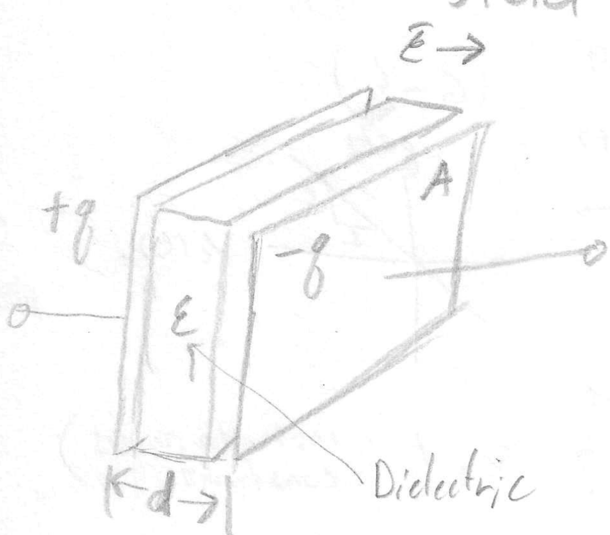
Chapter 6 Capacitance and Inductance

6-1 The Capacitor

a dynamic element involving the time variation of an electric field



We have a uniform electric field exist across the plates when there's a voltage across the capacitor.

We have charge separation, equal and opposite across the plates

we have $\bar{\mathcal{E}}(t) = \dfrac{q(t)}{\epsilon A}$

ϵ - permittivity of Dielectric
A - area of the plates
$q(t)$ - magnitude of charge on each plate

$\bar{\mathcal{E}}$ field $\frac{N}{C}$ or V/m

Note, the $\bar{\mathcal{E}}$ field can be written in terms of Voltage across the cap

$\bar{\mathcal{E}} = \dfrac{V_C(t)}{d}$ From Physics II and Fields and Waves

So we can write: $\dfrac{q(t)}{\epsilon A} = \dfrac{V_C(t)}{d} \rightarrow q(t) = \dfrac{\epsilon A}{d} V_C(t)$ (6-3)

We know from <u>Physics II</u> and <u>Fields and Waves</u>

$C = \dfrac{\epsilon A}{d}$ F (6-4) This is capacitance (Farads)

So we may write: $q(t) = C\, V_C(t)$ C (6-5)

(2)

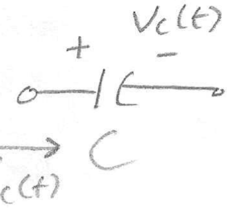
I–V Relationship

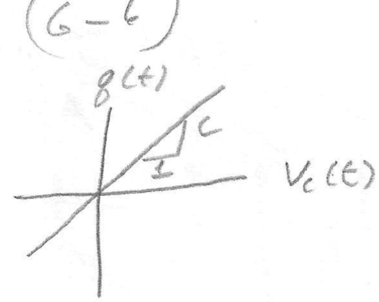
To Express the element constraint in terms of voltage and current. recall we know current is charge/second so differentiate

$$\frac{d}{dt} q(t) = \frac{d}{dt} C\, V_c(t)$$

We have $\quad i_c(t) = C \frac{d}{dt} V_c(t) \qquad (6-6)$

We assume the reference marks for current and voltage follow the passive sign convention





So $i_c(t) = 0$, for $V_c(t) = \text{Constant}$

and current is constant when $V_c = 0$? Yes! (instantaneous constant)

The capacitor is a dynamic element because the current is 0 unless the voltage is changing

→ discontinuous change in ~~current~~ Voltage would require an infinite current

→ So V_c must be a continuous function of time

To Express Voltage in terms of current, we are allowed to write

$$\int C \frac{d}{dt} V_c(t) = \int i_c(t)\, dt$$

(we multiplied both sides by dt and then integrate)

$$\int dV_c(t) = \frac{1}{C} \int i_c(t) dt$$

(this is egregious)

we assume at some time t_0 the voltage across the cap $V_c(t_0)$ is known

So voltage at a later time $t > t_0$

$$\rightarrow \int_{V_c(t_0)}^{V_c(t)} \underbrace{\cancel{\frac{d}{dt}}}_{dV_c} V_c(t) = \frac{1}{C} \int_{t_0}^{t} i_c(x)\, dx$$

or $\displaystyle\int_{V_c(t_0)}^{V_c(t)} dV_c(t) = \frac{1}{C} \int_{t_0}^{t} i_c(x) dx$

further

$$V_c(t) - V_c(t_0) = \frac{1}{C}\int_{t_0}^{t} i_c(x)dx$$

$$\rightarrow V_c(t) = V_c(t_0) + \frac{1}{C}\int_{t_0}^{t} i_c(x)\,dx \quad (6-7)$$

time t_0

In practice, t_0 is established by a physical event (closing switch) (start of a particular clock pulse)

Let's define $t_0 = 0$

$$V_c(t) = V_c(0) + \frac{1}{C}\int_{0}^{t} i_c(x)\,dx \quad (6-8)$$

Power and Energy   considering passive sign convention

$$P_c(t) = i_c(t)\,V_c(t) \quad (6-9)$$

we write

$p = \frac{d}{dt}W$

$$P_c(t) = C\,v_c(t)\,\frac{d}{dt}V_c(t) = \frac{d}{dt}\left[\frac{1}{2}C v_c^2(t)\right] \quad (6-10)$$

(we may write this)

→ Power can be '+' or '−'

absorbs Power   delivers power

→ As the Cap can deliver power, it can store it (Power)

Power is J/s (time rate change of Energy)

we can infer that $W_c(t) = \frac{1}{2}C\,V_c^2(t) + \text{constant}$ J

constant refers to stored energy at instant t (time), when $V_c(t) = 0$

→ $\bar{E}$ is 0 so stored energy is 0

$$\rightarrow W_c(t) = \frac{1}{2}C\,v_c^2(t) \quad (6-11)$$

(proportional to square so Energy can't be negative)

(4)

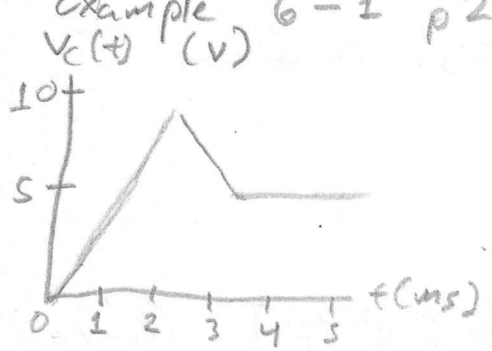
Capacitor stores power by absorbing
then returns power from stored energy

→ abrupt change in voltage → discontinuous change in energy

→ Power $= \frac{d}{dt} w$, if w is discontinuous
→ Power → ∞
(Not Possible)

Capacitor voltage is a <u>state variable</u> because it determines the energy state of the element.

