Fields and Waves I L11

Gauss's Law

We may take note that

$$\nabla \cdot \bar{E} = \frac{\rho_v}{\epsilon}$$

C/m³

Permittivity F/m

Divergence of the Electric Field
("flux of a vector field through a surface")

Also Note $\bar{D} = \epsilon \bar{E}$

$$\bar{E} = \frac{V}{m} = \frac{N}{C}$$

$$\bar{D} = \frac{F \cdot V}{m^2}$$

Consequently, we can say

$$\int_V (\nabla \cdot \bar{E})\, dV = \int_V \frac{\rho_v}{\epsilon}\, dV$$

So now we can apply Divergence Theorem

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→ Apply Divergence theorem to Left side

$$\rightarrow \int_V (\nabla\cdot\bar{E})\, dv = \oint \bar{E}\cdot d\bar{s}$$

→ The total divergence of the electric field in a volume = The total flux passing through the surface enclosing that volume

we can think the $\bar{E}$-field is the gradient of electric potential.

<u>Electric flux</u> is the measure of the $\bar{E}$-field through a given surface.

→ An $\bar{E}$-field does not flow

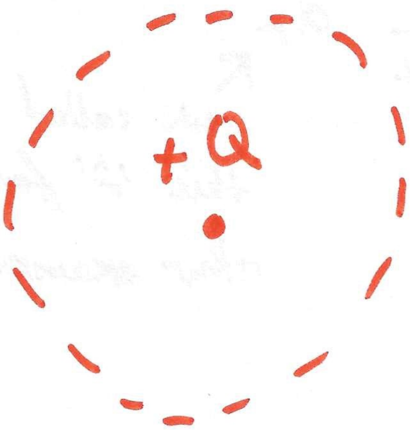
So then

$$\oint \bar{E}\cdot d\bar{s} = \int \frac{\rho_v}{\epsilon}\, dV$$

$$= \frac{1}{\epsilon}\int \rho_v\, dv \leftarrow$$ we assume ϵ is constant

Total electric field through closed surface = Total charge inside the volume

Look at a point charge



Sphere centered at point charge

known from Coulomb's law

$$\bar{E} = \frac{Q}{4\pi\epsilon R^2}\hat{R}$$

Total charge enclosed by surface

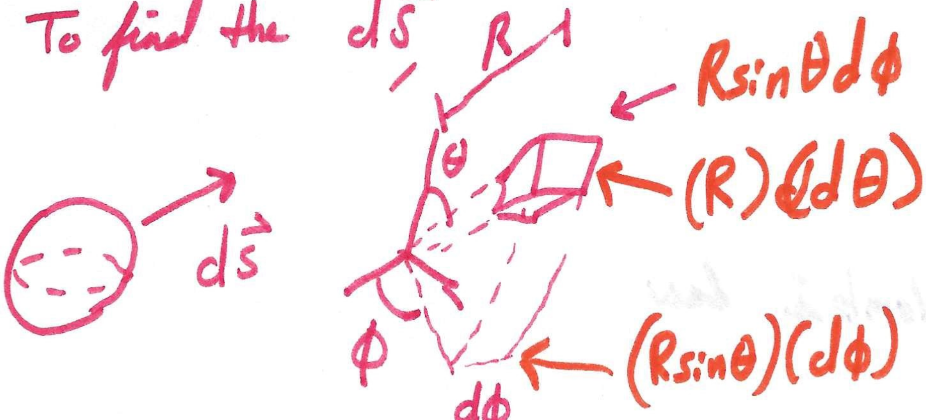
$$\oint \bar{E}\cdot d\bar{S} = \frac{1}{\epsilon}\int \rho_v dV$$

Example

Show that the electric field of a point charge satisfies Gauss's Law by integrating $\oint \bar{E} \cdot d\bar{s}$ over the surface of a sphere of radius a.

