DSP Lect 4

The Fourier series (Not derived here)

Every periodic continuous-time signal can be written as a sum of sinusoids

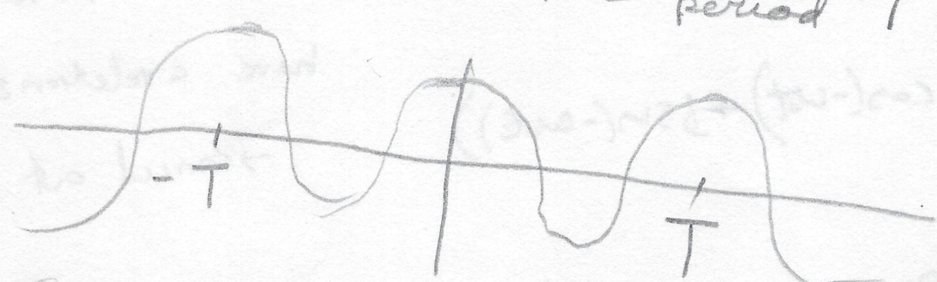
Assume we have a periodic signal $x(t)$

$$x(t+T) = x(t)$$

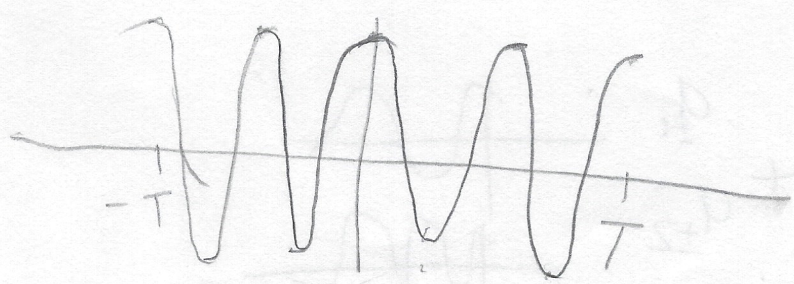
T = Period

0 to T is unique to the signal → repeated so many times

What other signals have period T



$\cos\left(\frac{2\pi}{T}t\right) = \cos(\omega_0 t)$ $\omega_0 = 2\pi/T$



$\cos(2\omega_0 t)$

$= \cos\left(\frac{2 \cdot 2\pi}{T}t\right)$

$\cos k\omega_0 t$, k an integer

$\sin(k\omega_0 t)$, (shifted cosine)

$k\omega_0$ is like a frequency

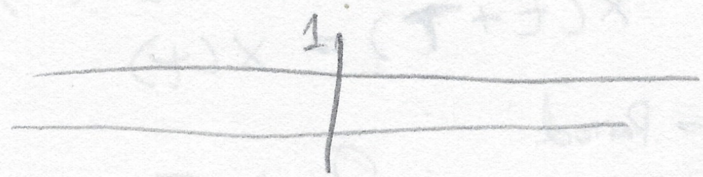
②

$e^{jk\omega_0 t} = \cos(k\omega_0 t) + j\sin(k\omega_0 t)$

Similarly, $\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$ is Periodic with Period T

→ Coefficient a changes amplitude and phase → $a_k \in \mathbb{C}$

$k=0$ $\quad a_0 \cdot$



$+ \; a_1 \cdot (\cos(\omega_0 t) + j\sin(\omega_0 t))$

$+ \; a_{-1} \cdot (\cos(-\omega_0 t) + j\sin(-\omega_0 t))$

Note a_1 and a_{-1} have a relationship → cancel out sin-

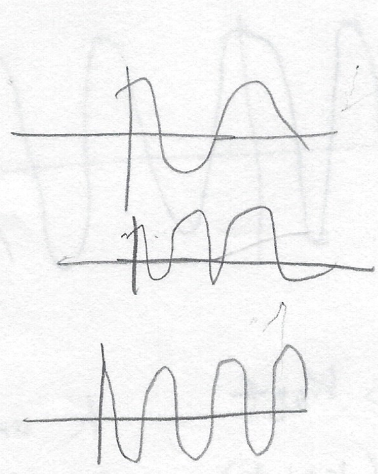
In signal processing, we can know a_-'s when know a_+'s (predicting)

