Gaussian surface Symmetry

We recall Gauss's Law

$$\oint_S \bar{E} \cdot d\bar{s} = \frac{1}{\epsilon} \int_V \rho \, dv$$

$$\oint_S \vec{D} \cdot d\bar{s} = \int_V \rho \, dv$$

$\vec{D} = \epsilon \bar{E}$

Permittivity

$$\epsilon_0 = 8.854 \times 10^{-12} \ \frac{F}{m}$$

$\epsilon = \epsilon_r \epsilon_0$

ϵ_r = relative constant

$\epsilon_r = 1$, 'free space'

Spherical Symmetry

$$\bar{E} \rightarrow E_R \hat{R}, \quad d\bar{s} \rightarrow R^2 \sin\theta \, d\theta \, d\phi$$

requirement

$$\rho(R,\theta,\phi) \longrightarrow \rho[R]$$

(Not a function of θ and ϕ)

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Ex $\rho = \rho_0 \; [C/m^3]$ $R < a$

and $\rho = \rho_{s0} \; C/m^2$ $R = b$

for $\nabla \cdot \bar{E} = \frac{\rho}{\epsilon}$ we know $\rho \rightarrow Q$ Point of charge origin

we have a charge density coming from a charge point

$\rho = \frac{Q}{\underbrace{4\pi b^2}_{\text{surface area of a spherical shell}}}$ ← Total charge

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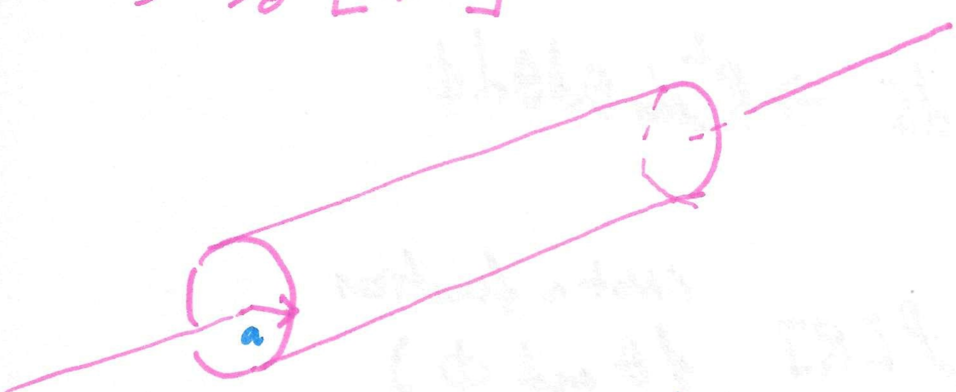
Cylindrical Example

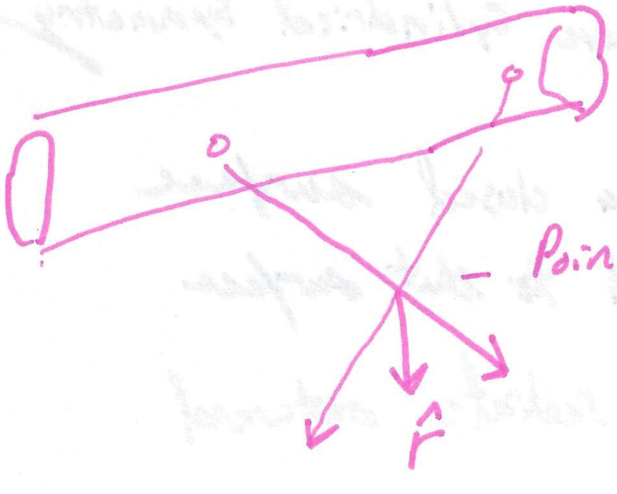
$\rho = \rho_0 \; [C/m^3]$ $r < a$

What's my Length? what are the Limits for z?



