Digital signal Processing — Lect 16

FIR filter design using least-squares

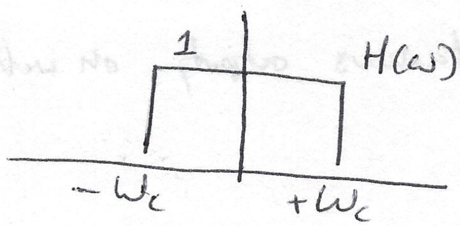
Filter design

$$x[n] \rightarrow \boxed{H} \rightarrow y[n]$$

$$y[n] = x[n] * h[n]$$

$$Y(\omega) = X(\omega)\,H(\omega)$$

$$|Y(\omega)| = |X(\omega)|\,|H(\omega)| \qquad \angle Y(\omega) = \angle X(\omega) + \angle H(\omega)$$

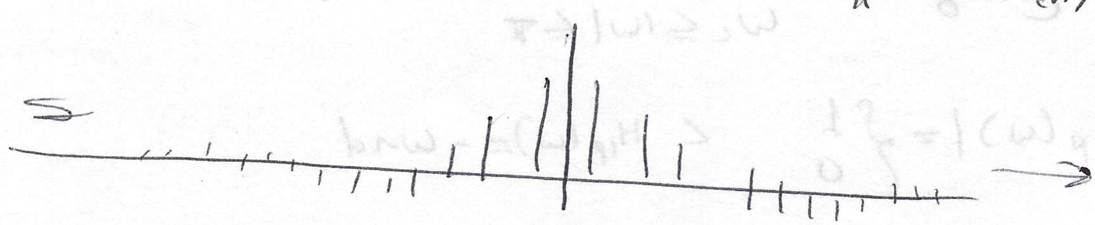


Ideal

$$H_{lp}(\omega) = \begin{cases} 1 & \omega < |\omega_c| \\ 0 & \omega_c < |\omega| < \pi \end{cases}$$

$\boxed{\text{NO Phase shift}}$

$$h_{lp}[n] = \frac{\sin \omega_c n}{\pi n} = \frac{\omega_c}{\pi}\,\frac{\sin(\omega_c n)}{\omega_c n} = \frac{\omega_c}{\pi}\,\mathrm{sinc}(\omega_c n) \qquad -\infty < n < \infty$$



- Infinite-Length Impulse response
- NOT CAUSAL

Think of pure delay filter

(2) Ideal Delay $h_d[n] = \delta[n - n_d]$

$$H_d(\omega) = e^{-j\omega n_d}$$

$|H_d(\omega)| = 1$, $\boxed{\angle H_d(\omega) = -\omega n_d}$ shift in phase is a linear function of ω

With a linear phase, you are delaying the output by some number of units $\rightarrow$ $\boxed{\text{NOT Distortion}}$ $\rightarrow$ Each value is moved by the same number of units. Not varied, if so, then its all uninterpreted $\rightarrow$ NON linear

$\rightarrow$ Shifted in $\boxed{\text{time}}$ Not instantaneous output, as well.

Linear Phase is desired

$$H_{lp}(\omega) = \begin{cases} e^{-j\omega n_d} & |\omega| < \omega_c \\ 0 & \omega_c \le |\omega| \le \pi \end{cases}$$

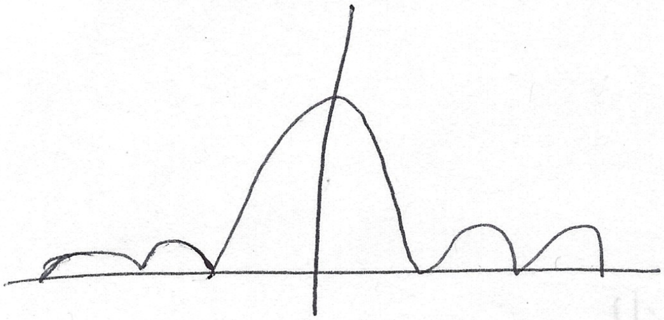
$$|H_{lp}(\omega)| = \begin{cases} 1 \\ 0 \end{cases} \qquad \angle H_{lp}(\omega) = -\omega n_d$$

$$\rightarrow h_{lp}[n] = \frac{\sin \omega_c (n - n_d)}{\pi (n - n_d)} = \frac{\omega_c}{\pi} \operatorname{sinc}(\omega_c (n - n_d))$$

