Percentor Complex Variables

#4, complex functions

Part I, Functions of a Complex Variable

Taylor Series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \cdots$$

We are jumping to question 3

3.) We have seen that there is a connection between the exponential function and the trigonometric functions sine and cosine, when working with complex numbers.

(a) Using the Taylor series for the exponential function, make the substitution $x = iz$. Prove algebraically that Euler's formula holds for all complex numbers z. That is, show $e^{iz} = \cos(z) + i\sin(z)$.

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$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

$$e^{iz} = 1 + iz + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \frac{(iz)^4}{4!} + \frac{(iz)^5}{5!}$$

$$= 1 + iz + -\frac{z^2}{2!} - \frac{iz^3}{3!} + \frac{z^4}{4!} + \frac{iz^5}{5!}$$

$$= \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots\right) + \left(iz - \frac{iz^3}{3!} + \frac{iz^5}{5!} - \cdots\right)$$

$$= \cos(z) \qquad + i\sin(z)$$

(b) explain why $e^{-iz} = \cos(z) - i\sin(z)$

even odd

$$e^{-iz} = e^{i(-z)} = \cos(-z) + i\sin(-z)$$

$$= \cos(z) - i\sin(z)$$

(c) Show that it makes sense algebraically to define $\cos(z) = \dfrac{e^{iz} + e^{-iz}}{2}$

$$e^{iz} \rightarrow \cos z + i \sin z$$

$$e^{-iz} \rightarrow \cos(z) - i\sin(z)$$

$$e^{iz} + e^{-iz} = 2\cos(z) \rightarrow \cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

(d) Show that it makes sense algebraically to define $\sin(z) = \dfrac{e^{iz} - e^{-iz}}{2i}$

$$e^{iz} - e^{-iz} = \cos(z) + i\sin(z) - [\cos z - i \sin z]$$

$$= 2i\sin(z)$$

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$

4.

(e) Prove, using the above definitions, that $\sin^2 z + \cos^2 z = 1$

$$\sin^2 z + \cos^2(z)$$

$$= \left(\frac{e^{iz}-e^{-iz}}{2i}\right)^2 + \left(\frac{e^{iz}+e^{-iz}}{2}\right)^2$$

$$= \frac{e^{2iz}-2e^{iz}e^{-iz}+e^{-2iz}}{4i^2} + \frac{e^{2iz}+2e^{0}+e^{-2iz}}{4}$$

$$= \frac{e^{2iz}-2+e^{-2iz}}{-4} + \frac{e^{2iz}+2+e^{-2iz}}{4}$$

$$= \frac{-e^{2iz}+2-e^{-2iz}+e^{2iz}+2+e^{-2iz}}{4}$$

$$= \frac{4}{4} = 1$$

#2 Let $f(z) = e^z$

(a) Evaluate $f(0)$, $f(2i\pi)$ and $f(8i\pi)$

$f(0) = e^0 = 1$

↳ $\cos 0 + i \sin 0 = 1$

$f(2i\pi) = e^{2i\pi} = \cos(2\pi) + i\sin(2\pi)$
$= 1$

$f(8i\pi) = 1$

(b) Make a conjecture about when $f(z) = 1$ and try to prove your conjecture

Every 2π rotation makes we land back at point $1 + 0i$

So any $2k\pi$ rotations ($k \in \mathbb{Z}$) should give $1 + 0i$

$\cos(z) + i\sin(z) = 1 + 0i$

$\rightarrow \cos(z) = 1$ and $\sin z = 0$

$\rightarrow \boxed{z = 0 + 2ik\pi}$

multiple solutions

e^z is not one to one like e^x is

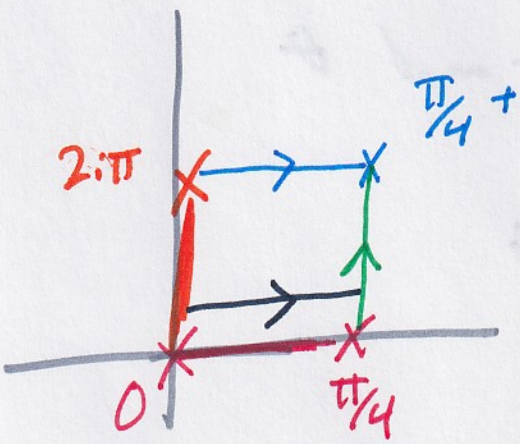
(c) Evaluate $f(\frac{\pi}{4})$ and $f(\frac{\pi}{4}(1+8i))$

