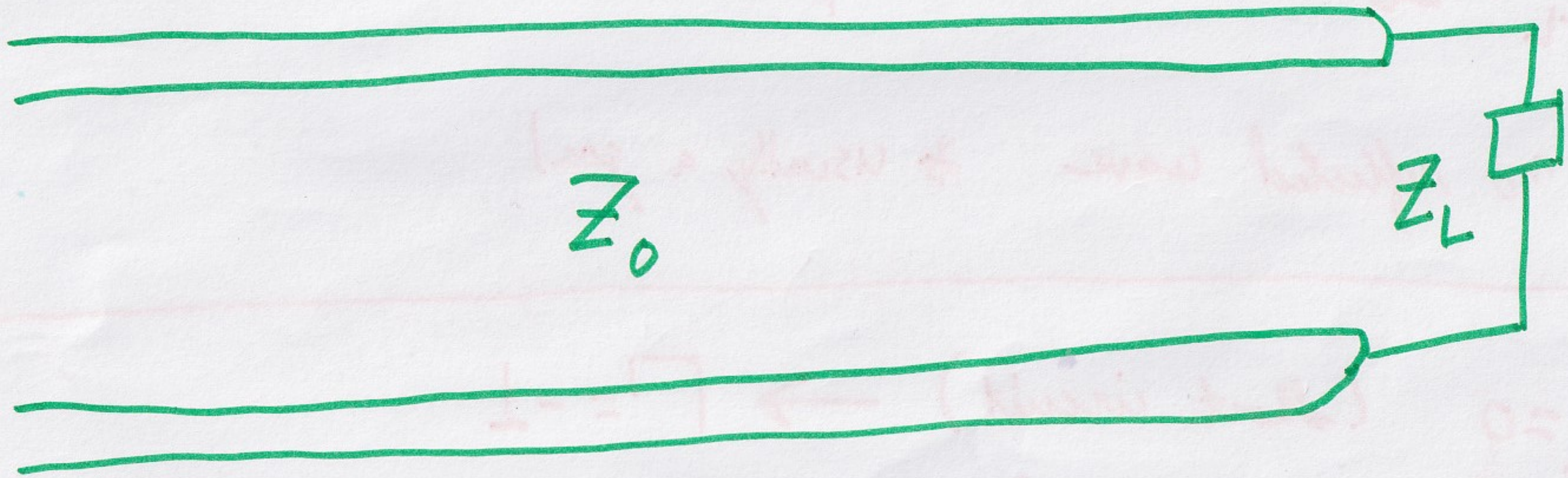
Fields and Waves I

L4

Input Impedance of a Transmission Line



Γ = reflection coefficient

$$\Gamma = \frac{|V_{back}|}{|V_{forward}|}$$ Amplitude of the forward and backward waves

