Pencenter complex variables #7

derivatives and the CR equations

$f(z) = |z|$ is not differentiable

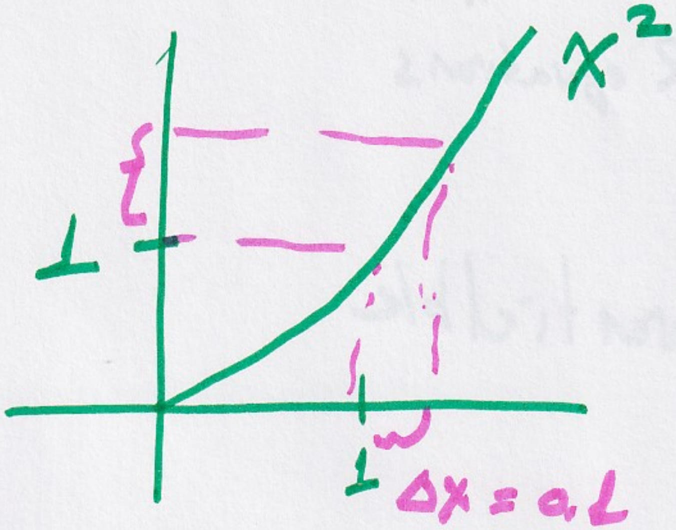
Week ___

2. In this question we will try to interpret the differentials dx, dy, Δx and Δy for a real-valued function. Let's consider $y = f(x) = x^2$

(a) if $x_0 = 1$ Evaluate $y_0 = f(x_0)$

$$y_0 = f(1) = 1^2 = 1$$

(b) If we make a small change to the x coordinate, called Δx, this will cause a small change in the y coordinate called Δy. If $\Delta x = 0.1$, determine Δy



$f(1.1) = (1.1)^2 = 1.21$

$\therefore \; 1.21 - 1 = 0.21$

Δy

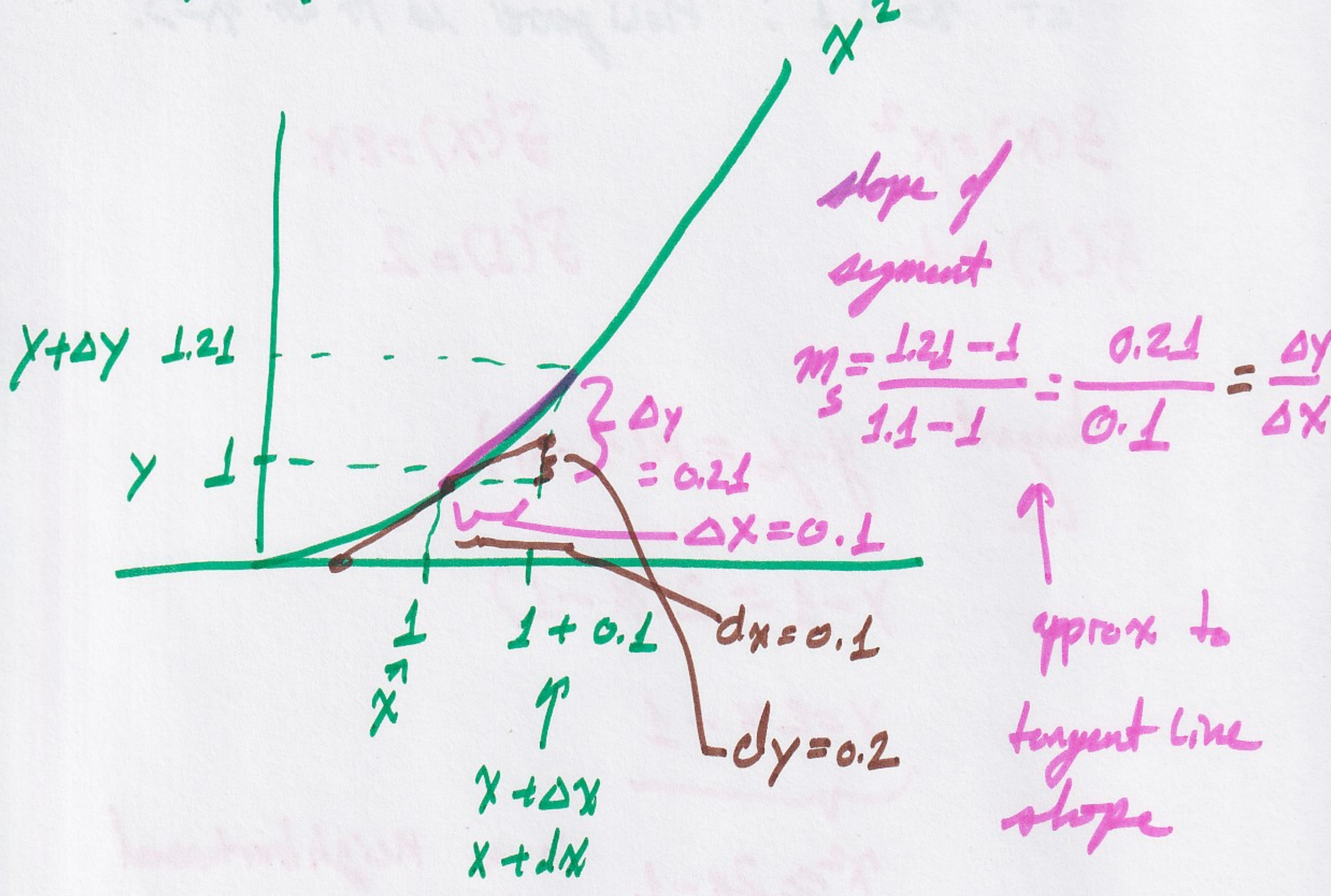
(c) we know that $\frac{dy}{dx} = 2x$. Using $x_0 = 1$ and $dx = 0.1$, evaluate dy

