Pencaster Complex Variables #6
Limits, continuity, and derivatives II

(d) Since the complex logarithm depends on the argument, we suspect that we may need a branch cut here as well. If we take $\text{Log}(z) := \ln|z| + i\,\text{Arg}(z)$ as our principal branch determine where the branch cut of the logarithm function is. Evaluate $\text{Log}(-i)$

$$\text{Log}(z) := \ln|z| + i\,\text{Arg}(z)$$

↳ Principal branch of Arg

$$-\pi < \theta \leq \pi$$

$$-1 = e^{-i\pi} \qquad i = e^{i\pi/2}$$

$$\text{Arg}(-i) = \pi + \frac{\pi}{2} = \frac{3\pi}{2}$$

Same

$$\Rightarrow -\pi + \pi/2 = -\frac{\pi}{2}$$

choose $\text{Arg}(-i) = \frac{-\pi}{2}$

so $Arg(-i) = \frac{-\pi}{2}$

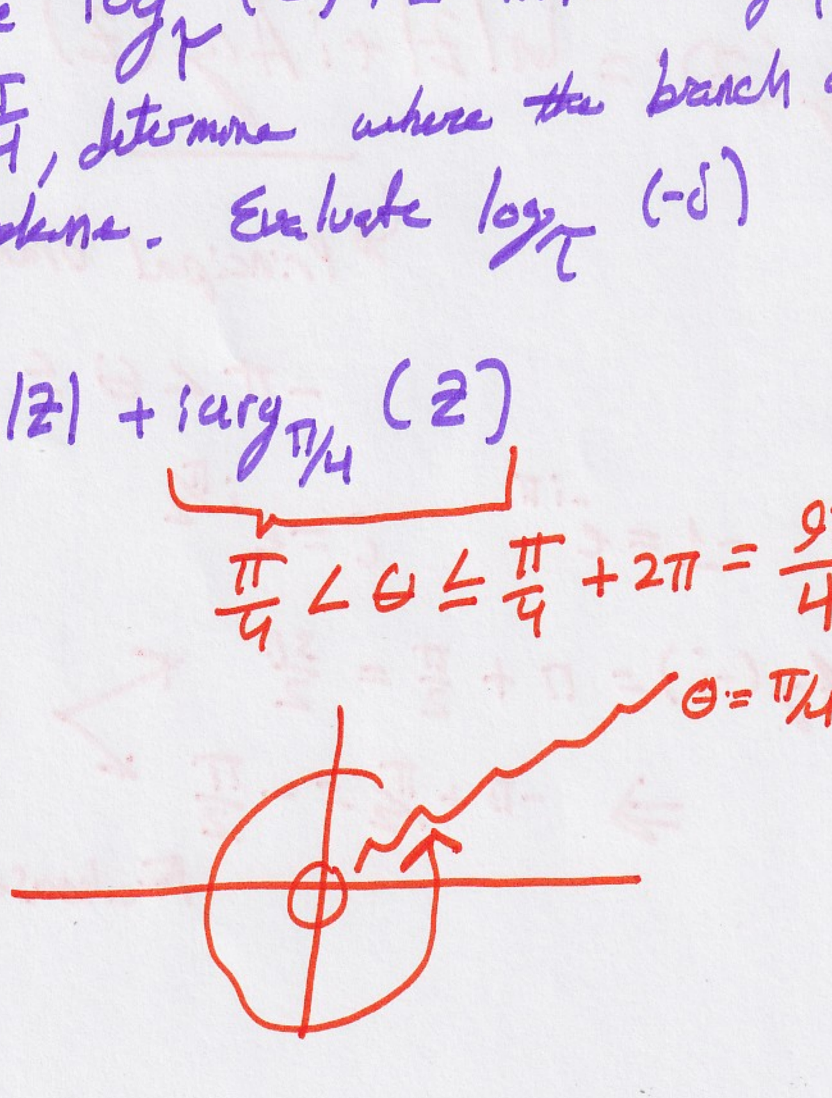
$$Log(-i) = \ln(1) + i\left(-\frac{\pi}{2}\right)$$

$$= -\frac{\pi i}{2} \rightarrow e^{-i\frac{\pi}{2}} = -i$$

$$Log(-i) = \frac{-i\pi}{2} \Leftrightarrow e^{\frac{-i\pi}{2}} = -i$$

(e) If we take $\log_\tau(z) := \ln|z| + i\arg_\tau(z)$ where $\tau = \frac{\pi}{4}$, determine where the branch cut is in the complex plane. Evaluate $\log_\tau(-8)$

$$\log_{\pi/4} := \ln|z| + \underbrace{i\arg_{\pi/4}(z)}_{\frac{\pi}{4} < \theta \leq \frac{\pi}{4} + 2\pi = \frac{9\pi}{4}}$$