See it →

$a_{\pm 1}$

$+ \; a_{\pm 2}$

$a_{\pm 3}$



Increasing ~~the~~ k → Larger frequency

→ for signals that we care about, the scale of the frequency (bigger the k) is in opposite of the a's (Larger the k, smaller the a must be)

→ certain sums of cosine are needed to add things up.

a_0 → DC constant term

→ represent signal as a fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$ also known as "synthesis" equation

a_k in terms of $x(t)$

n is a fixed integer

$$x(t) e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t}$$

$$= \sum_{k=-\infty}^{\infty} a_k e^{j(k-n)\omega_0 t}$$

integrate

$$\int_0^T x(t) e^{-jn\omega_0 t}\, dt = \int_0^T \sum_{k=-\infty}^{\infty} a_k e^{j(k-n)\omega_0 t}\, dt$$

$$= \sum_{k=-\infty}^{\infty} a_k \int_0^T e^{j(k-n)\omega_0 t}\, dt$$

$$= \int_0^T \cos((k-n)\omega_0 t)\,dt + j\int_0^T \sin((k-n)\omega_0 t)\,dt$$

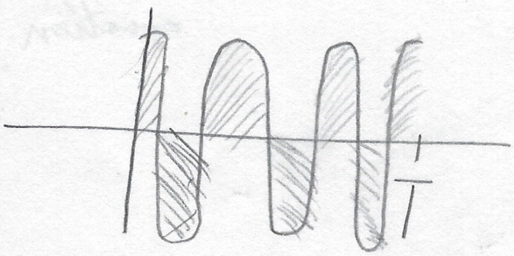
Say $k = n$

So

$$= \int_0^T 1\,dt + 0 = T$$

when $k \neq n$

So what are you integrating? is the same amount as



→ area above and below The 't' axis cancels out

recall $\omega_0 = \frac{2\pi}{T}$

→ Just do it directly, $\frac{\sin((k-n)\omega_0 t)}{(k-n)\omega_0}\Big|_0^T = 0$

apparently ignore ~~dt~~ imaginary part

So for $k=n$

$$\int_0^T x(t)\, e^{-jn\omega_0 t}\, dt = a_n T$$

So for a specific k

$$a_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

Analysis eq

$\{a_k\}$ are the Fourier Series

coefficients of $x(t)$ ("spectral coefficients")

→ if $x(t)$ is real

→ $\{a_k\}$ are complex, but there are patterns

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$x^*(t) = \sum_{k=-\infty}^{\infty} a_k^* e^{-jk\omega_0 t}$$

Note $\boxed{a_k = a_{-k}^*}$

If real $x(t)$ Then $\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k^* e^{-jk\omega_0 t}$

Let $k = -k$ so

$$\sum_{k=-\infty}^{\infty} a_k^* e^{-jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_{-k}^* e^{jk\omega_0 t}$$

still $a_k = a_{-k}^*$

Now write

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2A_k \cos(\theta_k + k\omega_0 t)$$

28:21

$a_k = |a_k| e^{j\theta_k}$

$|a_k| = A_k$

6

Alt

$$X(t) = a_0 + 2\sum_{k=1}^{\infty} B_k\cos(k\omega_0 t) - C_k\sin(k\omega_0 t)$$

Ex

$$x(t) = 5 + 2\cos(\omega_0 t)$$

$$a_k = \frac{1}{T}\int_0^T x(t)e^{-jk\omega_0 t}\,dt$$

$$x(t) = 5 + 2\left(\frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)\right)$$

$$= 5 + e^{j\omega_0 t} + e^{-j\omega_0 t}$$

$$a_0 e^{(0)j\omega_0 t} + a_1 e^{j\omega_0 t} + a_{-1}e^{-j\omega_0 t} + \dots$$

