DSP Lect 8

Z-transforms

continuous | discrete

$X(\omega)=\int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$ CTFT	$X(\omega)=\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$ DTFT
$X(s)=\int_{t=-\infty}^{\infty} x(t)e^{-st}\,dt$ Laplace	$X(z)=\sum_{n=-\infty}^{\infty} x[n]z^{-n}$ Z-transform

$z \in \mathbb{C}$

Recall:

$$e^{j\omega_0 n} \longrightarrow \boxed{H} \longrightarrow H(\omega_0)e^{j\omega_0 n}$$

z^n is "special" For Discrete-time LTI Systems

$x[n]=z^n$ for $z\in\mathbb{C}$

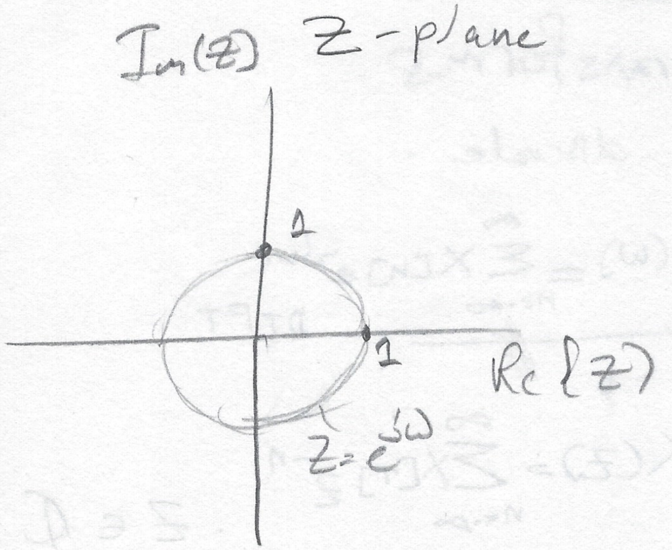
$$y[n]=\sum_{k=-\infty}^{\infty} h[k]x[n-k]=\sum_{k=-\infty}^{\infty} h[k]z^{n-k}$$

$$=z^n\underbrace{\sum_{k=-\infty}^{\infty} h[k]z^{-k}}=z^n H(z)$$

transfer function

Z-transform of the impulse response

(1) relationship between Z-transform and DTFT



$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$X(\omega) = X(z)\Big|_{z=e^{j\omega}}$$

Z transform has a region of convergence

→ certain z's ~~that~~ make ~~this~~ formula converge

* DTFT periodic 2π

unit circle is key for Discrete-time ~~systems~~

(Like $j\omega$ axis for continuous-time systems)

(when do poles ~~stay~~ inside the unit circle?)

Why Z-transform

DTFT doesn't always converge/exist

recall conditions

(sufficient: $\sum_{n=-\infty}^{\infty} |x[n]| < \infty$)

(not every signal satisfies this)

Z-transform is a more general case to consider

- (Z-transform may converge in places where the fourier transform does not exist)

- Notation is easier - polynomials in Z (rational functions of Z)

  $j\omega$ are not as clean as z

- Helps when designing filters
  → understanding what poles and zeros are doing
  → (Easier to design in the Z-world)

when does the Z-transform converge?

ROC.

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$z = re^{j\omega}$

$$\underline{X(z) = X(re^{j\omega})} = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$X(e^{j\omega}) \text{ DTFT} \quad = \sum_{n=-\infty}^{\infty} (x[n] r^{-n}) e^{-j\omega n}$$

$$= \text{DTFT}(x[n] r^{-n})$$

Z-transform converges if

$$\sum_{n=-\infty}^{\infty} |x[n] r^{-n}| < \infty$$

Maybe that if $r=1$ → not satisfied

what is the range of r for the sum to converge

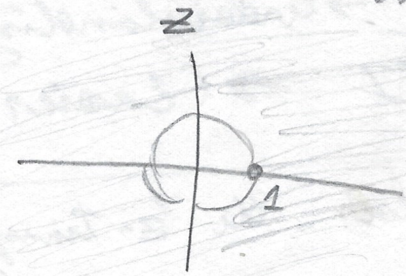
$u[n]$; $\sum_{n=-\infty}^{\infty} u[n] = \infty$ can't do it

look at $u[n] r^{-n}$ $r \in \mathbb{R}$

$$\sum_{n=-\infty}^{\infty} u[n] r^{-n} = \sum_{n=0}^{\infty} r^{-n} = \frac{1}{1-r^{-1}} \text{ if } \underline{|r|>1}$$

