The Credit Card Debt Puzzle

A buffer stock with continuous choices for consumption and borrowing

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Abstract

This paper examines how households in an expanded buffer stock model with a continuous choice for debt can explain the credit card debt puzzle - that some households choose to roll over credit card debt with high interest, while simultaneously holding liquid assets with low returns. Inspired by [Druedahl and Jørgensen, 2018], we test if the risk of becoming credit constrained can motivate the household to participate in precautionary borrowing. Specifically, we investigate whether households with positive net assets exhibit such behavior - the puzzle group. We solve a finite horizon model with backwards induction and perform counterfactual simulations to test the hypothesis. The counterfactual simulations show that the the puzzle group is larger, when the households is at risk of losing credit access, i.e., this model suggests that precautionary borrowing is an explanation to the credit card debt puzzle. Finally, the results are robust to changes in other structural parameters such as the utility discount factor and shock volatility.

Link to our Git repository.

Contents

1 Introduction

The standard buffer stock model in dynamic programming offers a way to study and analyze how households choose to consume during their life-cycle. We expand the model allowing the households to borrow in each period, where borrowing is modelled explicitly as a choice variable. This allows us to explore how credit can be used to smooth consumption over the life-cycle. Furthermore, we add a credit restriction to the model, such that the households risk losing access to credit at any point in time. Together with the income shocks of the classic buffer stock, the households in the model are thus exposed to two sources of income uncertainty. To mitigate these risks, the household could borrow before losing access to credit, i.e., borrow as a precaution. The purpose of this paper is to investigate if the buffer stock model expanded with debt produces this behavior.

To do so, we set up and solve the model from [Druedahl and Jørgensen, 2018], where the phenomenon known as the "credit card debt puzzle" is explored. It describes when some households choose to roll over credit card debt with high interest, while simultaneously holding liquid assets with low returns. These households are referred to as the "puzzle group". [Druedahl and Jørgensen, 2018] propose that this could be explained by the idea that some households participate in precautionary borrowing. As the households at any moment risk credit restrictions or lower income, they can choose to borrow as a precaution to hedge against the possibility of sudden credit loss. Importantly, the debt cannot be required to be repaid immediately, but only over time by the lender through installments. Therefore, the access to credit provides flexibility for economic agents to consumption smooth across time.

This paper aims to formulate and solve the expanded version of the buffer stock model by [Druedahl and Jørgensen, 2018] and find indications of precautionary borrowing for a puzzle group. We solve the model by value function iterations (VFI) and also attempt nested value function iterations (NFVI) as described in [Druedahl, 2021]. Although results from NVFI are limited due to time restrictions, speed comparisons reveal that NVFI is approximately 32 times faster than VFI.

By counterfactual simulations, we find that adding a 10% risk of being credit constrained results in a 9%-point increase in the size of the puzzle group compared to a model without any credit constraints. We find that approximately 5%-points can be attributed to precautionary borrowing. This result is robust to changes in the discount factor and shock sizes: A higher discount factor decreases precautionary borrowing, while increasing shocks sizes increases precautionary borrowing.

2 Model

When solving a buffer stock model, one could consider a finite- or an infinite-horizon case. On the one hand, the finite case includes life-cycle effects such as savings for the future. In the finite case, the choices of the household are affected by the fact that their lifetime is finite. On the other hand, infinite-horizon excludes any of such life cycle effects. When an infinite-horizon model has converged, the future has become so distant that the future utility is no longer considered by the household, and therefore the choice remains the same. At the point of convergence T, all prior periods $t = T-1, T-2, ...$ has the same value and the same choice. For this analysis, we consider the finite life-cycle case.

2.1 Bellman equation

In our consumption-savings model, the households maximize their expected discounted utility over their life-cycle. They achieve this by choosing optimal consumption, C_t, and debt, D_t, in each period, t. We use a constant relative risk aversion (CRRA) utility function:

$$u\left(C_t\right) = \frac{C_t^{1-\rho}}{1-\rho}$$

The optimal consumption and debt in each period correspond to the solution of the Bellman equation. In the Bellman equation, the households maximize their value in period t, $V(\boldsymbol{S}_t)$, given a set of state variables denoted by the vector $\boldsymbol{S}_t$. The value in period t is the sum of the instantaneous utility and discounted expected future utility. That is:

$$V(\boldsymbol{S}_t) = \max_{D_t, C_t} \frac{C_t^{1-\rho}}{1-\rho} + \beta \mathbb{E}[V(\boldsymbol{S}_{t+1})]$$

The state vector, $\boldsymbol{S}_t$, consists of the following.

Four *continuous* state variables:

- The end of period debt, D_{t-1}
- The end of period wealth, W_{t-1}
- The permanent income, P_t
- The market income, Y_t

Two *discrete* state variables:

- An unemployment indicator, $u_t \in \{0, 1\}$
- Credit constraint indicator, $x_t \in \{0, 1\}$

In each period the household choose consumption, C_t, and debt, D_t, which gives the recursive model:

$$V(\boldsymbol{S}_t) = \max_{D_t, C_t} \frac{C_t^{1-\rho}}{1-\rho} + \beta \mathbb{E}[V(\boldsymbol{S}_{t+1})] \tag{2.1}$$

s.t.

$$W_t = (1+r_w) W_{t-1} + Y_t - \underbrace{(r_d + \lambda) D_{t-1}}_{\text{Interest + installment}} + \underbrace{D_t - (1-\lambda) D_{t-1}}_{\text{New debt}} - C_t \tag{2.2}$$

$$M_t = (1+r_w) W_{t-1} + Y_t - (r_d + \lambda) D_{t-1} \tag{2.3}$$

$$N_t = W_t - D_t \tag{2.4}$$

$$D_t \leq \max \left\{ \underbrace{(1-\lambda) D_{t-1}}_{\text{Old contract}} , \underbrace{\mathbf{1}_{x_t=0} \cdot (\eta \cdot N_t + \varphi \cdot P_t)}_{\text{New contract}} \right\} \tag{2.5}$$

$$W_t, D_t, C_t \geq 0 \tag{2.6}$$

ρ is the risk aversion coefficient, β is the discount factor, r_d is the interest rate on debt, r_w is the return on assets, and λ is the minimum installment on the debt. We only consider the case, where the interest on debt is higher than that of assets, as households would otherwise borrow as much as possible. Finally, we have $\eta = 0$ to be able to compare our results to the standard buffer stock model. (2.1) is the Bellman equation, (2.2) the household's budget constraint, (2.3) the current cash on hand, (2.4) the net assets denoted by N_t, (2.5) the credit constraint, and finally (2.6) states that wealth, borrowing and consumption must be non-negative.