②

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{|V^-|}{|V^+|}$$

+ forward going

− backward going

$Z_L = Z_0$ $\quad \Gamma = 0$, Matched Load

no reflected wave * usually a goal

$Z_L = 0$ (short circuit) $\longrightarrow \Gamma = -1$

$Z_L = \infty$ (open circuit) $\longrightarrow \Gamma = +1$

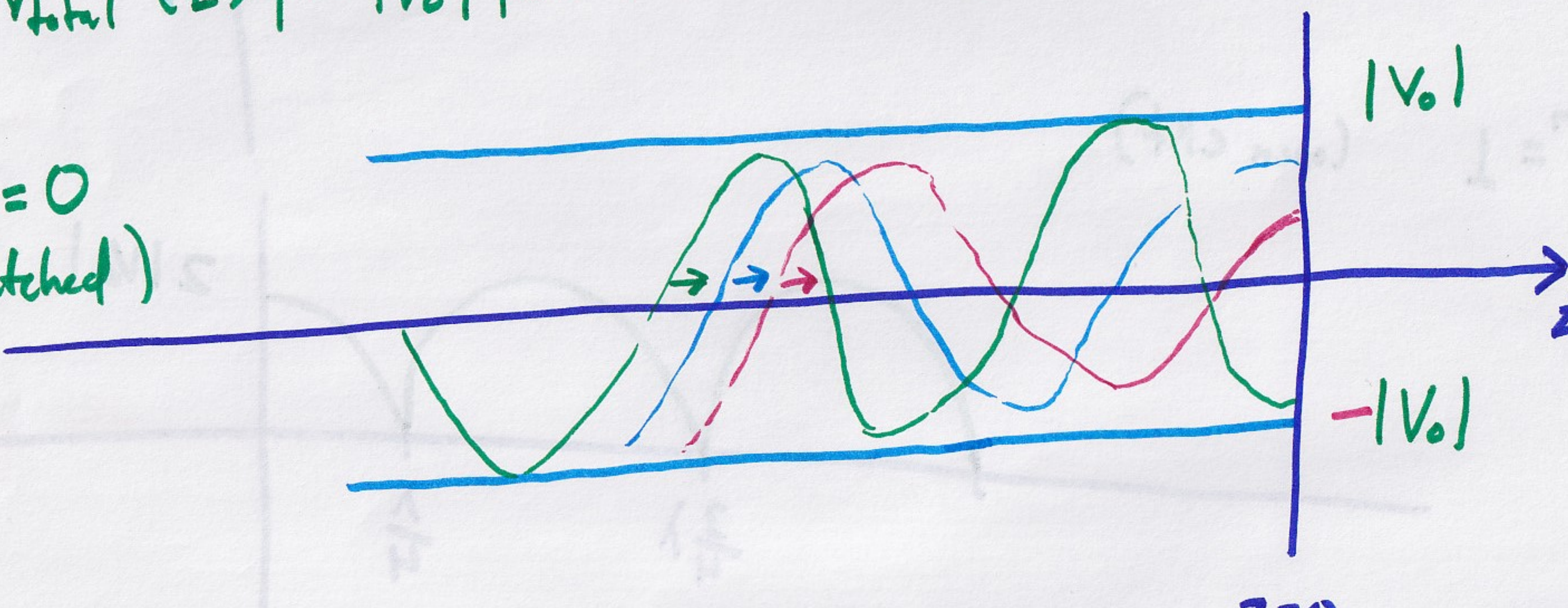
Standing wave pattern

max amplitude as a function of position

we looked at (voltage)

$$|V_{total}(z)| = |V_o| \, |(1 + \Gamma e^{+2j\beta z})|$$

$\Gamma = 0$
(matched)



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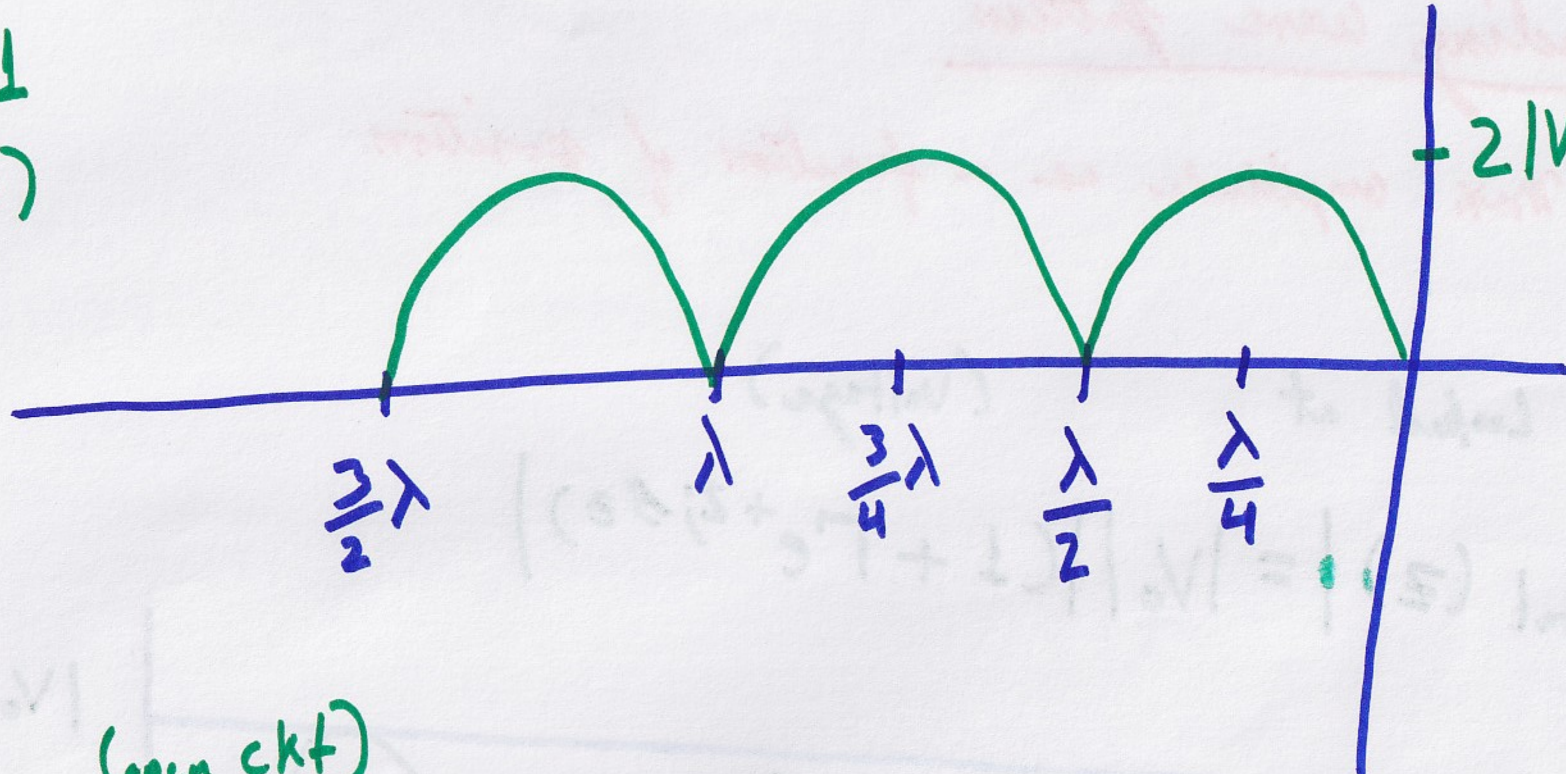
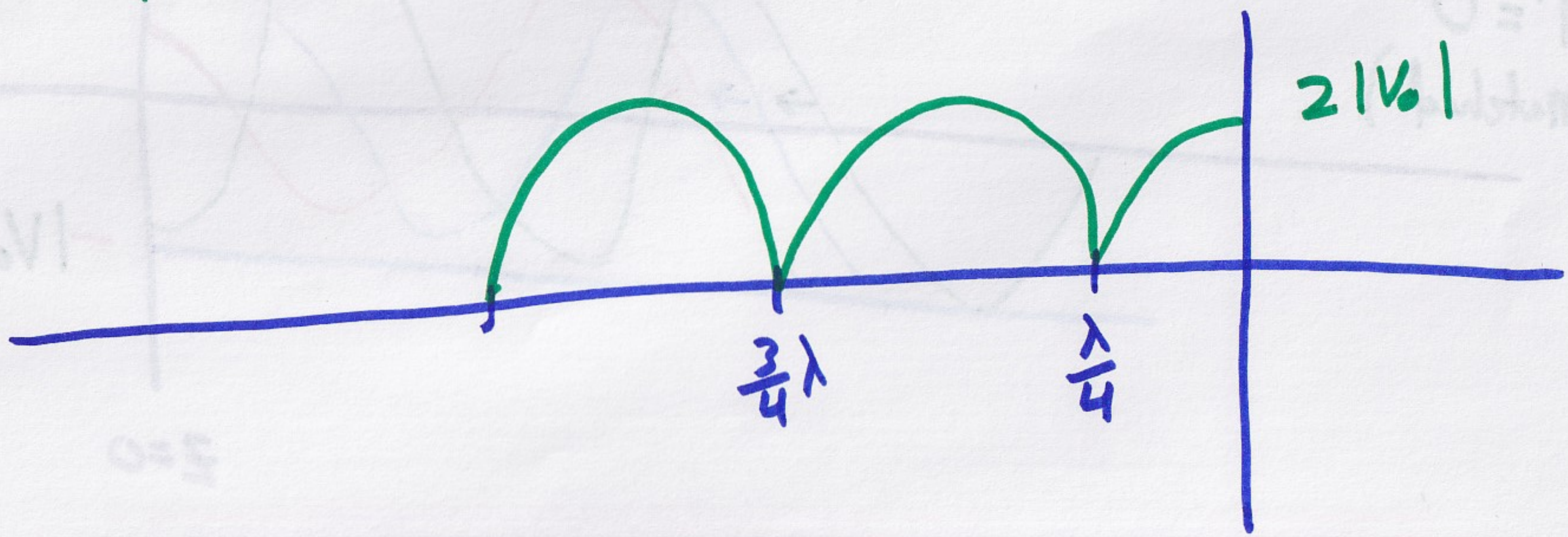
$\Gamma = -1$