$\log_{\pi/4}(-i)$

$= \ln(1) + i\left(\frac{3\pi}{2}\right)$

we have

$\frac{\pi}{4} < \theta \leq \frac{\pi}{4} + 2\pi$

So $-i = e^{i\frac{3\pi}{2}}$ or $e^{-i\frac{\pi}{2}}$

↑

Pick

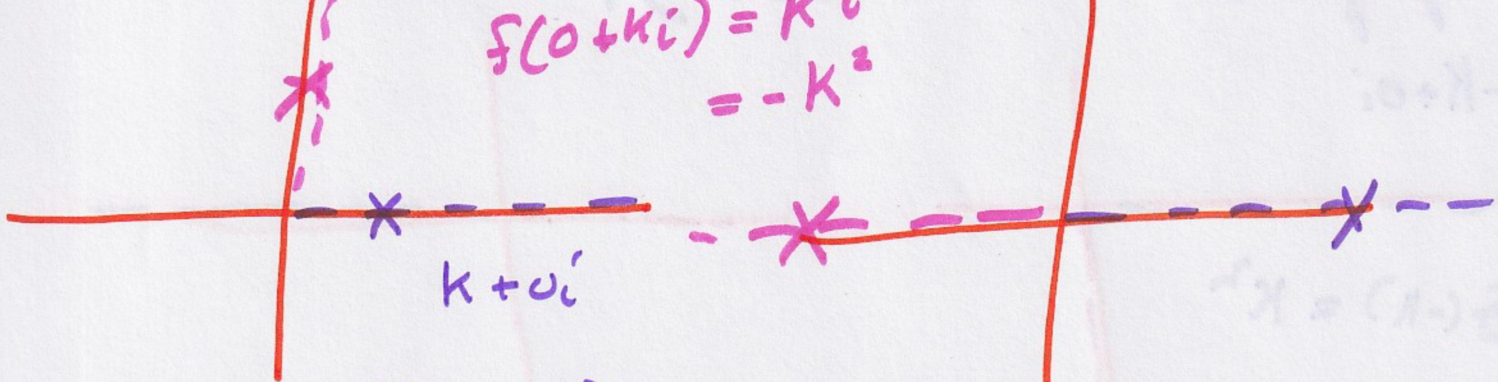
5. consider the function $f(z) = z^2$

(a) Describe the images of the rays $Re\{z\} \geq 0$ and $Im(z) \geq 0$ under f. Determine the image of quadrant one under f.

