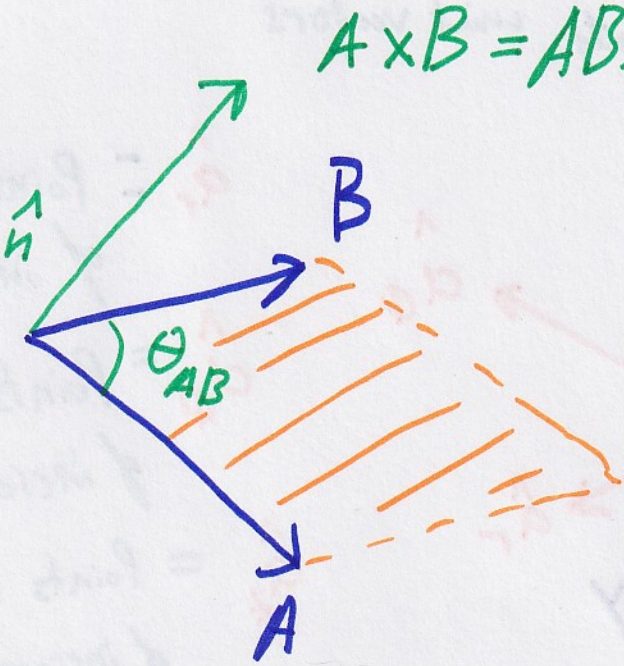
Fields and Waves I L9 part II

Vectors Curl Divergence

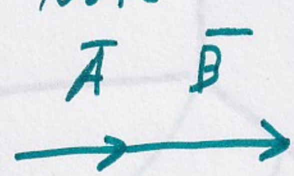
Cartesian coordinates Cross product

$$\bar{A} \times \bar{B} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$A \times B = AB \sin\theta \, \hat{n}$$



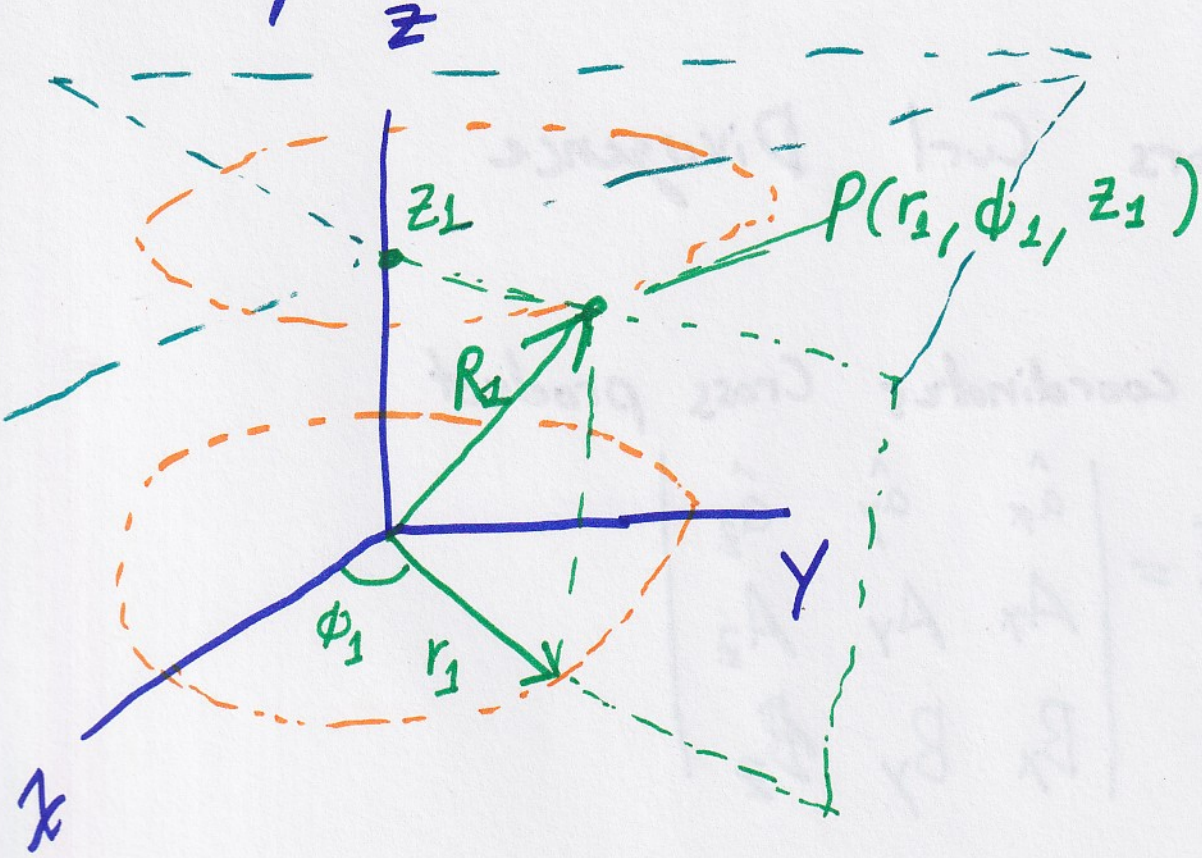
Note



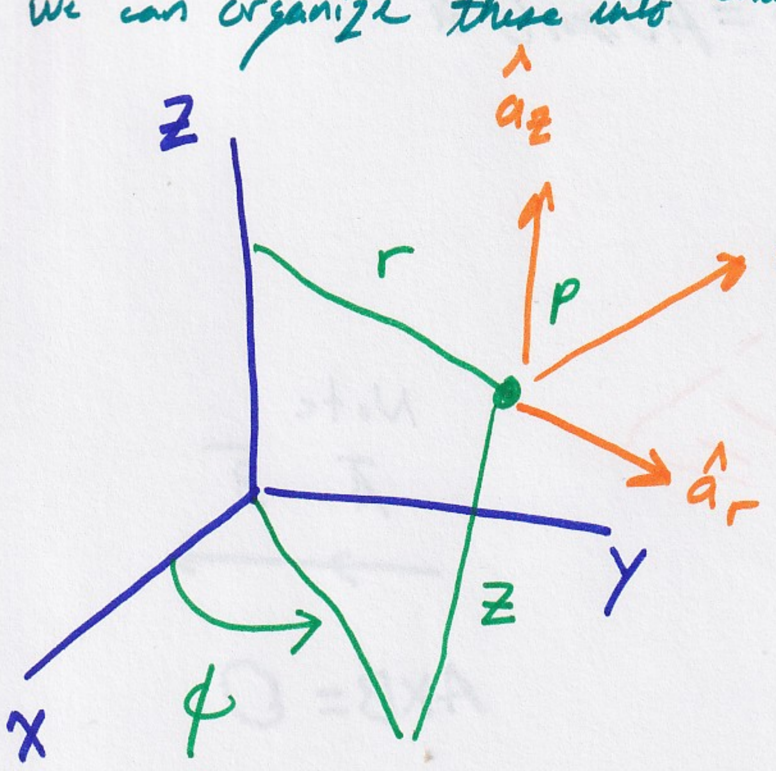
$$A \times B = 0$$

(cuz of the $\sin\theta$)

Recall from part I, Cylindrical coordinates



we can organize these into unit vectors



$\hat{a}_r$ = Points in direction of increasing r

$\hat{a}_\phi$ = Points in direction of increasing ϕ

$\hat{a}_z$ = Points in direction of increasing Z

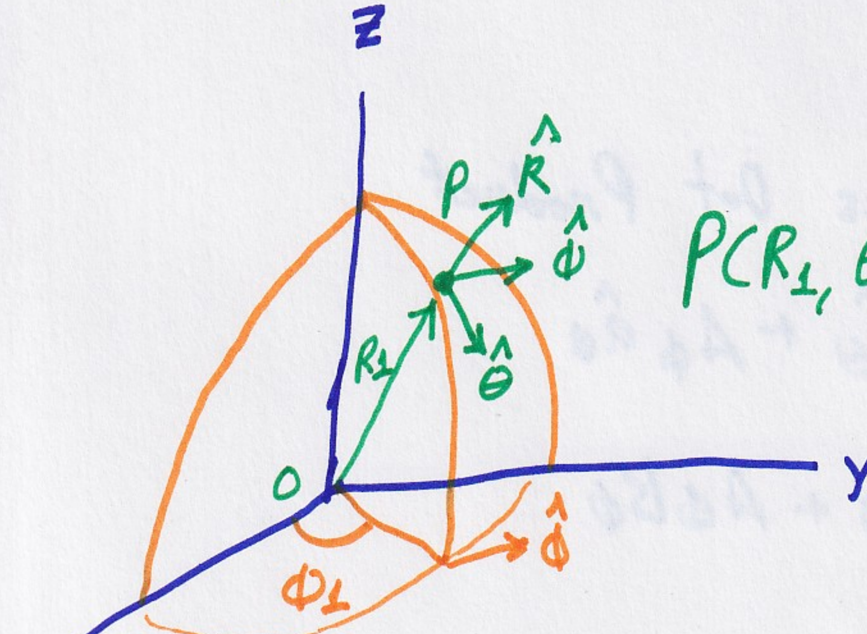
In cylindrical coordinates, both $\hat{a}_r$ and $\hat{a}_\phi$ are functions of ϕ

