LECTURE NO 5

22 Optimal Control

Design approaches :

- Classical techniques : Frequency response and root locus methods are effective but largely trial and error, with experiences very useful.
- Modern techniques : State space approach.

$$x(k+1) = f^k\left(x(k), u(k)\right) \qquad \text{... (22.1)}$$

$$J_i\left(x(i)\right) = \phi\left(N, x(N)\right) + \sum_{k=i}^{N-1} L^k\left(x(k), u(k)\right) \qquad \text{...(22.2)}$$

Let the plant be as given in (22.1) where the superscript k on f indicates that it can be time-varying.
Also, let the associated performance index be given in (22.2) where $[i, N]$ is the time interval of interest.

The optimal control problem is :

Find the control $u^*(k)$ on the interval $[i, N]$ that drives the system (22.1) along a trajectory $x^*(k)$ such that the performance index (22.2) is minimised.

Some examples of performance indices :

(a) Minimum time problems :
To find the control $u(k)$ to drive the system from given $x(0)$ to a desired final state ξ in minimum time.

$$J = N = \sum_{k=0}^{N-1} 1^k; \quad x(N) = \xi$$

$$\Rightarrow \quad \phi = N, L = 0$$

$$\left(\phi = 0; L = 1\right)$$

(b) Minimum fuel problems :
To find the control $u(k)$ to drive the system from given $x(0)$ to a desired final state ξ at fixed time *N* using minimum fuel.

$$J = \sum_{k=0}^{N-1} \left| u(k) \right|$$

$$\Rightarrow \phi = 0; L = \left| u(k) \right|; \quad x(N) = \xi$$

Since the fuel burnt is proportional to the magnitude of the control vector.

(c) Minimum energy problems :
To find the control $u(k)$ to minimise the energy of the final state and all intermediate states, and also of the control used to achieve this. *N* is fixed.

$$J = \frac{1}{2} s x^T(N) x(N) + \frac{1}{2}\sum_{k=0}^{N-1}\left(q x^T(N) x(N) + r u^T(k) u(k)\right)$$

q, r, s : are all scalars.

$$L = \frac{1}{2}\left(q x^T(N) x(N) + r u^T(k) u(k)\right)$$

$$\phi = \frac{1}{2} s x^T(N) x(N)$$

Minimising the energy corresponds in some sense to keeping the state and control close to zero. Choose
q large – if impt to keep the states small
r large – if impt to keep control energy small
s large – if impt to keep final state small

In general, **Q**, **R** and **S** are matrices.
(22.1) – relates to dynamics of the system.
(22.2) – is the performance index chosen to achieve the desired system response.

To achieve different control objectives, different indices J selected. In practice, it is usually necessary to do a control design with some trial J, compute the optimal $u^*(k)$ and run a simulation to see how the system response to this $u^*(k)$. If the system response is not acceptable, repeat with another J with different Q, R and S matrices.

22.1 Dynamic programming

Dynamic programming was developed by RE Bellman in the late 1950s. It can be used to solve control problems for non-linear, time-varying systems.

The optimal control is expressed in state variable feedback form.

22.1.1 Bellman's principle of optimality

Dynamic programming is based on Bellman's principle of optimality which states that :

An optimal policy has the property that no matter what the previous decisions (i.e. controls) have been, the remaining decisions must constitute an optimal policy with regards to the state resulting from those previous decisions.

Example 5.1 An aircraft routing example

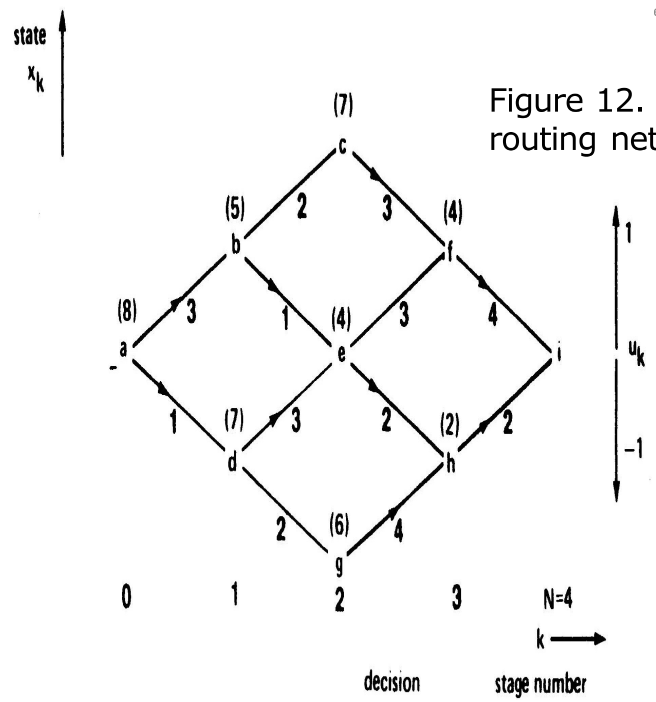
An aircraft can fly from left to right along the paths as shown in Figure 12.

