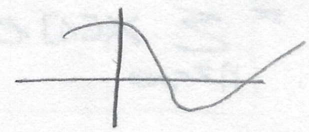
DSP Lect 7

Discrete-time Fourier transform

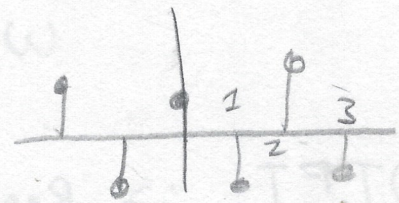
Continuous Time FT (CTFT)

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$



x(t)

Turn the integral to a sum → Sum over the Discrete values



$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

Discrete-time fourier transform → DTFT

→ still a continuous function of ω frequency domain is still continuous

Note! For the continuous time ~~fourier~~ transform we can have frequencies arbitrarily high

→ we can have a cosine of 5000, 10000 Hz

100K Hz

In discrete time there are fixed frequencies

set of fixed frequencies

② CTFTs Have a frequency range $(-\infty, \infty)$

DTFTs Have a frequency range width 2π

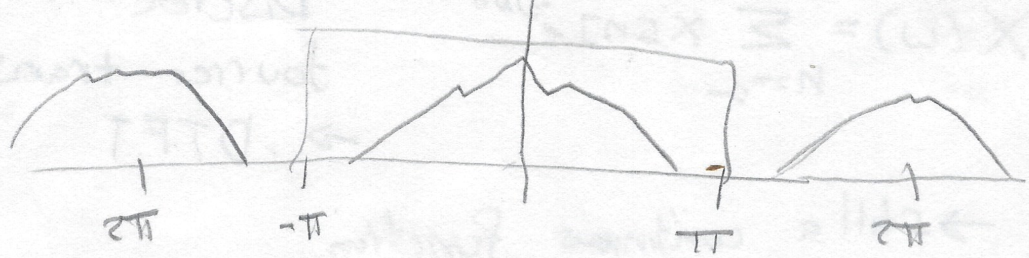
$$X(\omega+2\pi) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j(\omega+2\pi)n}$$

n is an integer

$$= \boxed{\sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}}\ \underbrace{e^{-j2\pi n}}_{1}$$

$= X(\omega)$ The DTFT can be evaluated at any ω, but it's 2π periodic

DTFT → Repeated copies of whats in the middle

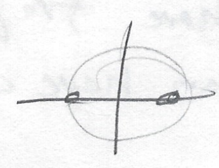


→ In practice we focus on the range $[-\pi, \pi]$

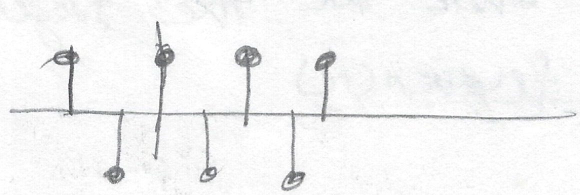
→ focus on whats drawn in the box

→ draw only that.

$$e^{j\pi n} = \cos(\pi n) + j\sin(\pi n)$$



$$= \pm 1 = (-1)^n$$



⇐ Highest frequency discrete time can make

Continuous time inverse fourier transform

→ CT IFT: $\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$

DT IFT

by analogy: $\frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega = x[n]$

Proof; consider $\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega$

$= \int_{-\pi}^{\pi} \sum_{m=-\infty}^{\infty} x[m]e^{-j\omega m}e^{j\omega n}d\omega$ → want to switch integral and sum

A sufficient condition (In order to do ~~this~~) to switch the order

$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$ must be true

So we can say

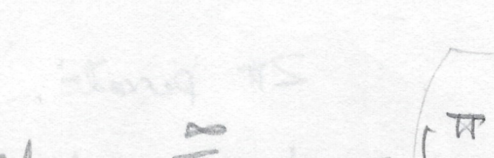
$\int_{-\pi}^{\pi} \sum_{m=-\infty}^{\infty} x[m]e^{-j\omega m}e^{j\omega n}d\omega = \sum_{m=-\infty}^{\infty} x[m]\boxed{\int_{-\pi}^{\pi} e^{j\omega(n-m)}d\omega}$

