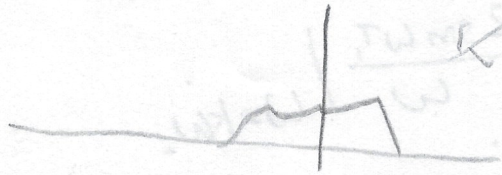
DSP Lect 5

FS Deals with periodic signals

FT Deals with more like an aperiodic signal

one in the middle



→ Imagine copies infinitely faraway

at the limit, FS coefficient become the Fourier transform

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T}$$

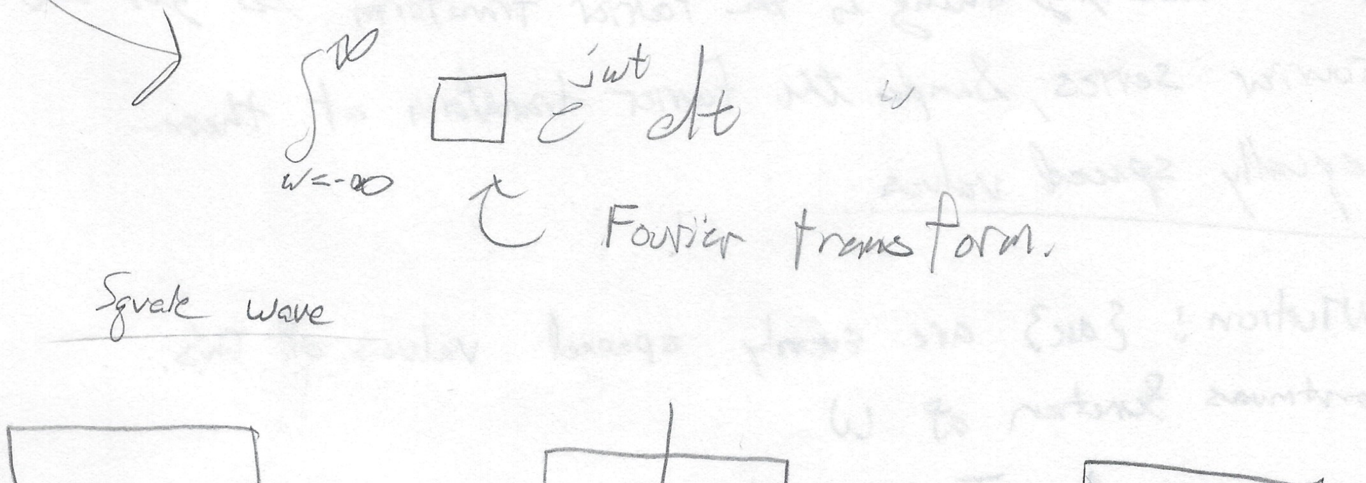
→ a bunch of sinusoids with ~~a~~ different frequencies

as $T \to \infty$, The sum becomes an Integral

$$\int_{w=-\infty}^{\infty} \boxed{} \, e^{j\omega t} \, dt$$

Fourier transform.

Square wave



(2)

$$x(t)=\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \qquad \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}\,dt$$

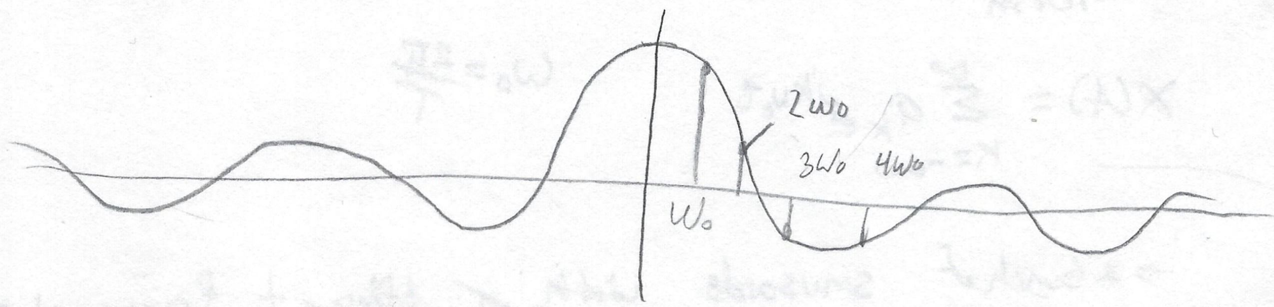
$$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T} \qquad = \frac{1}{T}\frac{-1}{jk\omega_0}\left(e^{-jk\omega_0 T_1} - e^{jk\omega_0 T_1}\right)$$

