Fields and Waves I L 10

Electric Fields

Electrostatics

Think Electric fields

Statics → No change with time

Charge

point charges [C] Coulombs

→ electrons (e^-)

→ protons (p^+)

Line charges

→ wire with Free charge

wire

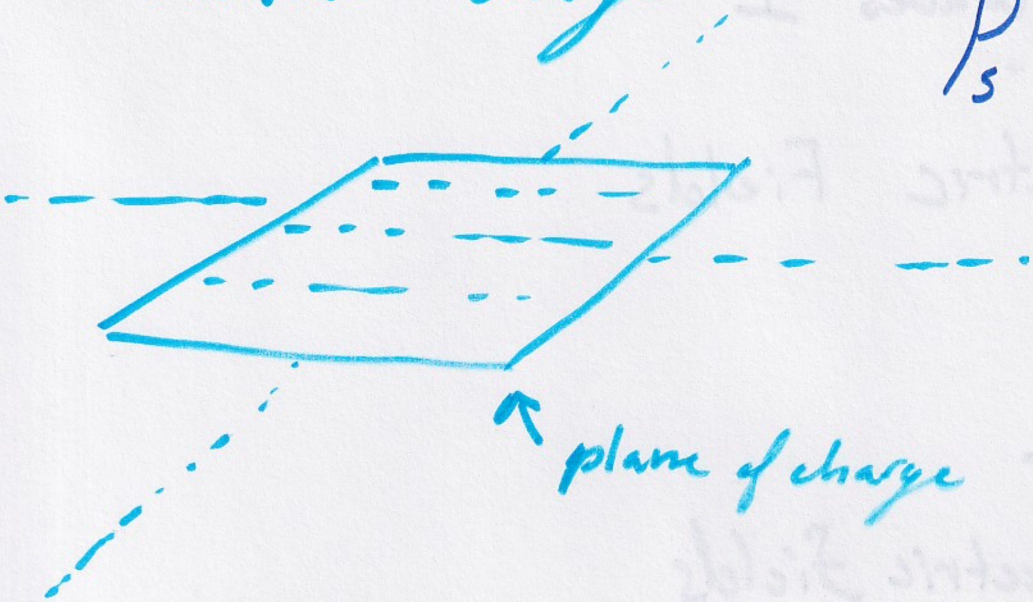
we have charge density $[C/m] = \rho_L$ (Line charge density)

Line charge

Surface charge

$\rho_s = [C/m^2]$

(surface charge density)



Ex)

Cylindrical shell of charge

(toilet paper tube

Volume charge

Charge density ρ_v

$[C/m^3]$

(volume charge density)

consequently, we can note that

$$Q = \int \rho_v \cdot dv = \int \rho_s \cdot d\vec{s} = \int \rho_\ell \cdot d\vec{\ell}$$

or $$Q = \sum_i Q_i$$

Example

a.) Find the ~~charge~~ charge in the sperical volume $0 \le r \le a$ containing a charge distribution $\rho = \rho_0 \, r^2/a^2$.

$$\rightarrow Q = \int \rho \, dv = \int_0^{2\pi} \int_0^{\pi} \int_0^{a} \rho_0 \frac{r^2}{a^2} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

(volume integral of a sphere)

$$= \frac{\rho_0}{a^2} 2\pi \left[-(-1-1) \right] \int_0^a r^4 dr = \frac{4\pi \rho_0}{a^2} \frac{a^5}{5}$$

$$= \frac{4\pi \rho_0 a^3}{5}$$ for $a = 2$ and $\rho_0 = 10^{-6}$ C/m^3 ?

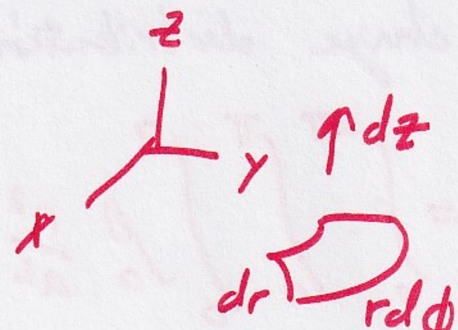
$$\rightarrow Q = 2.01 \times 10^{-5} \, C$$

b.) A surface charge on a disk increases linearly from $\rho_s = 0$ in the center to $\rho_s = 4\times10^{-6}\ C/m^2$ at the outer edge (where $r = 2m$). Find the total charge on the disk.

Since we are dealing with a surface charge, we'll use

$$Q = \int \rho_s \cdot d\bar{s}$$

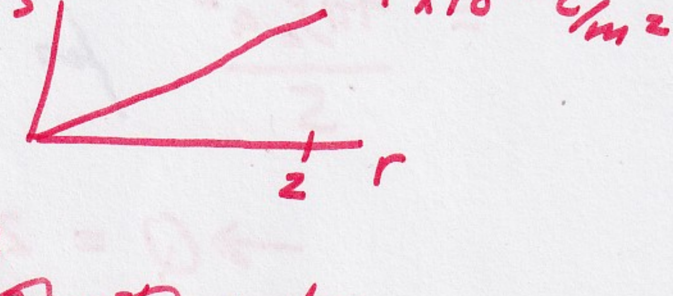
our $d\bar{s}$ is,



our normal is in the $\hat{z}$ direction! So $d\bar{s} = (dr)(rd\phi)$

$$\rightarrow Q = \int \rho_s \cdot d\bar{s} = \int_0^{2\pi}\int_0^2 \rho_s\, r dr d\phi$$

charge increases linearly,



$\frac{rise}{run}$

$$\cancel{\rightarrow Q = \int_0^{2\pi}\int_0^2 2\times10^{-6}\, r dr d\phi}$$

So write $\rho_s = 2\times10^{-6}\, r \ \frac{C}{m^2}$

where $r=2$, $\rho_s = 4\times10^{-6}\ C/m^2$

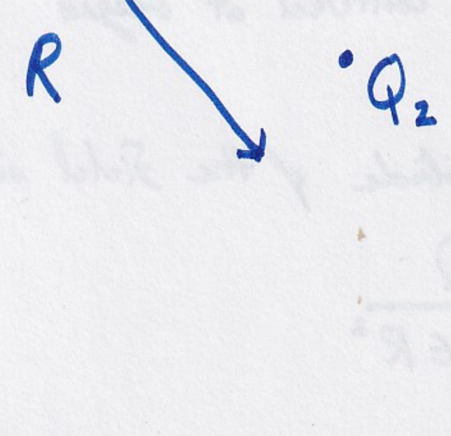
$$\rightarrow Q = \int_0^{2\pi}\int_0^2 \left(2\times10^{-6} r\right) r\,dr\,d\phi$$

$$= 2\times10^{-6}\int_0^{2\pi} \frac{r^3}{3}\Big|_0^2\, d\phi = \frac{2\times10^{-6}}{3}(8)(2\pi)$$

$$= \frac{32\pi\times10^{-6}}{3}\,C = 3.35\times10^{-5}\,C$$

Coulomb's Law

F (force), between point charges



$$\vec{F}_{12} = \frac{Q_1\,Q_2}{4\pi\epsilon_0 R^2}\,\hat{a}_R$$

Force on Q_2 by Q_1

$$\vec{F}_{21} = -\vec{F}_{12}$$

Coulomb's Law – $\overline{E}$ field

Electric field due to a point charge, centered at the origin

$\overline{E}$ of Q_1 is $\overline{E}_1 = \dfrac{Q_1}{4\pi \varepsilon_0 R^2} \hat{a}_R$