$\longrightarrow \int_{-\pi}^{\pi} \cos(\omega(n-m)) + j\sin(\omega(n-m))\,d\omega$ $\quad -j\cos(\omega(n-m))\Big|_{-\pi}^{\pi}$

$-j[\pm1 - \pm1]$

$n-m=0 \quad \int_{-\pi}^{\pi} 1\,d\omega = 2\pi$

$n-m\neq 0 \quad \int_{-\pi}^{\pi}$



(4) $\boxed{\text{DTFT}}$

DTFT:

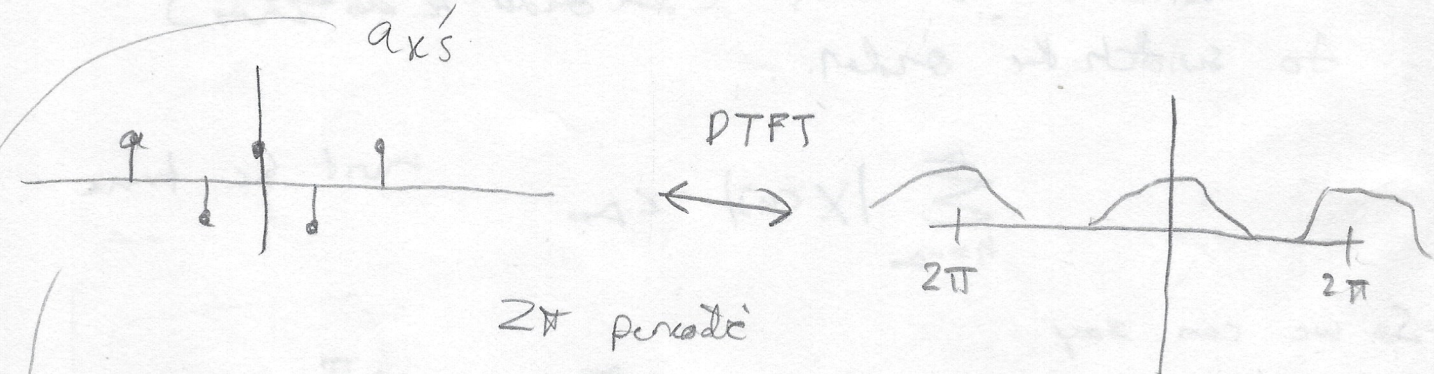
$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$\underbrace{x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega}$$

Looks like Fourier series expansion for a signal with period 2π

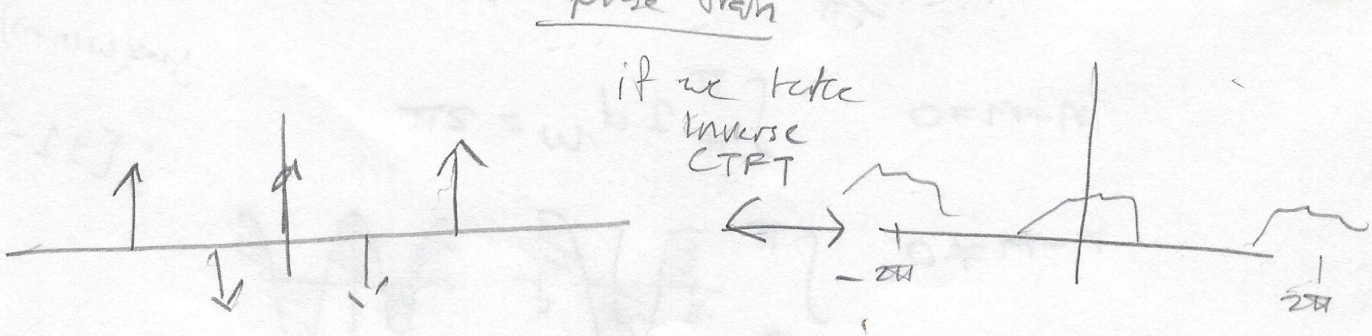
$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-j\underbrace{\frac{2\pi}{T}}_{\omega} kt} dt$$

1



Take the 2π periodic signal and go backwards and ~~compute~~ the Fourier series you get the a_ks

