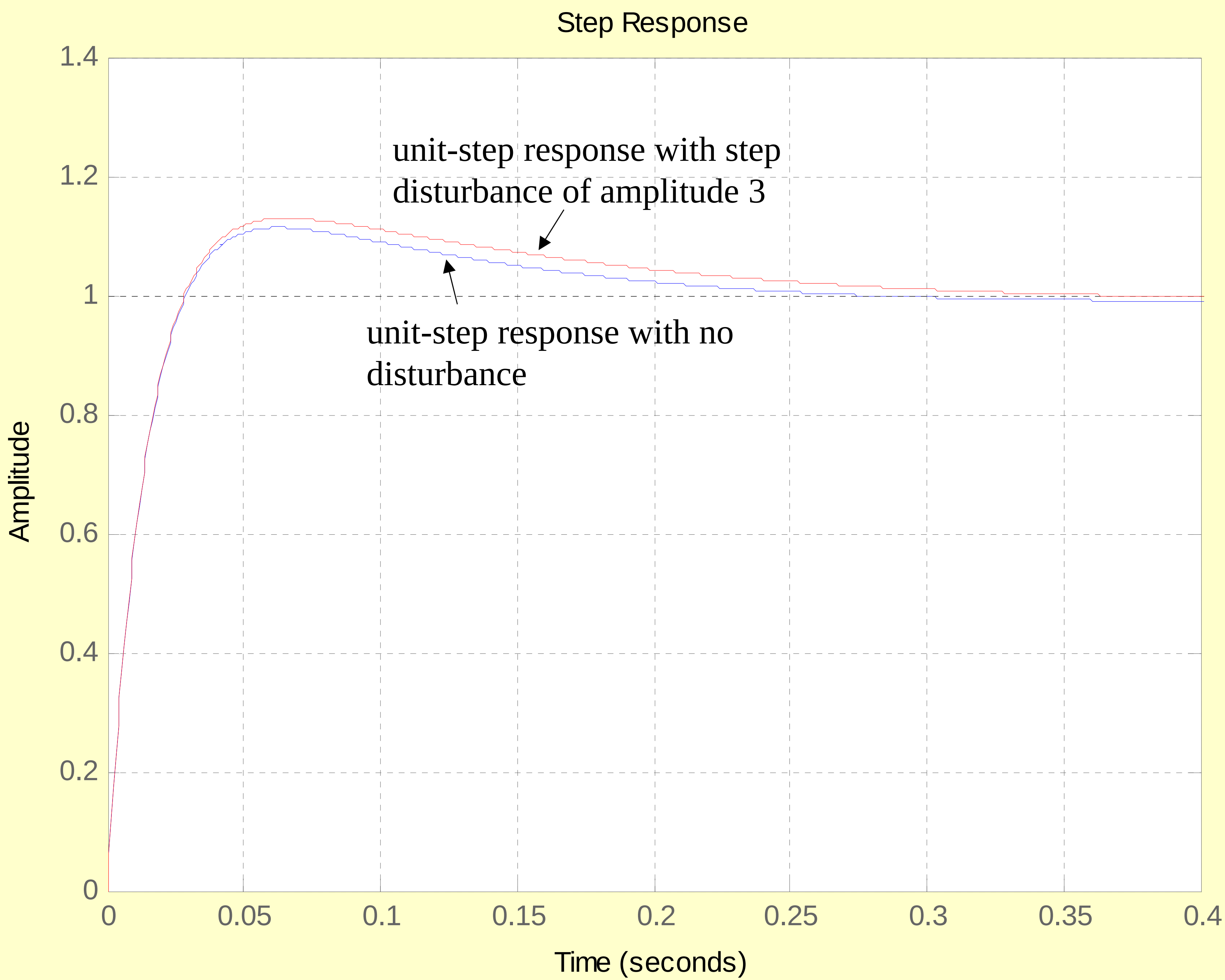
Suppose that we want a rise time $t_r = 0.5$ sec. For a 3rd order system, there is no straightforward way to relate t_r to ω_n.

Since we have designed our system to have a dominant 2nd-order system behavior, we can approximate t_r with the formula:

$$t_r = \frac{\pi - \theta}{\omega_n \sqrt{1-\zeta^2}}; \quad \theta = \cos^{-1}\zeta$$

With $\zeta = 0.8$ and $t_r = 0.5$ as specified, ω_n will be fixed as $\omega_n = 8.327$. Then, the values of K_P, K_I and K_D will be uniquely specified from the 3 previous equations, and we get $K_P = 95.7$, $K_I = 461.9$ and $K_D = 8.0$.

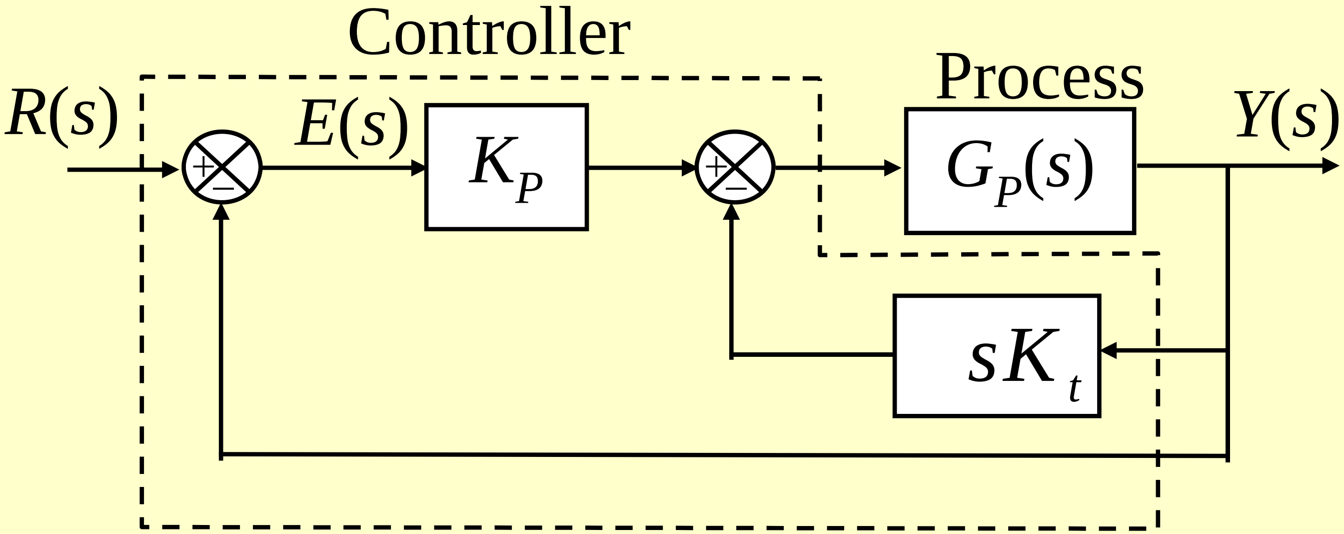
Note that the presence of the zeros will affect the actual performance of the system!!



Note that $t_r << 0.5$

c) Rate Feedback Control

(Tachogenerator Feedback Control)



- The controller is placed in the minor feedback loop.
- The time derivative of the output is fed back and compared with the input.
- This is often achieved, if the output is mechanical rotation, by using a tachogenerator.
- Rate feedback increases the damping factor in a similar manner as the derivative control.

Consider a process given by

$$G_P(s) = \frac{\omega_o^2}{s\left(s + 2\zeta_o\omega_o\right)}$$

With rate feedback, the open-loop transfer function of the overall system is

$$G'(s) = \frac{Y(s)}{E(s)} = \frac{K_P G_P(s)}{1 + sK_t G_P(s)}$$

$$= \frac{K_P\omega_0^2}{s\left(s + 2\zeta_o\omega_0 + K_t\omega_o^2\right)}$$

Therefore, $$\frac{Y(s)}{R(s)} = \frac{K_P G_P(s)}{1 + sK_t G_P(s) + K_P G_P(s)}$$

$$= \frac{K_P\omega_o^2}{s^2 + \left(2\zeta_o\omega_o + K_t\omega_o^2\right)s + K_P\omega_o^2}$$

Compare with using standard PD control:

$$\frac{Y(s)}{R(s)} = \frac{\left(sK_D + K_P\right)\omega_o^2}{s^2 + \left(2\zeta_o\omega_o + K_D\omega_o^2\right)s + K_P\omega_o^2}$$

With <u>rate</u> feedback, the un-damped natural frequency is

$$\omega_n' = \omega_o \sqrt{K_P}$$

while

$$\zeta'' = \frac{2\zeta_o\omega_o + K_t\omega_o^2}{2\omega_n'}$$

$$= \frac{\zeta_o}{\sqrt{K_P}} + \frac{K_t\omega_o}{2\sqrt{K_P}} = \zeta' + \frac{K_t\omega_o}{2\sqrt{K_P}}$$

That is, with rate feedback control, the damping ratio is increased by an amount $\dfrac{K_t\omega_o}{2\sqrt{K_P}}$, which is the same as with PD control.

The essential difference between rate feedback control and derivative control is the presence of the zero in the derivative control that gives a faster response to fast changing input.

Also, the steady-state errors of the two types of control methods are different (in this case with respect to a unit-ramp input).

With Derivative control:

$$E(s) = \frac{R(s)}{1+G(s)} = \frac{1}{s^2\left(1+G(s)\right)}; \qquad G(s) = \frac{\omega_o^2(sK_D + K_P)}{s\left(s+2\zeta_o\omega_o\right)}$$

$$e_{ss}(unit-ramp) = \lim_{s\to 0} sE(s) = \frac{1}{\lim\limits_{s\to 0} sG(s)} = \frac{2\zeta_o}{K_P\omega_o}$$

With Rate feedback control:

$$e_{ss}(unit-ramp) = \frac{1}{\lim\limits_{s\to 0} sG'(s)} = \frac{2\zeta_o + K_t\omega_o}{K_P\omega_o}$$

Clearly, rate feedback has a larger steady-state error.

Appendix 12.1: PID control of type-0 and type-2 2nd order systems

If $G_P(s) = \dfrac{\omega_o^2}{s^2 + 2\zeta_o\omega_o s + \omega_o^2}$, a type-0 2nd order system, with PID-controller, the OLTF is

$$G(s) = \frac{\omega_o^2}{s^2 + 2\zeta_o\omega_o s + \omega_o^2} \times \frac{K_D s^2 + K_P s + K_I}{s}$$

i.e. a type-1 system!