(short)



$\Gamma = 1$ (open ckt)



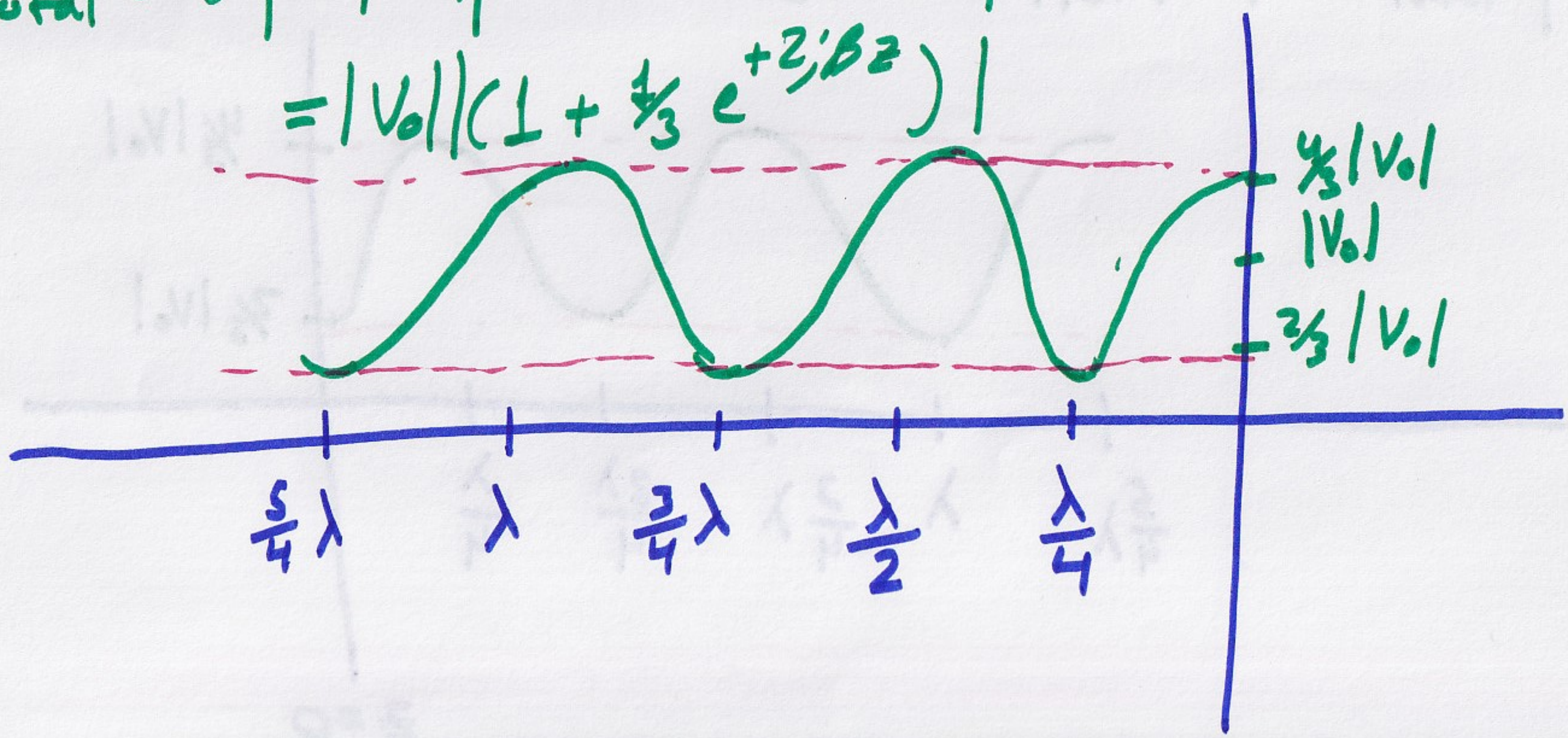
Example)

