DSP Inverse Z-transform ①

Lect 9 $X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$

$\alpha^n u[n] \overset{z}{\longleftrightarrow} \frac{1}{1-\alpha z^{-1}} \quad |z| > |\alpha|$

if $|\alpha| < 1$, DTFT exists → all good somehow

if $|\alpha| > 1$, wouldn't converge, will work for certain z

$\cos(\omega_0 n) u[n] \longleftrightarrow \frac{1-\cos(\omega_0) z^{-1}}{1 - 2\cos(\omega_0) z^{-1} + z^{-2}} \quad |z| > 1$

$r^n \sin(\omega_0 n) u[n] \longleftrightarrow \frac{r \sin(\omega_0) z^{-1}}{1 - 2r\cos(\omega_0) z^{-1} + r^2 z^{-2}} \quad |z| > r$

Notice, these z-transforms are formula's depend on z^{-1} (negative powers of z) $1/z$? why is that?

Say $x[n] = 0 \quad n < 0$

So $X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=0}^{\infty} x[n] z^{-n} = x[0] + x[1] z^{-1} + x[2] z^{-2} + \dots$

For causal signal, only negative powers of z

one could

$\frac{1-\cos(\omega_0) z^{-1}}{1 - 2\cos(\omega_0) z^{-1} + z^{-2}} \cdot \frac{z^2}{z^2} = \frac{z^2 - \cos(\omega_0) z}{z^2 - 2\cos(\omega_0) z + 1}$

easier to find the poles and zeros

Tables are not in this notation, so get use to it

② How to undo Z-transform

$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

Complex contour integral for some $|z| = r$ in the ROC

Kinda like integrals in 2D plane

Use patterns instead

$$X(z) = \frac{7 - 13z^{-1}}{1 - 2z^{-1} - 3z^{-2}} \quad |z| > 1$$

→ can't assume the time-domain signal is right-sided

↓

$$(1-3z^{-1})(1+z^{-1}) = 1 + z^{-1} - 3z^{-1} - 3z^{-2}$$

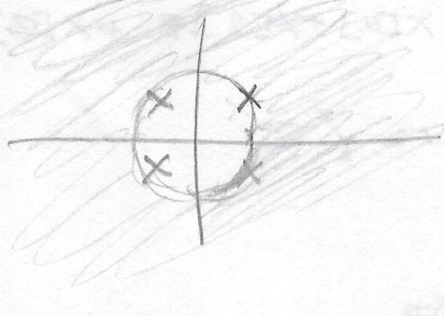
Think $1 - 2z - 3z^2$ or $z^2(1 - 2z^{-1} - 3z^{-2}) = z^2 - 2z - 3$

$$= (z+1)(z-3)$$

with $(z+1)(z-3) \times z^{-1} = (1 + z^{-1})(1 - 3z^{-1})$

RoC is a clue of what the inverse Z-transform is suppose to be

→ Learn to use the inverse transform



We know the RoC is bounded by the magnitudes of one of these poles

From RoC $|z| > 1$

one of the poles has magnitude 1

So now

$$X(z) = \frac{A}{1 - 3z^{-1}} + \frac{B}{1 + z^{-1}}$$

$$X(z) = \frac{(A+B) + (A-3B)z^{-1}}{(1-3z^{-1})(1+z^{-1})}$$ ③

$A+B=7$

$-A+3B=13$

$4B=20$

$A=2, B=5$

So now

$$X(z) = \frac{2}{1-3z^{-1}} + \frac{5}{1+z^{-1}}$$

Then

$$x[n] = 2\cancel{\left(\tfrac{1}{3}\right)^n}\,(3)^n u[n] + 5(-1)^n u[n]$$

So now

$$X(z) = \frac{3z}{z^2+2z+4}$$

$$\frac{-2 \pm \sqrt{4-4(4)}}{2} = -1 \pm \tfrac{1}{2}\sqrt{4-16}$$

$$= -1 \pm \tfrac{1}{2} i\, 2\sqrt{3}$$

$12 = 2\cdot 6$
$= 4\cdot 3$

$$= \frac{3z}{(-1+i\sqrt{3})(-1-i\sqrt{3})}$$

$\sqrt{3}$

-1

$\frac{2\pi}{6} + \frac{3\pi}{6}$

$2e^{i(\pi/3 + \pi/3)}$

$= 2e^{\pm i\,2/3\,\pi}$

from ~~the~~ ~~table~~, you see things that look familiar

$$\underset{\downarrow\, r}{2}\, e^{\pm i\, \underset{\downarrow\, \omega_0}{2/3\,\pi}}$$

