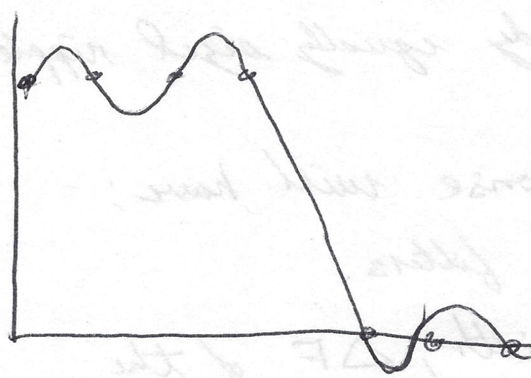
DSP Lect 17

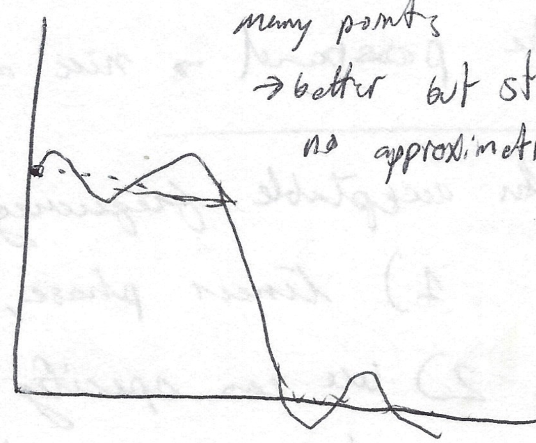
FIR filter design (chebyshev)

Last time FIR filter design

VIA LEAST-Squares / Interpolation

→ remember we specified the points, but we can't control what happens in between the points





→ How to force the filter not to deviate more then we asked for. in the pass band and stop band.

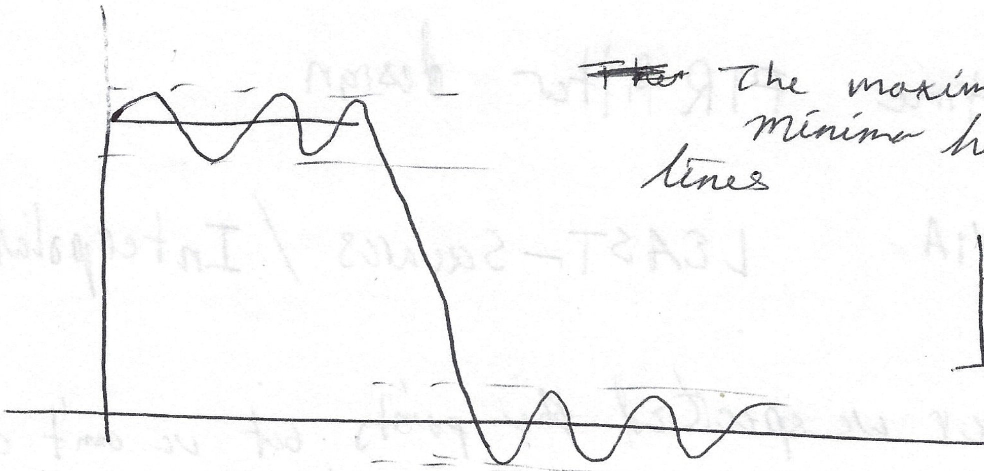
we want to

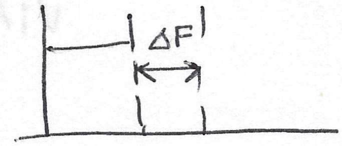
Chebyshev Approximation or Error criteria

$$\min \quad E = \max_{\omega \in [0,\pi)} \left| A(\omega) - A_d(\omega) \right|$$

Amplitude desired

② Fitlers that are optimal with respect to this criteria are called EQUIRIPPLE





→ No Extreme difference to where the filter might go in the passband → nice and steady equally sized ripple

An acceptable frequency response will have;

1) linear phase, FIR filters

2) We can specify the width, ΔF, of the transition band

if ΔF is small, we really need to book it, A Lot of filter Taps [length-N (N-tap) FIR Filter]

3) Deviation from 1 in the PASSBand of $\pm\delta_1$ δ_1 is thought of as some number.