$$= \frac{1}{T}\frac{2}{k\omega}\sin(k\omega_0 T_1)$$

and

$$Ta_k = \frac{2\sin k\omega_0 T_1}{k\omega_0} = \left.\frac{2\sin \omega T_1}{\omega}\right|_{\omega = k\omega_0}$$

To get a_k's, Take the continuous function of ω and Sample every ω_0 units



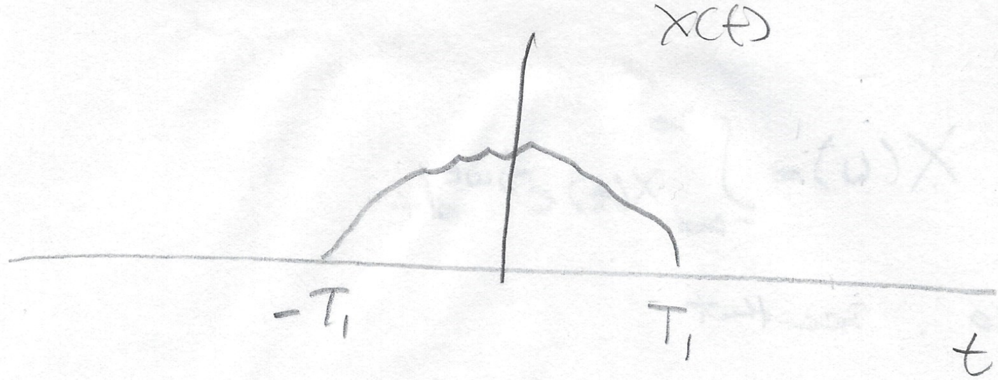
→ Sample every ω_0 to get a_k's

→ The underlying thing is the Fourier transform to get the fourier series, Sample the fourier transform at these equally spaced values.

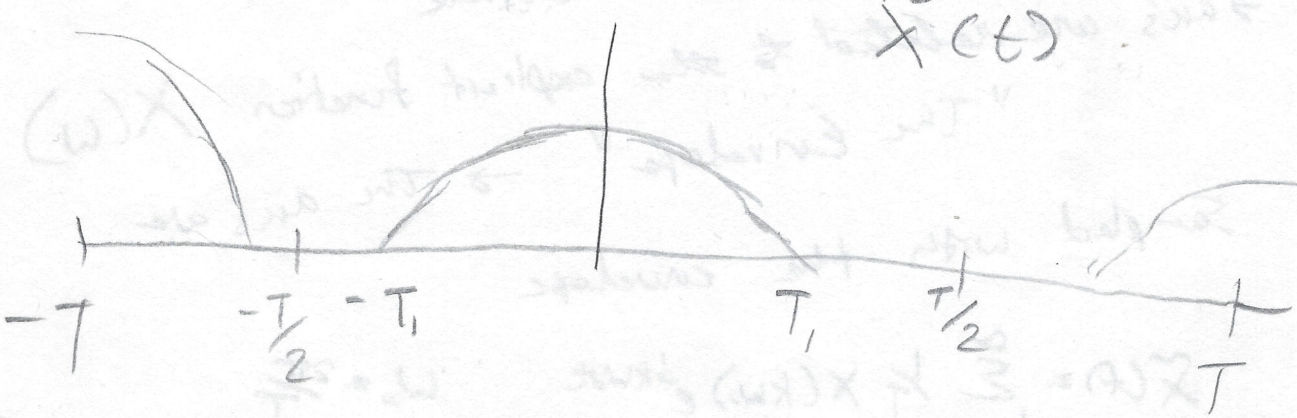
INTuition: $\{a_k\}$ are evenly spaced values of This continuous function of ω

As $T \to \infty$, $\omega_0 \to 0$, samples Get

Infinitely close.