$a_0 = 5$

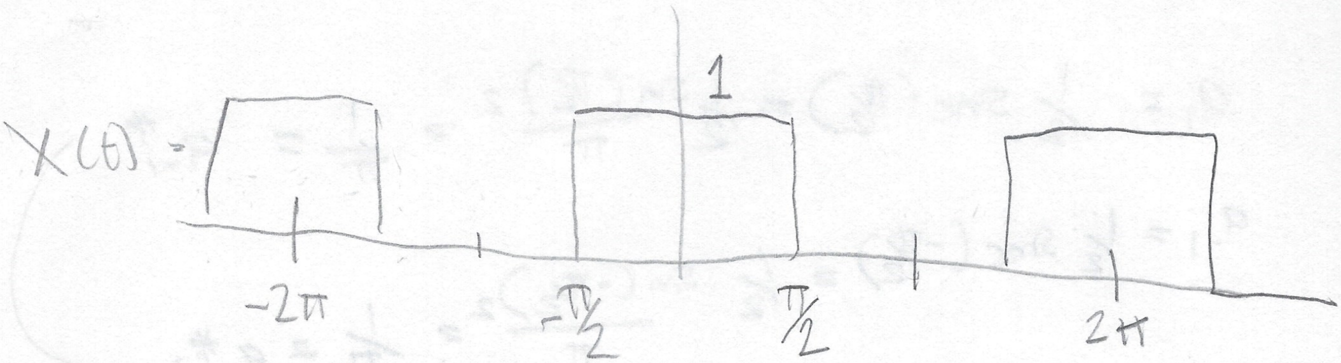
$a_1 = 1$

$a_{-1} = 1$

all other $a_k = 0$

And $a_1 = a_{-1}^*$

Pulse train



$T = 2\pi$, $\omega_0 = 2\pi/T = 1$

$$a_0 = 1/T \int_0^T x(t) e^{-j(0)\omega_0 t} dt = 1/T \int_0^T x(t) dt$$

$$= 1/2\pi \int_{-\pi}^{\pi} x(t) dt = 1/2\pi \int_{-\pi/2}^{\pi/2} dt = 1/2\pi \cdot \pi = 1/2$$

Average value of signal over one period

→ this signal is on half the time and off the other half

1 half the time, 0 o.w.

so 1/2

$$a_k = 1/T \int_{-T/2}^{T/2} x(t) e^{-jk\omega_0 t} dt$$

↳ still integrated over the period

$$= \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} e^{-jkt} dt = \frac{-1}{2\pi jk} \left(e^{-jk\pi/2} - e^{jk\pi/2}\right)$$

⑧

$$= \frac{1}{\pi k} \sin(k\pi/2) = \tfrac{1}{2} \operatorname{sinc}(k\pi/2)$$

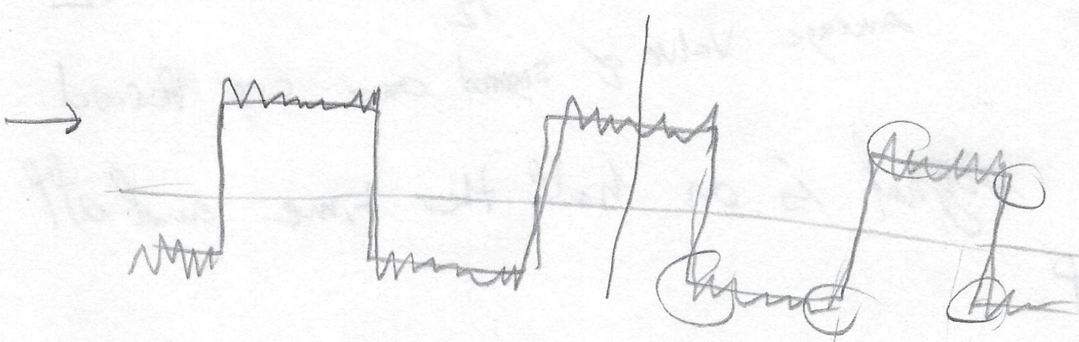
$$a_1 = \tfrac{1}{2} \operatorname{sinc}(\pi/2) = \frac{1}{2} \frac{\sin(\pi/2)\, 2}{\pi} = \frac{1}{\pi} = a_{-1}^*$$