$$Z_0 = 100\,\Omega \qquad Z_L = 200\,\Omega$$

$$\Gamma = \frac{200-100}{200+100} = 1/3$$

$$|V_{total}(z)| = |V_0|\,|(1+\Gamma e^{+2j\beta z})|$$

$$= |V_0|\,|(1 + 1/3\, e^{+2j\beta z})|$$

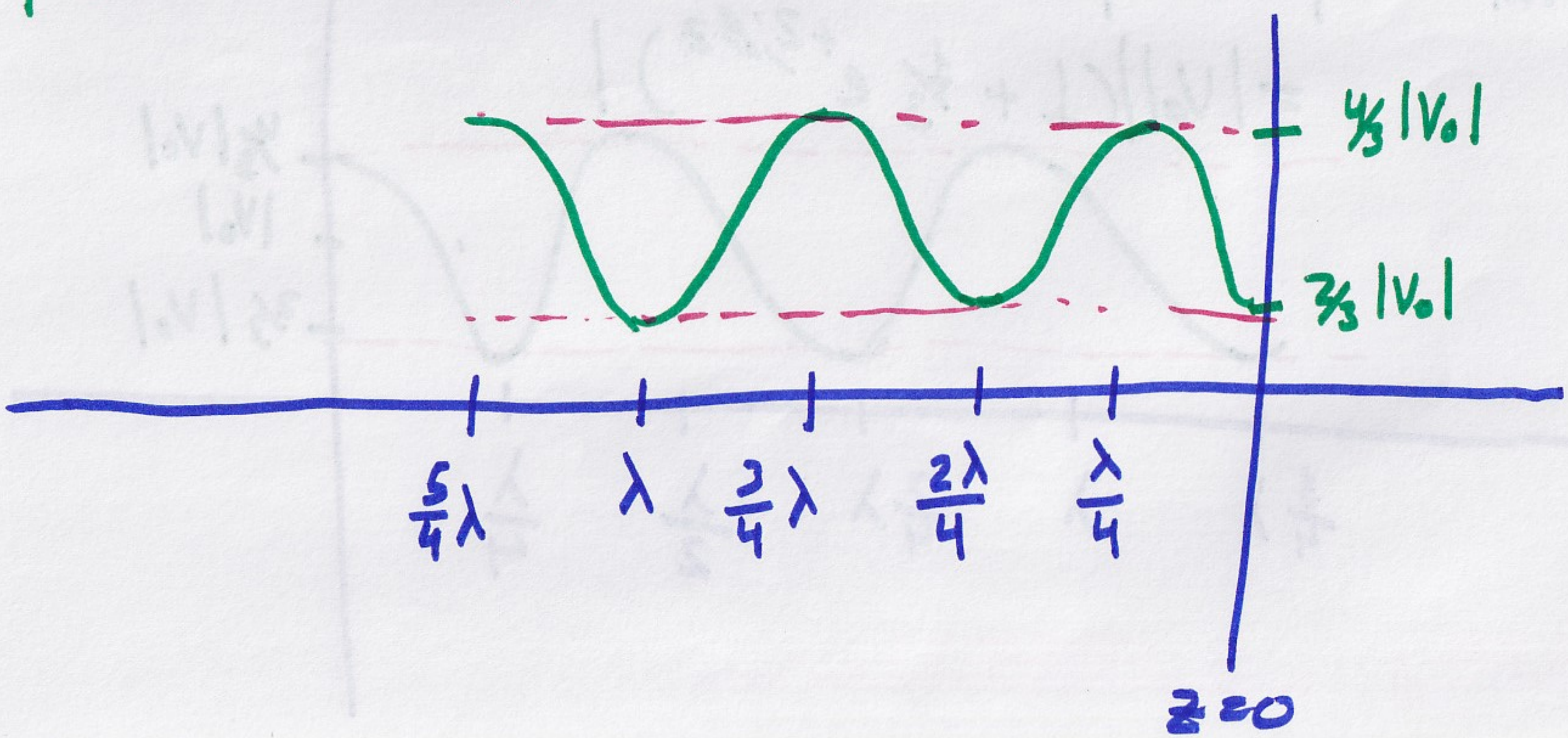


$V_{total}(z)$ is periodic with $n\frac{\lambda}{2}$ $\quad n \in N$

Example) $Z_0 = 100\,\Omega \quad Z_L = 50\,\Omega$

$$\Gamma = \frac{50-100}{50+100} = -\tfrac{1}{3}$$

$$|V_{total}(z)| = |V_0|\,|(1 - \tfrac{1}{3}e^{+2j\beta z})|$$



Position of Maxima / Minima depends on the phase of the reflection coefficient

So what's the impedance?