Example 6-1 p283



$C = 0.5\ \mu F$ Find current thru capacitor

we know $i = C\frac{d}{dt} V_c(t)$

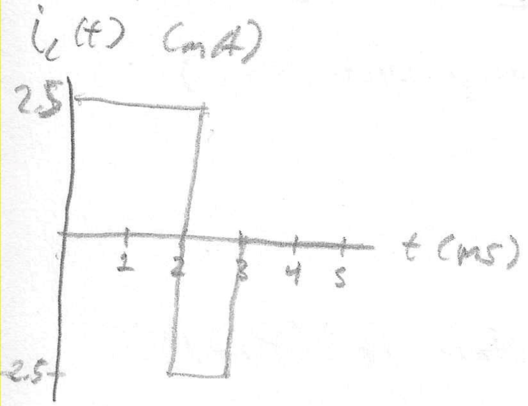
for $0 < t < 2ms$, $C\frac{d}{dt} V_c(t) = \frac{10}{2\times10^{-3}} (0.5\times10^{-6})$

→ $i_c(t) = C\frac{d}{dt} V_c(t) = 2.5\,mA$

for $2 < t < 3$, $\frac{d}{dt} V_c(t) = -5000\ V/s$

So $i_c(t) = (-5000)(0.5\mu F) = -2.5\,mA$

for $3ms < t$ $i_c(t) = 0$

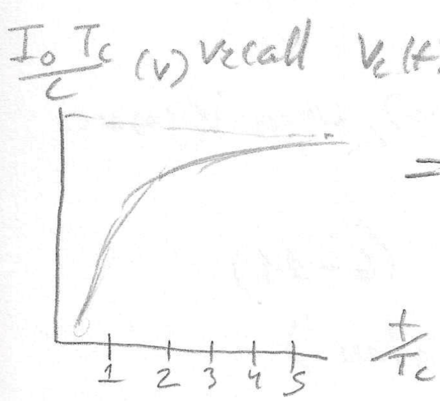


Example 6-2 $i_c(t) = I_o (e^{-t/T_c})\, u(t)\ A$

whats V_c if $V_c(0) = 0V$ (initial condition)

recall $V_c(t) = V_c(0) + \frac{1}{C}\int_0^t i_c(x)\,dx = \frac{I_o}{C}\int_0^t e^{-x/T_c}\,dx$

$= \frac{I_o T_c}{C}\left(-1\right)\left(e^{-t/T_c} - 1\right)\ V$



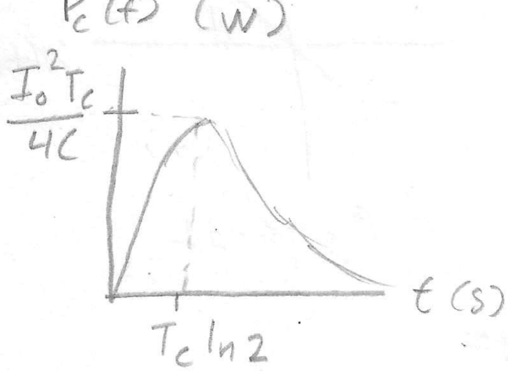
Example 6-4, Find power and energy

$$P_c(t) = i_c(t)\, v_c(t)$$

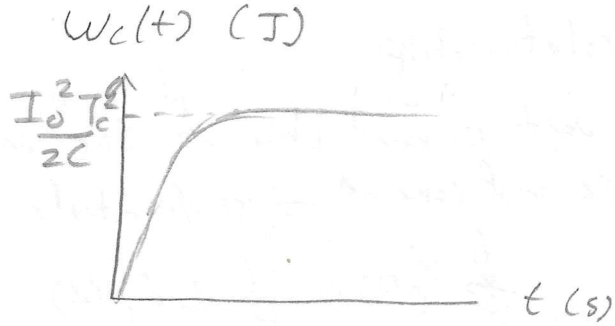
$$= I_0\left(e^{-t/T_c}\right) \frac{I_0 T_c}{C}\left(1 - e^{-t/T_c}\right)$$

$$= \frac{I_0^2 T_c}{C}\left(e^{-t/T_c} - e^{-2t/T_c}\right) \quad W$$

$$W_c(t) = \frac{1}{2} C v_c^2(t) = \frac{1}{2} C\left[\frac{I_0 T_c}{C}\left(1 - e^{-t/T_c}\right)\right]^2$$