Same equation

$$X(z) = \frac{3z^{-1}}{1+2z^{-1}+4z^{-2}}$$

← we want this to take the form of

$$\frac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$$

ROC $|z|>r$

(4) So then

$-(+2) = +2r\cos(\omega_0)$

$= +2 \cdot 2\left(-\frac{1}{2}\right)$

from

$$\frac{3z^{-1}}{1+2z^{-1}+4z^{-2}} = \frac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$$

recall $\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$

$\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$

$3z^{-1} = r\sin(\omega_0)z^{-1}$

$= 2\left(\frac{\sqrt{3}}{2}\right)z^{-1}\,(\sqrt{3})$

→ scalar of $\sqrt{3}$

So $\sqrt{3}\left(\frac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}\right)$ | $r=2$, $\omega_0 = \frac{2\pi}{3}$

$x[n] = \sqrt{3}\left(2^n \sin\left(\frac{2\pi}{3}n\right)u[n]\right)$

→ we implied this ~~to~~ be a right-sided signal

Say now we had

$$\frac{3z+5}{z^2+2z+4} \rightarrow Ar^n\sin(\omega_0 n)\,u[n] + Br^n\cos(\omega_0 n)\,u[n]$$

Bit more tedious

what about

$X(z) = 3z^{-2} + 5z^{-1} - \frac{1}{2} + 3z^3$ ROC $0 < |z| < \infty$

→ Some tricky shit indeed

use the definition of the z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = x[0] + x[1]z^{-1} + x[2]z^{-2} + \dots + x[-1]z + x[-2]z^2 + \dots$$

remember this kind of series stuf in cvars?

So then

$$X(z) = 3z^{-2} + 5z^{-1} - \frac{1}{2} + 3z^{3}$$

$$x[n] = 3\delta[n-2] + 5\delta[n-1] - \frac{1}{2}\delta[n] + 3\delta[n+3]$$

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what of e^{z} $|z| < \infty$

$X(z) = e^{z}$ → power series

$$= 1 + z + \frac{1}{2}z^{2} + \frac{1}{6}z^{3} + \cdots$$

$x[0]$ $x[-1]$ $x[-2]$ $x[-3]$

→ Left sided signal

$$= \sum_{n=0}^{\infty} \left(\frac{1}{n!}\right) z^{n}$$

$$\rightarrow x[n] = \begin{cases} 0 & n \geq 1 \\ \frac{1}{n!} & n \leq 0 \end{cases}$$

⑥ Long division

$$X(z) = \frac{1+2z^{-1}}{1+z^{-1}}$$

$$z+1 \overline{\smash{)}\, z+2} \quad 1 + \tfrac{1}{z} - \tfrac{1}{z^2}$$

$$z+1$$

$$1$$

$$-\left(1+\tfrac{1}{z}\right)$$

$$-\tfrac{1}{z}$$

$$-1-\tfrac{1}{z^2}$$

$$1+z^{-1} \overline{\smash{)}\, 1+2z^{-1}} \quad 1+\tfrac{1}{z}-\tfrac{1}{z^2}+\cdots$$

$$1+z^{-1}$$

$$z^{-1}$$

$$-\left(\tfrac{1}{z}+\tfrac{1}{z^2}\right)$$

$$-\tfrac{1}{z^2}$$

1) Linearity $\quad a x_1[n] + b x_2[n] \leftrightarrow a X_1(z) + b X_2(z)$

2) Time shift $\quad x[n-n_0] \leftrightarrow z^{-n_0} X(z)$

sorta like a time-shift corresponding to a phase shift in another domain

the added z^{-n_0} may introduce extra poles and zeros

So

$$X(z) = \frac{1+2z^{-1}}{1+z^{-1}} = \frac{1}{1+z^{-1}} + 2z^{-1}\frac{1}{1+z^{-1}}$$

$$x[n] = (-1)^n u[n] + 2(-1)^{n-1} u[n-1]$$

use big z's on top as delays

3) Scaling $a^n x[n] \longleftrightarrow X\left(\frac{z}{a}\right)$

a^n may bring the poles/zeros inward or outward.