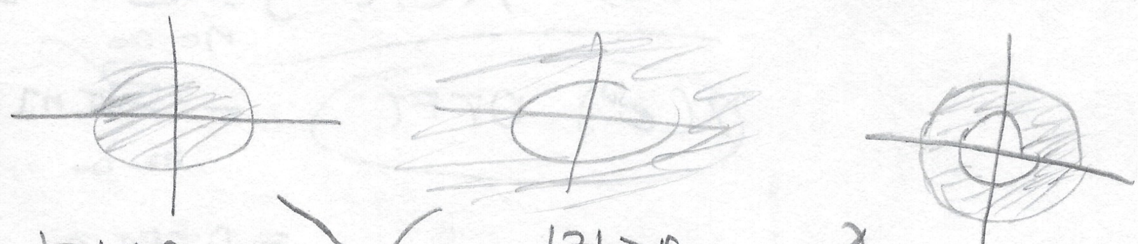
$$= \sum_{n=-\infty}^{\infty} (1/r)^n$$

The ROC of the z-transform of $u[n]$ is $|r|>1$



Convergence of the z-transform Depends only on $|z| = r$

ROCs



$|z| < r_2$ $|z| > r_1$

special cases of (generally look like this)

$r_1 < |z| < r_2$

when ROC includes unit circle

→ can talk about both the z-transform and DTFT

Stability

If the Z-transform of an impulse response, $h[n]$ for an LTI system converges on the unit circle (DTFT exists), then the system is stable (system is well-behaved)

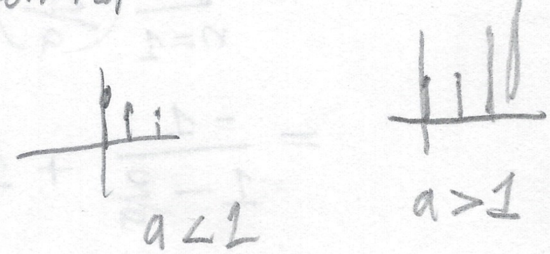
$$X(z) = \frac{N(z)}{D(z)}$$ ← Polynomials in z

$N(z) = 0 \Rightarrow X(z) = 0$ "zeros" of the transfer function

$D(z) = 0 \Rightarrow X(z) = \infty$ "poles"

EX Right-sided Exponential

$x[n] = a^n u[n]$



$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n$$

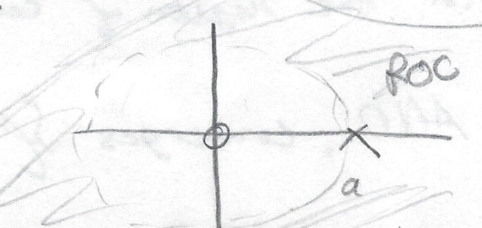
converges if $\left|\frac{a}{z}\right| < 1$

$|z| > |a|$

then $X(z) = \dfrac{1}{1 - a/z}$

$= \dfrac{z}{z-a}$

if $a < 1$
ROC includes unit circle
DTFT exists



o = zero
x = pole

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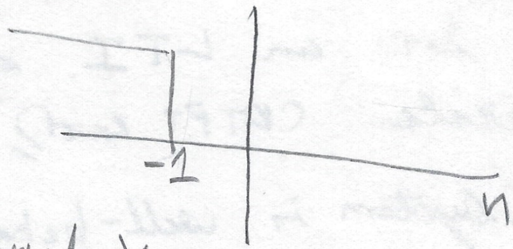
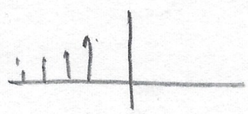
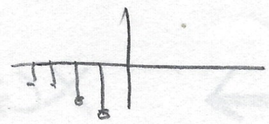
Left-side exponential

$x[n] = -a^n u[-n-1]$



Shift, flip around Y axis of $u[-(n+1)]$ → flip around $(-1) = n$

So  or 

$$X(z) = \sum_{n=-\infty}^{\infty} -a^n u[-n-1] z^{-n} = \sum_{n=-\infty}^{-1} -\left(\frac{a}{z}\right)^n$$

$$= \sum_{n=1}^{\infty} -\left(\frac{z}{a}\right)^n$$ converges if $\left|\frac{z}{a}\right| < 1$

$$= \frac{-1}{1-\frac{z}{a}} + 1$$

$-X(z) = \left(\frac{z}{a}\right) + \left(\frac{z}{a}\right)^2 + \dots$

$-X(z) - 1 + 1 = \cancel{+\left(\frac{z}{a}\right) + \left(\frac{z}{a}\right)^2 + \dots}$