Voltage grows and $i_c \to 0$ as $t \to$ larger



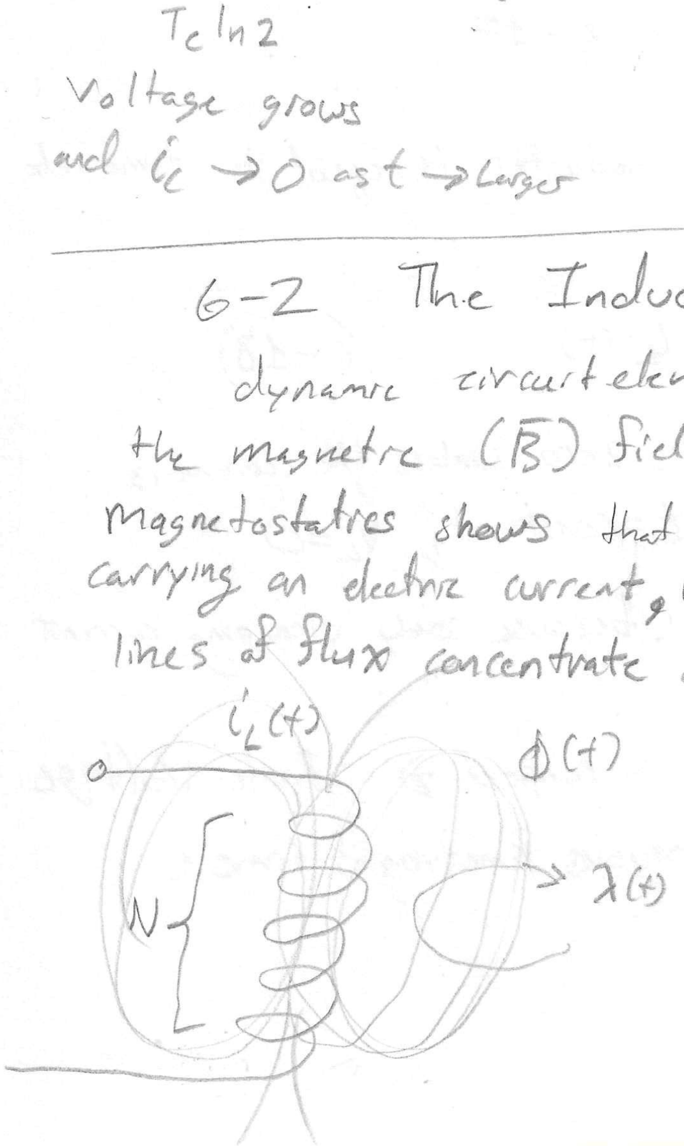
Capacitor stores energy

6-2 The Inductor

$\bar{H} = \frac{A}{m}$, $\bar{B} = \frac{N}{m}$

dynamic circuit element involving the time variation of the magnetic ($\bar{B}$) field produced by a current

Magnetostatics shows that ~~an~~ magnetic flux ϕ surrounds a wire carrying an electric current, when the wire is wound into a coil, the lines of flux concentrate along the axis of the coil



$\phi(t) = K_1 N i_L(t)$ Wb (V-s)

K_1 = a constant of proportionality

$\lambda(t) = N\phi(t)$ flux linkage (Wb)

N = # of turns (6-13)

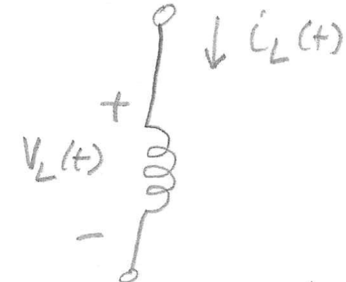
so $\lambda(t) = [K_1 N^2]\, i_L(t)$ (6-14

⑥

We have the <u>Inductance</u> L of the coil

$$\rightarrow \quad L = K_1 N^2 \ (H) \qquad (6-15)$$

Henrys



$$\rightarrow \quad \lambda(t) = L\, i_L(t) \qquad (6-16)$$

flux Linkage

Inductor's element constraint
→ Inductor is a Linear element

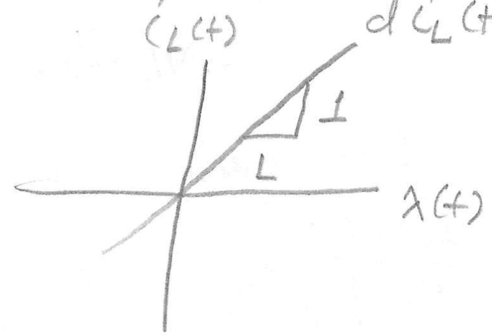


Fig 6-11

I-V relationship

We want element characteristic in terms of voltage and current → differentiate

$$\frac{d}{dt}\lambda(t) = \frac{d}{dt} L\, i_L(t) \qquad 6-17$$

Faraday's Law, voltage across the Inductor is equal to time rate of change of flux Linkage

$$\rightarrow V_L(t) = L \frac{d}{dt} i_L(t) \qquad (6-18)$$

→ Voltage across the inductor is zero unless the current is time varying, DC excitation, i = constant, $V_L = 0$

<u>Inductor is a Dynamic element</u> because only changing current produces a nonzero voltage

→ discontinuous change in current → Infinite Voltage

→ $i_L(t)$ must be a continuous function of time t

to express $i_L(t)$ in terms of $V_L(t)$

we can write

$$\int L \frac{d}{dt} i_L(t) = \int V_L(t)\,dt \qquad (6\text{-}19)$$

or $\boxed{\int d i_L(t) = \frac{1}{L}\int V_L(t)\,dt}$ (we multipled by dt and then Integrate)

assume $i_L(t_0)$ is known at time t_0,

$$\int_{i_L(t_0)}^{i_L(t)} d i_L(t) = \frac{1}{L}\int_{t_0}^{t} V_L(x)\,dx \qquad (6\text{-}20)$$

$$i_L(t) - i_L(t_0) = \frac{1}{L}\int_{t_0}^{t} V_L(x)\,dx \qquad (6\text{-}21)$$

$$\rightarrow i_L(t) = i_L(t_0) + \frac{1}{L}\int_{t_0}^{t} V_L(x)\,dx$$

t_0 may indicate the flip of a switch, we may write $t_0 = 0$

$$i_L(t) = i_L(0) + \frac{1}{L}\int_{t_0}^{t} V_L(x)\,dx \qquad (6\text{-}22)$$

→ i_L and V_L follow passive sign convention

Power and Energy

noting passive sign convention

$$P_L(t) = i_L(t)\,V_L(t) \quad W \qquad (6\text{-}23)$$

we may write (we can do this)

$$P_L(t) = (i_L(t))\left[L\frac{d}{dt} i_L(t)\right] = \frac{d}{dt}\left[\frac{1}{2} L\, i_L^2(t)\right] \qquad (6\text{-}24)$$

→ Power can be positive or negative inductor current and time derivative can have opposite signs

→ can absorb and deliver Power

→ can store energy

(8) To find the stored Energy

Power is the time rate of change of energy (J/s)

$(p = \frac{d}{dt} w)$ The quantity $[\frac{1}{2} L i_L^2(t)]$ represents the stored energy in the magnetic field of the inductor.

