DSP Lect 6

Frequency response

$$y(t) = x(t) * h(t) \qquad Y(\omega) = X(\omega)H(\omega)$$

why its true

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

$$Y(\omega) = \int_{-\infty}^{\infty} y(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau\right)e^{-j\omega t}dt$$

$$= \int_{-\infty}^{\infty} x(\tau)\left[\int_{-\infty}^{\infty} h(t-\tau)e^{-j\omega t}dt\right]d\tau$$

This Looks Like the FT of $h(t-\tau)$

$$= \int_{-\infty}^{\infty} x(\tau)\mathcal{F}(h(t-\tau))\,d\tau = \int_{-\infty}^{\infty} x(\tau)H(\omega)e^{-j\omega\tau}d\tau$$

for $h(t-\tau)$

$x = t-\tau$

$$\int_{-\infty}^{\infty} h(t-\tau)e^{-j\omega t}dt = \int_{-\infty}^{\infty} h(x)e^{-j\omega(x+\tau)}dx = e^{-j\omega\tau}H(\omega)$$

Note! for $F(k) = \int_{-\infty}^{\infty} f(x)e^{ikx}dx$, $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)e^{-ikx}dk$

translation is $F(k)e^{ik\tau}$ for above conventions

Continuing!

$$H(\omega)\int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}d\tau = H(\omega)X(\omega)$$

② Note! when using the convention I learned in C vars the translation is <u>opposite the sign</u>

→ which works out beautifully for convolution

when $F(k) = \int_{-\infty}^{\infty} f(x) e^{+ikx} dx$

recall

$$\delta(t) \to \boxed{\text{LTI system}} \to h(t) \quad \text{Impulse response}$$

$$F(h(t)) = H(\omega) \quad \text{frequency response}$$

Sometimes a system is written like

$$x(t) \to \boxed{h(t)} \to y(t)$$

$$x(t) \to \boxed{H(\omega)} \to y(t)$$

$$x(t) \to \boxed{h_1(t)} \to \boxed{h_2(t)} \to y(t)$$

$$x(t) \to \boxed{H_1(\omega)} \to \boxed{H_2(\omega)} \to y(t)$$

$$Y(\omega) = X(\omega) H_1(\omega) H_2(\omega) = X(\omega) H_2(\omega) H_1(\omega)$$

→ "Legal"

$$x(t) \to \boxed{H_2(\omega)} \to \boxed{H_1(\omega)} \to y(t)$$

$$x(t) \to \boxed{h_2(t)} \to \boxed{h_1(t)} \to y(t)$$

Changing order of LTI systems doesn't change result

→ So what does the frequency response mean?

$$X(t) = e^{j\omega_0 t}$$ (phase shift of 1)

↳ complex number phase shifted in the time domain

$$X(\omega) = 2\pi\delta(\omega-\omega_0)$$

→ time shift in frequency domain

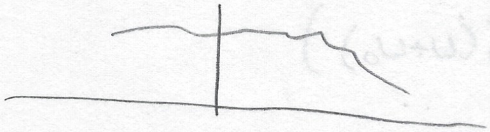
As a phase shift in the frequency domain is a ~~time shift~~ in the time domain.

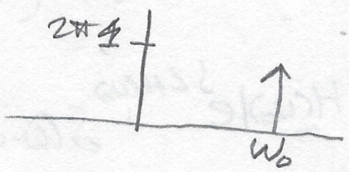
So now

$$Y(\omega) = H(\omega)X(\omega) = H(\omega)2\pi\delta(\omega-\omega_0)$$

$H(\omega)$ — freq response

$H(\omega)$





So then

$$H(\omega)2\pi\delta(\omega-\omega_0) = \underline{H(\omega_0)}\,2\pi\delta(\omega-\omega_0)$$

↳ constant

And so $F^{-1}\{Y(\omega)\} = y(t) = \underline{H(\omega_0)}\,\underline{e^{j\omega_0 t}}$

some complex scalar

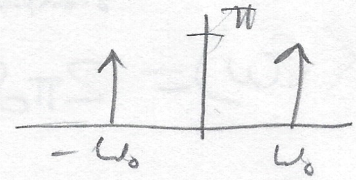
↳ original input signal

→ The LTI system cannot produce new frequencies that weren't present in the input