Take this signal and create periodic copies of it where there is a section(s) during the period where the value of $x(t)$ is zero



$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \qquad a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t)\, e^{-jk\omega_0 t}\, dt$$

And the signal within $(-T/2, +T/2)$ is the same so

$$= \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_{-\infty}^{\infty} x(t)\, e^{-jk\omega_0 t}\, dt$$

(4) Define

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

and also see that

$$a_k = \frac{1}{T}\int_{-\infty}^{\infty} x(t) e^{-jk\omega_0 t} dt = 1/T \, X(k\omega_0)$$

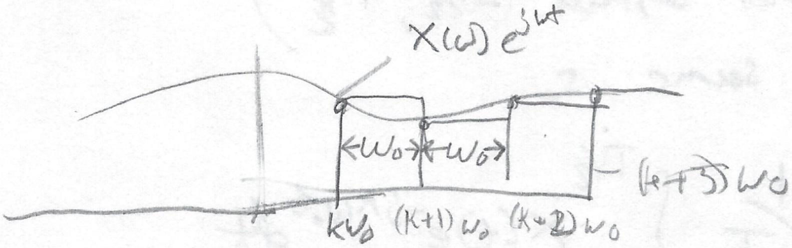
we picked the frequency $k\omega_0$ for the $X(\cdot)$
and now the integrals are identicle
→ a_k's are related ~~to the~~ explicit function $X(\omega)$
"The Envelope" → The a_k's are
sampled with the envelope

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} 1/T \, X(k\omega_0) e^{jk\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T}$$

$$\frac{\omega_0}{2\pi} = 1/T$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \left[X(k\omega_0) e^{jk\omega_0 t} \right] \omega_0$$

→ what does this sum mean?



The samples are summed up at equally spaced intervals of ω_0 (width is ω_0)

Take the spectral function and sample at $k\omega_0$

Each term of the sum can be thought of as a rectangle's area!

So take the Limit

$\lim_{\omega \to 0}$ of Both Sides

From before

$\frac{1}{2\pi} \sum_{n=-\infty}^{\infty} [X(n\omega_0) e^{jn\omega_0 t}] \omega_0$

inverse fourrier transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

fourrier transform

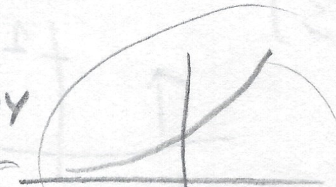
$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$x(t)$ continuous → Period → FS (discrete a_n)

→ aperiodic → FT (continuous $X(\omega)$)

When does this work?

1) $\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$ Finite Energy



aperiodic and ∞

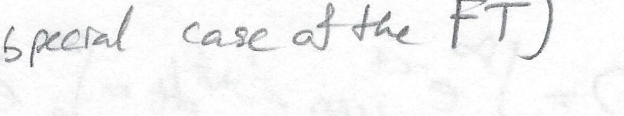
2) Finit # of Extrema (maxima, minima)

Not wiggling infinitely much

3) Finite # of Discontinuities

(FS a special case of the FT)

⊗ Exception; we do allow the FT of Periodic Signals (IT has a FS.)

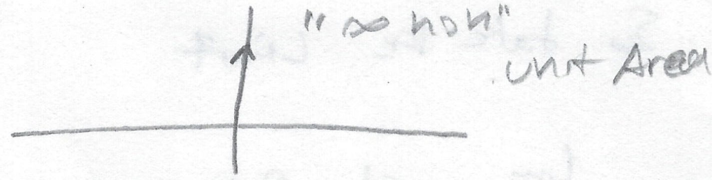


Area under signal is ∞

1st cond. Slides

⑥

1) $x(t) = \delta(t)$



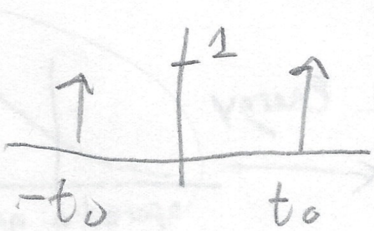
$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt$$

$$= e^{-j\omega(0)} = 1$$ FT of δ is constant!