The closed-loop C.E. is

$$q(s) = s\left(s^2 + 2_o\zeta_o\omega_o s + \omega_o^2\right) + \omega_o^2(K_D s^2 + K_P s + K_I)$$

We equate it to a "desired" C.E., i.e.

$$s^3 + \left(2_o\zeta_o\omega_o + \omega_o^2 K_D\right)s^2 + \left(\omega_o^2 + \omega_o^2 K_P s\right) + \omega_o^2 K_I$$

$$\equiv (s^2 + 2\zeta\omega_n s + \omega_\mathrm{n}^2)(s + p) = s^3 + (p + 2\zeta\omega_n)s^2 + (2p\zeta\omega_n + \omega_\mathrm{n}^2)s + p\omega_\mathrm{n}^2$$

Equating coefficients: $\omega_o^2 K_D + 2\zeta_o\omega_o = p + 2\zeta\omega_n$

$$\omega_o^2 + \omega_o^2 K_P = 2p\zeta\omega_n + \omega_n^2$$

$$\omega_o^2 K_I = p\omega_n^2$$

If $G_P(s) = \dfrac{\omega_o^2}{s^2}$, a type-2 2nd order system, with PID-controller, the

OLTF is

$$G(s) = \frac{\omega_o^2}{s^2} \times \frac{K_D s^2 + K_P s + K_I}{s}$$

i.e. a type-3 system!

The closed-loop C.E. is

$$q(s) = s^3 + \omega_o^2 (K_D s^2 + K_P s + K_I)$$

We equate it to a "desired" C.E., i.e.

$$s^3 + \omega_o^2 K_D s^2 + \omega_o^2 K_P s + \omega_o^2 K_I$$

$$\equiv (s^2 + 2\zeta\omega_n s + \omega_n^2)(s+p) = s^3 + (p + 2\zeta\omega_n) s^2 + (2p\zeta\omega_n + \omega_n^2) s + p\omega_n^2$$

Equating coefficients:

$$\omega_o^2 K_D = p + 2\zeta\omega_n$$

$$\omega_o^2 K_P = 2p\zeta\omega_n + \omega_n^2$$

$$\omega_o^2 K_I = p\omega_n^2$$

Summary 12. PI and PID Controllers

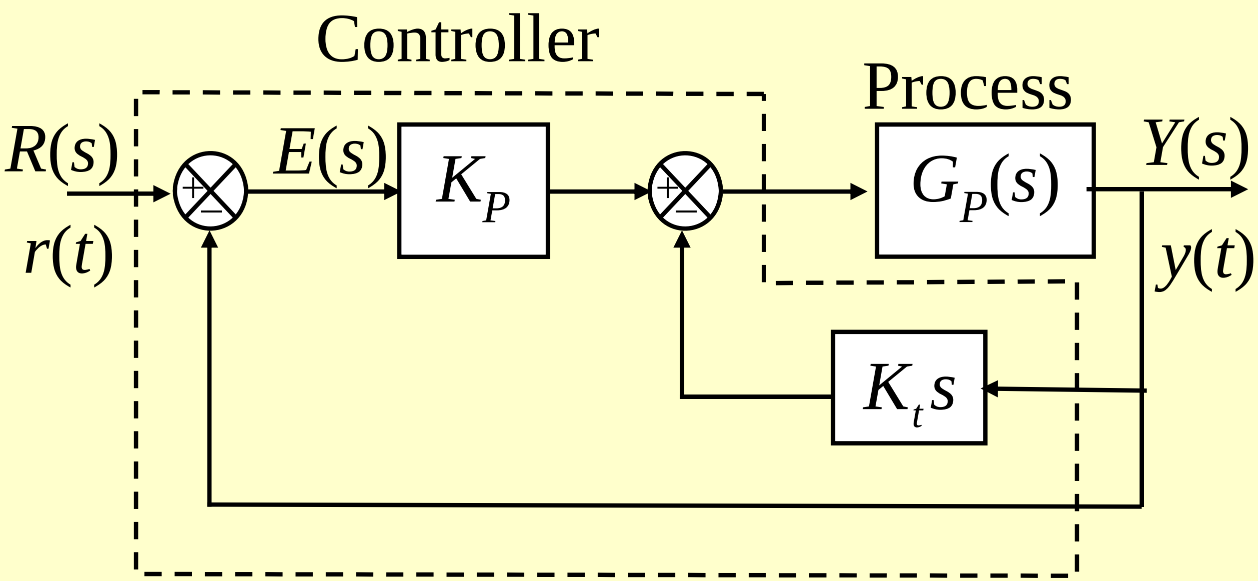
PI-controller: $G_c(s) = K_P + \frac{K_I}{s} = \frac{sK_P + K_I}{s}$

- It increases the system type by one and hence can eliminate the steady-state error of type-1 systems w.r.t. ramp input to zero.
- The system order is increased by 1, and can be unstable if K_P and K_I are not properly chosen.

PID Controller: $G_c(s) = K_P + \frac{K_I}{s} + sK_D = \frac{s^2 K_D + sK_P + K_I}{s}$

- To design suitable K_P, K_I and K_D, we equate the closed-loop characteristic equation to a desired one with desirable specifications.
- Meet steady-state error requirements first!
- Effects of K_P, K_I , K_D will depend on the specific system.

Rate Feedback Control



- Rate feedback increases the damping factor, just liked the PD control but it has a slower response and a poorer steady-state error performance.

Lecture 13: Root-Locus Analysis - Root Locus Plot

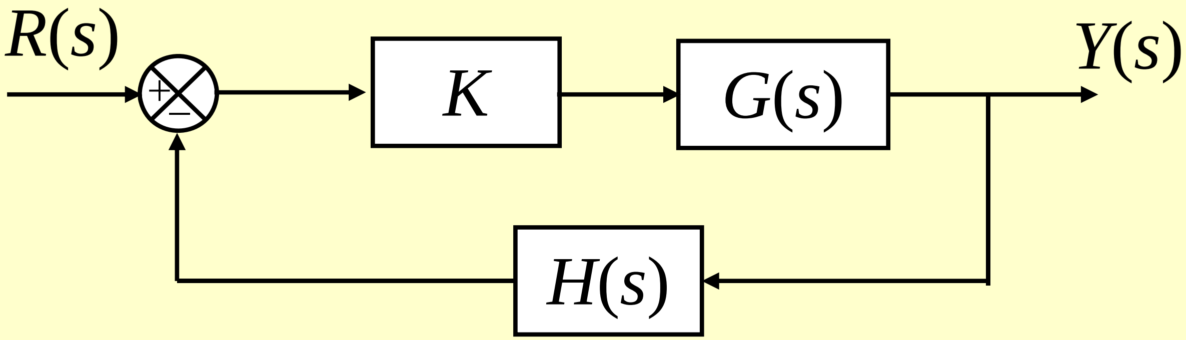
a) Basic Concepts

b) Mathematical Definition of Root-Locus

c) Rules for Constructing the Root-Locus

a) Basic Concepts

Consider a general feedback control system with a variable gain ***K***.



Root-Locus is a plot of the loci of the poles of the closed-loop transfer function when one of the system's parameters (***K***) is varied. Root locus allows us to visualize the changes in the closed-loop poles as the parameter ***K*** is increase from 0 to infinity.

A distinction is usually made between the following categories of root-locus:

(a) **Root-Locus Plot**:

The plot of the root loci when one parameter varies in positive values; this parameter is usually the forward gain K,

i.e. $0 \leq K < \infty$

(b) **Complementary Root-Locus Plot**:

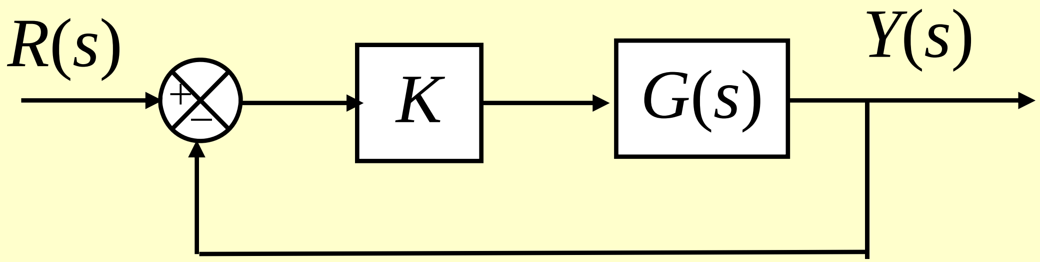
The plot is obtained for negative values of K,

i.e. $-\infty < K \leq 0$

(c) **Root Contours Plot**:

The plot is obtained when more than one parameter varies.

Example: Consider the system.



Suppose that $G(s) = \dfrac{1}{s(s+2)}$, find the roots of the C.E. for $0 \le K < \infty$ and plot these roots in the s-plane.

$$\frac{Y(s)}{R(s)} = \frac{KG(s)}{1+KG(s)} = \frac{K}{s^2+2s+K}$$

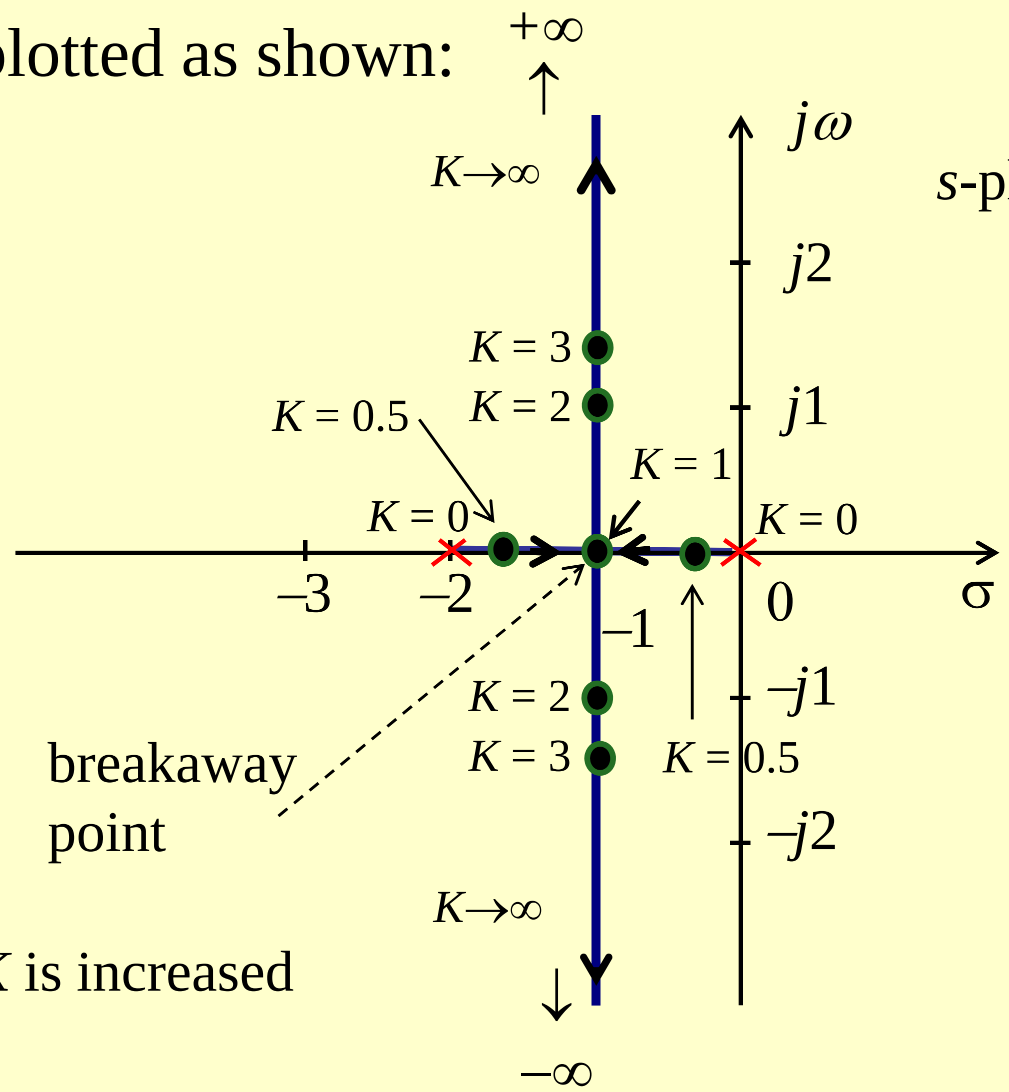
The C.E. is $s^2 + 2s + K = 0$. The roots of the C.E. are:

