Stochastic Dynamic User Equilibrium Using a Mixed Logit Modeling Framework

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***Abstract*— Traditional methods for stochastic user equilibrium are either probit-based or logit-based; both have weaknesses that limit their usefulness. This paper proposes a mixed logit algorithm for Stochastic Dynamic User Equilibrium (SDUE) which is capable of capturing required correlations at a reasonable cost. That is, the flexibility in specification of error terms provided by the mixed-logit allows the SDUE model to capture spatial and temporal correlations in unobserved factors. In addition, it does not require route enumeration. The capabilities of a mixed-logit model enable the development of a SDUE traffic assignment model with correct flow propagation. A mathematical programming problem is formulated for the SDUE whose solution is found by an iterative mixed-logit-based network-loading procedure which in theory is expected to calculate link flows at a more efficient rate compared to probit-based models.**

I. Introduction

Stochastic dynamic models for traffic assignment enable the study and analysis of vehicular traffic flows [1]. Traffic assignment methods based on Stochastic User Equilibrium (SUE) models use probabilistic frameworks to model variations in uncertainty or preferences. They can also be used to represent imperfect or incomplete knowledge of drivers about network conditions. Drivers' route choices are made based on these various sources of uncertainty and error. SUE models estimate traffic assignment using 'time-cost' functions based on a probability distribution rather than a single value for travel time.

Unlike the more simple and deterministic user equilibrium (UE) models, which assume that drivers have perfect information and homogeneous perceptions, the outstanding advantage of SUE models is that they allow for the possibility that travelers may have imperfect information and/or a range of different perceptions regarding travel costs. In the context of both stochastic and deterministic UE models, route choices are assumed to be a function only of perceived travel times, which can be modeled as flow-dependent random variables. This dependence is accounted for by assuming that the mean travel time for each link is a function of the flow on that link.

The probabilistic frameworks used by SUE models require an appropriate model form as well as the means to obtain parameter estimates for that model. Using any specific model depends on the assumptions underlying the specifications of the error terms. Traditional specifications of the error terms raise issues in terms of model accuracy and/or computational efficiency and/or tractability [2]. Recently, there have been significant developments regarding specification for flexible error terms [3, 4, 5]. Thus, innovative discrete-choice models are available, which provides new opportunities to study and/or improve the modeling of SUE algorithms.

Although the multinomial logit (MNL) framework is appealing given its simplicity and closed-form, it restricts the choice structure assuming independence from irrelevant alternatives (IIA) and homogeneous response [6]. Many practical choice situations, such as route choice, require modeling response heterogeneity and state dependence. In addition, temporal and serial correlation effects need to be considered explicitly to avoid in inconsistent estimators and erroneous inferences.

Another class of models, Generalized Extreme Value (GEV), can be used to address the independence assumption and capture correlation across subsets of alternatives. These models use a Type I extreme value error distribution [7, 6]. Examples of GEV models include the nested and the paired combinatorial logit [8]. As the number of alternatives increase, the number of possible nested structures increases exponentially thereby making this framework cumbersome to calibrate. This difficulty has limited the use of GEV models to route choice analysis [9].

Multinomial Probit (MNP) models provide a covariance matrix which enables to capture spatio-temporal correlations [7, 10, 11]. The MNP is able to represent unobservable factors resulting from general dynamic and/or stochastic processes. However, multi-dimensional integrals of the multivariate normal functions are computationally intensive. In addition, the chance of unstable variance-covariance parameters and the presence of flat log-likelihood functions create identification problems [12, 13, 14]. These limitations make the MNP problematic for use in SUE.

Advances in computing power have facilitated the application of the MNP. Monte-Carlo simulation has proven to be a suitable approach to estimate MNP models [13, 15] in addition to other alternative methods [2, 16]. Since Monte-Carlo simulation permit incorporation of more general error structures, this has facilitated the applications of variations of MNP models such as the generalized ordinal probit [17] and the space-time MNP [18].