2.2 Income shocks

The households face income shocks, which can be both positive/negative and transitory/permanent.

$$\begin{aligned}
Y_{t+1} &= \xi(u_{t+1}) P_{t+1} \\
P_{t+1} &= P_t \Gamma \psi_{t+1} \\
\xi(u_{t+1}) &= \begin{cases} \mu & \text{if } u_{t+1} = 1 \\ \frac{(\epsilon_{t+1} - u_\star \mu)}{(1-u_\star)} & \text{if } u_{t+1} = 0 \end{cases} \\
u_{t+1} &= \begin{cases} 1 & \text{with probability } u_\star \\ 0 & \text{else} \end{cases} \\
\epsilon_{t+1} &\sim \exp \mathcal{N}(-0.5\sigma_\xi^2, \sigma_\xi^2) \\
\psi_{t+1} &\sim \exp \mathcal{N}(-0.5\sigma_\psi^2, \sigma_\psi^2)
\end{aligned}$$

ξ and ψ are transitory and permanent log-normal income shocks respectively with mean 1. $u_\star$ is the unemployment rate, which is assumed constant throughout all periods. If the household becomes unemployed, i.e. $u_t = 1$, then the income is reduced by a factor of μ.

2.3 Borrowing

We base our modeling of the debt on [Druedahl and Jørgensen, 2018], however with some simplifying medications. We assume each household face the same probability, π, of being credit constrained in each period:

$$x_{t+1} = \begin{cases} 1 & \text{with probability } \pi \\ 0 & \text{else} \end{cases}$$

The credit constraint is determined by (2.5). The *old contract* is the maximum level of debt for the credit restricted households, while the *new contract* is only relevant for non-restricted households. The old contract is the existing amount of debt, which the household must make interest payments and installments on. The new contract specifies the amount on new debt a household is allowed to take. Since $\eta = 0$ by assumption, it only depends on a share of the household's permanent income, $\varphi \cdot P_t$. Thus, households with different income need the same share of collateral to borrow and thus have the same gearing.

The lender can at any time refuse the household access to credit but cannot force the household to repay the entire debt at once. This feature allows the debt to provide future liquidity to the households, as it will only be repaid in installments, λ. Therefore, it could in theory be optimal for the household to borrow, even if they have positive net assets, which defines the puzzle group:

$$\text{Puzzle group} \equiv \{D_t^*(\boldsymbol{S_t}) \mid D_t^*(\boldsymbol{S_t}) > 0 \vee W_t > 0\}$$

For $\lambda = 1$, borrowing would imply immediate repayment, where it will never be optimal for the household to borrow, as they would also have to pay interest on top of the installments. Net assets, N_t, would then simply be equal to the wealth, W_t, in this case. As a result, the traditional consumption of the buffer stock model by [Carroll, 1997], is contained within our model. Hence, we can simply reduce our model to only have consumption as the choice.

3 Solution algorithm

To solve the model, we perform value function iterations (VFI) and *nested* value function iterations (NVFI). We first present the case for VFI and later consider the case for NVFI.

3.1 Value function iteration

As our model has a finite horizon, we apply backwards induction to solve the Bellman equation for each of the periods $t = 0, 1, ..., T$ starting from the terminal period T. This procedure is almost equivalent to value function iterations except we stop at $t = 0$. Traditional value function iteration continues until convergence, i.e. when $||V_{t+1} - V_t|| < \epsilon$ for some $\epsilon > 0$.

When formulating and solving models in dynamic programming, there are implications of increasing the amount of state variables. In our case, we face a model with four continuous and two discrete state variables, while also having two shocks. To reduce complexity, we let M_t and D_{t-1} be our primary state variables, due to the household being mainly interested in their cash on hand, when considering future utility. We use W_{t-1} and D_{t-1} to calculate M_t in each period and assume that Y_t is known. In addition, we use the standard trick of normalizing our model variables with respect to P_t. These two choices simplify matters as the state variables, Y and P, then only enters in the expected future value. The normalized variables are denoted by lower cases. Our Bellman equation is now:

$$v\left(d_{t-1}, m_t\right) = \max_{d_t, c_t} u\left(c_t\right) + \beta \mathbb{E}(v\left(d_t, m_{t+1}\right))$$

If we instead wanted to solve the model over an infinite horizon, then the value function would be solved by iterating until reaching the stopping criterion. This would be done by using the contraction properties of the Bellman equation, which then gives the sequence:

$$V^{i+1} = kV^i, \quad i = 0, 1, 2, ...$$

where k is the operator for the contraction mapping which connects V^{i+1} and V^i. Then we start by an initial guess for the value function V_0, where for each iteration $i = 1, ..., n$ we compute for the value function, until we reach the stopping criterion ϵ, i.e. $||V^{i+1} - V^i|| < \epsilon$.

Gauss-Hermite quadrature

To solve our model, we need to choose how to compute the expected future value as it involves integration of the transitory and permanent shocks. We use Gauss-Hermite quadrature to compute the integral numerically, as we have shocks with log-normal distribution. The integral approximation of Gauss-Hermite is exact if the function is a low-order polynomial according to [Judd, 1998].

The numerical integration uses a finite number of nodes and weights for the integral, hence the general problem of numerical integration is the number of weights and nodes needed for the evaluation. When computing the expected value function numerically with n nodes and corresponding weights, we get:

$$\begin{aligned}\mathbb{E}_t\left[v_{t+1}(\boldsymbol{s}_{t+1})\right] &= \mathbb{E}_t\left[v_{t+1}\left(m_{t+1}, d_t, \xi_{t+1}, \psi_{t+1}\right)\right] \\ &= \int v_{t+1}\left(m_{t+1}, d_t, \xi_{t+1}, \psi_{t+1}\right) f\left(\xi_{t+1}, \psi_{t+1}\right) d\xi_{t+1} d\psi_{t+1} \\ &\approx \sum_{i=1}^{n} w_i v_{t+1}\left(m_{t+1}, d_t, \xi_i, \psi_i\right)\end{aligned}$$

We choose 8 nodes for the integral[1]. However, there is no general rule for the amount of nodes, but the higher the number of nodes, the slower computation time, as v_{t+1} must be evaluated at more points. However, per [Judd, 1998], for a function which is monotonically increasing with small high order derivatives, then the few points likely suffice.

Using Gauss-Hermite quadrature, the discretized shocks will be given by:

$$\Xi \otimes \Psi = \{\psi_1, \psi_2, ..., \psi_n\} \otimes \{\xi_1, \xi_2, ..., \xi_n\}$$

The sum of the weights, i.e. the probabilities, must sum to 1:

$$\sum_{\Xi \otimes \Psi} = \sum_{(\psi,\xi) \in \Xi \otimes \Psi} p\left(\psi, \xi\right) = 1$$