Intersections *a,b,c,* … represent cities.

The numbers represent the fuel required to complete each path.

The principle of optimality shall be used to solve the minimum fuel problem.

Figure 12. Aircraft routing network



To construct a state variable feedback that shows the optimal cost and the optimal control from any node to node *i,* let's define what is meant by state in this example.
At each stage $k = 0,1, \ldots, N-1,$ a decision is required. $N = 4$ is the final stage.
The current state is the node where we are making the current decision.
Thus, the initial state is $x_0 = a$. At stage 1, the state can be $x_1 = b$ or $x_1 = d$. Similarly, $x_2 = c, e$ or g and $x_3 = f$ or $h.$ The final state is constrained to be $x_N = x_4 = i$.

The control u_k at stage k can be

$$u_k = \pm 1 \Rightarrow \begin{cases} u_k = 1 & \textit{results in a move up} \\ u_k = -1 & \textit{results in a move down} \end{cases}$$

to stage $k + 1$.

To find the minimum fuel feedback control law using the principle of optimality, start at $k = N = 4$. No decision is required.

Decrement $k = 3$. If $x_3 = f$, the optimal (only) control is $u_3 = -1$ and cost = 4, place (4) above node f and place an arrowhead on path $f \rightarrow i$. If $x_3 = h$, the optimal (only) control is $u_3 = 1$, cost = 2.

Now decrement $k = 2$. If $x_2 = c$, then $u_2 = -1$ and cost = 4 + 3 = 7. If $x_2 = e$, a decision is required. If $u_2 = 1$ to go to f, then go via optimal path to i, cost = 4 + 3 = 7. On the other hand, if we apply $u_2 = -1$ at e to go to h, the cost = 2 + 2 = 4. Hence, at e, the optimal decision is $u_2 = -1$ with cost = 4.

If $x_2 = g$, there is only one choice with $u_2 = 1$ with a cost to go of 6.

By successively decrementing k and continuing to compare the control possibilities allowed by the principle of optimality, we can fill in the remainder of the control decisions (arrowheads) and optimal costs to go shown in the figure.
It should be realised that the only control sequences we are allowed to consider are those the last portions of which are optimal sequences.
When $k = 0$, either $u_0 = 1$ or $u_0 = -1$ yields the same cost to go of 8.

If we want to find the minimum fuel path from node *d* to the final destination i, just begins at *d* and follow the arrows. Suppose, in ignorance of the optimality principle, someone wanted to determine an optimal route from *a* to i by working forward. Then, the person at *a* would compare the costs of travelling to *b* and *d*, and decide to go to *d*. The next myopic decision would take him to *g*. From there on, there is no choice. The person must go via *h* to i. The net cost of this strategy is 1+2+4+2 = 9, which is not optimal.

22.2 Principle of optimality to discrete-time systems

$$x(k+1) = f^k\left(x(k), u(k)\right) \quad \text{... (22.3)}$$

$$J_i\left(x(i)\right) = \phi\left(N, x(N)\right) + \sum_{k=i}^{N-1} L^k\left(x(k), u(k)\right) \quad \text{...(22.4)}$$

Let the plant be as given in (22.3) and the associated performance index be in (22.4). The aim is to use Bellman's Principle of Optimality to find $u^*(k)$ that minimises (22.4).

Suppose we have computed the optimal cost $J^*_{k+1}\big(x(k+1)\big)$ from time $k+1$ to the terminal time *N* for all possible states $x(k+1)$ and that we have also found the optimal control sequences from $k+1$ to *N* for all $x(k+1)$.