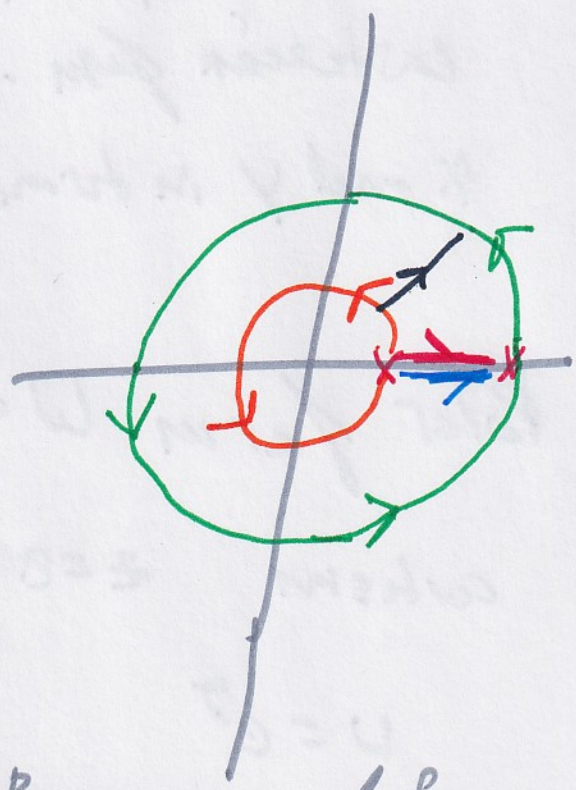
$e^z = \cos(z) + i\sin(z)$

$e^z = e^{\pi/4} = 2.19$

$e^{\frac{\pi}{4}(1+8i)} = e^{\pi/4} \cdot e^{\frac{8\pi i}{4}} = 2.19$

(d) Draw ~~two~~ Argand diagrams, the first containing the square with corners $0, \frac{\pi}{4}$, $\frac{\pi}{4}(1+8i)$ and $2i\pi$, and the second containing the image of this square under the mapping $f(z)$. What happens to horizontal lines under $f(z)$? What happens to vertical lines?





Horizontal Lines → Become rays of form $\varphi = \theta$

Vertical Lines → Become circles

$$\cancel{e^z}\; e^{x+iy} = e^x e^{iy} = e^x[\cos y + i\sin y]$$

5.) In this question we will try to intuitively deduce the complex logarithmic function. Suppose $w = e^z$, where w and z are complex numbers

(a) Using the equation $W = e^z$, replace W with its polar form, and replace z with its cartesian form. Deduce the values of x and y in terms of r and θ

Polar form $W = re^{i\theta}$

cartesian $z = x + iy$

$W = e^z$

$re^{i\theta} = e^{x+iy}$

$re^{i\theta} = e^x e^{iy}$

Like parts

$e^{i\theta} = e^{iy}$ $\quad r = e^x$

$\theta = y$ $\quad \ln(r) = x$

$(r > 0)$

(b) If we assume $w = e^{x+iy}$ then it makes algebraic sense to define the complex logarithm as $Log(w) := x + iy$ (here we use capital L to differentiate the complex logarithm from the real logarithm). Using your answer from part (a), define the complex logarithmic function. Discuss any potential issues.

$Log(w) = x + iy$

$w = re^{i\theta}$

$Log(w) = \ln(r) + i\theta$

If $w = 1$, we know $e^0 = e^{2i\pi} = e^{8i\pi} = 1$

Problem: many inputs mapped to 1

How do we map 1 back to?

$x^2 = 9 \longrightarrow$ ~~$x = 3$ or $x = -3$~~

$x = +3$ or $x = -3$

$x = \sqrt{9}$

which one?

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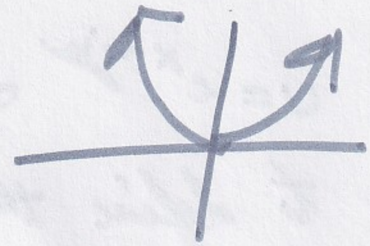
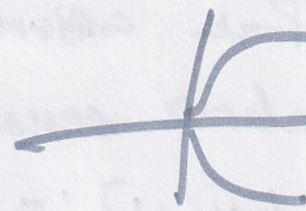
IDEA





We want to Select a region in complex plane where the Inputs all give unique outputs

$\sqrt{3} = 3$ principal root

$$\text{Log}(\omega) = \ln(r) + i\theta$$

issue

To Solve issue: take $-\pi < \theta \leq \pi$

Principle Branch of complex Logarithm.

(c) Evaluate Log(-1), Log(i) and Log(-i). Does the formula Log(wz) = Log(w) + Log(z) hold for all complex numbers? Why or why not?

$$Log(z) = \ln|z| + i\theta_z$$

$$Log(-1) = \ln(1) + i\pi$$
$$= 0 + i\pi$$
$$= i\pi$$

$$Log(i) = \ln(1) + i\,\pi/2$$
$$= i\pi/2$$

$$Log(-i) = \ln(1) + i\left(\frac{-\pi}{2}\right) = -i\frac{\pi}{2}$$

$$Log(-i) = Log(-1 \cdot i) \neq \underbrace{Log(-1) + Log(i)}_{\substack{i\pi + i\pi/2 \\ i3\pi/2}}$$