MRB MATH LIBRARY

DOCUMENTATION

This library is intended for use in lower-cost, slower microcontrollers not dedicated for mathematical calculations (e.g. ATmega/Arduino, ARM Cortex M0 and M4, ESP, Raspberry Pi pico) to speed up floating point operations. It can also be used in DSP to reduce cycle time, wherever cycle time is very critical, but the accuracy of calculations is not so critical. The MRB\_MATH library will also significantly speed up the execution time of math calculations on microcontrollers without FPU or without hardware support for the default math.h library. An additional function not included in the math.h library is the function for calculating the RMS and discrete Fourier transformation (DFT) of a signal.

In summary, this library allows faster calculations at the expense of lower precision and higher memory usage.

The documentation describes the functions contained in the MRB\_MATH library and compares them with their equivalents from the math.h library. Execution times and precision are compared. The results are summarized in graphs or tables. The possibility to parameterize each function is also described - the MRB\_MATH library makes it possible to increase the precision of a function at the cost of additional cycles or FLASH/RAM memory usage.

Functions included in the library

* **sin\_f**(float x) sinus from argument x
* **cos\_f**(float x) cosinus from argument x
* **fast\_invsqrt**(float x) fast inverse root square from argument x
* **fast\_sqrt**(float x) fast root square from argument x
* **RMS**(float x) root mean square function (with normal sqrt function)
* **fast\_RMS**(float x) fast root mean square function (with fast sqrt function)
* **rapid\_RMS**(float x) rapid root mean square fun. (approach without sqrt)
* **DFT**(float x) discrete Fourier transform

Execution speed overview

Table 1 Comparison of function execution time in terms of cycles on STM32 microcontroller with Cortex M-7 core

|  |  |  |
| --- | --- | --- |
| **Function / Library** | **<math.h>** | **“MRB\_MATH\_LIB.h”** |
| sinus | 762 cycles | 30 cycles |
| cosinus | 762 cycles | 32 cycles |
| sqrt | 674 cycles | 28 cycles |
| RMS | - | 694 cycles |
| fast RMS | - | 46 cycles |
| rapid RMS | - | 13 cycles |
| DFT | - | 25k cycles |

Library created by Maciej Brzycki ------ February 2024

# Trigonometric functions

Basic sine and cosine functions in the MRB\_MATH library are based on the look up table, which will be stored in the microcontroller's flash memory when the program is uploaded. While the program is running, it permanently occupies space in RAM memory. User can define the size of memory usage with *LUTSIZE* parameter. When equal to 1 - look up table will take about 4kB of memory (1000 float values), when 2 – 8kB and when 4 – 16kB.

**Important note**: the range of the function's arguments is not as large as that of the **sin** function from the math.h library. The argument of the **sin\_f** function **should be** in range from the range -2π to 4π. For wider range of arguments **sin\_f** and **cos\_f** will work, but computing time will be longer.

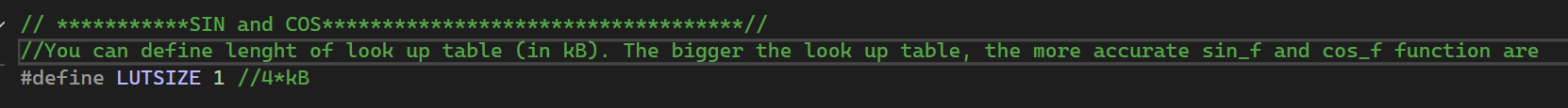


Figure 1 LUT size choice

A screenshot of a computer

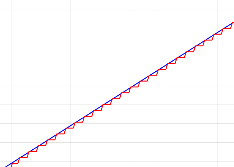
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Figure 2 LUT view - beginning of the declaration





Figure 3 Use of trigonometric functions

A graph of a function

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Figure 4 math.h sin(x) function and MRB\_MATH\_LIB.h sin\_f(x) comparison (LUTSIZE = 1)

A green graph with a white background

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Figure 5 Absolute error between sin and sin\_f functions (LUTSIZE = 1)

A graph of a graph

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Figure 6 cos and cos\_f comparison - zoom in

## Taylor series trigonometric functions

In library, there are also trigonometric functions that are estimated from Taylor series (*sin\_t* and *cos\_t*). The library calculates sinus based on first four elements of Taylor series, as this is sufficient to estimate values from -π and π.

*Sin\_t* formula was improved to do as few multiplications as possible and to not use divisions at all (F3, F5 etc. are 1 by factorial of particular number):



Figure 7 Implemented formula with 4 elements of Taylor series

Functions *sin\_t* and *cos\_t* are slower than trigonometric functions which use LUT (*sin\_f* and *cos\_f)*, but they occupy much less space in memory (as LUT is not declared). Important thing to notice is lacking precision for arguments near -π and π. For some arguments near those boundaries, value can exceed 1, what must be considered when testing the stability of the system. Functions *sin\_t* and *cos\_t* are always available, however user can define LUTSIZE as 0 and quit using the sin\_f and cos\_f functions, so lookup table won’t occupy memory.