For point charge $\vec{E} = \dfrac{Q}{4\pi\epsilon r^2}\hat{a}_r$ ← we called this '$\hat{r}$' for other examples

To find the $d\bar{s}$



so $d\bar{s} = [R\sin\theta(d\phi)][(R)(d\theta)]$

$= r^2 \sin\theta \, d\theta \, d\phi \Big|_{R^2 = r^2}$

→ $\oint \bar{E} \cdot d\vec{s} = \int_0^{2\pi}\int_0^{\pi} \dfrac{Q}{4\pi\epsilon r^2} r^2 \sin\theta d\theta d\phi$

$$= \frac{Q}{4\pi\epsilon} \int_0^{2\pi} \left(-\cos\theta \Big|_0^{\pi} \right) d\phi$$

$$= \frac{Q}{4\pi\epsilon} (4\pi) = \frac{Q}{\epsilon}$$

We know Gauss's Law states $\oint \vec{D} \cdot d\vec{s} = Q_{enclosed}$

and we just showed $\epsilon \left[\oint \vec{E} \cdot d\vec{s} \right] = Q$

because $\vec{E}$ is constant on the surface of the sphere

Re iterate!

The total field passing through a closed surface $=$ $\dfrac{\text{Total charge inside that surface}}{\epsilon}$

To Explain what just happened,

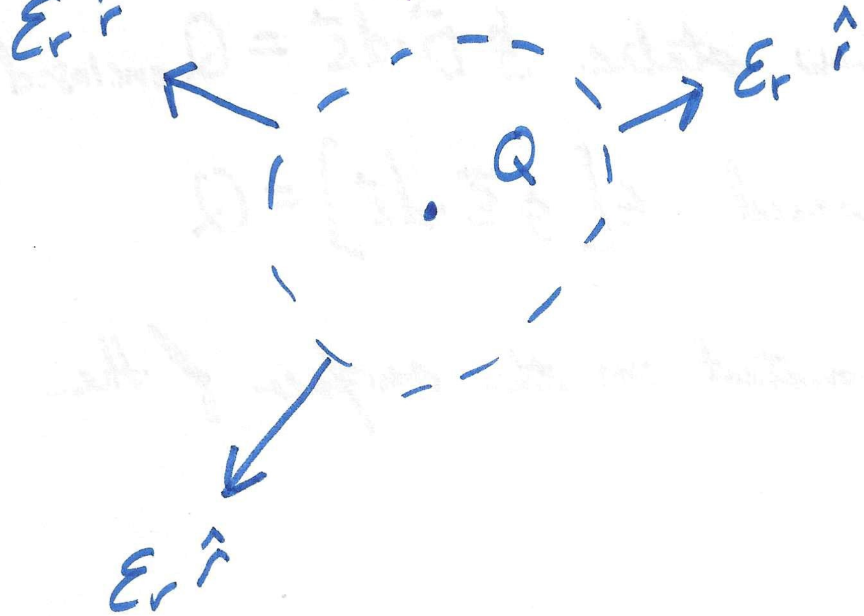
Point charge is centered at the Origin, charge Q

6 we have

* Spherical Symmetry

→ consider a spherical shell (surface)

→ on that surface, the field is constant in magnitude and radial in direction



Look at the Integral again

$$\oint \bar{E} \cdot d\bar{s} = \int \frac{\rho}{\epsilon} dV$$

* Spherical shell (any spherical shell)

$$\boxed{d\bar{s} = r^2 \sin\theta \, d\theta \, d\phi \ \hat{r}}$$

Also

* Spherical Symmetry

$$\bar{\mathcal{E}} = \mathcal{E}_r \hat{r}$$

↑ can be a constant or a function

to Look at $\bar{\mathcal{E}} \cdot d\bar{s} = (\mathcal{E}_r \hat{r}) \cdot (r^2 \sin\theta \, d\theta \, d\phi \, \hat{r})$

