Quantum gates and notation

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pyQuil & QUIL: Low-level quantum programming

Bloch sphere | Quantum gates | Quantum circuits | Notation

A two-dimensional quantum bit revisited

Bloch sphere | Quantum gates | Quantum circuits | Notation

We can write a two-dimensional **qubit** as the vector:

Global phase

$$|\psi\rangle = e^{i\gamma}\left(\alpha\,|0\rangle + \beta\,|1\rangle\right) = e^{i\gamma}\begin{bmatrix}\alpha \\ \beta\end{bmatrix}$$

Amplitudes

Ground state

Excited state

A two-dimensional quantum bit revisited

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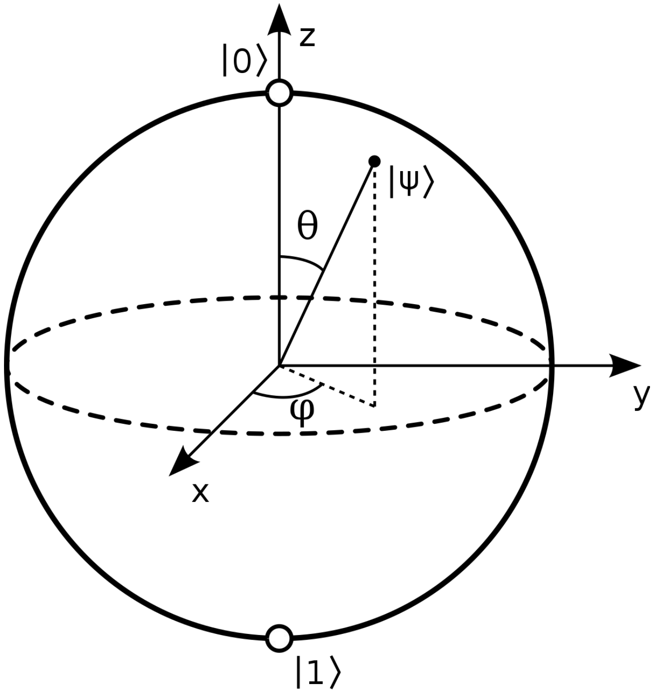
We can **always** drop the **global phase** since it's immeasurable:

~~Global phase~~

$$|\psi\rangle = \cancel{e^{i\gamma}}\left(\alpha\,|0\rangle + \beta\,|1\rangle \right) = \cancel{e^{i\gamma}} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$$\left| e^{i\gamma}\alpha \right|^2 = e^{i\gamma}\alpha e^{-i\gamma}\alpha^\dagger = e^{i\gamma - i\gamma}\alpha\alpha^\dagger = \alpha\alpha^\dagger$$

Bloch sphere



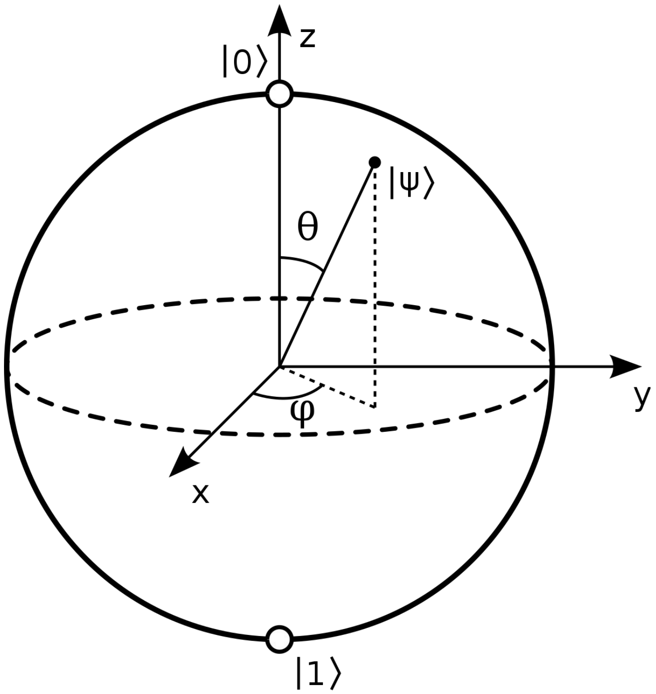
$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$$

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle$$

where $0 \leq \theta \leq \pi$ and $0 \leq \phi < 2\pi$

$$x = \sin(\theta)\cos(\phi)$$
$$y = \sin(\theta)\sin(\phi)$$
$$z = \cos(\theta)$$

Bloch sphere



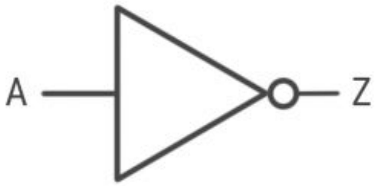
Quantum gates rotate the Bloch vector

The Bloch sphere provides a **visual tool** to understand quantum gate operations.