The optimal cost results when the optimal control sequences $u^*(k+1), u^*(k+2), \ldots, u^*(N-1)$ is applied to the plant with state $x(k+1)$.

Apply any arbitrary control $u(k)$ at time k and then use the known optimal control sequence from $k+1$ on, the resulting cost will be

$$L^k\left(x(k),u(k)\right)+J_{k+1}^*\left(x(k+1)\right) \qquad ...(22.5)$$

where $x(k)$ is the state at time k and $x(k+1)$ is given by (22.3).

According to Bellman's Principle, the optimal cost from time k onwards is equal to

$$J_k^*\left(x(k)\right)$$
$$=\min_{u(k)}\left(L^k\left(x(k),u(k)\right)+J_{k+1}^*\left(x(k+1)\right)\right)...(22.6)$$

The optimal control $u^*(k)$ at time k is the $u(k)$ that achieves the minimum of (22.6).

(22.6) is the principle of optimality for discrete-time systems. It allows one to optimise over one control vector at a time by working backwards from *N.*

22.3 Discrete-time linear quadratic regulator via dynamic programming

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k) \qquad \text{... (22.7)}$$

$$J_i = \frac{1}{2}\mathbf{x}^T(N)\mathbf{S}\left(N\right)\mathbf{x}(N)$$

$$+\frac{1}{2}\sum_{k=i}^{N-1}\left(\mathbf{x}^T(k)\mathbf{Q}\mathbf{x}(k) + \mathbf{u}^T(k)\mathbf{R}\mathbf{u}(k)\right) \qquad \text{...(22.8)}$$

$$\mathbf{S}\left(N\right) \geq 0; \mathbf{Q} \geq 0; R > 0$$

Let the plant be as given in (22.7) and the associated performance index be in (22.8).

A square symmetric matrix is positive definite > 0, if and only if all its eigenvalues are greater than 0.

Eigenvalues of a symmetric matrix are real. If some of the eigenvalues are 0 and the rest positive, the matrix is said to be positive semi-definite.

If the plant and weighting matrices are time-varying, the development to follow still holds.

The problem is to find the optimal control $u^*(k)$ on the interval $[i, N]$ that minimises J_i. The initial state $\mathbf{x}(i)$ is given.
At $k = N$,

$$J_N^* = \frac{1}{2}\mathbf{x}^T(N)\mathbf{S}\left(N\right)\mathbf{x}(N) \qquad \text{...(22.9)}$$

which is the penalty for being in state $\mathbf{x}(N)$ at time N. Now decrement k to N-1,

$$J_{N-1} = \frac{1}{2}\mathbf{x}^T(N)\mathbf{S}\left(N\right)\mathbf{x}(N) + \frac{1}{2}\mathbf{x}^T(N-1)\mathbf{Q}\mathbf{x}(N-1) + \frac{1}{2}\mathbf{u}^T(N-1)\mathbf{R}\mathbf{u}(N-1) \qquad \text{...(22.10)}$$

According to (22.6), need to find $\mathbf{u}^*(N-1)$ by minimising (22.10). Use (22.7) to write,

$$J_{N-1} = \frac{1}{2}\left(\mathbf{Ax}(N-1)+\mathbf{Bu}(N-1)\right)^T \mathbf{S}(N) \bullet$$

$$\left(\mathbf{Ax}(N-1)+\mathbf{Bu}(N-1)\right)+\frac{1}{2}\mathbf{x}^T(N-1)\mathbf{Qx}(N-1)$$

$$+\frac{1}{2}\mathbf{u}^T(N-1)\mathbf{Ru}(N-1) \qquad \ldots (22.11)$$

The minimum of J_{N-1} can be found by

$$\frac{\partial J_{N-1}}{\partial \mathbf{u}(N-1)} = 0$$

$$\Rightarrow \mathbf{B}^T\mathbf{S}(N)\left(\mathbf{Ax}(N-1)+\mathbf{Bu}(N-1)\right)$$

$$+\mathbf{Ru}(N-1) = 0 \qquad \ldots(22.12)$$

$$\Rightarrow \mathbf{u}^*(N-1) = -\left(\mathbf{B}^T\mathbf{S}(N)\mathbf{B}+\mathbf{R}\right)^{-1} \bullet$$

$$\mathbf{B}^T\mathbf{S}(N)\mathbf{Ax}(N-1) \ldots(22.13)$$

$$\mathbf{K}(N-1)$$

$$\triangleq \left(\mathbf{B}^T\mathbf{S}(N)\mathbf{B}+\mathbf{R}\right)^{-1}\mathbf{B}^T\mathbf{S}(N)\mathbf{A} \qquad \ldots(22.14)$$

$$\Rightarrow \mathbf{u}^*(N-1) = -\mathbf{K}(N-1)\mathbf{x}(N-1) \qquad \ldots(22.15)$$