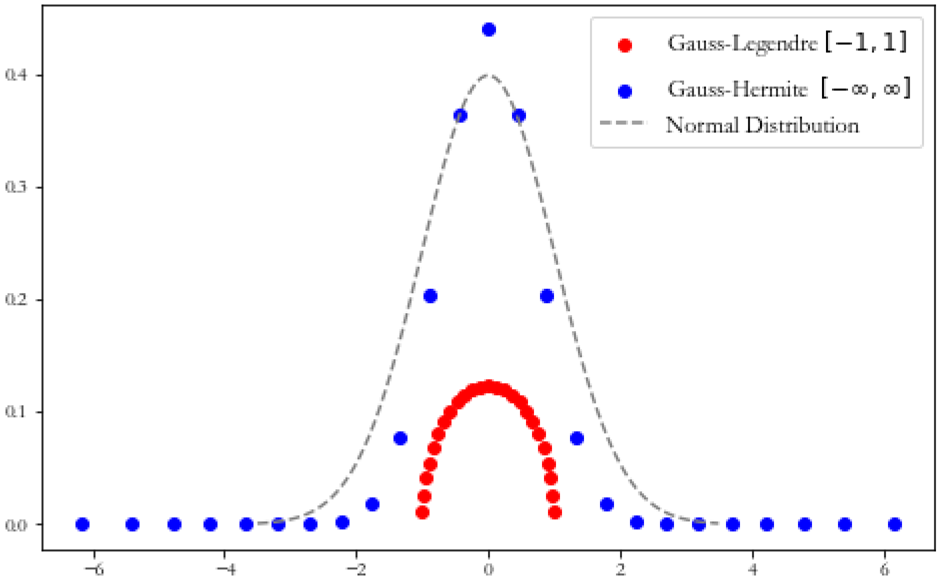
We could also have opted for Gauss-Legendre quadrature for the numerical integration, as they are useful for approximating many kinds of functions. However, as our shocks are assumed Gaussian distributed, it is natural to use the Hermite polynomials designed for Gaussian distributions instead of the more general Legendre approximation. The Gauss-Legendre quadrature has the weighting function given by $w\left(x\right) = 1$ and has a finite range $[-1, 1]$, while the Gauss-Hermite follows weights according to the normal distribution with $w\left(x\right) = e^{-x^2}$ and has the infinite range $[-\infty, \infty]$. Hence, we see that the weighting function for the Gauss-Hermite quadrature matches to the probability density function of the normal distribution:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

As we can see in Figure 1, Gauss-Hermite quadrature is a better fit for our log-normal shocks.

[1] We have used 8 nodes for the quadrature in the covered material and in [Druedahl and Jørgensen, 2018].

Figure 1: Quadrature nodes vs normal distribution



Note: We use 25 quadrature to highlight the differences.

Linear interpolation

When solving for the expected future value, v_{t+1}, we need to interpolate the state variables, m_{t+1}, and d_t. Interpolation, when chosen correctly, will reduce computational time and increase the model's accuracy by requiring less grid points for the approximation of the function.

We proceed with linear interpolation to approximate the expected future value function, which is only observed for a number of finite grid points. In other words, we aim to approximate the expected future value function following [Judd, 1998] such that

$$\hat{v}\,(m, d) \approx v\,(m, d)$$

where $\hat{v}$ is the estimated expected future value function and v is the "true" expected future value function. When interpolating a point in the state space, the first part is to find the neighbor points in the known grid:

$$\begin{aligned} m_i \leq m < m_{i+1} \\ d_j \leq d < d_{j+1} \end{aligned}$$

where i and j are the indexes of the m and d grid respectively. We use binary search for this initial step. Then, to compute the expected future value at a point in the state space, we calculate the slope at the neighbor points, where we first have d as fixed:

$$v_{i,j}(m,d) = v(m_i, d_j) + \frac{v(m_{i+1}, d_j) - v(m_i, d_j)}{m_{i+1} - m_i}(m - m_i)$$
$$v_{i,j+1}(m,d) = v(m_i, d_{j+1}) + \frac{v(m_{i+1}, d_{j+1}) - v(m_i, d_{j+1})}{m_{i+1} - m_i}(m - m_i)$$

Then, interpolating across d:

$$\hat{v}(m,d) = v_{i,j} + \frac{v_{i,j+1} - v_{i,j}}{d_{j+1} - d_j}(d - d_j)$$

The simple interpolation for cash-on-hand, m_{t+1}, and debt, d_t, allow us to approximate the expected future value function no matter the value of the post-decision state variables. Linear interpolation will not change the shape of the function, v, and is rather simple.

Alternative approximations like Chebyshev interpolation could yield worse results, due to the debt functions not behaving smoothly and contain clear kinks. The polynomial approximation would oscillate near the border of the defined ranges, which could cause severe problems regarding the accuracy. We therefore proceed with linear interpolation to approximate the expected future expected value.

3.2 The discretized model

After choosing state variables, deciding on a quadrature and a interpolation method, we present the final discretized model, where the total number of grid points to iterate over is manageable:

$$\text{Total grid points} = \underbrace{40}_{\text{Grid points for } d_{t-1}} \cdot \underbrace{40}_{\text{Grid points for } m_t} \cdot \underbrace{40}_{T \text{ periods}} \cdot \underbrace{2}_{\text{States of credit constraint, } x_t}$$

Full discretized model, where $\psi\Gamma$ is the permanent shock and income growth rate, that stems from the normalization:

$$\begin{aligned}
v\left(d_{t-1}, m_t\right) &= \max_{d_t, c_t} u\left(c_t\right) + \beta\Omega \\
\Omega &= (1-\pi)v_{x_t=0} + \pi v_{x_t=1} \\
v_{x_t=0} &= \sum_{\Xi\cdot\Psi} \omega_i \cdot v\left(d_t, m_{t+1}, \epsilon, \psi, 0\right) \\
v_{x_t=1} &= \sum_{\Xi\cdot\Psi} \omega_i \cdot v\left(d_t, m_{t+1}, \epsilon, \psi, 1\right) \\
u\left(c_t\right) &= \frac{c_t^{1-\rho}}{1-\rho} \\
w_t &= m_t + d_t + (1-\lambda)d_{t-1} - c_t \\
m_{t+1} &= \frac{1}{\psi\Gamma}\left(\left(1+r_w\right)w_t - d_t(r_d+\lambda)\right) + \xi \\
d_t &\leq \max\left\{\frac{1}{\psi\Gamma}\left(1-\lambda\right)d_{t-1}, \mathbf{1}_{(x_t=0)} \cdot \left(\eta \cdot n_t + \varphi\right)\right\} \\
n_t &= w_t - d_t \\
d_t, c_t, w_t &\geq 0 \\
\Xi \cdot \Psi &= \{\psi_b, \psi_g\} \cdot \{\xi_b, \xi_g\} \\
\sum_{\Xi\cdot\Psi} &= \sum_{(\psi,\xi)\in\Xi\cdot\Psi} p\left(\psi, \xi\right) = 1
\end{aligned}$$

3.3 Nested value function iteration

Inspired by [Druedahl, 2021], we also solve the model using *nested* value function iterations (NVFI). This provides a speed up relative to the standard value function iteration by utilizing the nesting structure present in the model.

New model formulation

When solving with NVFI, we consider an alternative model formulation such that the state variables are

$$\begin{aligned}
x_t &\in \{0, 1\} \\
\bar{d}_t &= (1-\lambda)d_{t-1} \\
\bar{n}_t &= n_{t|c_t=0} = m_t - \bar{d}
\end{aligned}$$

where the bar indicates *beginning of period* states. As [Druedahl and Jørgensen, 2018] points out, letting $\bar{n}_t$ be the state variable instead of m_t, speeds up the solution algorithm as the beginning of period debt, $\bar{d}_t$, only affects the feasible debt choices. Thus, we avoid having to account for the fact that $\bar{d}_t$ affects the cash-on-hand as in original model formulation of (2.2).

