Fields and Waves I L9

Vectors Curl Divergence

In electrostatics, we will need the math to be able to Interpret the following...

$\nabla \cdot \bar{B} = 0$ $\nabla \times \bar{H} = \frac{\partial}{\partial t} \bar{D}$

$\nabla \cdot \bar{D} = \rho$ $\nabla \times \bar{E} = 0$

Ex)

$\nabla \cdot \bar{A} \rightarrow$ Divergence ~~E~~ $\rightarrow$ Measure of slope (change) of a field (scalar)

$\nabla \times \bar{A} \rightarrow$ Curl Expression $\rightarrow$ Measure of rotation

These are vectors! Direction and Magnitude

scalars just have magnitude.

→ Both are a function of position

3 common coordinate systems

- Rectangle / cartesian
- Cylindrical
- Spherical

$$\bar{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z}$$

unit vectors, x, y, z direction respectfully

The unit ~~dire~~ vector for $\bar{A}$

$\hat{a}_x$ $\hat{a}_y$ $\hat{a}_z$

$\hat{i}$ $\hat{j}$ $\hat{k}$ Mechanical Engineers use this notetion

We can have

$$\bar{A} = A_x(x, y, z)\,\hat{x} + A_y(x, y, z)\,\hat{y} + A_z(x, y, z)\,\hat{z}$$

each term can be a function of position

We may also state

$$\nabla \cdot \bar{A} = \left[\frac{\partial}{\partial x}\hat{x}\cdot\bar{A}\right] + \left[\frac{\partial}{\partial y}\hat{y}\cdot\bar{A}\right] + \left[\frac{\partial}{\partial z}\hat{z}\cdot\bar{A}\right]$$

$$= \frac{\partial}{\partial x}A_x + \frac{\partial}{\partial y}A_y + \frac{\partial}{\partial z}A_z$$

