Advanced Calculus L 1

functions defined on $\mathbb{R}^n$

1/ Euclidean space $\mathbb{R}^n$

Algebraic Properties

$\mathbb{R}^n$ is the set of all points

$P = (x_1, \ldots, x_n)$ where $x_1, \ldots, x_n$ are real

define addition and scalar multiplication

If $P = (x_1, \ldots, x_n)$ $Q = (y_1, \ldots, y_n)$

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$$P+Q=(x_1+y_1, \dots, x_n+y_n)$$

and

$$\lambda P=(\lambda x_1, \dots, \lambda x_n)$$

define inner product by

$$(P,Q)=\sum_{k=1}^{n} x_k y_k$$ (dot product)

we define the Euclidean norm by

$$\|P\|=\sqrt{x_1^2+\dots+x_n^2}$$

(This is like the distance between points)

norm properties

1/ $\|P\| \geq 0$

2/ $\|\lambda P\|=|\lambda|\,\|P\|$ $\forall \lambda \in \mathbb{R}$

Cauchy - Schwarz inequality/

$$|\langle P, Q\rangle| \leq \|P\| \, \|Q\|$$

$$|P \cdot Q| \leq \|P\| \, \|Q\|$$

4'/ Triangle inequality

$$\|P + Q\| \leq \|P\| + \|Q\|$$

Proof

$$\forall \lambda, \quad (\|\lambda P + Q\|)^2 = (\lambda P + Q) \cdot (\lambda P + Q)$$

$$= |\lambda|^2 \|P\|^2 + 2\lambda (P \cdot Q) + \|Q\|^2 \geq 0$$

Let $\lambda = -\dfrac{P \cdot Q}{\|P\|^2}$ $\qquad P \neq 0$

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$$0 \le \frac{(P\cdot Q)^2}{\|P\|^2} - \frac{2\,|P\cdot Q|^2}{\|P\|^2} + \|Q\|^2$$

$$0 \le -\frac{|P\cdot Q|^2}{\|P\|^2} + \|Q\|^2$$

$$\frac{|P\cdot Q|^2}{\|P\|^2} \le \|Q\|^2 \rightarrow |P\cdot Q|^2 \le \left(\|P\|\,\|Q\|\right)^2$$

From Book

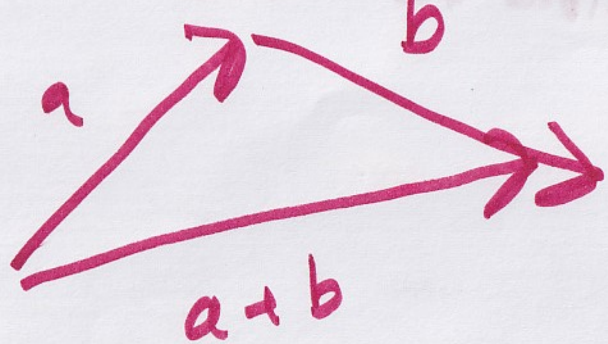
$$\|a+b\|^2 = (a+b)\cdot(a+b)$$

$$= \underline{a\cdot a + 2a\cdot b + b\cdot b}$$

$$\le a\cdot a + 2\|a\|\,\|b\| + b\cdot b$$

$$\le \underbrace{\|a\|^2 + 2\|a\|\,\|b\| + \|b\|^2}_{(\|a\| + \|b\|)^2}$$

$$\|a+b\|^2 \le (\|a\| + \|b\|)^2$$



⑥

Another way - method

$$0 \leq |\lambda|^2 \|P\|^2 + 2\lambda(P\cdot Q) + \|Q\|^2$$

we want the roots

If this has 2 real roots, it can be negative which contradicts the equality ($\leq$)

→ impossible to have 2 real distinct roots

$$(P\cdot Q)^2 - \|P\|^2\|Q\|^2 \leq 0$$

$$(P\cdot Q)^2 \leq \|P\|^2\|Q\|^2$$

$$P\cdot Q \leq \|P\|\,\|Q\|$$

$$\left|\sum_{k=1}^{n} x_k y_k\right| \leq \left(\sum_{k=1}^{n} x_n^2\right)^{1/2}\left(\sum_{k=1}^{n} y_k^2\right)^{1/2}$$

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2nd ; Topological properties of $\mathbb{R}^n$

i) Nieghbourhood

The set

$$B_\delta(P_0) = \{ P ; \; \|P - P_0\| < \delta \}$$

is called a ball at center P_0 and radius δ, or an open sphere at center P_0 and radius δ