for Length $= \infty$ $z = (-\infty, \infty)$

we can think of this as a section of charge



Point of Observation

we are gonna have a Lot of Influence from the section of charge on our observation point

we have Cylindrical Symmetry when

$$\rho(r, \phi, z) \longrightarrow \rho(r)$$

(charge density is only a function of distance from z-axis)

so we start with

$$\bar{E} \rightarrow E_r \, \hat{r}$$ (radiately outward from z-axis)

$$\oint \vec{E} \cdot d\vec{s} = \frac{1}{\epsilon} \int \rho \, dv$$

(1) Find the coordinate system
→ cylindrical

(2) Determine Symmetry, → we have cylindrical Symmetry

(3) Based on Symmetry, define a closed surface
such $\bar{E}$ is either $\perp$ or $\parallel$ to that surface

→ we know $\bar{E}$ will radiate outward

(4) Find $d\bar{s}$ for those surfaces

$$\rightarrow d\vec{s} \longrightarrow d\vec{s}_1,\ d\vec{s}_2,\ d\vec{s}_3$$

$d\vec{s}_1$ → end: $r\,dr\,d\phi\ -\hat{z}$

$d\vec{s}_2$ → curved: $r\,d\phi\,dz\ \hat{r}$

$d\vec{s}_3$ → end: $r\,dr\,d\phi\ \hat{z}$

(5) Find vector form $\vec{E}$ for that Symmetry

We have said earlier,

$$\vec{E} \rightarrow E_r\,\hat{r}$$

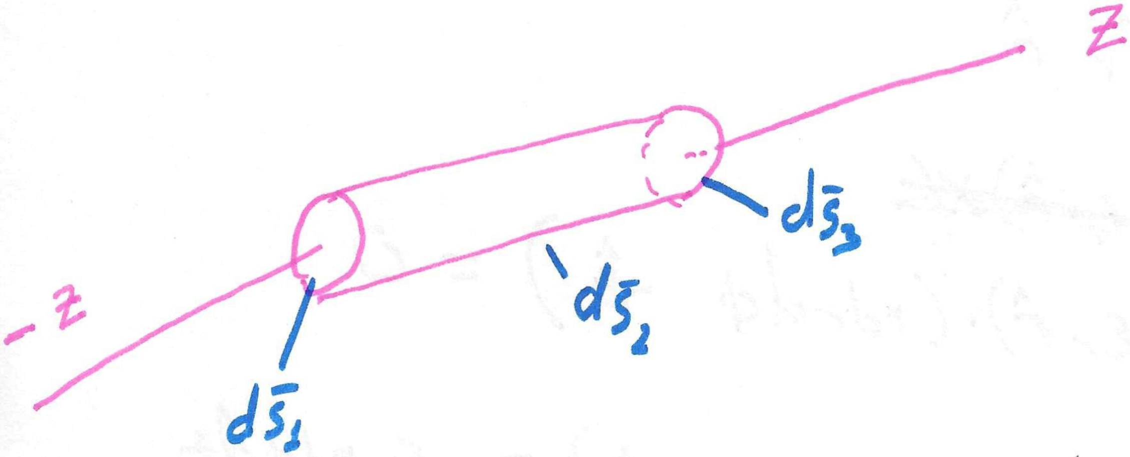
(6) Find dot Product

$$\vec{E}\cdot d\vec{s} = \cancelto{0}{\vec{E}\cdot d\vec{s}_1} + \vec{E}\cdot d\vec{s}_2 + \cancelto{0}{\vec{E}\cdot d\vec{s}_3}$$

we ~~have~~ can tell $\vec{\mathcal{E}}\cdot d\vec{S}_1$ and $\vec{\mathcal{E}}\cdot d\vec{S}_3$ are zero from step 3

so $\vec{\mathcal{E}}\cdot d\vec{S} = \mathcal{E}_r\, r\, d\phi\, dz$

To Examine further, Look at finite Cylindrical shell of length l



Flat surface at end 1 $\quad d\vec{S}_1$

Curved part $\quad d\vec{S}_2$

flat surface at end 2 $\quad d\vec{S}_3$

normal component for S_1, $-\hat{z}$ direction

$$d\vec{S}_1 \longrightarrow r\,dr\,d\phi \quad (-\hat{z})$$

normal for S_2, $\hat{r}$ direction

$$d\vec{S}_2 \longrightarrow r\,d\phi\,dz \quad (\hat{r})$$

normal for S_3, $+\hat{z}$ direction

$$d\bar{S}_3 \longrightarrow r\,dr\,d\phi \ (+\hat{z})$$

~~Ex~~ So to carry out the Integral in detail,

$$\oint \vec{\mathcal{E}}\cdot d\vec{s} = \int \vec{\mathcal{E}}\cdot d\vec{S}_1 + \int \vec{\mathcal{E}}\cdot d\vec{S}_2 + \int \vec{\mathcal{E}}\cdot d\vec{S}_3$$