Dot Product

$$\bar{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$$

$$\bar{B} = B_x\hat{x} + B_y\hat{y} + B_z\hat{z}$$

$$\bar{A}\cdot\bar{B} = A_xB_x + A_yB_y + A_zB_z$$

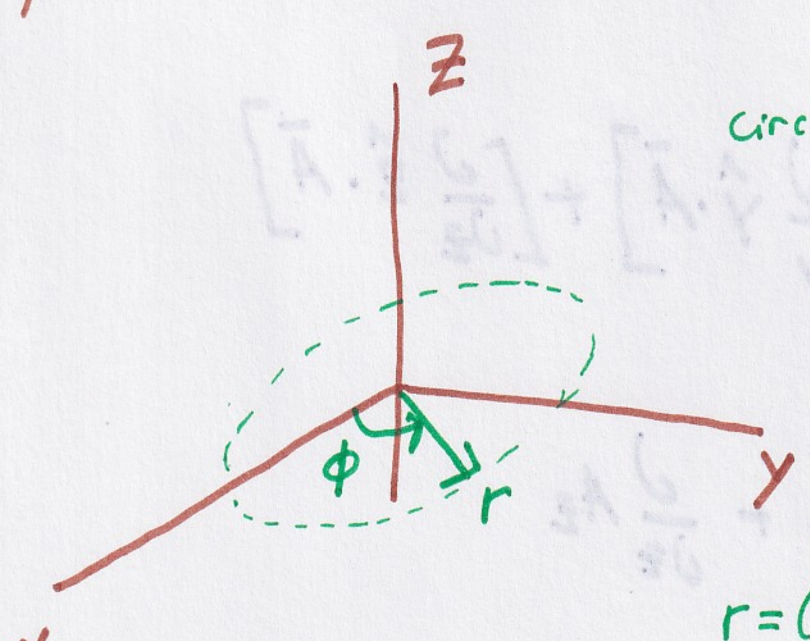
Projection of A onto B

<u>consequently</u>

$\hat{x} \cdot \hat{x} = 1 \qquad \hat{x} \cdot \hat{z} = 0$

$$|A| = (\bar{A} \cdot \bar{A})^{1/2}$$
$$= (A_x^2 + A_y^2 + A_z^2)^{1/2}$$

Cylindrical Coordinate System



circle about z-axis

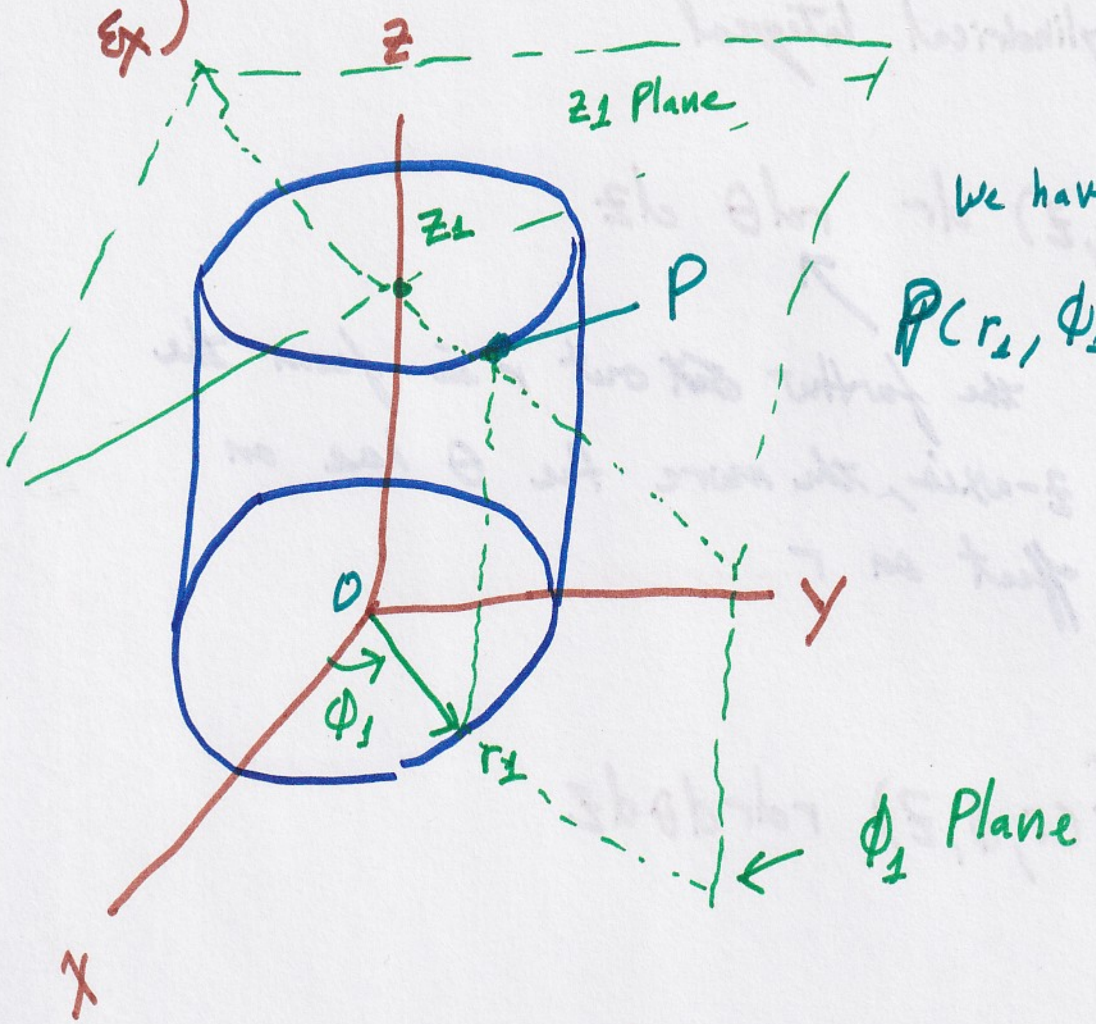
$$r = (x^2 + y^2)^{1/2}$$
(distance from z-axis)