And for $\bar{A} = A_r \hat{a}_r + A_\phi \hat{a}_\phi + A_z \hat{a}_z$

we can see that

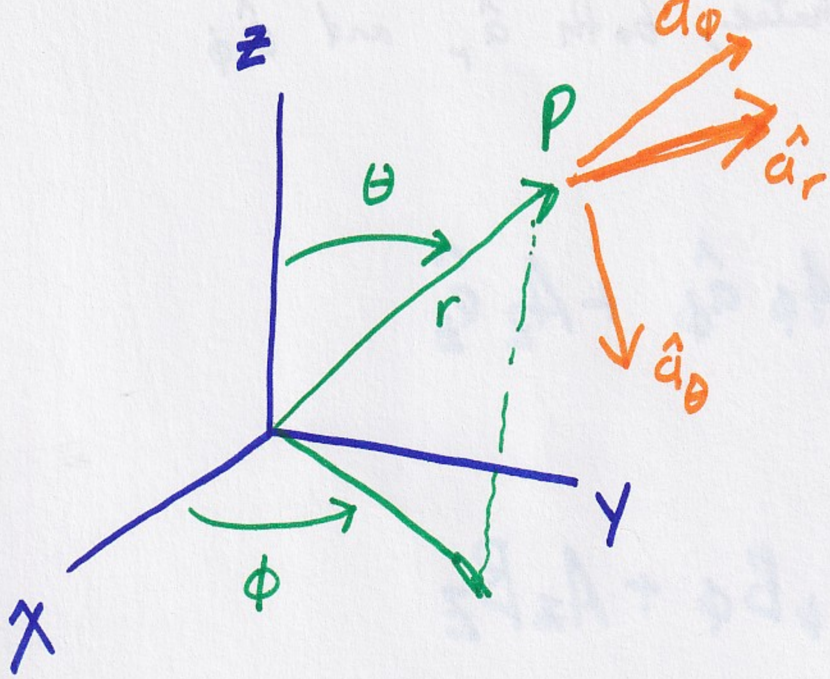
$$\bar{A} \cdot \bar{B} = A_r B_r + A_\phi B_\phi + A_z B_z$$

Look at Spherical coordinates



$P(R_1, \Theta_1, \phi_1)$

$$\overrightarrow{OP} = R_1 \hat{R} \quad \text{for } P(R_1, \Theta_1, \phi_1)$$



$\hat{a}_r$ = Points in the direction of increasing r

$\hat{a}_\theta$ = Points in the direction of increasing θ

$\hat{a}_\phi$ = Points in the direction of increasing φ

In spherical coordinates, $\hat{a}_r$, $\hat{a}_\theta$, and $\hat{a}_\phi$ are functions of φ and θ

Spherical Coordinates Dot Product

$$\bar{A} = A_r \hat{a}_r + A_\theta \hat{a}_\theta + A_\phi \hat{a}_\phi$$

$$\bar{A} \cdot \bar{B} = A_r B_r + A_\theta B_\theta + A_\phi B_\phi$$

Note

Cylindrical	Spherical
$\hat{a}_r$ $\hat{a}_\phi$ $\hat{a}_z$	$\hat{a}_r$ $\hat{a}_\theta$ $\hat{a}_\phi$

Example spherical coordinate systems

$$\bar{A} = r^3 \hat{a}_r + 4 \sin\theta \, \hat{a}_\phi$$

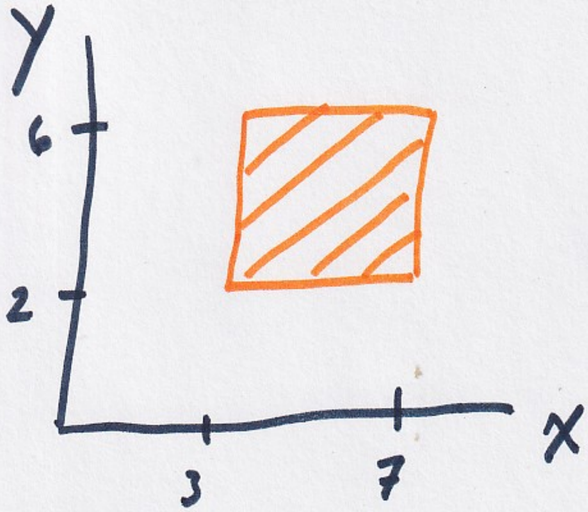
$$\bar{B} = 3\hat{a}_r + 2\hat{a}_\theta + \frac{12 \sin\theta}{r^3} \hat{a}_\phi$$

$$\bar{A} \cdot \bar{B} = r^3(3) + 0 \cdot 2 + \frac{(4)(12)\sin^2\theta}{r^3}$$

$$\bar{A} \times \bar{B} = \begin{vmatrix} \hat{a}_r & \hat{a}_\theta & \hat{a}_\phi \\ r^3 & 0 & 4\sin\theta \\ 3 & 2 & \frac{12\sin\theta}{r^3} \end{vmatrix} = -8\sin\theta \, \hat{a}_r - \left[12\sin\theta - r^3 \frac{12\sin\theta}{r^3}\right](\hat{a}_\theta) + 2r^3 \hat{a}_\phi$$

Differential calculus

Example



Length $\int_3^7 dx = x\Big|_3^7 = 7 - 3 = 4$

Area $\int_3^7 \int_2^6 dy\,dx = \int_3^7 4 \, dx = 16$

This is integration over 2 'delta' distances

$\rightarrow \underset{\Delta x}{\square}\,\Delta y$ or $\underset{dx}{\square}\,dy$

Differential lengths

1.) Rectangular Coordinates

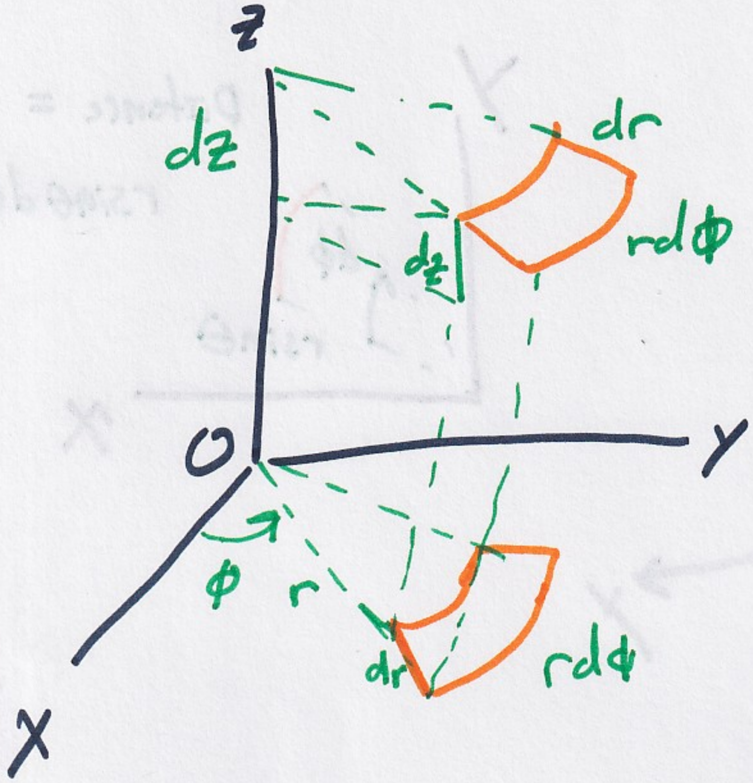
when you move a small amount in x-direction, the distance is $\underline{dx}$ (Units in 'meters')

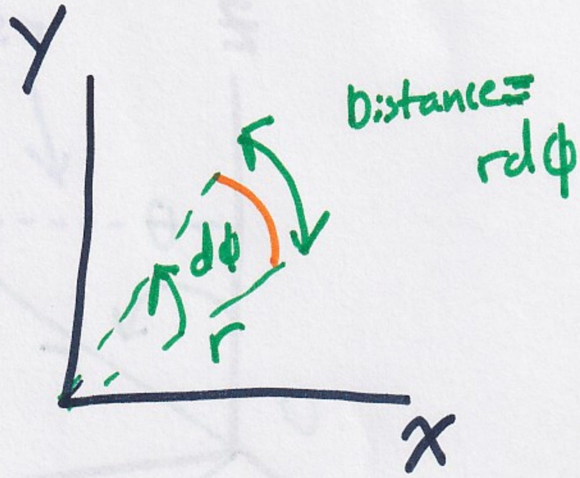
In a similar fashion we see that we can generate dy and dz

Differential Distances

(dx, dy, dz)

