IIR filters 18

$$y[n] = \sum_{m=0}^{M} b[m]x[n-m] - \sum_{k=1}^{N} a[k]y[n-k]$$

This part is the IIR filter

We are feeding back previous values of the output to create the current value of the output

h[n] can have infinite length

$$H(z) = \sum_{n=0}^{\infty} h[n]z^{-n} = \frac{\sum_{n=0}^{M} b[n]z^{-n}}{\sum_{n=0}^{N} a[n]z^{-n}} \quad \begin{matrix}\text{FIR} \\ \text{IIR}\end{matrix}$$

$$= \frac{B(z)}{A(z)} = \frac{b[0] + b[1]z^{-1} + b[2]z^{-2} + \dots + b[M]z^{-M}}{1 + a[1]z^{-1} + a[2]z^{-2} \dots + a[N]z^{-N}}$$

Differences

- Can't Do Linear phase

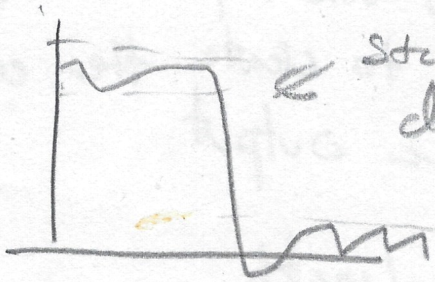
h[n] goes out infinitely in the right hand direction

We can't remove phase from the design process

② → so more complicated to design IIR filters

– Low orders of IIR filters

→ Low powers of M, N are sufficient to implement "TIGHT" specs compared to FIR



steep roll off is hard to do with low orders (N) of FIR filter

we could do this with an order 4 or 5 IIR filter.

DESIGN Process

– choose desired response → DTFT

→ class of filters (IIR order N in ner (NUM/DEN))

→ choose a Distance Measure Between

$H_{des}(\omega)$ and $H_{actual}(\omega)$

→ Find the optimal FILTER in the class

That ~~...~~ minimizes the ERROR

For IIR filters, commonly start by designing an analog filter → Digital

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Direct Digital IIR Filter Design

Prony's Method

Given a Desired Infinite Impulse response $h_{des}[n]$ $H_d[n]$ $0 \le n < \infty$

Find coefficients A, B s.t. we can approximate that response (IIR) as well as possible with this structure

$$y[n] = -\sum_{k=1}^{N} a[k]\, y[n-k] + \sum_{m=0}^{M} b[m]\, x[n-m]$$

$$H_d(z) = \sum_{n=0}^{\infty} h_d[n] z^{-n} = h_d[0] + h_d[1] z^{-1} + h[2] z^{-2} + \cdots$$

$$H(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

$a_0 = 1$

4

$$H_d = \frac{B(z)}{A(z)}$$

$$B(z) = H_d(z)\,A(z)$$

$$\cancel{(b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots)(h[0}$$

$$(\underline{1} + a_1 z^{-1} + a_2 z^{-2} + \cdots)(\underline{h[0]} + h[1] z^{-1} + h[2] z^{-2} \cdots) =$$

$$\underline{\underline{b_0}} + b_1 z^{-1} + b_2 z^{-2} + \cdots$$

$$\begin{array}{c}M+1\\ \\ K-M\end{array}\begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_M \\ \hline 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} h_d[0] & 0 & \cdots & 0 \\ h_d[1] & h_d[0] & 0 \cdots & 0 \\ h_d[2] & h_d[1] & h_d[0] \cdots & 0 \\ \vdots & & & \vdots \\ h_d[N] & h_d[N-1] & \cdots & h_d[0] \\ \hline h_d[N+1] & h_d[N] & \cdots & h_d[1] \\ h_d[K] & h_d[K-1] & \cdots & h_{K-N} \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ a_2 \\ \vdots \\ a_N \\ \hline \\ \end{bmatrix} \quad (N+1)\times 1$$

