Rencentro Complex Variables

class 5

Part II: Limits and Continuity

- We call L the limit of a sequence of complex numbers z_n and write $\lim_{n\to\infty} z_n = L$ provided that $\forall\ \epsilon > 0\ \exists$ a whole number ~~nubr~~ N s.t. $n > N$ implies $|z_n - L| < \epsilon$

- We call L the limit of $f(z)$ as z approaches a and write $\lim_{z\to a} f(z) = L$, provided that $\forall\ \epsilon > 0$ $\exists\ \delta > 0$ s.t. $|z-a| < \delta \longrightarrow |f(z) - L| < \epsilon$

'implies'

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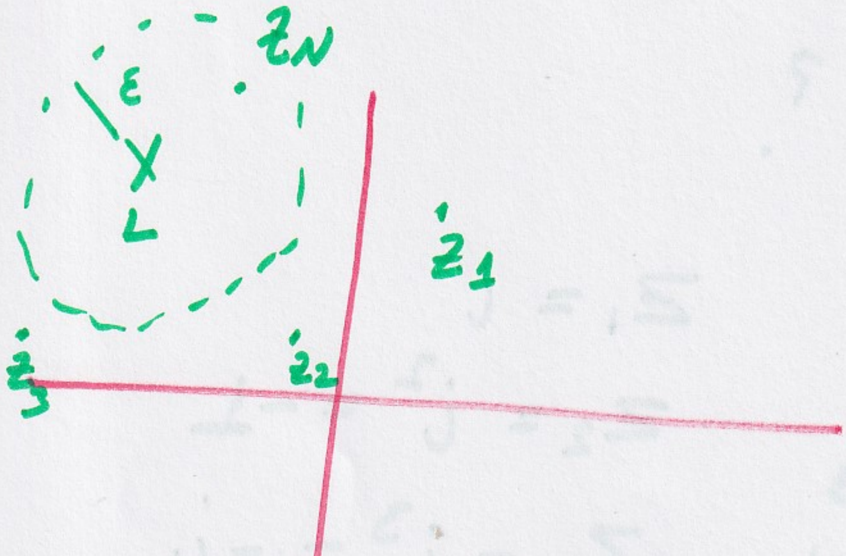
- A function $f(z)$ is said to be continuous at $z=a$ provided that $\lim_{z\to a} f(z) = f(a)$

- A relation $f: X \to Y$ is a function if it assigns each element of set X to exactly one element of Y

1.) In part (a), you may assume z_n is a generic sequence of complex numbers. In parts (b) and (c), a closed form will be explicitly given

(a) Use an Argand diagram to plot L, some values of z_n that satisfy $n < N$, some values of z_n that satisfy $n > N$, and the set $|z_n - L| < \varepsilon$



$|z_n - L| < \varepsilon$
↳ circle of radius ε (open)

when we get to Z_n, we are inside the open circle $|Z_n - L| < \varepsilon$

* <u>Note</u>: Since we are working in a 2D space, we may require convergence from <u>all</u> directions.

(b) Is the sequence convergent or divergent?

$$Z_n = i^n ?$$

(c) Is the sequence

$$Z_n = \left(\frac{i}{3}\right)^n ?$$

b) $n=1$ $\quad Z_1 = i$

$n=2$ $\quad Z_2 = i^2 = -1$

$n=3$ $\quad Z_3 = i^3 = -i$

$n=4$ $\quad Z_4 = i^4 = 1$

n even $\longrightarrow$ -1 or 1

n odd $\longrightarrow$ i or $-i$

$\lim_{x \to \infty} \sin(x) \longrightarrow ?$

$$\left.\begin{array}{l} \lim_{n \to \infty} \sin(2\pi n) \longrightarrow 0 \\ \lim_{n \to \infty} \sin(2\pi n + \pi/2) \longrightarrow 1 \end{array}\right\}$$ not the same no limit