The mixed logit model is one favorable variant of the MNP [4, 19, 5]. This model has been used to capture heterogeneous response behavior [5]. Using this formulation, Bhat modeled behavior in choosing joint modes and departure times [16]. To some extent, mixed logit combines the realistic structure of the probit with the tractability of the

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logit model. For each alternative, the mixed logit model enables the use of a combined structure for the error terms. That is, error terms include two components, one MVN distributed and one Gumbel-distributed. The Gumbel error terms are assumed to be independent and identical over time and alternatives. Unlike in the MNL model, at a given time, these errors only need be identically distributed across alternatives. Given that the normal error terms have a general variance-covariance structure, this limitation can be overcome. The closed-form of the logit likelihood function of the Gumbel errors contributes to computational tractability. Using Monte-Carlo simulation to integrate this conditional likelihood, the unconditional likelihood can be derived.

Advances in computer speed and simulation methods enable to benefit from mixed logits [2], particularly cross-sectional data [20, 21] and panel data [5, 16]. This paper proposes a new mixed logit-based algorithm, Stochastic Dynamic User Equilibrium (DSUE), that exploits the advantages of the mixed-logit framework in order to overcome all pre-restrictions and/or weaknesses of the more traditional logit, probit, and GEV-based SUE models.

II. Model Formulation

A. Route Choice Model

In stochastic route choices, some travelers may not use a minimum-cost route. This implies that perceptions regarding the minimum-cost route are different among travelers. Travelers may have the perception that they are using routes with the minimum cost, yet not all routes are in actually minimum cost.

Random Utility Maximization (RUM) often is used to estimate choice behavior [7, 6]. Under RUM, the probability that individual *n* choses the following sequence is:

$$P_n = Pr_n\{S_n^1, S_n^2, \dots\dots.., S_n^T\} = Pr_n\{U_n^{St} \geq U_n^{it}, \forall i \neq S_n^t, \forall t = 1, \dots\dots., T\} \quad (1)$$

where U_n^{St} is the utility of alternative *i* at time *t* for individual *n* and S_n^t is the associated choice.

The SDUE assignment model in this study used a Dynamic Mixed Logit (DML) structure to establish time-dependent route choices. Without loss of generality and using a mixed-logit model, the random utility is formed by the sum of deterministic and random terms. The random utility is the expected travel cost/time, U_n^{it}. The difference between the mixed logit and the probit model is in the assumptions about the distributions of the disturbance terms. It is assumed that the error terms in the probit model are MVN while in the mixed logit model, consideration is given to the variance structure. This structure is made up of at least two disturbance terms, that is, a Gumbel distributed term and a MVN term. An assumption about the Gumbel disturbance term is made such that they are spatially and temporally independent. Another assumption is the independent between these Gumbel and normal errors. Moreover, an assumption is made so that the Gumbel errors are identically distributed across alternatives for a given *t*.

According to this structure, in the SDUE assignment problem, the perceived travel cost along any given route is the actual (measure) travel time plus two kinds of error-term components. Using Bhat's description in [19], U_n^{it} can be described in the DML as follows:

$$U_n^{it} = V_n^{it} + \delta_n^{it} + \lambda_n^{it} \quad (2)$$

where V_n^{it} represents the actual cost of travel for route *i* at time *t* for individual *n*, and:

$$\delta_n^{it} = \varepsilon_n^{it} + \tau_n^{it} \quad (3)$$

where: $\varepsilon_n^{it} \sim MVN(0, \beta_1 . V^{i0})$ and $\tau_n^{it} \sim MVN(0, \beta_2 . V_n^{t,t-1,t-2,\dots})$, in which the βs are proportionality constants that could be used in future work to increase and/or reduce heterogeneity across different classes of drivers;

λ_n^{it} ~ independently Gumbel-distributed with a variance $\psi_t^2 = \pi^2 / 6\upsilon_t^2$, $t = 1, \dots T$; υ_t represents the Gumbel-scale parameter at time *t*. To obtain a zero value for the expected Gumbel disturbance term, the location parameter is reset.