$(K+1)\times 1$

$(K+1)(N+1)$ contains coefficients of h_d

(5)

$$\begin{array}{c} M+1 \\ K-M \end{array}\begin{bmatrix} b \\ 0 \end{bmatrix} = \overset{(M+1)\times(N+1)}{\begin{bmatrix} \multicolumn{2}{c}{H_1} \\ \hline \underbrace{h_1}_{(K-M)\times 1} & \underbrace{H_2}_{(K-M)\times N} \end{bmatrix}} \begin{bmatrix} 1 \\ a^* \end{bmatrix}\begin{array}{c} 1 \\ N \end{array}$$

$$a^* = \begin{bmatrix} a_1 \\ | \\ a_N \end{bmatrix}$$

$b = \cancel{H_1 a^*}$ $\quad b = H\begin{bmatrix} 1 \\ a \end{bmatrix}$

$0 = h_1 + H_2 \overset{N\times 1}{a^*}$ $\quad$ H_2: $(K-M)\times N$

if $K-M = N$

$K = M+N$

(IIR has $M+N+1$ values)

$0 = \beta + H_2 a$

$$\boxed{\begin{aligned} H_2 a^* &= -h_1 \; \beta \\ a^* &= -H_2^{-1} h_1 \\ b &= -H_1 H_2^{-1} h_1 \end{aligned}}$$

a and b that corresponds to the first K values of $h[n]$

$$a = -H_2^{-1} \beta$$

$$b = H_1 \begin{bmatrix} 1 \\ -H_2^{-1}\beta \end{bmatrix}$$

(b)

$$\begin{bmatrix} b \\ 0 \end{bmatrix} + e = \left[\begin{array}{c|c} \multicolumn{2}{c}{H_1} \\ \hline h_1 & H_2 \end{array}\right] \begin{bmatrix} 1 \\ a \end{bmatrix}$$

if $k \geq M+N$ — if square matrix, $e=0$

least-squares solution to minimize $\|e\|_2$

$$a = -(H_2^T H_2)^{-1} H_2^T \cdot \beta$$ pseudo-inverse

$$b = H_1 \begin{bmatrix} 1 \\ a \end{bmatrix}$$

We really want to minimize

$$\hat{e} = h[n] - h_d[n]$$

~~Frequen~~

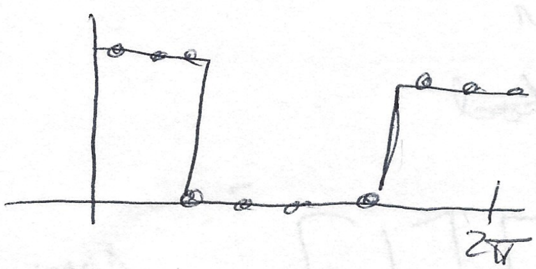
Frequency-Sampling Desing of IIR filters

We Have a desired response $H_d(\omega)$

Find the 'best' a, b To satisfy.

$$H(z) = \frac{B(z)}{A(z)} \quad \begin{matrix} M+1 \text{ unknowns} \\ N \text{ unknowns} \end{matrix} = \frac{\sum_{n=0}^{M} b[n] z^{-n}}{\sum_{n=0}^{N} a[n] z^{-n}}$$

$L = M + N + 1$ unknowns



SAMPLE $H_d(\omega)$ at $\frac{2\pi k}{L+1}$ $k = 0, 1 — L$ $a[0] = 1$

like the DFT of $h[n]$

obtain $H_d\left(\frac{2\pi k}{L+1}\right)$ $k = 0, 1, — L$

(2) Find the IDFT of $H_d\left(\frac{2\pi k}{L+1}\right)$ $k = 0, 1, —, L$

obtain $\{g[n] \;\; n = 0, —, L\} = IDFT\{H_d\left(\frac{2\pi k}{L+1}\right), k = 0, 1, — L\}$