$$s_1 = \begin{cases} -1+\sqrt{1-K} & \text{for} \quad K \le 1 \\ -1+j\sqrt{K-1} & \text{for} \quad K > 1 \end{cases} \quad ; \qquad s_2 = \begin{cases} -1-\sqrt{1-K} & \text{for} \quad K \le 1 \\ -1-j\sqrt{K-1} & \text{for} \quad K > 1 \end{cases}$$

The effect of K on the roots s_1 and s_2 is shown by the table below:

K	s_1	s_2	Remark
0	0	-2.0	
0.5	-0.293	-1.707	
1.0	-1.0	-1.0	breakaway point
2.0	$-1 + j1$	$-1 - j1$	
3.0	$-1 + j1.414$	$-1 - j1.414$	
4.0	$-1 + j1.732$	$-1 - j1.732$	
:	:	:	
:	:	:	
$\rightarrow\infty$	$-1 + j\infty$	$-1 - j\infty$	

The root-locus is plotted as shown:

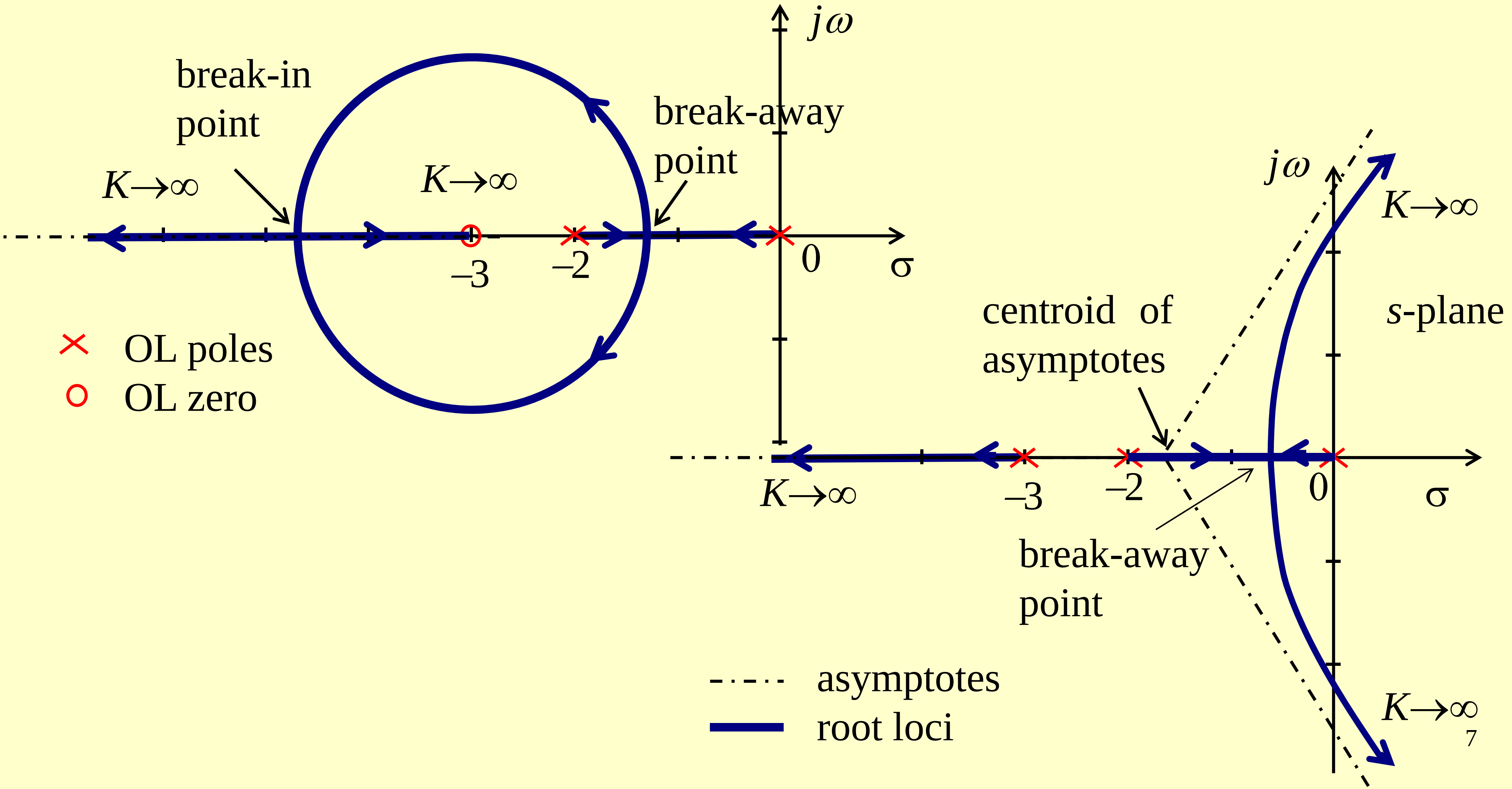


× OL poles

→ root-locus as K is increased

We shall develop a procedure to sketch the root-locus without computing the roots of the C.E. point by point.

The root loci plots of $1+G_p(s)=1+\dfrac{K(s+3)}{s(s+2)}=0$ and $1+G_p(s)=1+\dfrac{K}{s(s+2)(s+3)}=0$ are given below. Take note of the asymptotes and break-away (in) points.



b) Mathematical Definition of Root-Locus

Consider the closed-loop transfer function

$$\frac{Y(s)}{R(s)} = \frac{KG(s)}{1 + KG(s)H(s)}$$

The Characteristic Equation (C.E.) is

$$1 + KG(s)H(s) = 0$$

i.e. $KG(s)H(s) = -1$

Root-locus is defined by the conditions:

(I) **Magnitude Condition**:

$$\left| KG(s)H(s) \right| = 1$$

(II) **Angle Condition**:

$$\angle KG(s)H(s) = \pm(2n+1)\pi$$

where $n = 0, 1, 2, \ldots\ldots$ (an integer)

The open-loop transfer function may be expressed as

$$KG(s)H(s) = \frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

where $-z_i$'s are the zeros of OLTF
$-p_i$'s are the poles of OLTF

NB: If $-z_i$ (or $-p_i$) is complex, then $-z_{i+1}$ (or $-p_{i+1}$) is its complex conjugate

Hence, the C.E. $1 + KG(s)H(s) = 0$ is given by

$$(s+p_1)(s+p_2)\cdots(s+p_n) + K(s+z_1)(s+z_2)\cdots(s+z_m) = 0$$

When $K = 0$, the roots of the C.E. are:

$$-p_1, -p_2, \ldots\ldots, -p_n$$

When $K \to \infty$, the roots of the C.E. are:

$$-z_1, -z_2, \ldots\ldots, -z_m$$

In other words, as K varies from 0 to ∞, the roots of the C.E. traverse from the OL poles to the OL zeros, and $n - m$ of them to $|\infty|$.

With the OLTF $KG(s)H(s)$ given by

$$KG(s)H(s) = \frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

Magnitude Condition:

$$\left|KG(s)H(s)\right| = \frac{K\prod_{i=1}^{m}\left|s+z_i\right|}{\prod_{i=1}^{n}\left|s+p_i\right|} = 1 \qquad (1)$$

Angle condition:

$$\angle G(s)H(s) = \sum_{i=1}^{m}\angle(s+z_i) - \sum_{i=1}^{n}\angle(s+p_i)$$

$$= \pm(2j+1)\pi \quad ; j \text{ is an integer} \qquad (2)$$

Equations (1) and (2) define the root-locus. The root-locus of a system is a plot of all the values of s which satisfy eqns (1) and (2), and the plot involves steps as explained in Appendix 1. However, it can be readily plotted with a systematic procedure.

c) Rules for Constructing the Root-Locus

The OLTF is given by

$$KG(s)H(s) = \frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}; \quad (n \geq m)$$

The closed-loop C.E. is $(s+p_1)(s+p_2)\cdots(s+p_n) +$

$$K(s+z_1)(s+z_2)\cdots(s+z_m) = 0$$

i.e. n = number of finite poles of $G(s)H(s)$: $-p_i; \quad i = 1, 2, \ldots n$

m = number of finite zeros of $G(s)H(s)$: $-z_i; \quad i = 1, 2, \ldots m$

Rule 1 : $K = 0$, points on the root-locus are at the finite poles of $G(s)H(s)$.