V^{i0} is the free flow travel time on route *i* given by the summation of corresponding free flow link travel times and generates therefore correlation across routes sharing links and

$V_n^{t,t-1,t-2,\dots}$ are the actual travel times on route *i* for individual *n* across different time periods generating therefore correlation across time.

Given that this distributional assumption assumes independence only over logit error terms, temporal and spatial correlation is captured by using the MVN error terms.

Substituting (2) into (1) provides:

$$Pr_n\{S_n^t, t = 1, \dots T\} = \Pr\{(V_n^{St} + \delta_n^{St} + \lambda_n^{St}) - (V_n^{it} + \delta_n^{it} + \lambda_n^{it}) \geq 0, \forall i \neq S_n^t, t = 1, \dots, T\} \quad (4)$$

By conditioning on δ_n , (3) is given as:

$$Pr_n\{S_n^t, t - 1, \dots T\} = \int_{\delta_n} Pr_n\,\{S_n^t, t = 1, \dots T |\ \delta_n\} f(\delta_n) d\delta_n \quad (5)$$

where $f(\delta_n)$ is a probability density function of MVN distribution, with parameters expressed in (2).

The likelihood of observing a given choice sequence can be obtained by combining (3) and (4):

$$Pr_n\{S_n^t, t = 1, \dots T\} = \Pr\{(V_n^{St} + \delta_n^{St} + \lambda_n^{St}) - (V_n^{it} + \delta_n^{it} + \lambda_n^{it}) \geq 0, \forall i \neq S_n^t, t = 1, \dots, T |\ \delta_n\} f(\delta_n) \quad (6)$$

Conditional on δ_n, δ_n^{it} and δ_n^{St} can be assumed to be known. Hence, the conditional deterministic utility is:

$$W_n^{it} = V_n^{it} + \delta_n^{it}$$

Simplifying (5) yields:

$$Pr_n\{S_n^t, t = 1, \dots T\} = \int_{\delta_n} Pr_n\,\{(W_n^{St} + \lambda_n^{St}) - (W_n^{it} + \lambda_n^{it}) \geq 0, \forall i \neq S_n^t, t = 1, \dots T |\ \delta_n\} f(\delta_n) \quad (7)$$

The right-hand side (RHS) of (6) can be rewritten as:

$$Pr_n\{(W_n^{St} + \lambda_n^{St}) - (W_n^{it} + \lambda_n^{it}) \geq 0, \forall i \neq S_n^t, t = 1, \dots T |\ \delta_n\} = \prod_{t=1}^{T} Pr_n\{(W_n^{St} + \lambda_n^{St}) - W_n^{it} + \lambda_n^{it}) \geq 0, \forall i \neq S_n^t, |\ \delta_n\} \quad (8)$$

The spatio-temporal independence of the Gumbel disturbance terms is exploited in Equation (7). The probability expression on the RHS of this equation is expressed in a simple multinomial logit form, as follows:

$$Pr_n\{(W_n^{St} + \lambda_n^{St}) - (W_n^{it} + \lambda_n^{it}) \geq 0, \forall i \neq S_n^t, t = 1, \dots T | \ \delta_n\} = \frac{\exp({}^{\upsilon}{}_t W_n^{St})}{\sum_i \exp({}^{\upsilon}{}_t W_n^{it}} \qquad (9)$$

This is derived from two assumptions: (i) spatial and temporal independence of the disturbance terms, and (ii) identical distribution across alternatives for a given *t*. Using (7) and (8), and reconstituting for W_n^{it}, the likelihood of observing S_n^t , *t=1,… T*, can be expressed as:

$$Pr_n\{S_n^t, t = 1, \dots T\} = \int_{\delta_n} \prod_{t=1}^{T} \{\exp[v_t(V_n^{St} + \delta_n^{St})] / \sum_i \exp[v_t(V_n^{it} + \delta_n^{it})]\} f(\delta_n d\delta_n) \qquad (10)$$

The above formulation can be applied in cross-sectional models by considering only one time period.