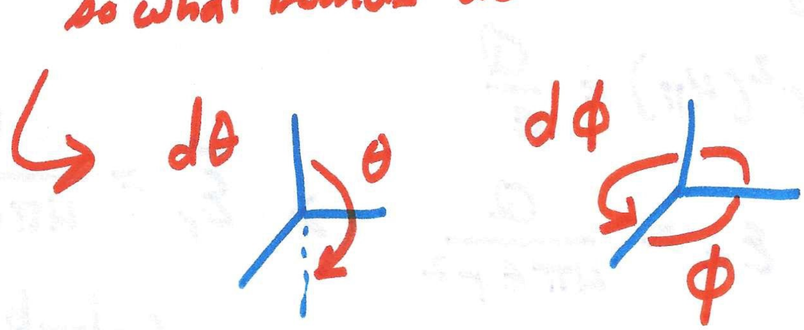
(dot product)

$$= \mathcal{E}_r r^2 \sin\theta \, d\theta \, d\phi$$

now we have

$$\iint \mathcal{E}_r r^2 \sin\theta \, \underline{d\theta \, d\phi}$$

so what bounds do we do?



Therefor

$$\int_0^{2\pi}\int_0^{\pi} \mathcal{E}_r r^2 \sin\theta \, d\theta \, d\phi = \mathcal{E}_r r^2 \int_0^{2\pi}\int_0^{\pi} \sin\theta \, d\theta \, d\phi$$

$$= \mathcal{E}_r r^2 \int_0^{2\pi} -\cos\theta \Big|_0^{\pi} d\phi$$

$$= \mathcal{E}_r r^2 \int_0^{2\pi} 2\, d\phi = \mathcal{E}_r r^2 4\pi \quad \text{(charge enclosed)}$$

Integral is done by taking advantage of spherical symmetry

$$\rightarrow \int_V \frac{\rho}{\epsilon} dv \Big|_{\text{point charge}} = \frac{Q}{\epsilon}$$ Just the charge enclosed

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So we make the connection

$$\oint_S \bar{\mathcal{E}} \cdot d\bar{s} = \int_V \frac{\rho}{\epsilon} dv$$

$$\mathcal{E}_r r^2 (4\pi) = \frac{Q}{\epsilon}$$

$$\rightarrow \mathcal{E}_r = \frac{Q}{4\pi \epsilon r^2} \rightarrow \vec{\mathcal{E}}_r = \frac{Q}{4\pi \epsilon r^2} \hat{r} \left[\frac{V}{m}\right]$$

Coulombs Law

Strategy for Solving Gauss's Law

for Left Side (LHS) [$\oint \bar{E} \cdot d\bar{S}$]

1.) Find Coordinate System

2.) Determine Symmetry

a) Spherical b) Cylindrical c) rectangle

3.) Based on Symmetry, define a closed surface such $\bar{E}$ is either $\perp$ or $\parallel$ to that surface

4.) Find $d\bar{s}$ for that surface (s)

