DSP Lecture 12 : (Incomplete) 2

The Cooley – Tukey and ~~Flood~~ Good – Thomas FFTs

zero padding up to the next power of 2 isn't bad, but for say a 5×10^6 DFT, well, zero padding may take a while.

The Cooley – Tukey FFT

DFT of length N

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$$

$n = 0, 1 — N-1$

$k = 0, 1, — N-1$

$W_N = e^{-\frac{2\pi}{N}j}$

our Assumption, N can be factored into two numbers

Assume $N = n_1 n_2$

Write $n = n_1 i + j$

$i = 0, 1, \cdots n_2 - 1$

$j = 0, 1, \cdots n_1 - 1$

$i + j$ are indices

Ex $N = 12$, $n_1 = 3$, $n_2 = 4$

$n = 3i + j$

$i = 0, 1 \cdots 3$

$j = 0, 1, 2$

when $j = 0$ we get

$n = 1, 2, 3$

when $j = 1$ we get

$n = 4, 5, 6, 7$

	$j=0$	$j=1$	$j=2$
$i=0$	0	4	8
	1	5	9
	2	6	10
$i=3$	3	7	11

(3) Write $k = n_2 a + b$ $\quad a \in 0, 1, \ldots n_1 - 1$

$b \in 0, 1, \ldots n_2 - 1$

$N = 12, \quad n_1 = 3 \quad n_2 = 4$

$k = 4a + b$

	b=0	b=1	b=2	b=3
a=0	0	1	2	3
a=1	4	5	6	7
a=2	8	9	10	11

So we can write,

$$X[k] = \sum_{i=0}^{n_2-1} \sum_{j=0}^{n_1-1} x[n_1 i + j] W_N^{k(n_1 i + j)}$$

$$X[n_2 a + b] = \sum_{i=0}^{n_2-1} \sum_{j=0}^{n_1-1} x[n_2 i + j] W_N^{(n_2 a + b)(n_1 i + j)}$$

Think that the input is a 2D array, and the output as a 2D array

$$W_N^{(n_2 a + b)(n_1 i + j)} = \cancel{W_N^{n_1 n_2 a i} W_N^{n_1 i b}} = \underset{=1}{W_N^{n_1 n_2 a i}} W_N^{n_1 i b} W_N^{n_2 a j} W_N^{b j}$$

$$W_N = e^{-\frac{2\pi}{N} j} \Rightarrow W_N^{n_1} = W_{n_1 n_2}^{n_1}$$

$$e^{-\frac{2\pi}{n_1 n_2} j n_1} = e^{-\frac{2\pi}{n_2} j} = W_{n_2}$$

$$= W_{n_2}^{ib} W_{n_1}^{aj} W_N^{bj}$$

$$X[n_2a+b] = \sum_{i=0}^{n_2-1} \underbrace{\boxed{\sum_{j=0}^{n_1-1} x[n_1 i + j] W_{n_1}^{aj}}}_{\text{Looks like a short DFT}} \underbrace{W_N^{bj}}_{\text{Twiddle factor}} W_{n_2}^{ib}$$

for this sum, i is a constant, we are changing j from 0 to n_1-1. We have a n_1-Length DFT

rewrite it more carefully

$$= \underbrace{\left[\sum_{j=0}^{n_1-1} \underbrace{\left[\sum_{i=0}^{n_2-1} x[n_1 i + j] W_{n_2}^{ib} \right]}_{\substack{\text{For } j \text{ Fixed} \\ n_2\text{-Length DFT}}} \underbrace{W_N^{bj}}_{\text{Twiddle factor for } j} W_{n_1}^{aj} \right]}_{n_1\text{-Length DFT}}$$

for every fixed j, I am doing a smaller DFT

We are doing ~~n_1-Length n_2 DFTs~~ n_1 n_2-Length DFTs

For the outside sum: Do n_2 n_1-length DFTs

We also need N multiplication by twiddle factors

Assuming Naive Smaller-length DFTs:

$$n_1(n_2^2) + n_2(n_1^2) + N$$

$$= N(n_1 + n_2 + 1) \quad \text{vs } N^2$$

④ We can write

In General, if $N = \prod_{i=1}^{\ell} n_i$

we have the number of multiplications $= N\left(\sum_{i=1}^{\ell} n_i\right)$

Last time if $N = 2^{v}$

we showed Radix-2 FFT had

$\mathcal{O}(N \log_2 N)$ multiplications

$\log_2 N = v$ $\qquad \underbrace{2 + 2 + \cdots}_{\text{add 2 } v \text{ times}} = 2v$

The C-T FFT is like a generalization with the same kind of efficiency, (But N can be anything)

