Interpretation of Regression Coefficients with Log-Transformed Variables

	Functional Form	Relationship	Interpretation	Exact Method	Approximate Method[a]
Case 1[b]	$\ln(y) = \alpha + \beta x$	x possesses a linear relationship with $\ln(y)$	a change in x will produce a % change in y	a change in x of $x_2 - x_1$ results in a $\left(e^{\beta(x_2-x_1)} - 1\right) * 100\%$ change in y	a change in x of $x_2 - x_1$ results in a $\beta(x_2 - x_1) * 100\%$ change in y
Case 2	$y = \alpha + \beta\ln(x)$	$\ln(x)$ possesses a linear relationship with y	a % change in x will produce a change in y	a change in x of $\left(\frac{x_2}{x_1} - 1\right) * 100\%$ results in a $\beta \ln\left(\frac{x_2}{x_1}\right)$ change in y	a change in x of 1% results in a $\frac{\beta}{100}$ change in y
Case 3	$\ln(y) = \alpha + \beta\ln(x)$	$\ln(x)$ possesses a linear relationship with $\ln(y)$	a % change in x will produce a % change in y	a change in x of $\left(\frac{x_2}{x_1} - 1\right) * 100\%$ results in a $\left(\left(\frac{x_2}{x_1}\right)^{\beta} - 1\right) * 100\%$ change in y	a change in x of 1% results in a $\beta\%$ change in y

[a]The approximate method uses log differences as a consistent estimator of percent changes. Bias increases as the percent changes increase. Therefore, this method should generally only be used with small % increases (ie, 1-10%).

[b]The simplest method for this case is to assume a $\Delta x = 1$. If the results are not easily interpretable (large or very small percentages), you can adjust Δx by orders of magnitude to fit the scale of your data (ie., multiply or divide by 10, 100, 1000, etc.).

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$\boxed{\text{Case 1:}}$ $\ln(y) = \alpha + \beta x$

$$\% \Delta y = \left(\frac{y_2}{y_1} - 1\right)$$

approximation $\% \Delta y = \ln\left(\frac{y_2}{y_1}\right)$

$\rightarrow$ works when $y_2 - y_1$ is relatively small

① rewrite as $f(y)$

$$y = e^{(\alpha + \beta x)}$$

$$y = e^{\alpha} e^{\beta x}$$

② solve

$$\frac{y_2}{y_1} - 1 = \frac{e^{\alpha} e^{\beta x_2}}{e^{\alpha} e^{\beta x_1}} - 1$$

$$= \frac{e^{\beta x_2}}{e^{\beta x_1}} - 1$$

$$= e^{\beta x_2 - \beta x_1} - 1$$

$$= e^{\beta (x_2 - x_1)} - 1$$

"A change in x of $(x_2 - x_1)$ leads to a $[e^{\beta (x_2 - x_1)} - 1] \times 100\%$ change in y."

① solve

$$\ln(y_2) - \ln(y_1) = (\alpha + \beta x_2) - (\alpha + \beta x_1)$$

$$\ln\left(\frac{y_2}{y_1}\right) = \beta x_2 - \beta x_1$$

$$= \beta (x_2 - x_1)$$

"A change in x of $(x_2 - x_1)$ leads to a $[\beta (x_2 - x_1)] \times 100\%$ change in y." (for sufficiently small % changes in y).

$\boxed{\text{Implication of Functional Form:}}$

There is a linear relationship between changes in x and percent changes in y.

$\boxed{\text{Case 2:}}$ $y = \alpha + \beta \ln(x)$

$$\Delta y = y_2 - y_1$$

$$y_2 - y_1 = (\alpha + \beta \ln(x_2)) - (\alpha + \beta \ln(x_1))$$

$$= \beta(\ln(x_2) - \ln(x_1))$$

$$= \beta \ln\left(\frac{x_2}{x_1}\right)$$

"An $\left[\frac{x_2}{x_1} - 1\right] \times 100\%$ change in x leads to a $\beta \ln\left(\frac{x_2}{x_1}\right)$ change in y."

approximation:

$$\ln\left(\frac{1.01}{1}\right) = .00995$$

1% increase $\sim$ 1%

"A 1% change in x leads to a $\beta/100$ change in y."

$\boxed{\text{Implication of Functional Form:}}$

There is a linear relationship between <u>percent</u> changes in x and changes in y.

$\boxed{\text{Case 3:}}$ $\ln(y) = \alpha + \beta \ln(x)$

$$\% \Delta y = \frac{y_2}{y_1} - 1$$

① rewrite

$$y = e^{\alpha + \beta \ln(x)}$$
$$= e^{\alpha} e^{\beta \ln(x)}$$
$$= e^{\alpha} \left(e^{\ln(x)}\right)^{\beta}$$
$$= e^{\alpha} x^{\beta}$$

② solve

$$\frac{y_2}{y_1} - 1 = \frac{e^{\alpha} x_2^{\beta}}{e^{\alpha} x_1^{\beta}} - 1$$
$$= \frac{x_2^{\beta}}{x_1^{\beta}} - 1$$
$$= \left(\frac{x_2}{x_1}\right)^{\beta} = 1$$

"A $\left[\frac{x_2}{x_1} - 1\right] \times 100\%$ change in x leads to a $\left[\left(\frac{x_2}{x_1}\right)^{\beta} - 1\right] \times 100\%$ change in y."

approximation!

$$\ln\left(\frac{y_2}{y_1}\right) = (\alpha + \beta \ln(x_2)) - (\alpha + \beta \ln(x_1))$$
$$= \beta(\ln(x_2) - \ln(x_1))$$
$$= \beta\left(\ln\left(\frac{x_2}{x_1}\right)\right)$$
$$= \beta(\underset{\substack{\uparrow \\ 1\% \text{ increase in } x}}{0.01})$$

$$\ln\left(\frac{y_2}{y_1}\right) \times 100\% = \beta(0.01) \times 100\% = \beta$$

"A 1% increase in x leads to a β% increase in y." (for sufficiently small % changes in y).

Implications for Functional Form:

There is a linear relationship between percent changes in x and percent changes in y.