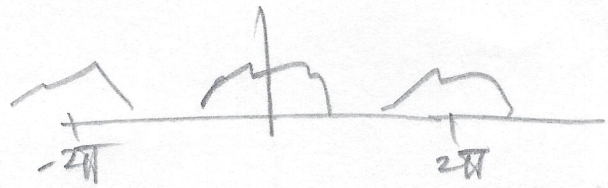
→ If we convert this to an impulse train



$\boxed{\text{DTFT}}$

→ when taking the fourier transform we should get a finite number

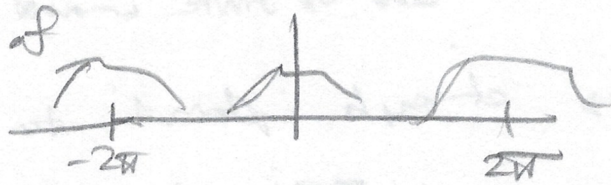
but here



we get ∞ (summing over an infinite number copy of 'stuff')

Exception for periodic signals

If we take FT of



we get an impulse train where the height of each impulse corresponds to the Fourier Series coefficients.

	Continuous	Discrete
periodic	FS	? (DFT) for periodic
NON periodic	FT	DTFT

[DTFT]

6 When does the DTFT work?

If $\sum_{n=-\infty}^{\infty} |x[n]| < \infty$

$$\lim_{N\to\infty} \sup_{w} |X(w) - X_N(w)| = 0$$

As N gets Big, $X_N(w) = \sum_{n=-N}^{N} x[n] e^{-jwn}$ — adding up a finite amount of Fourier Series

Partial sum — add up finite amount of terms

→ converges at each ~~point~~ to $X(w)$

→ want approx of FT to be as close as possible to the true thing, we get there by adding enough terms at every point

If instead we have

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$$

(Looser cond.)

(Signals can satisfy this but not the previous conditions)

we have as a convergence

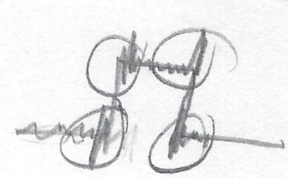
→ $$\lim_{N\to\infty} \int_{-\pi}^{\pi} |X(w) - X_N(w)|^2 dw = 0$$

* mean square convergence

may not be true exactly (at every point) but the difference gets smaller.

$\boxed{\text{DTFT}}$

A reason for that is related to the Gibbs phenomenon (last time)



$$\lim_{N\to\infty} \sup_{\omega} |X(\omega)-X_N(\omega)| \text{ is violated}$$

when integrating the Gibbs, the difference gets smaller and smaller

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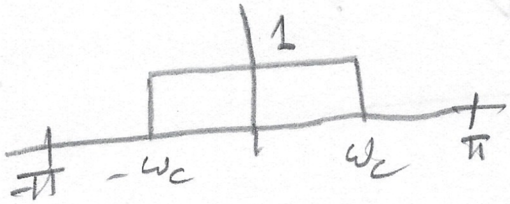
LPF

$X(\omega) =$



$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} \, d\omega$$

$$= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega n} \, d\omega = \frac{1}{2\pi j n} e^{j\omega n}$$

$$= \frac{1}{2\pi j n} \left(e^{j\omega_c n} - e^{-j\omega_c n} \right) = \frac{1}{\pi n} \sin(\omega_c n)$$

$$= \frac{\omega_c}{\pi} \operatorname{sinc}(\omega_c n)$$

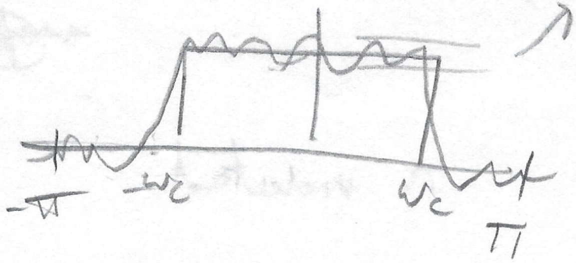
Satisfies $\sum | \, |^2 < \infty$

Not $\sum | \, | < \infty$

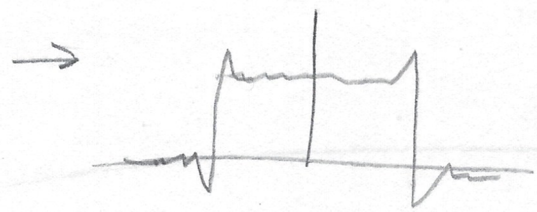
→ If we consider these partial approximations in the time domain (start in the middle and add additional terms)