So Ex $N = 15$ $\quad n_1 = 5,\ n_2 = 3$

original x vector

$x(n)$

① Do 5 Length-3 DFTs

DFT's on the columns

Sort into 3x5 array

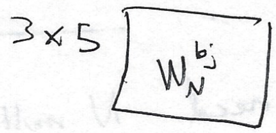
$i=0$	0	1	2	3	4
$i=1$	5	6	7	8	9
$i=2$	10	11	12	13	14
	$j=0$				$j=4$

i, j

[matlab reshape]

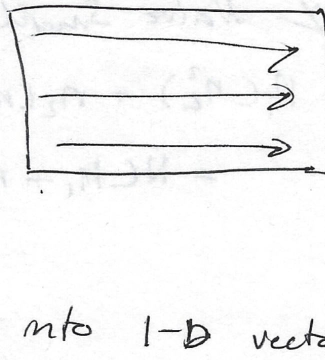
② multiply The 3x5 array by a 3x5 array W_N^{bj} $\quad b = 0, 1, 2$ $\quad j = 0, \ldots 4$

Twiddle factor matrix



③ 3 Length-5 DFTs

DFT's along the rows



④ PUT 2-D elements back into 1-D vector

Take them out in a different order than put back in

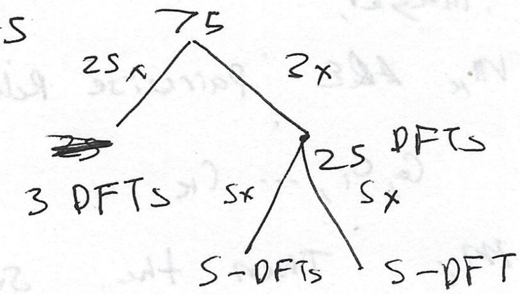
Take them out like this

0	3	6	9	12
1	4	7	10	13
2	5	8	14	14

which gives us our $X[k]$'s

use operation M(:) puts back into a vector.
colon operator

Ex $N = 75$



How to get rid of TWIDDLE Factors?

Good-THOMAS FFT. Uses some Abstract Math

Number Theory, Abstract algebra

1) IF a and b are Integers, Not both zero.
Then the Greatest Common Divisor $(a,b) = \gcd(a,b)$ Exists

There are Integers m_0 and n_0 so that

$$(a,b) = m_0 a + n_0 b$$

IF $(a,b) = 1$ a and b are relatively prime, And there exist
(if Greatest Common Divisor) is 1) integers m_0, n_0 so that

$$1 = m_0 a + n_0 b$$

⑥ $3, 4 \Rightarrow \gcd = 1$

$(\ )3 + (\ )4 = 1$

$-5 \qquad 4$ yes!

$4 - 3 = 1$ ∞ set that m_0, n_0 can work

CHINESE REMAINDER THEOREMS

1) Given a set of Integer,

$m_0, m_1, \ldots m_k$ ARE Pairwise Relatively prime, AND

A SET of Integers $c_0, c_1, \ldots c_k$

with $0 \le c_i < m_i$, Then the System

$c_i = c \mod m_i \qquad i = 0, \ldots k$

Has at most one solution for c in the

Interval $0 \le c < \prod_{i=0}^{k} m_i$

Ex $m_0 = 3 \quad m_1 = 4 \quad m_2 = 5$

$3 \cdot 4 \cdot 5 = 60$

$9 \mod 4 = 1$

$3 \mod 5 = 3$

Find X so that $X = 1 \mod 3$

$X \in [0, 1, \ldots 59]$

$= 2 \mod 4$

$= 0 \mod 5$

all of these are satisfied

$\cancel{0}, \cancel{5}, 10, \cancel{15}, \cancel{20}, 25, \cancel{30}, \cancel{35}, 40, \cancel{45}, \cancel{50}, 55$

$9+1 \quad 24+1 \quad 39+1 \quad 54+1$

1 mod 3 must be a multiple of 3 plus 1

$1 \mod 3 = 1$

$2 \mod 4 = 2$

$0 \mod 5 = 0$

$10 \to 8 + 2$

Answer is 10

We were not going to find multiple answers

$(1,2,0) \xrightarrow{\text{Exists}} 10$

2) Let $M = \prod_{i=0}^{K} m_i$ Be a Product of Relatively Prime ~~factors~~ Integers

$M=60$

m_i	M_i
3	20
4	15
5	12

Let $M_i = \frac{M}{m_i}$

Let N_i Satisfy

$$\underbrace{N_i M_i + n_i m_i}_{\text{Relatively Prime}} = 1$$

Then $C_i = C \bmod m_i$ is uniquely Solved by

$i = 0, 1, \dots K$

$$C = \left(\sum_{i=0}^{K} C_i N_i M_i \right) \bmod M.$$

48:44

$M=60,$