Nesting

To utilize the nesting structure of the model, we divide the household's problem into two subproblems: One part for the case where the household keeps the debt from previous period and a second part where the household adjusts the debt, i.e., the old and new contract. We refer to these as the *keeper's* and *adjuster's* problem. Thus, the household's value function corresponds the maximum over the value function when *not* adjusting the debt, v_t^{Keep}, and the value function when adjusting the debt, v_t^{Adj}:

$$v_t(\bar{d}_t, \bar{n}_t, x_t) = \max\left\{v_t^{\text{Keep}}(\bar{d}_t, \bar{n}_t), v_t^{\text{Adj}}(\bar{d}_t, \bar{n}_t, x_t)\right\}$$

The adjuster's and keeper's problem are

$$\begin{aligned}
v_t^{\text{Keep}}(\bar{d}_t, \bar{n}_t) &= \max_{c_t} u(c_t) + \beta v_{t+1}(\bar{d}_{t+1}, \bar{n}_{t+1}) \\
&\text{s.t.} \\
n_t &= \bar{n}_t - c_t \\
\bar{d}_{t+1} &= \frac{1}{\Gamma\psi}(1-\lambda)\bar{d}_t \\
\bar{n}_{t+1} &= \frac{1}{\Gamma\psi}((1+r_a)n_t - (r_d - r_a)\bar{d}_t) + \xi \\
c_t &\in \left[0, \max(0, \bar{n}_t + \bar{d}_t)\right]
\end{aligned}$$

$$\begin{aligned}
v_t^{\text{Adj}}(\bar{d}_t, \bar{n}_t, x_t) &= \max_{c_t, d_t} u(c_t) + \beta v_{t+1}(\bar{d}_{t+1}, \bar{n}_{t+1}, x_t) \\
&\text{s.t.} \\
n_t &= \bar{n}_t - c_t \\
\bar{d}_{t+1} &= \frac{1}{\Gamma\psi}(1-\lambda)d_t \\
\bar{n}_{t+1} &= \frac{1}{\Gamma\psi}((1+r_a)n_t - (r_d - r_a))d_t + \xi \\
d_t &\in \left[\max(-\bar{n}_t, 0), \max(\bar{d}_t, \mathbf{1}_{x_t=0} \cdot (\eta \cdot \bar{n}_t + \varphi))\right] \\
c_t &\in \left[0, \max(0, \bar{n}_t + d_t)\right]
\end{aligned}$$

We only need the *post-decision states*, n_t and d_t, to calculate the expected future value function. Therefore, we write this as the post-decision value function, w_t:

$$\begin{aligned}
w_t(d_t, n_t, x_t) &= v_{t+1}(\bar{d}_{t+1}, \bar{n}_{t+1}, x_{t+1}) \\
w_t(d_t, n_t) &= v_{t+1}(\bar{d}_{t+1}, \bar{n}_{t+1}, x_{t+1})
\end{aligned}$$

where the credit constraint indicator, x_t, can be left out as it does not affect x_{t+1}. This allows us to rewrite the keeper's problem by conditioning on $d_t = \bar{d}_t$, as the debt by assumption remains unchanged.

$$
\begin{aligned}
v_t^{\text{Keep}}(\bar{d}_t, \bar{n}_t) = \max_{c_t} u(c_t) + \beta w_t(d_t, n_t) \\
\text{s.t.} \\
\bar{d}_t &= d_t \\
n_t &= \bar{n}_t - c_t \\
\bar{d}_{t+1} &= \frac{1}{\Gamma\psi}(1-\lambda)\bar{d}_t \\
\bar{n}_{t+1} &= \frac{1}{\Gamma\psi}((1+r_a)n_t - (r_d - r_a)\bar{d}_t) + \xi \\
c_t &\in \big[0, \max(0, \bar{n}_t + \bar{d}_t)\big]
\end{aligned}
$$

As such, the keeper's problem can be nested inside the adjuster's problem as written below. The interpretation of this reformulation is that the adjuster's problem, where the household chooses both consumption and debt, becomes sequential: Firstly, the household chooses debt, d_t, and secondly chooses consumption, c_t, afterwards.

$$
\begin{aligned}
v_t^{\text{Adj}}(\bar{d}_t, \bar{n}_t, x_t) = \max_{d_t} v_{t+1}^{\text{Keep}}(d_t, \bar{n}_t) \\
\text{s.t.} \\
d_t \in \big[\max(-\bar{n}_t, 0), \max(\bar{d}_t, \mathbf{1}_{x_t=0} \cdot (\eta \cdot \bar{n}_t + \varphi))\big]
\end{aligned}
$$

The algorithm

For every time period, t, the following steps are performed in the NVFI algorithm.

- **Step 1:** Compute the post-decision value function, w_t, over a grid of post-decision state variables, d_t and n_t. That is for each post-decision state, calculate the *next* pre-decision states $\bar{d}_{t+1}$ and $\bar{n}_{t+1}$ and then interpolate v_{t+1}. Of course, for the terminal period T, this step is superficial as $v_{t+1} = 0$.
- **Step 2:** Solve the keeper's problem over a grid of pre-decision states $\bar{d}_t$, $\bar{n}_t$ and x_t. That is for each pre-decision state and each feasible consumption choice: 1) Calculate the resulting post-decision states d_t and n_t, and 2) interpolate the post-decision value function, w_t, from step 1 to get the value, v_t^{Keep}. Then, solve the Bellman equation to get c_t^{Keep}.
- **Step 3:** Solve the adjuster's problem over the grid of pre-decision states $\bar{d}_t$, $\bar{n}_t$ and x_t by interpolating the keeper's function from step 2 to obtain the optimal debt choice. That is for each pre-decision state and each feasible debt choice, interpolate the keeper's function to get v_t^{Adj}. Then, solve the Bellman equation from to get d_t^{Adj}. Finally, interpolate c_t^{Keep} to get $c_t^{\text{Adj}} = c_t^{\text{Keep}}(d_t^{\text{Adj}}, \bar{n}_t)$.

The speed up relative to VFI lies in 1) that the dimensionality of the adjuster's problem is reduced, as there is only one effective choice variable, d_t, and 2) that the pre-computation of the post-decision value function implies less interpolations in total. When pre-computing the post-decision value function, we do not need to interpolate for each combination of c_t and d_t to get the expected future value, and thus the number of total interpolations is lower.