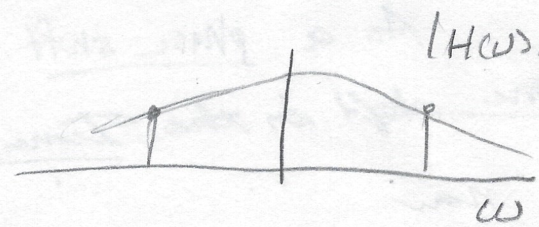
④ Suppose $h(t)$ is real

$$x(t) = \cos(\omega_0 t)$$

So then, $H(\omega) = (H(-\omega))^*$



$$Y(\omega) = X(\omega)H(\omega)$$



$$= \pi\left(\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right) H(\omega)$$

$H(\omega) = (H(-\omega))^*$

$H(-\omega) = (H(\omega))^*$

$$= \pi\left(H(\omega_0)\delta(\omega-\omega_0) + H(-\omega_0)\delta(\omega+\omega_0)\right)$$

$$= \pi\left(H(\omega_0)\delta(\omega-\omega_0) + (H(\omega_0))^*\delta(\omega+\omega_0)\right)$$

$$= \pi\left(|H(\omega_0)|e^{j\angle H(\omega_0)}\delta(\omega-\omega_0) + |H(\omega_0)|e^{-j\angle H(\omega_0)}\delta(\omega+\omega_0)\right)$$

$$\Rightarrow \quad y(t) = |H(\omega_0)| \cos(\omega_0 t + \angle H(\omega_0))$$

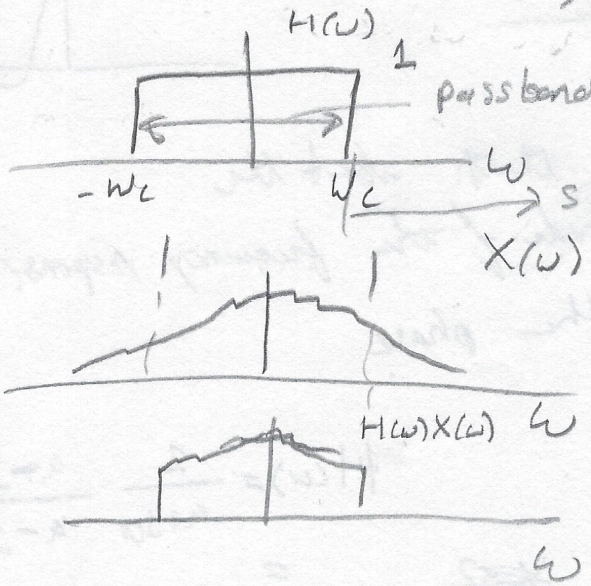
Same cosine affected by amplitude and a new phase

$\cos(3t) \rightarrow \boxed{} \rightarrow \cos 5t$ Not an LTI system as There is a different frequency Not from the original signal

To mute a note

We often/usually interpret $H(\omega)$ as what the system does to frequencies of the input signal "Filter"

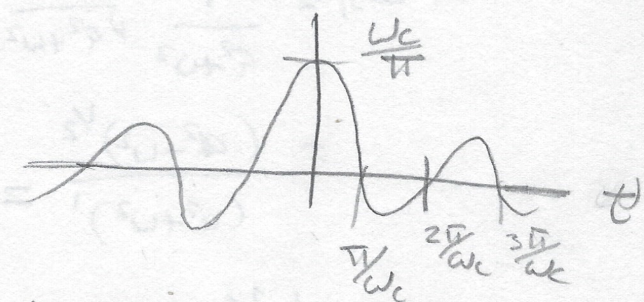
Important one; Low Pass filter



W_c: cut off frequency

So we can see that

$$h(t) = \frac{\omega_c}{\pi}\,\text{sinc}(\omega_0 t)$$



There are some issues with $h(t) = \frac{\omega_c}{\pi}\,\text{sinc}(\omega_0 t)$

→ It's not causal. We are looking into the future to filter the signal (existing signal in $(-t)$) which corresponds to the impulse response sectioned ~~to the~~ left of ~~the~~ y-axis.