The Bloch sphere **only applies to single qubit states**. Visualizing multi-qubit states is out of the scope of this tutorial.

Classical NOT Gate

NOT



Input	Output
0	1
1	0

Quantum X (NOT) Gate

Input	Output
0	1
1	0

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$$\underset{}{?} \times \underset{|0\rangle}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} \rightarrow \underset{|1\rangle}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}$$

Single-qubit quantum gates

Quantum X (NOT) Gate

Input	Output
0	1
1	0

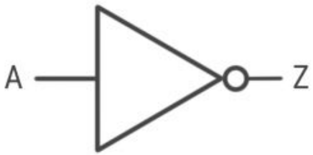
$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \times \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{|0\rangle} \rightarrow \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{|1\rangle}$$

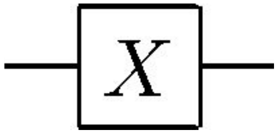
Single-qubit quantum gates

X (NOT) Gate

NOT → X



→



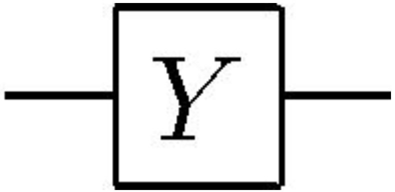
$$\mathbf{X}\,|0\rangle = |1\rangle$$

$$\mathbf{X}\,|1\rangle = |0\rangle$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \times \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \beta \\ \alpha \end{bmatrix}$$

Single-qubit quantum gates

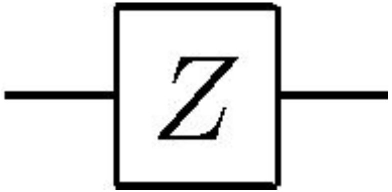
Pauli Y



$$Y|0\rangle = i|1\rangle$$
$$Y|1\rangle = -i|0\rangle$$

$$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \times \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} -i\beta \\ i\alpha \end{bmatrix}$$

Pauli Z



$$Z|0\rangle = |0\rangle$$

$$Z|1\rangle = -|1\rangle$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \times \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha \\ -\beta \end{bmatrix}$$

Properties of single-qubit gates

$$|\langle \Psi \mid \Psi \rangle|^2 = 1$$

The **Bohr rule** states that the inner product **must equal to unity.**

Properties of single-qubit gates

$$|\Psi'\rangle = U\,|\Psi\rangle$$

$$|\langle \Psi' \mid \Psi' \rangle|^2 = 1$$

Quantum gates **must preserve normalization.**

Properties of single-qubit gates

Bloch sphere | **Quantum gates** | Quantum circuits | Notation

$$|\Psi'\rangle = U\,|\Psi\rangle \qquad\qquad |\langle\Psi' \mid \Psi'\rangle|^2 = 1$$

$$|\langle\Psi|\,U^\dagger U\,|\Psi\rangle|^2 = 1$$

$$U^\dagger U = \mathbb{I}$$

The only requirement for a quantum gate is **unitarity.**

Hence, there are infinitely many quantum gates!

Properties of single-qubit gates

Quantum gates must be **unitary:**

$$UU^\dagger = U^\dagger U = \mathbb{I}$$

Example:

$$S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \qquad S^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}$$

Hermitian conjugate

$$SS^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -i^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbb{I}$$

Properties of single-qubit gates

Some quantum gates (e.g. Pauli gates) are **hermitian:**

$$U = U^\dagger$$

Example:

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = X^\dagger = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Elementary single-qubit gates

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Hadamard	$-\boxed{H}-$	$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$	from pyquil.gates import H
Pauli-X	$-\boxed{X}-$	$\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$	from pyquil.gates import X
Pauli-Y	$-\boxed{Y}-$	$\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}$	from pyquil.gates import Y
Pauli-Z	$-\boxed{Z}-$	$\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$	from pyquil.gates import Z
Phase	$-\boxed{S}-$	$\begin{bmatrix}1 & 0\\ 0 & i\end{bmatrix}$	from pyquil.gates import S
$\pi/8$	$-\boxed{T}-$	$\begin{bmatrix}1 & 0\\ 0 & e^{i\pi/4}\end{bmatrix}$	from pyquil.gates import T

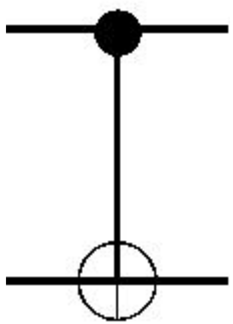
\+ **many more!**

A lot of quantum gates are **already implemented** in pyQuil.