sin fluctuates between $1, -1$

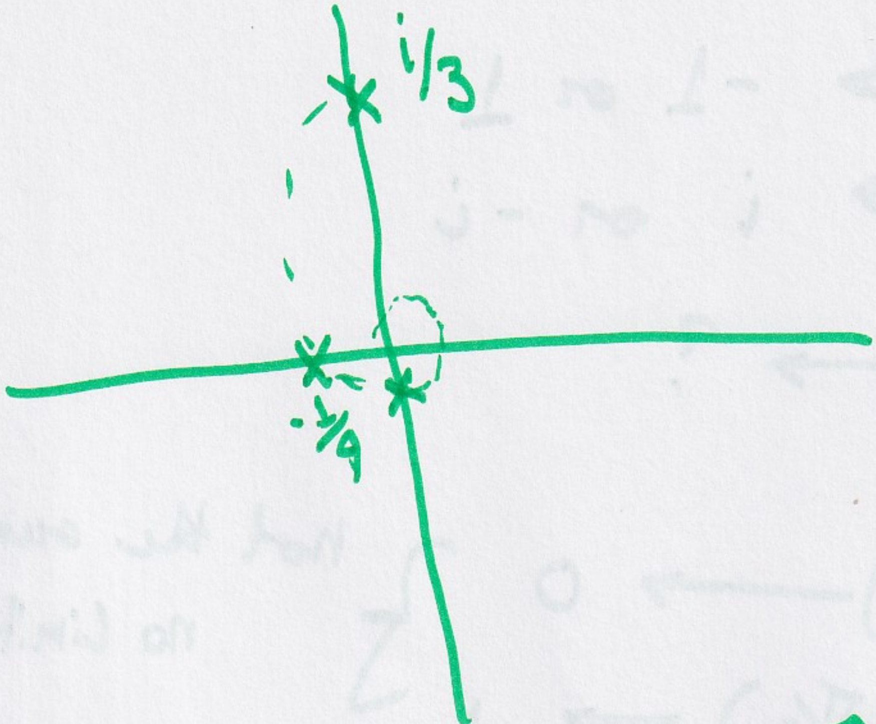
$\lim_{x \to 0} \sin(x) = 0$

There is a difference between complex limits of sequences and complex limits of sequences

c) $z_n = (i/3)^n$

$z_1 = i/3$

$z_n = (i/3)^n$



Looks like we are spiralling toward origin.

Suspect $L=0$

$$z_2 = \left(\frac{i}{3}\right)^2 = \frac{i^2}{9} = -\frac{1}{9}$$

$$z_3 = \left(\frac{i}{3}\right)^3 = -\frac{i}{27}$$

$$\left(\frac{i}{3}\right)^n = \left(\frac{1}{3}\right)^n (i)^n$$

rotation by $\pi/2$

decrease of modulus by 3 each time

$$\left(\frac{i}{3}\right) = \left(\frac{1}{3}\right)^n e^{n\pi/2}$$

(d) make a conjecture about convergence of complex sequences of the form

$z_n = a^n \qquad a \in \mathbb{C}$

$|a| < 1 \longrightarrow$ converges to $L = 0$

$|a| > 1 \longrightarrow$ diverges to ∞

$|a| = 1 \longrightarrow a = 1$ ok

$\qquad\quad \searrow a = e^{i\theta}$

$a^n = (re^{i\theta})^n$

$= r^n \underbrace{e^{in\theta}}_{|e^{in\theta}| = 1}$

→ So the sequence is completely determined by r^n for $|a| \neq 1$

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$r > 1 \rightarrow$ divergence to ∞