Unit vector pointing away from Q_1

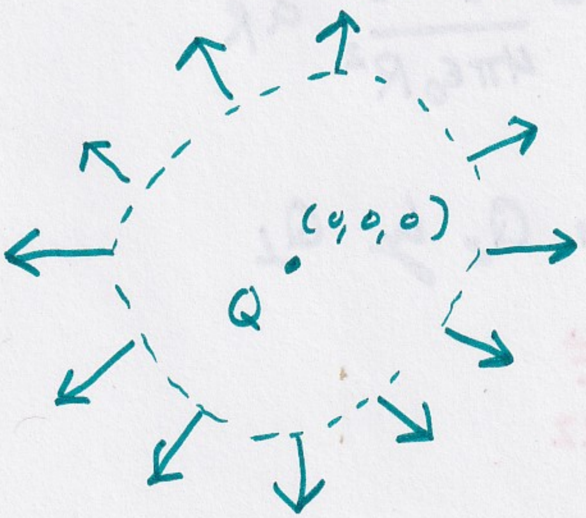
Then, $\vec{F}_{12} = Q_2 \cdot \vec{E}_1$

→ Maxwell's equations are written with <u>E-field</u>

Take Another Look

$$\vec{E} = \frac{Q}{4\pi \epsilon R^2} \hat{R}$$

(capital R means spherical representation)



sphere centered at origin

The magnitude of the field is

$$\frac{Q}{4\pi \epsilon R^2}$$

direction is radially outward $\hat{R}$

$$\vec{E} = \frac{Q}{4\pi \epsilon R^2} \hat{R}$$

Q is charge point (total) [C]

ϵ permittivity. $[F/m] = \left[\frac{C}{V\text{-}m}\right]$

— material response to E ($\vec{D}$) field

$\epsilon = \epsilon_r \epsilon_0$ ← permittivity of free space (vacuum)

↑

relative value

$\bar{D} = \epsilon \bar{E}$ $\vec{D}$ is the electric displacement field

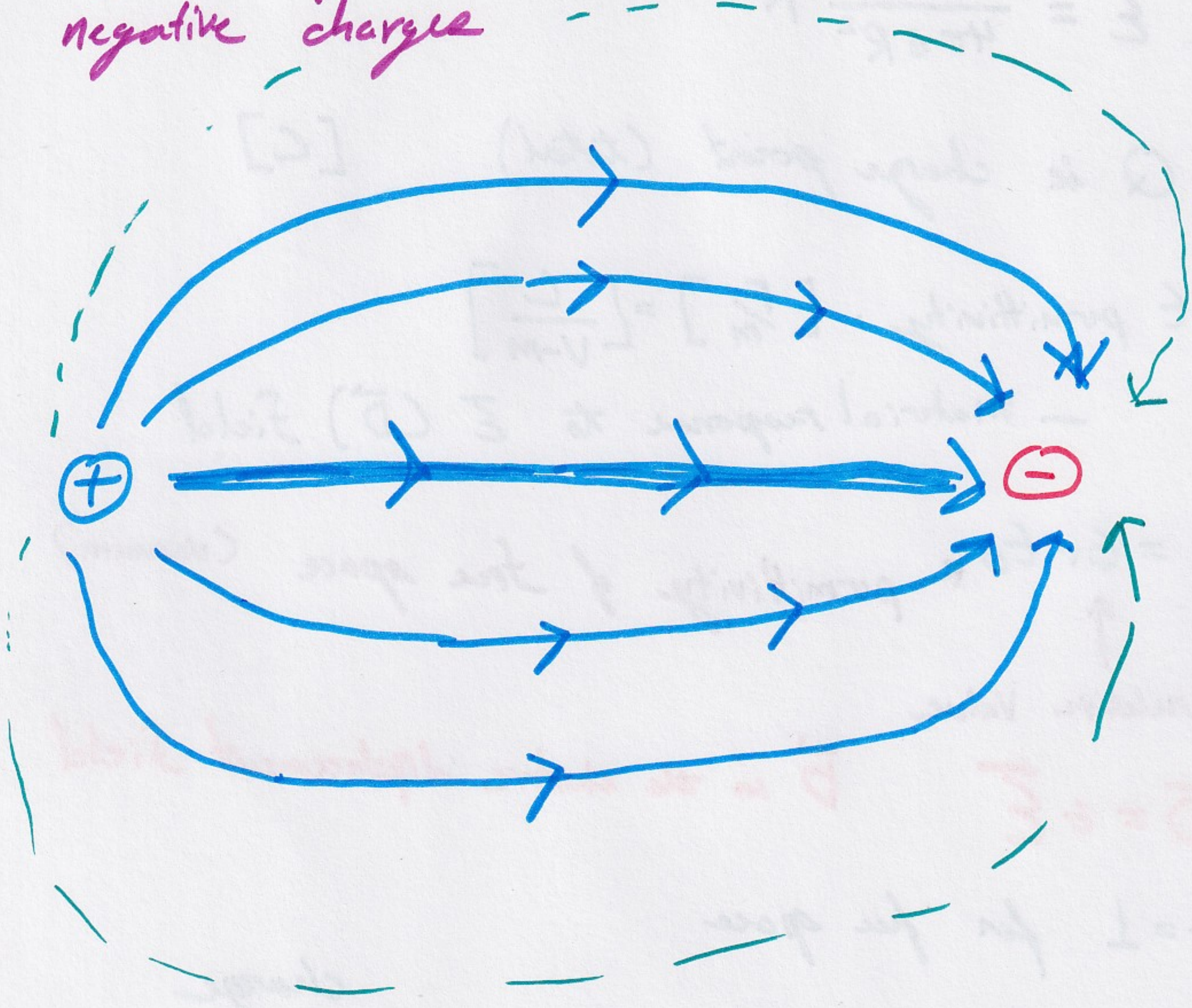
$\epsilon_r = 1$ for free space

R ⟶ radial distance from point ~~charge~~ charge

Electric Field Lines

$\vec{E}$-Field Lines

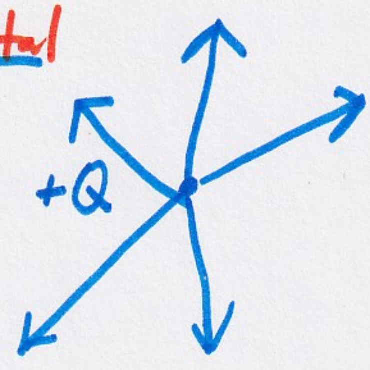
→ begin on positive charges and end on negative charges