$$\vec{\mathcal{E}} = \mathcal{E}_r \, \hat{r}$$

$\vec{\mathcal{E}}\cdot d\vec{S}_1$: $\cancel{(\mathcal{E}_r \hat{r}) \times (}$

$$(\mathcal{E}_r \hat{r})\cdot(r\,dr\,d\phi \ -\hat{z}) = 0$$

$\vec{\mathcal{E}}\cdot d\vec{S}_2$: $(\mathcal{E}_r \hat{r})\cdot(r\,d\phi\,dz \ \hat{r}) = \mathcal{E}_r \, r\,d\phi\,dz$

$\vec{\mathcal{E}}\cdot d\vec{S}_3$: $(\mathcal{E}_r \hat{r})\cdot(r\,dr\,d\phi \ +\hat{z}) = 0$

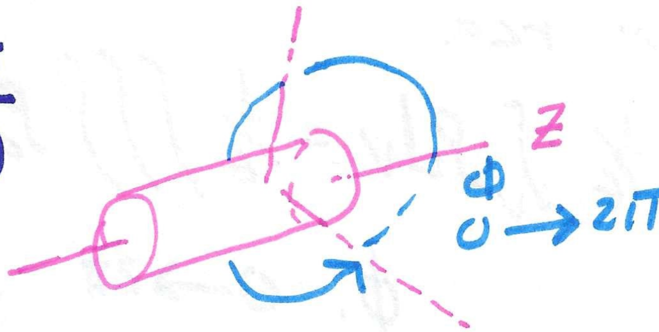
$\vec{\mathcal{E}}\cdot d\vec{S}_3 = 0 = \vec{\mathcal{E}}\cdot d\vec{S}_2$ is because the field is parallel to $d\vec{S}_3$'s surface

For these kinds of Symmetrical problems,

→ Field lines that contribute must be perpendicular to the surface (parallel to the normal)

7 Integrate ~~RHS~~ (LHS)

$\oint \bar{E} \cdot d\bar{S}$



~~So~~ $\iint \bar{E} \cdot d\bar{S}_2$

$\rightarrow \iint E_r \, r \, d\phi \, dz$

$= \int_0^{l} \int_0^{2\pi} E_r \, r \, d\phi \, dz$

$= E_r \, r (2\pi)(l)$

we have arbitrary radius r

we know

$$\rho = \begin{cases} \rho_0 \quad C/m^3 & r < a \\ 0 & r > a \end{cases}$$

→ We have Two Solution Regions

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(8) Integrate the charge distribution over the defined volume

$$\int_V \frac{\rho}{\epsilon} dV \quad (RHS)$$

for $r < a$

$$\frac{1}{\epsilon}\int_V \rho \, dV = \frac{1}{\epsilon}\iiint \rho_o \, r \, dr \, d\phi \, dz$$

$\phi: 0 \rightarrow 2\pi \quad z: 0 \rightarrow l$

$r: 0 \rightarrow r$

$$\frac{1}{\epsilon}\int_0^l \int_0^{2\pi} \int_0^r \rho_o \, r \, dr \, d\phi \, dz$$

$$= \frac{1}{\epsilon} \underbrace{\rho_o \frac{l(2\pi) r^2}{2}}_{} = \frac{Q}{\epsilon}$$

This is Total charge inside a cylindrical shell with arbitrary radius $r < a$

(9) set ~~RHS =~~

LHS = RHS and find Expression for the field component.

$$\mathcal{E}_r \, 2\pi r l = \frac{1}{\epsilon} \pi r^2 l \rho_o$$

$$\mathcal{E}_r = \frac{1}{\epsilon} \frac{r \rho_o}{2} \hat{r} \ \ V/m \qquad r < a$$

for $a < r$

$$\oint \bar{\mathcal{E}} \cdot d\bar{s} \rightarrow \mathcal{E}_r \, 2\pi r l \qquad r > a$$