$$\rightarrow R \,||\, \frac{1}{j\omega C}$$

$$\rightarrow \left(\frac{1}{R} + j\omega C\right)^{-1} = \frac{1}{\frac{1}{R} + j\omega C} = \frac{R}{1 + j\omega RC} = \frac{R - j\omega R^2 C}{1 + \omega^2 R^2 C^2}$$

$$\frac{1}{Z}\,\frac{Z^*}{Z^*} = \frac{Z^*}{\|Z\|^2}$$

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Example, $Z_L = 100 - j50$ $\quad Z_0 = 100\,\Omega$

$$\Gamma_L = \frac{50 - j50 - 100}{50 - j50 + 100} = -0.2 - j0.4$$

Phasor form

$$\Gamma_L = |\Gamma_L| \angle \Gamma_L$$

$$|\Gamma_L| = \left(\Gamma_{L\,real}^2 + \Gamma_{L\,imaginary}^2 \right)^{1/2}$$

$$\angle \Gamma_L = \tan^{-1}\left(\frac{\Gamma_I}{\Gamma_R}\right)$$ (Know what quadrant you're in)

So $\Gamma_L = 0.447 \angle -117^\circ$

$= 0.447 \angle -0.65\pi = 0.447\, e^{-j0.65\pi}$

$$|V_{total}(z)| = |V_0||(1 + 0.447 e^{-j0.65\pi} e^{+2j\beta z})|$$
$$= |V_0||(1 + 0.447 e^{j(2\beta z - 0.65\pi)})|$$

Find the Location on where is the maximum

so to start,

$$2\beta z - 0.65\pi = 0, 2\pi, 4\pi, \ldots$$
$$= 2\pi n \quad n \in N$$

→ Location of Voltage maxima on the T-Line

recall $\beta = \frac{2\pi}{\lambda}$

so $2\left(\frac{2\pi}{\lambda}\right)z - 0.65\pi = 0$

$$\rightarrow \quad z = 0.65\pi \frac{\lambda}{4\pi} + n\frac{\lambda}{2}$$

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So now we have something called

SWR → standing wave ratio

$$SWR = \frac{|V_{max}|}{|V_{min}|} = \frac{|V_0|(1+|\Gamma|)}{|V_0|(1-|\Gamma|)}$$

when the line is matched, $SWR = 1$

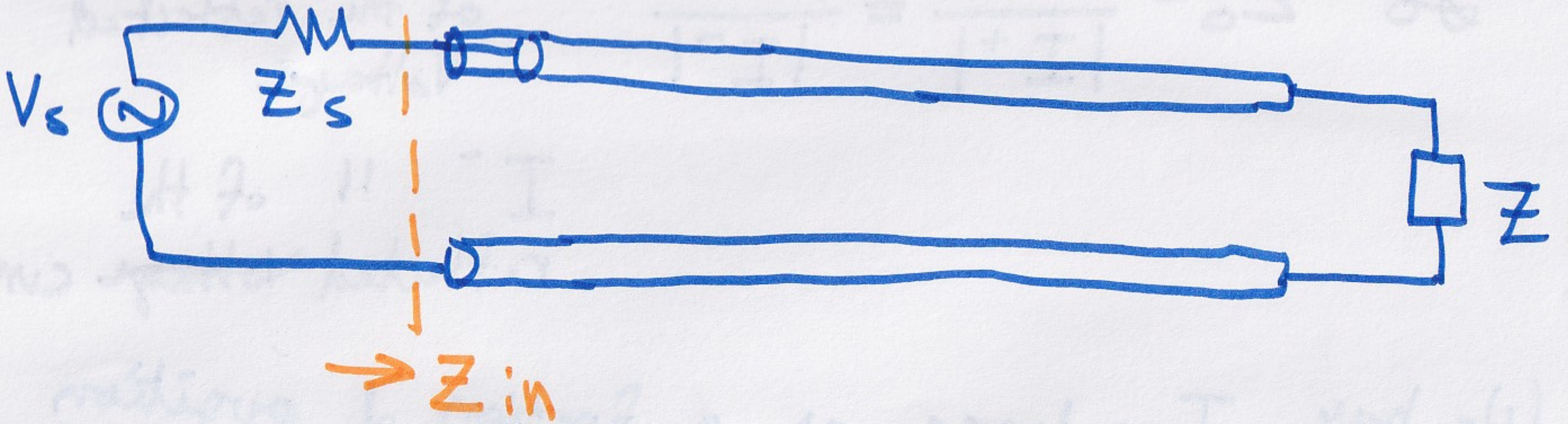
consequently, if we have open/short circuit Load