Substitute (22.15) into (22.11), the optimal cost to go from $k = N-1$,

$$J_{N-1}^{*} = \frac{1}{2}\mathbf{x}^T(N-1)\mathbf{S}\left(N-1\right)\mathbf{x}(N-1) \quad ...(22.16)$$

$$\mathbf{S}\left(N-1\right)$$

$$\triangleq \left[\mathbf{A}-\mathbf{BK}(N-1)\right]^T \mathbf{S}\left(N\right)\left[\mathbf{A}-\mathbf{BK}(N-1)\right]$$
$$+\mathbf{K}^T(N-1)\mathbf{RK}(N-1)+\mathbf{Q} \quad ...(22.17)$$

Now decrement to $k = N - 2$. Then,

$$J_{N-2} = \frac{1}{2}\mathbf{x}^T(N-1)\mathbf{S}\left(N-1\right)\mathbf{x}(N-1)$$
$$+\frac{1}{2}\mathbf{x}^T(N-2)\mathbf{Q}\mathbf{x}(N-2)$$
$$+\frac{1}{2}\mathbf{u}^T(N-2)\mathbf{R}\mathbf{u}(N-2) \quad ...(22.18)$$

(22.18) are the admissible costs for *N-2* since these costs are optimal from *N-1* onwards. To determine $\mathbf{u}^*(N-2)$, according to (22.6), we must minimise (22.18) in a similar way as before.

If we continue to decrement *k* and apply the optimality principle, the results for each $k = N-1, N-2, \ldots, 1, 0$ are

$$\mathbf{K}(k) = \left(\mathbf{B}^T\mathbf{S}(k+1)\mathbf{B} + \mathbf{R}\right)^{-1}\mathbf{B}^T\mathbf{S}(k+1)\mathbf{A}$$

$$\mathbf{S}(k) = \left[\mathbf{A} - \mathbf{B}\mathbf{K}(k)\right]^T\mathbf{S}(k+1)\left[\mathbf{A} - \mathbf{B}\mathbf{K}(k)\right] + \mathbf{K}^T(k)\mathbf{R}\mathbf{K}(k) + \mathbf{Q}$$

$$\mathbf{u}^*(k) = -\mathbf{K}(k)\mathbf{x}(k);\ J_k^* = \frac{1}{2}\mathbf{x}^T(k)\mathbf{S}(k)\mathbf{x}(k)$$

Example 5.2 Consider

$$\begin{cases} x(k+1) = 2x(k) + u(k) \\ J = \sum_{k=0}^{2}\left(x^2(k) + u^2(k)\right) \end{cases}$$

$$\Rightarrow A = 2; B = 1; Q = 2; R = 2; S(3) = 0$$

$$K(2) = \left(B^T S(3) B + R\right)^{-1} B^T S(3) A = 0$$

$$S(2) = \left[A - BK(2)\right]^T S(3)\left[A - BK(2)\right] + K^T(2)RK(2) + Q = 2$$

$$K(1) = \left(B^T S(2) B + R\right)^{-1} B^T S(2) A = 1;$$

$$S(1) = 6;$$

$$K(0) = 1.5;$$

$$S(0) = 8$$

The optimal gain schedule is

$$\{K(0) \quad K(1)\} = \{1.5 \quad 1\}$$

And, the minimum cost (from slide 31) is

$$J = \frac{1}{2} x^T(0) S(0) x(0) = 4x^2(0)$$

23 Solution of the discrete Riccati difference equation

The difference equations employed in the design of optimal control systems are :

$$\mathbf{K}(k) = \left(\mathbf{B}^T \mathbf{S}(k+1) \mathbf{B} + \mathbf{R}\right)^{-1} \mathbf{B}^T \mathbf{S}(k+1) \mathbf{A} \qquad \text{...(23.1a)}$$

$$\mathbf{S}(k) = \left[\mathbf{A} - \mathbf{B}\mathbf{K}(k)\right]^T \mathbf{S}(k+1) \left[\mathbf{A} - \mathbf{B}\mathbf{K}(k)\right] + \mathbf{K}^T(k) \mathbf{R} \mathbf{K}(k) + \mathbf{Q} \qquad \text{...(23.1}b\text{)}$$