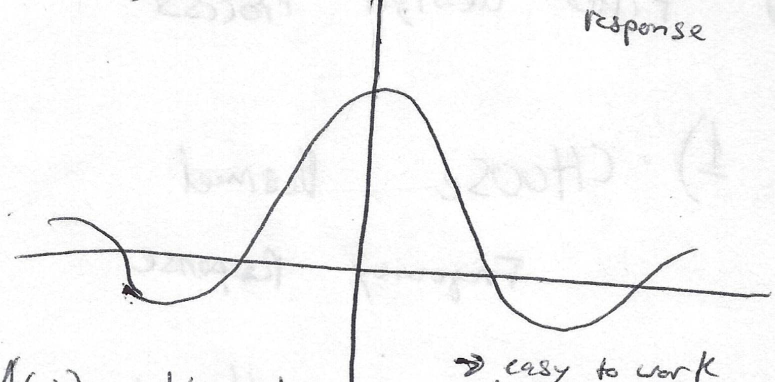
sinc shifted in time

Group delay $= -\frac{d}{d\omega} \arg H(\omega)$

$|H(\omega)|$ Magnitude respons



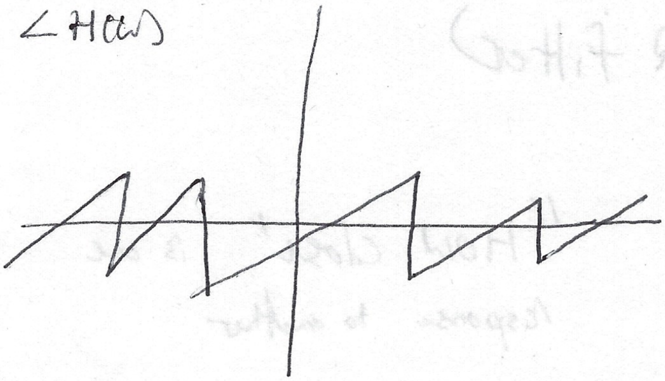
$A(\omega)$ Amplitude response



$A(\omega) = \pm|H(\omega)|$

→ easy to work with, is continuous

$\angle H(\omega)$

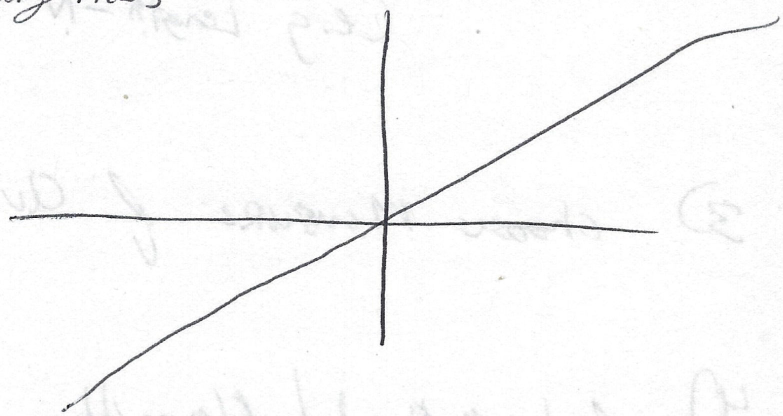


Angle wraps around, always between 0 and 2π

→ Not desireable to work with from an analysis P.O.V. Due to the discontinuities

→ derivative is not continuous

arg $H(\omega)$



Some times called "phase unwrapping,"

13.02

So if the Group delay is close to a constant, then we ~~have~~ are close to linear phase.