DTFT

8 In the frequency domain, we get

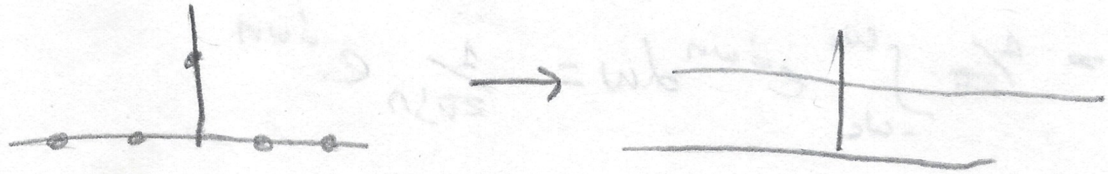


Gibbs phenomenon - Like
→ don't converge at every - point



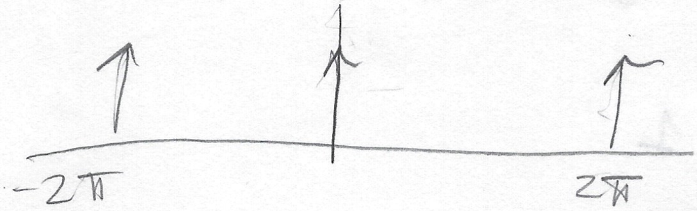
$$x[n] = \delta[n]$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} \delta[n]\, e^{-j\omega n} = 1$$



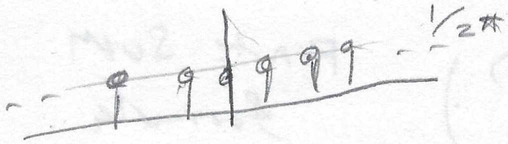
what if $X(\omega) = \delta(\omega)$?

Note > we can't have a delta sitting alone at 0 like in the continuous fourier transform, cause everything in the discrete - frequency domain world must be periodic
(time)

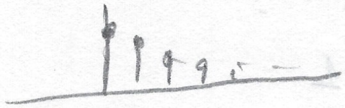


Inverse FT?
$\delta(\cdot)$ can't be by itself

$$X[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}\,d\omega = \frac{1}{2\pi}$$



$x[n] = a^n u[n]$ $\quad |a|<1$



remember the conditions is satisfied
→ can take FT

$$X(\omega) = \sum_{n=-\infty}^{\infty} a^n u[n] e^{-j\omega n} = \sum_{n=0}^{\infty} a^n e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (ae^{-j\omega})^n = \frac{1}{1-ae^{-j\omega}}$$