~~circuit load~~

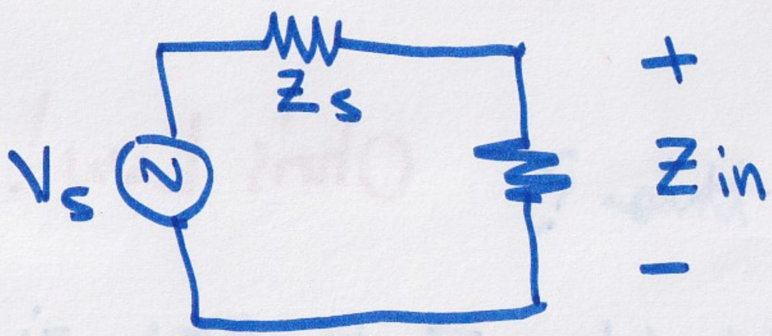
$$SWR = \frac{2}{0} \rightarrow \infty \quad \text{not good}$$

$$1 \leq SWR < \infty \qquad |\Gamma| \leq 1$$

Look at the Circuit



Zin is the Input impedance of the T-Line



$$V_{in} = \frac{Z_{in}}{Z_{in} + Z_S} V_S$$

Vin is the Total voltage at the input of the T-line

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So $Z_0 = \dfrac{|V^+|}{|I^+|} = \dfrac{|V^-|}{|I^-|}$

V^- is the Amplitude of the reflected Voltage

I^- " of the reflected ~~voltage~~ current

We have Impedance as a function of position

$Z(z)$

Z ← Impedance

(z) ← position

But how do we get this? Ohm's Law!

$$Z(z) = \frac{V_{total}(z)}{I_{total}(z)} = \frac{V_0\left(1 + \Gamma e^{+2j\beta z}\right)e^{-j\beta z}}{I_0\left(1 - \Gamma e^{+2j\beta z}\right)e^{-j\beta z}}$$

Note!

$Z(z)$ is the ratio of the total voltage (incident and reflected wave voltages) to the total current at any point z on the line.

This is in contrast with the characteristic impedance of the line Z_0, which relates the voltage and current of each of the two waves individually

$$Z_0 = \frac{V_0^+}{I_0^+} = \frac{-V_0^-}{I_0^-}$$

So then,

$$Z(z) = \frac{V_0 \left(1 + \Gamma e^{+2j\beta z}\right)}{\frac{V_0}{Z_0}\left(1 - \Gamma e^{+2j\beta z}\right)}$$

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$$Z(z) = Z_0 \frac{(1+\Gamma e^{+2j\beta z})}{(1-\Gamma e^{+2j\beta z})}$$

Interesting, we have no voltage dependency but we have frequency dependency

$$\beta = \frac{2\pi}{\lambda} = \frac{\omega}{v_p}$$

So we may write,

$$Z(z) = Z_0 \frac{Z_L + jZ_0 \tan(\beta z)}{Z_0 + jZ_L \tan(\beta z)}$$

Also, for a Known Line length, l

$$Z_{in} = Z_0 \frac{Z_L + jZ_0 \tan(\beta l)}{Z_0 + jZ_L \tan(\beta l)}$$

$l \rightarrow$ length of T-line

$Z_0 \rightarrow$ characteristic Impedance

$Z_L \rightarrow$ Load impedance

$$\beta = \frac{2\pi}{\lambda} = \frac{\omega_p}{V_p} = \frac{2\pi f}{V_p}$$

Frequency controls wave length

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The Input Impedance of an Oscilloscope

- typically we want a Large input Impedance for the oscilloscope

Ex) $c = v_p = 3 \times 10^3$ m/s   freq = 1 mHz

$\lambda f = c$ $\lambda = 300$ [m]

Coaxial ~~cables~~ connectors for the oscilloscope ~ 1 m

$$Z_{in} = Z_0 \frac{Z_L + j Z_0 \tan\left[\left(\frac{2\pi}{300}\right)(1)\right]}{Z_0 + j Z_L \tan\left[\left(\frac{2\pi}{300}\right)(1)\right]}$$

Scope is in parallel with R_2, Ideally we want the scope to be an open ~~cir~~ circuit

For $l << \lambda$

$$Z_{in} = Z_0 \frac{Z_L + j Z_0 \tan\left[\left(\frac{2\pi}{300}\right)(1)\right]}{Z_0 + j Z_L \tan\left[\frac{2\pi}{300}(1)\right]}$$

$$= Z_0 \frac{Z_L}{Z_0} = Z_L$$ (In this case, Z_L would be the impedance of the oscilloscope

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The Ideal input impedance for an oscilloscope is

$$Z_{scope} \rightarrow \infty$$

So if the Load is ∞ (open)

$$Z_{in} = Z_0 \frac{\infty + j Z_0 \tan(\beta l)}{Z_0 + j \infty \tan(\beta l)}$$

$$= Z_0 \frac{\infty}{j \infty \tan(\beta l)} = Z_0 \frac{1}{j \tan(\beta l)}$$

So open ckt Load

$$Z_{in_{oc}} = -\frac{j Z_0}{\tan(\beta l)}$$

So if $l = 1$ and $l << \lambda$

$Zin_{oc} \rightarrow \infty$ (to be expected)