→ Design Amplitude response to get what I want

(4) Filter design Process

1) CHOOSE Desired Frequency Response

2) Choose Allowable class of filters (e.g Length-N FIR filter)

3) choose Measure of Quality "How close" is one response to another

4) Apply method / Algorithm to find the OPTIMAL values

5) choose the "best" realization of the filter

→ we want real Filters that are causal, Digital

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N}$$

IF $\{a_i\} = 0$, FIR filter (Finite Impulse response)

Otherwise IIR filter (Infinite Impulse Response)

In General

norm or distance

(5)

$$\min_{a,b} \|E(z)\| = \left\| H_{des}(z) - \frac{b_0 + b_1 z^{-1} + \dots b_M z^{-M}}{1 + a_1 z^{-1} + \dots + a_N z^{-N}} \right\|$$

set of a and b

Filters are real and causal plus give them linear phase

We'll restrict our attention to linear phase filters

Consider $h[n]$, A length N filter. Assume Linear phase

$$\arg H(\omega) = \Theta(\omega) = K_1 + K_2\omega$$

$$H(\omega) = \sum_{n=0}^{N-1} h[n] e^{-j\omega n} \qquad M = \frac{N-1}{2}$$

$$= e^{-j\omega M} \sum_{n=0}^{N-1} h[n] e^{-j\omega n} e^{j\omega M}$$

$$\boxed{h[0] e^{j\omega M}}$$

$$= e^{-j\omega M} \sum_{n=0}^{N-1} h[n] e^{j\omega(M-n)}$$

$$h[N-1] e^{j\omega(M-(N-1))}$$

$$= \boxed{h[N-1] e^{-j\omega M}}$$

$$= e^{-j\omega M} \Big((h[0] + h[N-1]) \cos\omega M$$

$$+ j(h[0] - h[N-1]) \sin\omega M$$

$$+ (h[1] + h[N-2] \cos\omega(M-1)$$

$$j(h[1] - h[N-2] \sin\omega(M-1) + \dots$$

6 When can this be put into the form

$$H(\omega) = A(\omega)\, e^{j(K_1 + K_2\omega)}$$

Real Amplitude   Linear phase

Need to get rid of sin terms

IF $h[n] = h[N-n-1]$, Then all the sin terms drop away

$$H(\omega) = e^{-j\omega M}\Big(\big(h[0] + h[N-1]\big)\cos\omega M + \big(h[1] + h[N-2]\big)\cos\omega(M-1) + \cdots \Big)$$

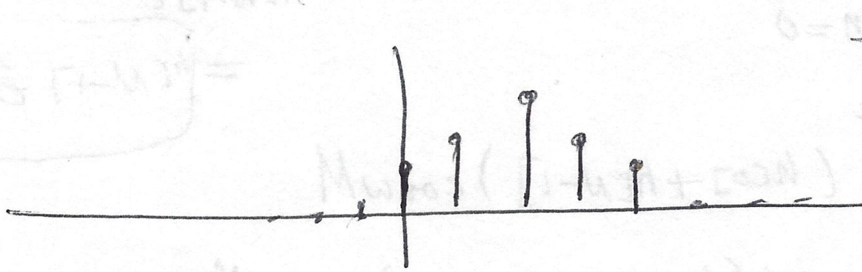
$$A(\omega) = \sum_{n=}^{M-1} 2h[n]\cos\omega(M-n) + h[M]$$

iF N is ODD

All this Means is that $h[n]$ is Symmetric around ITs middle Element

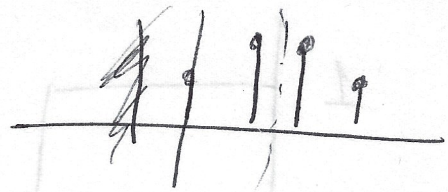
$N=5$, $M = \frac{N-1}{2} = 2$

odd-length filter symmetric about the middle, → we have Linear phase $e^{-j\omega M}$



IF N is even, There's A slightly Different Formula for $A(\omega)$

Symmetric, but no middle element to flip around



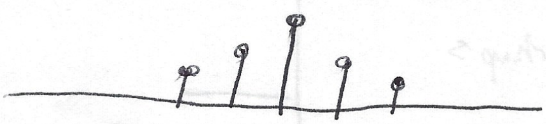
$$A(\omega) = \sum_{n=0}^{\frac{N}{2}-1} 2h[n]\cos(\omega(N-n))$$