real signal ↦ to complex FT

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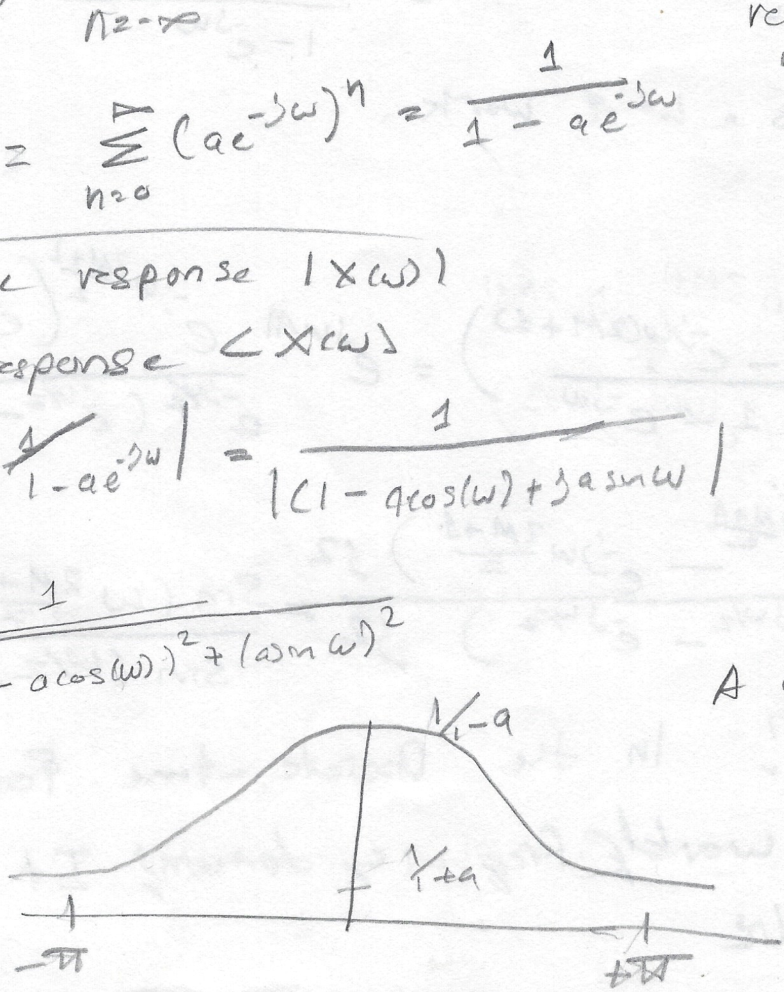
magnitude response $|X(\omega)|$

phase response $\angle X(\omega)$

$$|X(\omega)| = \left|\frac{1}{1-ae^{-j\omega}}\right| = \frac{1}{|(1-a\cos(\omega)+ja\sin\omega|}$$

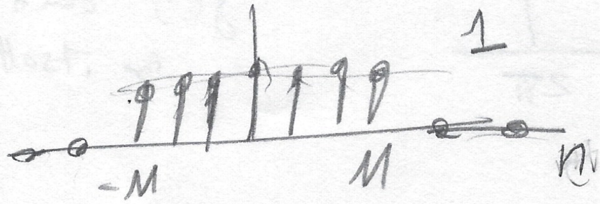
$$= \frac{1}{\sqrt{(1-a\cos(\omega))^2+(a\sin\omega)^2}}$$

A crude LPF



(10) [DTFT]

Pulse in the time domain



$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = \sum_{n=-M}^{M} e^{-j\omega n} = e^{j\omega M} \sum_{n=0}^{2M} e^{-j\omega n}$$

$$= e^{j\omega M}\left(\frac{1-e^{-j\omega(2M+1)}}{1-e^{-j\omega}}\right)$$

Finite Sum formula (recall earlier learned equations)

$\omega=0 \quad X(\omega)=2M+1$

or try l'Hospitals rule

$$\frac{e^{j\omega M} - e^{-j\omega 2M + j\omega + j\omega M}}{1-e^{-j\omega}} = \frac{e^{j\omega M} - e^{-j\omega M - j\omega}}{1-e^{-j\omega}}$$