We choose grids for the state variables in accordance with the maximum and minimum attainable net assets and debt levels from the defined choice sets. However, the borders of the state space can change slightly over time due to interest rates on net assets and debt.

$$
\begin{aligned}
d_{\min} &\equiv 0 \\
d_{\max} &= \eta \cdot \bar{n}_{\max} \\
n_{\min} &= \bar{n}_{\max} - c_{\max} = \bar{n}_{\max} - (\bar{n}_{\max} - \eta \cdot \bar{n}_{\max}) = -\eta \cdot \bar{n}_{\max} = -d_{\max} \\
n_{\max} &\equiv \bar{n}_{\max}
\end{aligned}
$$

The entire model is written as

$$
\begin{aligned}
v\left(\bar{d}_t, \bar{n}_t\right) &= \max_{d_t, c_t} u\left(c_t\right) + \beta\Omega \\
\Omega &= (1-\pi) v_{x_t=0} + \pi v_{x_t=1} \\
v_{x_t=0} &= \sum_{\Xi \cdot \Psi} \omega_i \cdot v\left(\bar{d}_{t+1}, \bar{n}_{t+1}, \epsilon, \psi, 0\right) \\
v_{x_t=1} &= \sum_{\Xi \cdot \Psi} \omega_i \cdot v\left(\bar{d}_{t+1}, \bar{n}_{t+1}, \epsilon, \psi, 1\right) \\
u\left(c_t\right) &= \frac{c_t^{1-\rho}}{1-\rho} \\
n_t &= \bar{n}_t - c_t \\
\bar{d}_{t+1} &= \frac{1}{\Gamma\psi}(1-\lambda) d_t \\
\bar{n}_{t+1} &= \frac{1}{\Gamma\psi}((1+r_a) n_t - (r_d - r_a) d_t) + \xi \\
d_t &\in \left[\max(-\bar{n}_t, 0), \max(\bar{d}_t, \mathbf{1}_{x_t=0} \cdot (\eta \cdot \bar{n}_t + \varphi)\right] \\
c_t &\in \left[0, \max(0, \bar{n}_t + d_t)\right] \\
\Xi \otimes \Psi &= \{\psi_1, \psi_2, ..., \psi_n\} \otimes \{\xi_1, \xi_2, ..., \xi_n\} \\
\sum_{\Xi \otimes \Psi} &= \sum_{(\psi, \xi) \in \Xi \otimes \Psi} p\left(\psi, \xi\right) = 1
\end{aligned}
$$

3.4 Assignment of parameter values

Solving our model requires assigning values to the model parameters regardless of the chosen algorithm. We strive to set the parameters such that the results are economic insightful. The chosen parameters also play a part in the counterfactual simulations, which we will return to later. We assign parameters according to [Druedahl and Jørgensen, 2018] to compare our results with their buffer stock model and examine how households consume and borrow with high interest debt. Note that the interest on savings is negative, as it is the real interest rate.

Table 1: Model parameters

Parameter	Value	Description
Income		
Γ	1.02	Annual income growth
$u_\star$	0.07	Risk of unemployment
σ_ψ^2	$0.01 * \frac{4}{11}$	Permanent income shocks
σ_ξ^2	0.04	Transitory income shocks
μ	0.30	Value of negative income shock
Borrowing/savings		
r_w	-0.0148	Interest on savings
$r_d - r_w$	0.1236	Spread between interest on debt and savings
φ	0.74	Maximum debt allowed w.r.t. income
η	0.00	Maximum debt allowance w.r.t. net assets
λ	0.03	Installment
Credit risk		
$\pi_{x=1}^{\text{lose}}$	0.10	Risk of credit constraint
Consumer utility		
β	0.90	Discount factor
ρ	3.00	CRRA utility factor

4 Results

4.1 Solution methods and results

We solve the model discretely with Gauss-Hermite quadrature shocks with $T = 40$ and 40 grids point in each grid. Firstly, we solve it with VFI. Secondly, we consider an attempt to solve the model with NVFI. The run times can be seen in Table 2.

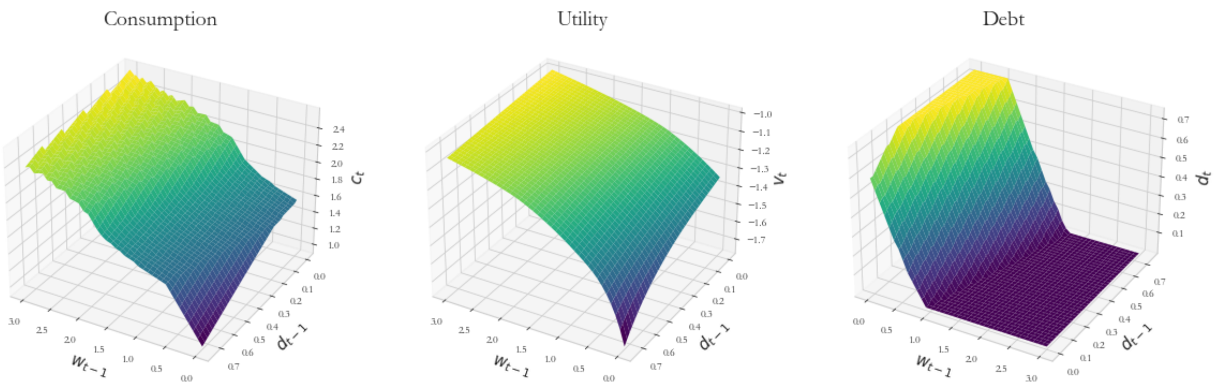
Table 2: Solution methods

Solution method	Solution time
VFI	79 min. 33 seconds
NVFI	2 min. 35 seconds

4.2 Economic results

At every given point in time, each household needs to decide the optimal amount of consumption and debt. To see how the households react for any given states, we plot the optimal consumption, value and debt for each w_{t-1} and d_{t-1} in Figure 2 at time $t = 20$. We plot the choice functions for non-credit constrained households, i.e. $x_t = 0$, while the choices for credit constrained households can be seen in Figure 11 in the Appendix.

Figure 2: Optimal consumption, utility and debt at $t = 20$ at $x_t = 0$



Note: It should be noted that w_{t-1} and d_{t-1} for optimal debt is reversed in order to get a viewable graph.

Consumption

The households' choices seem reasonable compared to a standard buffer stock model, as consumption increases with diminishing effects, when the initial wealth increases. This is due to the CRRA utility function which has diminishing returns with regards to consumption. In addition, consumption decreases, when the level of debt increases. This is due to the cost of debt arising from a interest rate of debt significantly higher than the interest rate of assets. If possible, households prefer to consume less and repay debt.

It should be noted that consumption is approximately one and not zero when $w_{t-1} = 0$ and $d_{t-1} = \varphi$. In each period, the households gets income growth of approximately one, hence households can still consume even if $w_{t-1} = 0$.

Debt

The optimal choices for debt are quite distinct from the choice for consumption. We consider three different groups.