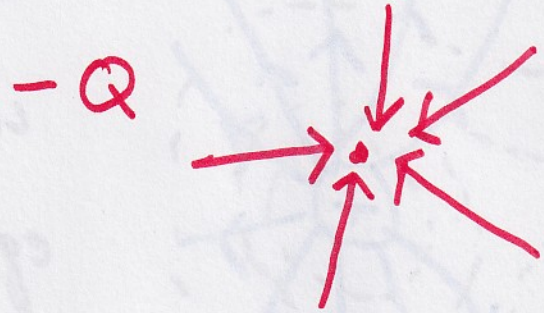
Magnitude of field lines vary with position

Exception is if we have $\underline{+Q_{total}}$

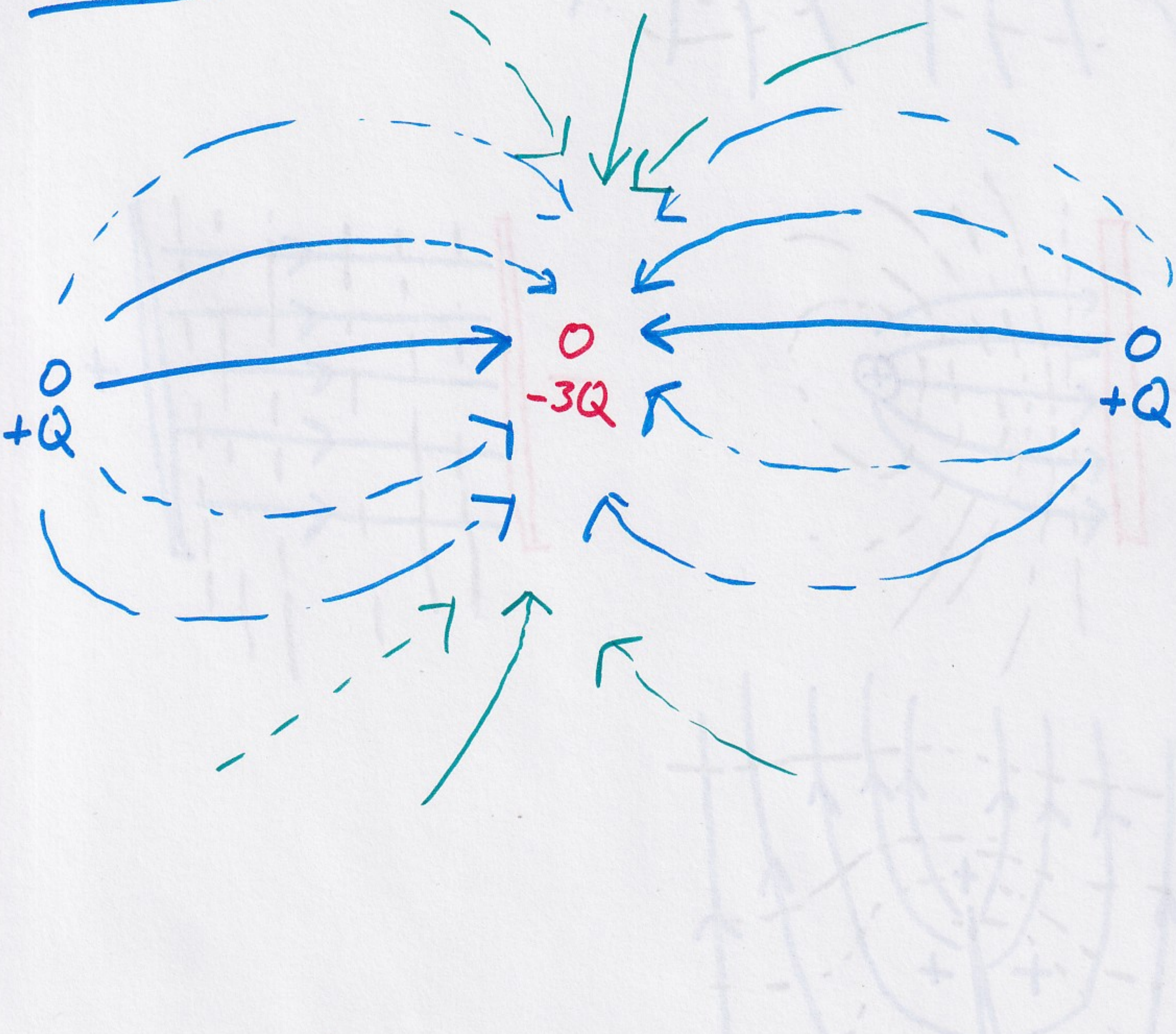
then field lines <u>go to infinity</u>



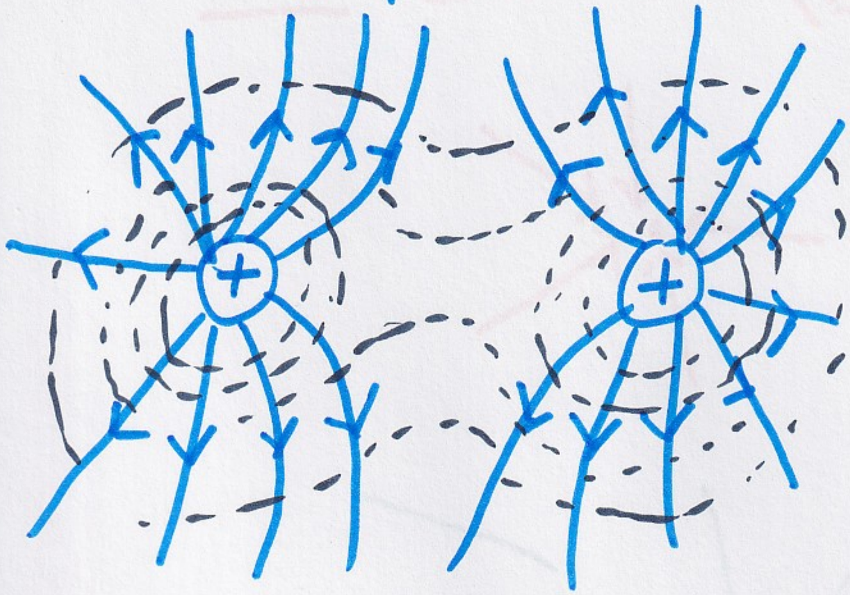
If we have $\underline{-Q_{total}}$, field lines $\underline{\text{'start'}}$ $\underline{\text{at Infinity}}$