Symmetric filter yields linear phase whether N is odd or even

Interested in N = odd

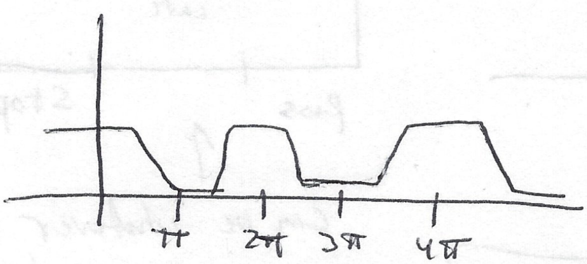
TYPE I

N is odd



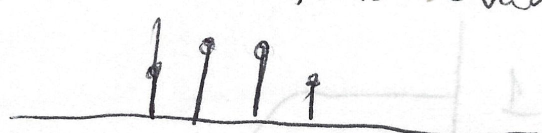
"Low Pass"

$A(\omega)$



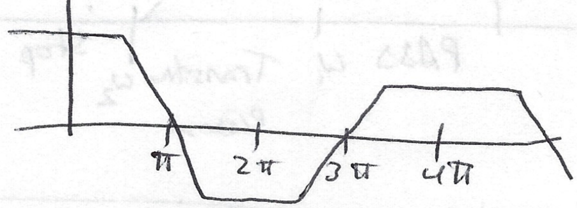
TYPE II

N is Even



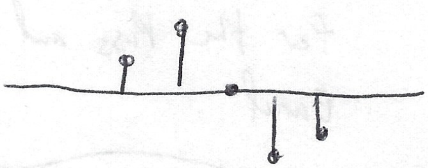
$A(\omega)$

"Low Pass"

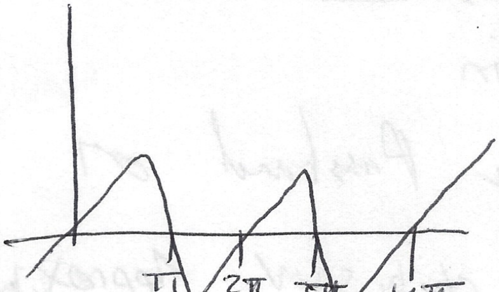


Not 2π period, but is periodic for $|H(\omega)|$ 2π

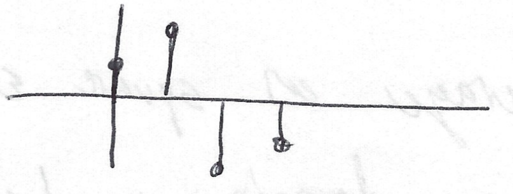
TYPE III N ODD



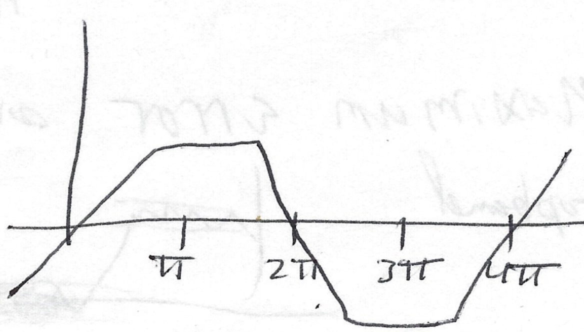
"Band Pass"



TYPE IV N EVEN

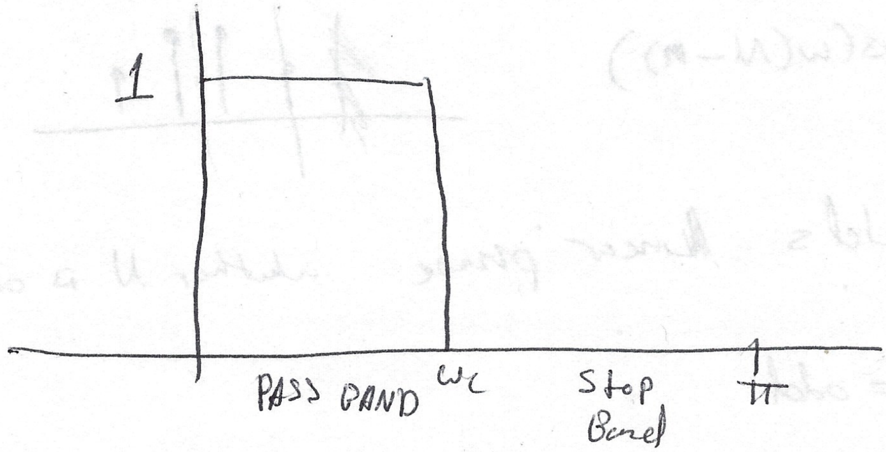


"High Pass"