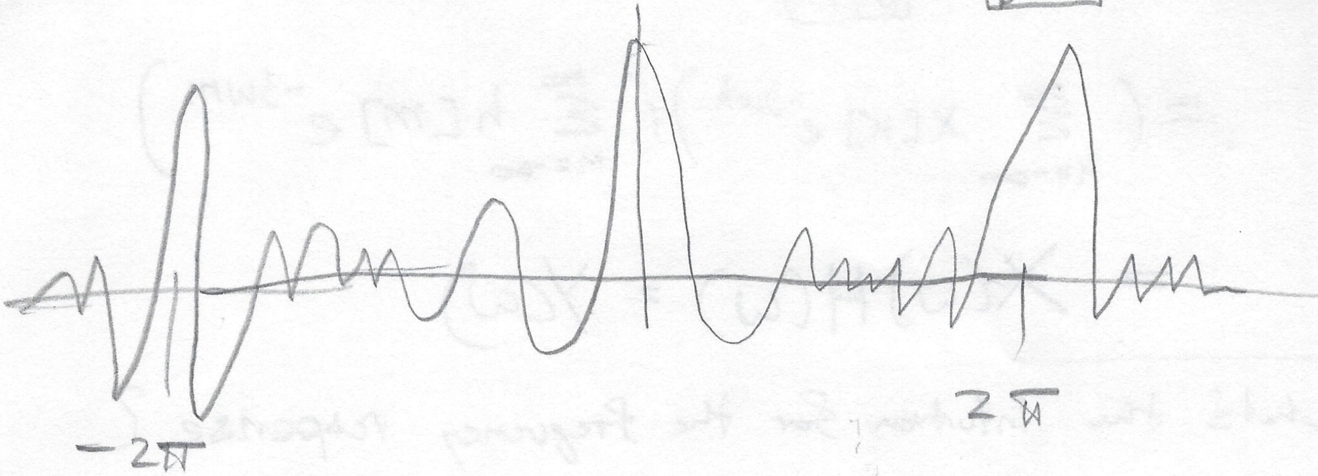
→ forget it, it's a lot of work

So if $\omega \neq 0$

$$\rightarrow \quad e^{j\omega M}\left(\frac{1-e^{-j\omega(2M+1)}}{1-e^{-j\omega}}\right) = e^{j\omega M} \frac{e^{-j\omega\frac{2M+1}{2}}\left(e^{j\omega\frac{2M+1}{2}} - e^{-j\omega\frac{2M+1}{2}}\right)}{e^{-j\omega/2}\left(e^{j\omega/2} - e^{-j\omega/2}\right)}$$

$$= 1 \, \frac{\left(e^{j\omega\frac{2M+1}{2}} - e^{-j\omega\frac{2M+1}{2}}\right) j2}{\left(e^{j\omega/2} - e^{-j\omega/2}\right) j2} = \frac{\sin\left(\omega\frac{2M+1}{2}\right)}{\sin(\omega/2)}$$

→ No Sinc? In the Discrete-time Fourier ~~transfer~~ world (frequency domain) It must be periodic



- Periodic version of the sinc functions
  → kinda like what happens when we set sinc functions and shifted them to be centered somewhere around 2π intervals

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Want to take DTFT of the impulse response

Use DTFT to study LTI ~~systems~~

→ Convolution property still holds

$$y[n] = x[n] * h[n]$$

$$Y(\omega) = \sum_{n=-\infty}^{\infty} y[n] e^{-j\omega n} = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x[k] h[n-k] e^{-j\omega n}$$

$$= \sum_{k=-\infty}^{\infty} x[k] \sum_{n=-\infty}^{\infty} h[n-k] e^{-j\omega n}$$

let $m = n-k$, $n = m+k$

$$= \sum_{k=-\infty}^{\infty} x[k] \sum_{m=-\infty}^{\infty} h[m] e^{-j\omega(m+k)}$$

$\boxed{DTFT}$

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$$= \left(\sum_{k=-\infty}^{\infty} x[k] e^{-j\omega k} \right) \left(\sum_{m=-\infty}^{\infty} h[m] e^{-j\omega m} \right)$$

$$= X(\omega) H(\omega) = Y(\omega)$$

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what's the Intuition for the frequency response?

take subcomponent frequency of the input and modulated by some amplification ~~or attenuations~~ and shifts by some phase

value of ~~frequency response~~ at a fixed frequency tells us ~~that~~ How are cosines and sines of that frequency are being shifted rounded

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$$x[n] = A e^{j\omega_0 n}$$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k] x[n-k]$$

$$= \sum_{k=-\infty}^{\infty} h[k] A e^{j\omega_0 (n-k)} = A e^{j\omega_0 n} \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega_0 k}$$

$$= H(\omega_0) A e^{j\omega_0 n}$$

$$A e^{j\omega_0 n} \rightarrow A \underbrace{|H(\omega_0)|}_{\text{Amplitude Scaling}} e^{j(\underbrace{\angle H(\omega_0)}_{\text{Phase shifting}} + \omega_0 n)}$$