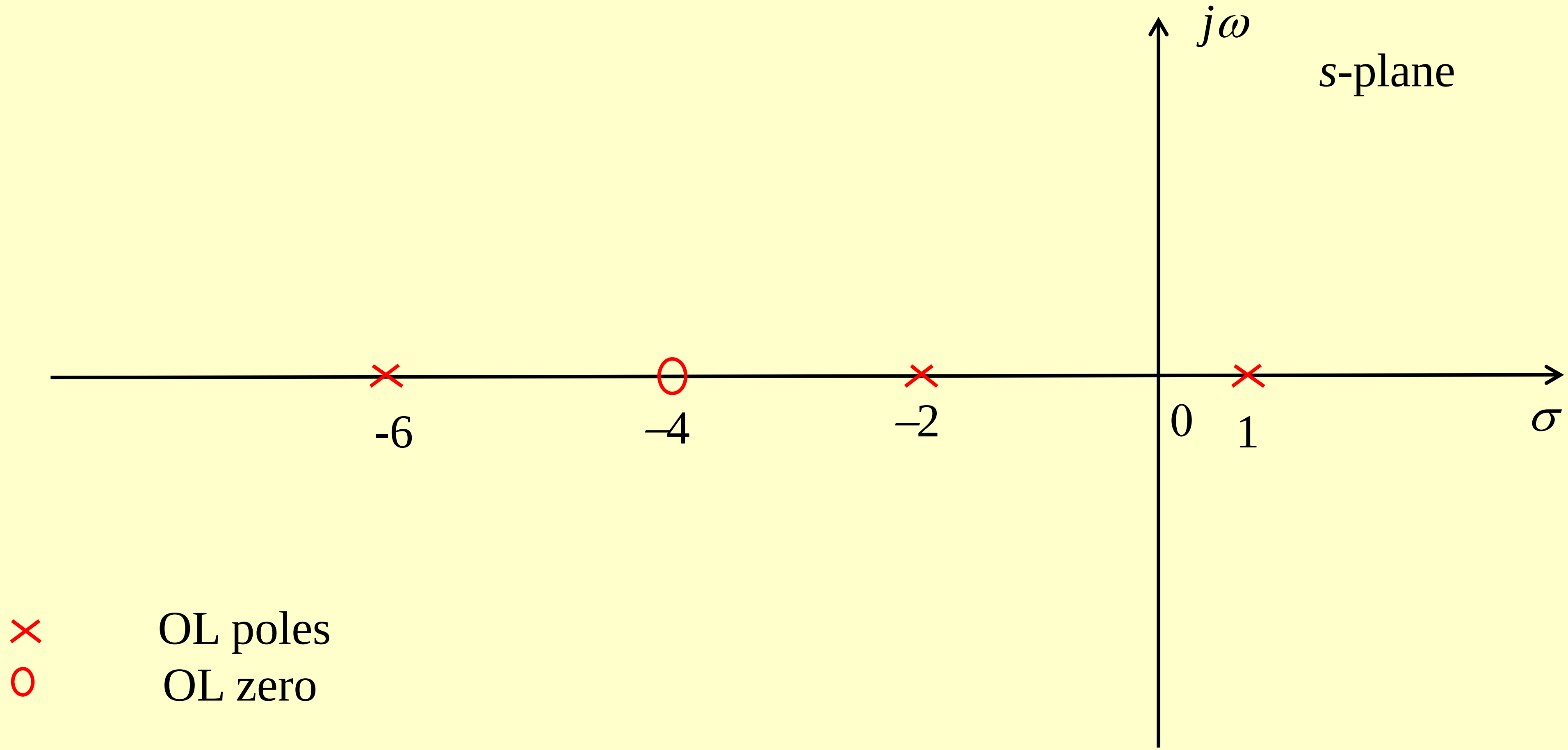
Rule 2 : $K \rightarrow +\infty$, points on the root-locus are at the finite zeros of $G(s)H(s)$ plus (n - m) of them go to infinity.

Rule 3 : The number of branches of the root loci is equal to n.

Rule 4 : The root loci are symmetrical with respect to the real-axis of the s-plane.

<u>*Example*</u>: $KG(s)H(s) = \dfrac{K(s+4)}{(s-1)(s+2)(s+6)}$

$n = 3$ and $m = 1$



(The explanations for the following rules 5-8, and 10 are given in the Appendix 2.)

Rule 5: There are $|n-m|$ asymptotes. As $|s| \to \infty$ the root loci are asymptotic to straight lines (asymptotes) with angle with the real-axis given by

$$\theta_j = \frac{(2j+1)\pi}{n-m}; \qquad j = 0,1,\ldots,(|n-m|-1)$$

Rule 6: The $|n-m|$ asymptotes intersect on the real axis at the point given by:

$$\sigma_C = \frac{\sum(\text{finite\_poles\_of\_}G(s)H(s)) - \sum(\text{finite\_zeros\_of\_}G(s)H(s))}{(n-m)}$$

i.e. $$\sigma_C = \frac{\sum_{i=1}^{n}(-p_i) - \sum_{i=1}^{m}(-z_i)}{(n-m)}$$

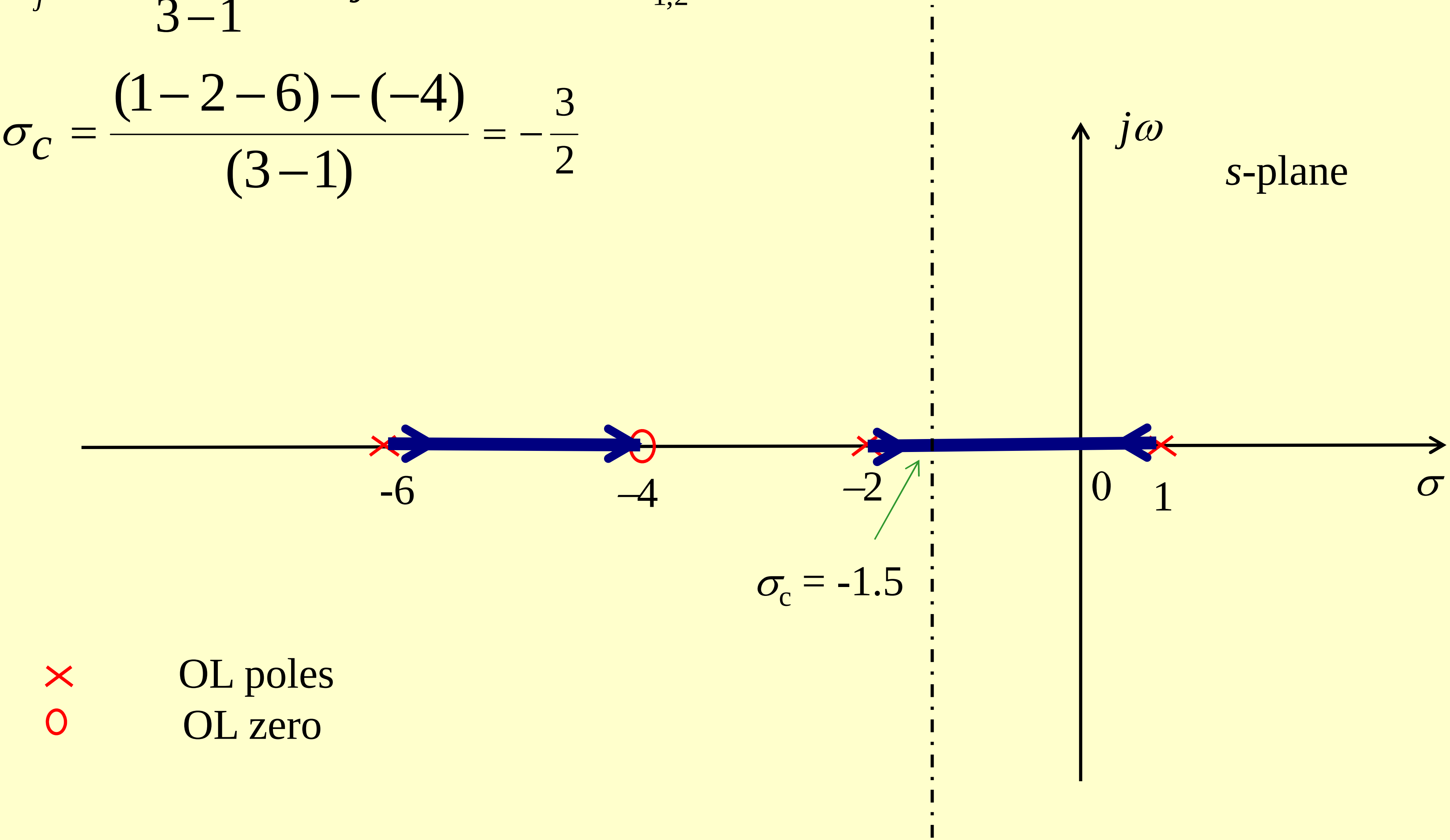
Rule 7: Root loci are found on a section of the real axis only if the total number of real poles and zeros of $G(s)H(s)$ to the right of the section is ODD.

$$Example: \quad KG(s)H(s) = \frac{K(s+4)}{(s-1)(s+2)(s+6)}$$

$n - m = 2$

$$\theta_j = \frac{(2j+1)\pi}{3-1}; \quad j = 0,1; \quad \text{i.e.} \quad \theta_{1,2} = 90^\circ, -90^\circ$$

$$\sigma_C = \frac{(1-2-6)-(-4)}{(3-1)} = -\frac{3}{2}$$



Rule 8:

(a) The angle of departure from complex poles are given by

$$\theta_d = 180° + \theta'$$

where θ' is the angle of $G(s)H(s)$ at that pole with the angle contribution from the pole itself ignored.

(b) The angles of arrival at complex zeros are given by:

$$\theta_a = 180° - \theta''$$

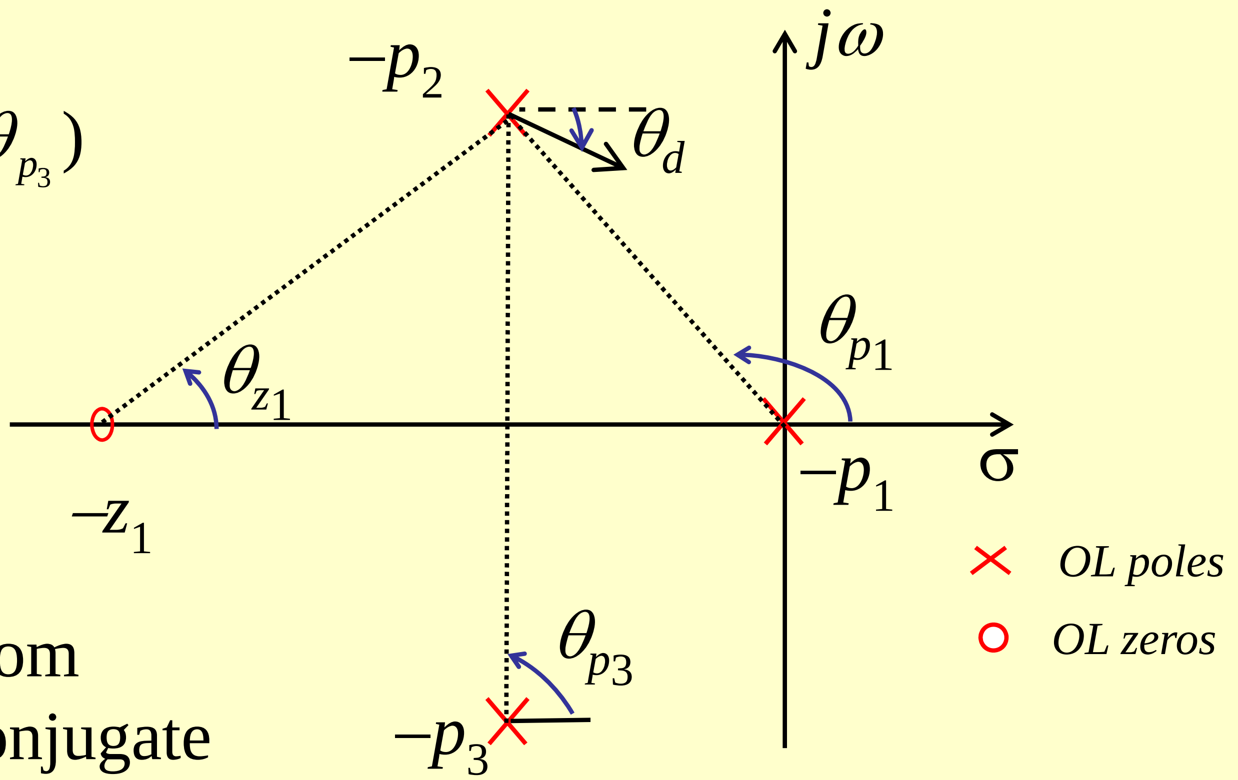
where θ'' is the angle of $G(s)H(s)$ at that zero with the angle contribution from the zero itself ignored.