5.) Find vector form $\vec{E}$ for that symmetry

6.) Find dot product

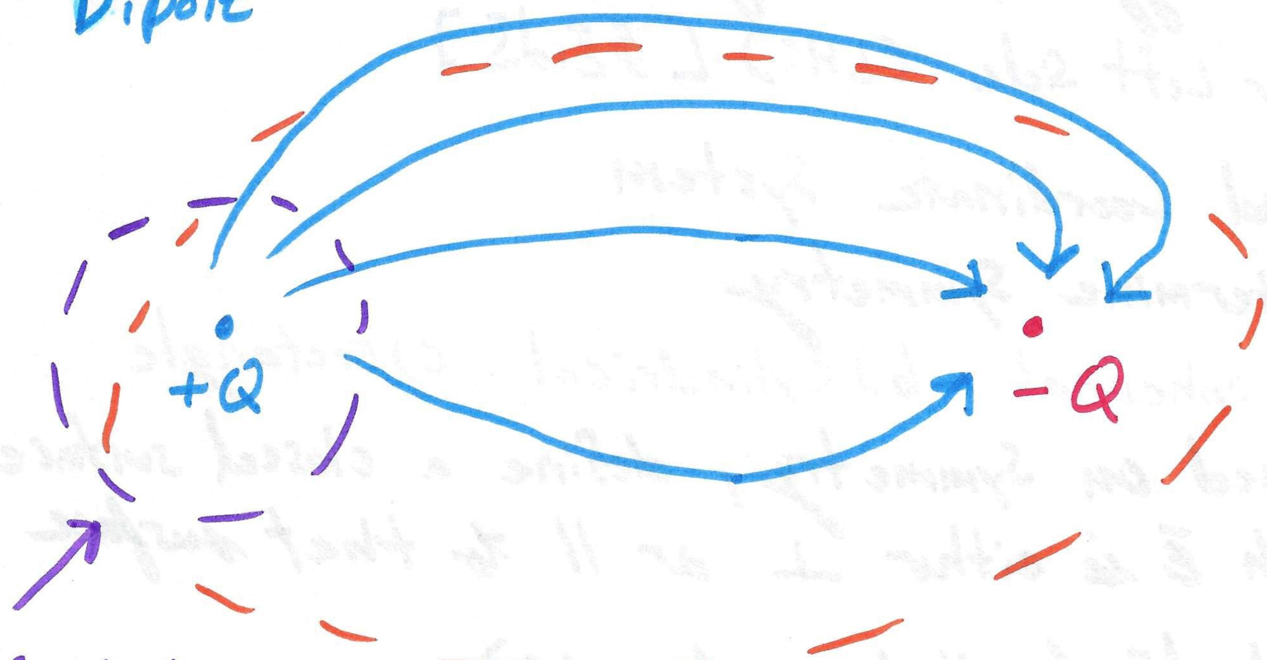
7.) Integrate

for Right side (RHS) [$\frac{1}{\epsilon}\int_V \rho \, dV$]

8.) Integrate the charge distribution over the defined Volume

9.) Set Left and Right equal, and find the expression for the field component

Dipole



$\oint_S \vec{E} \cdot d\vec{s} \neq 0$

for this shell $\quad \int \frac{\rho}{\epsilon} dV = 0$

Zero total charge!

$+Q - Q = 0$

We may also say

Total Electric flux is zero

$\oint \vec{E} \cdot d\vec{s} = 0$

The E-field on the surface is not zero but the Total flux on the shell is zero as $\vec{E}$-field goes inward, then outward

Spherical charge density

$$\rho_V = \begin{cases} \rho_0 & \text{for } R<a \\ 0 & \text{else} \end{cases}$$ ← some radius a

Two 'regions' of interest

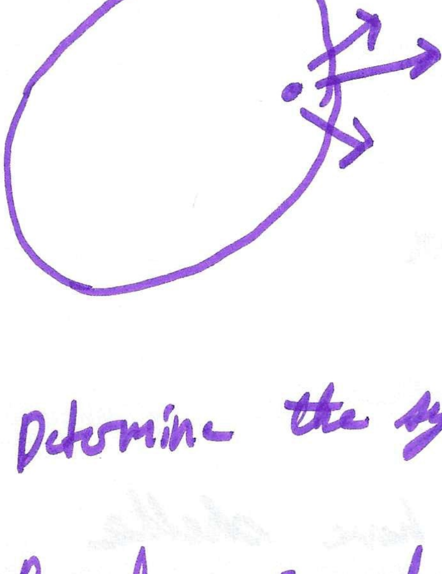
$R<a$ } find a solution for each
$a<R$ } region independently