2. Cylindrical Coordinates

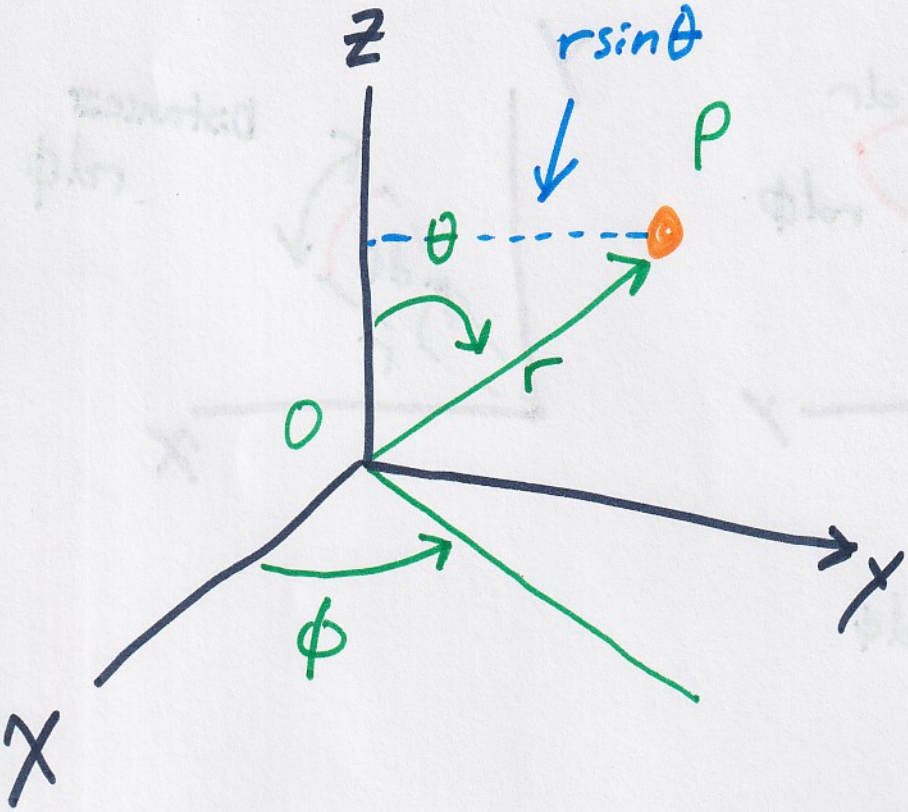


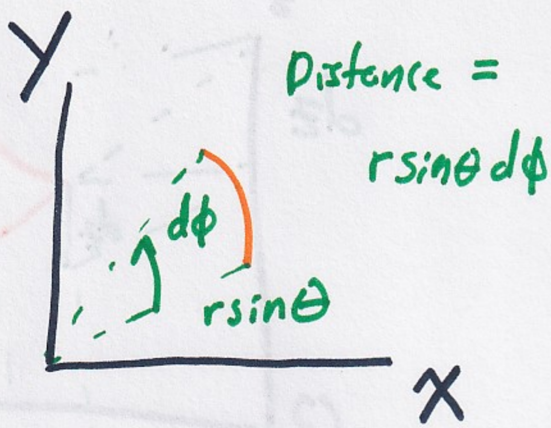


Differential distances

$(dr, rd\phi, dz)$

3.) Spherical Coordinates





Differential Distances

$(dr,\ r\,d\theta,\ r\sin\theta\, d\phi)$

Representation of differential lengths $\cancel{d\bar{l}}$ in the 3 coordinate systems $d\bar{l}$

rectangular

$$d\bar{l} = dx \cdot \hat{a}_x + dy \cdot \hat{a}_y + dz \cdot \hat{a}_z$$

cylindrical

$$d\bar{l} = dr \cdot \hat{a}_r + r \cdot d\phi \cdot \hat{a}_\phi + dz \cdot \hat{a}_z$$

spherical

$$d\bar{l} = dr \cdot \hat{a}_r + r \cdot d\theta \cdot \hat{a}_\theta + r \cdot \sin\theta \cdot d\phi \cdot \hat{a}_\phi$$

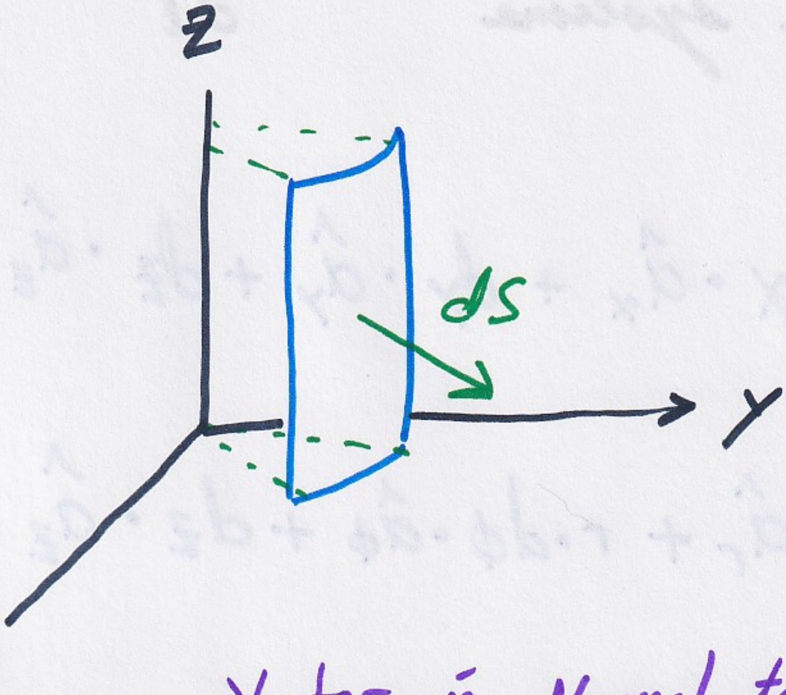
Differential surface and volumes

Example of surface differentials

$$ds = dx\, dy\, \hat{a}_z \quad \text{or} \quad ds = r d\phi \cdot dz\, \hat{a}_r$$

So how do we set ~~up~~ this up?