4) Deviation from 0 in the stop band of $\pm\delta_2$

THE Amplitude Response of A TYPE 1 Linear Phase FIR filter CAN Be written

$$A(\omega) = \sum_{k=0}^{r-1} c_k \cos \omega k \qquad r = \frac{N+1}{2}$$

r is the M

The Approximation problem

Given
- A set of frequency bands IN $[0,\pi]$
- A DESIRED real valued $A_{des}(\omega)$
- A Positive weight Function $W(\omega)$
  → how seriously do we care about approximating the ~~desired~~ desired Amplitude response in each band
- The Form of $A(\omega)$

$$A(\omega) = \sum_{k=0}^{r-1} c_k \cos(\omega k) \qquad r = \frac{N+1}{2}$$

Find $\{c_k\}$

Solve Using the Alternation Theorem

IF $A(\omega)$ is a sum of r cosines

Then a Necessary And Sufficient Condition For $A(\omega)$ to Be the Unique, Best Weighted Chebyshev Approximation To $A_{des}(\omega)$ on the Given BANDS is that

(4)

$$\mathcal{E}(\omega) = W(\omega)\,|A(\omega) - A_d(\omega)|$$

Exhibit AT LEAST $r+1$ Extremal

frequencies in the Given BANDs

Extremal frequency is saying we have achieved the maximum error

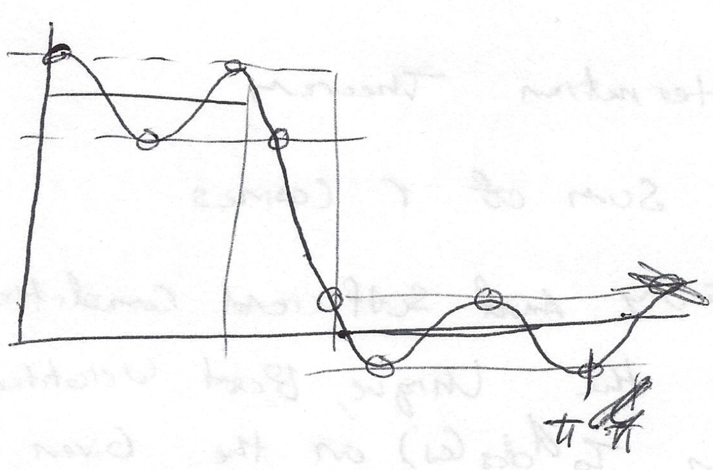
Looking back, if we count the number of times we hit the dotted line in the chebyshev thing we should have at least $r+1$ of them.

Extremal frequencies

$\{\omega_1, \omega_2, ---, \omega_{r+1}\}$ such that

$$\mathcal{E}(\omega_i) = \max \mathcal{E}(\omega)$$

$$\mathcal{E}(\omega_i) = -\mathcal{E}(\omega_{i+1}) \quad i = 1 \ldots r$$



$N=13$

$r = 7$ 7 cosines

We have $r+1$ places where we are exactly hitting

Remember, we are only Looking at the error specified within the bands

So we need 8 etremal frequencies ⇒ OPTIMAL

REMEZ EXChange AlGorithm

Iterative way, → we iteratively guess what those frequencies might be, and we refine them until we find something that satisfies our constraints

LEMMA Radke Say "Baby" Theorem

$$\hookrightarrow \quad \mathcal{E}(\omega) = A_d(\omega) - \sum_{k=0}^{r-1} c_k \cos(\omega k)$$

CAN Be Made to take on the values $\pm\delta$ for any given set $\{\omega_1, \dots \omega_{r+1}\}$

We can Say

$$A_d(\omega_i) = \sum_{k=0}^{r-1} c_k \cos \omega_i k + (-1)^i \delta \qquad i = 1 \dots r+1 \qquad (*)$$

This has a unique solution for $\{c_k, k=0, \dots r-1\}$ and δ,

(6)

Given $T_0 = \{\omega_1, \ldots \omega_{r+1}\}$ Initial Guesses for the Etremal frequencies

1) Solve the linear Equations in (*)
Solution has an error that oscillates with Amplitude $\underline{\underline{\delta_K}}$ on T_K.
Extremal error

2) Interpolate ~~that~~ to find frequency response on all of $[0, \pi]$

3) Search $[0, \pi]$ To see if/where the magnitude of Error $> \delta_K$

4) IF MAX error $= \delta_K$, DONE, ELSE
TAKE $r+1$ maximal error points as T_{K+1}

We choose some points, make the frequency response do what I want exactly at those points, seeing if those really are actually where the error is the greatest, If Not, we just figure out what the greatest error points are in my current set and keep on iterating

EX]

[Side Note: Sum of cosines $\Longleftrightarrow$ Sum of Polynomials using A change of variables]

$$S(x) = A(\cos^{-1}(x))$$

$$= \sum_{k=0}^{r-1} c_k \cos(\underbrace{k\cos^{-1}(x)}) \quad \text{if } k \text{ is an integer, this thing here turns out to be a Polynomial in } S$$

$\rightarrow$ Chebyshev Polynomials

$$= \sum_{k=0}^{r-1} d_k x^k$$

TRY To Approximate x^2 "$A_{des}(\omega)$"

By $\underline{d_0 + d_1 x}$ over $[0,1]$ using chebyshev error

role of "$A(\omega)$", $\underline{2 \text{ functions}}$

We have a class of functions given by $d_0 + d_1 x$ we try to do as well as possible to approximate some new given functions

i.e. $$\min_{d_0, d_1} \max_{[0,1]} \left| x^2 - (d_0 + d_1 x) \right|$$

Two numbers we set to change around

Interval of error

2 functions $\Rightarrow$ 3 extremal Points

If we find a place where the error hits this extremes at these three points, then we are done.