$-X(z) + 1 = 1 + (z/a) + (z/a)^2 \dots$

$-X(z) + 1 = \frac{1}{1 - z/a}$

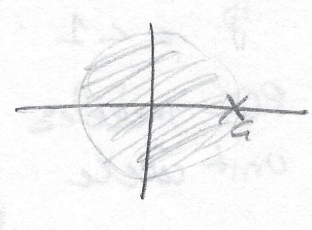
$$= \frac{-a}{a-z} + \frac{a-z}{a-z}$$

$$= \frac{-z}{a-z} = \frac{z}{z-a}$$ (Same as previous)

Same z-transform?

No, because the z-transform has the z function and the region of convergence

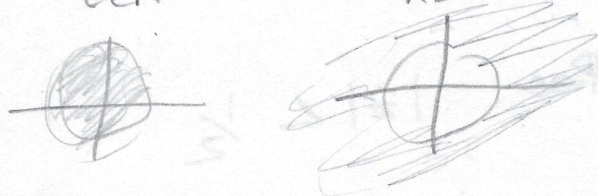
AND converges if $\underline{|z| < |a|}$



⊛ need to know ROC if asked to take the inverse z transform

In a lot of problems, we deal with right-sided signals → so we can tell the ROC

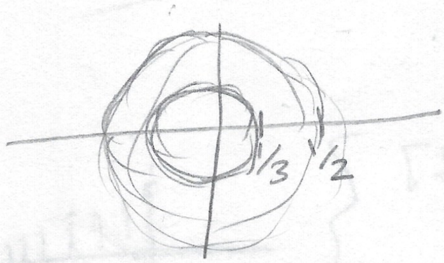
Left Right



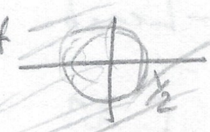
$$X[n] = \left(\tfrac{1}{2}\right)^n u[n] + \left(\tfrac{-1}{3}\right)^n u[n]$$ both right side

$$\frac{z}{z-1/2} + \frac{z}{z+1/3} = \frac{z^2 + \frac{2}{6}z + z^2 - \frac{3}{6}z}{(z-\frac{1}{2})(z+\frac{1}{3})} = \frac{2z^2 - \frac{1}{6}z}{(z-\frac{1}{2})(z+\frac{1}{3})}$$

ROC $|z| > \frac{1}{2}$ $\quad$ $|z| > \frac{1}{3}$



See that



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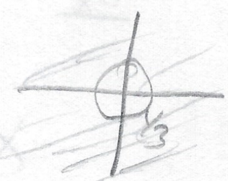
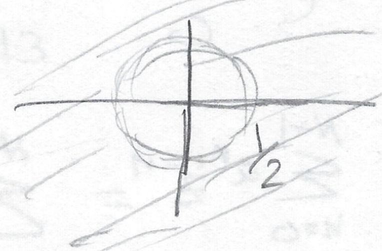
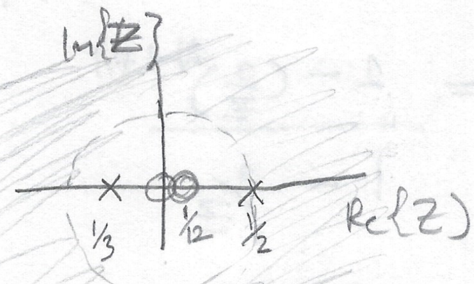