4.) Time-reversal $x[-n] \longleftrightarrow X\left(\frac{1}{z}\right)$

$\Rightarrow$ check out in text later

5.) Convolution

$x[n] * h[n] \longleftrightarrow X(z)H(z)$

6.) Differentiation $nx[n] \longleftrightarrow -z\frac{d}{dz}X(z)$

7.) Initial value theorem

$$x[0] = \lim_{z \to \infty} X(z)$$

DC value

$$H(z) = \frac{N(z)}{D(z)} \quad \text{Polynomials}$$

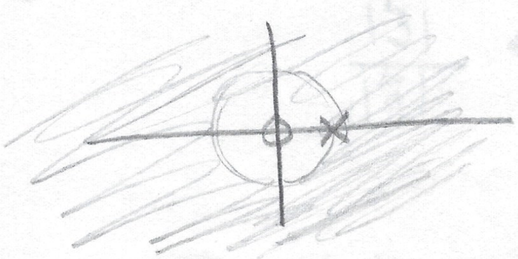
Transfer function

$N(z) = 0 \rightarrow H(z) = 0$ zeros

$D(z) = 0 \rightarrow H(z) = \infty \rightarrow$ poles

$h[n] = \left(\frac{1}{3}\right)^n u[n] \qquad H(z) = \frac{1}{1 - \frac{1}{3}z^{-1}} \quad |z| > \frac{1}{3}$

$$= \frac{z}{z - \frac{1}{3}}$$

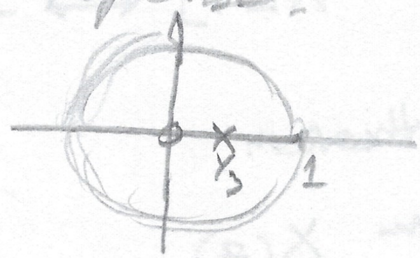


⑧ for laplace, when you know the poles/zeros in the s-domain → You can make inferences about what's happening in the frequency domain (and time)

when the ROC includes the unit circle, we have the DTFT

from $-\pi$ to π we can Note the frequency response.

Look at

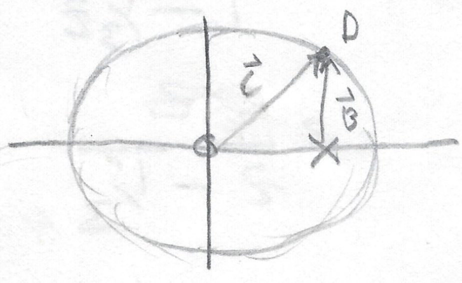


So what's happening in the frequency domain

evaluate $X(Z)$ as we move around the unit circle

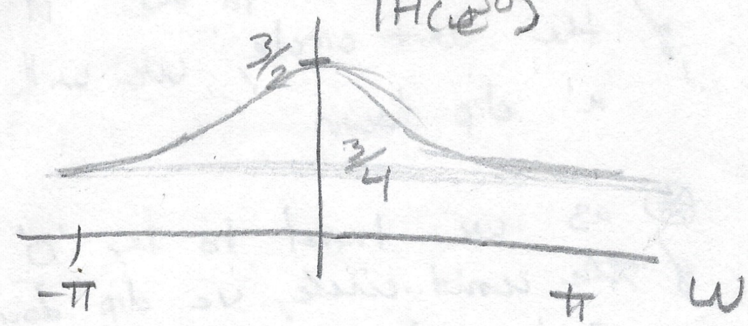
$$|H(e^{j\omega_0})| = \left|\frac{N(e^{j\omega})}{D(e^{j\omega})}\right| = \frac{\prod\left(\begin{array}{l}\text{Length of vector}\\ \text{from each zero to } e^{j\omega_0}\end{array}\right)}{\prod\left(\begin{array}{l}\text{Length of vector}\\ \text{from each pole to } e^{j\omega_0}\end{array}\right)}$$

So if we want to know what's going on with the FT at point D get Length of $\vec{C}$ over $\vec{B}$



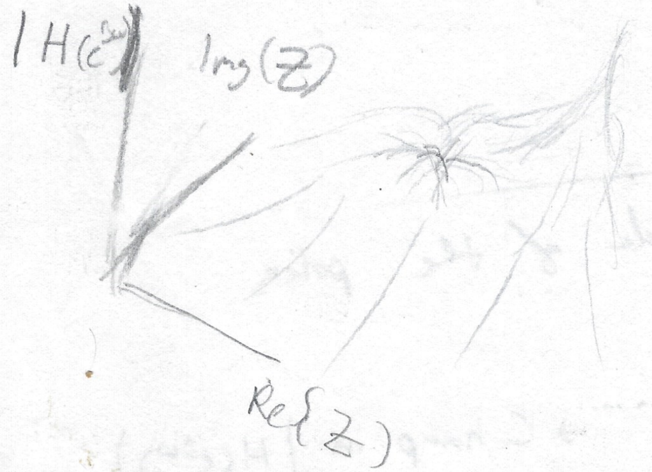
$$\rightarrow \frac{|\vec{C}|}{|\vec{B}|}$$

with this, we can make a crude plot



for $\omega = 0 \rightarrow \frac{1}{2/3}$

as we move on the unit circle, the ratios of the vectors get smaller. zero vector is constant, pole vector increasing