$$\underline{Example}: \ Consider \ KG(s)H(s) = \frac{K(s+z_1)}{s(s+p_2)(s+p_3)} \quad ; \qquad K \geq 0$$

The angle of departure from complex pole $-p_2$:

$$\theta_d = 180° + \theta'$$

where $\theta' = \theta_{z_1} - (\theta_{p_1} + \theta_{p_3})$



The angle of departure from complex pole $-p_3$ (the conjugate of $-p_2$) will be $-\theta_d$!

<u>Rule 9</u>: The points where the root loci intersect the imaginary axis are obtained using the Routh-Hurwitz criterion.

<u>Rule 10</u>: The breakaway/breakin points are points where two or more branches of the root locus depart from or arrive at a main branch.

The angle of departure/arrival of breakaway/breakin point is given by

$$\theta_{ba} = \frac{(2j+1)180^\circ}{k}; \; j = 0, 1, \cdots, k-1$$

where k = number of loci leaving or approaching the breakaway/breakin point.

Suppose that the C.E. is expressed as

$$1+KG(s)H(s)=1+K\frac{B(s)}{A(s)}=0 \qquad (5)$$

Then, $K=-\dfrac{A(s)}{B(s)}$ and

$$\frac{dK}{ds}=-\frac{B(s)A'(s)-A(s)B'(s)}{B^2(s)} \qquad (6)$$

The breakaway/breakin points are given by

$$\frac{dK}{ds}=0 \qquad (7)$$

i.e. the breakaway/breakin points are given by the roots of

$$A(s)B'(s)-A'(s)B(s)=0 \qquad (8)$$

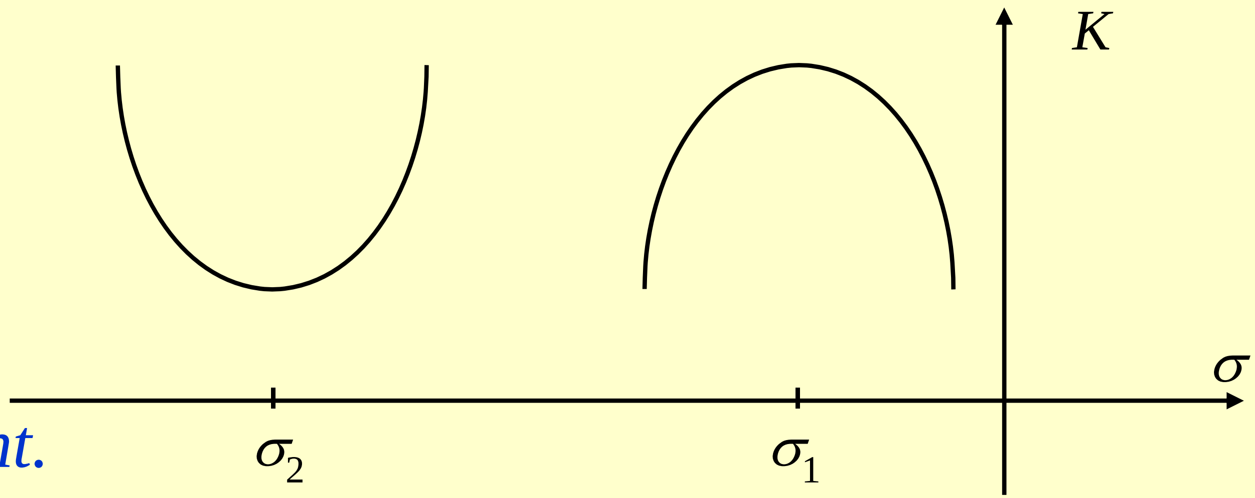
NB: **NOT** all roots of equation (8) correspond to the actual breakaway/breakin points.

With condition (7), we can also use eqn. (6) to derive a numerical method to obtain the breakaway/breakin points. For breakaway(in) points on the real axis, $s = \sigma$, so we have

$$K = -\frac{A(\sigma)}{B(\sigma)} \tag{9}$$

We plot a graph of K for various values of σ between the selected points where the breakaway(in) point is expected.

Then σ at which K is maximum (or minimum) is the breakaway (or breakin) point.



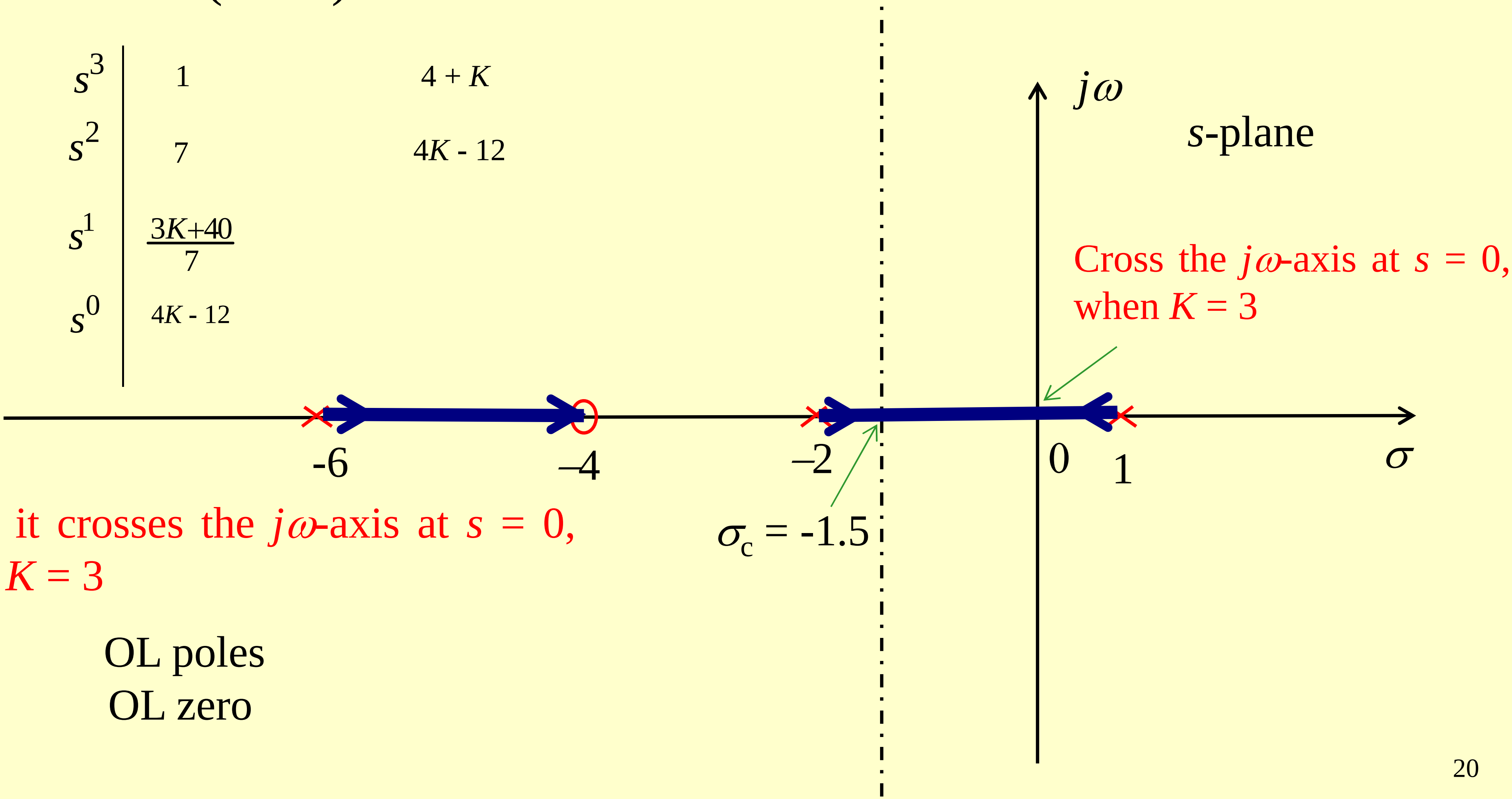
σ_1 is the breakaway point.
σ_2 is the breakin point.