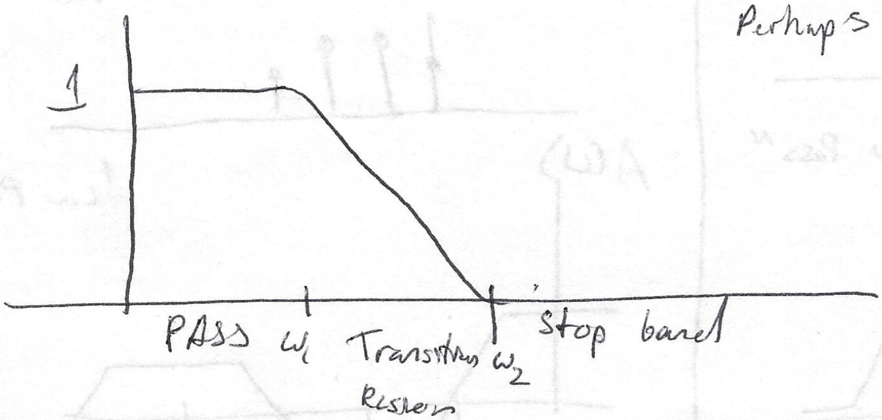


8 $\boxed{\text{TYPE I}}$

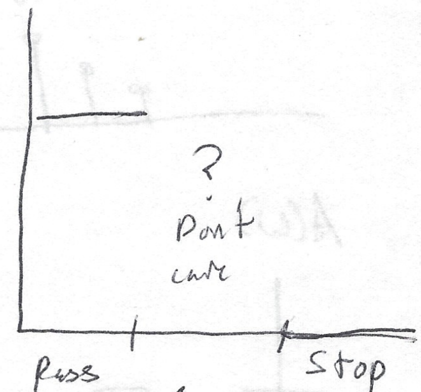
We have a Desired Amplitude Response $A(\omega)$



Ideal response



Perhaps

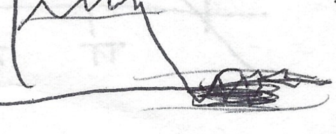


Can be whatever as long as we get good characteristics for the Pass and stop Band

Approximation criteria

1) Average or square error in Frequency domain $\Rightarrow$ Least-squares Approximation

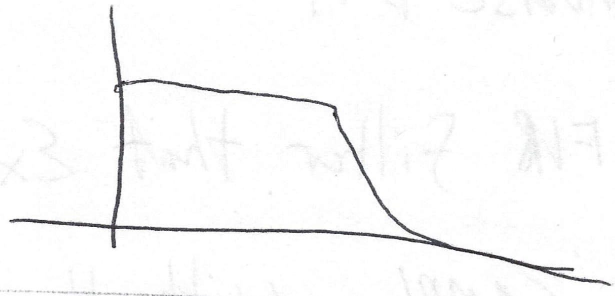
2) Maximum error over the Passband or stopband