Guess: $T_0 = \{ \frac{1}{4}, \frac{1}{2}, 1 \}$

⑧ Make the Error Oscillate on these three points

$$x_i^2 = \underline{d_0} + \underline{d_1} x_i + (-1)^i \underline{\delta} \qquad T_0 = \{ \tfrac{1}{4}, \tfrac{1}{2}, 1 \}$$

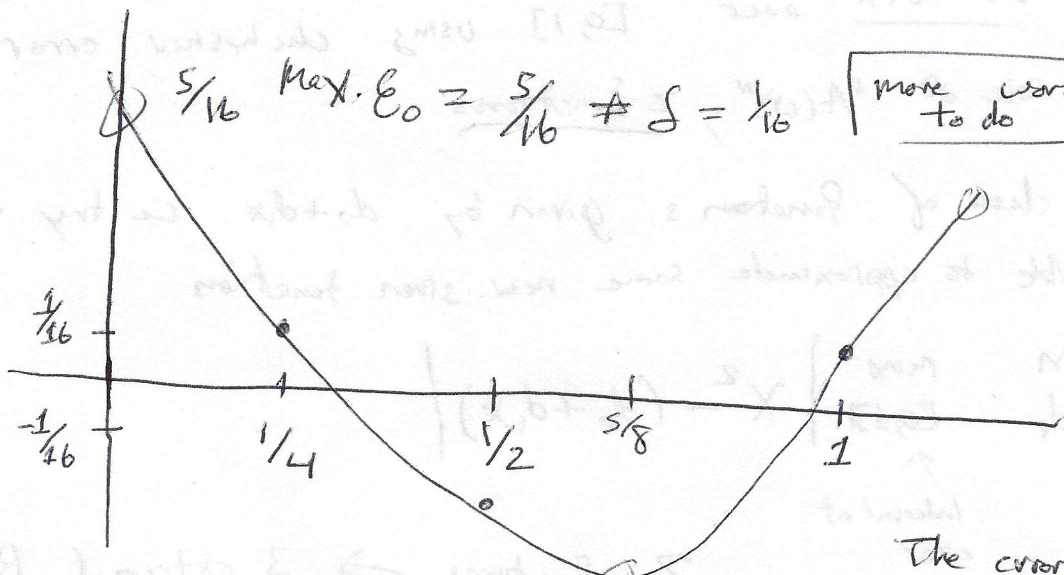
$$\begin{bmatrix} 1 & \frac{1}{4} & 1 \\ 1 & \frac{1}{2} & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} d_0 \\ d_1 \\ \delta \end{bmatrix} = \begin{bmatrix} \frac{1}{16} \\ \frac{1}{4} \\ 1 \end{bmatrix}$$

$$d_0 = -\tfrac{5}{16} \qquad d_1 = \tfrac{5}{4} \qquad \delta = \tfrac{1}{16}$$

plot the corresponding function and compare it to the desired function

$$\mathcal{E}_0(x) = x^2 - (d_0 + d_1 x)$$

$\max \mathcal{E}_0 = \frac{5}{16} \neq \delta = \frac{1}{16}$ $\boxed{\text{more work to do}}$



We were able to make the error equal to these constant values at these three points