$\sim 0 + ki$

$f(0+ki) = k^2 i^2$

$= -k^2$



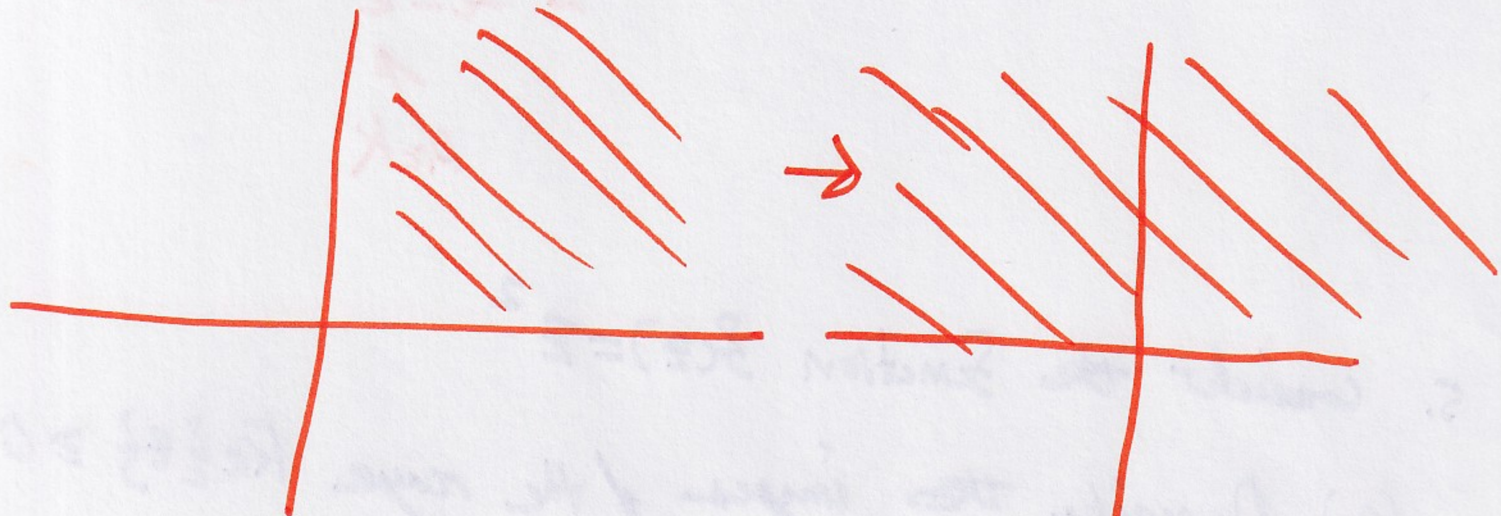
$k + 0i$

$f(k + 0i)$

$= k^2$

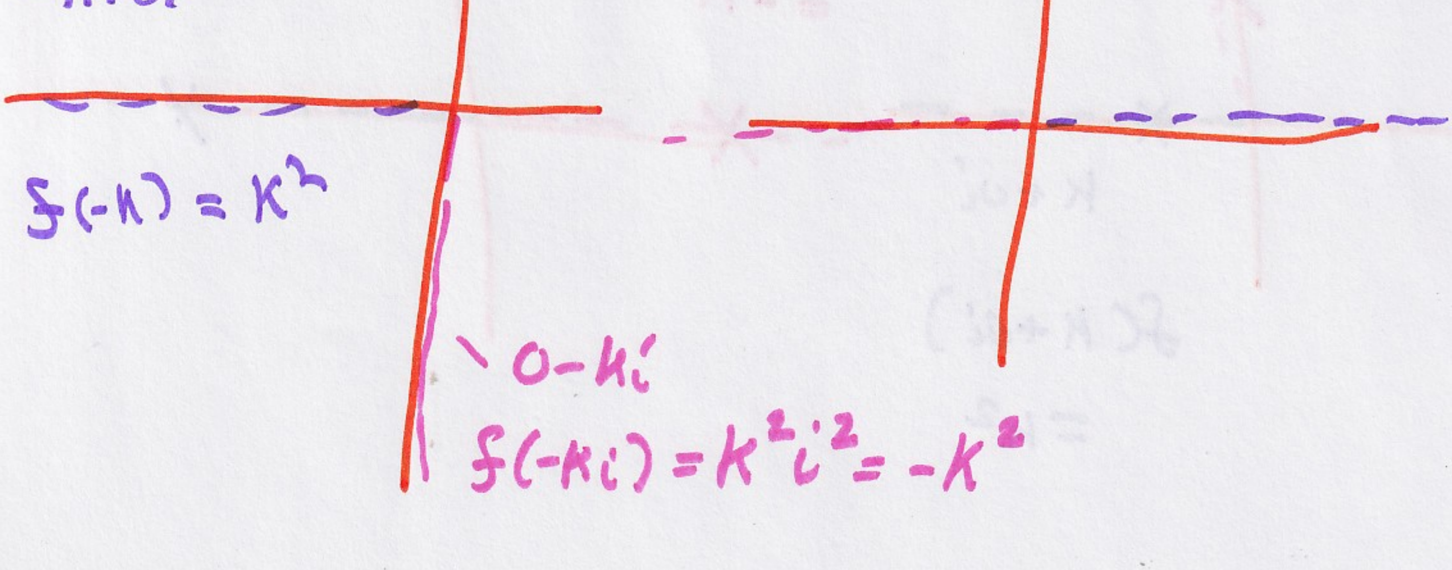
$z = re^{i\theta}$

$\hookrightarrow z^2 = r^2 e^{i2\theta}$ ← angle gets doubled!