ii) the set

$$\overline{B}_\delta(P_0) = \{ P ; \|P - P_0\| \leq \delta \}$$

is called a closed sphere of center P_0 and radius δ

iii) the set

$$N'(P_0) = \{ P ; 0 < \|P - P_0\| < \delta \}$$

is called a deleted neighborhood of P_0

Center point is removed

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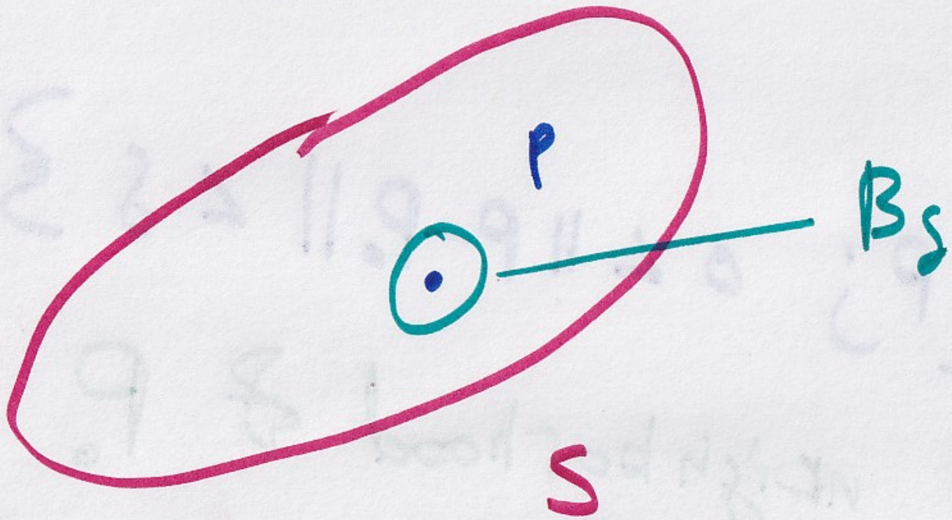
2/ open set ~ Let $S \in \mathbb{R}^n$

A point P of S is called an interior point if there is a neighborhood $B_\delta(P)$ entirely contained in S



The set of all interior points of S is denoted by S^0

If $S^0 = S$, then we say that S is open

3/ closed set

Let $S \subseteq \mathbb{R}^n$ ~ A point P is a point of accumulation of S if every neighborhood has at least one point of S (in S) other than P itself

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The set of all points of accumulation is denoted by $\overline{S}$

if $\overline{S} = S$ then we say that S is closed

(i.e S is closed if it contains all of its points of accumulation)

Note: Point of accumulation,
limit point
cluster point

→ are the same concept

Review of Continuity of functions $F : \mathbb{R}^n \to \mathbb{R}^m$

~~Def~~ Def: Suppose $F : \mathbb{R}^n \to \mathbb{R}^m$ is defined on a domain D and A is a point of accumulation of D

(14)

we say that F is continuous at A if

Given $\ni \varepsilon > 0$, $\exists\ \delta > 0$ s.t.

$$\| F(P) - F(A) \| < \epsilon \quad \text{whenever}$$

$$\| P - A \| < \delta \quad P \in D$$

(P is arbitrary)

or we write $\lim\limits_{P \to A} F(P) = F(A)$

Ex) $f: \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = \frac{xy}{1+x^2+y^2} \qquad (x,y) \in \mathbb{R}$$

$P_0 = (0,0)$ Is f continuous at $(0,0)$?

$$|f(P) - f(0,0)| = |f(P)| = \frac{|xy|}{1+x^2+y^2}$$

arbitrary point

Polar coordinates

$$|r^2 \cos(\theta)\sin(\theta)| \le r^2$$

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$$|f(P)-0| \leq \frac{r^2}{1+r^2} \leq r^2 < \delta^2 = \epsilon$$

$$\delta = \sqrt{\epsilon}$$

if $\|P\| < \delta = \sqrt{\epsilon}$

then $|f(P)-0| < \epsilon$

f is continuous at $(0,0)$

Ex)

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = \bar{0} \end{cases}$$

$$|f(P) - 0| = \frac{|xy|}{x^2+y^2} \qquad \begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

$$f(P) = \frac{r^2 \cos\theta \sin\theta}{r^2} = \cos\theta\sin\theta = \frac{\sin 2\theta}{2}$$

$$\lim_{P \to (0,0)} f(P) \stackrel{?}{=} 0$$

Tempting to say continuous

if go to origin by $\pi/4$ get $1/2$

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Thm 1 $f: \mathbb{R}^n \to \mathbb{R}^m$ is continuous at A

$$\Longleftrightarrow$$

For <u>every</u> sequence, (P_k) converging to A we have $\lim_{k \to \infty} F(P_k) = F(A)$

Proof:

"$\Rightarrow$" Assume F is continuous at A

Given $\epsilon > 0$, $\exists\ \delta > 0$ s.t.

$\cancel{\| F(P) - F(A)}$

$\| F(P) - F(A) \| < \epsilon$

when

$$\|P - A\| < \delta$$

Now $P_k \to A$.

$$\exists \; k_0 \in \mathbb{N} \text{ s.t.}$$

$$\| P_k - A \| < \delta, \; \forall \; k \geq k_0$$

then $$| F(P_k) - F(A) | < \epsilon$$

$$\therefore \; F(P_k) \to F(A)$$

as $k \to \infty$

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"$\Leftarrow$" By contradiction

Suppose F not continuous at A

Now $(P_n) \to A$

$F(P_k) \to F(A)$

$\exists\ \varepsilon_0 > 0$ s.t.

$\cancel{\| F(A) - F}$ $\quad \| F(P_n) - F(A) \| \geq \varepsilon_0 > 0$

with $\| P - A \| < \delta \quad \forall\ \delta > 0$

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Take $\delta = \frac{1}{k}$

if $k \to \infty$ $\frac{1}{k} \longrightarrow 0$ duh!!

$\| P_k - A \| < \frac{1}{k}$, $\| F(P_k) - F(A) \| \geq \epsilon_0 > 0$

This says $P_k \to A$

But $F(P_k) \not\longrightarrow F(A)$

$\longrightarrow$ contradiction