(same for $r < a$)

and

$$\frac{1}{\epsilon} \int_v \rho \, dv = \frac{1}{\epsilon} \int_0^l \int_0^{2\pi} \int_0^a \rho \, r dr d\phi dz$$

$$+ \frac{1}{\epsilon} \int_0^l \int_0^{2\pi} \int_a^r \rho \, r dr d\phi dz$$

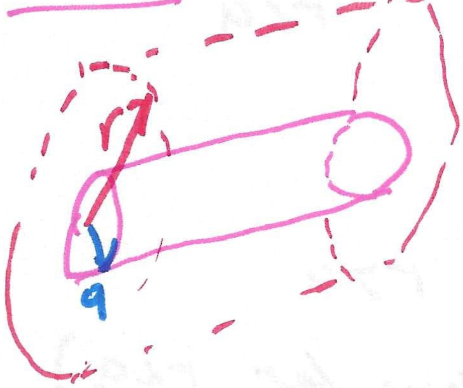
we know $\rho_{r<a} = \rho_o \, , \ \rho_{a<r} = 0$

So

$$\frac{1}{\epsilon}\int_v \rho\, dv = \frac{1}{\epsilon}\int_0^l\int_0^{2\pi}\int_0^a \rho_0 \, r dr\, d\phi\, dz$$

$$= \frac{1}{\epsilon}\rho_0 \frac{a^2}{2} 2\pi l$$

Visual



So

$$\mathcal{E}_r\, 2\pi r l = \frac{1}{\epsilon}\frac{a^2}{2}\pi l\, 2\rho_0$$

$$\mathcal{E}_r = \frac{1}{\epsilon}\,\frac{a^2\rho_0}{2r}\,\hat{r}\ \frac{V}{m}$$

$\underline{a < r}$

observations

$\rho \longrightarrow \rho_L \ \frac{C}{m}$ Line charge along z-axis

We have $\bar{\mathcal{E}} \propto \frac{1}{r}$ outside charge distribution, $\nabla\cdot\bar{\mathcal{E}} = 0$

(No $\bar{\mathcal{E}}$-field passing through a closed surface)

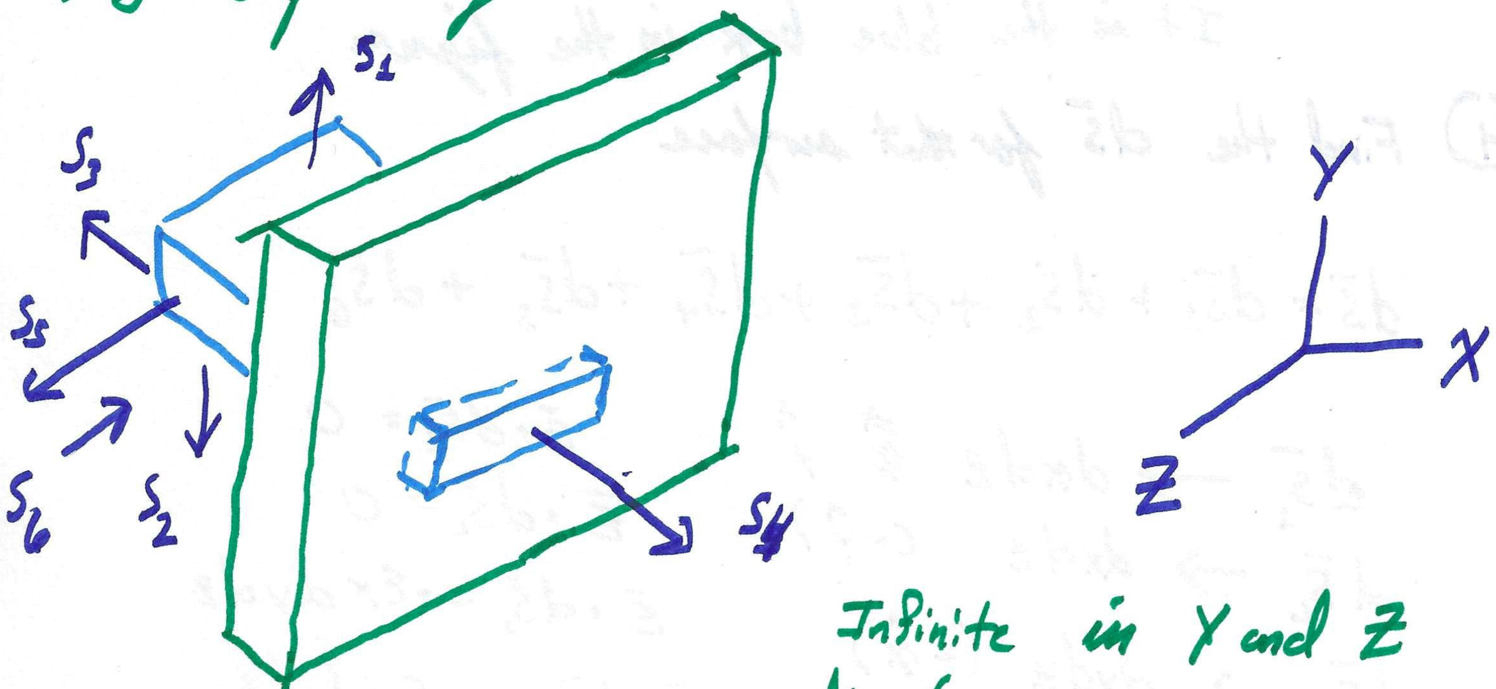
Field falls off as $1/r$ for Cylindrical Symmetry