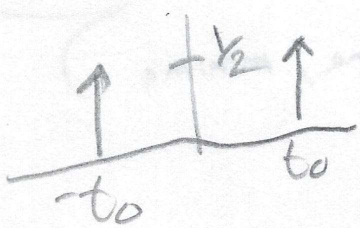
2) $x(t) = \delta(t - t_0)$

$$X(\omega) = \int_{-\infty}^{\infty} \delta(t - t_0) e^{-j\omega t} dt = e^{j\omega t_0}$$ "different phase"

3)



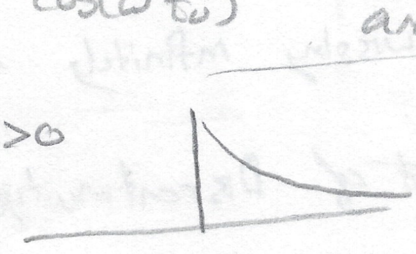
$$e^{j\omega t_0} + e^{-j\omega t_0} = 2\cos(\omega t_0)$$



$\rightarrow \cos(\omega t_0)$

* Think, the height of the δ on the cartesian plane is represented by it's area

4) $x(t) = e^{-at} u(t)$ $a > 0$



area is finite
→ Take FT

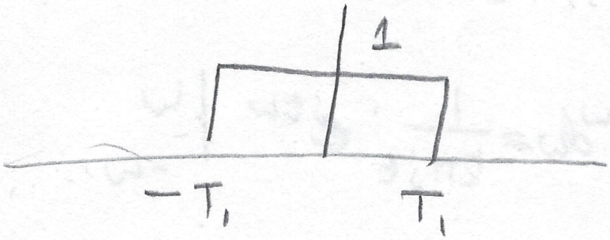
$$X(\omega) = \int_{-\infty}^{\infty} e^{-at} u(t) e^{-j\omega t} dt = \int_0^{\infty} e^{-(a+j\omega)t} dt$$

$$= \frac{-1}{a+j\omega} e^{-(a+j\omega)t} \Big|_0^{\infty} = \frac{-1}{a+j\omega}(0 - 1)$$

FT evaluates along $j\omega$ axis when the ROC includes the $j\omega$ axis.

Laplace > FT > FS Terms of Generality

5)



$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$$

$$= \int_{-T_1}^{T_1} e^{-j\omega t}\, dt$$

$$= \frac{-1}{j\omega} e^{-j\omega t}\Big|_{-T_1}^{T_1} = \frac{-1}{j\omega}\left(e^{-j\omega T_1} - e^{j\omega T_1}\right)$$

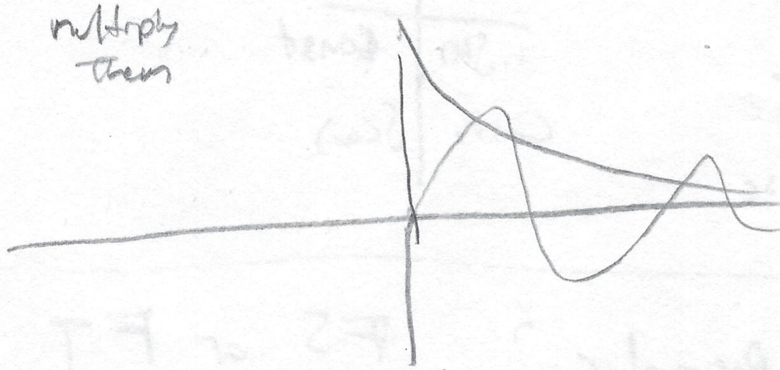
$$2T_1 \operatorname{sinc}(\omega T_1) = \frac{2\sin(\omega T_1)}{\omega} = \frac{2T_1 \sin \omega T_1}{\omega T_1}$$