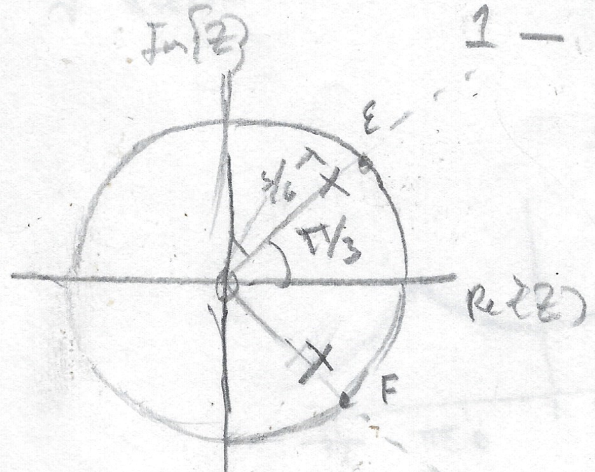


$$h[n] = \left(\frac{5}{6}\right)^n \sin\left(\frac{\pi}{3} n\right) u[n]$$

$r = 5/6$

$\omega_0 = \pi/3$ → freq of the time signal

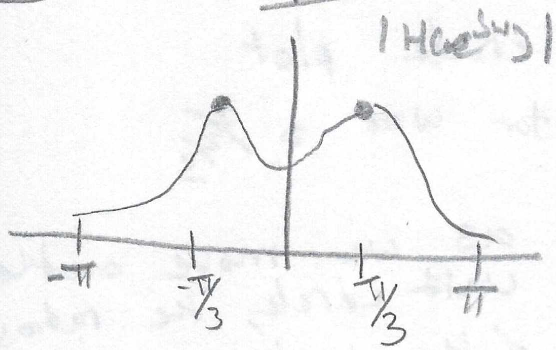
$$H(z) = \frac{\frac{5}{6}\sin\left(\frac{\pi}{3}\right) z^{-1}}{1 - \frac{10}{6}\cos\left(\frac{\pi}{3}\right) z^{-1} + \left(\frac{5}{6}\right)^2}$$



at points E, F we are near the extremen cause by the poles (the sticking up parts)

→ Local maximum of the frequency response

So plot



⊛ as we travelled to the 'π' end of the unit circle, we will find a dip down

⊛ as we travel to the '0' end of the unit circle, we dip down, but not as deep

→ crude Band Pass

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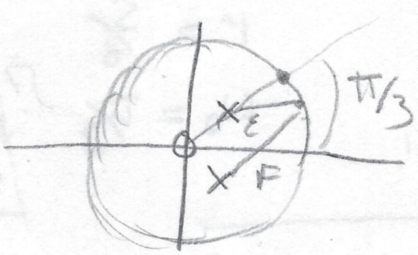
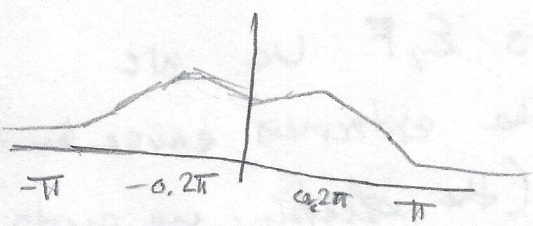
if we change the magnitude of the poles
→ (move them inward)
the 'tent poles' will be less → change in $|H(e^{j\omega})|$
the local maxima may move inwards (toward angle 0)

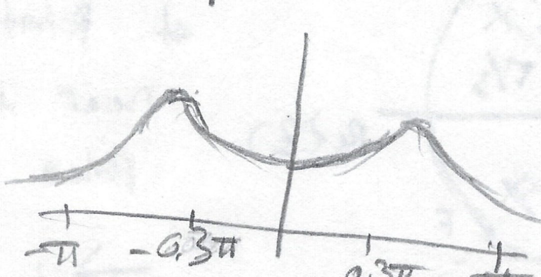


Product of vectors $\overline{E}$ and $\overline{F}$ may not be the local maxima at the same location any more with a decreased r