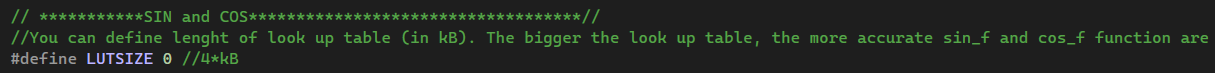


Figure 8 No lookup table. Using only Taylor series based trigonometric functions

A blue and red line

Description automatically generatedA graph of a function

Description automatically generated

Figure 9 sin and sin\_t comparison

# A graph of a function Description automatically generated

Figure 10 Difference between sin and sin\_t functions

Table 2 Mean absolute difference between sin and sin\_f (or cos and cos\_f) with 1e-5 angle step.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **LUTSIZE(n\*4kB)** | **1n** | **2n** | **4n** | **0 (Taylor series)** |
| **Mean Absolute error** | 0.000326 | 0.000163 | 0.000082 | 0.000582 |

# Square root function

The fast square root function is based on inverse square root Quake’s algorithm. The Quake algorithm avoids directly calculating the square root. Instead, it uses a fast approximation, combined with Newton’s method for refinement. The key trick is in using bit-level manipulation of the floating-point number to produce a rough initial guess.

A computer screen with text and numbers

Description automatically generated

Figure 11 Quake algorithm

**Step-by-step Quake’s algorithm description:**

**Approximation using a "magic number":**

* The number *x* is first reinterpreted as an integer (using *\*(long int\*)&x*), which allows bit manipulation – e.g. float number *1.0f* would be reinterpreted as *1065353216*.
* This integer is then modified using a "magic constant" **0x5f3759df**, which was empirically determined. The expression *i = 0x5f3759df - (i >> 1)* performs the bit manipulation to get an initial approximation of .
* This step essentially tricks the floating-point representation into giving an initial estimate that’s close to the correct result.

**Convert the bits back to a float:**

* After the bit manipulation, the result is cast back into a floating-point value (using *\*(float\*)&i*), so now *y* is a rough approximation of ​.

**Refinement using Newton's method:**

The result is then refined with one iteration of Newton’s method to improve the approximation. This step reduces the error in the approximation. The formula used is:

After that, result of Quake’s algorithm is multiplied by input value, so division is avoided again (which increases execution speed a lot). Instead function does:

A computer screen with text

Description automatically generated with medium confidence

Figure 12 Inverse-Inverse square root

User can adjust the function by changing *SQRT\_ACCURACY* parameter. Increasing SQRT\_ACCURACY also increases linearly execution speed of the function. The relative error (relative to **sqrt** function from the math.h library) and the execution time of the function for a given parameter value are summarized below. The error was presented in relative form due to the very large range of values tested by the **fast\_sqrt** function.

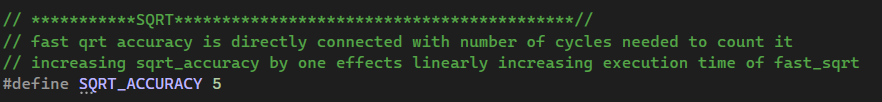


Figure 13 Adjusting fast quare root accuracy - in default equal to 5

A graph of a function

Description automatically generated

Figure 14 sqrt from math.h and fast\_sqrt comparison

A graph with a line

Description automatically generated

Figure 15 Relative error [%] of fast\_sqrt function in comparison to sqrt function from math.h library

Table 3 Mean relative error of fast\_sqrt function with SQRT\_ACCURACY parameter change

|  |  |  |
| --- | --- | --- |
| **SQRT\_ACCURACY** | **Mean relative error** | **Function execution speed** |
| 2 | 1,6e-4 [%] | 17 cycles |
| 3 | 3,5e-6 [%] | 21 cycles |
| 5 | 2,3e-6 [%] | 26 cycles |
| 10 | 2,2e-6[%] | 44 cycles |
| 20 | 2,1e-6[%] | 90 cycles |

# Root mean square function

As is this one of the most commonly needed tools in digital signal processing, the root mean square function (RMS) has also been added to MRB\_MATH library. Three variants of the RMS function are included in this library: normal, fast and rapid. The normal variant is based on the sqrt function included in the math.h library. The fast variant is based on the fast\_sqrt function contained in the MATH\_MRB library. The rapid variant uses mathematical relationships assuming a perfectly sinusoidal waveform, so it does not use division or root operations. The accuracy of the various variants and their execution times for different signals is summarized below.