Ex) $l = \frac{\lambda}{4}$ in Length

$$\tan(\beta l) = \frac{\sin(\beta l)}{\cos(\beta l)} = \frac{\sin\left[\frac{2\pi}{\lambda}\frac{\lambda}{4}\right]}{\cos\left[\frac{2\pi}{\lambda}\frac{\lambda}{4}\right]} = \frac{\sin(\pi/2)}{\cos(\pi/2)}$$

$$= \frac{1}{0} \rightarrow \infty$$

So then

$$Zin_{oc} = -j\frac{Z_0}{\tan(\beta l)} \quad \text{at } l = \frac{\lambda}{4}$$

$$\rightarrow Zin_{oc} = 0$$

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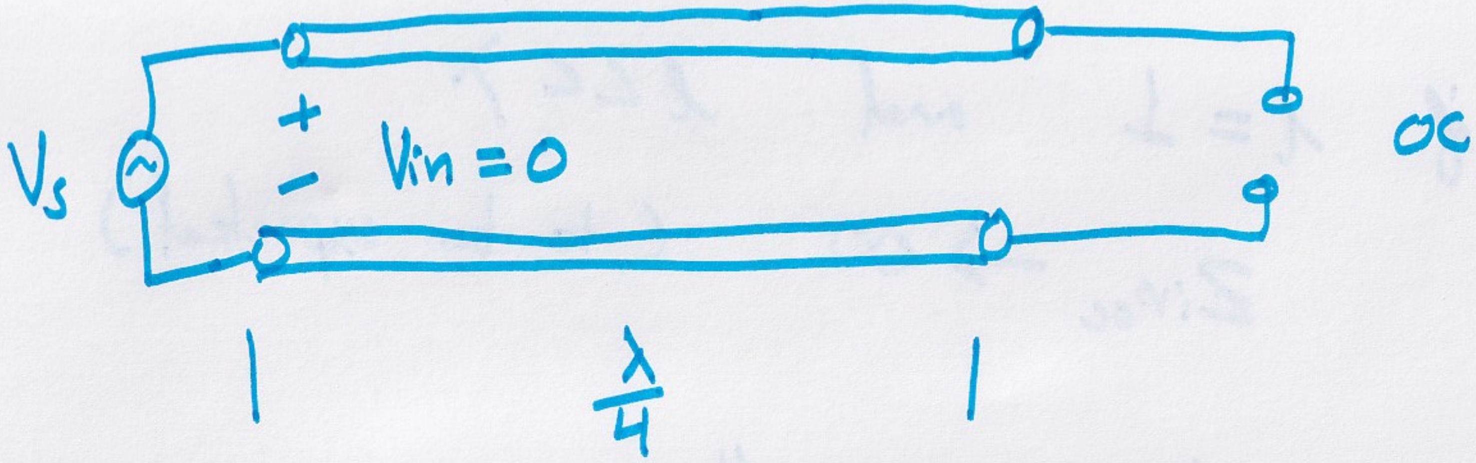
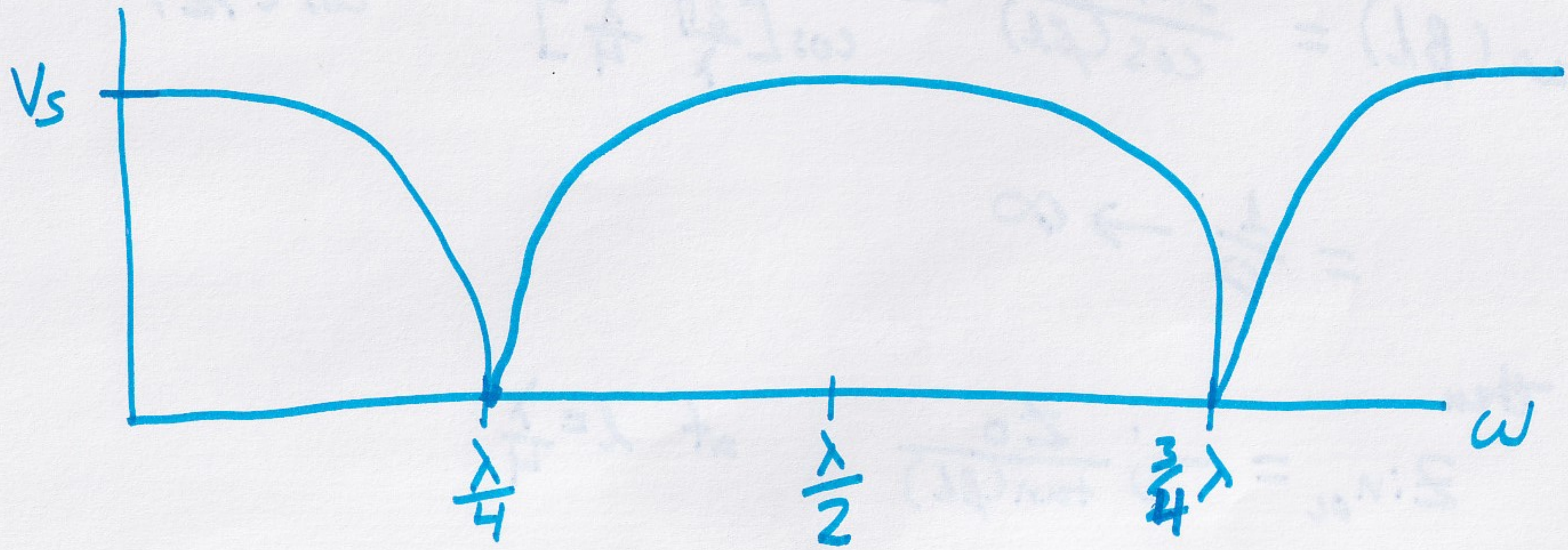


If we plot Vin



This is changing the frequency,

So if we change the frequency so that the wavelength is a third of what it was originally, (increase the frequency 3 fold)