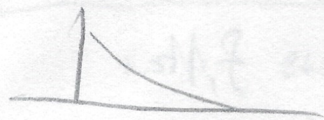
→ The signal extends outward infinitely from the origin, so trying to create a huge delay ~~cause some problems~~ to capture thousands of samples for the filter is a lot for precision. (Gotta stop somewhere)

→ The rippleness of the sinc function will leak out in the output which is unwanted.

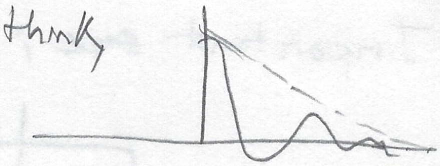
* We will need to sacrifice in the frequency domain for a better time-domain filter.

(6) So what a good approximation?

Consider $h(t)=e^{-at}u(t)$ $a>0$



and think



$H(\omega)=\frac{1}{a+j\omega}$ first, think about the magnitude of the frequency response

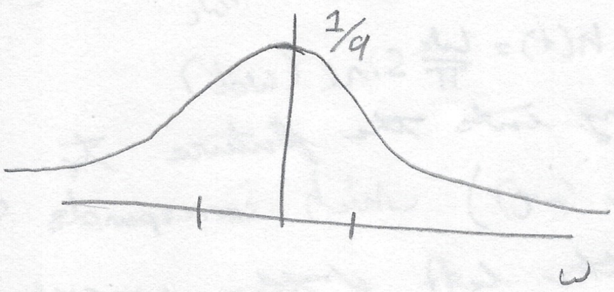
also the phase

$|H(\omega)| = \frac{1}{a^2+\omega^2}\sqrt{a^2+\omega^2}$

$= \frac{(a^2+\omega^2)^{1/2}}{(a^2+\omega^2)^{1}} = \frac{1}{(a^2+\omega^2)^{1/2}}$

$H(\omega) = \frac{1}{a+j\omega}\,\frac{a+j\omega}{a-j\omega}$

$= \frac{a-j\omega}{|a^2+\omega^2|}$



<u>H(ω) is a complex #</u>

So Look at $|H(\omega)|$

→ This is how the input is affected by the magnitude!

→ Essence of LPF

The width depends on $\underline{a}$

1) Compute $X(\omega)$ from $x(t)$

2) multiply $X(\omega)H(\omega) = Y(\omega)$

3) Take Inverse FT to get $y(t)$

$x(t) = e^{-5t}u(t), \quad h(t) = e^{-3t}u(t)$

$X(\omega) = \frac{1}{5+j\omega}$ $\quad H(\omega) = \frac{1}{3+j\omega}$

$Y(\omega) = \frac{1}{(5+j\omega)(3+j\omega)}$ $\quad H(\omega) = \frac{1}{3+j\omega}$

$Y(\omega) = \frac{A(3+j\omega) + B(5+j\omega)}{(5+j\omega)(3+j\omega)} = \frac{(3A+5B) + (A+B)j\omega}{(\quad)(\quad)}$

$3A + 5B = 1 \quad A + B = 0 \quad 2B = 1 \quad B = \frac{1}{2} \quad A = -\frac{1}{2}$

$B = -A$

$Y(\omega) = \frac{-1/2}{5+j\omega} + \frac{1/2}{3+j\omega}$

$y(t) = -\frac{1}{2}e^{-5t}u(t) + \frac{1}{2}e^{-3t}u(t)$

$x(t) = e^{-t}u(t)$ $\quad H(\omega) = \frac{2+j\omega}{(1+j\omega)(3+j\omega)}$

$X(\omega) = \frac{1}{1+j\omega}$

So $Y(\omega) = \frac{2+j\omega}{(1+j\omega)^2(3+j\omega)}$

$Y(\omega) = \frac{A}{(1+j\omega)^2} + \frac{B}{(1+j\omega)} + \frac{C}{(3+j\omega)}$

$\rightarrow \frac{A(3+j\omega) + B(1+j\omega)(3+j\omega) + C(1+j\omega)^2}{(\quad)^2(\quad)}$