Representation of differential surface element

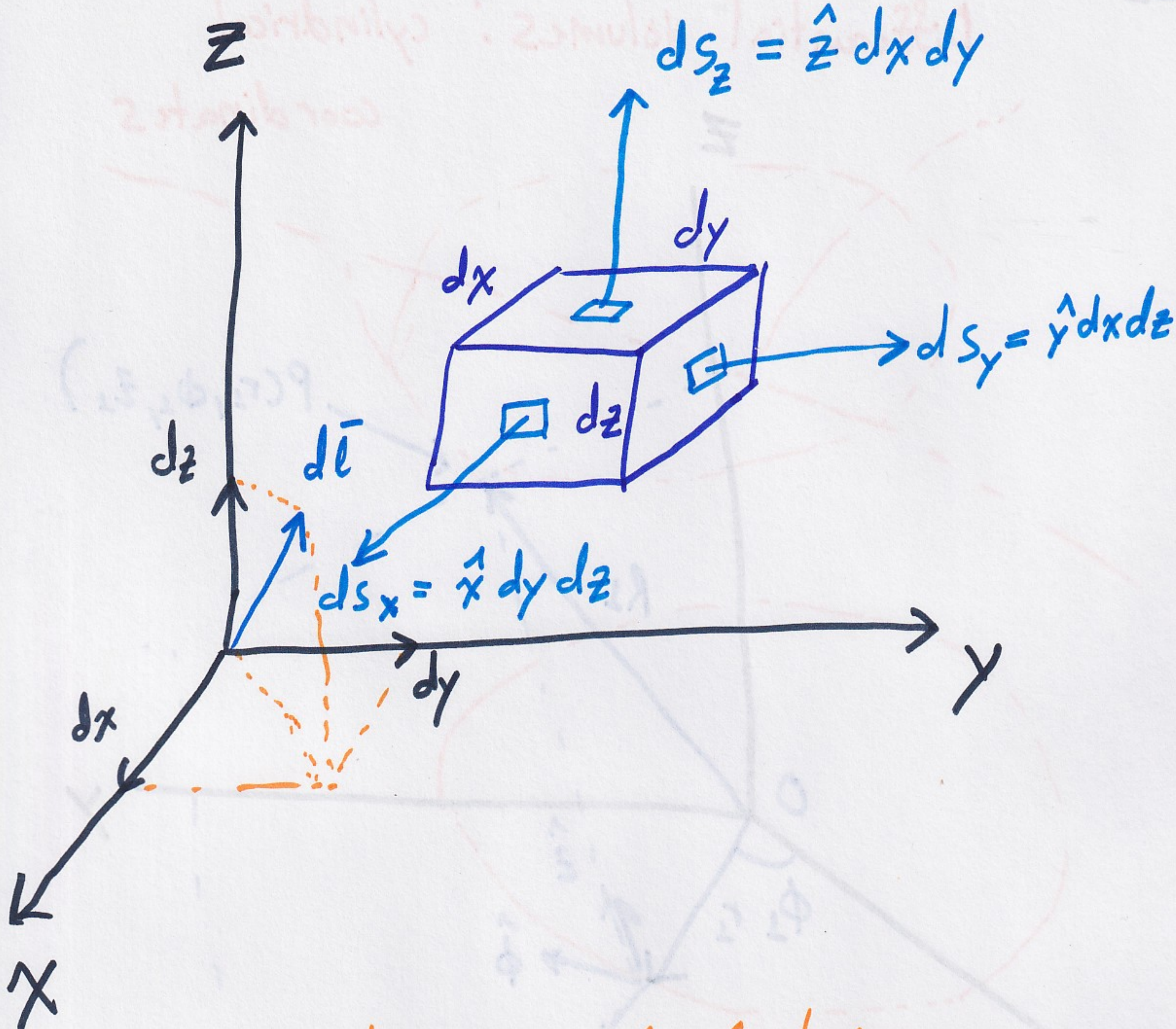


Vector is Normal to surface

$$ds = dx\,dy\,\hat{a}_z$$

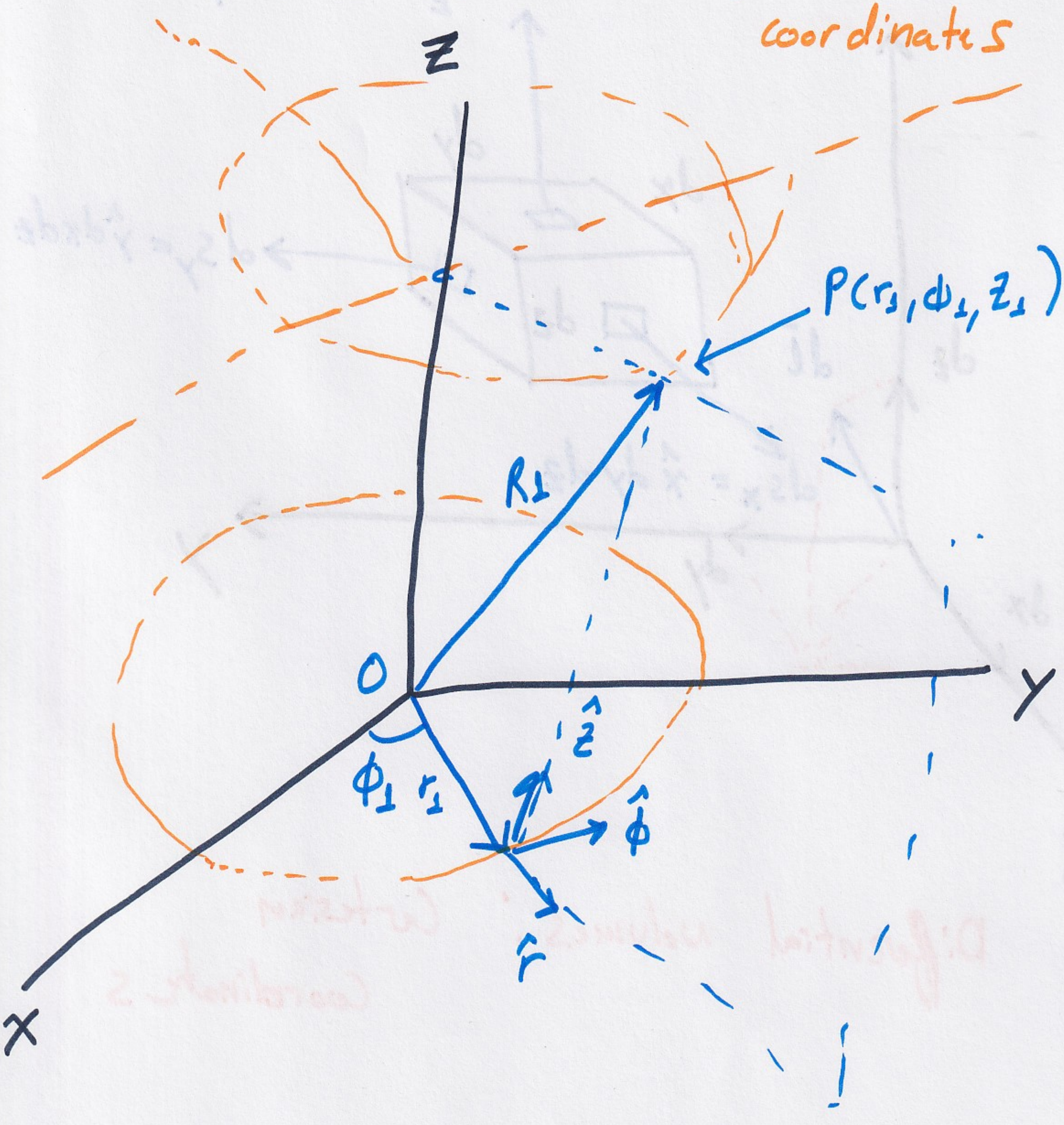
Differential Volume (a Scalar)

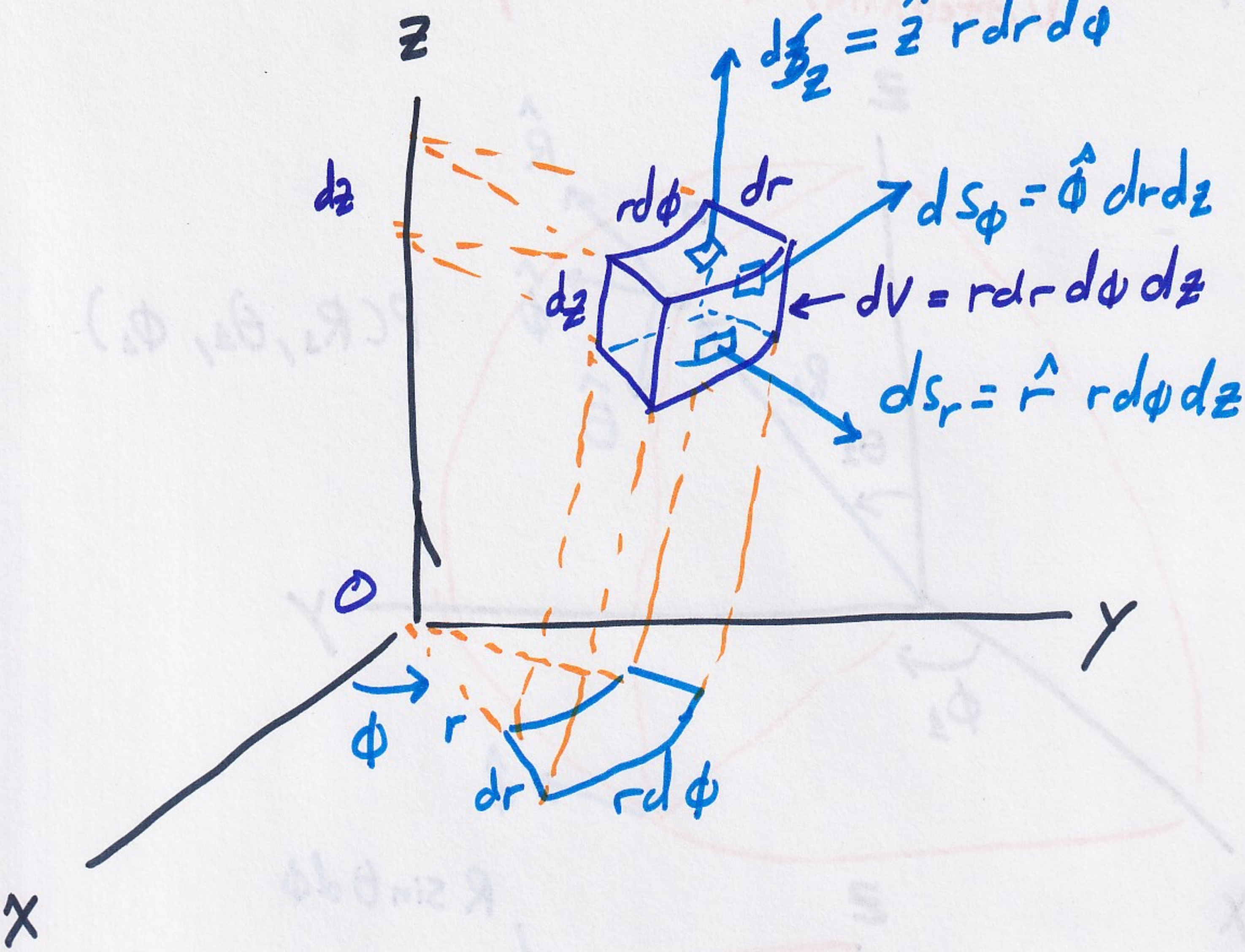
$$dv = dx\,dy\,dz$$



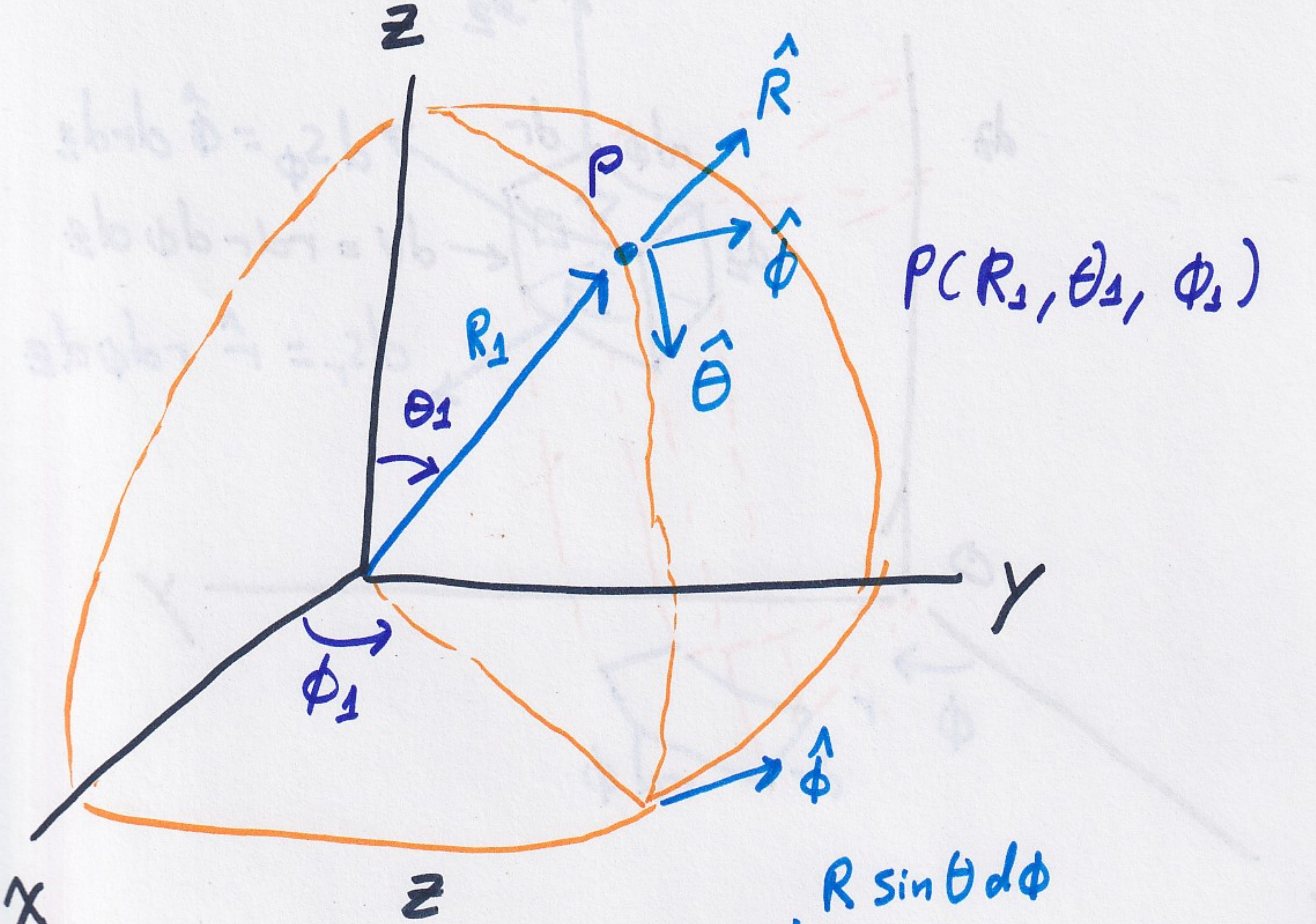
Differential volumes: Cartesian Coordinates

