ADAPTIVE FIITERING

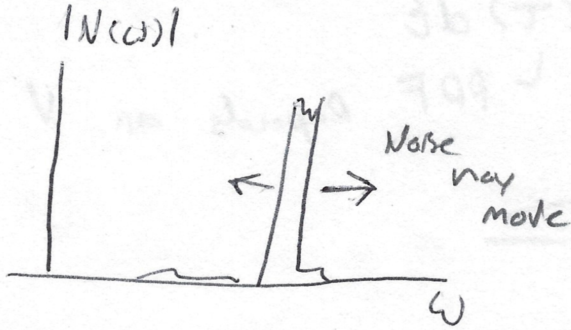
$$x[n] \longrightarrow \boxed{H} \longrightarrow y[n]$$

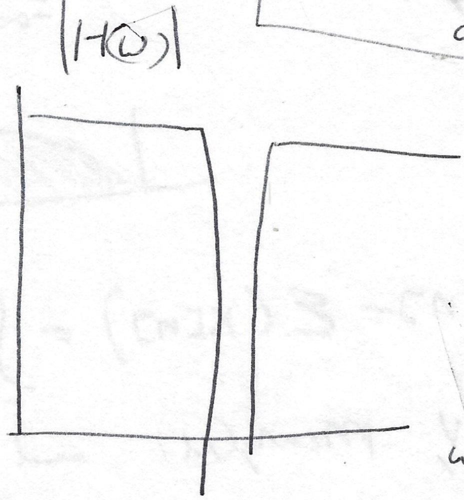
└ LTI

↳ FIR filter with N Taps

Say we have a system with some Sort of Noise.

Spectrum of the noise





filter to Notch out the Noise

Band stop Fil

We don't know where the noise is, but we have a probability description of where it may be located. We want to remove the Noise by a filter

→ We want to design a filter to move with the Noise

→ Say we have a filter of N-taps we want to update these N-taps at every point in time for optimal performance → (update the Coefficients)

→ (Adaptive Filters)

(2)

- Rate of Convergence
- Quality of Tracking
- Robustness
- Complexity

Now, we think about $x[n]$ as stochastic

We describe $x[n]$ with a CDF!

$$\underbrace{P(x[n] < t)}_{\text{CDF}} = \int_{-\infty}^{t} \underbrace{p_n(\tau)}_{\text{PDF}}\, dt$$

PDF depends on n

t

$$\mu[n] = E(x[n]) = \int_{-\infty}^{\infty} \tau\, p_n(\tau)\, d\tau$$

Single RV, mean, (μ) and variance, σ^2

Multiple RVs auto-covariance (covariance)

$$c(n, n-k) = E\big((x[n]-\mu[n])(x[n-k]-\mu[n-k])\big)$$

Auto-correlation

$$r(n, n-k) = E(x[n]\, x[n-k])$$

A stochastic process is stationary if the statistical properties are time-invariant.

For Stationary Process.

$\mu[n] = \mu$ for all n

$r(n, n-k) = r(n+m, n+m-k) = r(k)$ "lag"

$c(n, n-k) = c(k)$ only depends on the Difference in time. → not the time we look at it.

The above are not sufficient for the process to be stationary

A Process is Wide-SENSE STATionary (WSS)

if $\mu[n] = \mu$ $\forall n, k$

$r(n, n-k) = r(k)$ > are both true

A stationary process is WSS

$$WSS \Longleftrightarrow \mathcal{E}\{|x[n]|^2\} < \infty \quad \forall n$$

Correlation MATRIX for WSS

$$V[n] = [x[n], x[n-1], \ldots x[n-(M-1)]]^T \quad M \times 1$$

$$R = \mathcal{E}(\underset{m \times 1}{V[n]} \cdot \underset{1 \times m}{V[n]^T}) \quad m \times m$$

$$R_{ij} = \mathcal{E}(x[n-i] \cdot x[n-j]) = (r(|i-j|))$$

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$$R = \begin{bmatrix} r[0] & r[1] & & & r[M-1] \\ r[1] & r[0] & r[1] & & \\ & & \ddots & \ddots & \\ & & & & r[1] \\ r[M-1] & & & r[1] & r[0] \end{bmatrix}$$