energy at time t, $w_L(t) = \frac{1}{2} L i_L^2(t) + \text{constant}$ J

Constant = 0 because energy stored at instant t at which $i_L(t) = 0$, so we have

$$w_L(t) = \frac{1}{2} L i_L^2(t) \quad J \qquad (6\text{-}25)$$

energy stored is never negative

Inductor returns stored energy when delivering power

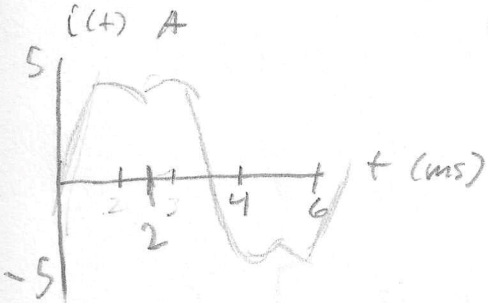
(6-25) implies inductor current is a continuous function of time because an abrupt change in current causes a discontinuity in energy. → which would mean infinite Power

→ current is called the state variable of the inductor because it determines the energy state of the element

---

Example 6-6 p290

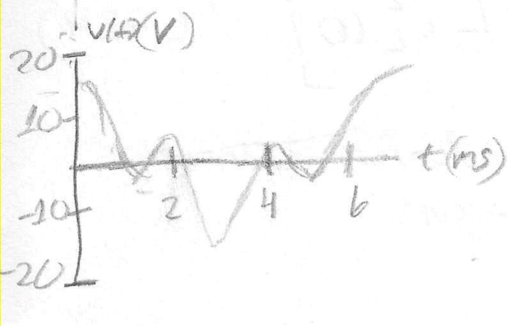
$L = 2\ mH \quad i_L(t) = 4 \sin 1000t + 1 \sin 3000t$ A



$v_L(t)$? $\quad v_L(t) = L \frac{d}{dt} i_L(t)$

$= 0.002 [4(1000) \cos 1000t + 3000 \cos 3000t]$

$= 8 \cos 1000t + 6 \cos 3000t$ V



There are two sinusoids at different f's

1 $Krad/s$ AC component is 4X bigger than 3 $Krad/s$

AC responces of energy storage elements depend on frequency ~ we can create frequency-selective filters

Exercise 6-6, for $t>0$, $v_L(t) = 20e^{-2000t}$ V, $i_L(0)=0$

$L = 4\,mH$

(a) current for $t>0$?

$$i_L(t) = \frac{1}{L}\int_{t_0}^{t} v_L(x)\,dx$$

$$= \frac{1}{L}\int_{t_0}^{t} 20e^{-2000x}\,dx = \frac{1}{L}\frac{20\,(-1)}{2000}\left[e^{-2000x}\right|_0^t$$

$$= -\frac{20}{2000\,L}\left[e^{-2000t} - 1\right]$$

$$= 2.5\left[1 - e^{-2000t}\right] \text{ A}$$

(b) power $t>0$?

$$P_L(t) = \frac{d}{dt}\left(\frac{1}{2}L\,i_L^2(t)\right) = \frac{L}{2}\frac{d}{dt}\left((2.5)(1-e^{-2000t})\right)^2$$

$$= \frac{L}{2}(2.5)^2\frac{d}{dt}\left[1 - e^{-2000t} - e^{-2000t} + e^{-4000t}\right]$$

$$= \frac{L}{2}(2.5)^2\frac{d}{dt}\left[1 - 2e^{-2000t} + e^{-4000t}\right]$$

$$= \frac{L}{2}(2.5)^2\left[4000\,e^{-2000t} - 4000\,e^{-4000t}\right]$$

$$= 50\left[e^{-2000t} - e^{-4000t}\right] \text{ W}$$

(c) Energy $t>0$?

$$W_L(t) = \frac{1}{2}L\,i_L^2(t) \text{ J}$$

$$= \frac{L}{2}\left[1 - 2e^{-2000t} + e^{-4000t}\right] \text{ J}$$

(10)

Principle of Duality

KVL	KCL
LOOP	Node
Resistance	Conductance
Voltage Source	Current Source
Thevenin	Norton
Short circuit	Open circuit
Series	Parallel
Capacitance	Inductance
Flux Linkage	charge

if every electrical term in a correct statement about circuit behavior is replaced by its dual, then the result is another correct statement.

Capacitor

1.) $i_C = 0$ unless V_C is changing
Cap acts like open circuit to DC excitations

2.) $V_C(t)$ is continuous
if V_C is discontinuous
$P_C \to \infty$, $i_C \to \infty$

3.) Cap absorbs Power when charging (storing Energy), releases Energy when delivering Power

Inductor

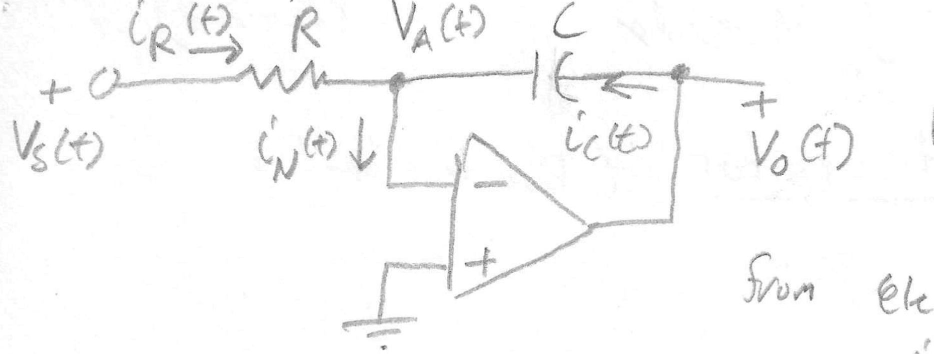
1.) $V_L = 0$ unless i_L is changing
Coil acts like short circuit for DC excitations

2.) $i_L(t)$ is a continuous function of time, if i_L is discontinuous;
$P_L \to \infty$, $V_L \to \infty$

3.) Coil absorbs power from the circuit when storing energy. Then delivers power to the circuit and releases the energy

Energy is conserved
both are passive elements

6-3 Dynamic OP AMP Circuits



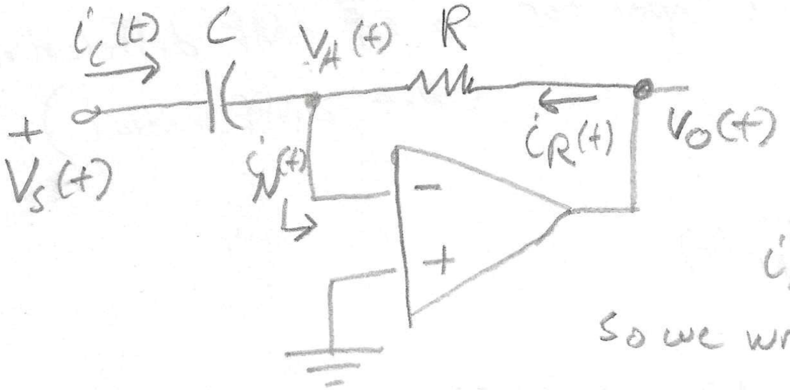
What's the Input-output?