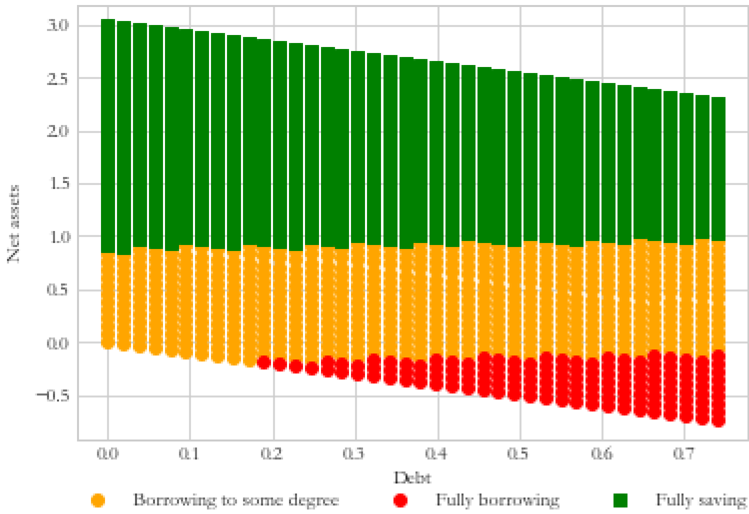
Group 1: Fully saving: Rich households (high wealth) avoid taking on any kind of debt. This aligns with the fact that debt is costly due to interest payments. At a certain level of wealth, the current utility from increasing debt is offset by the discounted future disutility, arising from high interest payments.

Group 2: Fully borrowing: The poor households (low wealth) borrow. This could be to 1) increase consumption today, or 2) to smooth future consumption. Hence, it can be beneficial for them to take on maximal debt to smooth consumption.

Group 3: Borrowing to some degree: When initial wealth is sufficient, but not too high, the households choose to borrow to some degree. This behavior can be attributed to either the desire to consume more today or the need for precautionary borrowing. Since there is a 10% risk that households become credit constrained in any future period, we would expect households to borrow to some degree to smooth out current and future consumption.

The group that both borrow and save is defined as the ”puzzle” group by [Druedahl and Jørgensen, 2018]. In our model, we define the puzzle group as households that obtain between 0.01 and 0.73 debt while simultaneously holding positive net assets. Remember, $\varphi = 0.74$. It is striking that a large share of the ”borrowing-to-some-degree” group holds *positive* net assets. To study this group, we plot net assets, n_t, compared to chosen debt, d_t, at time $t = 20$. This group can be seen in Figure 3 as a part of the households that ”borrow to some degree”.

Figure 3: Degrees of borrowing and saving compared to wealth



Note: Saving: $d_t = 0.00$. Borrowing: $d_t > 0.73$. Borrowing to some degree: $0.01 < d_t < 0.73$.

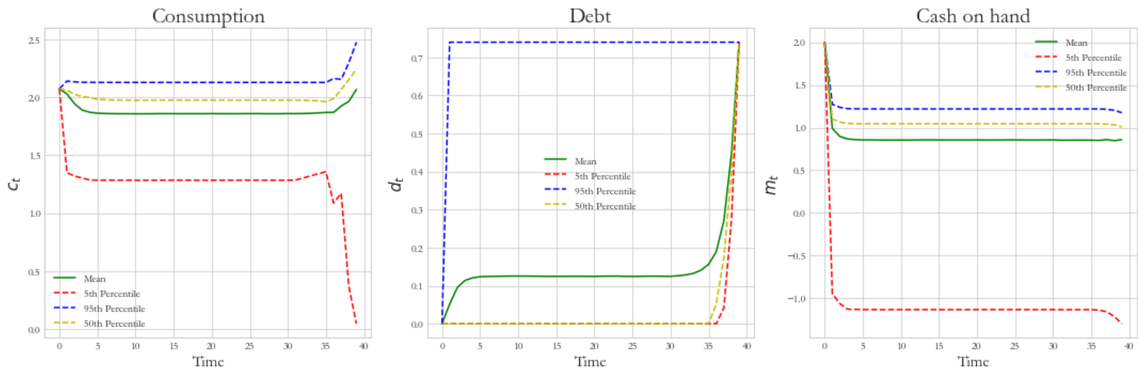
As soon as households have negative net assets, they cannot repay their initial debt. In other words, they are insolvable. However, the interesting group consist of those who have positive net assets but enough to repay their debt. It seems suboptimal to incur debt with high interest, while having low interest assets. To determine the extent to which households in the group smooth consumption or borrow precautionary, we apply a simulation based study.

4.3 Simulations and counterfactuals

Procedure

To simulate the model, we generate a sample of 1,000,000 households and study their behavior. At each period, t, the individual household get their own permanent and transitory income shocks, and they get an annual income growth of 2%. Furthermore, they pay interest of remaining debt and negative interest rate from holding assets. We apply the credit constraint by selecting 10% of the households in each period at random and limit their maximal amount of debt. To be specific, we set their maximum current amount of debt as their previous level of debt. We also ensure that each credit constrained household optimize their consumption and debt according to the choice functions in Figure 11 in the Appendix. The initial starting values are set to $w_{t-1} = 2$ and $d_{t-1} = 0.74$. Alternative combinations were possible, but changes in the initial values does not significantly affect the solution since the model converges quickly. We simulate the households for 40 periods and plot their mean, median, 5. and 95. percentile in Figure 4. The results are kept normalized.

Figure 4: Simulated consumption, debt and cash on hand over time



Note: Allocation of consumption, debt and cash on hand for the 5-percentile, average, median and 95-percentile household.

Results

The typical household consume approximately 1.9 every period, while the 5. and 95. percentile range from 1.28 to 2.13. Since households are assumed to die at age 40, there is no incentive to save and therefore consumption increases in the final periods from period 35 to 40, due to life-cycle effects. Since the income shocks are symmetric, one could expect the 5. and 95. percentile to also be symmetric around the mean. However, due to diminishing returns on consumption, high income households will consume relative less at a given income shock. Furthermore, low-income households are required to take on debt to maintain current consumption, which will decrease their income in the next period, hence decreasing consumption even more. The results are normalized such that there is no income growth over the life-cycle and therefore also no consumption growth.

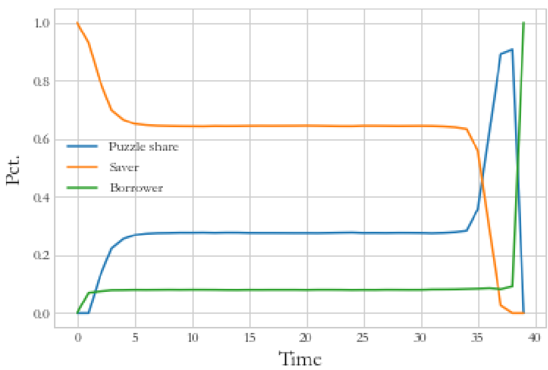
For debt, we find that the average household borrow approximately 0.125 each period, while the median does not borrow. This is likely due to the mechanics described earlier, where rich households do not need to take on debt and poor households maximize their debt. This is also confirmed by the 95-percentile line at 0.74. This means that at least 5% of households maximize debt at any given period.

One limitation of our model is how households die at time 40. Since they die, they do not have to repay any debt taken at period 39. Therefore, each household will increase their level of debt at the end of their life cycle and maximize it at the last period. This is also what we can see. But since we have $\beta = 0.90$ value, it is safe to assume that at $t = 20$, the future is so distant for the households that there are no life-cycle effects present. In other words, the households find a steady state from approximately period 5 to 30 in which we can analyze changes in consumption and debt when performing counterfactuals in our model.