Multi-qubit quantum gates

Controlled-NOT (CNOT) gate

Control qubit



Target qubit

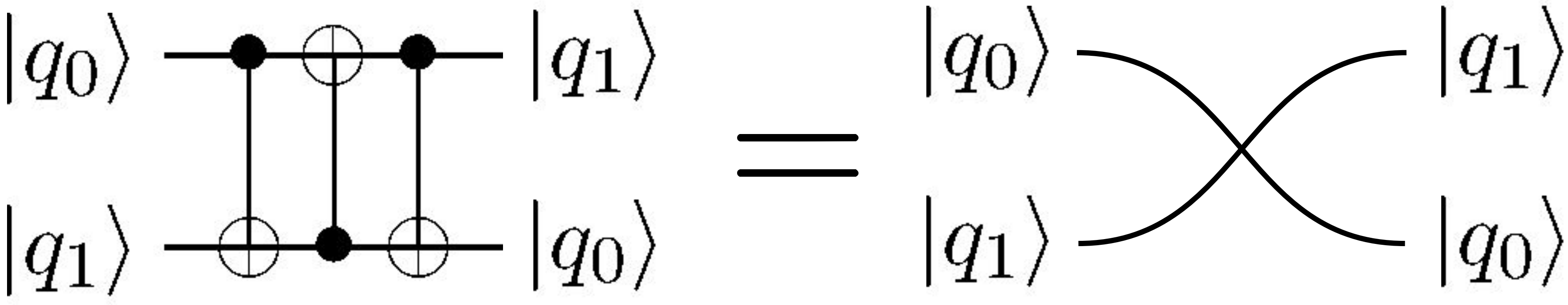
```
from pyquil.gates import CNOT
```

Target qubit	Control qubit	Output
0	0	00
0	**1**	**11**
1	0	10
1	**1**	**01**

Essential for **entangling** two qubits!

Multi-qubit quantum gates

SWAP gate

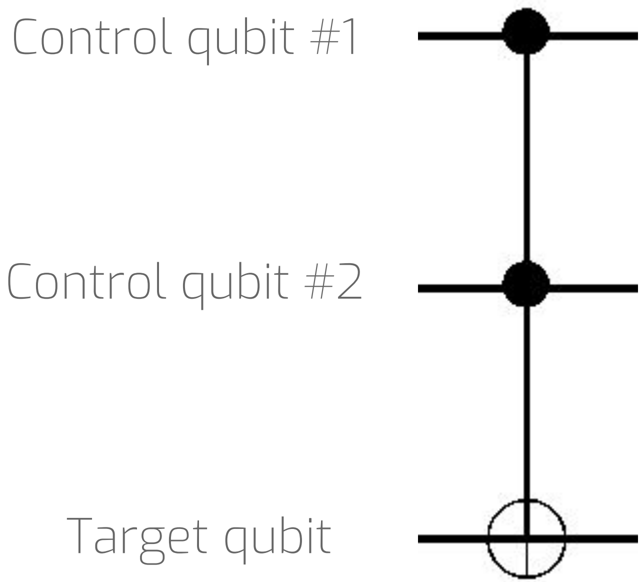



```python
from pyquil.gates import SWAP
```

The SWAP operation allows us to **move qubits on the quantum processor**!