KCL at node A

$$i_R(t) + i_C(t) = i_N(t)$$

from element constraints

$$i_C(t) = C\frac{d}{dt}\left[V_o(t) - V_A(t)\right]$$



$$i_R(t) = \frac{1}{R}\left[V_s(t) - V_A(t)\right]$$

Ideal OP AMP constraints

$$i_N(t) = 0, \quad V_A(t) = 0$$

so we write

$$\frac{V_s(t)}{R} + C\frac{d}{dt}V_o(t) = 0$$

rewrite

$$\frac{d}{dt}V_o(t) = -\frac{1}{RC}V_s(t)$$

(It's egregious, but we can (for some reason) multiply by dt and then Integrate → just Like Cap and inductor)

$$\rightarrow \int dV_o(t) = -\frac{1}{RC}\int V_s(t)\,dt$$

Assume we know $V_o(t)$ at $t_o = 0$, $\displaystyle\int_{V_o(0)}^{V_o(t)} dV_o(t) = -\frac{1}{RC}\int_0^t V_s(x)\,dx$

Note, $V_A(t) = V_R(t) + V_s(t)$

KVL: $V_o(t) = V_C(t) + V_A(t) \rightarrow V_o(t) = V_C(t)$

$$\rightarrow V_o(t) = V_o(0) - \frac{1}{RC}\int_0^t V_s(x)\,dx$$

as $V_o(t) = V_C(t) \rightarrow V_C(0) = 0$

So

$$V_o(t) = -\frac{1}{RC}\int_0^t V_S(x)\,dx \qquad (6\text{-}26)$$

→ Inverting integrator (Due to '−')

$$\frac{1}{RC} = s^{-1}$$

Interchanging resistor and capacitor → OP AMP differentiator (Not differential)

p294

KCL at A

$$i_R(t) + i_C(t) = i_N(t)$$

Element constraints

$$i_C(t) = C\frac{d}{dt}\left[V_S(t) - V_A(t)\right]$$

$$i_R(t) = \frac{1}{R}\left[V_O(t) - V_A(t)\right]$$

Ideal OP AMP constraints: $i_N(t) = 0$, $V_A(t) = 0$

So we write, $\frac{V_O(t)}{R} + C\frac{d}{dt}V_S(t) = 0$

$$\rightarrow V_O(t) = -RC\frac{d}{dt}V_S(t) \qquad 6\text{-}27$$

RC = Seconds!

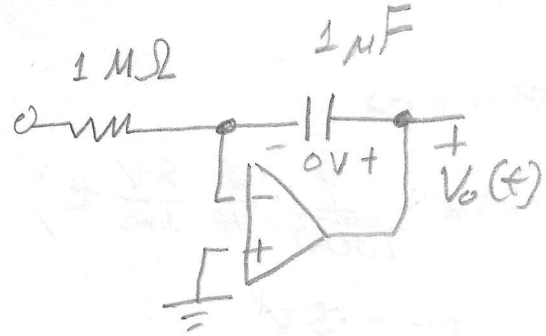
Note, Inductor versions aren't interesting because coils are big and have resistance (resistance losses)

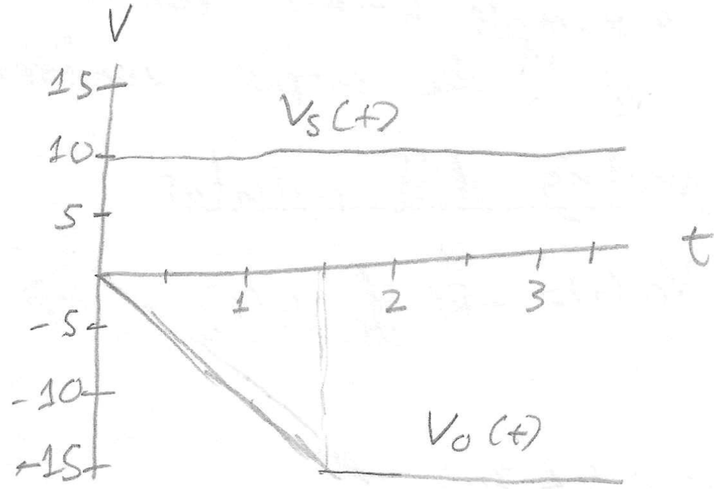
$V_1(t) \longrightarrow \boxed{K} \longrightarrow \boxed{\int} \xrightarrow{V_o(t)}$ $K = -\frac{1}{RC}$

$V_1(t) \longrightarrow \boxed{K} \longrightarrow \boxed{\frac{d}{dt}} \xrightarrow{V_o(t)}$ $K = -RC$

p295

Example 6-9





$V_{cc} = 15V, -V_{cc}$

$V_s(t) = 10u(t)\ V \quad V_o(t)?$

$$V_o(t) = V_o(0) - \frac{1}{RC}\int_0^t V_s(x)\,dx = 0 - \frac{1}{RC}\int_0^t 10\,dx \qquad R*C = 1\ s$$