The present SDUE algorithm does not requires route enumeration. It exploits the characteristics of the Dial's logit-based UE algorithm [22], which only uses expected link travel times. Using the above DML formulation, link travel times for individual *n* at time *t* can be expressed in the DML formulation as:

$$T_n^{at} = C_n^{at} + \varepsilon_n^{at} + \tau_n^{at} + \lambda_n^{at} \qquad (11)$$

where C_n^{at} is the actual link travel time, *a*, that individual *n* would experience if his/her traverse link a at time period *t*;

$$\varepsilon_n^{at} = \eta_{1,n}^{at} . \sqrt{\beta_1 . C^{a0}} \qquad (12)$$

where $\eta_{1,n}^{at} \sim N(0,1)$, β_1 is a proportionality constant and is the travel time for free-flow along link *a*. Thus, $\varepsilon_n^{at} \sim N(0, \beta_1 . C^{a0})$ is similar to probit-based SUE models [23]. Hence, the ε_n^{at} terms are condition to be normal distributed with *0* mean and variance proportional to the free flow travel time:

$$\tau_n^{at} = abs\left(\eta_{2,n}^{at}\right) . \beta_2 . \{\frac{(t-1)!}{\sum_{L=1}^{t-1} \prod_{\substack{K=1 \\ K \neq 1}}^{t-1} (t-k)} \left[\sum_{r=1}^{t-1} \frac{C^{ar}}{(t-r)}\right] - C^{at} \qquad (13)$$

where $\eta_{2,n}^{at} \sim N(o, 1)$, β_2 is a proportionality constant; C^{at} and C^{ar} are the link travel times during intervals *t* and *r*, respectively. Thus, $\tau_n^{at} \sim N(0, \beta_2 . C^{at-1, t-2, t-3, \dots})$. The τ_n^{at} terms are condition to be normal distributed with *0* mean and variance proportional to a weighted deviation in travel times across time interval. The λ_n^{at} have an independent Gumbel distribution with variance, $\sigma_t^2 = \pi^2 / 6\mu_t^2$ t= 1,…T, where μt represents the Gumbel-scale parameter at time t. To obtain a zero value for the expected Gumbel disturbance term, the location parameter is reset. Hence, the expected route travel times can be calculated as the summation of expected link travel times as follows:

$$U_n^{it} = \sum_a T_n^{at} . \zeta_{od}^{ai} \qquad (14)$$

where ζ_{od}^{ai} is an incidence variable to ensure that will include only those links along the i^{th} route. Hence, U_n^{it} can be expressed as:

$$U_n^{it} = \sum_{a \in R^{od}} \{(C_n^{at} + \varepsilon_n^{at} + \tau_n^{at} + \lambda_n^{at}) . \zeta_{od}^{ai}\} \qquad (15)$$

which is equivalent to:

$$\sum_{a \in R^{od}} (C_n^{at} . \zeta_{od}^{ai}) + \sum_{a \in R^{od}} (\varepsilon_n^{at} + \tau_n^{at}) . \zeta_{od}^{ai} + \sum_{a \in R^{od}} (\lambda_n^{at} . \zeta_{od}^{ai}) \qquad (16)$$

Given that in this study, the user's utilities only consist of travel time, $V_n^{it} = \sum_{a \in R^{od}} (C_n^{at} . \zeta_{od}^{ai})$.

Substituting this result and using (3):

$$U_n^{it} = V_n^{it} + \delta_n^{it} + \lambda_n^{it}$$

which is exactly the Equation (2). This result shows how we can apply the mixed-logit framework in the SDUE problem and how we can specify the constituting elements of utility.

Interpretation for (15) is as follows. The expected or perceived travel time (cost) for individual *n* along route *i* at time period *t* is equal to the actual contemporary travel time that the individual would experience along route *i* if network conditions at time period *t* did not change while traveling from the current position to the destination plus random deviation terms due to unobserved factors and spatial and correlations in some of those unobserved factors. The first deviation term, δ_n^{it}, is normally distributed. It consists of two components, one that captures the spatial correlation due to overlapping links across alternative routes; the other captures the temporal correlation caused by previous perceptions that might influence contemporary perceptions.