$H(z) = \frac{B(z)}{A(z)}$ $\quad B(z) = H(z) A(z) \Rightarrow B(\omega) = H(\omega) A(\omega)$

product in frequency domain

circular convolution In the time domain

$\Rightarrow b = \underbrace{h}_{g[n], \; n = 0, —, L} \circledast a$

$$
\begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_M \\ \vdots \end{bmatrix}
=
\begin{bmatrix}
g_0 & g_L & g_{L-1} & \cdots & g_1 \\
g_1 & g_0 & g_L & \cdots & g_2 \\
g_2 & & \ddots & & \vdots \\
\vdots & & & \ddots & g_{L-1} \\
\vdots & & & & g_L \\
g_L & & & & g_0
\end{bmatrix}
\begin{bmatrix} 1 \\ a_1 \\ \vdots \\ a_N \\ 0 \\ \vdots \\ 0 \end{bmatrix}
$$

$$g_i = \mathrm{IDFT}\left\{ H_d\left(\frac{2\pi k}{L+1}\right) \right\}$$

→ we get a matrix similar ~~to the prony method~~

$$
\begin{bmatrix} b \\ 0 \end{bmatrix}
=
\begin{bmatrix} G_1 \\ \hline \alpha \mid G_2 \end{bmatrix}
\begin{bmatrix} 1 \\ a \end{bmatrix}
$$

Same structure as prony

$$a = -G_2^{-1}\alpha$$

$$b = G_1 \begin{bmatrix} 1 \\ a \end{bmatrix}$$

EXTEND to having more samples of $H_d(\omega)$

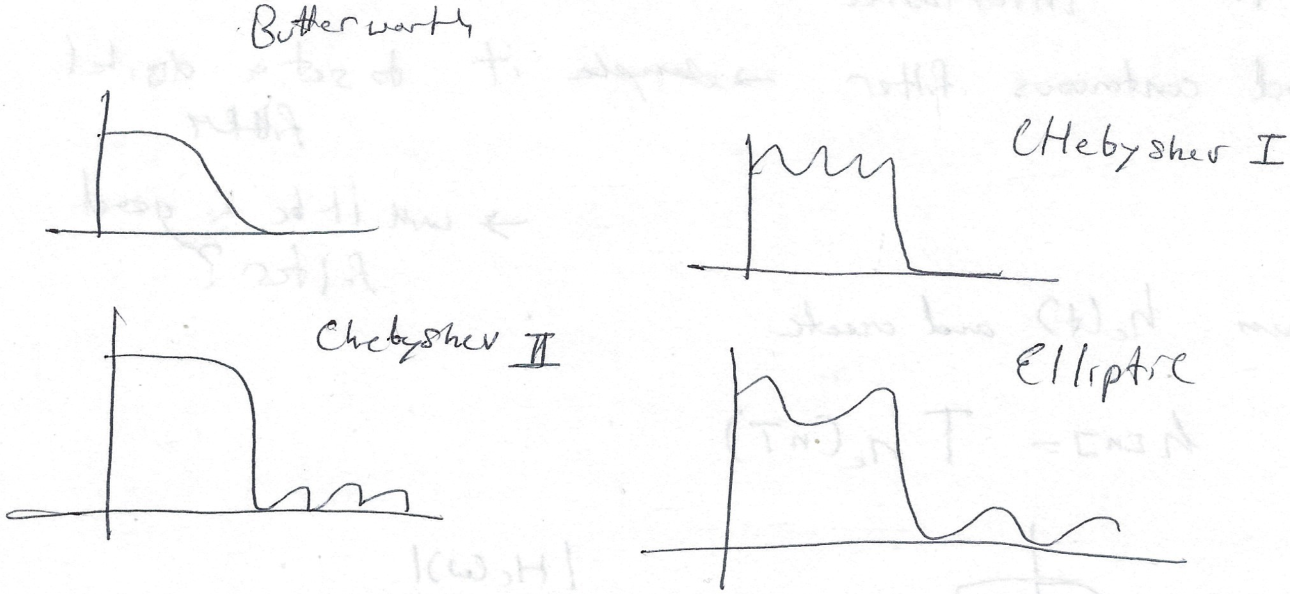
Than $M+N+1$ ⟹ least squares Design

(like FIR, EXTENSION of Prony solution)