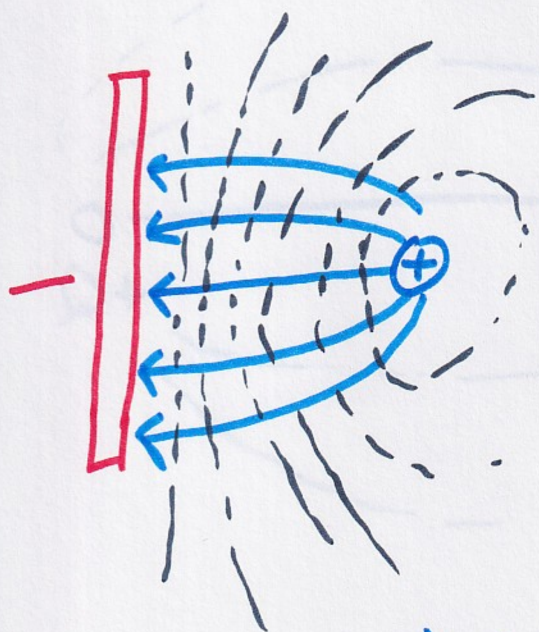
Example

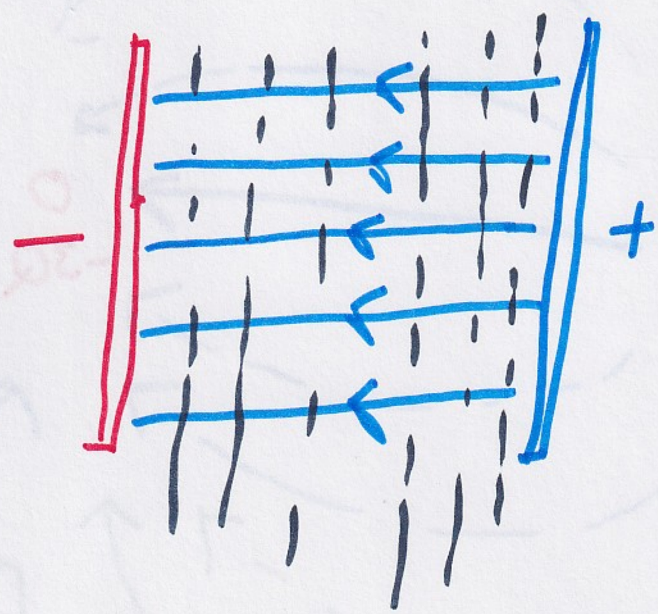


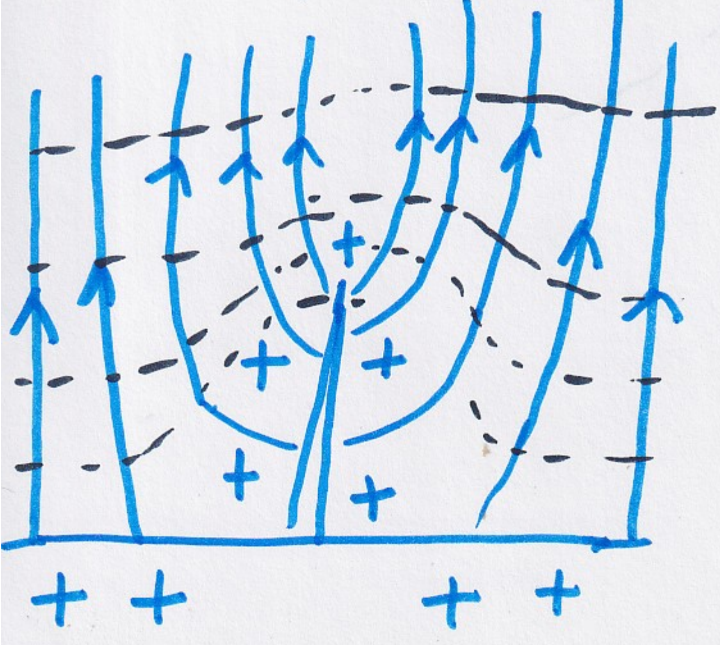
other examples

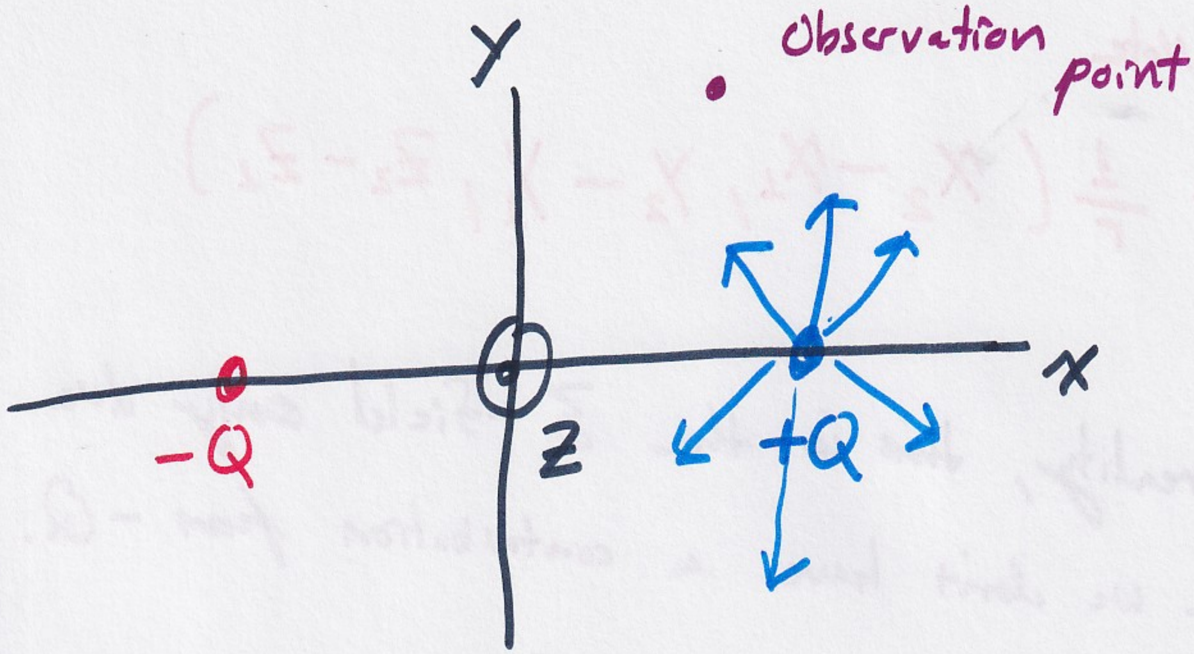


The black dashed lines are the equipotential Lines









Our observation point is at (X_2, Y_2, Z_2)