What component has the field?
→ Co-axial

Rectangle charge distribution

$$\rho = \begin{cases} \rho_0 & -a < x < a \\ 0 & \text{else} \end{cases}$$

⇒ Slab of charge



Infinite in Y and Z direction

① $\oint \bar{E} \cdot d\bar{s} = \int_V \frac{\rho}{\epsilon} dV$
(LHS) (RHS)

Find coordinate System → Rectangle

② Determine Symmetry → We'll have Rectangle Symmetry if applicable

(3) Based on symmetry, define a closed surface such $\bar{E}$ is either $\perp$ or $\parallel$ to that surface

→ We are going with a box (our closed surface). This box has six sides enclosing surface.

It is the blue box in the figure

(4) Find the $d\bar{s}$ for that surface

$$d\bar{s} = d\bar{s}_1 + d\bar{s}_2 + d\bar{s}_3 + d\bar{s}_4 + d\bar{s}_5 + d\bar{s}_6$$

$d\bar{s}_1 \rightarrow dxdz\ \hat{y}$ $\qquad$ $\bar{E} \cdot d\bar{s}_1 = 0$

$d\bar{s}_2 \rightarrow dxdz\ (-\hat{y})$ $\qquad$ $\bar{E} \cdot d\bar{s}_2 = 0$

$d\bar{s}_3 \rightarrow dydz\ (-\hat{x})$ $\qquad$ $\bar{E} \cdot d\bar{s}_3 = -E_x\, dydz$

$d\bar{s}_4 \rightarrow dydz\ \hat{x}$ $\qquad$ $\bar{E} \cdot d\bar{s}_4 = +E_x\, dydz$

$d\bar{s}_5 \rightarrow dxdy\ \hat{z}$ $\qquad$ $\bar{E} \cdot d\bar{s}_5 = 0$

$d\bar{s}_6 \rightarrow dxdy\ (-\hat{z})$ $\qquad$ $\bar{E} \cdot d\bar{s}_6 = 0$