Substitute (23.1a) into (23.1b),

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$$\mathbf{S}(k) = \mathbf{A}^T\mathbf{S}(k+1)\mathbf{A} + \mathbf{Q} - \mathbf{A}^T\mathbf{S}(k+1)\mathbf{B}\left[\mathbf{B}^T\mathbf{S}(k+1)\mathbf{B} + \mathbf{R}\right]^{-1} \bullet \mathbf{B}^T\mathbf{S}(k+1)\mathbf{A} \qquad ...(23.2)$$

(23.2) is the discrete Riccati equation.

For the infinite time problem as $N \to \infty$, the following conditions are required for the asymptotically stability of the closed-loop system :

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The pairs $(\mathbf{A}, \mathbf{B})$ and $(\mathbf{A}, \mathbf{C})$ must be completely controllable and observable, respectively, for any $(n \text{ x } n)$ matrix $\mathbf{C}$ such that $\mathbf{C}\mathbf{C}^T = \mathbf{Q}$.
Through some lengthy manipulations involving Hamiltonians, generalised eigenvalues, etc., for the steady state solution as $N \to \infty$ such that the gains in $\mathbf{K}(k)$ have become constant values, it must then be true in (23.2) that $\mathbf{S}(k) = \mathbf{S}(k+1)$ = constant = $\hat{\mathbf{S}}$.

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(23.2) then becomes

$$\hat{\mathbf{S}} = \mathbf{A}^T\hat{\mathbf{S}}\mathbf{A} + \mathbf{Q} - \mathbf{A}^T\hat{\mathbf{S}}\mathbf{B}\left[\mathbf{B}^T\hat{\mathbf{S}}\mathbf{B} + \mathbf{R}\right]^{-1}\mathbf{B}^T\hat{\mathbf{S}}\mathbf{A} \qquad ...(23.4)$$

(23.4) is the discrete algebraic Riccati equation.

The solution $\hat{\mathbf{S}}$ can be found by recursion or by the eigenvalue-eigenvector method.

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The recursive method is discussed here : Set *N* to a large value and recursively calculate the matrix elements of $\hat{\mathbf{S}}$ (using a computer) until they are constant values. The computer solution requires setting a tolerance level ε, so that the difference between every element in $\mathbf{S}(k)$ and the corresponding element of $\mathbf{S}(k+1)$ is < ε.

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Example 5.3

$$\mathbf{x}(k+1)=\begin{bmatrix}0 & 1\\ -1 & 1\end{bmatrix}\mathbf{x}(k)+\begin{bmatrix}0\\ 1\end{bmatrix}u(k);\mathbf{x}(0)=\begin{bmatrix}1\\ 1\end{bmatrix}$$

Find the optimal control $u^*(k)$ such that the following performance index is minimised,

$$J=\sum_{k=0}^{7}\left(x_1^2(k)+u^2(k)\right)$$

$$\Rightarrow N=8;\mathbf{Q}=\begin{bmatrix}2 & 0\\ 0 & 0\end{bmatrix};R=2;\mathbf{S}(8)=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

(23.1a) and (23.1b) are solved recursively.

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$$\mathbf{S}(7)=\begin{bmatrix}2 & 0\\ 0 & 0\end{bmatrix};\qquad \mathbf{K}(7)=\begin{bmatrix}0 & 0\end{bmatrix}$$

$$\mathbf{S}(6)=\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix};\qquad \mathbf{K}(6)=\begin{bmatrix}0 & 0\end{bmatrix}$$

$$\mathbf{S}(5)=\begin{bmatrix}3 & -1\\ -1 & 3\end{bmatrix};\qquad \mathbf{K}(5)=\begin{bmatrix}-0.5 & 0.5\end{bmatrix}$$

$$\mathbf{S}(4)=\begin{bmatrix}3.2 & -0.8\\ -0.8 & 3.2\end{bmatrix};\qquad \mathbf{K}(4)=\begin{bmatrix}-0.6 & 0.4\end{bmatrix}$$

$$\mathbf{S}(3)=\begin{bmatrix}3.23 & -0.922\\ -0.922 & 3.69\end{bmatrix};\qquad \mathbf{K}(3)=\begin{bmatrix}-0.615 & 0.462\end{bmatrix}$$