$dy = 2x\,dx$

$= 2(1)(0.1) = 0.2$

dy is different to Δy

(d) Plot $f(x)$, (x_0, y_0), $(x_0+\Delta x, y_0+\Delta y)$, $\Delta x = dx$, Δy, and dy on the same set of axes.



$m_T = \frac{0.2}{0.1}$

But as $\Delta x \to 0$, $\Delta y \to dy$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

(f) Explain why $x^2 \simeq 2x - 1$ in a neighborhood of $x = 1$. How "good" is this approximation at $x = 1.1$? How "good" is it at $x = 3$?

$f(x) = x^2$ $\qquad f'(x) = 2x$

$f(1) = 1$ $\qquad f'(1) = 2$

tangent line $\quad y - y_0 = m(x - x_0)$

$y - 1 = 2(x - 1)$

$$\underbrace{y = 2x - 1}$$

$x^2 \approx 2x - 1$ in a neighborhood of $x = 1$

	x^2	$2x-1$	
1.1	1.21	$2(1.1)-1 = 1.2$	close
3	9	$2(3)-1 = 5$	not close

$$\frac{d}{dz}|z| = \lim_{\Delta \to 0} \frac{|z+\Delta| - |z|}{\Delta}$$

$\leftarrow$ real

$\uparrow$ complex

3. In this question, we will try to interpret the differentials dz, dw, Δz and Δw for a complex-valued function. Let's consider $w = f(z) = z^2$.

(a) if $z_0 = 1 + i$. Evaluate $w_0 = f(z_0)$

$$f(1+i) = (1+i)^2$$
$$= 1 + 2i + i^2 = 2i$$

(b) if we make a small change in a neighborhood of z, called Δz, we expect this will cause a small change in the neighborhood of w called Δw. If $\Delta z = 0.1$, determine Δw

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$z + \Delta z = 1 + i + 0.1 = 1.1 + i$