Example: $$KG(s)H(s) = \frac{K(s+4)}{(s-1)(s+2)(s+6)}$$

$$1 + KG(s)H(s) = (s-1)(s+2)(s+6) + K(s+4) = 0$$

i.e. $$s^3 + 7s^2 + (4+K)s + 4K - 12 = 0$$

s^3	1	$4 + K$
s^2	7	$4K - 12$
s^1	$\frac{3K+40}{7}$	
s^0	$4K - 12$	



Thus, it crosses the $j\omega$-axis at s = 0, when K = 3

× OL poles
o OL zero

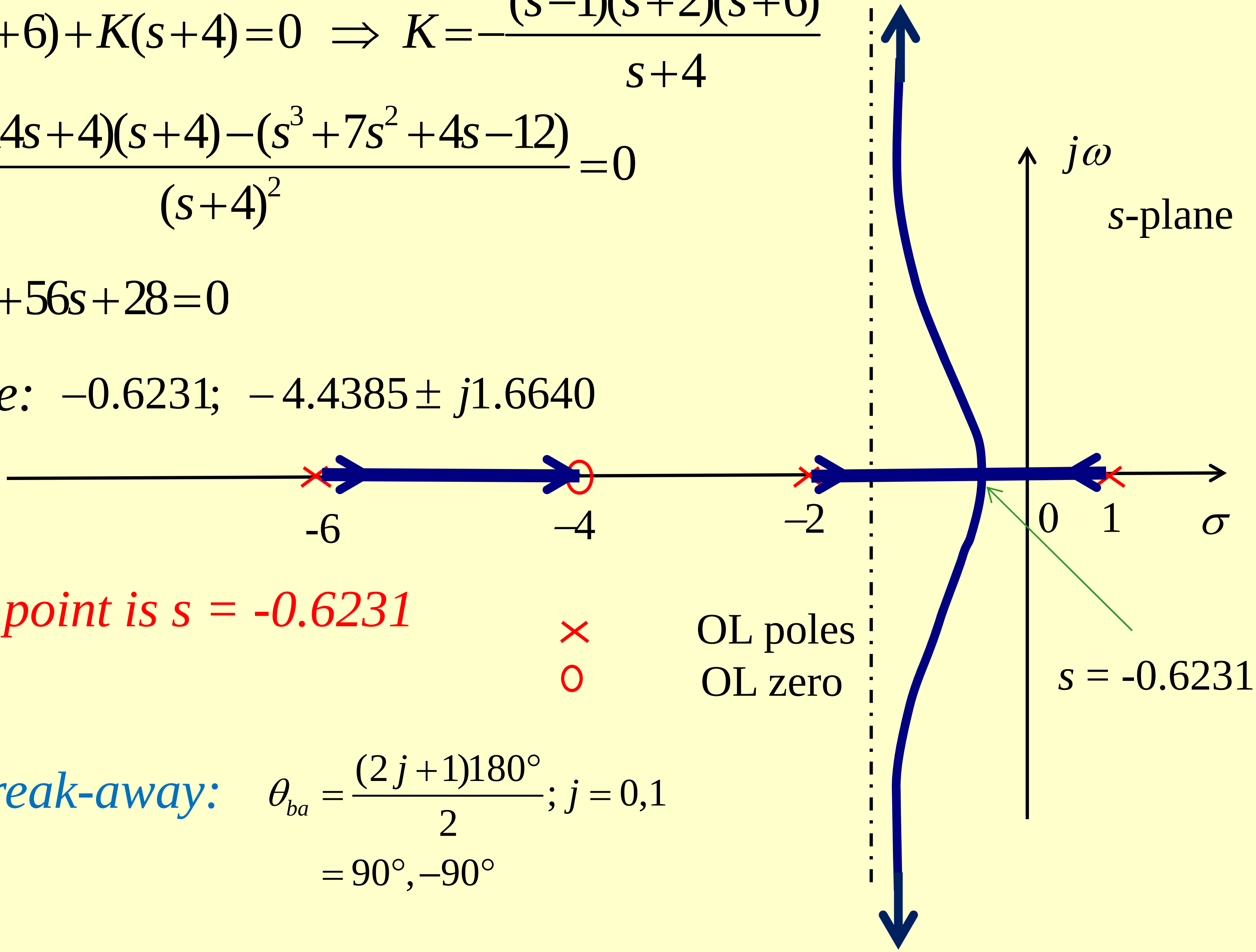
$$Example: \quad KG(s)H(s) = \frac{K(s+4)}{(s-1)(s+2)(s+6)}$$

$$(s-1)(s+2)(s+6)+K(s+4)=0 \;\Rightarrow\; K=-\frac{(s-1)(s+2)(s+6)}{s+4}$$

$$\frac{dK}{ds}=-\frac{(3s^2+14s+4)(s+4)-(s^3+7s^2+4s-12)}{(s+4)^2}=0$$

$$\Rightarrow\; 2s^3+19s^2+56s+28=0$$

$$The\ roots\ are: \; -0.6231; \; -4.4385 \pm j1.6640$$



Break-away point is s = -0.6231

Angles of break-away: $\theta_{ba} = \dfrac{(2j+1)180°}{2};\ j=0,1$

$= 90°, -90°$

<u>Rule 11</u>: The value of K at a point s_1 on the root locus is obtained by applying the Magnitude Condition and is obtained from

$$|K_1| = \frac{1}{|G(s_1)H(s_1)|}$$

$$= \frac{\prod_{j=1}^{n} |s_1 + p_j|}{\prod_{j=1}^{m} |s_1 + z_j|}$$

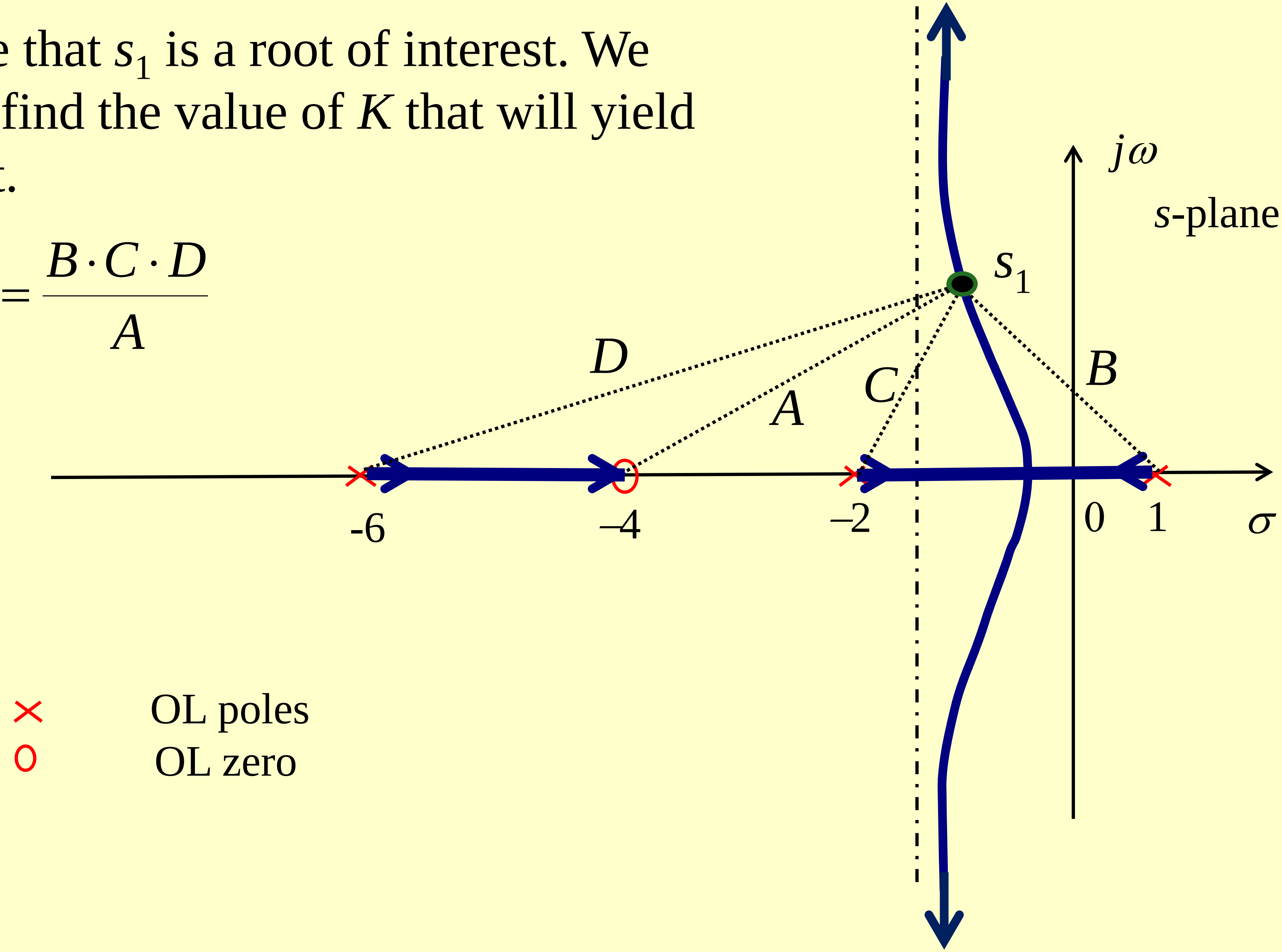
i.e.

$$|K_1| = \frac{\text{product\_of\_vector\_lengths\_from\_OLTF\_poles}}{\text{product\_of\_vector\_lengths\_from\_OLTF\_zeros}}$$

Example: $KG(s)H(s) = \dfrac{K(s+4)}{(s-1)(s+2)(s+6)}$

Suppose that s_1 is a root of interest. We want to find the value of K that will yield this root.

$$|K| = \frac{B \cdot C \cdot D}{A}$$



Appendix 1:

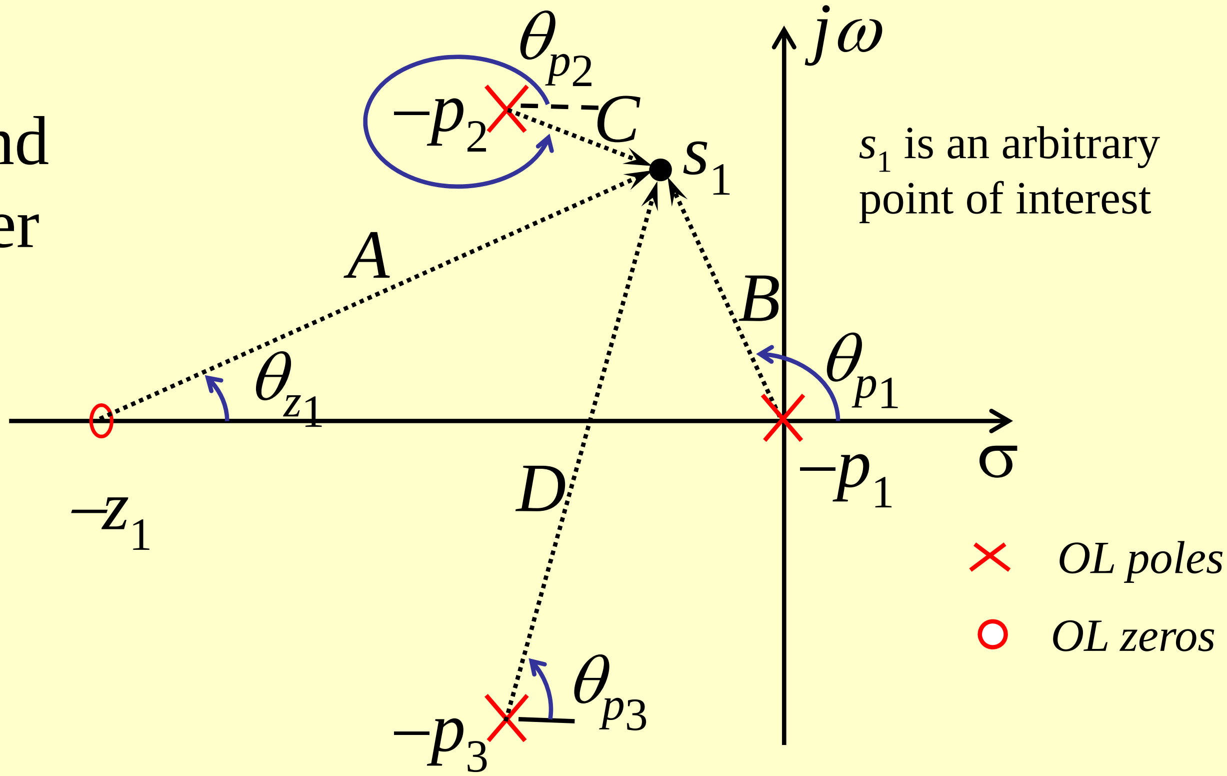
Given the OL poles and zeros of $KG(s)H(s)$, the root-locus plot of the closed-loop system involves:

(a) A search for points in the s-plane which satisfy equation (2).