To find $\vec{E}$, find $R<a$

point of observation

① Spherical coordinates

 $\hat{R}$



$\hat{R}$ / spherical symmetry

② Determine the symmetry, its spherical

③ Based on symmetry, define the closed space

– The spherical shell is our closed surface

$\bar{E}$ is $\parallel$ to surface normal

④ find the $d\vec{s}$ for that surface

$$d\vec{s} = R^2 \sin\theta \, d\theta \, d\phi \, \hat{R}$$

⑤ Vector form of $\vec{E}$ for that symmetry

$$\vec{E} \rightarrow \vec{E}_R \hat{R}$$

⑥ Find dot product

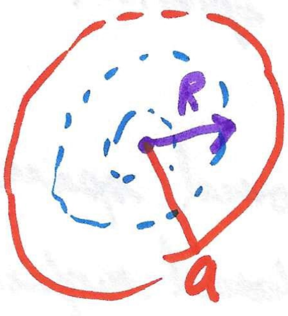
$$\vec{E} \cdot d\vec{s} = E_r R^2 \sin\theta \, d\theta \, d\phi$$

⑦ Integrate RHS

$$\oint_S \vec{E} \cdot d\vec{s} = \int_0^{2\pi} \int_0^{\pi} E_R R^2 \sin\theta \, d\theta \, d\phi$$

$$= E_R R^2 4\pi = 4\pi R^2 E_R$$

→ This is our solutions for $R < a$



~~we~~ we have shells within a

(8) Integrate charge distribution over defined volume!

$$\int_V \frac{\rho}{\epsilon}\,dV = \iiint \frac{\rho_0}{\epsilon} \underbrace{R^2 \sin\theta\, dR\, d\theta\, d\phi}_{dV \text{ for sphere}}$$

arbitrary sphere between 0 and a

boundaries:

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{r} \frac{\rho_0}{\epsilon} R^2 \sin\theta\, dR\, d\theta\, d\phi$$

recall $dR \rightarrow R$ direction

$R\,d\theta \rightarrow \theta$ direction

$R\sin\theta\, d\phi \rightarrow \phi$ direction

so $\displaystyle\int_V \frac{\rho}{\epsilon}\,dV = 4\pi \frac{\rho_0}{\epsilon}\int_0^r R^2\, dR$

$$= 4\pi \frac{\rho_0}{\epsilon} \frac{r^3}{3}$$

$$\frac{4\pi \rho_0 r^3}{3}$$

Total charge enclosed by a shell centered at origin with ~~radius~~ radius $r < a$

9.) set LHS equal to RHS, then find the Expression for the field component

$$\oint_S \vec{\mathcal{E}} \cdot d\vec{s} = \int_V \frac{\rho}{\epsilon}\, dV$$

$$4\pi R^2 \mathcal{E}_R = \frac{4}{3}\pi R^3 \frac{\rho_0}{\epsilon} \rightarrow \mathcal{E}_R = \frac{1}{3}\frac{\rho_0 R}{\epsilon}$$

so $\vec{\mathcal{E}} = \frac{1}{3}\frac{\rho_0 R}{\epsilon}\hat{R}$ for $R < a$

The field gets bigger as you move away from the origin

So what's the ~~shell~~ $\mathcal{E}$-field outside the shell?

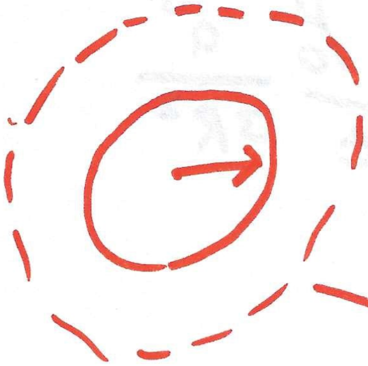
Steps 1-7 are the same spherical symmetry

$a(t) = a(-t)$
$\to a_K \quad \to a_K$

for $R > a$

$\oint \vec{\varepsilon} \cdot d\vec{s} = \varepsilon_R \, 4\pi R^2$

remember $\oint \vec{\varepsilon} \cdot d\vec{s}$ is the total flux passing through the surface enclosing that volume



Spherical shell
This guy contains all charge

Basically, for $R > a$, $\oint \vec{\varepsilon} \cdot d\vec{s}$ will be the same

for (8) Integrate the charge distribution over the defined volume...