In calculating route choices of individuals, choice probabilities were evaluated many times to eliminate bias, where each time value represents a different individual. In this framework, travel time is the average travel time of many users, and therefore assumed to be independent of the error term components. However, for reasons mentioned above, the expected travel deviation of a given individual is correlated over time and space; these correlations are captured by a correlation in errors within the same individual (12) and (13).

It is reasonable to assume that an individual's decision is dependent on previous decisions, but independent of future decisions. This is consistent with the causality requirement. Furthermore, it is reasonable to assume that the present choice of route is not going to chance in the future if network conditions do not chance across periods because people do not chose a route expecting to change it in the future. If network conditions change, it is possible that people will change their current route. Hence, it is required to recalculate route choices at the beginning of each time period.

Given these assumptions, and if network conditions are continuously updated and explicitly capture spatial and temporal correlation in unobserved factors, it is possible to estimate an individual's choices at the beginning of each time period *t*, eliminating the need for a priori knowledge regarding future decisions, as implied by (10). A procedure to implement the SDUE paradigm using this framework is proposed below.

B. Stochastic Dynamic User Equilibrium

The SDUE is defined as a dynamic generalization of SUE [24]: ‘‘At each instant, no traveler believes that he or she can improve his or her perceived travel cost by unilaterally changing route.’’ SDUE can be described as follows [25]:

$$Pr_n^{od}\{S^t\} = \frac{f_S^{od}(t)}{q^{od}(t)} \quad S \in R^{od}, \forall od, \forall t \tag{17}$$

$$\sum_{S \in R^{od}} f_S^{od}(t) = q^{od}(t) \ \ \forall od. \forall t \tag{18}$$

$$f_S^{od}(t) \geq 0 \quad \forall od, \forall t \tag{19}$$

where:

$$Pr_n\{S^t\} = Pr_n(U^{St} \leq U^{it} \forall i \in R^{od} | \ V^t) \ \forall S \in R^{od}, \forall t \tag{20}$$

$q^{od}(t)$ and $f_S^{od}(t)$ represent respectively, at time t from origin o to destination d, the demand and flow on route S.

The probability $Pr_n\{S^t\}$ in (17) is associated with the smallest perceived cost on route S, U^{St}, which is dependent on the route cost pattern at t, V^t. The probabilities from both (17) and (20) should be the same at SDUE. This condition can be obtained if there is no change in perceived cost pattern following the loading of the dynamic stochastic network. Ran and Boyce proposed the above SDUE as a variational inequality [26].

III. Solution

A. Dynamic Stochastic Network Loading

Considering the error terms structure of (10) and the choice probability structure of (9), the stochastic assignment can be calculated as the average of many logit-based loadings conditional upon the error terms, which are draw (for each iteration k) from a normal distribution with mean 0 and a variance proportional to the free-flow travel time. The following algorithm, DYNASTOCH, was proposed by Ran and Boyce [26] and is summarized here for completeness.

1) *Step* 0*: Initialization*

(a) Set h = 1

(b) Define A_i as the set of downstream nodes of all links leaving node i.

(c) Define B_i as the set of upstream nodes of all links arriving at node i.

2) *Step* 1*: Calculation of link likelihoods*

(a) Compute the expected link travel times $T_{a.h}$ without including the Gumbel error terms, using (11).

(b) Calculate the minimum instantaneous travel time $C_{jd}^{*t,h}$ for travelers departing node j for iteration h during the current interval t.

(c) Compute the minimum instantaneous travel time $C_{jd}^{*t,h}$ for travelers departing node i at iteration h and interval t.

(d) Calculate the link likelihood $L_{ij}^{t,h*t,h}$ for each link (i ,j) for iteration h during the current time interval t:

$$L_{ij}^{t,h} = \begin{cases} e^{\theta\{C_{id}^{*t,h} - C_{jd}^{*t,h} - T_{ij,h}\}} & \text{if } C_{jd}^0 < C_{id}^0 \\ 0 & \text{otherwise} \end{cases} \tag{21}$$

where C_{id}^0 represents travel time from i to d at zero flow.