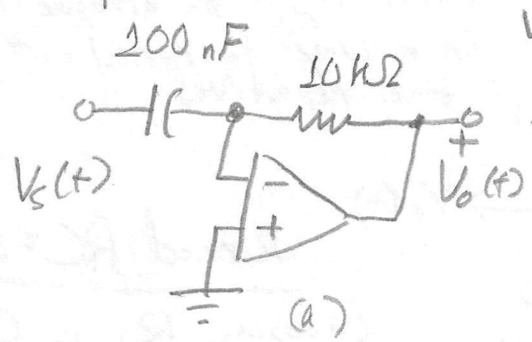
$$= -\int_0^t 10\,dx = -10t\ V \quad t > 0$$

slope $= -10\ V/s$ total Integrator

output is unbounded until $-V_{cc}$ if $-V_{cc} = -15$

$t = 1.5$ seconds til saturate

Example 6-10 p296



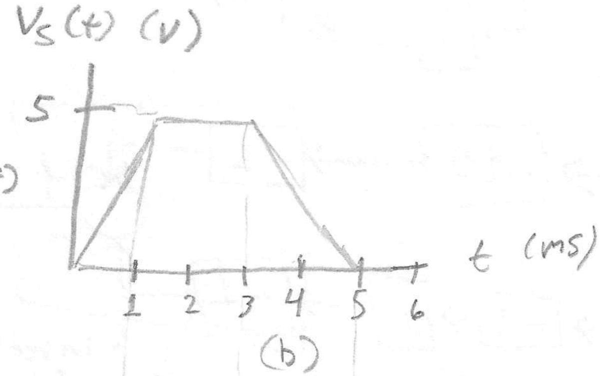
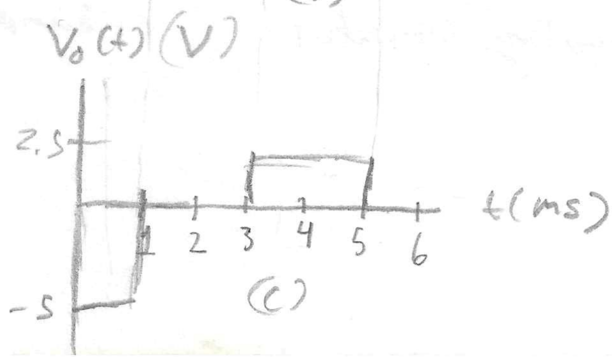


Fig 6-19(a)



Example 6-10 continued
what's the output waveform

<u>inverting differentiator</u>

$$V_o(t) = -RC \frac{d}{dt} V_s(t)$$

$0 < t < 1\,ms$

$$\rightarrow V_o(t) = -\frac{1}{1000}\left(\frac{d}{dt}\frac{5V}{1ms}t\right)$$
$$= -5V$$

$1ms < t < 3ms$

$$V_o(t) = -\frac{1}{1000}\frac{d}{dt}(5) = 0\,V,$$

$3ms < t < 5ms$

$$V_o(t) = -\frac{1}{1000}\frac{d}{dt}\frac{-5t}{(5-3)ms}$$
$$= +2.5\ V$$

$t > 5ms$, $V_o(t) = 0$

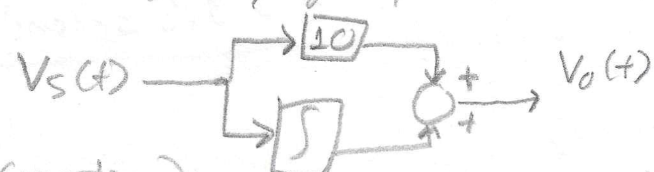
Design Example 6-11

Design op Amp ~~circuit~~ to Implement

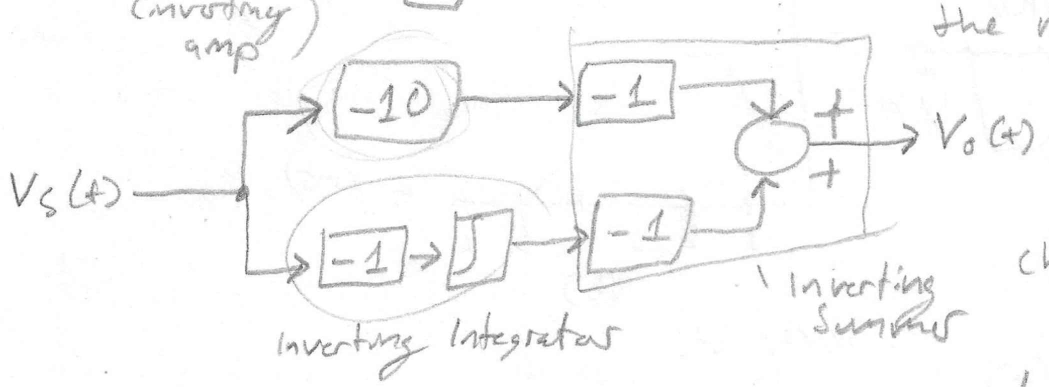
$$V_o(t) = 10\,V_s(t) + \int_0^t V_s(t)\,dt$$