Point charge $+Q$ is at (X_1, Y_1, Z_1)

So what's the E field at the observational point?

$$\vec{E}_{X_2, Y_2, Z_2} = \frac{Q}{4\pi\epsilon r^2}\hat{r}$$

where

$$r = \sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2 + (Z_2 - Z_1)^2}$$

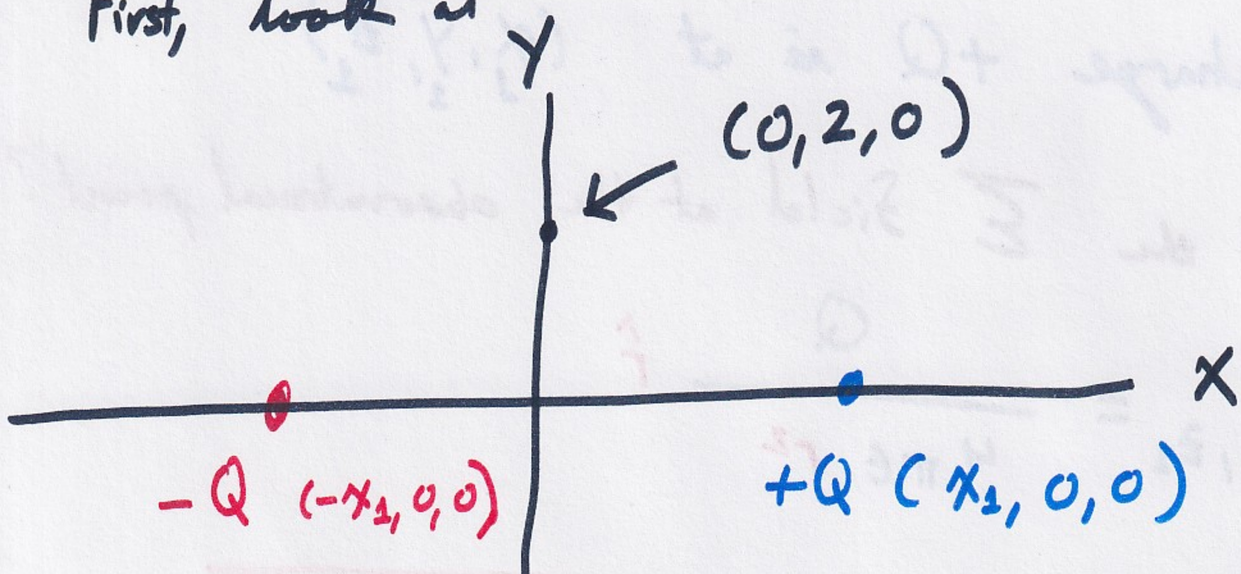
Distance from point charge to observational point and Note

$$\hat{r} = \frac{1}{r}\left(x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1 \right)$$

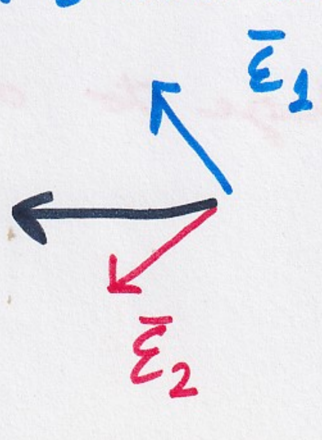
But in reality, this is the $\bar{E}$-field only due to +Q. we don't have a contribution from -Q.

So what's the Total $\bar{E}$-field?

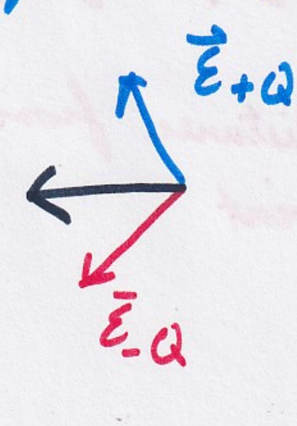
First, look at



we will have a $\bar{E}$-field contribution from +Q and -Q



or



To figure out the $\bar{\mathcal{E}}$-field, use superposition!

$$\bar{\mathcal{E}}_{total} = \bar{\mathcal{E}}_{+Q} + \bar{\mathcal{E}}_{-Q}$$ This is the Idea!

Strictly Look at $(0,2,0)$
→ observe $\bar{\mathcal{E}}_1$ and $\bar{\mathcal{E}}_2$