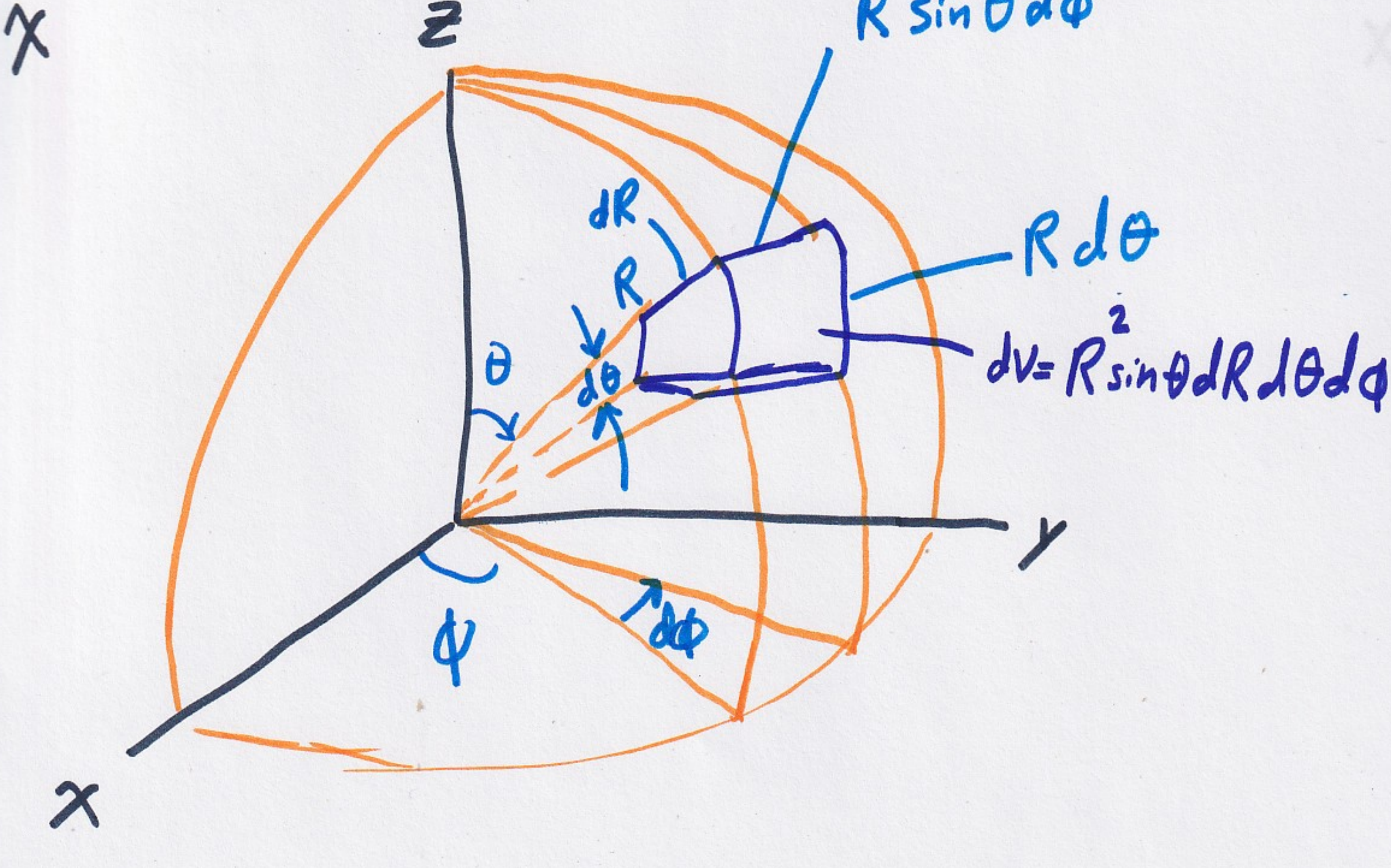
Differential volumes: cylindrical coordinates




Z
$d\cancel{s}_z = \hat{z}\, r\,dr\,d\phi$
dz
$r\,d\phi$
dr
$dS_\phi = \hat{\phi}\, dr\,dz$
dz
$dV = r\,dr\,d\phi\,dz$
$ds_r = \hat{r}\, r\,d\phi\,dz$
O
Y
ϕ
r
dr
$r\,d\phi$
X

Differential volumes : spherical coordinates





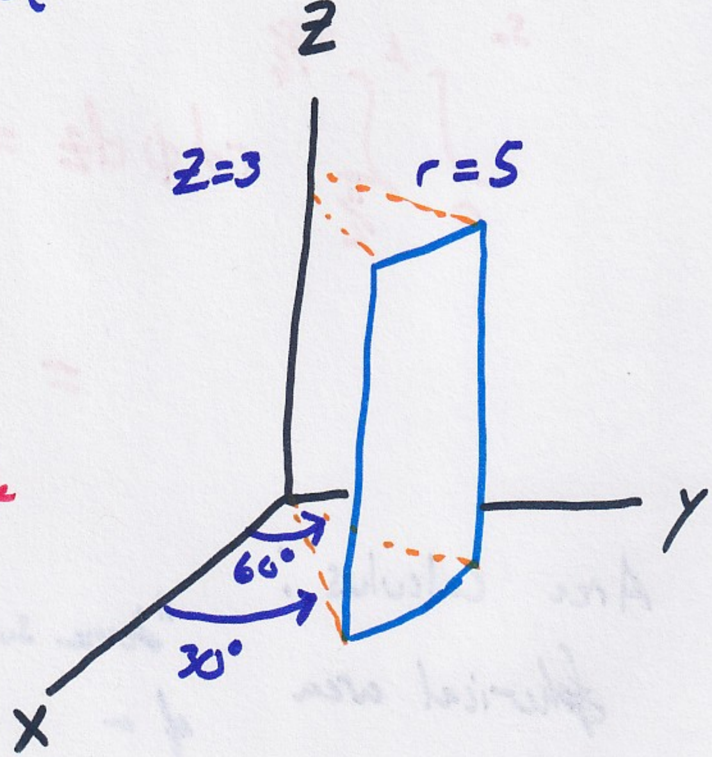
Integrals Calculations

Area calculus:

- find the following area
- Cylindrical area

-$r=5$, $30° < \phi < 60°$

$0 < z < 3cm$

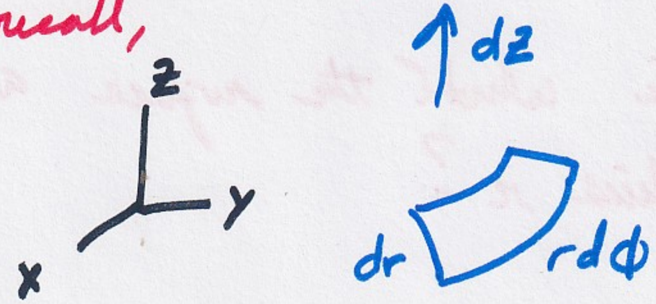


Basically, what's the surface area with radius r and length l.

so choose the normal to the surface

$$d\bar{s} = ds_r = \hat{r}\, r\, d\phi\, dz$$

we choose the normal (from the cylindrical coordinates) because recall,



Area is l(w)

so $(r d\phi)(dz)$

So we set up the Integral

$$30 \times \frac{\pi}{180} = \frac{\pi}{6} \quad , \quad 60 \times \frac{\pi}{180} = \frac{2\pi}{6}$$

So

$$\int_0^{l} \int_{\pi/6}^{2\pi/6} r\,d\phi\,dz = r\left[\frac{2\pi}{6} - \frac{\pi}{6}\right] l = \frac{r\pi}{6} l$$

$$= \frac{(5)\pi}{6}(3) \quad cm^2$$

Area Calculus:

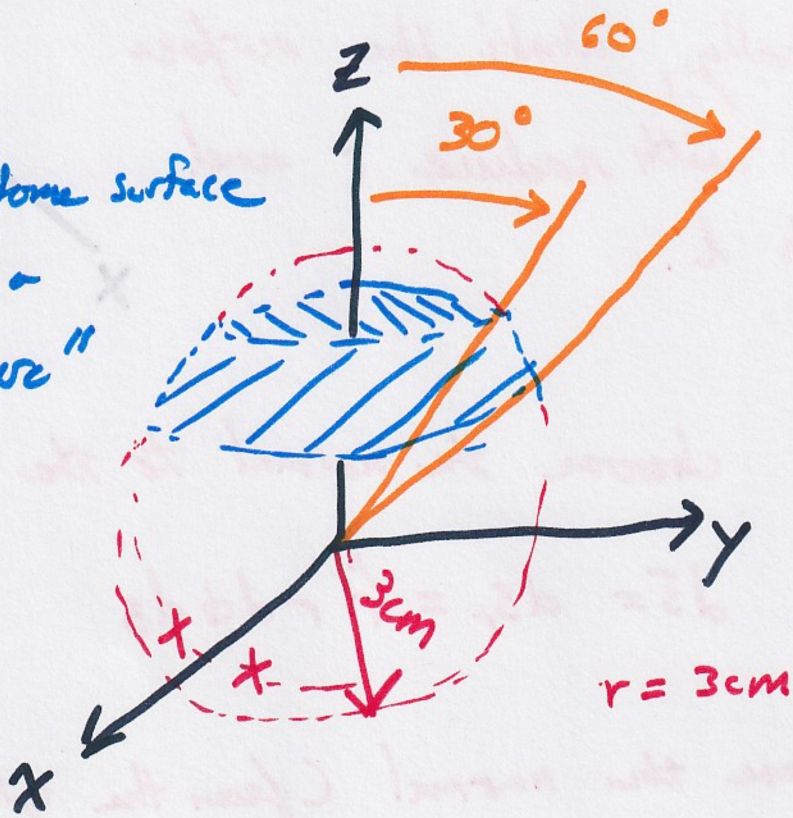
spherical area

$30^\circ < \theta < 60^\circ$

$0 < \phi < 2\pi$

$r = 3cm$

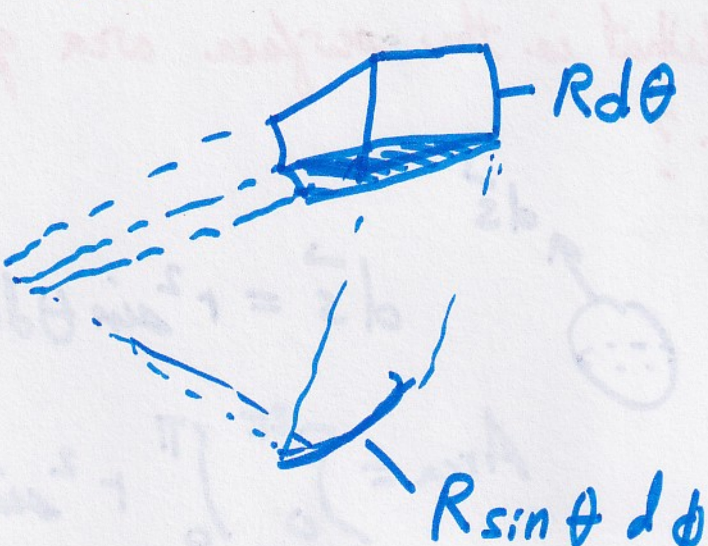
"Some surface of a sphere"



So the Question is what's the surface area of a sphere of radius r?

The Idea

recall



So we can say $d\bar{s} = (R\sin\theta\, d\phi)(R\, d\theta)\ \hat{a}_r$

$$= R^2 \sin\theta\, d\theta\, d\phi\ \hat{a}_r$$

direction going outward

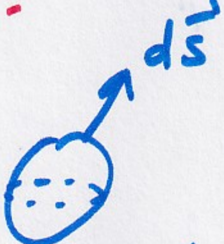
$$\text{Area} = \int_{\phi}\int_{\theta} R^2 \sin\theta\, d\theta\, d\phi$$

$$= \int_0^{2\pi}\int_{\frac{\pi}{6}}^{2\pi/6} R^2 \sin\theta\, d\theta\, d\phi$$

$$= \int_0^{2\pi} R^2\left(-\left[\cos\tfrac{\pi}{3} - \cos\tfrac{\pi}{6}\right]\right) d\phi$$

$$= R^2(0.366)\, 2\pi \quad = 9(0.366)\, 2\pi \text{ cm}^2$$

a) What is the surface area of a sphere of radius r?



$$d\vec{s} = r^2 \sin\theta \, d\theta \, d\phi \, \hat{a}_r$$

$$\text{Area} = \int_0^{2\pi} \int_0^{\pi} r^2 \sin\theta \, d\theta \, d\phi$$

$$= r^2 \left(-\cos\theta \Big|_0^{\pi} \right) 2\pi = \underline{4\pi r^2}$$

b) What is the surface area of the side of a cylinder with radius r and length l?

(some write $\vec{ds}$)

$$d\vec{s} = r \, d\phi \, dz \, \hat{a}_r$$

$$\text{Area} = \int_0^{l} \int_0^{2\pi} r \, d\phi \, dz = r 2\pi l$$

c) what is the volume of a sphere of radius r?

"Think" volume!

$$\underbrace{(R\sin\theta\, d\phi)(R\,d\theta)}_{\text{from surface area}}(dR)$$

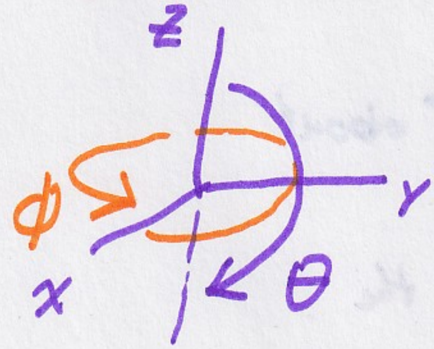
we are going outward from the origin to R (radius)

$$\text{Volume} = \int_0^{2\pi}\int_0^{\pi}\int_0^{r} r^2 \sin\theta \, dr\, d\theta\, d\phi$$

$$= \frac{r^3}{3}\left(-[-1-1]\right)2\pi = \frac{4\pi}{3}r^3$$

1st we go out to radius

2nd we then integrate from +Z axis to −Z axis



3rd we ~~go~~ integrate from X 360° to X again

d) What is the volume of a cylinder of radius r and length l?

so

$$\underbrace{(r\,d\phi)(dz)}_{\text{from surface area}}\,(dr)$$

we are going away from z (axis) to radius r

so $dV = r\,dr\,d\phi\,dz$

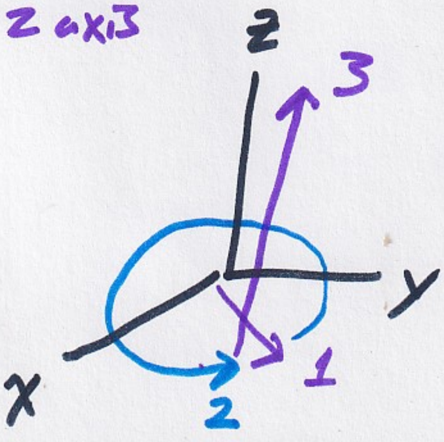
$$\text{Volume} = \int_0^{l}\int_0^{2\pi}\int_0^{R} r\,dr\,d\phi\,dz$$

$$= \frac{R^2}{2}(2\pi)\,l = \pi R^2 l$$

1st, Integrate from z axis to r

2nd Integrate 360° about ~~the~~ X axis

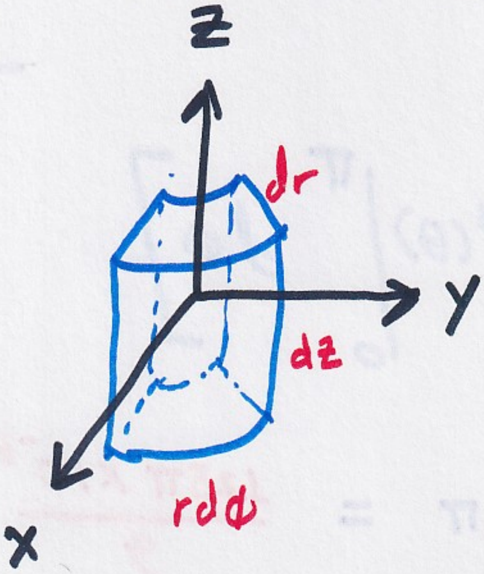
3rd Integrate along the z axis



Volume in cylindrical coordinates

What is the volume with boundaries

$1 < r < 3$, $0 < \phi < \pi/3$, $-2 < z < 2$



$$\rightarrow \int dV = \int_{-2}^{2}\int_{0}^{\pi/3}\int_{1}^{3} r\,dr\,d\phi\,dz$$

$$= \frac{1}{2}\left[3^2-1^2\right]\left(\frac{\pi}{3}\right)\left[2--2\right]$$

$$= \frac{16\pi}{3}$$

The electric charge density inside a sphere is given by $4\cos^2(\theta)$. Find the total charge Q contained in a sphere of radius 2cm.

So recall $dV = r^2 \sin\theta \, dr\, d\theta\, d\phi$

recall density $\rho = \frac{m}{V}$ or for charge, $\rho = \frac{Q}{V}$

But charge depends on the variable θ

$\rightarrow$ So put $4\cos^2(\theta)$ in the integral

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{0.02} 4\cos^2(\theta)\, r^2 \sin(\theta)\, dr\, d\theta\, d\phi$$