Solution is not unique

Draw block diagram, gain, integrator, summer

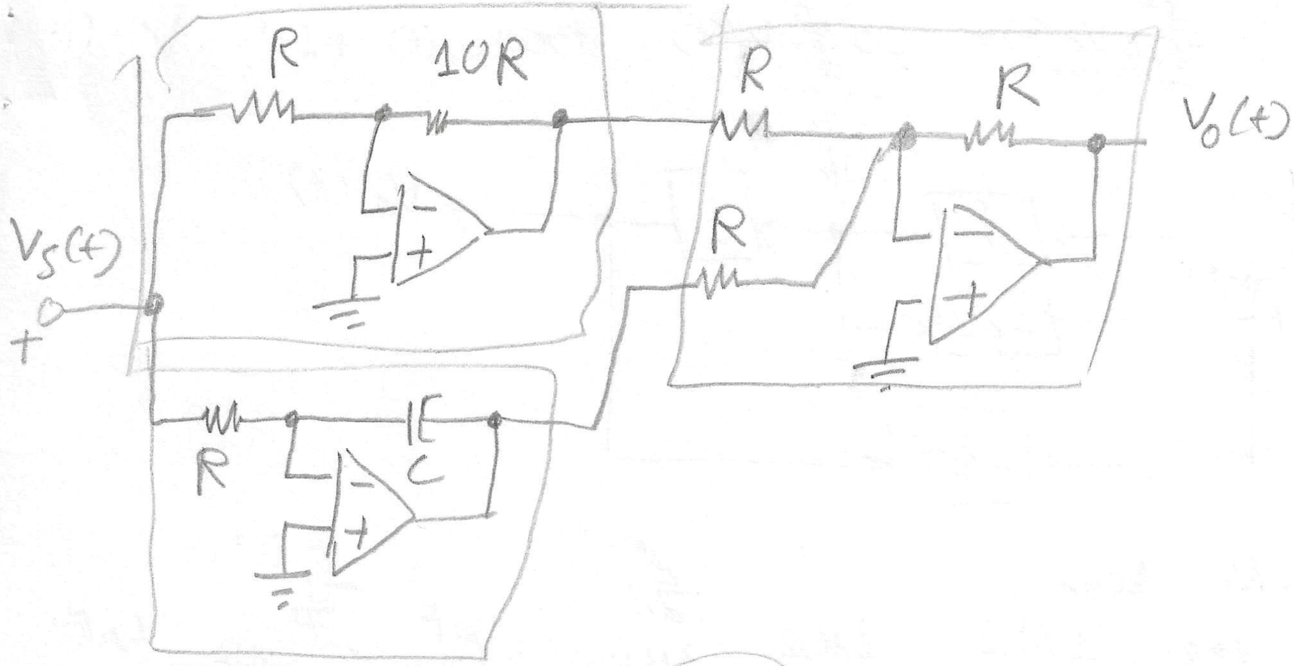


The circuits we know are inverting, so arragne it in a way to cancel out the negatives



<u>We need RC = 1</u>

choosing Rs or C depends on factors Like accuracy, Internal resistance of source, <u>output load</u>



Design 6-12 p248

Differential, integral, and integrodifferential equations can be solved using dynamic circuits.

→ Second order system

$$5 \frac{d^2}{dt^2} V_O(t) + 100 \frac{d}{dt} V_O(t) + 5000 V_O(t) = 250 V$$

Develop a block-diagram representation of the eq and then design a circuit to solve the equation. [design a circuit w/ output $V_o(t)$]

Highest order derivative, coefficient of 1

$$\frac{d^2}{dt^2} V_o(t) = -20 \frac{d}{dt} V_o(t) - 1000 V_o(t) + 50V$$

Highest order derivative is sum of the 3 inputs

Strategy

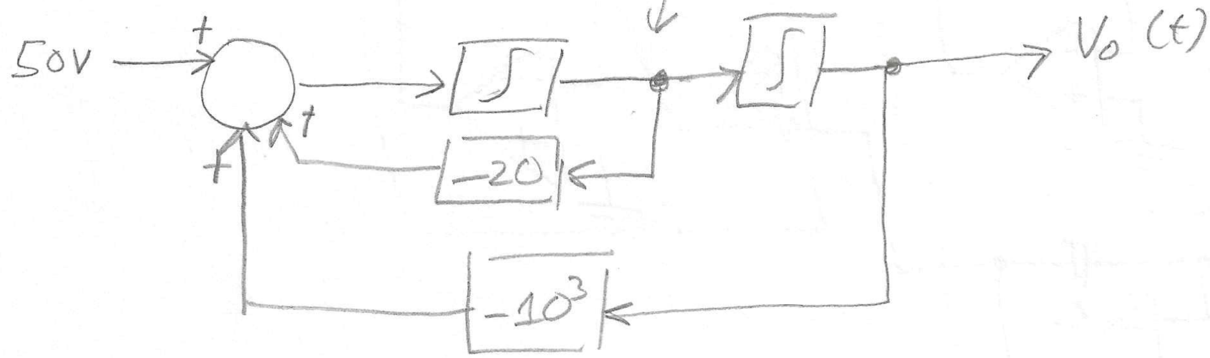
$$\int \frac{d^2}{dt^2} V_o(t) = \frac{d}{dt} V_o(t) \quad (1)$$

$$\int \frac{d}{dt} V_o(t) = V_o(t) \quad (2)$$

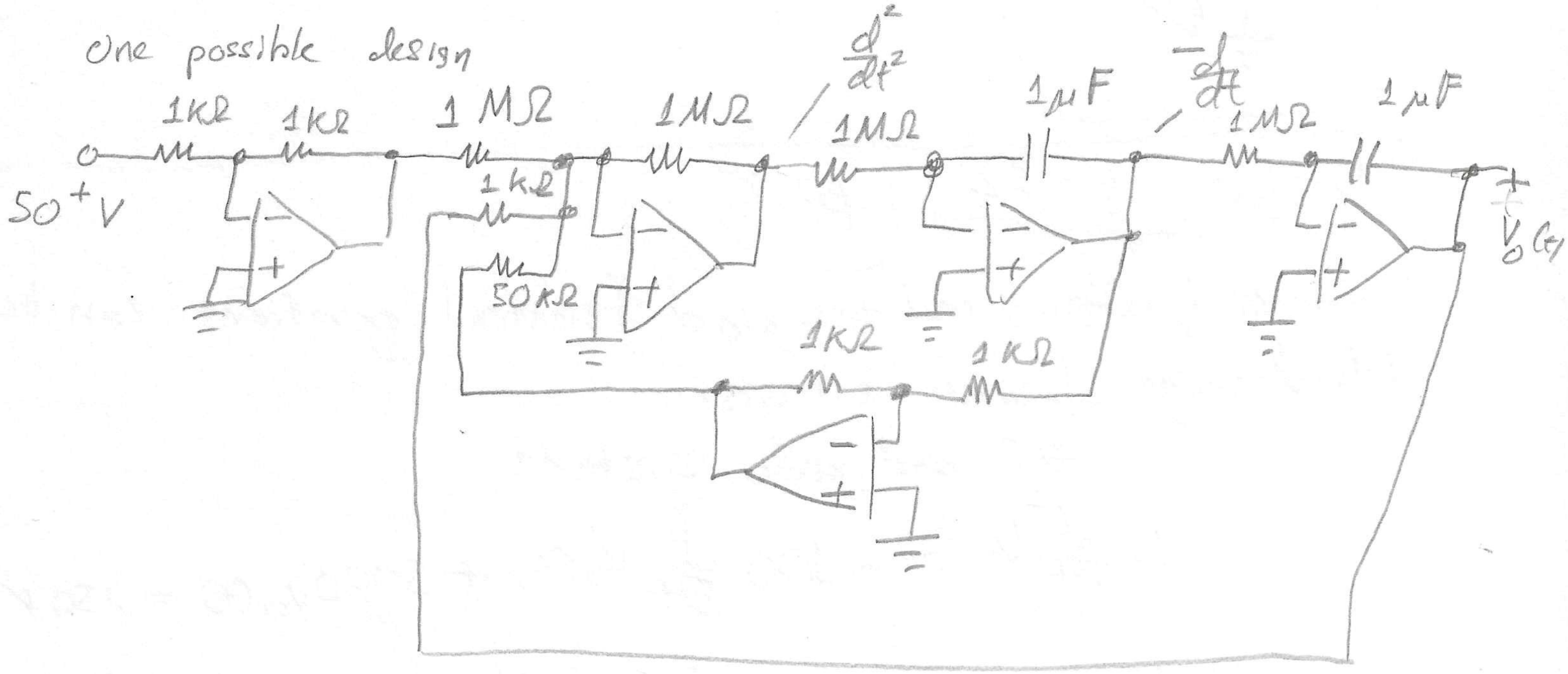
outputs of integrators → Scale them → Feed back into Summer