$$\vec{\mathcal{E}}_1 = -\mathcal{E}_{1x}\hat{x} + \mathcal{E}_{1y}\hat{y}$$

$$\vec{\mathcal{E}}_2 = -\mathcal{E}_{2x}\hat{x} + -\mathcal{E}_{2y}\hat{y}$$

These '$\mathcal{E}_{1x}$' are magnitudes
→ $\mathcal{E}_{1x} > 0$

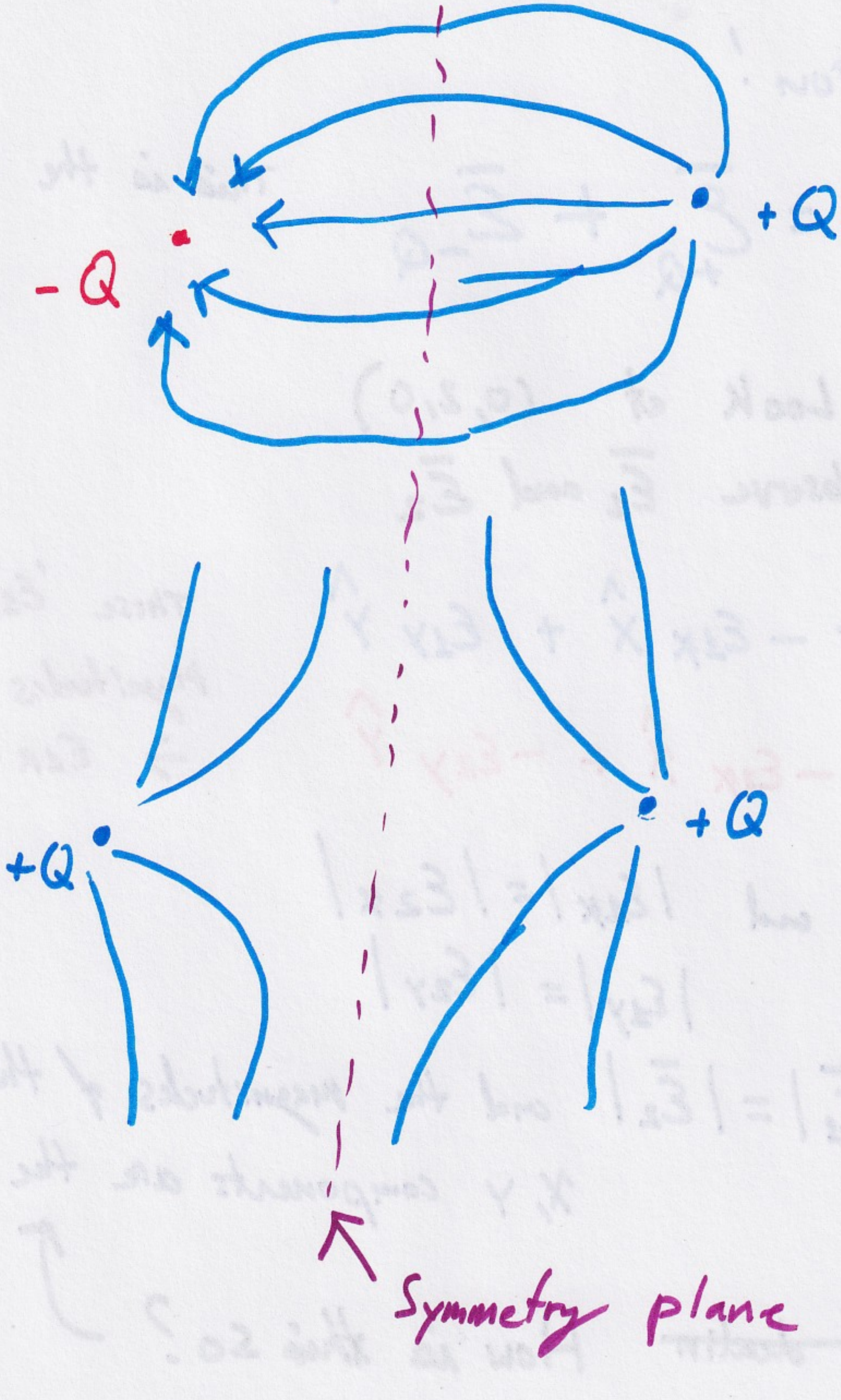
and $|\mathcal{E}_{1x}| = |\mathcal{E}_{2x}|$
$|\mathcal{E}_{1y}| = |\mathcal{E}_{2y}|$

→ $|\bar{\mathcal{E}}_1| = |\bar{\mathcal{E}}_2|$ and the magnitudes of the x, y components are the same

~~We are dealin~~ How is this so?

What's going on?

we have a Dipole (pair of +/− charges)



~~some~~ notes about Dipoles

→ for two +Qs
If the charges are very far away, the Analysis looks like a single point

Here's How it plays out,

$$\bar{E}_{(0,2,0)} = \Big\{ \frac{+Q}{4\pi\epsilon(5^2+2^2)} \frac{(-5,2)}{(5^2+2^2)^{1/2}}$$

$$+ \frac{-Q}{4\pi\epsilon(5^2+2^2)} \frac{(5,2)}{(5^2+2^2)^{1/2}} \Big\}$$

$$= \left[\frac{Q}{4\pi\epsilon(5^2+2^2)} \right] \left[\frac{(-5-5)\hat{x} + (2-2)\hat{y}}{(5^2+2^2)^{1/2}} \right]$$

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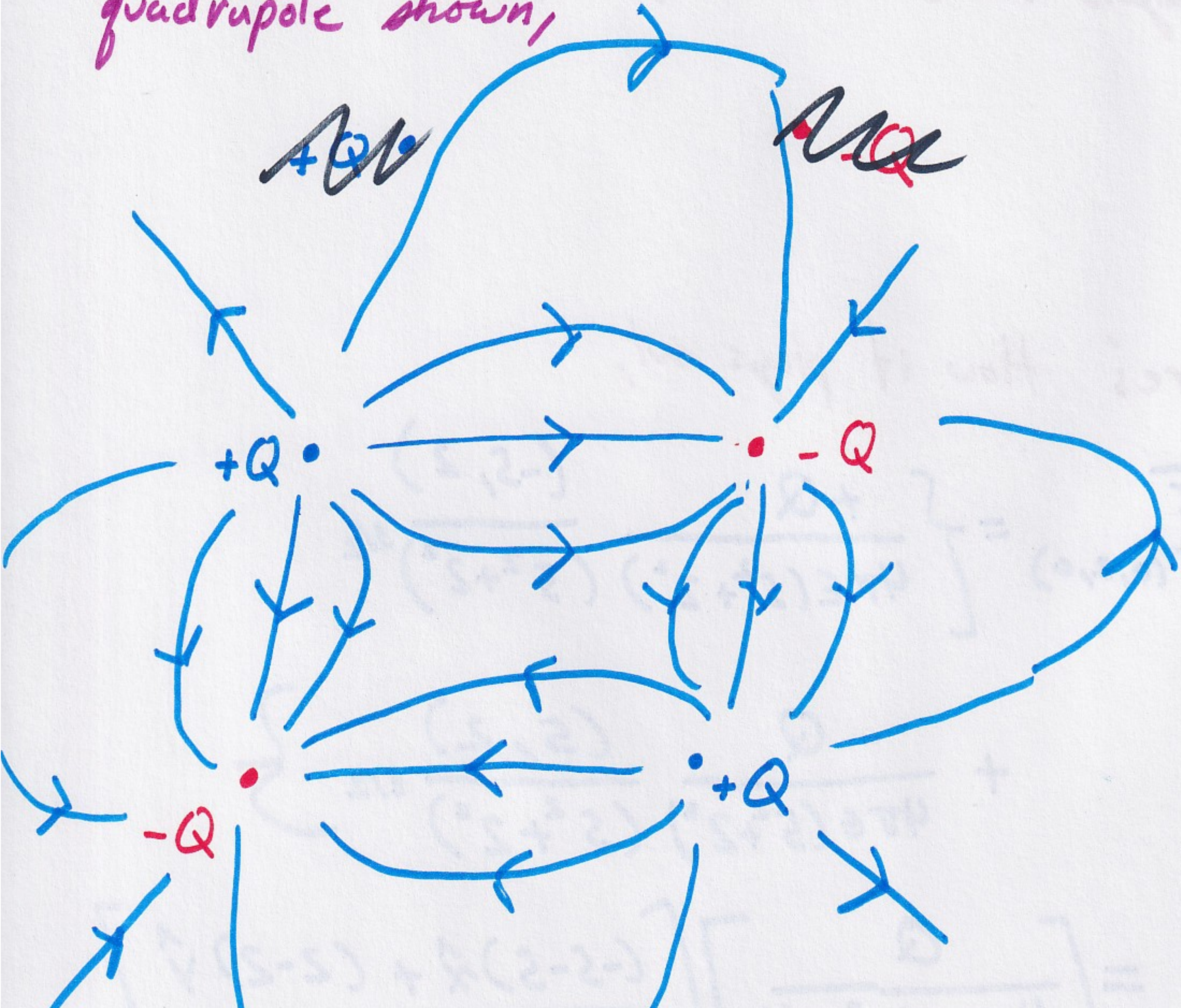
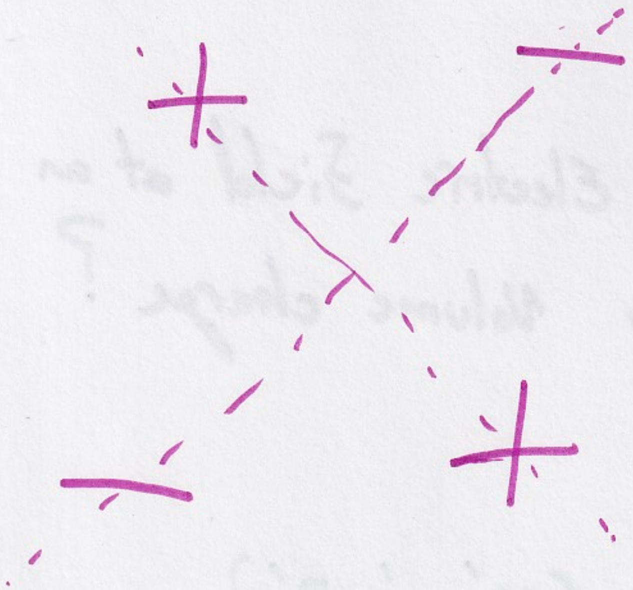
Another Example