3) *Step* 2*: Backward pass*

Scanning all nodes j from the destination d in ascending sequence with respect to $C_{jd}^{*t,h}$, compute $W_{jd}^{t,h}$, the weight for each link (i,j), for iteration h and interval t:

$$W_{jd}^{t,h} = \begin{cases} L_{jd}^{t,h} & \text{if j = d} \\ L_{jd}^{t,h} \ \sum_{m \in A_j} W_{mj}^{t,h} & \text{otherwise} \end{cases} \tag{22}$$

where A_j is the set of links starting from node j. Stop when the origin node o is reached.

4) *Step* 3*: Forward pass*

Scan, starting with the origin o, all nodes i in descending sequence with respect to $C_{id}^{*t,h}$. When each node i is considered during each iteration h at interval t, compute the inflow to each link (i,j) using:

$$X_{ij}^{t,h} = \begin{cases} q_{of}^t \ \frac{W_{ij}^{t,h}}{\sum_{m \in A_i} W_{iim}^{t,h}} & if \ i = 0 \\ \left[\sum_{m \in B_i} X_{mi}^{t,h}\right] \frac{W_{ij}^{t,h}}{\sum_{m \in A_i} W_{im}^{t,h}} & \text{otherwise} \end{cases} \tag{23}$$

where B_i is the set of links ending at node i.

5) *Step* 4*: Flow averaging*

Calculate the flow average using many iterations or until convergence is reached. Flow averaging can be calculated using:

$$X_{ij}^{t,h} = \frac{\left[(h-1)X_{ij}^{t,(h-1)} + X_{ij}^{t,h}\right]}{H}, \forall i \rightarrow j \tag{24}$$

where H is the current number of iterations.

This flow averaging procedure intends to capture correlations across different routes, similar to probit-based SUE models. Since the variance in expected travel times is constant across time intervals, this procedure is expected to capture unobserved correlations across time intervals. By using this mechanism, the integral in (9) can be solved numerically.

6) *Step* 5*: Stopping test*

If h arrives at a previously specified value, or link flows satisfy a convergence requirement, stop; else, set $h = h + 1$ and proceed to *Step* 1.

The DML-based network-loading algorithm described here applies to special cases that have only one O-D pair. However, it may easily be extended to general applications that have several O-D pairs.

B. Diagonalization Method

The variational inequality initially proposed by Ran and Boyce [26] is difficult to evaluate directly. They proposed a similar framework under the diagonalization method, such that the objective functions (25)–(27) are defined using route flows and costs:

$$min. Z(f) = \sum_t \sum_{od}\{-q^{od}(t) G^{od}\left[U^{od}(t)\right] + \sum_s f_S^{od}(t) U_S^{od}\left(f_S^{od}(t)\right) - \sum_{S \in R^{od}} \int_0^{f_S^{od(t)}} U_S^{od}(w) dw\} \tag{25}$$

subject to $\sum_{S \in R^{od}} f_S^{od}(t) = q^{od}(t) \quad \forall od, \forall t$ (26)

$$f_S^{od}(t) \geq 0 \quad \forall od, \forall t \tag{27}$$

where $G^{od}[U^{od}(t)$ is the dissatisfaction or the expected perceived cost of travel.

Similar to [28], the dissatisfaction for the logit model can be computed as:

$$G^{od}[U^{od}(t) = -\frac{1}{0} In \sum_{S \in R^{od}} \exp[-\theta U_S^{od}(t)] \tag{28}$$

C. Stochastic network loading-base dynamic equilibrium algorithm

The diagonalization method from the previous section requires a priori knowledge of future network conditions. In practice, this information is very difficult to obtain, and it might be impossible to obtain with acceptable reliability for real-time operations under congestion conditions [29]. This paper proposes an alternative algorithm that does not require *a priori* knowledge of future network conditions and finds equilibrium during each interval *t*. The dynamic equilibrium algorithm developed in this study uses a mixed-logit-based method for stochastic network loading, described in Section A of the Solution Algorithm, to find the desired equilibrium conditions. The solution to the SDUE can be calculated using the following algorithm.