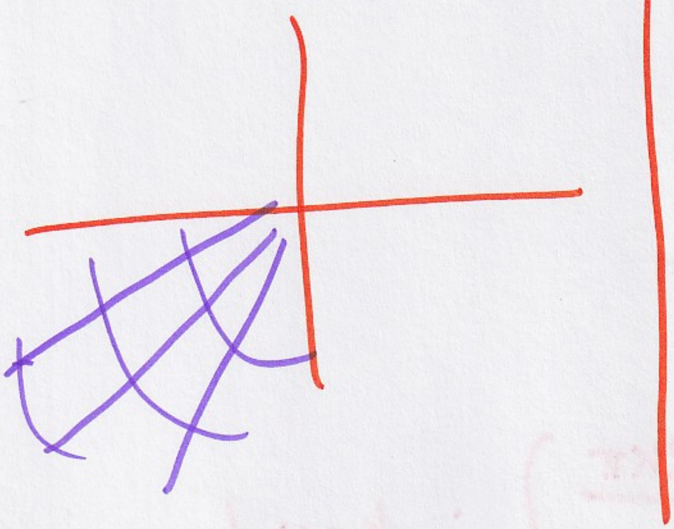


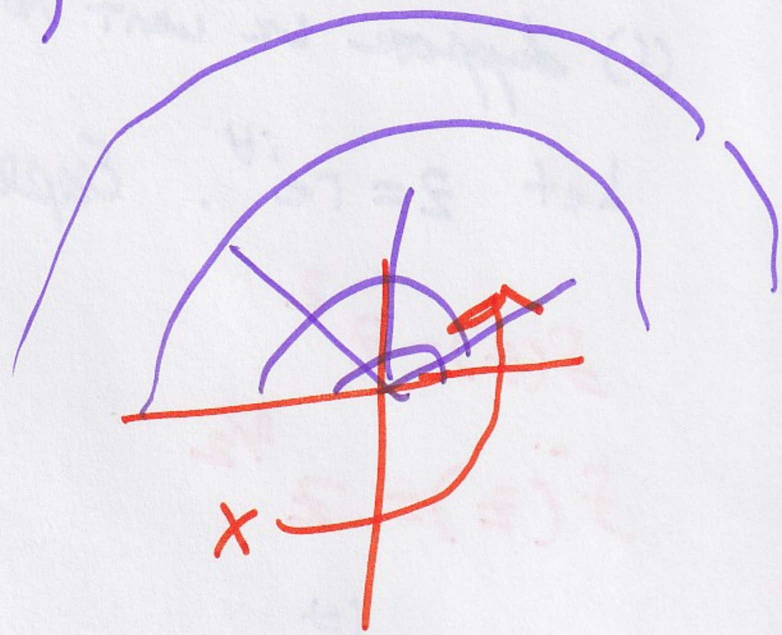
(b) Describe the images of the rays $\mathrm{Re}(z) \le 0$ and $\mathrm{Im}(z) \le 0$ under f. Determine the images of quadrant three under f.