Sketch the $\overline{E}$-field lines for the electric quadrupole shown,



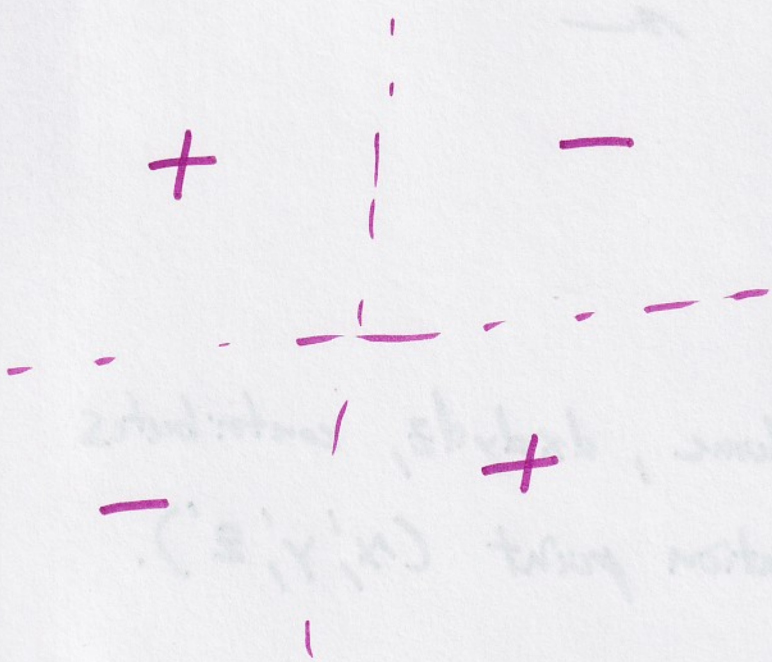
Sketch the planes for which you expect the field to be symmetric.

So we have



Symmetry plane:

charge is symmetric about the lines, $\vec{E}$ is symmetric about the lines too



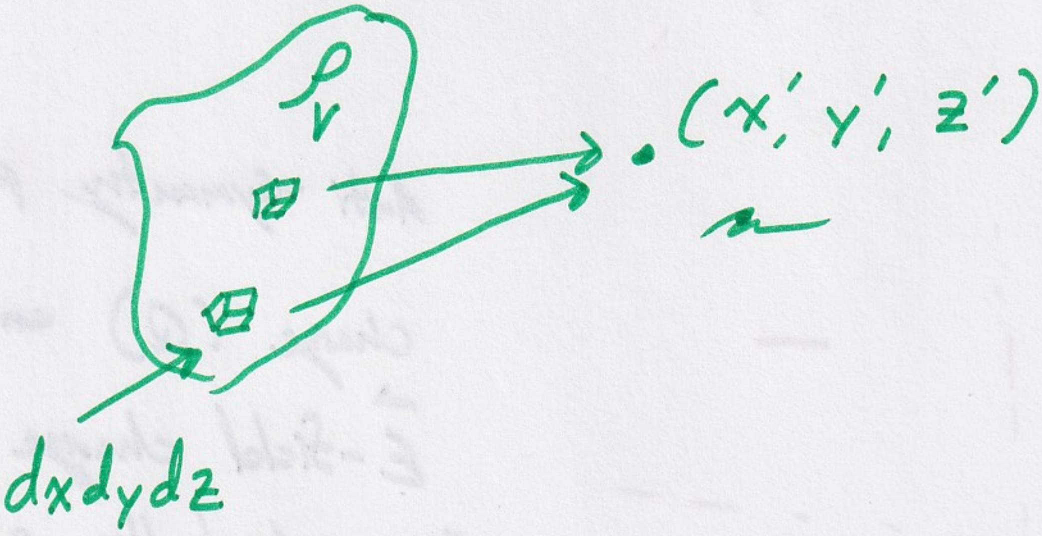
Anti-symmetry planes

charge (Q) and $\vec{E}$-field change sign about the planes

Why should we care about Symmetry?

→ Because without the concepts of symmetry and anti-symmetry, few problems can be solved by hand.

How do we find the Electric Field at an arbitrary point due to Volume charge?