Multi-qubit quantum gates

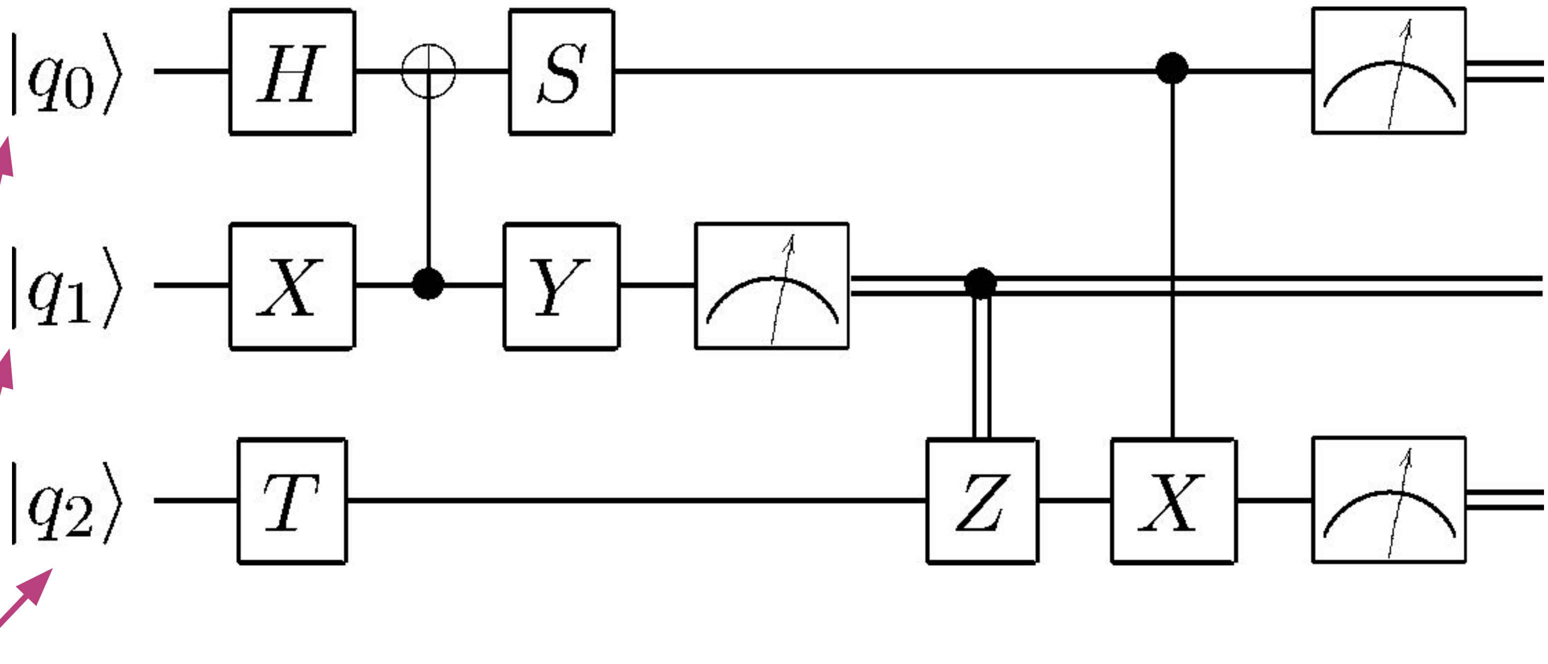
Toffoli (CCNOT) gate



Target qubit	Control qubit #1	Control qubit #2	Output
0	1	0	010
0	0	1	001
0	**1**	**1**	**111**