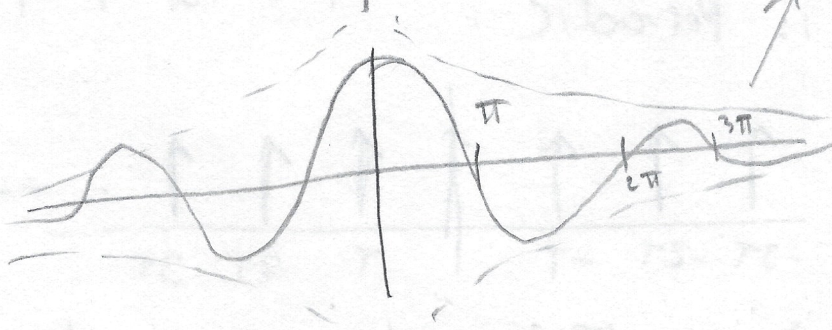
Zero's become functions of T_1

$\operatorname{sinc}(\omega)$

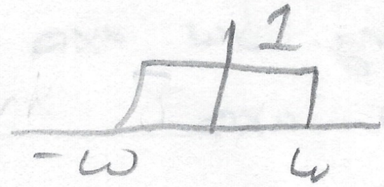
$\frac{1}{\omega}\sin(\omega)$

multiply them





(8) what at a pulse in the frequency domain

$X(j\omega) =$ 

a) $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{jt\omega} d\omega$

$$= \frac{1}{2\pi} \int_{-\omega}^{\omega} e^{jt\omega} d\omega = \frac{1}{2\pi j t} e^{jt\omega} \Big|_{-\omega}^{\omega}$$

$$= \frac{1}{\pi t} \cdot \frac{1}{2j} \left(e^{jt\omega} - e^{-jt\omega} \right) = \frac{\sin(t\omega)}{\pi t}$$

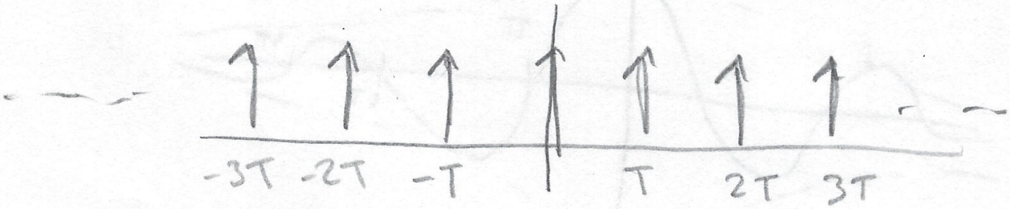
$$= \frac{\omega}{\pi} \operatorname{sinc}(t\omega)$$

Duality

T	F
Pulse	Sinc
Sinc	Pulse

T	F
$\delta(t)$	Const
Const.	$\delta(\omega)$

$x(t)$ is Periodic? FS or FT



$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$ impulse train

FS: $a_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jk\frac{2\pi}{T}t} dt = \frac{1}{T}$ For all K

$$\frac{1}{T} e^{-jk\frac{2\pi}{T}T} = \frac{1}{T}$$

FT: $X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{j\omega t}\, dt$

$$= \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \delta(t-kT)\, e^{j\omega t}\, dt = \sum_{k=-\infty} \cancel{e^{j\omega kT}}$$

No! start with FS

$F^{-1}\{\delta(\omega)\} = \frac{1}{2\pi}$

$$X(\omega) = \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \underbrace{\frac{1}{T} e^{j\omega_0 k t}}_{\text{shifted constant}}\, e^{j\omega t}\, dt$$