- Symmetric
- Equal Diagonals (Toeplitz)
- Non-Negative Definite

$X^T R X \geq 0$ for any $X \in R^m$

SPECIAL MODES FOR STOCHASTIC SIGNALS

Parametric Modes: Moving Average (MA)

Autoregressive (AR)

ARMA

Noise

White Gaussian Noise Model

$V[n]$ Has PDF $\frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2\sigma^2}(t-\mu)^2}$

'White' indicates $\mu = 0$

$\mathcal{E}(V[n]) = 0$

$$\mathcal{E}(V[n]V[n-k]) = \begin{cases} \sigma_v^2 & k=0 \\ 0 & k \neq 0 \end{cases} \rightarrow \sigma_v^2 \delta_{nk}$$

Fires when $n=k$

Moving Average model (MA)

Signal is generated by a linear combination of Noise values

$$X[n] = V[n] + b_1 V[n-1] + b_2 V[n-2] + \cdots + b_k V[n-k]$$

Where $\{b_i\}$ Constants

$\{V[n]\}$ is a White Gaussian Noise Process i.e.) FILtering White Noise with an FIR

Filter

$$X[n] = \sum_{l=0}^{l=n} V[n-l]\, b[l] = \underline{V} * \underline{b}$$

(X100 time convoluting the filter with the noise

$$V[n] \longrightarrow \boxed{\begin{matrix}FIR\\ \{b_i\}\end{matrix}} \longrightarrow \boxed{X[n]} \longrightarrow \boxed{H(k)} \rightarrow Y$$

Auto Regressive Model (AR)

$$X[n] = -a_1 X[n-1] - a_2 X[n-2] - \cdots - a_M X[n-M] + V[n]$$

X[n] = previous linear combinations of the signal + some Noise

error term assumed to be gaussian

$\{a_i\}$ constants $\quad a_0 = 1$

$\{V[n]\}$ WGN Process

Since the current value of X depends on previous values of X, X is regressing on itself by a

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$$V[n] = \sum_{i=0}^{M} a[i]\, X[n-i] = X * a$$

IIR FILTER Applied to Noise

X[n] can be viewed as the output of applying an IIR to WGN process.

(6) current value of signal related to the previous value of the signal offset by a little bit of error.

ARMA model is a combo of both

Generally, we have ~~A~~RMA MODEL:

$$X[n] + a_1 X[n-1] + a_2 X[n-2] + \dots a_M X[n-M]$$
$$= V[n] + b_1 V[n-1] + \dots b_K V[n-K]$$

equivalently $V[n] \rightarrow \boxed{H(z)} \rightarrow X[n]$ $\quad H(z) = \dfrac{B(z)}{A(z)}$

Apply this to noise

How to estimate the Parameters of an AR Process? $a_1 \dots a_M$, σ_v^2

(Also the order of the mode M.)

we look at the input signals that we get and we try and figure out, are these input signals well modeled by an MA process or AR Process, if well modeled by an AR process, How do we know how many previous values we should look at

The Noise variance, the parameters

→ <u>Yule walker</u>

Given $x[n]$, find $\{a_i\}$, σ_v^2, (M)