→



find intersection

ROC: $|z| > \frac{1}{2}$



ROC is related to the poles

Poles play a huge role

⑧ consider

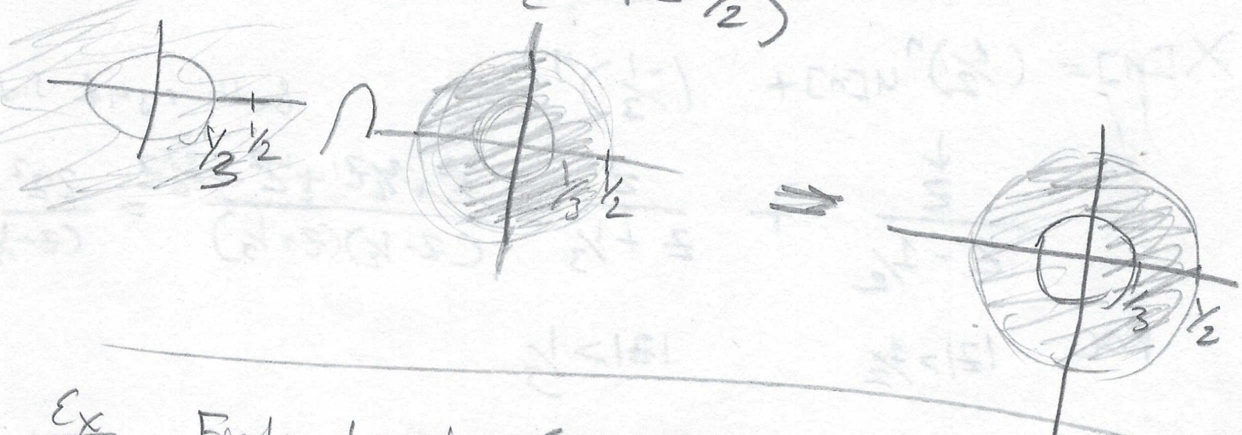
$$x[n] = \left(-\tfrac{1}{3}\right)^n u[n] - \left(\tfrac{1}{2}\right)^n u[-n-1]$$

$$X(z) = \frac{2z\left(z-\frac{1}{12}\right)}{\left(z+\frac{1}{3}\right)\left(z-\frac{1}{2}\right)}$$

for $\left(-\frac{1}{3}\right)^n u[n] \longrightarrow$ ROC: $|z| > \frac{1}{3}$

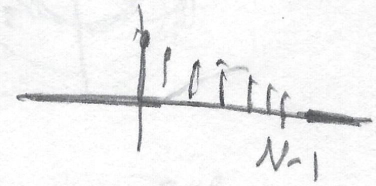
for $-\left(\frac{1}{2}\right)^n u[-n-1] \rightarrow$ ROC: $|z| < \frac{1}{2}$

So $\left\{|z| > \frac{1}{3}\right\} \cap \left\{|z| < \frac{1}{2}\right\}$



Ex   Finite-Length exponential

$$x[n] = \begin{cases} a^n & n \in [0, N-1] \\ 0 & \text{Else} \end{cases}$$



$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} \left(\frac{a}{z}\right)^n = \frac{1-\left(\frac{a}{z}\right)^{N-1+1}}{1-a/z}$$

finite number of terms, formula is always valid

$$= \frac{z^N - a^N}{z^N - a z^{N-1}}$$

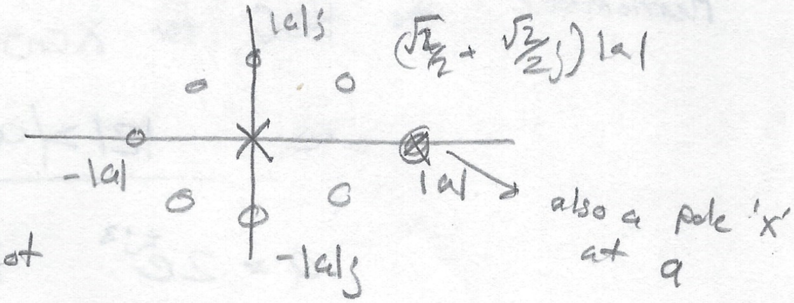
$$= \frac{z^N - a^N}{z^{N-1}(z-a)}$$

$N-1$ poles at 0

1 pole at $z=a$

The zero's come at $z^N = a^N$

Say $N=8$



7 poles at ⊗

Where there is a pole on top of a zero, they cancel, terms vanish

→ 7 poles at origin   7 zeros scattered

ROC is $|z|>0$ (whole z-plane but $z=0$)

So for any value of z, we can set the z-transform

generally true when input signal is finite length

→ Problems occur when input signal is ∞ long

$$x[n] = 2^n \cos(3n)u[n] = 2^n \tfrac{1}{2}(e^{j3n} + e^{-j3n})u[n]$$

$$= \tfrac{1}{2}\left[(2e^{3j})^n + (2e^{-3j})^n\right]u[n]$$

$$X(z) = \tfrac{1}{2}\left[\frac{1}{1-\frac{2e^{3j}}{z}}\right] + \tfrac{1}{2}\left[\frac{1}{1-\frac{2e^{-3j}}{z}}\right]$$

$$= \tfrac{1}{2}\left[\frac{z}{z-2e^{3j}}\right] + \tfrac{1}{2}\left[\frac{z}{z-2e^{-3j}}\right]$$

$$= \tfrac{1}{2}\left[\frac{z(z-2e^{-3j}) + z(z-2e^{3j})}{(z-2e^{3j})(z-2e^{-3j})}\right] = \tfrac{1}{2}\left[\frac{2z^2 - 2[2\cos(3j)]z}{z^2 - 2z(2\cos(3j)) + 4}\right]$$

$$= \frac{z^2 - 2\cos(3)z}{z^2 - 4\cos(3)z + 4}$$

ROC?