NoTe: $H_d(\omega)$ should be consistent with the real $h_d[n]$

So that $a[n]$, $b[n]$ real, & No garantee that designed filters are stable

Design digital filters from Analog IIR ~~filters~~



Closed form solutions exist for such types of filters in continuous time

How to convert these ~~to~~ discrete ~~time~~?

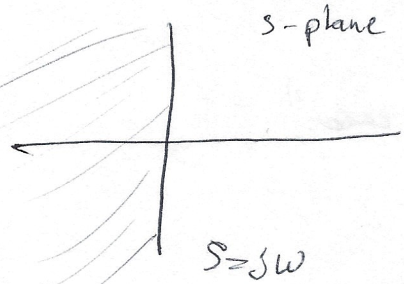
$H_c(s)$ $\xrightarrow{T}$ $H(z)$, $h[n]$

$h_c(t)$

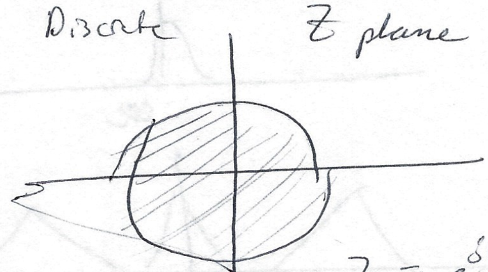
Sample to get discrete

Discrete Z plane

s-plane



$s = j\omega$



$z = e^{j\omega}$

Map s-plane to z-plane

→ conformal Mapping

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2 MAIN Approaches

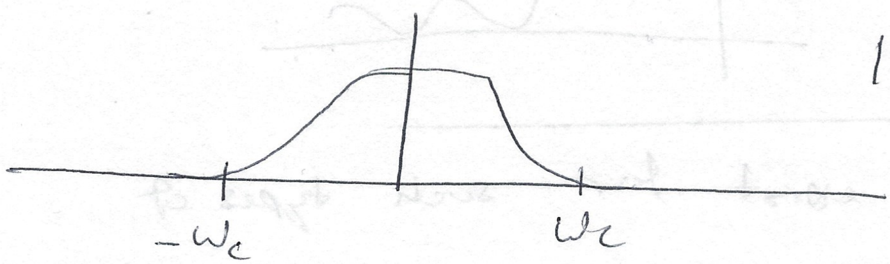
1) IMpulse Invariance

good continuous filter → sample it to get a digital filter

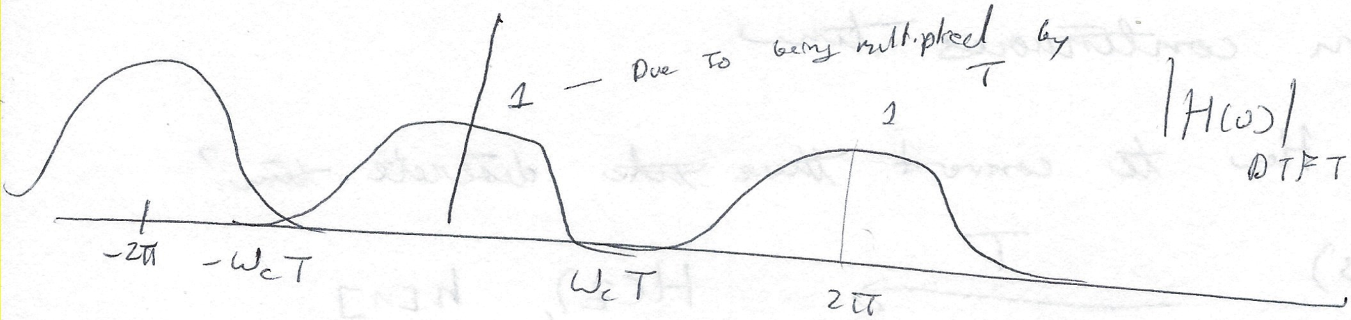
→ will it be a good filter?