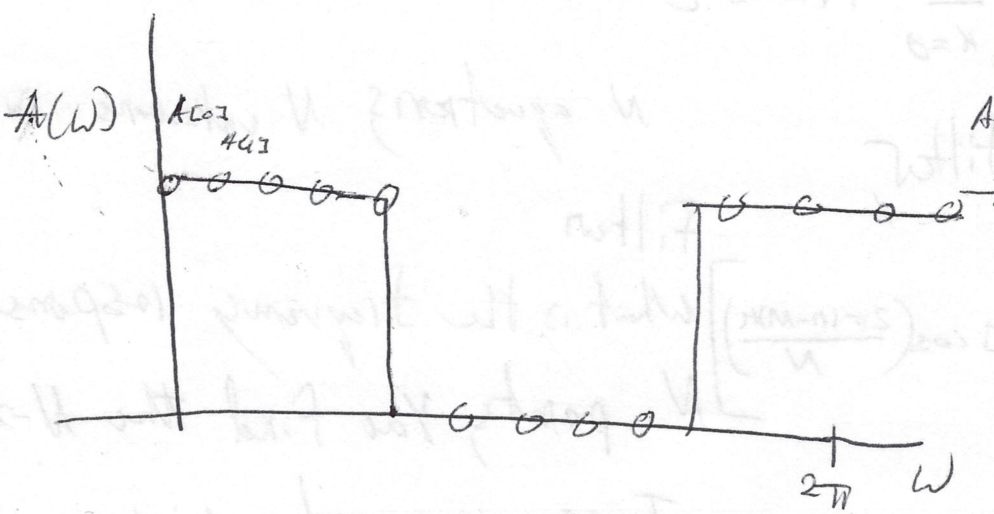
$\Longrightarrow$ chebyshev Approximation

3) Taylor series Approximation To Desired Response/

Maximally flAT $\rightarrow$ Butterworth



frequency Sampling design of FIR Filters



we can do this with any Amplitude response desirable $\rightarrow$ Today, LPF

SAMPLE N (N = ~~[illegible]~~ length of Filter)

Equally-spaced values of $A_d(\omega)$

$$\omega = \frac{2\pi k}{N} \quad k = 0, 1, \cdots N-1$$

these samples are the samples I would have gotten if we had taken the DFT

10 The filter is like the ideal DTFT, when taking the DFT, it's like sampling the DTFT at equally spaced points.

→ What is the impulse response corresponding to hitting those samples

TAKE The inverse DFT

We can get an FIR filter that Exactly interpolates these SAMPLES with the IDFT

$$h[n] = \frac{1}{N}\sum_{k=0}^{N-1} H[k]\, e^{\frac{2\pi j k n}{N}}$$

For Type I filter,

$$h[n] = \frac{1}{N}\left[A[0] + \sum_{k=1}^{M} 2A[k]\cos\left(\frac{2\pi(n-M)k}{N}\right)\right]$$

N equations N unknowns in the filter

What is the frequency response at N points, You find the N-filter Taps → Exact solution

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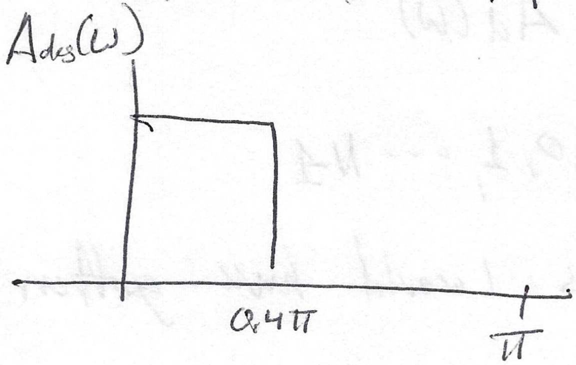
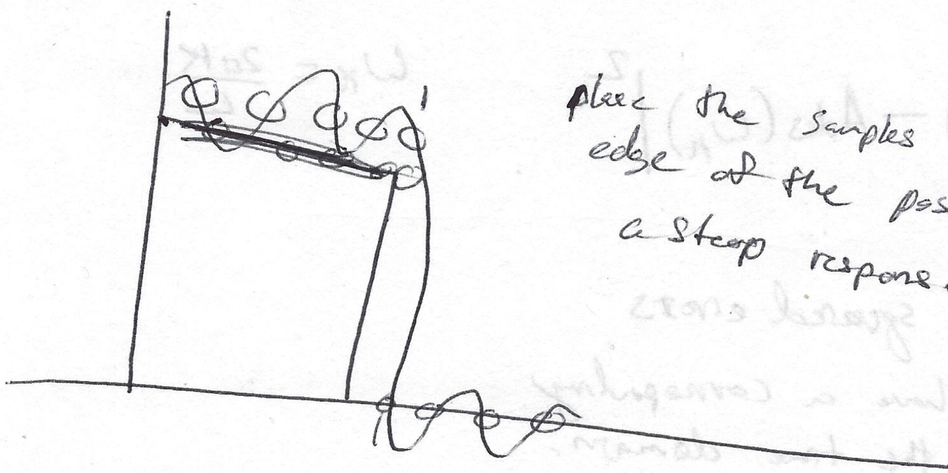
Ex: $N=15$ $M=\frac{N-1}{2}=7$