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$$\mathbf{S}(2) = \begin{bmatrix} 3.297 & -0.973 \\ -0.973 & 3.729 \end{bmatrix}; \quad \mathbf{K}(2) = \begin{bmatrix} -0.651 & 0.481 \end{bmatrix}$$

$$\mathbf{S}(1) = \begin{bmatrix} 3.301 & -0.962 \\ -0.962 & 3.75 \end{bmatrix}; \quad \mathbf{K}(1) = \begin{bmatrix} -0.652 & 0.485 \end{bmatrix}$$

$$\mathbf{S}(0) = \begin{bmatrix} 3.305 & -0.97 \\ -0.97 & 3.777 \end{bmatrix}; \quad \mathbf{K}(0) = \begin{bmatrix} -0.6538 & 0.486 \end{bmatrix}$$

$$u^*(k) = -\mathbf{K}(k)\mathbf{x}(k); \mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow u^*(0) = -\mathbf{K}(0)\mathbf{x}(0) = 0.1678$$

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$$\Rightarrow \mathbf{x}^*(1) = \mathbf{A}\mathbf{x}(0) + \mathbf{B}u(0) = \begin{bmatrix} 1 \\ 0.1678 \end{bmatrix}$$

$$\Rightarrow u^*(1) = -\mathbf{K}(1)\mathbf{x}(1) = 0.5708$$

$$\Rightarrow \mathbf{x}^*(2) = \mathbf{A}\mathbf{x}^*(1) + \mathbf{B}u^*(1) = \begin{bmatrix} 0.1678 \\ -0.2614 \end{bmatrix}$$

$$\Rightarrow u^*(2) = -\mathbf{K}(2)\mathbf{x}^*(2) = 0.235$$

k	0	1	2	3	4
$u^*(k)$	0.1678	0.5708	0.235	-0.071	-0.115

k	5	6	7
$u^*(k)$	0.0358	0	0

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$$\mathbf{S} = \begin{bmatrix} 3.308 & -0.972 \\ -0.972 & 3.78 \end{bmatrix} > 0; \mathbf{K} = \begin{bmatrix} -0.654 & 0.486 \end{bmatrix}$$

For large *N*, it can be shown that the Riccati equation and the constant optimal control approaches the above steady-state solutions.

For *N* = 8, the finite-time problem already has solutions rapidly approaching these values.

Note that the pair $(\mathbf{A}, \mathbf{B})$ is completely controllable and there is a (2 x 2) matrix $\mathbf{C}$ such that $\mathbf{C}\mathbf{C}^T = \mathbf{Q}$ e.g. $\mathbf{C} = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix}$ and the pair $(\mathbf{A}, \mathbf{C})$ is completely observable. The closed-loop system is asymptotically stable as $N \to \infty$.

Example 5.4 An infinite time problem

$$\mathbf{x}(k+1)=\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}\mathbf{x}(k)+\begin{bmatrix}1\\ 0\end{bmatrix}u(k) \qquad ...(23.5a)$$

$$J=\frac{1}{2}\sum_{k=0}^{\infty}\left(\mathbf{x}^T(k)\mathbf{Q}\mathbf{x}(k)+ru^2(k)\right)$$

$$\mathbf{Q}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix};r=1;\mathbf{x}(0)=\begin{bmatrix}1\\ 0\end{bmatrix}.$$

Determine the optimal controller and optimal trajectories $\mathbf{x}^*(k)$ for $k=1,2$ and 3.
Comment on the stability of the closed-loop system.

$$\mathbf{A}=\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix};\mathbf{B}=\begin{bmatrix}1\\ 0\end{bmatrix}; \qquad u(k)=-\mathbf{K}\mathbf{x}(k)$$

where $\mathbf{K}=\left(\mathbf{B}^T\mathbf{S}\mathbf{B}+\mathbf{R}\right)^{-1}\mathbf{B}^T\mathbf{S}\mathbf{A}$

with $\mathbf{R}=r=1,$ and $\mathbf{S}>0$ solves,

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$$\mathbf{S} = \mathbf{A}^T\mathbf{S}\mathbf{A} + \mathbf{Q} - \mathbf{A}^T\mathbf{S}\mathbf{B}\left[\mathbf{B}^T\mathbf{S}\mathbf{B} + \mathbf{R}\right]^{-1}\mathbf{B}^T\mathbf{S}\mathbf{A} \quad ...(23.5)$$