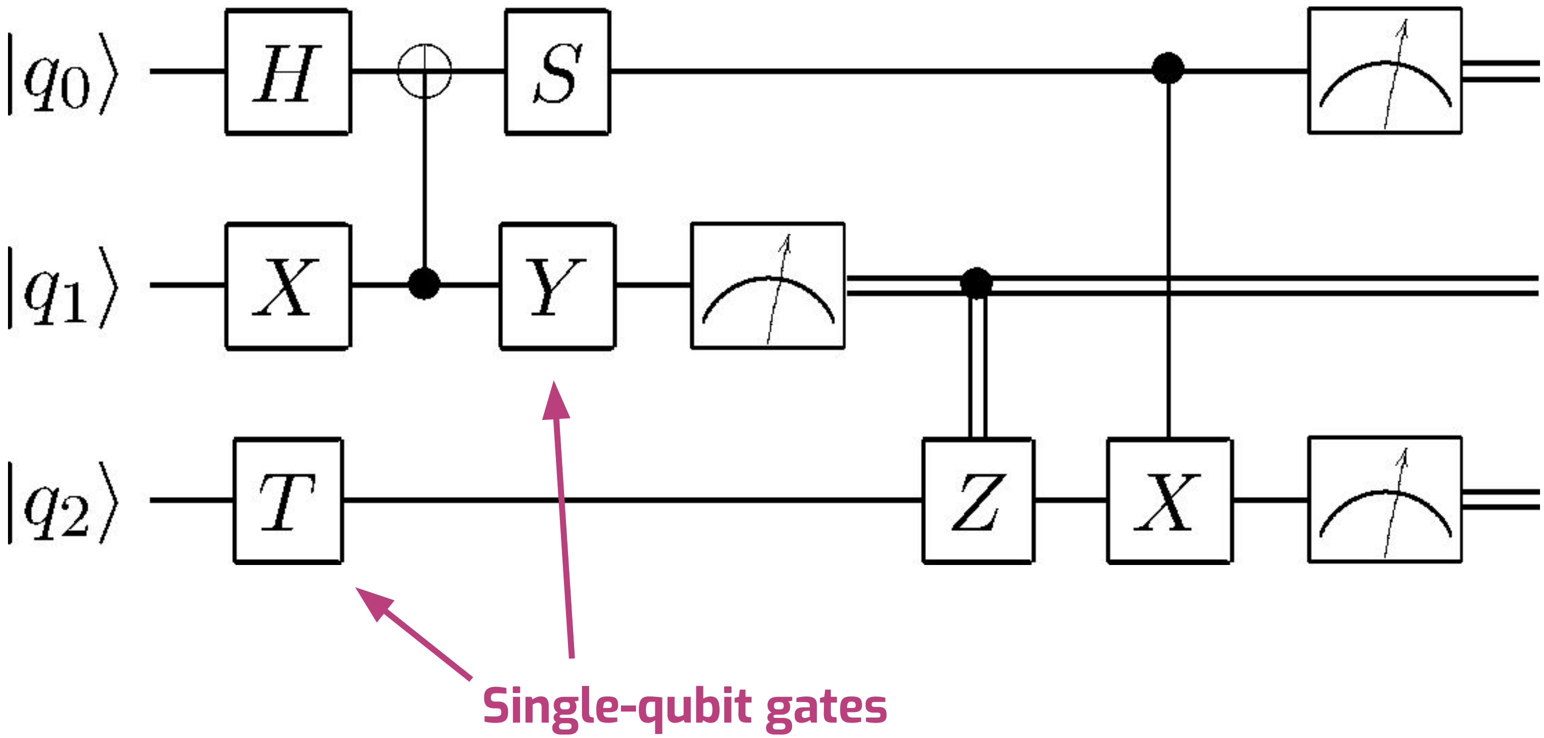
```
from pyquil.gates import CCNOT
```

Quantum circuits

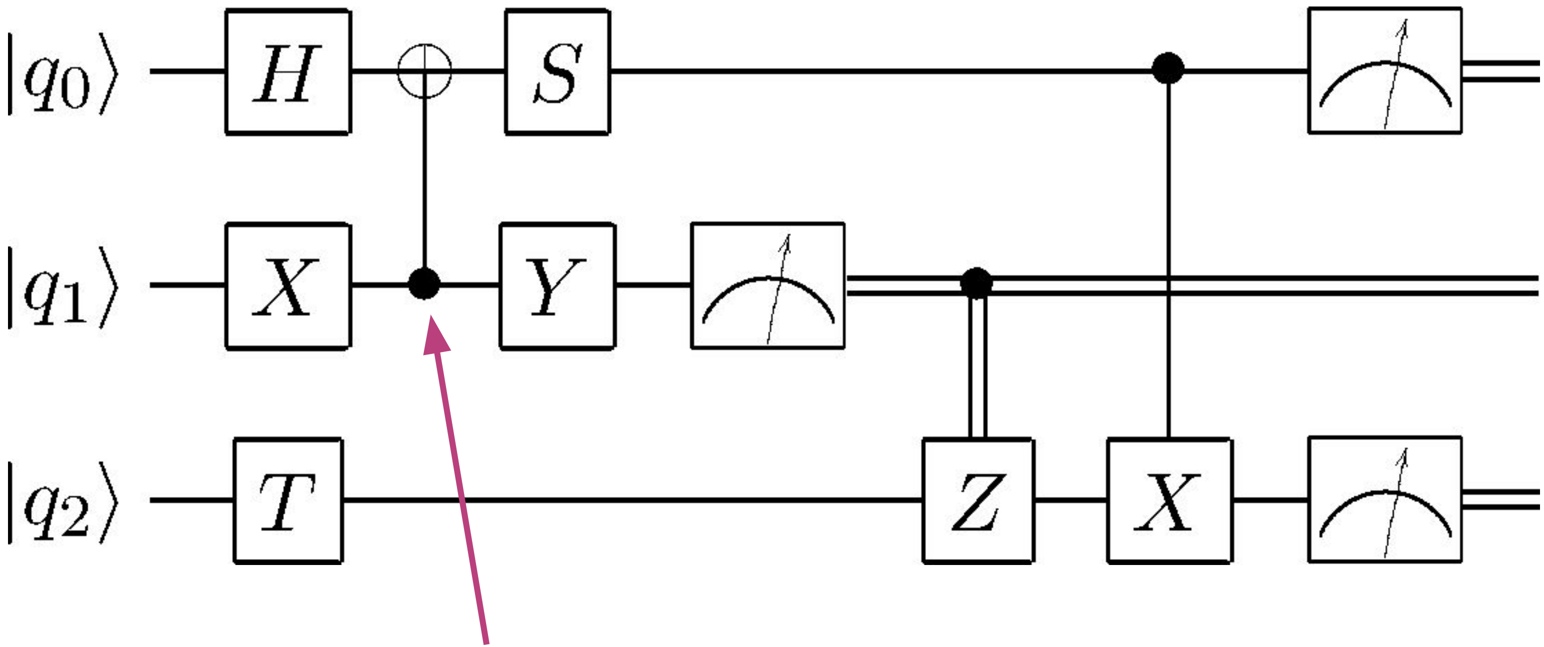


Initial qubit state ($|000\rangle$)