$f(1.1+i) = (1.1+i)^2$

$= 1.21 + 2.2i + i^2$

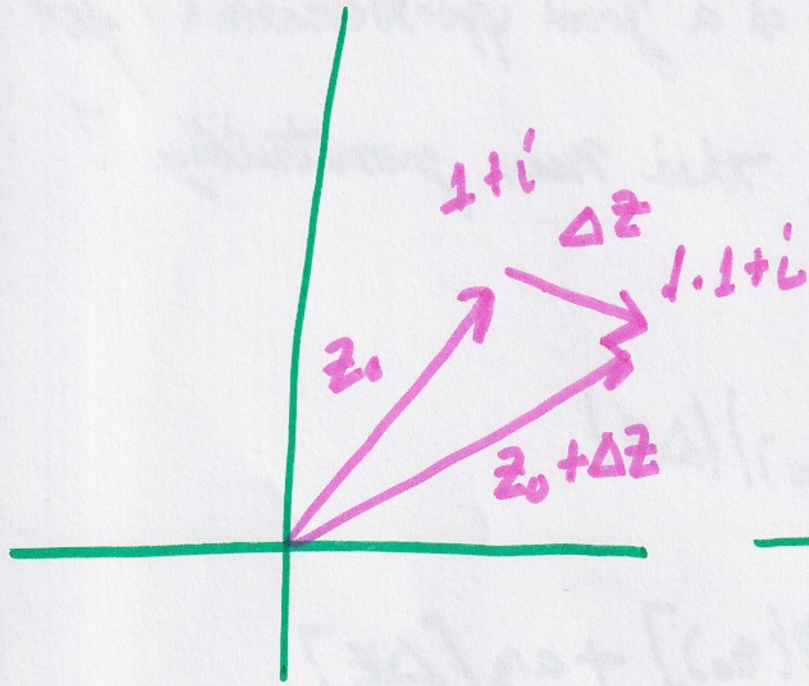
$= 0.21 + 2.2i$

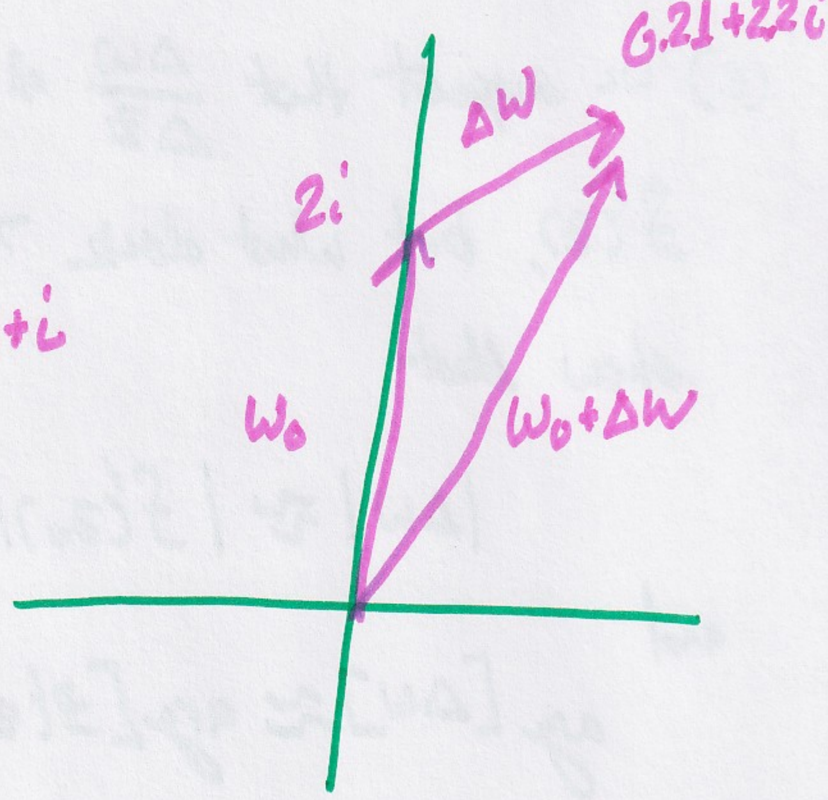
How is this different from $f(1+i)$?

$\Delta W = \underbrace{0.21 + 2.2i}_{W_0 + \Delta W} \; \underset{W_0}{\overset{W}{-2i}}$

$= 0.21 + 0.2i = \Delta W$

(c) Plot z_0, Δz and $z_0 + \Delta z$ on the same Argand diagram as a triangle. Plot w_0, Δw and $w_0 + \Delta w$ on a second Argand ~~diagramme~~ as a triangle. It may be helpful to think of these complex numbers as vectors





z plane

w plane

(d) We know that $f'(z) = 2z$ (you can show this using the limit definitions if you want).

Evaluate $f'(z_0)$

$$f'(z) = 2z$$

$$f'(1+i) = 2(1+i) = 2 + 2i$$

(e) We suspect that $\frac{\Delta w}{\Delta z}$ is a "good approximation" for $f'(z)$, but what does this mean geometrically?

Show that

$$|\Delta w| \approx |f'(z_0)||\Delta z|$$

and

$$\arg[\Delta w] \approx \arg[f'(z_0)] + \arg[\Delta z]$$

Thus, small changes of Δz in the neighborhood of $z_0 = 1+i$ get stretched by a factor of $|f'(z_0)|$ and rotated by $\arg[f'(z_0)]$.

$$\frac{\Delta w}{\Delta z} = \frac{0.21 + 0.2i}{0.1} = 2.1 + 2i$$

$$|\Delta w| = |0.21 + 0.2i| = 0.29$$

$$|f'(z_0)||\Delta z| \underset{(2)}{=} |1+i||0.1| \underset{(2)}{=} \sqrt{2} \cdot 0.1 = \cancel{2.82}$$

0.282

$$\operatorname{Arg}\Delta w \simeq \pi/4$$

close!

$$\operatorname{Arg} f'(z_0) + \operatorname{Arg}\Delta z = \pi/4 + 0 = \pi/4$$

Δz vectors are getting stretched and twisted into Δw

This action / $\Delta z \Rightarrow \underline{|f'(z_0)|}\, e^{i\pi/4}$

stretch by modulus $f'(z_0)$ and twisted by $\pi/4$

Part II: The Cauchy-Riemann Equations

We will develope a set of partial differential equations that serve as a necessary and sufficient condition for a complex-valued function to be differentiable with respect to a complex variable. Despite being named after Augustin Cauchy and Bernhard Riemann, the equations initially appeared in the work of d'Alembert and Euler.