The error doesn't take the maximum at these three points

New ones

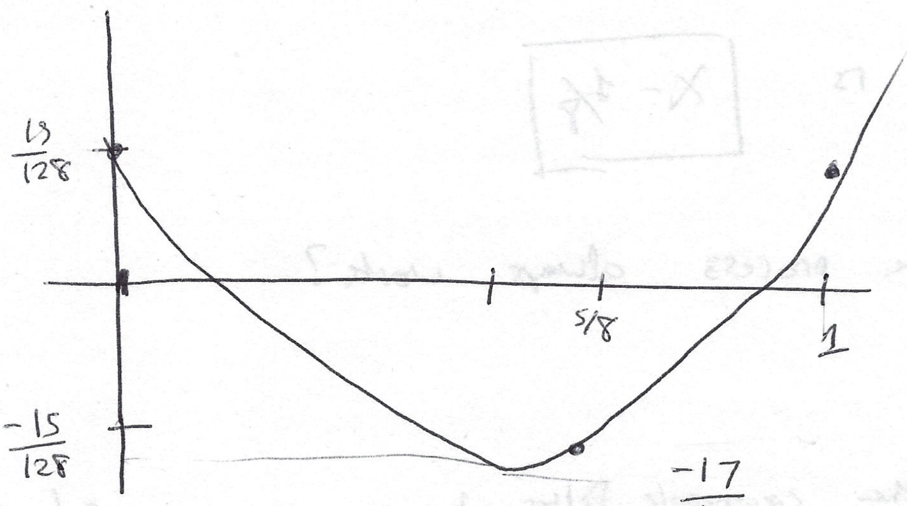
$$T_1 = \{ 0, \tfrac{5}{8}, 1 \}$$

Make the new error Take the biggest values at these places

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 5/8 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} d_0 \\ d_1 \\ \delta \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{25}{64} \\ 1 \end{bmatrix}$$

$d_0 = \frac{-15}{128}$, $d_1 = 1$, $\delta = \frac{15}{128}$

$\mathcal{E}_1(x) = x^2 - \left(\frac{-15}{128} + x\right)$



getting closer, but the error at 1/5 is larger than at 5/8

$\max \mathcal{E} = \frac{17}{128} > \frac{15}{128}$

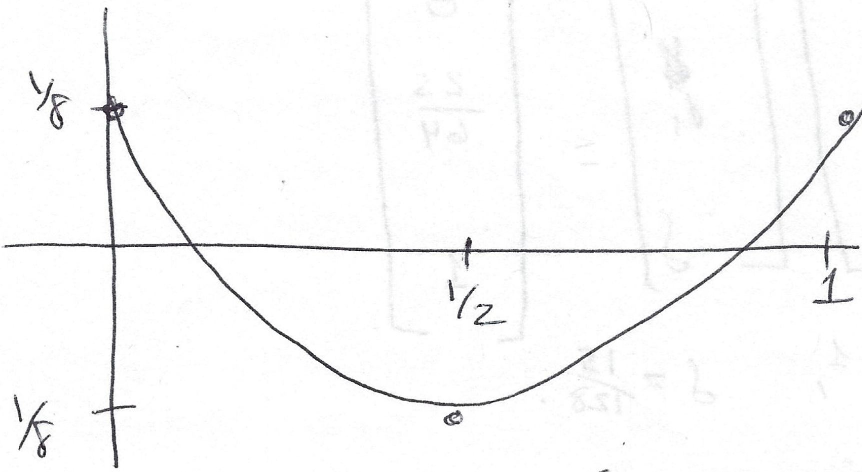
$T_2 = \{0, \frac{1}{2}, 1\}$

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & \frac{1}{2} & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} d_0 \\ d_1 \\ \delta \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{4} \\ 1 \end{bmatrix}$$

$d_0 = -\frac{1}{8}$ $\quad d_1 = 1$ $\quad \delta = \frac{1}{8}$

(10)

$$\mathcal{E}_2(x) = x^2 - (-1/8 + x)$$



we are done because the function doesn't get any worse than 1/8

So since $\max \mathcal{E}_2 = 1/8 = \delta$. Done!

Best Approximation is $\boxed{x - 1/8}$

does this iterative process always work?

→ Yes, and No.

Yes, always get the equiripple filter to converge on a set of Extremal frequencies

→ How do you force that equiripple filter to satisfy the contraints in the passband and the stopband given in the beginning → may not be possible to satisfy those contraints given the number of filter taps that you asked for.

Does REMEZ Always work?

Yes, IT Always converges to AN ~~equivalent~~ Equiripple Approximation.

But it may not have the Passband / stopband characteristics needed for a Given N.

Heuristic!

Given: ~~N Filter Length~~

F_p, F_s Pass Band / Stopband Edges 43.49

δ_1 Deviation in Passband from 1

δ_2 Deviation in Stopband from 0

$$N \approx \frac{-20\log_{10}\sqrt{\delta_1 \delta_2} - 13}{14.6(F_s - F_s)} + 1$$

need a larger N for a steeper passband

45

MatLaB stuff

FIR advantages

- can achieve exactly Linear phase
- efficient Implementations
- Easily designed with Linear prog methods
→ always stable

FIR disadvantages

may Need very Large N to achieve good approximations To Some Frequency Responses

Disadvantages

may ~~need~~ many operations per output ~~to save~~

~~frequency~~ responses

Storage of many coefficients

-Delay may be large

Zero Locations for Linear phase FIR filters

IF z is a zero, then $\bar{z}$ is a zero and so is

$\frac{1}{z}, \frac{1}{\bar{z}}$