Quantum circuits

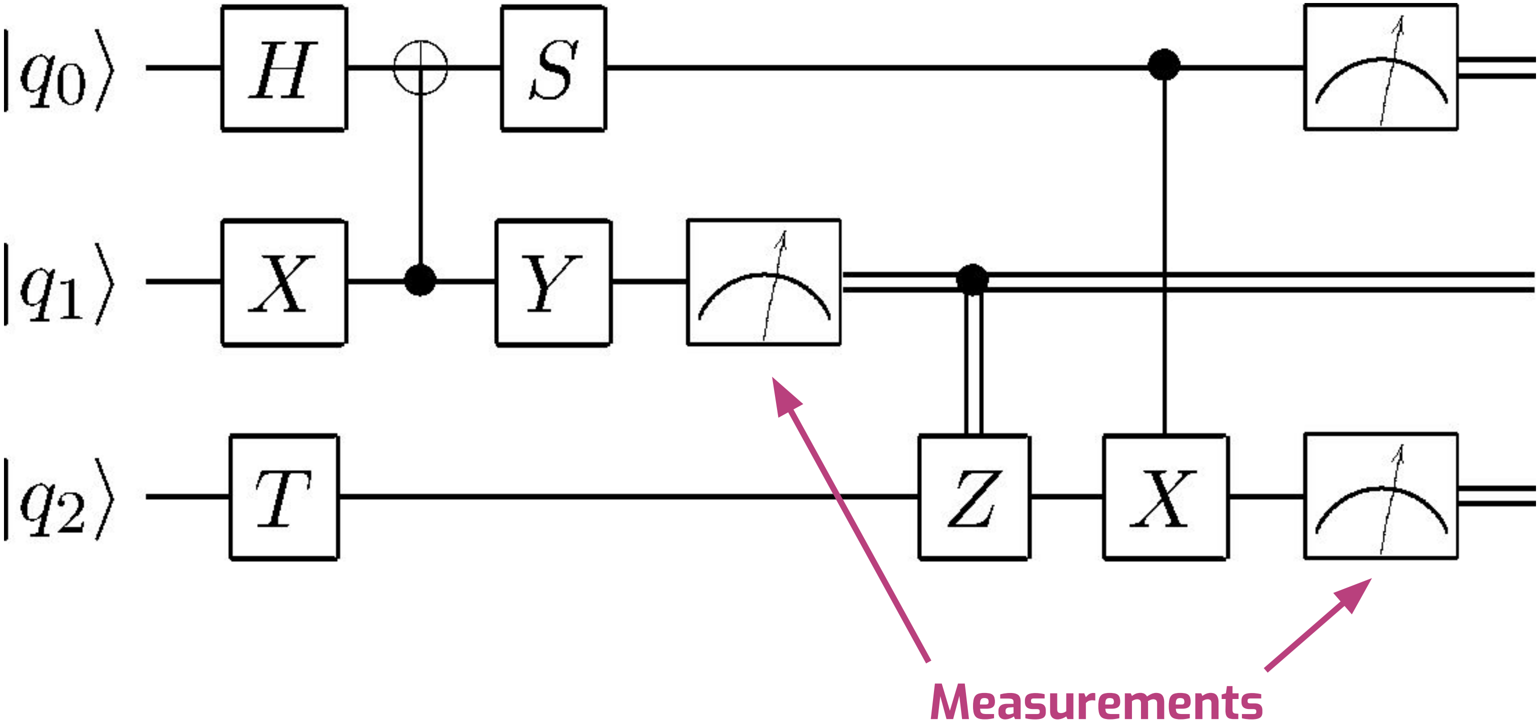


Quantum circuits

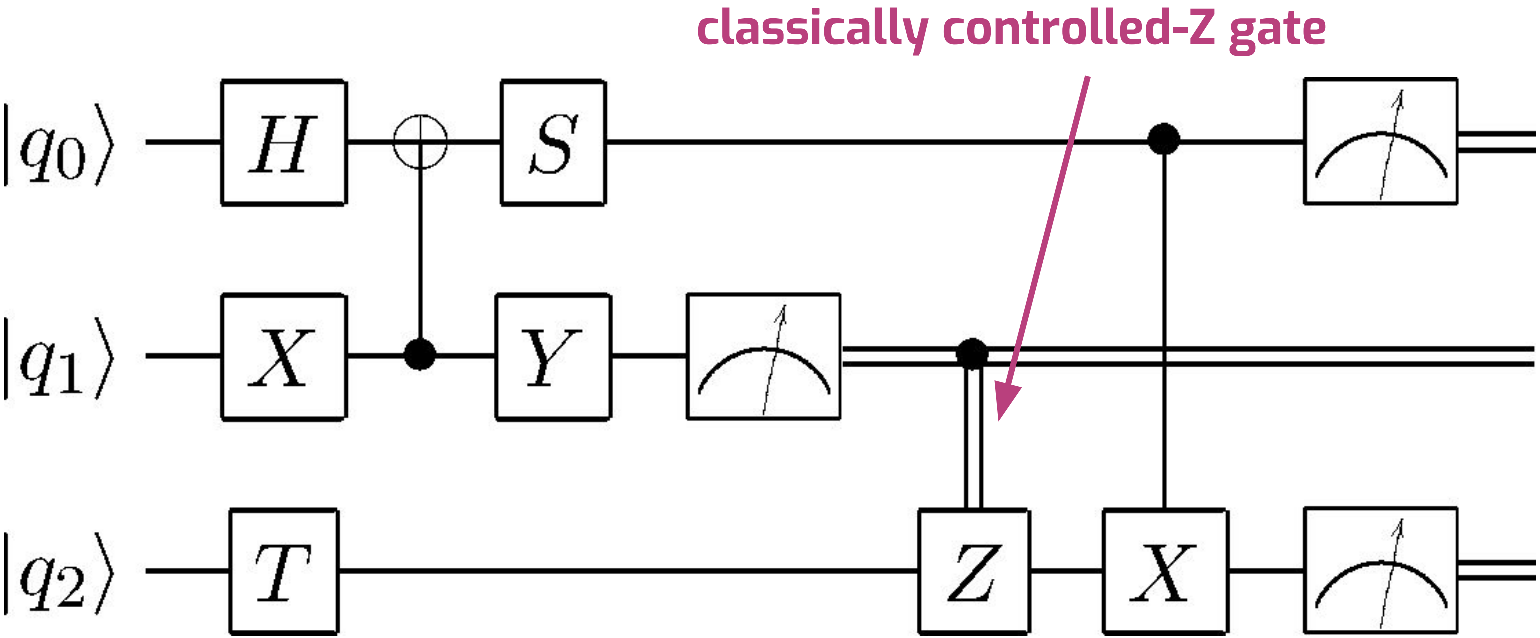


controlled-NOT (CNOT)

Quantum circuits



Quantum circuits



Rigetti's shared memory makes such operations very easy!

Notation

$$|000\rangle$$

Is the **super lazy** physicist's version of

$$|0\rangle\,|0\rangle\,|0\rangle$$

which is the **lazy** physicist's version of

$$|0\rangle \otimes |0\rangle \otimes |0\rangle$$

Notation

$$|\underset{\text{Qubit } \mathbf{2}}{0}\;\underset{\text{Qubit } \mathbf{1}}{0}\;\underset{\text{Qubit } \mathbf{0}}{0}\rangle$$

This is a **common convention** in quantum computing.
You will explore the reasoning behind this in the next tutorial.

