Finite Element Method for Stokes Flow in a 2D Channel

Abstract

This document provides a tutorial-style introduction to the Finite Element Method (FEM) as applied to incompressible, viscous Stokes flow in a 2D channel under the influence of a pressure gradient. It presents theoretical formulations, the concept of mesh generation, shape functions, and multiple approaches to FEM including the Direct Matrix Method, the Weighted Residuals (Galerkin) Method, and the Variational (Rayleigh-Ritz) Method.

1 Introduction: Stokes Flow

Stokes flow refers to low Reynolds number flow dominated by viscous forces. The governing equations for incompressible Stokes flow (assuming steady-state and neglecting body forces), and after some slight rearrangement are:

$$\mu \nabla^2 \mathbf{u} = \nabla p, \quad \text{in } \Omega \tag{1}$$

$$\tag{2}$$

$$\nabla \cdot \mathbf{u} = 0, \quad \text{in } \Omega \tag{3}$$

where:

- $\mathbf{u} = (u, v)^T$ is the velocity field,
- p is the pressure,
- μ is the dynamic viscosity.

We assume a pressure gradient $\frac{dp}{dz}$ driving the flow through a 2D channel. This gradient is projected as a forcing term in the momentum equations.

2 Mesh Generation in FEM

The domain Ω is partitioned into smaller subdomains called **finite elements** (typically triangles or quadrilaterals in 2D). This process is called **meshing**. Each node in the mesh corresponds to a point where the unknowns (e.g., velocity, pressure) are computed.

- Each element has its own local coordinate system.
- The global problem is constructed by assembling contributions from all elements.
- The quality and refinement of the mesh influence the accuracy of the solution.

3 Shape Functions

Within each element, the unknowns are approximated using **shape functions**. These are interpolation functions defined over the element:

$$\mathbf{u}_h(x, y) = \sum_{i=1}^{N_u} \mathbf{N}_i(x, y)\, \mathbf{u}_i, \quad p_h(x, y) = \sum_{j=1}^{N_p} \phi_j(x, y)\, p_j$$

- $\mathbf{N}_i$: vector-valued shape functions for velocity.
- ϕ_j: scalar shape functions for pressure.
- N_u, N_p: number of velocity and pressure degrees of freedom per element.

A popular and stable element pair is the Taylor-Hood element: quadratic (P_2) for velocity, linear (P_1) for pressure.

4 1. Direct Matrix Method

This method directly constructs the global system from the discretized weak form. For Stokes flow, the system has a block structure:

$$\begin{bmatrix} \mathbf{K} & \mathbf{G}^T \\ \\ \mathbf{G} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_p \\ \\ \mathbf{0} \end{bmatrix}$$

Where:

- $\mathbf{K}$ is the stiffness matrix from $\mu \nabla^2 \mathbf{u}$,
- $\mathbf{G}$ is the divergence matrix enforcing incompressibility,
- $\mathbf{f}_p$ arises from the pressure gradient: $\mathbf{f}_p = \int_\Omega \left(\frac{1}{\mu} \frac{dp}{dz} \right) \boldsymbol{N}_i \, d\Omega$.

5 2. Weighted Residuals (Galerkin) Method

We begin by forming the weak form of the governing equations:

Multiply Eq. 1. by test function $\boldsymbol{v}$ and integrate:

$$\int_\Omega \mu \nabla \mathbf{u} : \nabla \boldsymbol{v} \, d\Omega - \int_\Omega p(\nabla \cdot \boldsymbol{v}) \, d\Omega = \int_\Omega \left(\frac{dp}{dz} \right) \cdot \boldsymbol{v} \, d\Omega$$

$$\int_\Omega q(\nabla \cdot \mathbf{u}) \, d\Omega = 0$$

These expressions define the bilinear and linear forms for FEM discretization. The shape functions are substituted into these integrals to construct the matrix system.

6 3. Rayleigh-Ritz (Variational) Approach

We define a functional $\Pi[\mathbf{u}, p]$ whose minimization yields the solution:

$$\Pi[\mathbf{u}, p] = \frac{1}{2} \int_\Omega \mu \nabla \mathbf{u} : \nabla \mathbf{u} \, d\Omega - \int_\Omega \left(\frac{dp}{dz} \right) \cdot \mathbf{u} \, d\Omega + \int_\Omega p \, \nabla \cdot \mathbf{u} \, d\Omega$$

Stationarity conditions with respect to $\delta \mathbf{u}$ and δp yield the same weak form as in the Galerkin method.

7 Boundary Conditions

To solve the problem, appropriate boundary conditions are required:

- **No-slip on walls**: $\mathbf{u} = \mathbf{0}$ on boundaries.
- **Pressure or velocity** specified at inlet/outlet to drive the flow.

8 Conclusion

The Finite Element Method is a powerful tool for simulating incompressible flows such as Stokes flow in a 2D channel. By creating a mesh, choosing appropriate shape functions, and formulating the problem using one of the three classical FEM approaches, one can obtain accurate approximations of fluid motion under a pressure gradient. This framework generalizes well to more complex Navier-Stokes problems and 3D domains.