Let $\mathbf{S} = \begin{bmatrix} S_{11} & S_{12} \\ S_{12} & S_{22} \end{bmatrix}$. $(23.5) \Rightarrow \begin{bmatrix} S_{11} & S_{12} \\ S_{12} & S_{22} \end{bmatrix}$

$$= \begin{bmatrix} \left(S_{11} + 2S_{12} + S_{22} + 1\right) & \left(S_{11} + S_{12}\right) \\ \left(S_{11} + S_{12}\right) & \left(S_{11} + 1\right) \end{bmatrix}$$

$$- \frac{1}{\left(S_{11} + 1\right)} \begin{bmatrix} \left(S_{11} + S_{12}\right)^2 & S_{11}\left(S_{11} + S_{12}\right) \\ S_{11}\left(S_{11} + S_{12}\right) & \left(S_{11}\right)^2 \end{bmatrix}$$

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$$S_{11} = S_{11} + 2S_{12} + S_{22} + 1 - \frac{\left(S_{11} + S_{12}\right)^2}{\left(S_{11} + 1\right)} \quad ...(23.6)$$

$$S_{12} = S_{11} + S_{12} - \frac{S_{11}\left(S_{11} + S_{12}\right)}{\left(S_{11} + 1\right)} \quad ...(23.7)$$

$$S_{22} = S_{11} + 1 - \frac{\left(S_{11}\right)^2}{\left(S_{11} + 1\right)} \quad ...(23.8)$$

$(23.7) \quad \Rightarrow \quad S_{12} = 1$

$(23.6) \quad \Rightarrow S_{22} = S_{11} - 2 \quad ...(23.9)$

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Substitute (23.9) into (23.8),

$$S_{11}^2 - 3S_{11} - 3 = 0 \Rightarrow S_{11} = 3.7913; -0.7913$$

Choose $S_{11} = 3.7913$.

$$\Rightarrow \begin{bmatrix} S_{11} & S_{12} \\ S_{12} & S_{22} \end{bmatrix} = \begin{bmatrix} 3.7913 & 1 \\ 1 & 1.7913 \end{bmatrix} > 0$$

$$\Rightarrow \mathbf{K} = \left(\mathbf{B}^T \mathbf{S}\mathbf{B} + \mathbf{R}\right)^{-1} \mathbf{B}^T \mathbf{S}\mathbf{A} = \begin{bmatrix} 1 & 0.7913 \end{bmatrix}$$

$$\Rightarrow u(k) = -\begin{bmatrix} 1 & 0.7913 \end{bmatrix} \mathbf{x}(k) \qquad \text{... (23.10)}$$

$$\mathbf{x}(k+1) = \left[\mathbf{A} - \mathbf{B}\mathbf{K}\right]\mathbf{x}(k) = \begin{bmatrix} 0 & 0.2087 \\ 1 & 0 \end{bmatrix} \mathbf{x}(k)$$

$$\lambda_i \begin{bmatrix} 0 & 0.2087 \\ 1 & 0 \end{bmatrix} = \pm 0.4568$$

The closed-loop state equation is given and the closed-loop system is stable.

Also, the optimal trajectories are,

$$\mathbf{x}(N) = \left[\mathbf{A} - \mathbf{BK}\right]^N \mathbf{x}(0)$$

$$\Rightarrow \mathbf{x}(1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \mathbf{x}(2) = \begin{bmatrix} 0.2807 \\ 0 \end{bmatrix}; \mathbf{x}(3) = \begin{bmatrix} 0 \\ 0.2807 \end{bmatrix}$$

$$\mathbf{x}(30) = \begin{bmatrix} 0.6206 \text{ x } 10^{-10} \\ 0 \end{bmatrix}, \mathbf{x}(N) \to \mathbf{0} \text{ as } N \to \infty$$

Use (23.5a) and (23.10) iteratively to compute the optimal trajectories $\mathbf{x}^*(k)$ with given $\mathbf{x}(0)$.