Notation

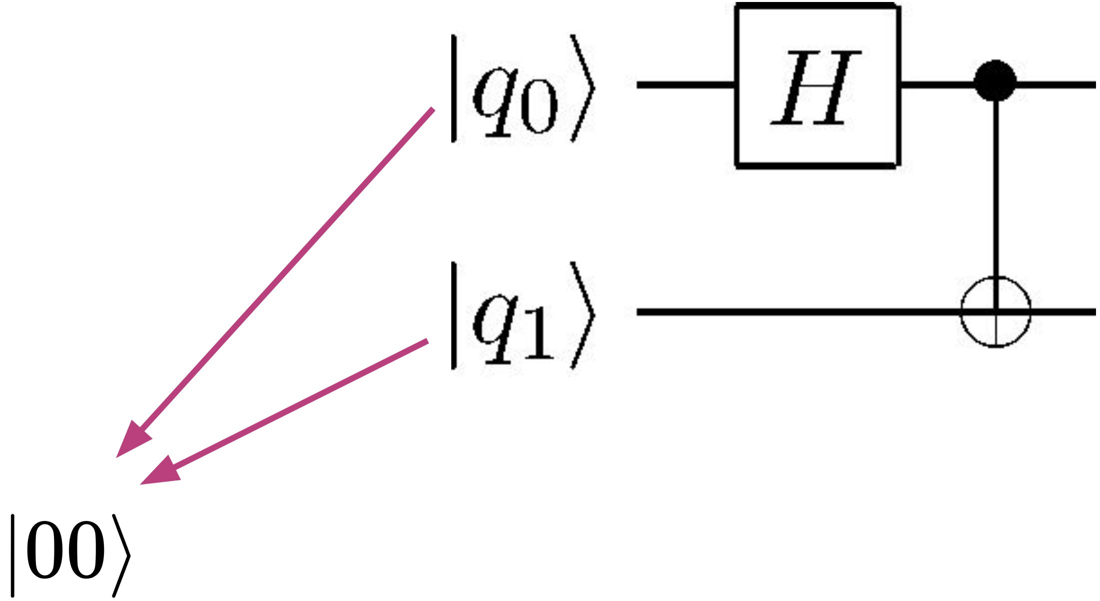
$$Z_0 \otimes Z_1 \otimes Z_2$$

is the **lazy** physicist's version of

$$\sigma_0^Z \otimes \sigma_1^Z \otimes \sigma_2^Z$$

Pen & Paper quantum computing

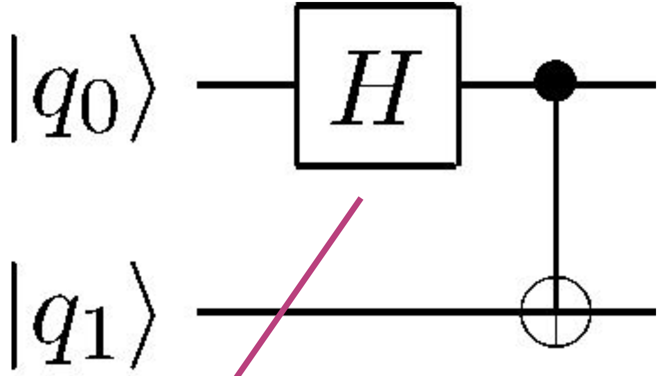
Bloch sphere | Quantum gates | Quantum circuits | Notation



To finish it up, let's put it all together and
walk through a **quantum circuit with pen & paper!**

Pen & Paper quantum computing

Bloch sphere | Quantum gates | Quantum circuits | Notation

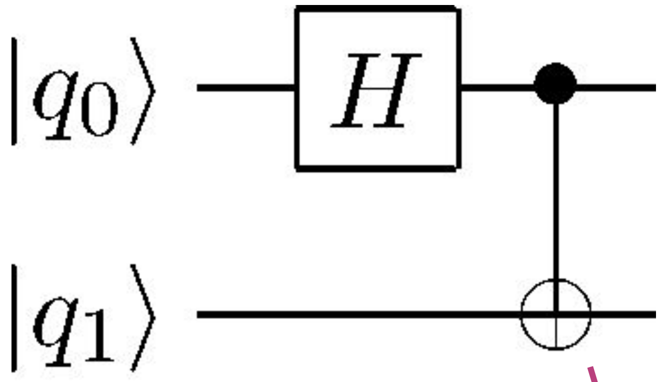


$$|00\rangle \quad \rightarrow \quad (\mathbb{I} \otimes H)\,|00\rangle$$

In a multi-qubit system don't forget the **tensor product with the identity matrix!**

Pen & Paper quantum computing

Bloch sphere | Quantum gates | Quantum circuits | Notation



$$|00\rangle \quad \rightarrow \quad (\mathbb{I} \otimes H)\,|00\rangle \;\rightarrow\; \mathrm{CNOT}(0,1)(\mathbb{I} \otimes H)\,|00\rangle$$

Pay attention to the **order of operations!**

Pen & Paper quantum computing

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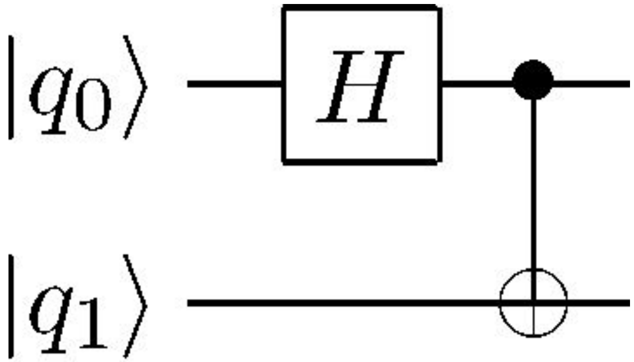

$$\mathrm{CNOT}(0,1)(\mathbb{I}\otimes H)\,|00\rangle = \mathrm{CNOT}(0,1)\left[\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|01\rangle\right]$$

$$= \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle = |\Phi^{+}\rangle$$

This is a maximally entangled state. It's often called the **first Bell state.**

Now, please start working through the exercises in the Jupyter Notebook for **Tutorial 2**