However, Cauchy and Riemann's work on complex function theory in the mid-nineteenth century aptly caused this set of equations to be known as the Cauchy-Riemann equations.

- A complex function is entire if it is differentiable in the entire complex plane, and analytic in an open set D if it has a derivative at every point in D

- If $u(x,y)$, where $x(r,\theta)$ and $y(r,\theta)$, then $u_r = u_x x_r + u_y y_r$ by the chain rule

1.) The following is an outline of a proof for determining the Cauchy-Riemann equations. Work through each step, filling in the blanks.

(a) If we consider $z_0 = x_0 + iy_0$, we can substitute into our complex function to get

$$f(z_0) = f(x_0, y_0) = \underline{u(x_0, y_0) + i\,v(x_0, y_0)}$$

$$f(x,y) = \underbrace{u(x,y)}_{\text{real}} + i\,\underbrace{v(x,y)}_{\text{Imaginary}}$$

$$f(z) = z^2$$

$$f(x,y) = (x+yi)^2$$

$$= x^2 - y^2 + 2ixy$$

$$= (x^2 - y^2) + i(2xy)$$

$$f(x_0, y_0) = u(x_0, y_0) + iv(x_0, y_0)$$

$$f(1+i) = (1+i)^2 = 2i$$

$$f(1,1) = (1^2 - 1^2) + i\,(2\cdot 1\cdot 1)$$

$$= 2i$$

(b) we assume f is differentiable, so that the limit as $\Delta z \to 0$ must be true from all paths. Specifically, let's approach along the horizontal path $(x_0+\Delta x, y_0) \to (x_0, y_0)$. This gives

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$$

$$= \lim_{\Delta x \to 0} \frac{u(x_0+\Delta x, y_0) + iv(x_0+\Delta x, y_0) - u(x_0, y_0) - iv(x_0, y_0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{u(x_0+\Delta x - u(x_0, y_0)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x_0+\Delta x, y_0) - v(x_0, y_0)}{\Delta x}$$

① $u(x_0+\Delta x, y_0)$ ② $iv(x_0, y_0)$

③ $$\frac{u(x_0+\Delta x)-u(x_0,y_0)}{\Delta x}$$

④ $$\frac{v(x_0+\Delta x, y_0)-v(x_0,y_0)}{\Delta x}$$

(c) Next, we approach along the vertical line $(x_0, y_0+\Delta y) \to (x_0,y_0)$. This gives

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$$

$$= \lim_{\Delta y \to 0} \frac{u(x_0, y_0+\Delta y)+iv(x_0,y_0+\Delta y)-u(x_0,y_0)-iv(x_0,y_0)}{i\Delta y}$$

$$= \lim_{\Delta y\to 0} \frac{v(x_0,y_0+\Delta y)-v(x_0,y_0)}{\Delta y} - i\lim_{\Delta y \to 0}\frac{u(x_0,y_0+\Delta y)-u(x_0,y_0)}{\Delta y}$$

$$= v_y(x_0,y_0) - i\,u_y(x_0,y_0)$$

① $u(x_0, y_0+\Delta y)$ ② $iv(x_0, y_0+\Delta y)$

③ $$\frac{v(x_0, y_0+\Delta y)-v(x_0, y_0)}{\Delta y}$$

④ $$\frac{u(x_0, y_0+\Delta y)-u(x_0, y_0)}{\Delta y}$$

(d) Equating our results from (b) and (c) we see that $u_x = v_y$ and $u_y = -v_x$

2. Let's develop the polar form of the Cauchy-Riemann equations

(a) Express x and y as functions of r and θ, where $r > 0$ is the modulus and $-\pi < \theta \leq \pi$ is the argument.

$$z = x + iy = re^{i\theta}$$
$$= r[\cos\theta + i\sin\theta]$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

(b) Use the chain rule to determine u_r and v_θ. Use the Cartesian form of the Cauchy-Riemann equations to express u_r in terms of r and v_θ

$u(x,y)$ → $x(r,\theta)$, $y(r,\theta)$

$v(x,y)$ → $x(r,\theta)$, $y(r,\theta)$

$$u_r = u_x \cdot x_r + u_y \cdot y_r$$

$\cancel{\#}$ $V_\theta = V_x \cdot X_\theta + V_y \cdot Y_\theta$

$u_r = u_x \cos\theta + u_y \sin\theta$

$\cancel{V_\theta = V_x r\sin}$

$\cancel{V_\theta = -V_x r\sin}$

$V_\theta = -V_x r\sin\theta + V_y r\cos\theta$

CR equations

$U_x = V_y$ and $U_y = -V_x$

$r \cdot U_r = V_\theta$

$U_r = \frac{1}{r} V_\theta$ CR polar Equations