In the same way, if h[n] is real

$$\cos(\omega_0 n + \phi) \rightarrow \underbrace{|H(\omega_0)|}_{\text{Amp}} \cos(\omega_0 n + \phi + \underbrace{\angle H(\omega_0)}_{\text{phase shift}})$$

can't creat new frequencies

only amp./attenu, phase shift

$h[n] = (1/3)^n u[n]$

$x[n] = 2e^{j\pi/3 n}$ (single sinusoid at freq $\pi/3$

So what happening $H(\pi/3)$?

$$H(\omega) = \frac{1}{1 - \frac{1}{3}e^{-j\omega}}, \quad H(\pi/3) = \frac{1}{1 - \frac{1}{3}e^{-j\pi/3}}$$

$$= \frac{1}{1 - \frac{1}{3}(\frac{1}{2} - \frac{\sqrt{3}}{2}j)} = \frac{1}{\frac{5}{6} + \frac{\sqrt{3}}{6}j}$$

So whats

$|\cdot|$, $\angle|\cdot|$

Now

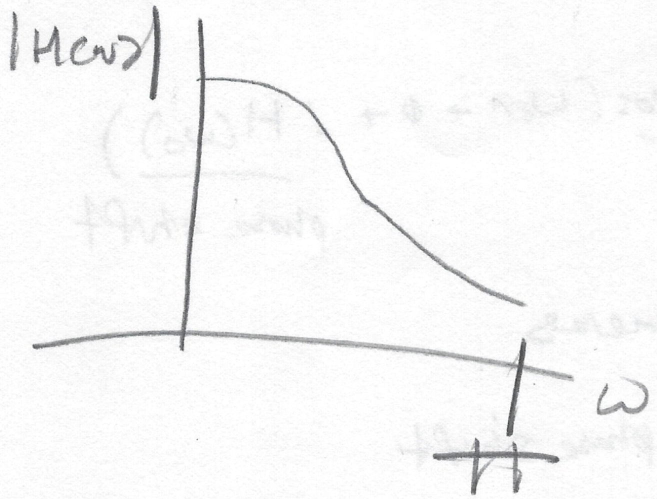
$$2|H(\pi/3)| e^{j(\frac{\pi}{3}n + \angle H(\frac{\pi}{3}))}$$

When you have a combination of cos, sine

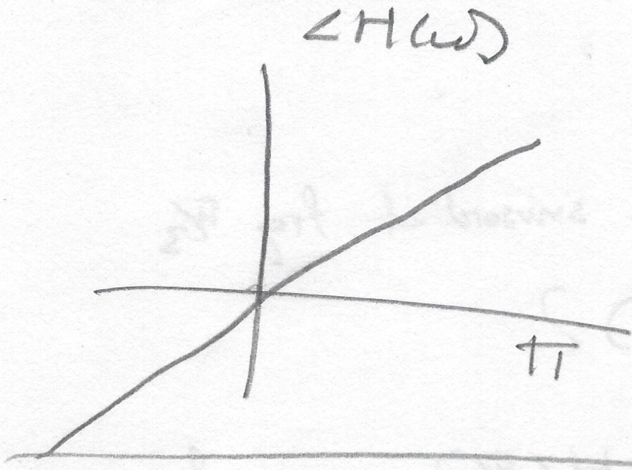
in the input, can only get that attenuated or Ampl. and phase shift