– Equivalent formulas for Type II, III, IV

or when we TAKE the samples spaced DIFFERENTLY



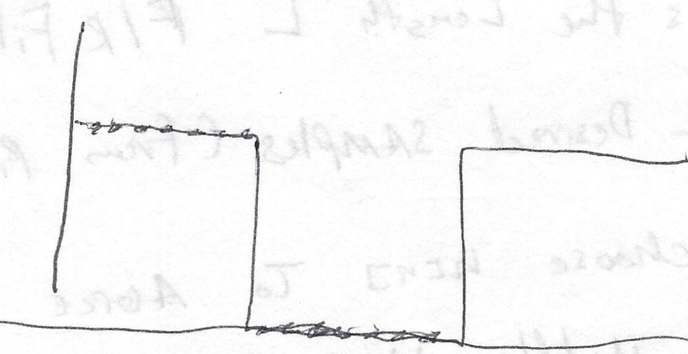
place the samples very close to the edge of the pass/stop band to get a steep response

OR USE N – Non-uniformly spaced samples
(But then can't use IDFT)

still have N equations though

Instead what about minimizing error at L > N samples?

→ Length N filter



if # of samples is bigger than # of filter coefficients

All equations can't be satisfied

more equations than unknowns

→ least squared solutions

(12) Instead of interpolating through all of those dots we approx a solution, minimize the sum of squared errors at those dots

$$\cancel{\mathcal{E} = \sum_{k=0}^{L-1} |A(\omega_k)|}$$

$$\mathcal{E} = \sum_{k=0}^{L-1} \left| A(\omega_k) - A_{des}(\omega_k) \right|^2 \qquad \omega_k = \frac{2\pi k}{L}.$$

this sum of squared errors turns out to have a corresponding interpretation in the time domain.

→ recall Parsevals theorem

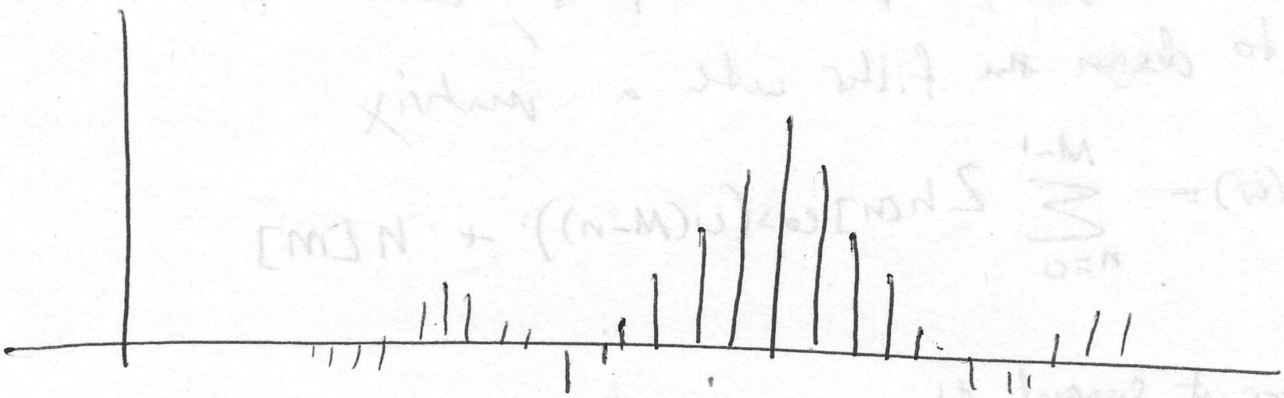
The Error by Parsevals theorem may be written this way,

$$E = \sum_{n=0}^{L-1} \left| h[n] - h_d[n] \right|^2$$

where $h_d[n]$ is the Length L FIR Filter that Goes through the L Desired SAMPLES (from previous method)