1) *Step* 0*: Initialization*

Set t = 0, m = 0 to perform loading of the stochastic dynamic network with free-flow travel time, and find a feasible link-flow vector X_m^t and route-flow vector $f_m(t)$.

2) *Step* 1*: Sub-problem*

Using the descent feasible-direction algorithm shown in [23, 27], a sub-problem of the diagonalization method can be solved: Minimize for the current time interval t. This results in a new vector for the link-flow X_m^t.

3) *Step* 1.1: *Direction finding*

Load the network using the DYNASTOCH algorithm summarized in Section A taking into consideration the link flow vector X_m^t. This yields the auxiliary route-flow vector $f_m(t)$ and auxiliary link-flow vector y_m^t.

4) *Step* 1.2*: Line search*

The parameter $\emptyset$ is computed as the value at the minima of equation (29) at the current iteration m and time interval t.

$$\text{Arg min } Z[f_m(t) - \emptyset(t)\{f_m(t) - f_m(t)\}] \tag{29}$$

$$0 \leq \emptyset \leq 1$$

then, update the route flow as:

$$f_m(t) = f_m(t) - \emptyset(t)\{f_m(t) - f_m(t)\} \tag{30}$$

Find the corresponding link-flow vector X_m^t for $f_{m+1}(t)$.

5) *Step* 1.3*: Convergence*:

If *m* arrives at a previously selected number or meets the convergence criteria, go to Step 2; else, set *m* = *m* + 1 and go to Step 1.1 [1].

6) *Step* 2*: Move period:*

set t = t + 1 and go to Step 1.

From Step 1.1, as the number of interactions increases, the gap between the current and auxiliary flow pattern diminishes [30]. This is because, to find the search direction, the network is loaded following a stochastic dynamic approach. This implies that, in equilibrium, the auxiliary flow pattern will be identical from both the stochastic dynamic network loading and the current flow pattern. Hence, a pure network-loading method seems appropriate to solve (30), given that the network-loading method used in this study actually converges towards the equilibrium state.

For a pure network-loading method, $\emptyset$ = *1.0* irrespective of the iteration number *m*. This implies that, for each iteration, the link flow pattern is not updated as the value found between the current and auxiliary flow. Instead, the auxiliary flow pattern replaces the current flow pattern. Hence, loading of the network in a stochastic dynamic approach can be performed in successive steps regardless of the size of the move parameter $\emptyset$. This methodology is applicable to both reactive and predictive assignment since the move size can be calculated without consideration of the mathematical formulation [1].

IV. CONCLUSION

A mathematical programming formulation and solution algorithm were proposed for the SDUE assignment problem. The proposed solution algorithm does not require direct evaluation of the objective function. The stochastic network-loading algorithm is expected to exploit the advantages of the mixed-logit model in terms of computational efficiency, flexibility, and accuracy. These advantages have been shown by several authors [4, 19, 5, 31] in various context. They enable the development of a robust SDUE assignment models that are applicable in general transportation networks while ensuring correct flow propagation. The flexibility in error terms specification provided by the mixed-logit model enables the proposed SDUE model to explicitly capture spatial and temporal correlations in unobserved factors in a single framework. To the best knowledge of the authors, this is the first study proposing the use of the mixed-logit model to solve the SDUE problem. The computational efficiency and the prediction accuracy of the mixed-logit model is expected to enable deployment of the SDUE model for the modeling or large scale networks or for real-time operations such as those proposed in [32, 33, 34].

Future work is essential to test the envisioned capabilities of the proposed SDUE model. However, all details required for implementation of the proposed model are provided in this paper. Another potential extension of this work is to include additional factors that influence route choice decisions. It is well known that an individual's route decisions are dependent of several factors in addition to travel time.

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