$$v[n] \rightarrow \boxed{IIR\ \{a_i\}} \rightarrow x[n]$$

with 0 mean

σ_v^2

$$x[n] + a\,x[n-1] + \cdots\ a_n x[n-M] = v[n]$$

$$x[n]x[n-l] + a_1 x[n-1]\,x[n-l] + \cdots\ a_M x[n-M]\,x[n-l] = v[n]\,x[n-l]$$

we multiplied on $x[n-l]$

choose l

Take Expected value of both sides

$$\mathcal{E}(x[n]\,x[n-l]) = r(l]$$

$$r[l] + a_1 r[l-1] + \cdots + a_M r[l-M] = 0$$

$$\mathcal{E}(v[n]\,x[\underline{n-l}]) = 0 \quad \text{if } l>0$$

all of this happens before $v[n]$ is generated

noise term

Generate M equations

$l = 1, \text{—}, M$

noise is independent of $x[n-l]$

(8) $r[l] + a_1 r[l-1] + a_2 r[l-2] + a_3 r[l-3] + \dots a_M r[l-M] = 0$

$l=1 \quad r[1] + a_1 r[0] + a_2 r[1] + \dots + a_M r[M-1] = 0$

$r[-1] = r[1]$

due to $\mathcal{E}(x[2]x[3]) = \mathcal{E}(x[2]x[1])$

$l=2 \quad r[2] + a_1 r[1] + a_2 r[0] + \dots + a_M r[M-2] = 0$

$l=M \quad r[M] + a_1 r[M-1] + a_2 r[M-2] + \dots + a_M r[0] = 0$

$$l=1 \quad \begin{bmatrix} r[0] & r[1] & \dots & \cancel{r[M-1]} \\ r[1] & r[0] & \dots & r[M-2] \\ & & & \\ & & & r[1] \\ r[M-1] & \dots & r[1] & r[0] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_M \end{bmatrix} = \begin{bmatrix} -r[1] \\ -r[2] \\ \\ \\ -r[M] \end{bmatrix}$$

Like the Toeplitz matrix

(Move $r[1]$ to the RHS)

→ we are generating the autocorrelation matrix.

Compactly $\underbrace{R}$ autocorrelation matrix (Involves $r[0]$ — $r[M-1]$)

$Ra = r$ ← ' of Autocorrelation from $r[1]$ to $r[M]$

$a = R^{-1} r$

Initially, we don't know the autocorrelations

- How to estimate them?

$$\hat{r}[i] = \frac{1}{N}\sum_{j=1}^{N} x[j]\,x[j-i]$$

for a given realization of data

So what of the variance? what if $i=0$

Yule-Walker Equations

$$x[n]x[n] + a_1 x[n]x[n-1] + \dots a_M x[n]x[n-M] = v[n]x[n]$$

$\mathcal{E}$(both sides)

$$r[0] + a_1 r[1] \dots \quad a_M r[M] = \sigma_v^2$$

$$\mathcal{E}(v[n]x[n]) = \mathcal{E}\left(v[n]\left(-\sum_{i=1}^{M} a[i]\,x[n-i] + v[n]\right)\right)$$

independent with previous $x[n-i]$

$$= \mathcal{E}(v[n]v[n]) = \sigma_v^2$$

once we estimated the a's

$$\sigma_v^2 = \sum_{k=0}^{M} a_k r[k] \quad (a_0 = 1)$$ assumption

$\mathcal{E}[v[n]\,v[n-i]] = 0$, $i>0$

How do we know M?

Note: $x[n-i]$ and $v[n]$ are ind. for $i>0$

$x[n-i]$ only depends on $v[n-k]$ for all $k \geq i$

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What is a reasonable M?

Let $\{X[0], \ldots X[n]\}$ be Data from experiment.

Let $\hat{\theta}_m$ be estimated parameters from Yule-Walker assuming Model Order $= M$.

$$L(\hat{\theta}_m) = \log(P(\{x[i]\} | \hat{\theta}_m))$$

Likelyhood

prob of seeing that data give parameters $\hat{\theta}_m$

maximize this log likelihood as a function of m, more parameters, better the data can fit the model.

Look for max value over some range some model order
or

$$\underset{M=0,1,2\ldots}{\arg\max} \; L(\hat{\theta}_m) - m$$

PENALize larger model orders

fit with the data assuming model order m

Trade off to fritting the data and having a really long model

→ Look at video