Counterfactuals

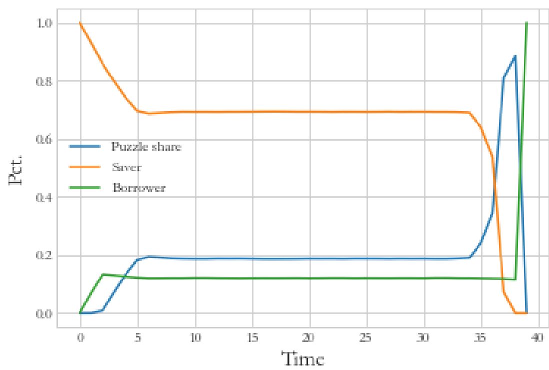
We perform a simple counterfactual to find the share of our puzzle group that participate in precautionary borrowing. First, we calculate the share of savers, borrowers and puzzle group from our simulation with a 10% risk of being credit constraint in each period. Then, we perform the exact same simulation with a 0% chance of being credit constrained in a given period. By setting up such a counterfactual, we can hold all other things equal except a change in the chance of households being credit constrained. Any change in the size of the puzzle group will then only be attributed to a change in credit constraint. In other words, the change in the size of the puzzle group will then only be caused by households that borrow precautionarily. The results of the two simulations can be seen in Figure 5 and 6.

Figure 5: Shares with 10% chance of credit constraint



Note: $\pi^{\text{lose}}_{x=1} = 0.10$

Figure 6: Shares with 0% chance of credit constraint



Note: $\pi^{\text{lose}}_{x=1} = 0.00$

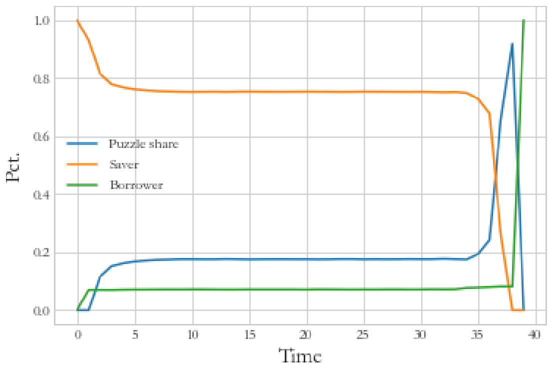
With the 10% risk of credit constraint each period, we get the following shares in period 20: 64.4% savers, 8.0% borrowers and 27.6% puzzle. This closely aligns with the empirical observations from [Telyukova, 2013], who identifies the groups: 68% savers, 5% borrowers and 27% puzzle share. Based on this, our model appears appropriate. Note that we again observe end of life-cycle effects.

When we completely remove the risk of being credit constrained, we instead have: 69.3% savers, 12.0% borrowers and 18.7% puzzle share. Therefore, removing the credit constraint increases savers and borrowers by respectively 5.1%-points and 4%-points, while decreasing the puzzle group by 8.9%-points. The increase in borrowers is expected, since some previously constrained households now are allowed to borrow, thus removed from the puzzle group. However, the increase in savers can likely fully be contributed to no more precautionary borrowing. In other words, restricting access to credit should not affect households not already borrowing. They will then only borrow if they are afraid of being credit constrained in the next period.

By modelling credit constraint in its simplest form, we find clear indications of precautionary borrowing. Giving households a 10% chance of being credit constrained will increase the share of households that borrow precautionary by 5%. Therefore, precautionary borrowing is a large part of the explanation behind the credit card debt puzzle.

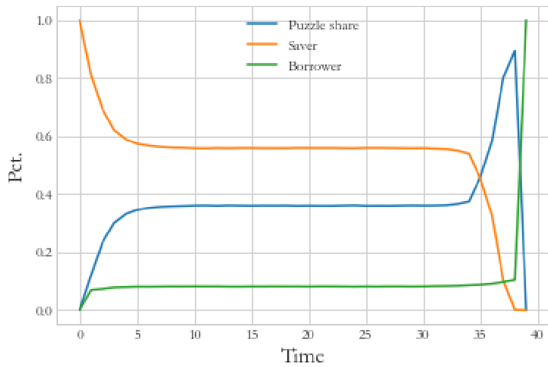
Robustness check

Figure 7: Shares with high beta value



Note: $\beta = 0.99$

Figure 8: Shares with larger shocks



Note: $\sigma_\xi^2 = 0.05$ and $\sigma_\psi^2 = 0.10$

To further explore the mechanics of our model, we perform a few more counterfactuals to check how the puzzle group respond to parameter changes. We test for an increase in the discount factor, β, and larger income shocks, σ_ψ^2 and σ_ξ^2.

Figure 7 shows that a higher value for β implies a lower puzzle group. This is because the households discount future interest payments less, and thus the cost of debt increases. So, households that chose to borrow under $\beta = 0.90$ now incur a higher cost of debt under $\beta = 0.99$, and therefore become savers.

Figure 8 shows that the puzzle group increases when the shocks increase. A higher shock rate implies more volatile income for the households, and therefore the motive for precautionary borrowing increases. As such, the size of the puzzle group increases for more volatile income shocks, ξ and ψ.

Therefore, changes in the model's structural parameters yield expected results. Hence, we find the model robust.

4.4 Preliminary results from NVFI

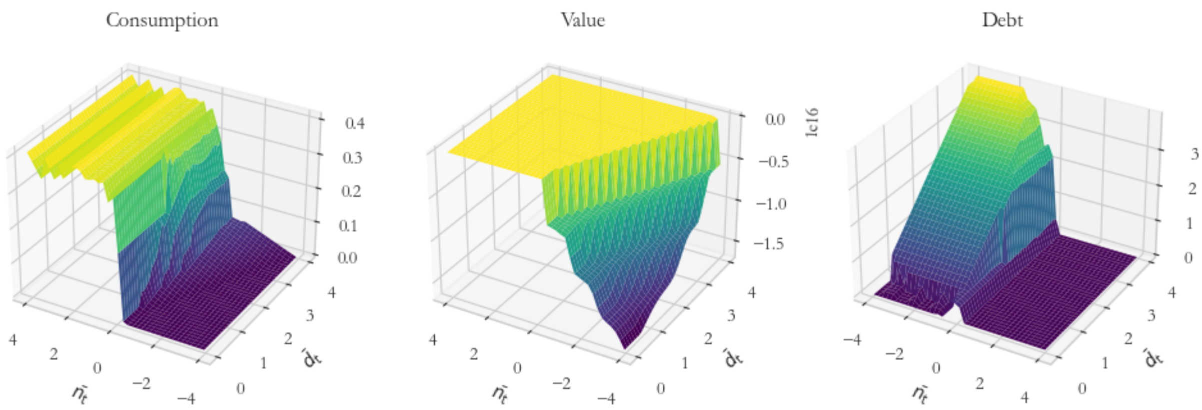
This part of the paper is still work in progress. Therefore, the results from the nested value function iteration are limited. However, it does to produce some economic relationships that seem reasonable in Figure 9 and as noted in Table 2, it is extremely fast even when solving for 40 periods and grids. We allow for $\eta > 0$ when solving the NVFI.