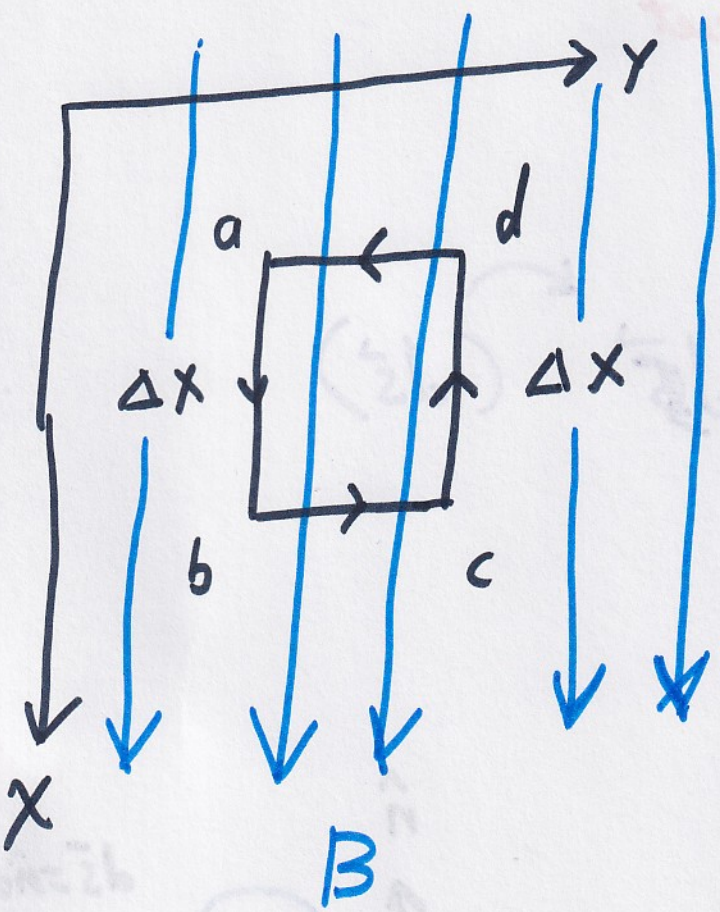
$$= 4\frac{r^3}{3}\Big|_0^{0.02}\left[\int_0^{2\pi}\int_0^{\pi} \cos^2(\theta)\sin(\theta)\, d\theta\, d\phi\right]$$

$$= \frac{4}{3}\left[8\times10^{-6}\right]\left[\int_0^{2\pi} -\frac{\cos^3(\theta)}{3}\Big|_0^{\pi} d\phi\right]$$

$$= \frac{32\times10^{-6}}{3}\left[\frac{-(-1-1)}{3}\right] 2\pi = \frac{128\pi\times10^{-6}}{9}\, C$$

The curl, gradient and divergence operators

Note: $\oint$ implies a <u>closed Loop</u> Integral



$\oint \vec{B} \cdot dl$ measures circulation or Curl of B

Example of a uniform field B in the x direction

Circulation $= \oint \vec{B} \cdot dl$

$$= \int_a^b B\hat{x} \cdot \hat{x}\,dx + \int_b^c B\hat{x} \cdot \hat{y}\,dy + \int_c^d B\hat{x} \cdot \hat{x}\,dx$$

$$+ \int_d^a B\hat{x} \cdot \hat{y}\,dy = 0$$

The curl operator

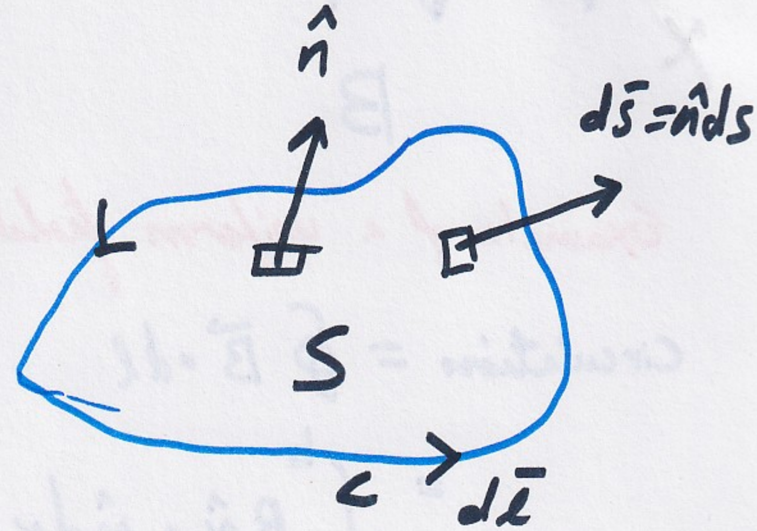
Note: $\nabla \times \vec{B}$ yields a vector

This phenomenon does not give us a cross ~~product~~ product

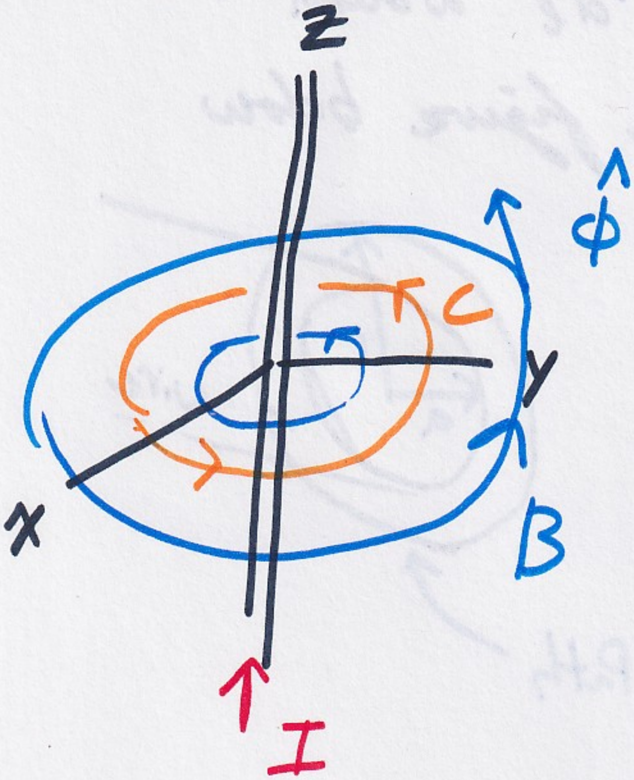
Stokes's Theorem

$$\oint \bar{B} \cdot d\vec{\ell} = \iint (\nabla \times \bar{B}) \cdot d\vec{s} \quad (d\vec{s})$$

This surface integral has its surface enclosed by the line on the LHS (Left hand side)



Current flow in a wire



We can write,

$$\nabla \times \bar{B} = \mu_0 J$$

(curl of $\bar{B}$)

and

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I$$

Circulation

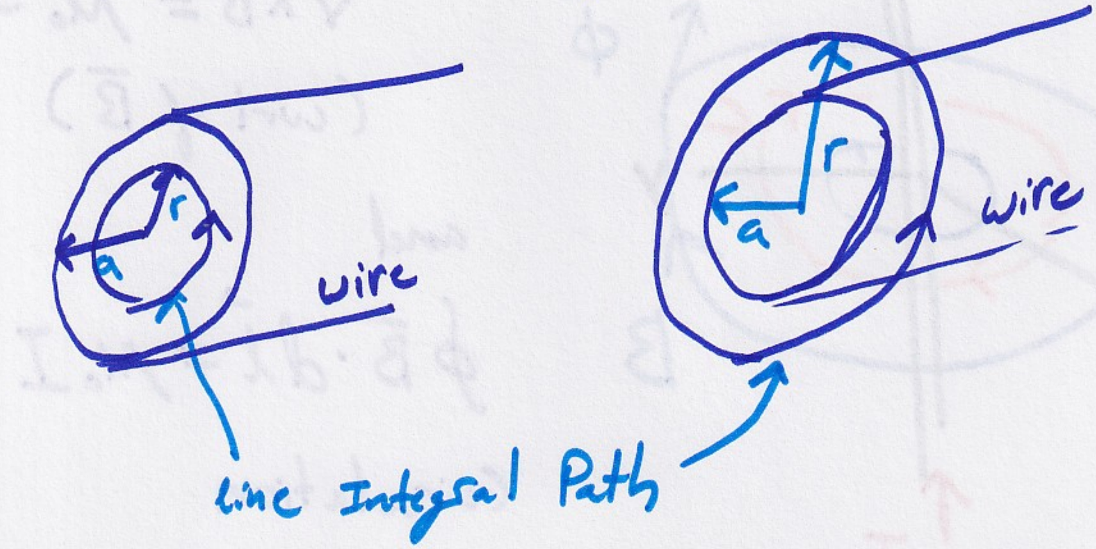
Rotation / Curl operator Examples

The magnetic field of a straight wire of radius a which has a constant current density J_0, is given by:

$$\bar{B} = \frac{\mu_0 J_0 r}{2} \hat{a}_\phi \qquad r < a \text{ (inside the wire)}$$

$$\bar{B} = \frac{\mu_0 J_0 a^2}{2r} \hat{a}_\phi \qquad r > a \text{ (outside the wire)}$$

a) Calculate $\oint \bar{B} \cdot d\bar{l}$ around the 2 Paths shown in the figure below



for $r < a$

$$\oint \vec{B} \cdot d\vec{l} = \int_0^{2\pi} B_\phi \, r \, d\phi = \int_0^{2\pi} \frac{\mu_0 J_0 r}{2} r \, d\phi$$

$$= \frac{\mu_0 J_0 r^2}{2} 2\pi = \mu_0 J_0 \pi r^2$$

for $r > a$

$$\oint \vec{B} \cdot d\vec{l} = \int_0^{2\pi} \frac{\mu_0 J_0 a^2}{2r} r \, d\phi = \mu_0 J_0 \pi a^2$$

b. Calculate $\nabla \times \overline{B}$ for both regions

$$\nabla \times \overline{B} = \frac{1}{r}\begin{vmatrix} \hat{a}_r & \hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ 0 & rB & 0 \end{vmatrix} = \frac{1}{r}\left[\hat{a}_r\left(-\frac{\partial}{\partial z} rB\right) - \hat{a}_\phi(0) + \hat{a}_z\left(\frac{\partial}{\partial r} rB\right)\right]$$

$$= \frac{1}{r}\frac{\partial}{\partial r} rB \; \hat{a}_z$$

for $r < a$

$$\nabla \times \vec{B} = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\mu_0 J_0 r}{2} \hat{a}_z$$

$$= \frac{\mu_0 J_0}{2r}\frac{\partial}{\partial r} r^2 \hat{a}_z = \mu_0 J_0 \hat{a}_z$$

for $r > a$

$$\nabla \times \vec{B} = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\mu_0 J_0 a^2}{2r} \hat{a}_z = 0$$

Example

Calculate $\int \nabla\times\vec{B}\cdot d\vec{s}$ over the two shaded surfaces and compare to the previous result.