(b) The determination of the values of K using equation (1).

Example: Consider $KG(s)H(s) = \dfrac{K(s+z_1)}{s(s+p_2)(s+p_3)}$; $K \geq 0$

The poles and zero are shown in the diagram, and we want to check whether s_1 is on the root locus.



If s_1 is a point on the Root-Locus, it must satisfy the following two conditions:

Magnitude Condition:

$$\frac{|s_1 + z_1|}{|s_1||s_1 + p_2||s_1 + p_3|} = \frac{1}{|K|} \qquad \text{(a)}$$

Angle Condition:

$$\angle(s_1 + z_1) - \left(\angle s_1 + \angle(s_1 + p_2) + \angle(s_1 + p_3)\right)$$

$$= \pm(2j+1)\pi$$

$$\Rightarrow \theta_{z_1} - (\theta_{p_1} + \theta_{p_2} + \theta_{p_3}) = \pm(2j+1)\pi \qquad \text{(b)}$$

If s_1 satisfies equation (b), then the value of K at that point is obtained from eqn (a), i.e.

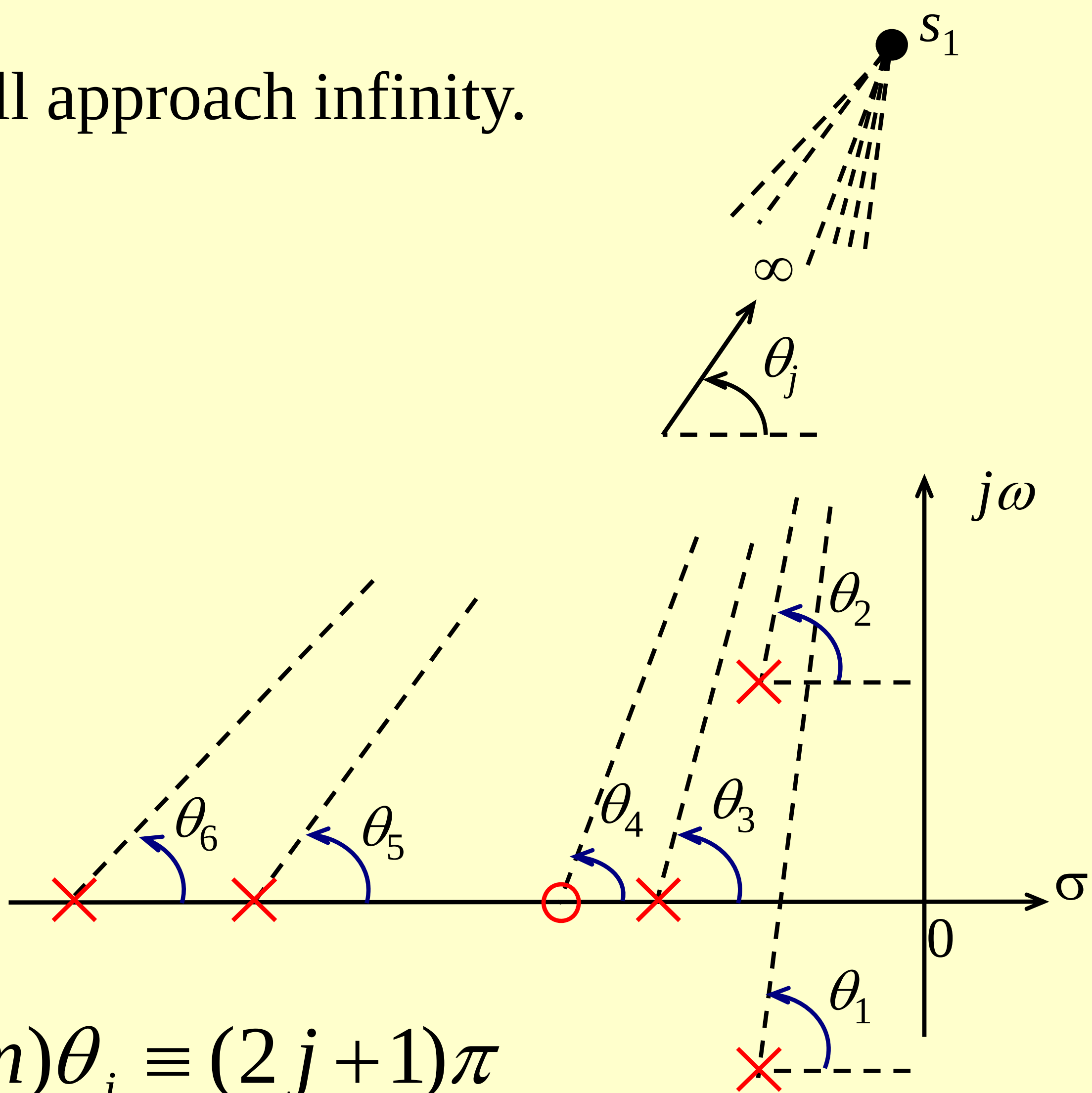
$$|K| = \frac{B \cdot C \cdot D}{A}$$

Next, repeat for new value of s_1 and so on.

Appendix 2: Explanations of rules 5-8, and 10.

Rule 5: As $K \to +\infty$, $|n - m|$ loci will approach infinity.

i.e. There are $|n - m|$ asymptotes.



If $|s_1| \to \infty$

$\Rightarrow \theta_j \approx \theta_1 \approx \theta_2 \approx \theta_3 \approx \theta_4 \approx \theta_5 \approx \theta_6$

By the Angle Condition:

$$\angle G(s_1)H(s_1) = (n-m)\theta_j \equiv (2j+1)\pi$$

$$\therefore \quad \theta_j = \frac{(2j+1)\pi}{n-m}; \qquad j = 0, 1, \ldots, (|n-m|-1)$$

Rule 6: From the OLTF

$$KG(s)H(s) = \frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)} = \frac{K\left[s^m + \left(\sum_{i=1}^{m} z_i\right)s^{m-1} + \cdots\cdots + \left(\prod_{i-1}^{m} z_i\right)\right]}{s^n + \left(\sum_{j=1}^{n} p_j\right)s^{n-1} + \cdots\cdots + \left(\prod_{j-1}^{n} p_j\right)}$$

Divide the denominator and the numerator by the numerator term,

$$KG(s)H(s) \approx \frac{K}{s^{n-m} + \left(\sum_{j=1}^{n} p_j - \sum_{i=1}^{m} z_i\right)s^{n-m-1}} \qquad \text{(A1)}$$

Consider the following function

$$KP(s) = \frac{K}{\left(s-\sigma_c\right)^{n-m}}$$

$$= \frac{K}{s^{n-m} - (n-m)\sigma_c s^{n-m-1} + \cdots\cdots} \qquad \text{(A2)}$$

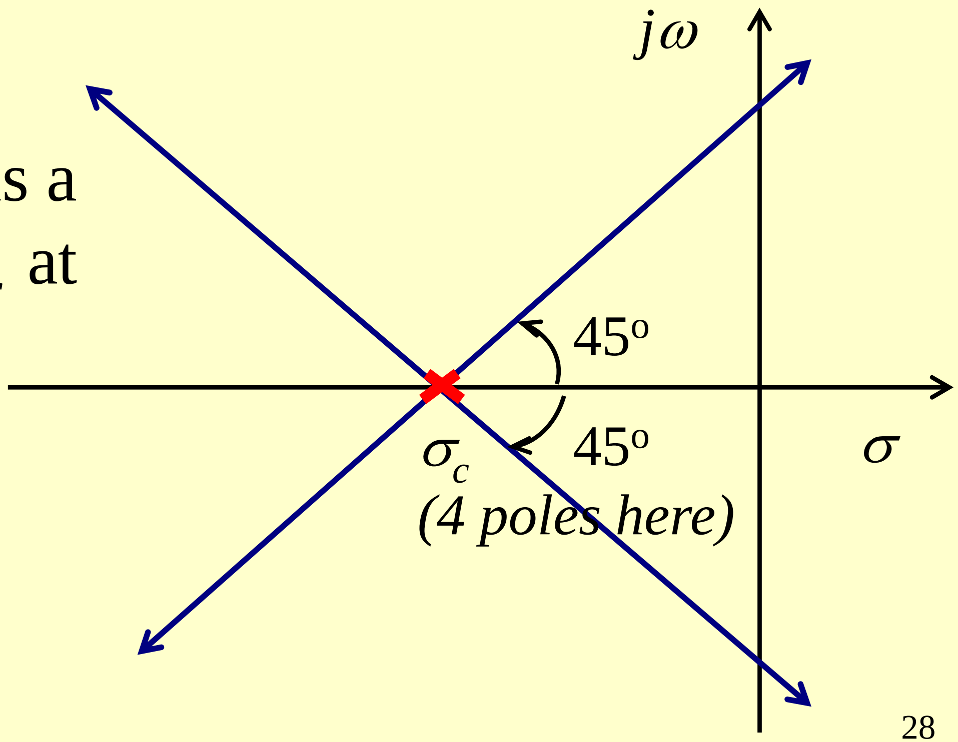
The C.E. $1 + KP(s) = 0$ has $(n - m)$ root locus branches which are straight lines passing through the point $s = \sigma_c$ and having angles,

$$\theta_j = \frac{(2j+1)\pi}{n-m} \qquad ; j = 0,1,\ldots,(n-m-1)$$

The case of $n-m=4$ is illustrated below. It can be readily verified that every point on the 4 straight lines satisfies the angle condition.