Given $h_c(t)$ and create

$$h[n] = T\,h_c(nT)$$

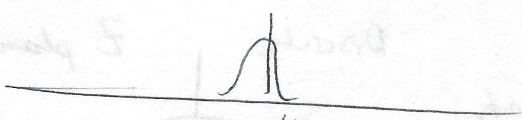


$|H_c(\omega)|$

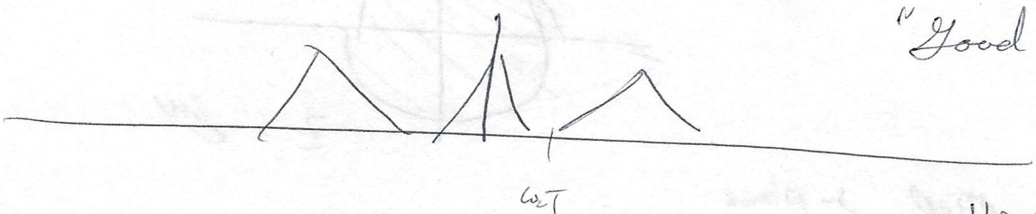


$|H(\omega)|$

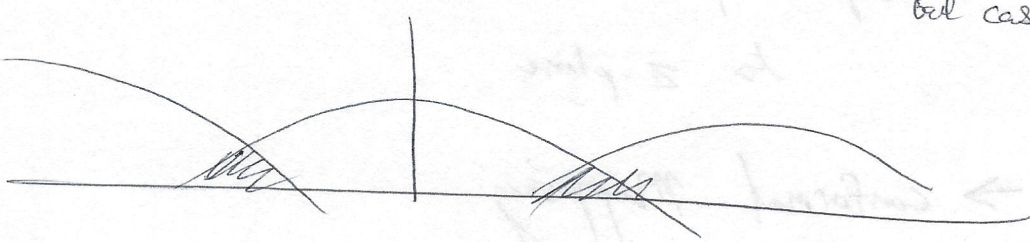
DTFT of $h[n]$



"Good case"

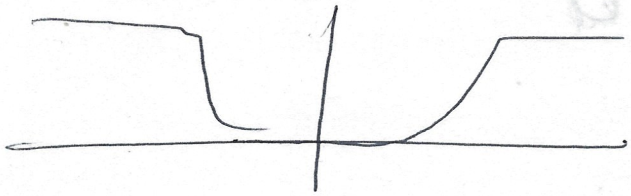


"Bad case"