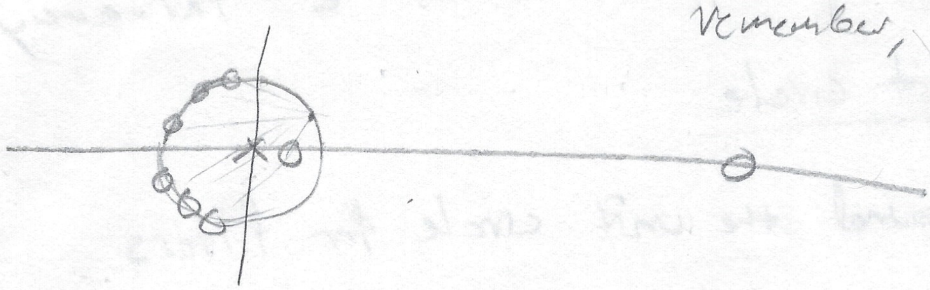
$\omega_0 = 1/3$



$\omega_0 = 11/12$

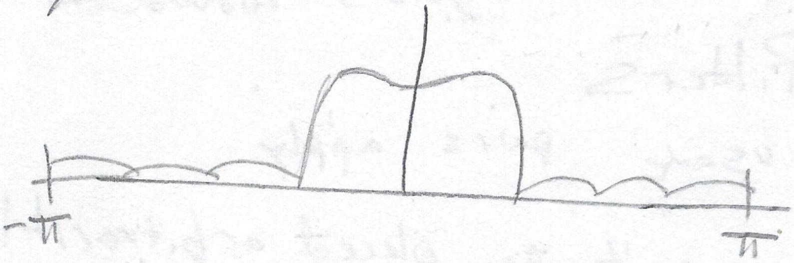


See and predict

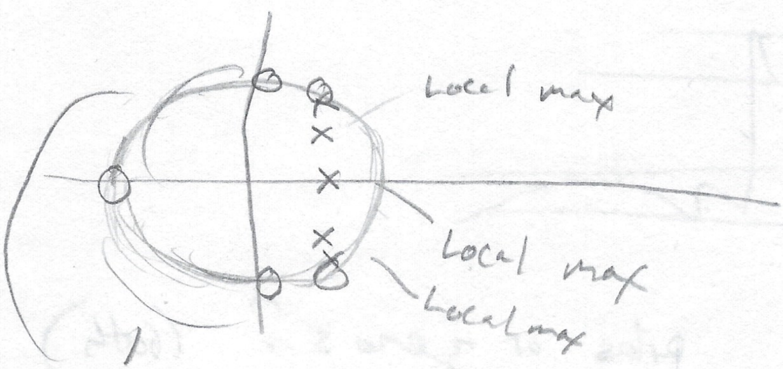


remember, each zero vector is multiplied by each other the divided by the pole vector multiplied by each other

maybe



→ Low passy filter



not much uplifted → pretty much nailed down

omg!!!

→ So this is why we care about the z-transform

(12) we don't really care for values of z far-away from the unit circle

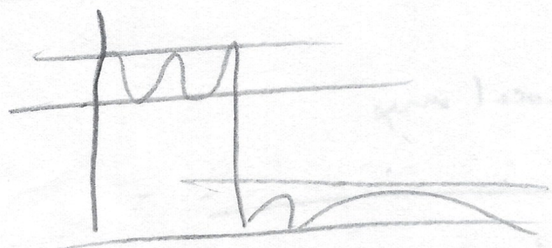
→ on and around the unit circle for filters

push the poles and zeros around to design filters

- complex conjugate pairs apply

poles / zeros can't be placed arbitrarily

think of tolerances



Taps — Lmso poles or zeros? (both)

can't have everything, there are trade-offs

Z-transforms are used to solve

Difference equations

$$y[n] + \tfrac{1}{4} y[n-1] = x[n] + \tfrac{1}{5} x[n-1]$$

what is $H(z)$?

$$Y(z) + \tfrac{1}{4} z^{-1} Y(z) = X(z) + \tfrac{1}{5} z^{-1} X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + \frac{1}{5} z^{-1}}{1 + \frac{1}{4} z^{-1}} \qquad |z| > \tfrac{1}{4}$$

right-sided

what is the response to a unit step?

(step response)

$$X(z) = \frac{1}{1 - z^{-1}} \qquad Y(z) = \frac{1 + \frac{1}{5} z^{-1}}{(1 + \frac{1}{4} z^{-1})(1 - z^{-1})}$$

unit step

$$= \frac{A}{(1 + \frac{1}{4} z^{-1})} + \frac{B}{1 - z^{-1}}$$

unilateral z-transform

→ only causal signals