The user must set the parameters of the measured RMS signal in the preprocessor directives before using any of this functions (this must be done in the preprocessor, in order to avoid using the malloc() function to create a sample buffer). Important information is signal measuring frequency, base frequency for which RMS should be calculated and buffer size (should be counted as *measuring frequency divided by base frequency*). *RMS\_HANDLERS* corresponds to number of buffer arrays that are declared for this scope. Each signal should have different buffer. That means, e.g. for three phase system, there are three current measurements and three voltage measurements, so *RMS\_HANDLERS* should be equal to 6.

A screen shot of a computer

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Figure 16 Setting parameters of RMS functions

A screen shot of a computer

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Figure 17 Approach for use of RMS function with multiple occurrences

Table 4 Execution speed comparison for different RMS functions

|  |  |
| --- | --- |
| **Execution speed** | |
| **RMS** | 694 cycles |
| **fast\_RMS** | 46 cycles |
| **rapid\_RMS** | 13 cycles |

A graph with a green line and blue line

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Figure 18 Comparison of transient state of normal RMS and rapid RMS for ideal sinusoidal signal

A graph showing a graph of a graph

Description automatically generated with medium confidence

Figure 19 Comparison for noisy signal with third harmonic injected

A graph with a line going up

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Figure 20 Relative error of rapid\_RMS in comparison to normal RMS

# Discrete Fourier transform function

The last tool in the library is a discrete Fourier transform function designed for real time and non-real time computing on a microcontroller. Samples can be provided each cycle or can be buffered and given to DFD in selected time. Before using, user have to adjust preprocessor parameters: *DFT\_HANDLERS, SAMPLING\_FREQ,* *BASE\_FREQ, MAX\_HARMONIC* and *DFT\_RESULUTION*. Macro *MAX\_HARMONIC* corresponds with the highest frequency to which algorithm will count. The more harmonics we want to calculate, the longer computing will be needed. *DFT\_ RESULUTION* sets the difference between samples in **DFT** results. With accuracy equal to one, results are received with accuracy to one harmonic (results table will look like: h1, h2, h3 etc.). With accuracy equal to two results table will look like: h1, h1.5, h2 etc. Along with increasing resolution, the buffer array is getting bigger and computing time increases. By default, this function uses trigonometric function from math.h (sinf and cosf) as they are more optimized for wider range of angle arguments than MRB\_MATH functions – by definition the range of angles for Fourier transform would be up to . If your microcontroller doesn’t have FPU you should uncomment *#define NO\_FPU* line.

*DFT\_HANDLERS* corresponds to the number of buffer arrays that are declared for this scope. Each signal should have a different buffer. That means if user wants to make online DFT for 3 signals, for optimal memory usage, this parameter should be equal to 3. To store DFT results user also must declare array with proper length (at least half the buffer plus one).

A screen shot of a computer

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Figure 21 Definition of DFT parameters

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Figure 22 Multiple occurrence of DFT function and result array definition (offline use of DFT)

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Figure 23 Online use of DFT function

DFT is very costly in terms of computing. Execution speed of DFT function highly depends on buffer size (which is determined by the sampling frequency and resolution) and amount of harmonics which will be calculated. The **DFT** function called in program with a sampling frequency most of the time collects samples into a buffer and will perform calculations only after it is full. For a frequency of 50 Hz, the most costly actions for the processor will be performed every 20ms. After the first buffer is full, the buffer is cycled and the next calculation will be performed for a completely new set of samples.

Table 5 Exuction speed of DFT - comparison for different sampling and resolution parameters

|  |  |
| --- | --- |
| Buffer size | Execution speed |
| 20 (1kHz sampling, DFT resolution 1) | 25k cycles |
| 100 (5kHz sampling, DFT resolution 1) | 48k cycles |
| 200 (10kHz sampling, DFT resolution 2) | 139k cycles |

Figures 19 and 20 shows Fourier spectrums under various test conditions. Figure 19 shows Fourier spectrum of tested 50,5Hz sinusoidal signal contaminated with 2nd , 3rd , 5th and 7th harmonics. Test results were similar for all frequency sampling conditions.

Second figure presents difference in results between different resolutions. Signal for second test was additionally contaminated with non-integer harmonics (1.1n, 1.2n, 1.5n, 1.6n and 1.8n) every with an amplitude equal to 5% of 1st harmonic amplitude.

A graph with a line

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Figure 24 Fourier spectrum - DFT test for non-ideal 50,5Hz sinusoidal signal and 2,3,5,7n harmonics

A graph of a graph

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Figure 25 Fourier spectrum - DFT test for different DFT\_RESOLUTION parameters