$$a_{-1} = \tfrac{1}{2} \operatorname{sinc}(-\pi/2) = \frac{1}{2} \frac{\sin(-\pi/2)\, 2}{\pi} = \frac{1}{\pi} = a_1^*$$

$$a_2 = \tfrac{1}{2} \operatorname{sinc}(\pi) = 0$$

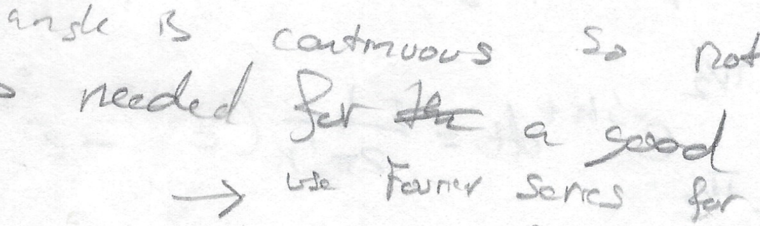
real signal

on computer programs adding more terms of the series yields an approximation of the pulse train.



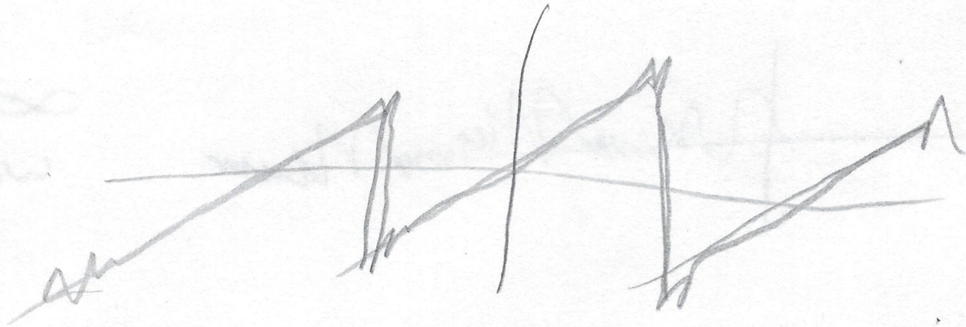
→ overshoot at the edges

Signal is not continuous



Triangle is continuous so not a whole lot of terms needed for ~~the~~ a good approx.

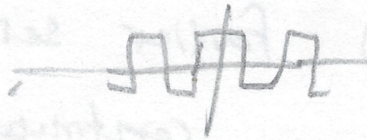
→ use Fourier Series for compression



→ to phase shift multiply a_k by complex Number magnitude 1

→ if train is purely odd, 

→ purely sin's

if train is even purely, 

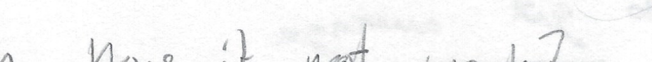
→ purely cosines

→ even functions made an Even functions

→ all to do with phase shifts

Notes and Properties of the fourier Series

When does it not work?

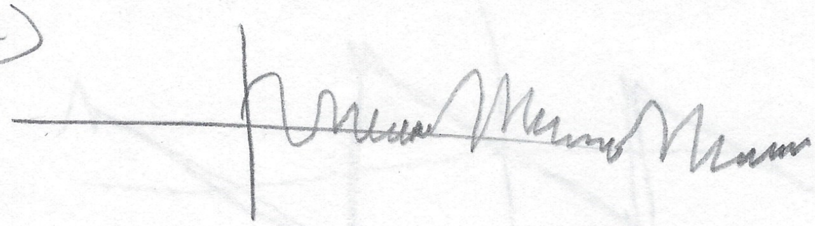
1) 

maybe you can for the guy

∞ Area under the curve

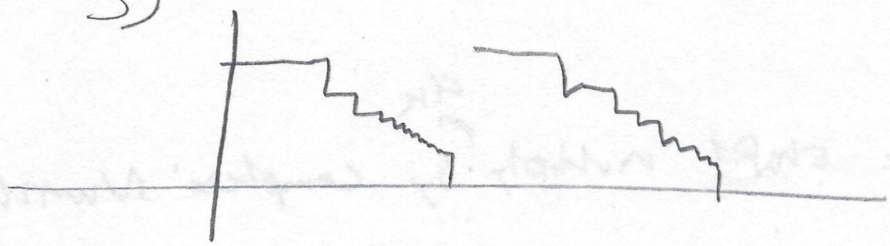
→ Periodic, but Asymptotic

(10) 2)