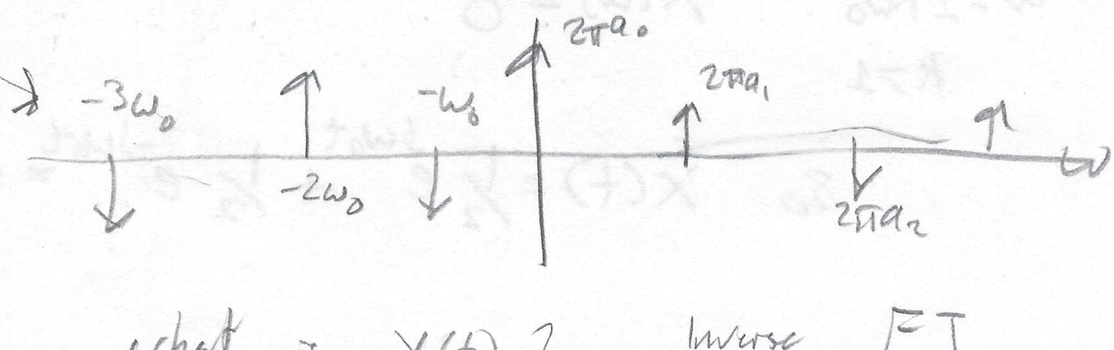
$$= \frac{2\pi}{T} \sum_{k-\infty}^{\infty} \delta(\omega - k\omega_0)$$

START more Generally

Suppose $x(t)$ is a periodic signal,
So it can be represented as $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

$$X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k\, \delta(\omega - k\omega_0)$$

Let's Suppose we have a signal whose Fourier transform looks like



what is $x(t)$? Inverse FT

(10)

Suppose $X(\omega) = \delta(\omega - k\omega_0)$

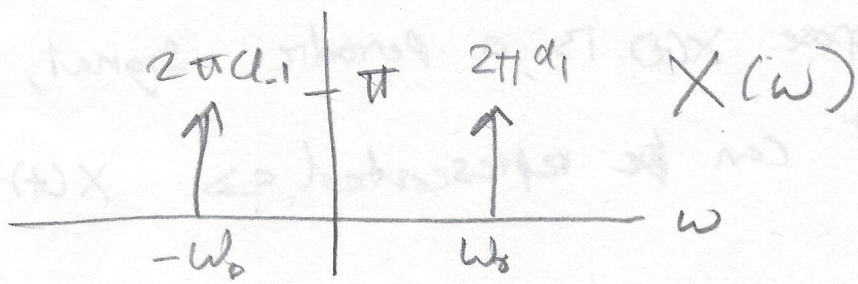
$$X(t) = \frac{1}{2\pi} \int \delta(\omega - k\omega_0) e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi} e^{jk\omega_0 t} \quad \rightarrow \text{ So For } X(\omega) = \sum 2\pi a_k \delta(\omega - k\omega_0)$$

$$X(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

Fourier Series of $X(t)$ for a Periodic Signal with $T = \frac{2\pi}{\omega_0}$

Ex



$\omega = 0 \Rightarrow \cancel{X}(\omega) = 0$

$\omega = \pm\omega_0 \quad X(\omega) = \pi\delta(\omega)$

$\omega = \pm k\omega_0 \quad X(\omega) = 0$
$k > 1$

$a_0 = 0$

$a_\pm = \frac{1}{2}$

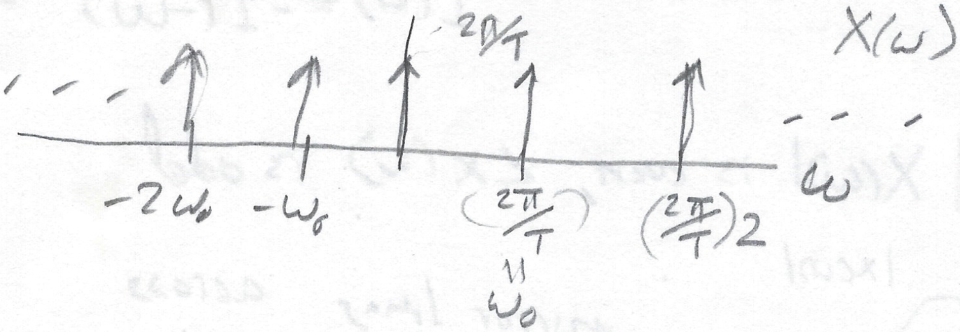
$a_{\pm k} = 0 \quad k > 1$

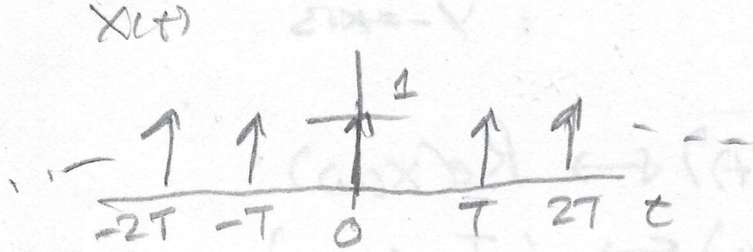
So $X(t) = \frac{1}{2} e^{j\omega_0 t} + \frac{1}{2} e^{-j\omega_0 t} = \cos(\omega_0 t)$