(5) Find vector form $\vec{E}$ for that symmetry

→ $E_x\ \hat{x}$ $\quad$ we have $\bar{E}$ in x direction

(6) Find dot Product

$\bar{E} = E_x\ \hat{x}$

⑦ Integrate

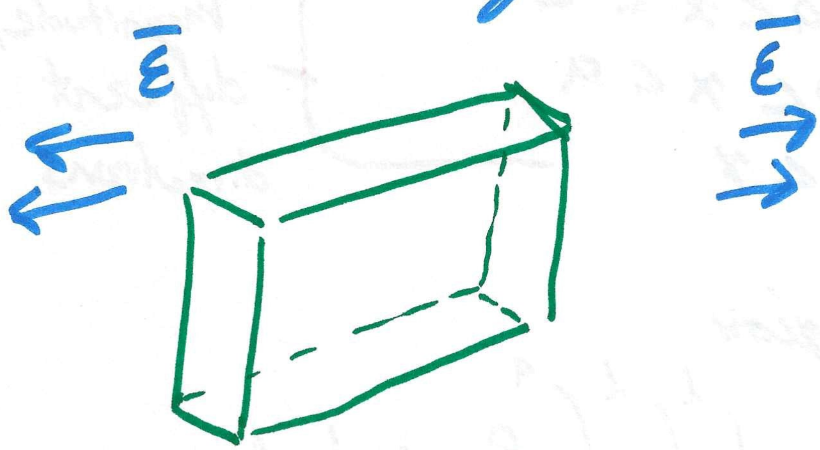
Left surface

$$\oint \bar{\varepsilon}\cdot d\bar{s} = -\iint \varepsilon_x \, dy\, dz$$

$$+\iint \varepsilon_x \, dy\, dz$$

Right Surface

Look at the Symmetry



so $\oint \bar{\varepsilon}\cdot d\bar{s} = 2\iint \varepsilon_x \, dy\, dz$ right surface

↑

Symmetry

Dimensions: l^3, $\Delta x = \Delta y = \Delta z = l$

so $\oint \bar{\varepsilon}\cdot d\bar{s} = 2 l^2 \varepsilon_x$

(8) Integrate the charge distribution over the defined volume

$$\frac{1}{\epsilon}\int_v \rho\, dv = \frac{1}{\epsilon}\iiint \rho_0\, dx\, dy\, dz$$

Look at Regions of $\bar{E}$

same magnitude and different directions $\left\{\begin{array}{l} x < -a \\ -a < x < 0 \\ 0 < x < a \\ a < x \end{array}\right\}$ same magnitude, different directions

so for $x > a$ region

$$\frac{1}{\epsilon}\int_v \rho\, dv = \frac{1}{\epsilon}\int_0^l \int_0^l \int_{-a}^{a} \rho_0\, dx\, dy\, dz$$

$$= \frac{2al^2}{\epsilon}\rho_0$$ we enclose all the $\frac{\text{charge}}{\epsilon}$

(9) $\oint \bar{E}\cdot d\bar{s} = \frac{1}{\epsilon}\int \rho\, dv$ set them equal

$$2l^2 E_x = \frac{2al^2}{\epsilon}\rho_0 \longrightarrow E_x = \frac{a}{\epsilon}\rho_0$$

Here's the $\mathcal{E}$-field and the regions

$$\bar{\mathcal{E}} = \begin{cases} -\frac{a}{\epsilon}\rho_0 \hat{x} & x < -a \\ -\frac{|x|}{\epsilon}\rho_0 \hat{x} & -a < x < 0 \\ \frac{|x|}{\epsilon}\rho_0 \hat{x} & 0 < x < a \\ \frac{a}{\epsilon}\rho_0 \hat{x} & a < x \quad \cancel{a<x} \end{cases}$$

$\left.\begin{matrix} x<-a \\ -a<x<0 \end{matrix}\right\}$ $-\hat{x}$ direction

$\left.\begin{matrix} 0<x<a \\ a<x \end{matrix}\right\}$ $+\hat{x}$ direction

Another Example,

Now shrink the cube so that we're inside the domain, (slab)

$|x| < a$

Two l's as this is a cube

$$\frac{1}{\epsilon}\int \rho \, dV = \frac{1}{\epsilon}\int_0^l \int_0^l \int_{-x}^{x} \rho_0 \, dx\, dy\, dz$$