∞ wiggly

3)



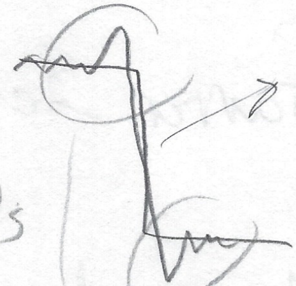
∞ Discontinuities

→ finite # of Discontinuities is okay

→ what happens at one?

a) Fourier series converges at every continuous point

b) Converges to the average value at every Discontinuity



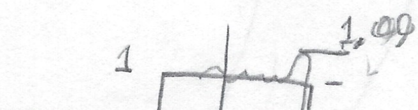
Slope get very large

adds

subtracts

→ to get average

GIBBS phenomenon



we can never get the over shoot Below ≈ 9% of the Height of the ~~Discontinut~~ Discontinuity

Note at every continuous point the series converges, so the width of the error gets smaller

$X(t)$ periodic with period T

$X(t) \Longleftrightarrow \{a_k\}$ → Let this represent the Fourier series

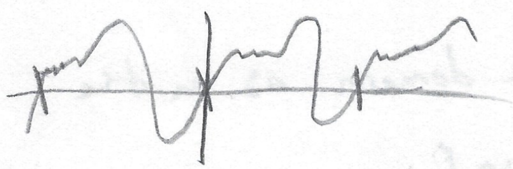
1) Linearity : 2 signals with given F.S., same T (Period)

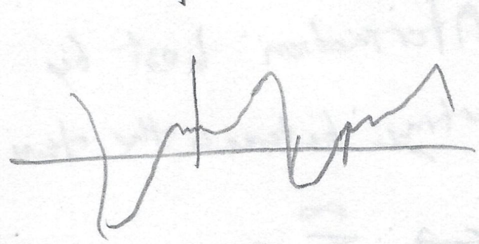
$X(t) \Longleftrightarrow \{a_k\}$

$Y(t) \Longleftrightarrow \{b_k\}$

$\alpha X(t) + \beta Y(t) \Longleftrightarrow \{\alpha a_k + \beta b_k\}$

2) time shifting





context of frequency domain, the signal shouldn't change, that's only the phase of the f.S. coefficients

$X(t) \rightarrow \{a_k\}$

and $Y(t) = X(t - t_0)$ $\quad Y(t) \Longleftrightarrow a_k e^{-jk\omega_0 t_0}$

Phase shift of F.S. coefficients

Magnitude remains

12 Differentiation

$x(t) \leftrightarrow \{a_k\}$ $\qquad$ $x'(t) \leftrightarrow \{jk\omega_0 a_k\}$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$x'(t) = \sum_{k=-\infty}^{\infty} a_k jk\omega_0 e^{jk\omega_0 t}$$

4) Parseval's theorem

Look at avg power of the signal

Avg PWR of signal

$$\frac{1}{T}\int_0^T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

Power of F.S. Coefficients

Same amount of power in the time domain as in the frequency domain

(No information lost by converting between the two domains)

5) Convolution

$x(t) \leftrightarrow \{a_k\}$

$y(t) \leftrightarrow \{b_k\}$

$$x(t)y(t) \leftrightarrow \sum_{l=-\infty}^{\infty} a_l b_{k-l} = a * b$$

Convolution of the two vectors of F.S. coefficients

$$\int_0^T x(\tau)y(t-\tau)\,d\tau \leftrightarrow \underbrace{T a_k b_k}_{\text{coefficients}}$$