That is, the root locus of $1 + KP(s) = 0$ is a set of $(n - m)$ lines drawn from $s = \sigma_c$ at angles

$$\theta_j = \frac{(2j+1)\pi}{n-m} \qquad ; j = 0,1,\ldots,(n-m-1).$$



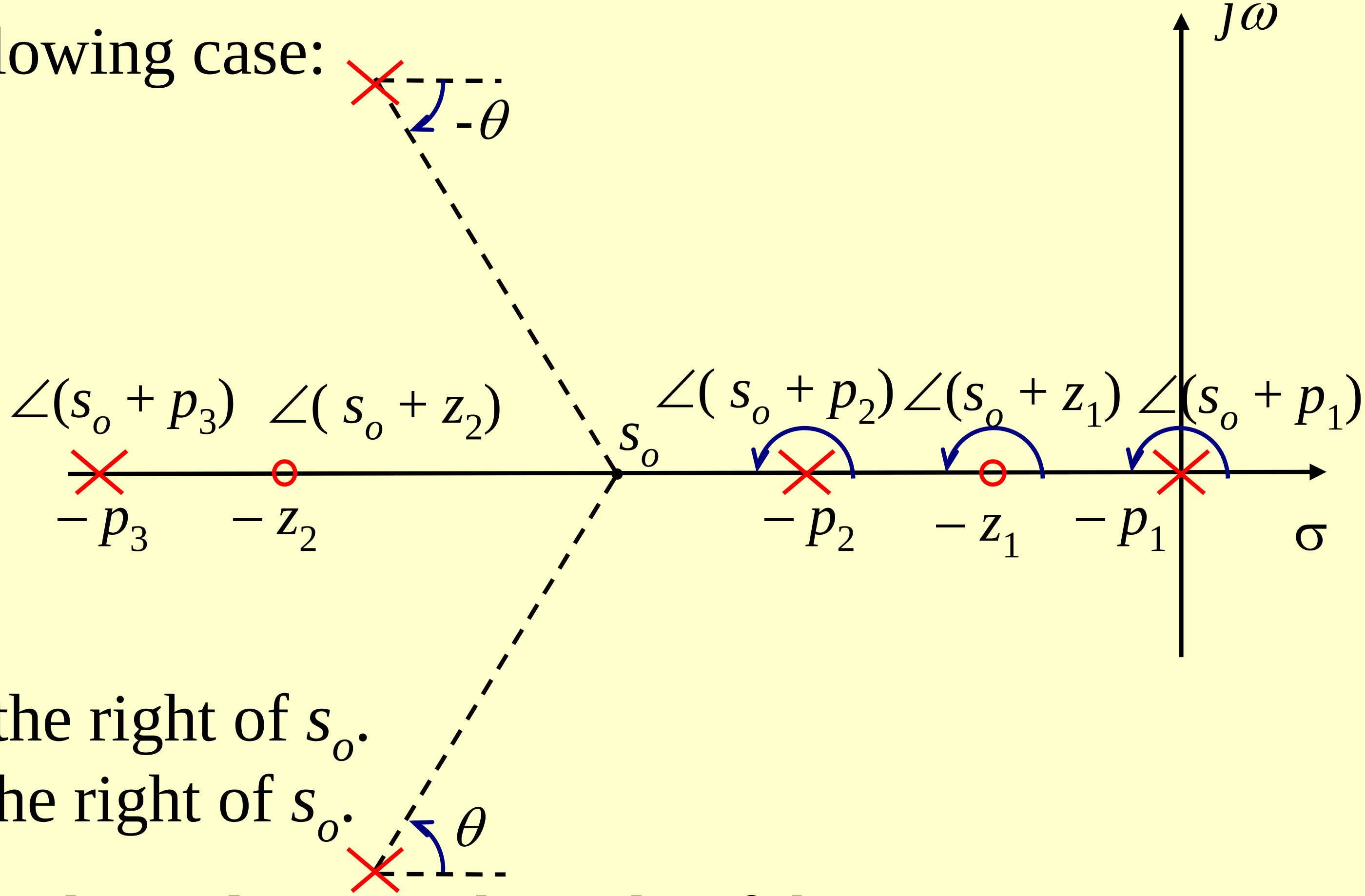
- The $1+KG(s)H(s)$ behaves in the same manner as $1+KP(s)$ for values of $s \to \infty$, because the first two higher order terms of their denominators are identical.
- $1+KG(s)H(s)$ approaches $1+KP(s)$ for values of $s \to \infty$. Therefore, the branches of $1 + KG(s)H(s) = 0$ which tend to infinity approach the straight line root locus branches of $1 + KP(s) = 0$.
- The straight line root loci of $1 + KP(s) = 0$ act as asymptotes to the $(n - m)$ branches of $1+KG(s)H(s) = 0$.
- From eqns (A1) and (A2)

$$-(n-m)\sigma_c = \sum_{j=1}^{n} p_j - \sum_{i=1}^{m} z_i$$

i.e. $$\sigma_c = \frac{\sum_{j=1}^{n}(-p_j) - \sum_{i=1}^{m}(-z_i)}{(n-m)}$$

- the <u>centroid</u> of the asymptotes.

Rule 7: Consider the following case:



m_R : Number of zeros on the right of s_o.
n_R : Number of poles on the right of s_o.

- The poles and zeros on the real axis to the right of this point s_o contribute an angle of 180° each.
- The poles and zeros to the left of this point s_o contribute an angle of 0° each.
- The net angle contribution of a complex conjugate pole or zero pair is always zero.

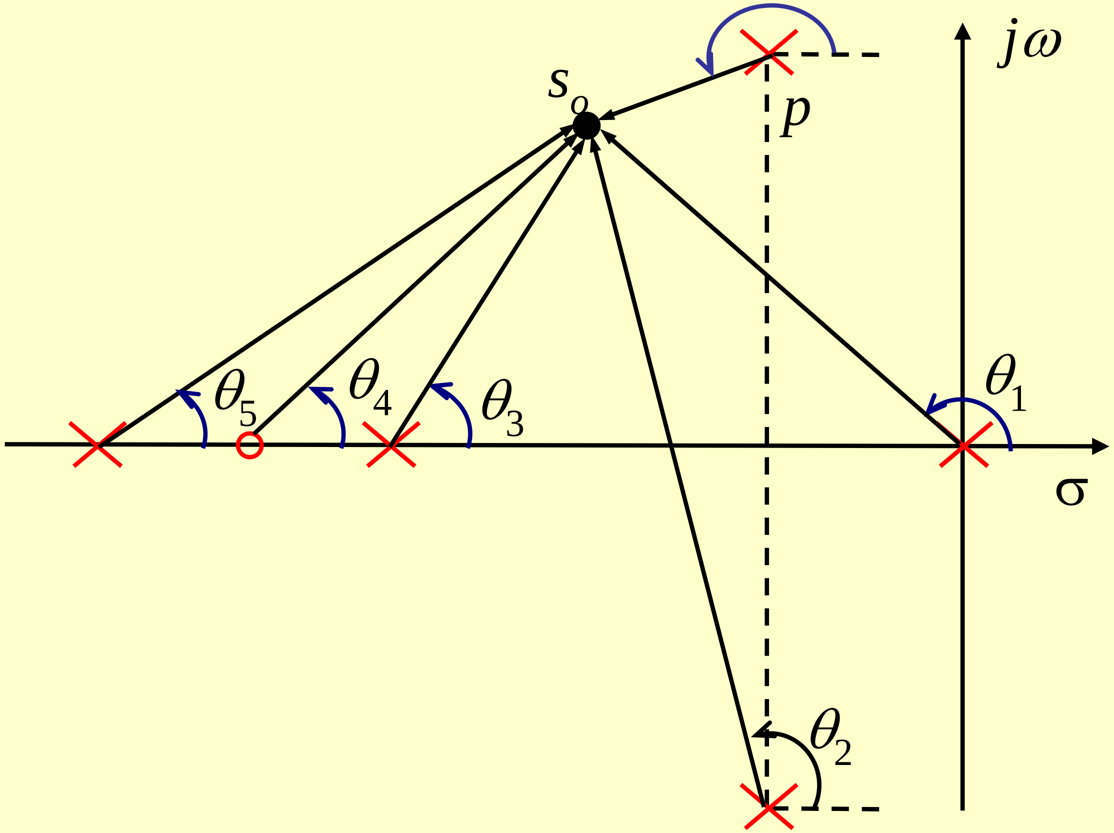
So,

$$\angle G(s)H(s) = (m_R - n_R)180^\circ = \pm(2j+1)180^\circ$$

i.e. $m_R - n_R$ (and hence $m_R + n_R$) must be an odd number.

<u>Rule 8</u>: Consider the following case:

$\theta_p \to \theta_d \quad (s_o \to p)$



- At point s_o, the net angle contribution of all other poles and zeros at this point is:

$$\theta' = \theta_4 - \left(\theta_1 + \theta_2 + \theta_3 + \theta_5\right)$$

- In the limit as the point s_o on the root locus approaches p, θ_P equals the angle of departure of the root locus from the pole p, ($\theta_P \rightarrow \theta_d$).

From the Angle Condition:

$$\theta' - \theta_d = -(2j + 1)180^\circ = -180^\circ$$

or $\quad \theta_d = 180^\circ + \theta'.$

Similar explanation applies to angle of arrival.

Rule 10: The Characteristic Equation is

$$1 + KG(s)H(s) = 0$$

If C.E. has a multiple root at $s = -b$ of multiplicity r, i.e.

$$1 + KG(s)H(s) = (s + b)^r A_1(s)$$

where $A_1(s)$ does not contain the factor $(s + b)$, then

$$\frac{d}{ds}\left[1 + KG(s)H(s)\right] = (s+b)^r A_1^{'}(s) + A_1(s) r (s+b)^{r-1}$$

$$= (s+b)^{r-1}\left[rA_1(s) + (s+b)A_1^{'}(s)\right]$$

At $s = -b$, RHS = 0 (only if $r \geq 2$)

$$\Rightarrow \quad \frac{d}{ds}\left[1 + KG(s)H(s)\right] = 0 \qquad \text{(A3)}$$

The C.E. can be expressed as

$$1 + KG(s)H(s) = 1 + K\frac{B(s)}{A(s)} = 0 \qquad \text{(A4)}$$

So,

$$\frac{d}{ds}\left[1 + KG(s)H(s)\right] = K\frac{A(s)B'(s) - A'(s)B(s)}{A^2(s)}$$

From eqn (A3), the roots are the breakaway points, therefore the breakaway points are also given by the roots of

$$A(s)B'(s) - A'(s)B(s) = 0 \qquad \text{(A5)}$$

From eqn (A4),

$$K = -\frac{A(s)}{B(s)} \qquad \text{(A6)}$$

Hence

$$\frac{dK}{ds} = -\frac{B(s)A'(s) - A(s)B'(s)}{B^2(s)}$$

$$= \frac{A(s)B'(s) - B(s)A'(s)}{B^2(s)}$$

With eqn (A5), we have

$$\frac{dK}{ds} = 0 \qquad \text{(A7)}$$

That is, the breakaway(in) points can also be computed from the roots of (A7), or through a numerical method.

Warning: **NOT** all roots of equation (A5) (or (A7) correspond to the actual breakaway point. (Verify with Angle Condition!!!)