We get $V_{in} = 0$ at $3f$ with Length $l = \frac{\lambda}{4}$

where λ is the original wave length

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for the short circuit load

$\rightarrow Z_{in}$

$Z_L = 0$ (short circuit load)

$$Z_{in} = Z_0 \frac{0 + jZ_0 \tan(\beta l)}{Z_0 + j0 \tan(\beta l)}$$

$$= jZ_0 \tan(\beta l)$$

For lengths $l = 0, \frac{\lambda}{2}, \lambda, \frac{3}{2}\lambda, \ldots$

$\longrightarrow Z_{in} = 0$

For a matched Load $(Z_L = Z_0)$

$$Z_0 = Z_{in}$$

$$Z_{in} = Z_0 \frac{\cancel{Z_L}^{Z_0} + jZ_0 \tan(\beta\ell)}{Z_0 + j\cancel{Z_L}^{Z_0} \tan(\beta\ell)}$$

$$= Z_0 (1)$$

Whenever $\tan(\beta\ell) \to 0$ for $\ell = n\lambda$

$$Z_{in} = Z_0 \frac{Z_L + jZ_0 \cancel{\tan(\beta\ell)}^{0}}{Z_0 + jZ_L \cancel{\tan(\beta\ell)}^{0}}$$

$$= Z_L$$

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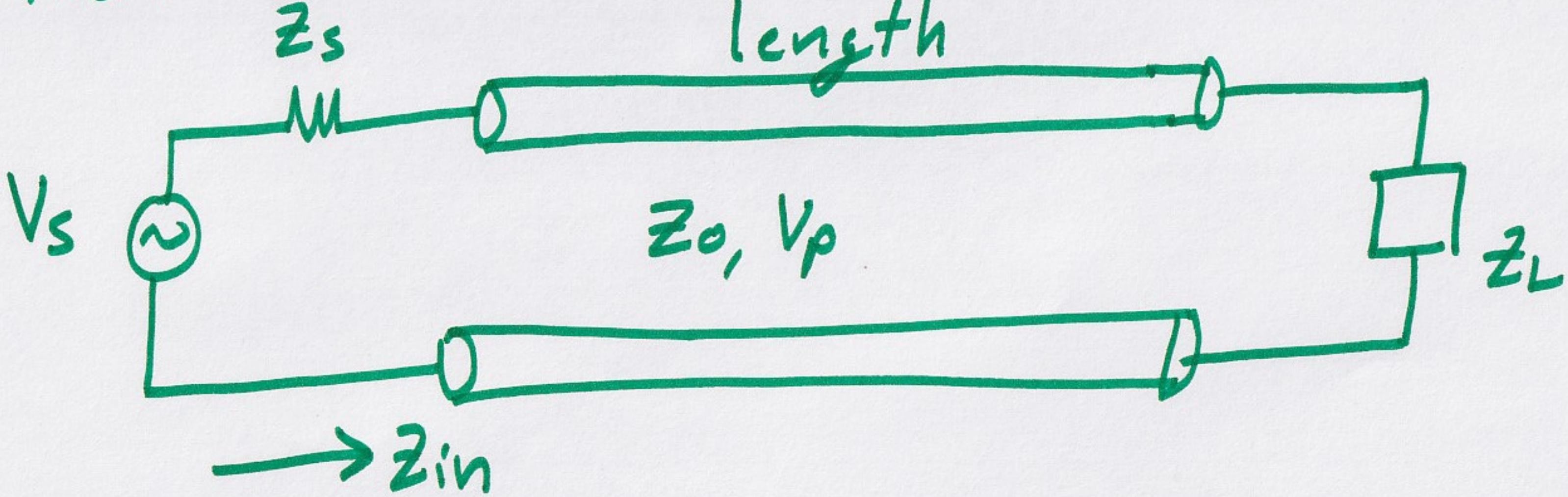
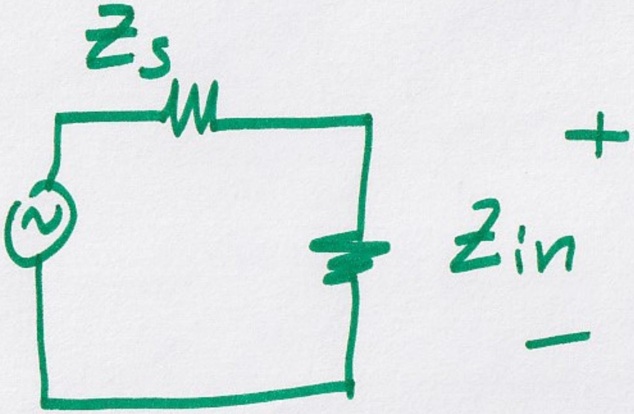
whenever $\tan(\beta l) \rightarrow \infty$ for $l = n\frac{\lambda}{2} + \frac{\lambda}{4}$

$$Z_{in} = Z_0 \frac{Z_L + jZ_0 \cancel{\tan(\beta l)}\rightarrow\infty}{Z_0 + jZ_L \cancel{\tan(\beta l)}\rightarrow\infty}$$

$$= Z_0 \frac{jZ_0(\infty)}{jZ_L(\infty)} = \frac{Z_0^2}{Z_L}$$

So then





$$V_{in} = \frac{Z_{in}}{Z_s + Z_{in}} V_s$$

V_{in} is the voltage of Z_{in} and that voltage is at the source end of the Line

$$V_{in}\Big|_{source} = V_{total}(-\ell) = V_0^+\left(1 + \Gamma e^{+2j\beta z}\right)e^{-j\beta z}$$

we evaluate V_{total} at $z = -\ell$