Optimal filter now (continuous)
doesn't garuntee good filter in discrete

Aliasing can be a big problem
especial when designing a highpass filter

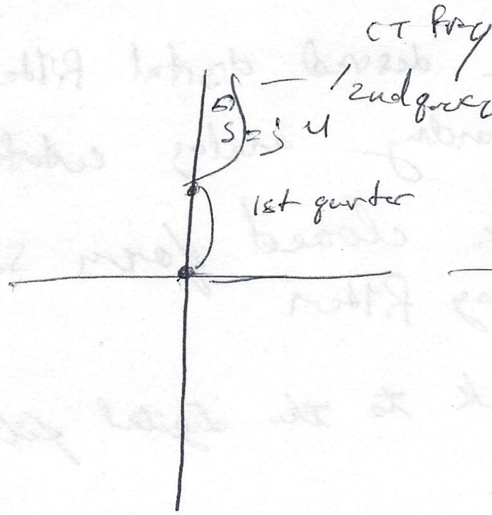


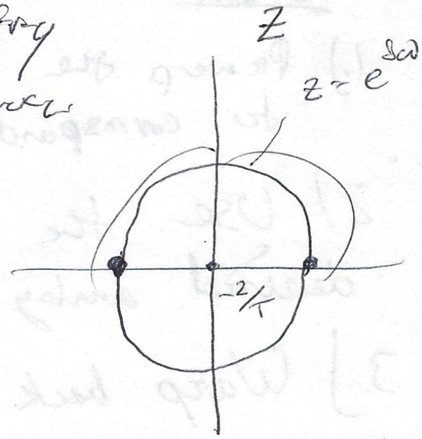
Bilinear Transformation; consider a trasformation of the s-plane into z-plane

$$S = \frac{2}{T} \frac{z-1}{z+1} \qquad z = \frac{2/T + S}{2/T - S}$$

$H_c(s)$ replace s with $\frac{2}{T} \frac{z-1}{z+1}$, treat it as $H(z)$

S	z	$\omega \in (0, 2\pi)$
0	1	0
$\pm\infty$	-1	π
$\frac{2}{T} j$	j	$\pi/2$
$-2/T$	0	N/A





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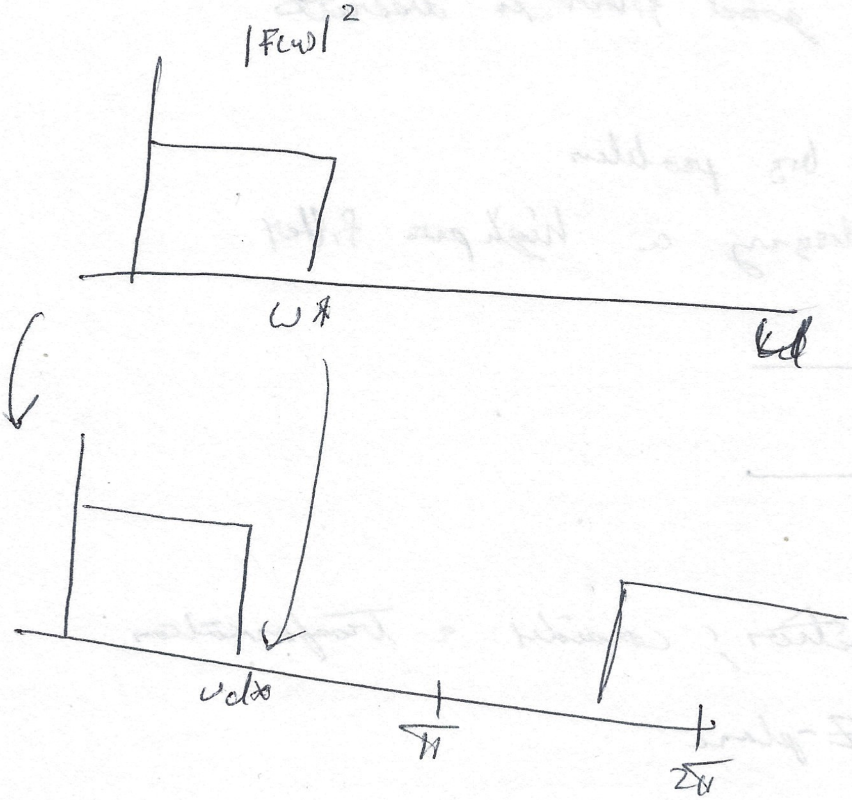
Analog filter