we have r, ϕ, z

$0 < r < \infty$

$0 < \phi < 2\pi$

$-\infty < z < \infty$



We have function

$P(r_1, \phi_1, z_1)$

Vector $\overline{OP} = r_1 \hat{r} + Z_1 \hat{z}$

for cartesian coordinates we set up the integral as

$$\int_z \int_y \int_x f(x,y,z)\, dx\, dy\, dz$$

Incremental in position

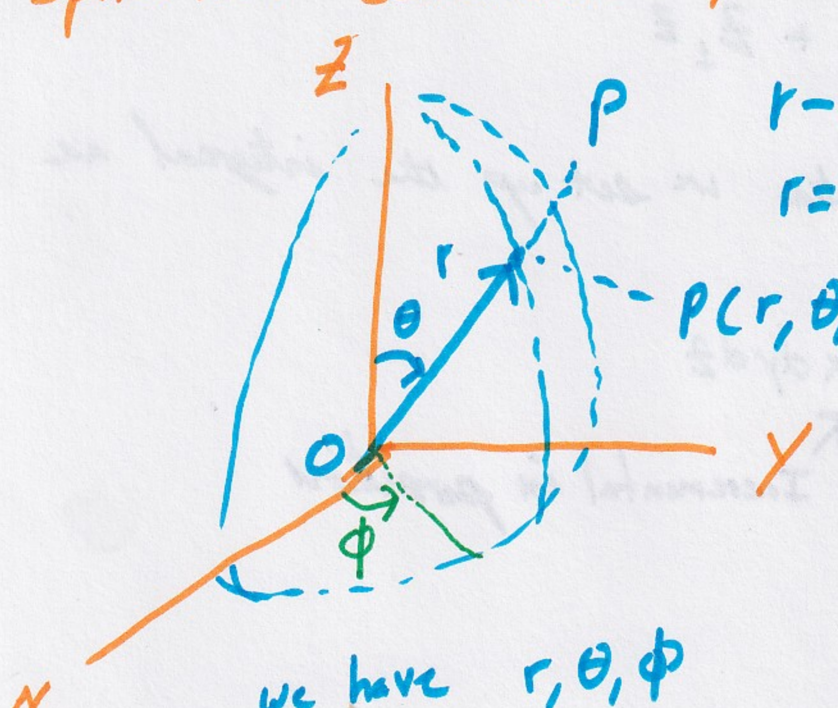
To set up the cylindrical Integral

$$\iiint f(r,\phi,z)\, dr \quad r d\theta \quad dz$$

the further ~~out~~ out r is from the z-axis, the more the θ has an effect on r

$$\int_z \int_\phi \int_r f(r,\phi,z)\, r\,dr\,d\theta\,dz$$

Spherical Coordinate System



$P(r,\theta,\phi)$ (positron)

r – distance from the origin

$r = (x^2+y^2+z^2)^{1/2}$

we have r, θ, ϕ

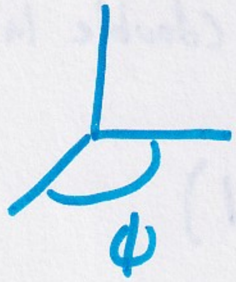
$$\overline{OP} = r\hat{r} \text{ for } P(r,\theta,\phi)$$

So how do we set up the Integral

write

$$\iiint f(r,\theta,\phi)\ dr\ \ r\,d\theta\ \ d\phi$$

$$d\phi \rightarrow (r\sin\theta\, d\phi)$$



change in ϕ could significantly change ~~θ~~ r and θ position

$$\int_{\phi}\int_{\theta}\int_{r} [f(r,\theta,\phi)]\ r^2 \sin\theta\ dr\, d\theta\, d\phi$$

$0 < \phi < 2\pi$

$0 < \theta < \pi$

$0 < r < \infty$

How do we set the Volume and Surface Integral?

$$\int_V (\nabla \cdot \bar{A})\, dV = \int_S \bar{A} \cdot d\bar{S}$$

Surface Integral (double Integral)

Volume Integral (implies triple Integral)

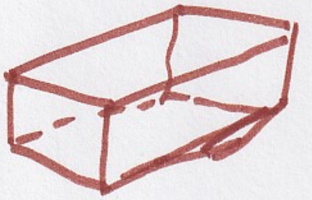
So what's going on?

$\nabla \cdot \bar{A} \rightarrow$ divergence of a field (scalar)

so $\int_V \nabla \cdot \bar{A}\, dV \rightarrow$ gives Total Divergence within that volume

So what do we do about dV and $d\bar{S}$?

The volume is defined by a surface, S

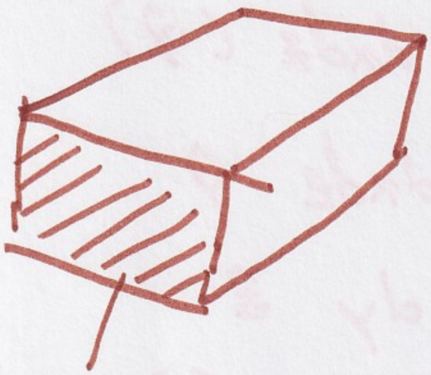


Volume of cube $(\Delta X)(\Delta Y)(\Delta Z)$

$S \rightarrow$ six outer faces of cube form a closed surface.