Now that we calculated $\nabla\times\vec{B}$, we can now use Stokes's Theorem to see that the result is the same as $\oint \vec{B}\cdot d\vec{l}$!

$r<a$

$$\int \nabla\times\vec{B}\cdot d\vec{s} = \int_0^{2\pi}\int_0^{R} \mu_0 J_0 \hat{a}_z \cdot r\,dr\,d\phi\, \hat{a}_z$$

$$= \mu_0 J_0 \frac{R^2}{2}\int_0^{2\pi} d\phi = \mu_0 J_0 \pi R^2$$

same as $\oint \vec{B}\cdot d\vec{l}$ for $r<a$

$r>a$ we note that the loop ~~and~~ encloses the surface with $r<a$ and $r>a$

so we are left with

$$\int \nabla\times\vec{B}\cdot d\vec{s} = \int_0^{2\pi}\int_0^{a} \mu_0 J_0\, r\,dr\,d\phi + \int_0^{2\pi}\int_0^{R} 0\cdot r\,dr\,d\phi$$

This is the $r>a$ contribution

so ultimately

$$= \mu_0 J_0 \pi a^2$$

we have stokes's theorem

$$\oint \vec{B} \cdot d\vec{l} = \int \nabla \times \vec{B} \cdot d\vec{s}$$

Gradient Operator

Gradient measures change in a scalar field

- result is a vector pointing in the direction of Increase

Ex Cartesian System

$$\nabla f = \frac{\partial f}{\partial x} \cdot \hat{a}_x + \frac{\partial f}{\partial y} \cdot \hat{a}_y + \frac{\partial f}{\partial z} \cdot \hat{a}_z$$

Note $\nabla \times \nabla f = 0$ Always

if $F = \nabla f$, $\nabla \times F = 0$ & $\oint F \cdot dl = 0$

because of stokes

Gradient examples

Compute the gradient of the following functions

a. $f = 8a^2\cos\phi + 2rz$ (cylindrical)

$$\nabla f = \frac{\partial f}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial f}{\partial \phi}\hat{a}_\phi + \frac{\partial f}{\partial z}\hat{a}_z$$

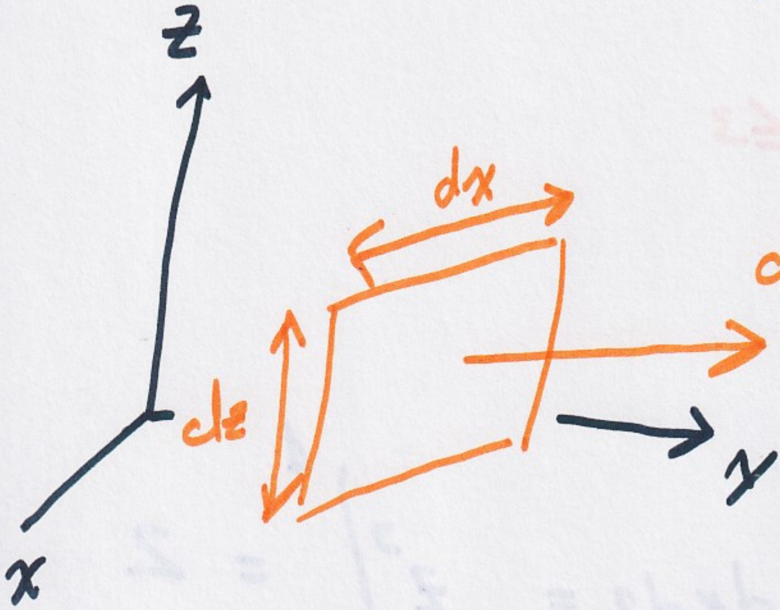
$$= 2z\,\hat{a}_r + \frac{1}{r}(-8a^2\sin\phi)\,\hat{a}_\phi + 2r\,\hat{a}_z$$

b. $f = \dfrac{a\cos 2\theta}{r}$ (spherical)

$$\nabla f = \frac{\partial f}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{a}_\phi$$

$$= -a\cos 2\theta\,(r^{-2})\,\hat{a}_r + \frac{1}{r}\frac{a}{r}(-2\sin 2\theta)\,\hat{a}_\theta + 0$$

Surface Integrals



$d\vec{s} = dx \cdot dz \cdot \hat{a}_y$

all 3 coordinates are involved

$\int \bar{A} \cdot d\bar{s}$ measures flux of A

through a surface

Example Fluid FLOW

for $v \parallel d\bar{s}$, we have fluid flow

but $v \perp d\bar{s}$, no flow

Hence $v \cdot ds$ measures flux

Example

$y=2$

$x=0\sim 3$, $0\le x\le 3$

$-1\le z\le 1$

$\bar{A} = xy\cdot\hat{a}_x + z^2\cdot\hat{a}_y$

so $\int \vec{A}\cdot d\vec{s} = \int_{-1}^{1}\int_{0}^{3} z^2\,dx\,dz = z^3\Big|_{-1}^{1} = 2$

we have the contribution from $\hat{a}_y$ only

Understand ... Divergence operator

$\int \bar{A}\cdot d\bar{s}$ measures flux thru any surface

$\oint \bar{A}\cdot d\bar{s}$ measures flux thru closed surfaces

$\nabla\cdot\bar{A}$ is a 'local' measure of flux property

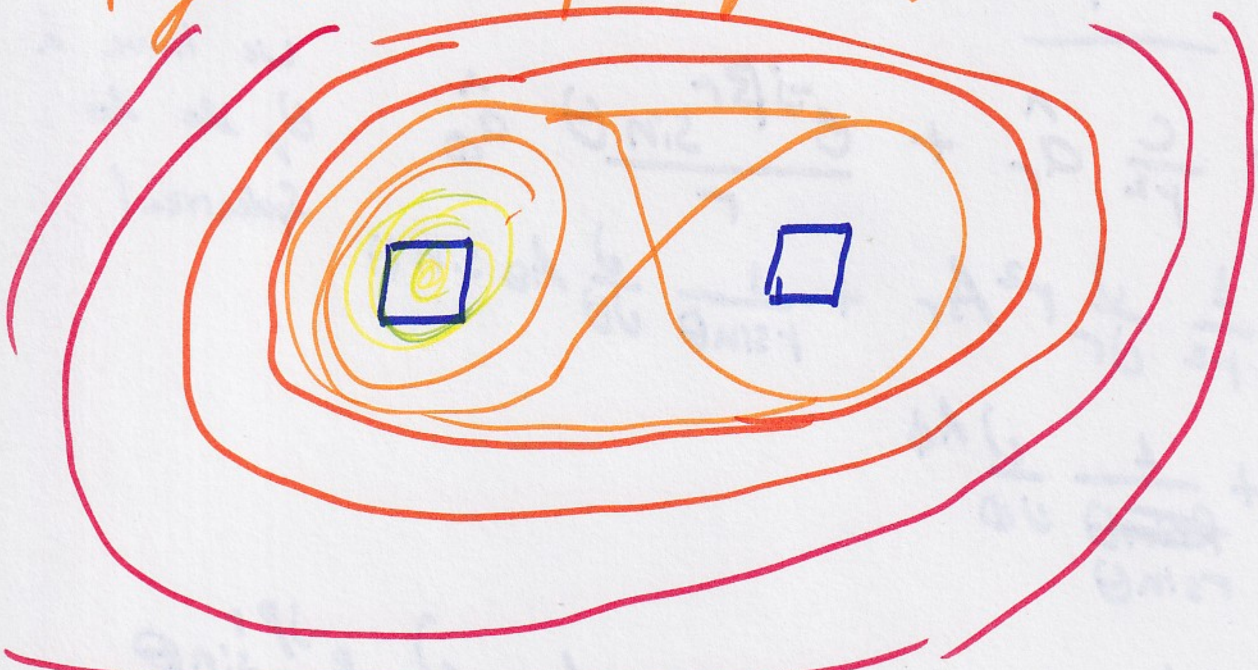
→ We will get a scalar

~~Divergence~~ Divergence Theorem

$$\oint \vec{A} \cdot d\vec{s} = \int (\nabla \cdot \bar{A}) \cdot dV$$

Volume Integral on right (RHS)
is the volume enclosed by Surface on LHS

Physical Example of Divergence Operator



we have 2 chips, Temperatures in colour

Integral heat flux through the surface of a chip

=

Amount of heat included in its volume

Calculate $\nabla\cdot\bar{A}$ for each of the vectors below

a) $\bar{A} = x^2 y\,\hat{a}_x + c^2 x\,\hat{a}_z$ $\quad$ c, β are constants

$$\nabla\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$= \frac{\partial}{\partial x} x^2 y + 0 + \frac{\partial}{\partial z} c^2 x$$

$$= \underline{2xy}$$

b) $\bar{A} = \frac{c}{r^2}\hat{a}_r + \frac{e^{-j\beta r}\sin\theta}{r}\hat{a}_\phi$ $\quad$ we have a θ, so do spherical

$$\nabla\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 A_r + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta} A_\theta \sin\theta$$

$$+ \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} A_\phi$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{c}{r^2} + 0 + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\frac{e^{-j\beta r}\sin\theta}{r}$$

$$= 0$$

Summary

Curl $(\nabla \times \bar{B}) \rightarrow$ vector

Measures the circulation of a vector field

Gradient $\nabla f \longrightarrow$ Vector

Measures the change in a scalar field

Divergence $\nabla \cdot A \longrightarrow$ Scalar

Measures the flux of a vector field through a surface