Debt: For negative values of $\bar{n}$ (beginning of period net assets), the households is by assumption not allowed to take on new debt and must be satisfied with the existing level of debt, $\bar{d}$ (beginning of period debt). This is reflected in the figure. For high positive levels of net assets, the households refrain from taking on debt - likely due to the significant cost of debt. However, the puzzle group does not appear, as no households are taking on debt while holding positive net assets.

Consumption: Consumption is increasing in net assets, which is in line with basic economic intuition. For negative $\bar{n}$, the choice set for consumption is zero by assumption - unless the household is borrowing enough. However, consumption is small at approximately 0.4 which is not in line with economics intuition. This is also reflected in the figure.

Value: The value function behaves abnormally and is subject for further investigation.

Figure 9: Optimal consumption, utility and debt at $t = 20$ at $x_t = 0$



Note: It should be noted that $\bar{n}_t$ for optimal debt is reversed in order to get a viewable graph. $\eta = 0.80$.

5 Discussion

5.1 Credit constraint and unemployment

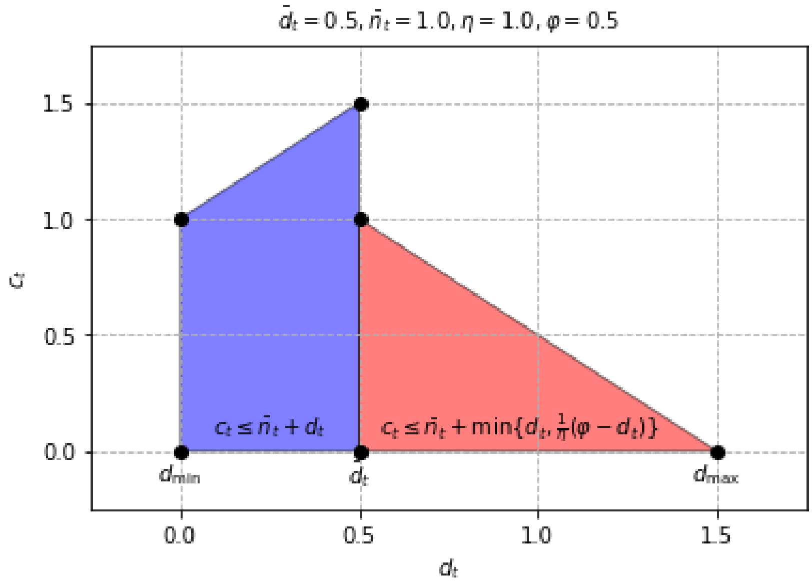
In our model, credit restrictions and unemployment are rather simple, where both occur with a given probability in each period. In [Druedahl and Jørgensen, 2018], unemployment increases the risk of credit restriction, and the credit restriction follows a first order Markov chain. The process is calculated using a transition matrix, which is based on empirical data. Hence, credit restrictions can be carried over between periods in contrast to our model. Both unemployment and credit constraint affect the puzzle groups and the savings/debt behavior of the households.

Therefore, expanding our framework to include persistence in the credit constraint could affect the size of the puzzle group. However, our simple framework yields substantial results closely aligned with empirical data, and a more complicated model is therefore not necessarily better.

5.2 Implications of non-convex choice sets

Throughout most of the analysis, we set $\eta = 0$. This imply that all household could take on the same amount of debt, when normalized for P_t. If we had chosen $\eta > 0$, then we would allow for different households to take on different amounts of debts, and the choice sets for debt would no longer be convex. Rich households (high n_t) would be able to take on more debt relative to poorer households (low n_t). This means that the choice set for c_t will not be convex and the results will not look like a standard buffer stock model. The choice set of c_t given d_t can be seen in Figure 10.

Figure 10: Non-convex choice sets



Note: This figure is identical to Figure 3.1 from [Druedahl and Jørgensen, 2018].

With our current model, the choice set for c_t is outlined as the blue area in the above figure. Setting $\eta = 1$, the red area would be added to our choice set. The optimal choice function for the households would likely have kinks in it, and it would be considerably harder to interpret the results against normal buffer stock models.

6 Conclusion

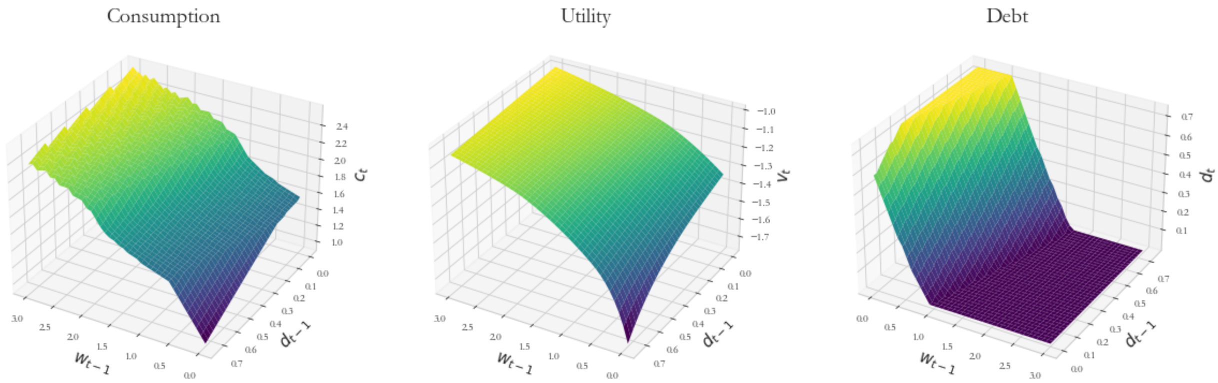
This paper explores a generalized buffer stock model with debt as an additional choice variable. Inspired by [Druedahl and Jørgensen, 2018], we test if the risk of becoming credit constrained can motivate the household to participate in precautionary borrowing. Specifically, we investigate whether households with positive net assets exhibit such behavior.

The motivation is that empirical data show that some households choose to roll over credit card debt with high interest, while simultaneously holding liquid assets with low returns. This is often called the credit card debt puzzle. As the households at any moment risk credit restrictions, an explanation for this 'puzzle group' could be precautionary borrowing to hedge against the possibility of sudden credit loss.

We solve the finite horizon model with backwards induction and perform counterfactual simulations to test the hypothesis. The counterfactual simulations show that the size of the puzzle group is greater, when the household is at risk of losing credit access, i.e., this model suggests that precautionary borrowing is an explanation to the credit card debt puzzle. Our result is robust to changes in other parameters such as the utility discount factor and shock volatility.

7 Appendix

Figure 11: Optimal consumption, utility and debt at $t = 20$ at $x_t = 1$



Note: It should be noted that w_{t-1} and d_{t-1} for optimal debt is reversed in order to get a viewable graph

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