For periodic impulse train, Period T $\quad a_k = 1/T$

$\Rightarrow$ The ~~Impulse~~ Fourier transform of impulse train

we get impulses $\times$ a_k spaced apart by ω_0





wider ~~apart~~ the pulses are in the time domain, closer they are in the frequency domain

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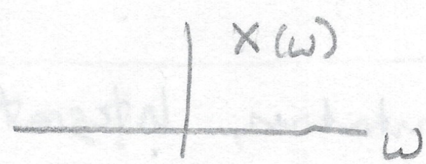
FT properties

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$X(j\omega) \quad X(s) \quad s \to j\omega$



1) Linearity

$x(t) \leftrightarrow X(\omega) \qquad y(t) \leftrightarrow Y(\omega)$

$a x(t) + b y(t) \leftrightarrow a X(\omega) + b Y(\omega)$

2) Time Shift: $\quad x(t - t_0) \leftrightarrow X(\omega) e^{-j\omega t_0}$

$|X(\omega) e^{-j\omega t_0}| = |X(\omega)|$

(12)

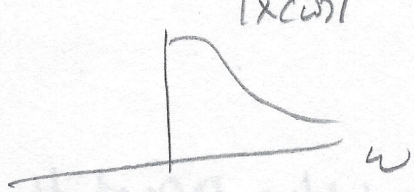
Symmetry Properties

$X(\omega) = R(\omega) + j I(\omega)$

IF $x(t)$ is Real, $R(\omega) = R(-\omega)$ real pt even

$I(\omega) = -I(-\omega)$ Img, odd

$|X(\omega)|$ is even, $\angle X(\omega)$ is odd



mirror Imag across y-axis

can show $Ev(x(t)) \leftrightarrow Re(X(\omega))$

$Od(x(t)) \leftrightarrow j\,Im(X(\omega))$

$x(t)$ Real, even, $X(\omega)$ real, Even

$x(t)$ Real, odd $X(\omega)$ Imaginary, odd

stem from ~~Property 1~~ Symetry Properties

Differentiation, Integration

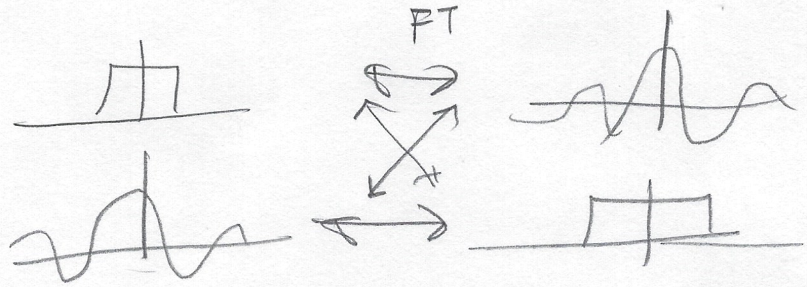
$x'(t) \leftrightarrow j\omega X(\omega)$ $\boxed{sX(s)}$

$\int_{-\infty}^{t} x(\tau)d\tau \leftrightarrow \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega)$

Time-scaling

$x(at) \leftrightarrow \frac{1}{|a|}X(\frac{\omega}{a})$

Duality



Parseval

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |x(t)|^2 d\omega$$

Key Principle <u>Convolution:</u>

If $y(t) = x(t) * h(t)$

$Y(\omega) = X(\omega) H(\omega)$

Impulse response

$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t)$

$\boxed{H(\omega)}$

Frequency response

USE FT in practice