$-K + 0i$

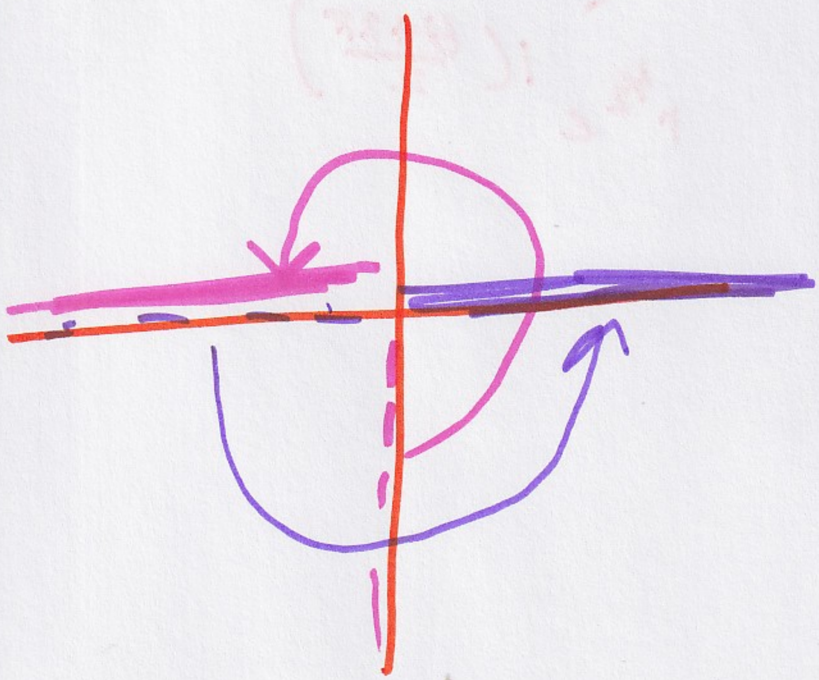


test point, $re^{i(\frac{5\pi}{4})}$





modulus is squared! $r^2 e^{i(\frac{5\pi}{4})2}$



(1) Suppose we want to define a inverse $f^{-1}(z) = z^{1/2}$.

Let $z = re^{i\theta}$. Express $z^{1/2}$ in polar form.

$$f(z) = z^2$$

$$f^{-1}(z) = z^{1/2}$$

$$z = re^{i\theta}$$

$$z^{1/2} = \left\{ r^{1/2} e^{i\left(\frac{\theta + 2k\pi}{2}\right)} \right. ; \quad k = 0, 1$$

$k = 0$

$$r^{1/2} e^{i\theta/2}$$

Solution on principle branch

$$r^{1/2} e^{i\left(\frac{\theta + 2\pi}{2}\right)}$$

(e) By definition, we have $z^{1/2} = e^{\log(z)/2}$. Show that this expression is equivalent to $|z|^{1/2} e^{i\frac{Arg(z)}{2}}$ when we take the principal branch of the logarithm function. Determine a branch cut for the principal value of the square root function.

$$z^{1/2} = e^{\log(z)/2}$$

$$= e^{[\ln|z| + i\,Arg(z)]/2}$$

Picked Principal Branches

$$= e^{\frac{\ln|z|}{2}} \cdot e^{i\,Arg(z)/2}$$

$$= e^{\ln(|z|^{1/2})}$$

$$= |z|^{1/2} \cdot e^{i\frac{Arg(z)}{2}}$$

8

(f) Express $f^{-1}(-i)$ in Cartesian form

$\sqrt{-i}$

$z = -i$

$|-i| = 1$

$Arg(z) = \frac{-\pi}{2}$

$* \ Arg \rightarrow -\pi < \theta \le \pi$

$* \ Log(z) := \ln|z| + i\,Arg(z)$

$\left.\begin{array}{l} \log(z) \\ \arg(z) \end{array}\right\}$ general function

$$\underline{z^{1/2} = |z|^{1/2} \cdot e^{i\frac{Arg(z)}{2}}}$$

gives value on principal branch

$\sqrt{-i}$?

$$\sqrt{-i} = |z|^{1/2} \cdot e^{i\frac{Arg(z)}{2}}$$

$$= \sqrt{1} \cdot e^{-i\pi/4}$$

$$= \cos(-\pi/4) + i\sin(-\pi/4)$$

$$= \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$$

Part I: Differentiability and Analytic Functions

In this section we will explore when complex functions have a complex derivative. It turns out that if $f'(z)$ exists, we can think of it as an <u>amplitwist</u>: a magnification followed by a rotation. Only a special class of functions have the property we desire: $w=f(z)$ can be viewed in a small neighborhood as an amplitwist! These are exactly the analytic functions and we will spend a significant time discussing these complex functions

- For a complex function $f(z)$ defined in a neighborhood of z_0, we define the derivative of f at z_0 to be the limit

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$$

if it exists

1. Consider the function $f(z) = |z|$

(a) Determine an expression for the difference quotient in the limit definition of the derivative.

$$f(z) = |z| = \sqrt{x^2+y^2}$$

$$\lim_{\Delta z \to 0} \frac{|z_0 + \Delta z| - |z_0|}{\Delta z}$$

(b) consider $\Delta z \to 0$ along the horizontal axis so that $\Delta z = \Delta x$. Show that the limit of the difference quotient give $\dfrac{Re(z_0)}{|z_0|}$