So $\int_V \frac{\rho}{\epsilon} dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \frac{\rho_0}{\epsilon} R^2 \sin\theta \, dR \, d\theta \, d\phi$

$+ \int_0^{2\pi}\int_0^{\pi}\int_a^{r} \frac{\rho_0}{\epsilon} R^2 \sin\theta \, dR \, d\theta \, d\phi$

The charge density is zero for $R > a$
(As stated earlier in the Example)

$$\int_V \frac{\rho}{\epsilon} dv = \frac{\rho_0}{\epsilon} \frac{4\pi a^3}{3}$$

9 set LHS = RHS and find $\bar{E}$

$$E_R 4\pi R^2 = \frac{\rho_0}{\epsilon} \frac{4\pi a^3}{3} \rightarrow E_R = \frac{\rho_0}{\epsilon} \frac{a^3}{3R^2}$$

E_R falls off for $1/R^2$

At Schenectady this Looks
like a point charge.

So we have

$$\bar{E} = \frac{\rho_0}{\epsilon} \frac{a^3}{3R^2} \hat{R} \quad \text{for } a < R$$

$$\bar{E} = \begin{cases} \frac{\rho_0}{\epsilon} \frac{R}{3} \hat{R} & R < a \\ \frac{\rho_0}{\epsilon} \frac{a^3}{3R^2} \hat{R} & a < R \end{cases}$$

Look at Divergence

$$\nabla \cdot \bar{E} = \frac{\rho}{\epsilon}$$ ⟵ charge density

for $R < a \Rightarrow \nabla \cdot \bar{E} = \frac{\rho_0}{\epsilon}$

To show this

$$\nabla \cdot E = \frac{\partial}{\partial R} E_R = \frac{1}{R^2} \frac{\partial}{\partial R} \left(R^2 E_R\right)$$

$$= \frac{1}{R^2} \frac{\partial}{\partial R} R^2 \left(\frac{\rho_0}{\epsilon} \frac{R}{3}\right) = \frac{1}{R^2} \frac{\partial}{\partial R} \frac{\rho_0}{\epsilon} \frac{R^3}{3}$$

$$= \frac{1}{R^2} \frac{\rho_0}{\epsilon} R^2 = \frac{\rho_0}{\epsilon}$$

Let's use a real number

Spherical Symmetry, $\rho_s = \rho_0$ at $R = 6$

charge distribution is a spherical shell C/m^2

~~to~~

$$\bar{\mathcal{E}} = \begin{cases} 0 & R < b \\ \underline{?} & b < R \end{cases}$$

from before, we note

$$\oint \bar{\mathcal{E}} \cdot d\bar{s} = 4\pi R^2 \mathcal{E}_R$$

To get $\int_V \frac{\rho}{\epsilon} dV \Rightarrow [4\pi b^2][\frac{\rho_0}{\epsilon}]$

Surface area of a spherical shell

uniform surface charge density

Set LHS = RHS

$$4\pi R^2 \mathcal{E}_R = 4\pi b^2 \frac{\rho_0}{\epsilon}$$

$$\mathcal{E}_R = \frac{b^2}{R^2} \frac{\rho_0}{\epsilon}$$

~~the~~ surface charge density forces charge discontinuity

Electric Field, $\vec{\mathcal{E}}$

Displacement Field, $\vec{D}$

$$\vec{D} = \epsilon \vec{\mathcal{E}}$$

permittivity

$$\nabla \cdot \vec{\mathcal{E}} = \frac{\rho}{\epsilon}$$

$$\nabla \cdot \vec{D} = \rho$$ ← not dependent on materials

Find $\vec{D}$ and divide by ϵ