(14) Magnitude Response

When $h[n]$ is Real, $|H(\omega)|$ is Even



When $h[n]$ is real $\angle H(\omega)$ is odd

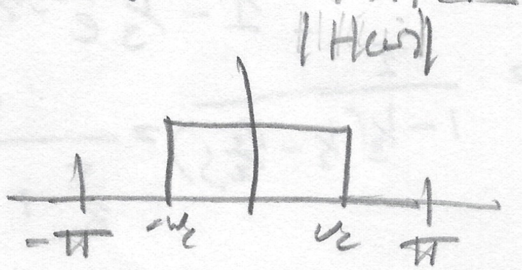


filters classified by their magnitude responses

"Ideal" Filters

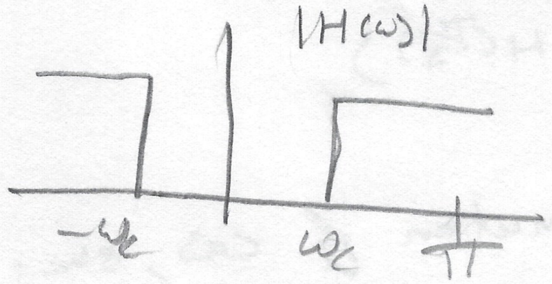
LPF

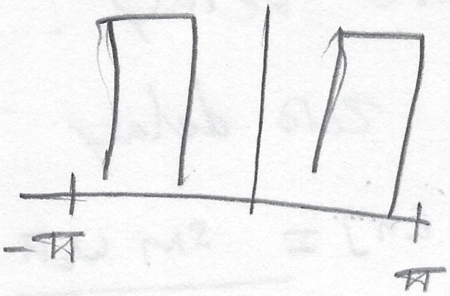
cutoff freq ω_c



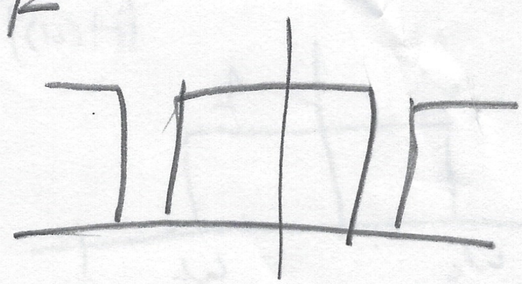
Keep in mind the frequencies are bounded here

HPF



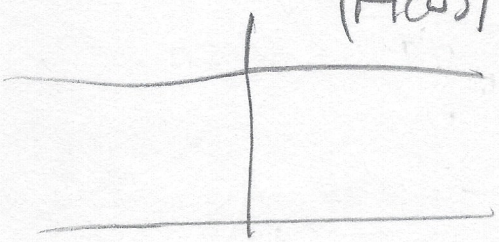


BPF



Band stop

$|H(\omega)|$



All pass filter
'phases' of cosines, sines
<u>changes</u> may be important

Phase Response

why is $\angle H(\omega) = -c\omega$ Desireable (Linear)

1) Say $|H(\omega)|$ is piecewise constant

piecewise constant

$Y(\omega) = X(\omega)H(\omega)$ $(|H(\omega)| = 1)$

$= X(\omega)\,|H(\omega)|\,e^{j\angle H(\omega)}$

$= X(\omega)\,e^{-jc\omega}$ for Linear phase filter

$\angle H(\omega) = -c\omega$

$Y[n] = X[n-c]$

Linear phase delays output

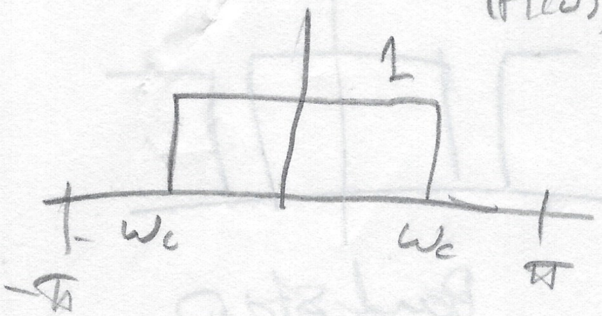
all cosines and sines
Delayed same amount

No Distortion

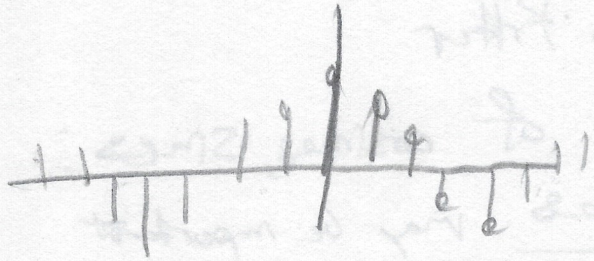
$\boxed{\text{DTFT}}$

why not have zero delay?

$|H(\omega)| = H(\omega)$ zero delay



$$h[n] = \frac{\sin \omega_c \pi n}{\pi n}$$



Non causal

[Not Absolutely Summable → $\Sigma |\ | = \infty$, Not stable and ∞ Long

→ this $h[n]$ is not desireable

→ can come close to an Ideal LPF but with Linear phase

delay in output 'fine'