We need Surface Vectors

- Find the component normal to the surface

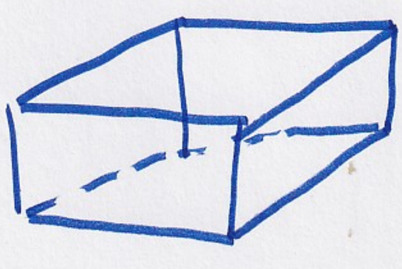


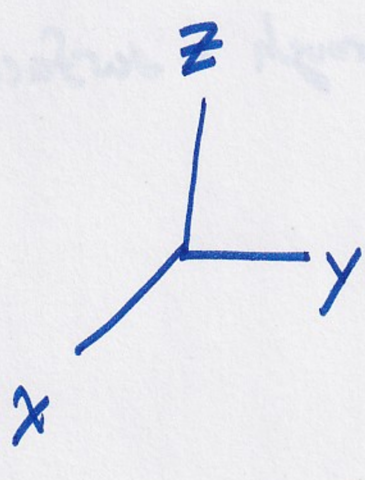
Surface # 1 has a normal component in the $\hat{x}$ direction

don't move in $\hat{x}$ direction to remain on surface (need to integrate surface)

Integrating along surface 1, S_1

$$d\bar{S}_1 = dy\,dz\,\hat{x}$$





6 faces

S_1 front $\hat{x}$ $\quad d\bar{S}_1 = dy\,dz\ \hat{x}$

S_2 back $-\hat{x}$ $\quad d\bar{S}_2 = dy\,dz\ (-\hat{x})$

S_3 Left $-\hat{y}$ $\quad d\bar{S}_3 = dx\,dz\ (-\hat{y})$

S_4 Right $\hat{y}$ $\quad d\bar{S}_4 = dx\,dz\ \hat{y}$

S_5 Top $\hat{z}$ $\quad \bar{S}_5 = dx\,dy\ \hat{z}$

S_6 bottom $-\hat{z}$ $\quad \bar{S}_6 = dx\,dy\ (-\hat{z})$

The Idea here is that

Vectors passing through Surfaces

=

Divergence

→ Measures the flux of a vector field through a surface

$$\int_V (\nabla \cdot \bar{A}) dV = \oint_S \bar{A} \cdot d\bar{S}$$

closed surface

$$\int_V \left(\frac{\partial}{\partial x} A_x + \frac{\partial}{\partial y} A_y + \frac{\partial}{\partial z} A_z \right) dV$$

$$= \int_{S_1} \bar{A} \cdot d\bar{S}_1 + \int_{S_2} \bar{A} \cdot d\bar{S}_2 + \dots + \int_{S_6} \bar{A} \cdot d\bar{S}_3$$

Lets start with the first term

$$\int_V \frac{\partial}{\partial x} A_x dV$$

$$= \int_{z_1}^{z_2} \int_{y_1}^{y_2} \int_{x_1}^{x_2} \frac{\partial}{\partial x} A_x \, dx dy dz$$

$$= \int_{z_1}^{z_2} \int_{y_1}^{y_2} \left(A_x \Big|_{x_2} - A_x \Big|_{x_1} \right) dy dz$$

(This is the Idea)

When we Look at the surface Integrals

$$\int_{S_1} \bar{A}\cdot d\bar{S}_1 = \int_{S_1} \bar{A}\cdot(dydz\,\hat{x})\Big|_{x=x_1}$$

$$= \int_z\int_y A_x\, dydz\Big|_{x=x_1}$$

$$= \int_z\int_y A_x\Big|_{x=x_1} dydz$$

$$\int_{S_2} \bar{A}\cdot d\bar{S}_2 = \int_{S_1} \bar{A}\cdot\left[dydz\,(-\hat{x})\right]\Big|_{x=x_2}$$

$$= -\int_{S_1} A_x\, dydz\Big|_{x=x_2}$$

$$= -\int_z\int_y A_x\Big|_{x=x_2} dydz$$

So $\int \nabla\cdot\bar{A}\,dV = \oint \bar{A}\cdot d\bar{S}$

<u>Divergence Theorem</u>

not expected to derive This

~~This~~ There is a second part to This Lecture with extensive Examples. It's a good Idea to review it.