(10) Poles at $z = 2e^{\pm 3j}$ zeros at $z = 0$

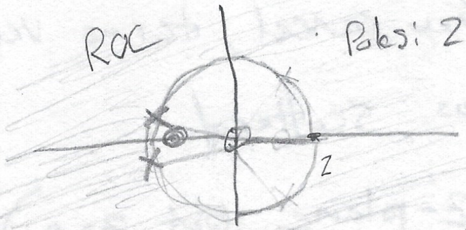
$z = 2\cos 3$

remember the ROC for $x[n] = a^n u[n]$ 58:08

was $|z| > |a|$

$a = 2e^{\pm j3}$ $a \in \mathbb{C}$

$|a| = 2$ ROC $|z| > 2$



circle of radius 2

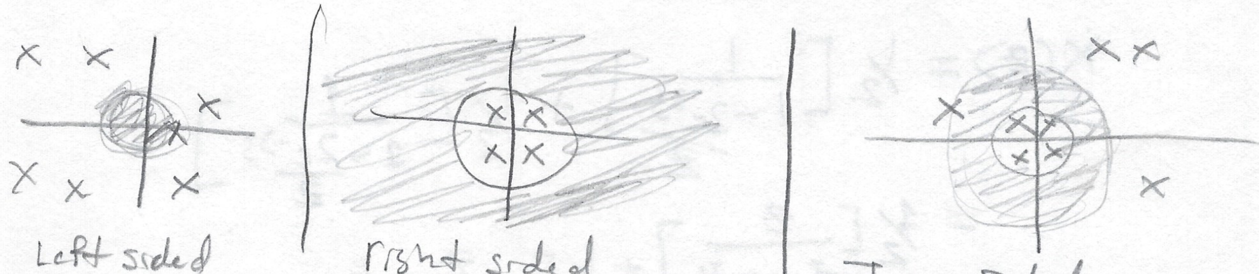
Rules of ROC

1) ROC is a ring or Disc centered at 0.

2) ROC contains no poles

3) If $x[n]$ is finite length ROC is entire z-plane (Except possibly $z=0$, or $z=\infty$).

4)



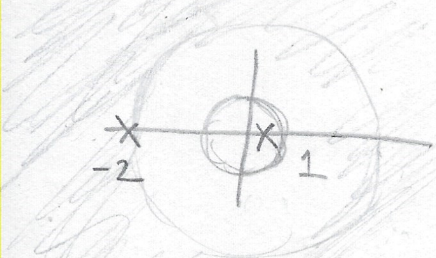
Left sided $x[n]$ | Right sided $x[n]$ | Two-Sided $x[n]$

ROC bounded by the poles

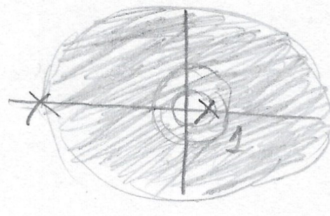
Remember, No pole in ROC

5) ROC tells us: Is the system stable?
→ For a Transfer Function $H(z)$ Is The System Causal?

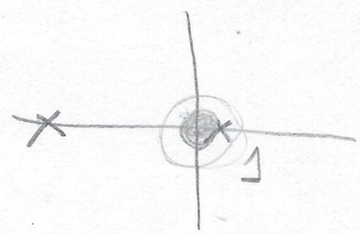
STABLE $\Rightarrow$ ROC includes unit circle
CAUSAL $\Rightarrow$ impulse response is right-sided



Causal
Not Stable
(unit circle not included)



stable
Not causal
(ROC doesn't go out to ∞)



Not Stable
Not causal

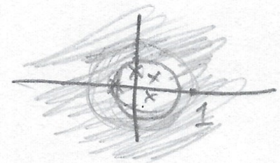
$\frac{|z|}{|a|} > 1$

a^n

if $a > 1$ $(a)^n \to \infty$ as $n \to \infty$

if $a \leq 1$ $(a)^n$ may go to 0

Most Desirable Scenario
ROC Looks Like



ROC extends outwards from Largest-magnitude Pole, all poles inside unit circle

→ wanted especially in control systems