← for symmetry

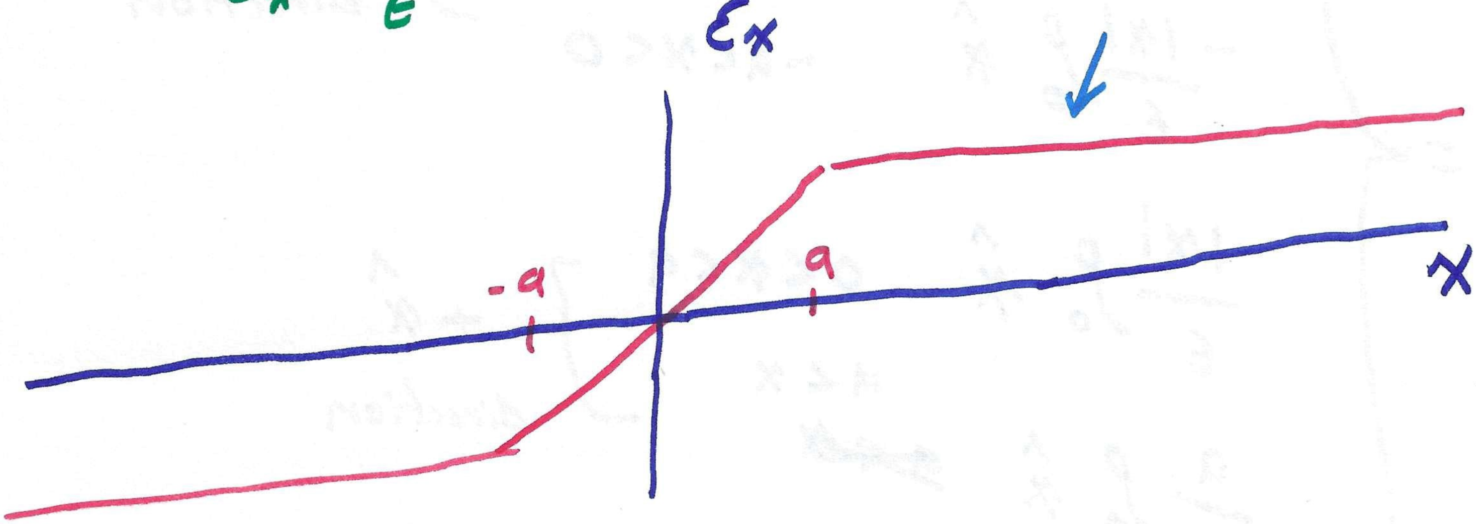
$$= \frac{1}{\epsilon} l^2 2x \rho_0$$

Then

$$2 \mathcal{E}_x l^2 = \frac{1}{\epsilon} 2x\, l^2 \rho_0$$

$$\mathcal{E}_x = \frac{1}{\epsilon} x \rho_0$$

Artifact of ∞ Length in 2 Dimensions



Another Example

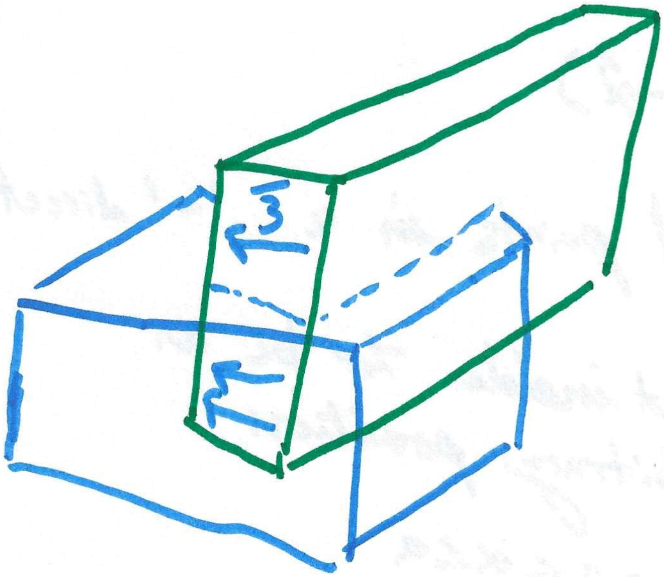
$$\rho = x \frac{C}{m^3} \qquad -a < x < a$$

→ non uniform charge distribution

consequently, Total charge is zero

$$Q_{Total} = 0 \longrightarrow \bar{\mathcal{E}} \text{ outside } \rho \text{ is zero}$$