Analog filter

Digital filter

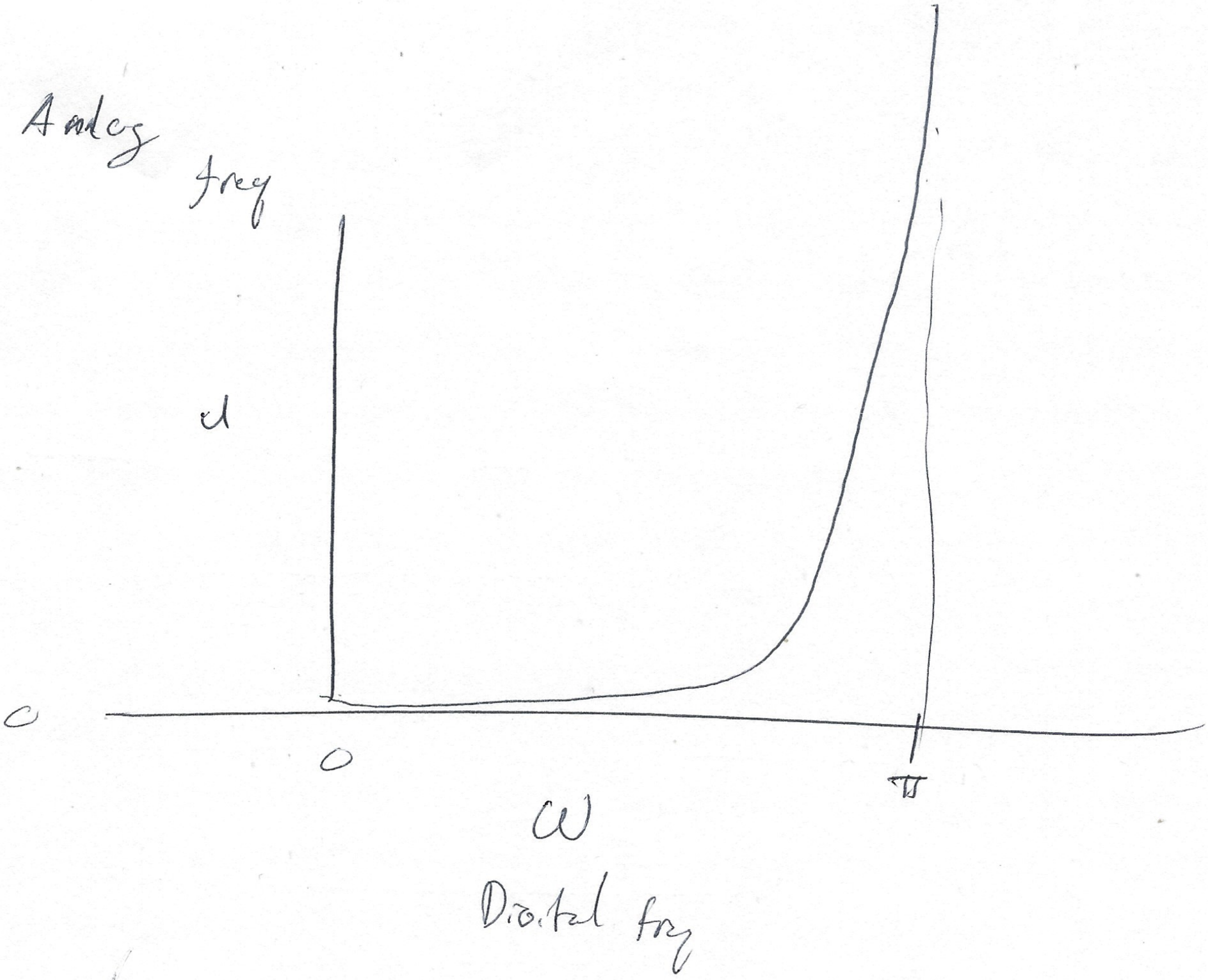
Design analog filter s.t. the cutoff will map to the desired location in the discrete world.

Idea

1.) Prewarp the desired digital filter cutoff frequencies to corresponding analog cutoffs

2) Use the closed form solution to obtain the desired analog filter

3.) Warp back to the digital filter with bilinear ~~transformation~~



fda tool → 56 ms