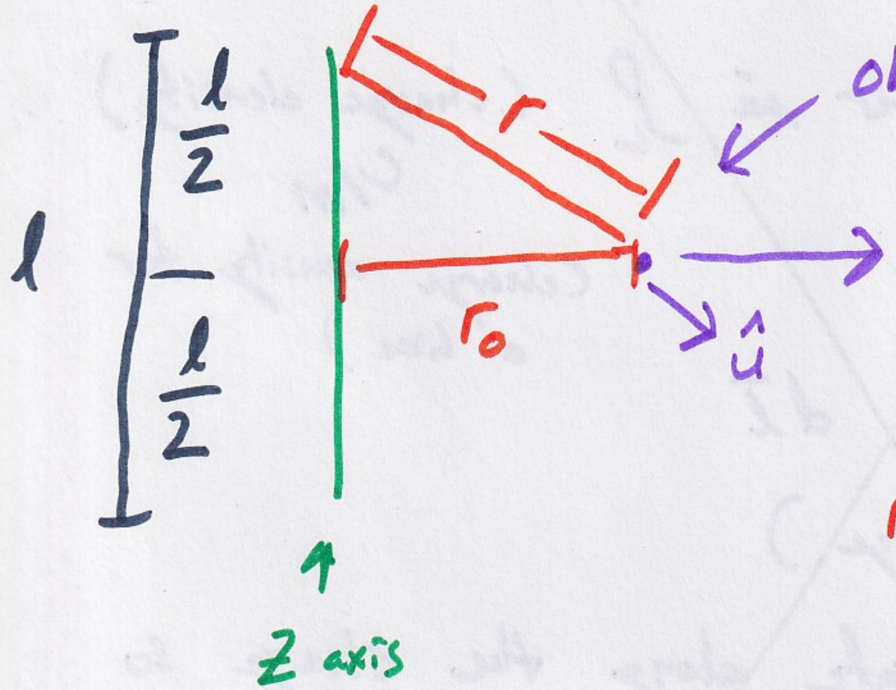


Every little piece of the Volume, dxdydz, contributes some $\bar{E}$-field to the observation point (x', y', z').

Look at Line Charge

$$\rho_L = \frac{Q}{l} \left[C/m \right]$$

l = length of Line of charge



$r_0 \rightarrow$ ~~some~~ distance from Z-axis

Every point on this Line is charge and it's contributing to the $\bar{E}$-field at the observation point. So How do we find the $\bar{E}$-field?

The $\bar{E}$-field of a Line of charge can be found by superposition the point-charge-fields of infinitesmal charge elements

We know the $\bar{E}$-field surrounding a charge point, Q is

$$\bar{E} = \frac{Q}{4\pi \epsilon r^2} \cancel{X} \hat{u}$$

We can say that The $\bar{E}$-field at the observation ~~point~~ point due to a small charge in the line, dQ is

$$d\bar{E} = \frac{dQ}{4\pi \epsilon r^2} \cancel{X} \hat{u}$$

we can restate dQ in terms of charge density,

$$dQ = \rho_L \, dz$$

We can say this because the $d\bar{l}$ is across the line of charge and we have stated the line of charge is ~~across~~ accross the z-axis. $d\bar{l} \rightarrow dz$

From the symmetry of the configuration, we can see that the $d\bar{E}$ components in the $\hat{r}$ direction are zero.

This is because the upper half of the Line charge apply a 'downward' force while the Lower half applies of force of equal magnitude in the 'upward' direction.

→ The top and Lower parts of the segment contribute equally to the $\bar{E}_{total}$ at the point of Observation. (Because of Symmetry)

~~Ø~~ Consequently from this revelation,

$$r = (r_0^2 + z^2)^{1/2}$$

$$\rightarrow \bar{E} = 2\int_0^{l/2} \frac{\rho_L}{4\pi\epsilon(r^2+z^2)} \frac{r_0}{(r_0^2+z^2)^{1/2}} dz$$

~~Because $d\bar{E} =$~~

Because $$d\bar{E}_r = \frac{\rho_L}{4\pi\epsilon r^2} \frac{r_0}{r} dz = d\bar{E}_r$$

$$\cancel{\left(dQ = \rho_L \frac{r_0 dz}{r}\right)}$$

To explain why, we just point out,

$d\bar{E}$ is the '$\hat{r}$' component of our configuration

So we write $d\bar{E}_r = d\bar{E}\cos\theta = d\bar{E}\,\frac{r_0}{r}$ $\left(\frac{x}{r}\right)$

So to perform the Integral

$$\bar{E} = \frac{2\rho_L}{4\pi\epsilon}\int_0^{l/2}\frac{r_0}{(r_0^2+z^2)^{3/2}}\,dz$$

$$= \frac{2\rho_L}{4\pi\epsilon}\,r_0\left[\frac{1}{r_0^2(r_0^2+l^2)^{1/2}} - \frac{1}{r_0(r_0^2)^{1/2}}\right]$$

(Skipped a lot of steps for this tricky Integral)

But if we actually do it,

$z = r_0\tan(u)$

$dz = r_0\sec^2 u\,du$

we have u-sub then 'u' ($\sec^2 u\,du$)

$$\bar{E} = \frac{2\rho_L r_0^2}{4\pi\epsilon}\int \frac{\sec^2 u\,du}{(r_0^2 + r_0^2\tan^2 u)^{3/2}}$$

$$= \text{'' ''}\int \frac{\sec^2 u\,du}{[r_0^2(1+\tan^2 u)]^{3/2}}$$

$$= \text{'' ''}\int \frac{\sec^2 u\,du}{[r_0^2(\sec^2 u)]^{3/2}}$$

$$= \text{'' ''}\int \frac{1}{r_0^3}\frac{\sec^2 u\,du}{\sec^3 u\,du} = \text{'' ''}\int \frac{1}{r_0^3}\cos u\,du$$

we know

$\cancel{z = r_0 \tan u \rightarrow \frac{z}{r_0} = \tan u \rightarrow \arctan\frac{z}{r_0} = u}$

$$z = r\tan u \rightarrow \frac{z}{r} = \tan u \rightarrow \arctan\frac{z}{r} = u$$

continuing...

$$= \text{'' ''}\frac{1}{r_0^3}\sin u\,\Big|$$

boundaries yet to be determined

$$= \frac{2\rho_L r_0^2}{4\pi\epsilon} \frac{1}{r_0^3} \sin\left(\arctan\left(\frac{z}{r}\right)\right)\Bigg|_0^{l/2}$$

(we go from u sub back to z)

$$\left(\sin\theta = \frac{y}{r} \right)$$

$$= \frac{2\rho_L r_0^2}{4\pi\epsilon} \frac{1}{r_0^3} \left[\frac{z}{(r_0^2+z^2)^{1/2}} \right]_0^{l/2}$$

So we have

$$\bar{E} = \frac{\rho_L}{4\pi\epsilon} \frac{1}{r_0} \left[\frac{l}{(r_0^2+(\frac{l}{2})^2)^{1/2}} \right]$$

And if we let $l \longrightarrow \infty$

$$\longrightarrow \quad \bar{E} = \frac{\rho_L}{4\pi\epsilon r_0} \quad \text{(Infinite Line)}$$

for the Problems of

Finite Straight Line, ~~Discs~~

Discs

Finite circles,

we can do all of these with Coulomb's Law

because we are dealing with Indefinitely small points

Also depends ~~of~~ on Free space