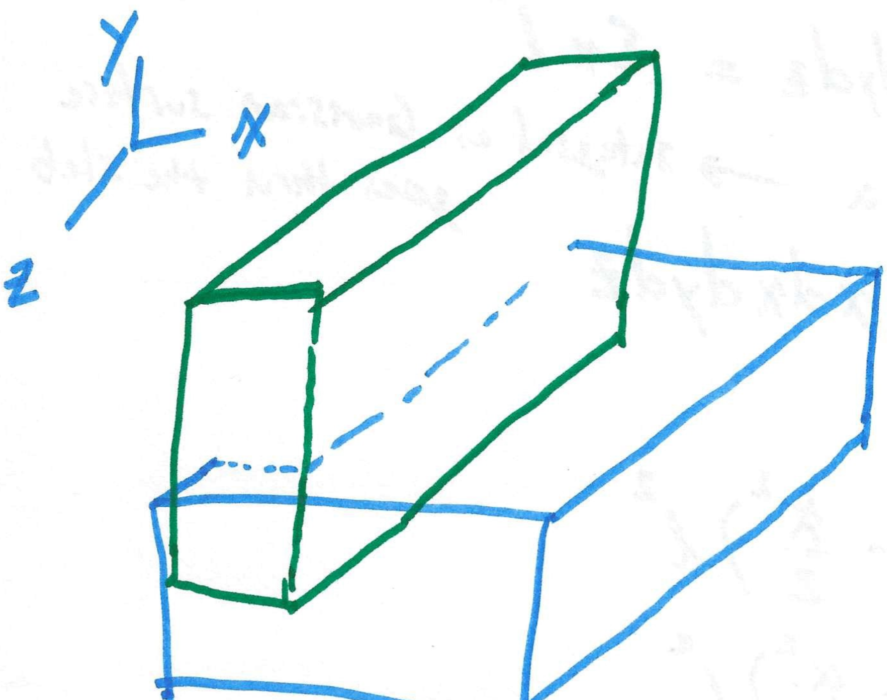
as x can be 'negative'



every positive charge has a negative charge

$\bar{E}$-field goes from '+' to '−' so thats why $\bar{E}$ goes in the $-\hat{x}$ direction

$\bar{E} \cdot d\bar{s} = 0$ on every position



The Gaussian surface is inside the slab of charge (ρ)

We can say that we have 6 surfaces

4 are trivial $\bar{E} \cdot d\bar{s} \rightarrow 0$

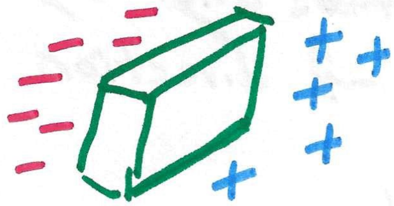
1 is trivial, $\bar{E} = 0$ on the surface

But the surface inside the slab has a nonzero contribution

$$\int \bar{E} \cdot d\bar{s} \neq 0$$

$$d\bar{s} \rightarrow dydz\ (-\hat{x})$$

inside the slab, field points in a '-' direction



set inside to be an arbitrary position

$$-a < x < a$$

→ Integral about Gaussian surface

$$\oint_S \bar{E} \cdot d\bar{s} = \int_0^l \int_0^l E_x\, dydz = E_x l^2$$

→ Integral as Gaussian surface goes thru the slab

$$\frac{1}{\epsilon} \int \rho\, dV = \frac{1}{\epsilon} \int_0^l \int_0^l \int_x^a x\, dx\, dy\, dz$$

$$= \frac{1}{\epsilon} \left(\frac{a^2}{2} - \frac{x^2}{2} \right) l^2$$

$$E_x l^2 = \frac{1}{\epsilon} \left(\frac{a^2}{2} - \frac{x^2}{2} \right) l^2$$

$$\bar{E} = \begin{cases} -\frac{1}{\epsilon}\left(\frac{a^2}{2} - \frac{x^2}{2}\right)\hat{x} & -a < x < a \\ 0 & x < -a \text{ or } a < x \end{cases}$$

The '−' sign comes from the fact $\bar{E}$ flows in the $(-\hat{x})$ direct because $\bar{E}$ goes from '+' to '−'