$= \frac{(3A+3B+C) + (A+4B+2C)j\omega + (B+C)j\omega^2}{(\quad)^2(\quad)}$

$3A + 3B + C = 2 \qquad A + 4B + 2C = 1$

$B + C = 0$

$C = -B$

$\rightarrow 3A + 2B = 2, \quad A + 2B = 1 \qquad 2A = 1 \qquad A = \frac{1}{2}, \; B = \frac{1}{4}, \; C = -\frac{1}{4}$

So $$Y(\omega) = \frac{1/2}{(1+j\omega)^2} + \frac{1/4}{(1+j\omega)} + \frac{-1/4}{(3+j\omega)}$$

$$y(t) = \tfrac{1}{2} t e^{-t} u(t) + \tfrac{1}{4} e^{-t} u(t) - \tfrac{1}{4} e^{-3t} u(t)$$

$$y(t) = \frac{d}{dt} x(t) \qquad Y(\omega) = j\omega X(\omega)$$

$$\frac{d^2}{dt^2} y(t) + 4\frac{dy(t)}{dt} + 3y(t) = \frac{d}{dt} x(t) + 2x(t)$$

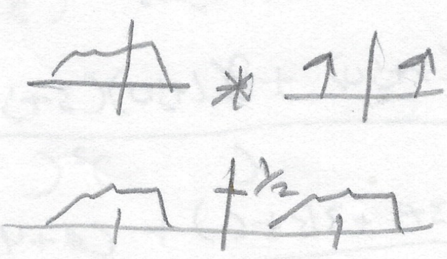
$$(j\omega)^2 Y(\omega) + 4j\omega Y(\omega) + 3Y(\omega) = j\omega X(\omega) + 2X(\omega)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{2 + j\omega}{(1+j\omega)(3+j\omega)}$$

Note multiplication in the time domain is convolution in the frequency domain

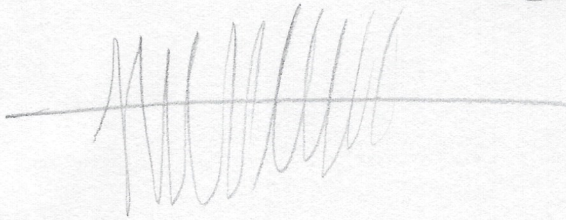
$$z(t) = x(t)y(t) \longleftrightarrow Z(\omega) = \frac{1}{2\pi} X(\omega) * Y(\omega)$$

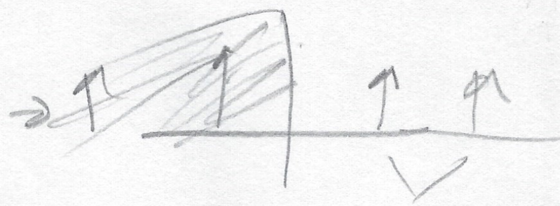
$x(t)\cos(\omega_0 t)$



$\boxed{\text{modulation property}}$

2 cosines

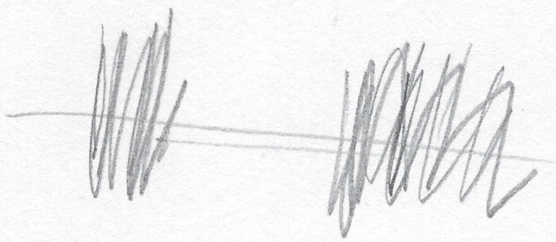


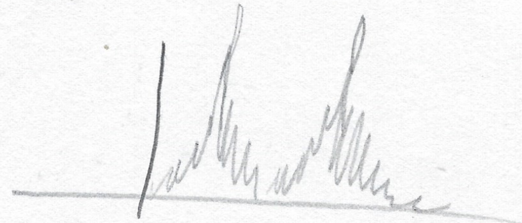


2 cosines

→ Low pass filter to get rid of one of the cosines

Same 2 cosines multiplied by 2 pulses





Sinc functions convolved with the impulses

A musical instrument gives you resonant frequencies → strong frequency peaks in the frequency domain

FT operates on ∞ long continuous signal

→ ∞ long discrete time signal

→ finite long discrete signal

→ use filters ~~and~~ Notches to get rid of unwanted Noise

→ to clean up the signal, there may need to be compromise.