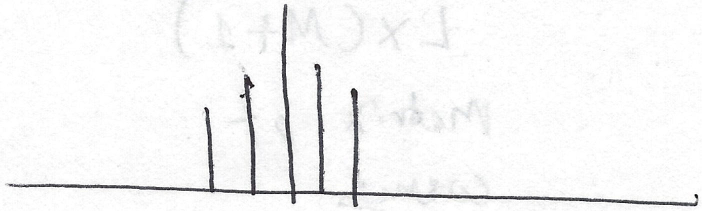
To minimize this, choose $h[n]$ to Agree with $h_{des}[n]$ in the Middle N Filter TAPs (like truncating $h_{des}[n]$)

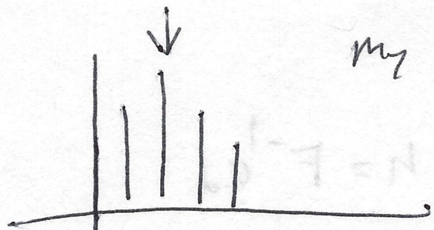
I could Design My Great Length L filter, by Symmetry Looks Like this



This filter would minimize the error, Exactly interpolates the L points

I want to find a Length 5 filter that exactly, that tries to minimize my error on those L points → Match at the middle 5 ~~errors~~ units





my New best filter

It is easy to design these Least squared filters, 1st find the best Long, L, filter and chop out the middle N pieces needed.

(14) MATLAB

With Nonequally spaced Samples, easier to think about how to design the filter with a matrix

$$A(\omega) = \sum_{n=0}^{M-1} 2h[n]\cos(\omega(M-n)) + h[M]$$

Vector of Resampled a_d
Amplitude responses

$$\begin{bmatrix} \\ \\ \\ \\ \end{bmatrix} = \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \begin{bmatrix} h[0] \\ h[1] \\ \vdots \\ h[M] \end{bmatrix}$$

$L \times 1$

$L \times (M+1)$
matrix of cosines

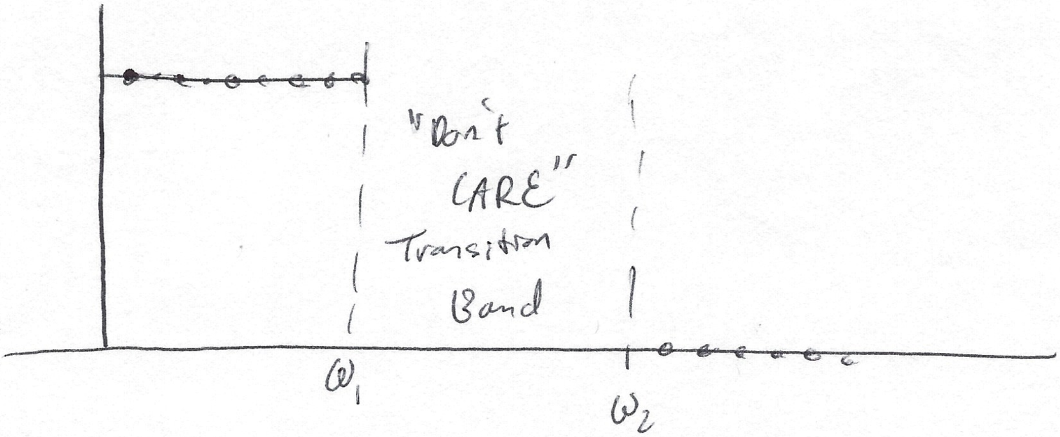
$M+1$
filter Taps

$a_d = Fh$

If F is square, $h = F^{-1} a_d$

Otherwise the Least-squares Solution is

$$h = \underbrace{(F^T F)^{-1} F^T}_{\text{Pseudo inverse}} a_d \rightarrow \text{MATLAB } \boxed{h = F \backslash a_d}$$



$$\mathcal{E} = \sum \underbrace{W_k}_{} \left| A(\omega_k) - A_d(\omega_k) \right|^2$$

↳ Different weights at different points

$$\hat{h} = (F^T W F)^{-1} F^T \underbrace{W a_d}_{}$$

↳ Diagonal weight matrix

→ you can treat each of these points ~~in the~~ differently, you can force better approximation in the pass band over the stop ~~band~~ band