(16)

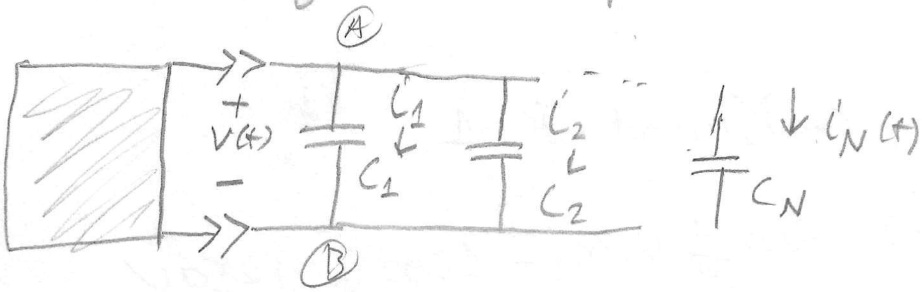
$$\frac{d^2}{dt^2} V_o(t) = -20\frac{d}{dt}V_o(t) - 1000\,V_o(t) + 50 \quad V$$



One possible design



6-4 Equivalent Capacitance and Inductance



KCL at (A)

$i = i_1 + i_2 + \dots + i_N(t)$

They are parallel, KVL

$V_1 = V_2 = \dots = V_N$

So then $i_K(t) = C_K \frac{d}{dt} V(t)$

and Now KCL: $i(t) = C_1 \frac{d}{dt}V(t) + C_2\frac{d}{dt}V(t) + \dots C_N \frac{d}{dt}V(t)$

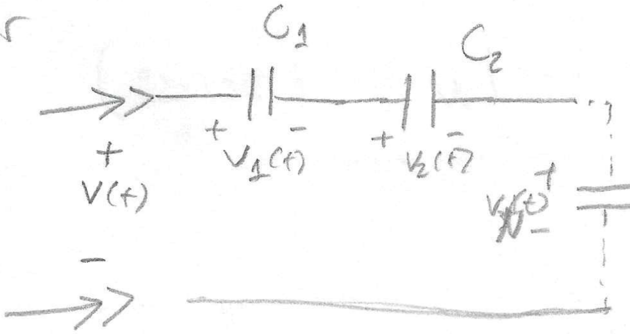
$$= (C_1 + C_2 + \dots + C_N)\frac{d}{dt}V(t)$$

We see $C_{eq} = C_1 + C_2 + \cdots C_N$ (6-28)

(for parallel caps)

initial voltage $V(0)$ is the same across the caps

For



KVL

$V(t) = V_1(t) + V_2(t) \cdots V_N(t)$

We know,

$i_1 = i_2 = i_3 \cdots = i_N = i(t)$

So we know I-V relationship

$$V_k(t) = V_k(0) + \frac{1}{C_k}\int_0^t i(x)\,dx$$

Substitute into KVL,

$$V(t) = V_1(0) + \frac{1}{C_1}\int_0^t i_1(x)\,dx + V_2(0) + \frac{1}{C_2}\int_0^t i_2(x)\,dx + \cdots + V_N(0) + \frac{1}{C_N}\int_0^t i_N(x)\,dx$$

$$\rightarrow V(t) = \left[V_1(0) + V_2(0) \cdots + V_N(0)\right] + \left(\frac{1}{C_1} + \frac{1}{C_2} \cdots \frac{1}{C_N}\right)\int_0^t i(x)\,dx$$

From observation

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots \frac{1}{C_N} \quad \text{(series)}$$

The initial voltage on C_{eq} is the sum of the initial voltages on each of the original N caps

(18) if we replace cap for coils

KVL $v(t) = L_1 \frac{d}{dt} i(t) + L_2 \frac{d}{dt} i(t) + \cdots L_N \frac{d}{dt} i(t)$

$$= (L_1 + L_2 + \cdots L_N) \frac{d}{dt} i(t)$$

so

$$L_{eq} = L_1 + L_2 + \cdots L_N \quad \text{(series)}$$

---

for parallel,

KCL $i(t) = i_1 + i_2 + \cdots i_N$

$$= i_1(0) + \frac{1}{L_1}\int_0^t v(x)\,dx + i_2(0) + \frac{1}{L_2}\int_0^t v(x)\,dx$$

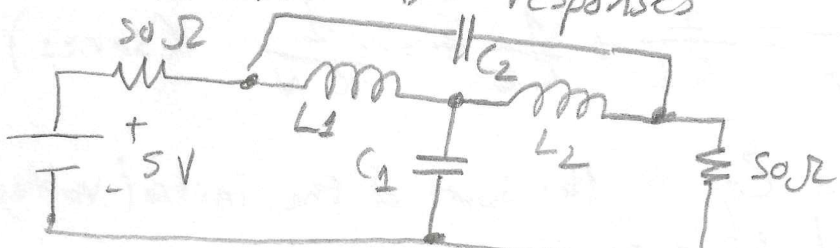
$$+ \cdots + i_N(0) + \frac{1}{L_N}\int_0^t v(x)\,dx$$

$$= [i_1(0) + i_2(0) + \cdots + i_N(0)] + \left(\frac{1}{L_1} + \frac{1}{L_2} + \cdots + \frac{1}{L_N}\right) \cdots \int_0^t v(x)\,dx$$

so $\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \cdots \frac{1}{L_N}$ (parallel

---

A Note about DC responses



This gives us initial Operating Points (ICs) initial conditions for transient responses that begin at t=0