$r < 1 \rightarrow$ convergence $(L=0)$

$\rightarrow$ if $|a| = 1$

(1) $a = 1 \rightarrow \{1, 1, 1, \ldots\}$

converges to $L = 1$

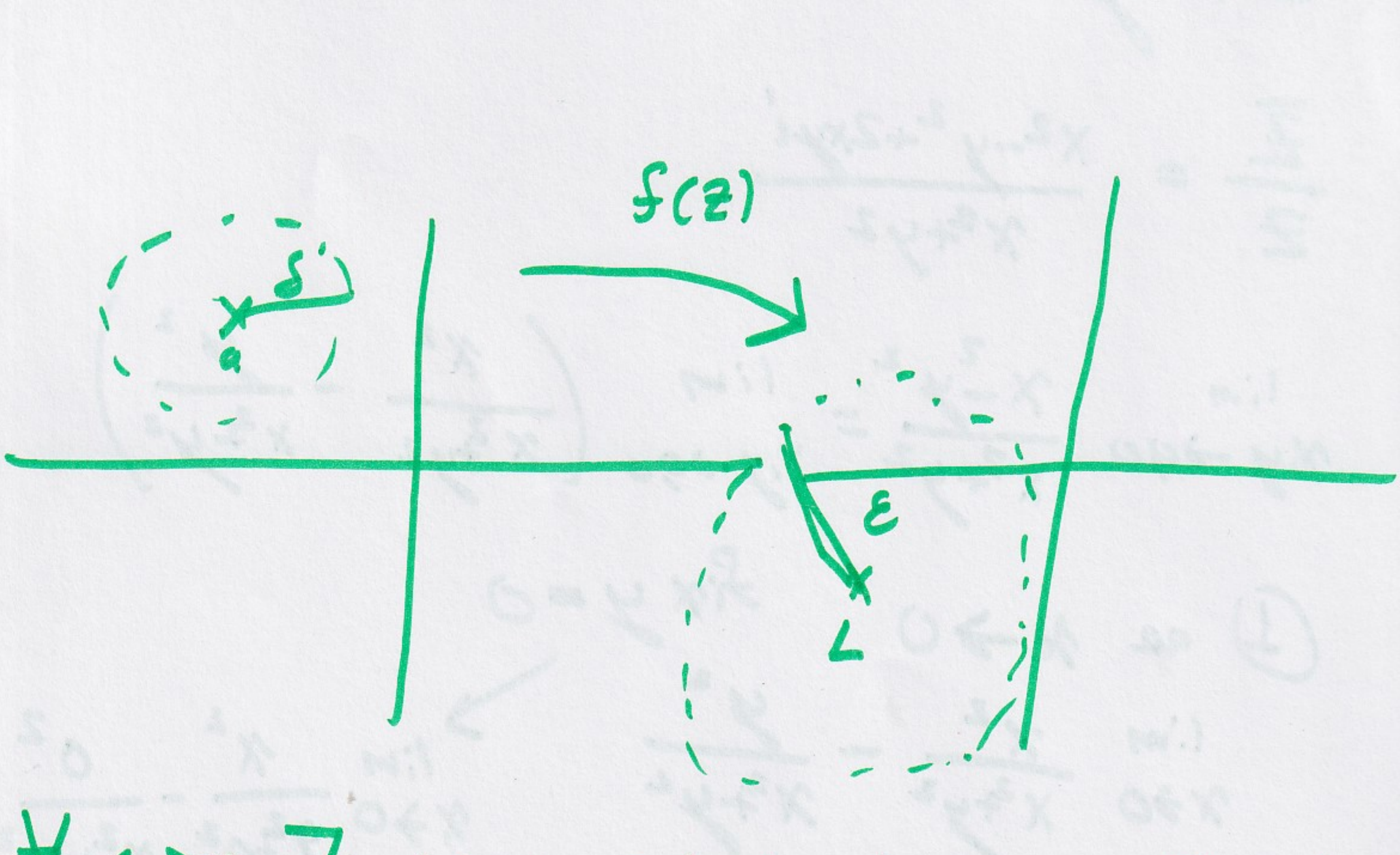
(2) $a \neq 1$ rotates around unit circle

$a = e^{i\theta}$ $(\theta \neq 0)$

Limit ~~is one~~
Does not exist

2.) In part (a) you may assume $f(z)$ is a generic complex function. In parts (b) and (c), a specific function will be given.

(a) Use two Argand diagrams to help visualize the limit of a function. On the first diagram, plot a and the set $|z-a|<\delta$. On the second diagram, plot L and the set $|f(z)-L|<\varepsilon$. Discuss the connection between your diagrams.



$\forall\, \varepsilon>0\ \exists\ \delta>0$ s.t. $|z-a|<\delta$ implies $|f(z)-L|<\varepsilon$

#2 is a HW problem

3.) Show that the following limits do not exist

(a) $\lim_{z \to 0} \frac{\bar{z}}{z}$

Goal: find two paths $z \to 0$, but limits come out different

$z = x + iy$

$\bar{z} = x - iy$

$$\frac{\bar{z}}{z} = \frac{x^2 - y^2 + 2xyi}{x^2 + y^2}$$

$$\lim_{x,y \to 0,0} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{x,y \to 0,0} \left(\frac{x^2}{x^2+y^2} - \frac{y^2}{x^2+y^2} \right)$$

① as $x \to 0$ fix $y = 0$

$$\lim_{x \to 0} \frac{x^2}{x^2+y^2} - \frac{y^2}{x^2+y^2} \to \lim_{x \to 0} \frac{x^2}{x^2+0^2} - \frac{0^2}{x^2+0^2}$$

$$= \lim_{x \to 0} \frac{x^2}{x^2} = 1$$

$$\lim_{y \to 0} \frac{0^2}{0^2+y^2} - \frac{y^2}{0^2+y^2}$$

$$= \lim_{y \to 0} \frac{-y^2}{y^2} = -1$$

$$\lim_{z \to 0} Re\left(\frac{\bar{z}}{z}\right)$$ ~~ONE~~ Limit Does Not Exist

$$\therefore \lim_{z \to 0} \frac{\bar{z}}{z}$$ ~~ONE as well~~ Limit Does Not Exist

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Polar:

$$\lim_{z \to 0} \frac{\bar{z}}{z} = \lim_{r \to 0} \frac{re^{-i\theta}}{re^{i\theta}}$$

$$= \lim_{r \to 0} e^{-2i\theta} = e^{-2i\theta}$$

so this limit depends on the argument

$Arg(z) = \theta \in [-\pi, \pi]$

So Limit Does Not Exist

(b) $\lim_{z \to 0} f(x+iy)$, where $f(x+iy) = \frac{x^2}{x^2+y^2} + \underline{2i}$

$\lim_{z\to 0} 2i = 2i$

$z = re^{i\theta}$

$|z|^2 = r^2 = x^2+y^2$

$x = r\cos\theta$

$y = r\sin\theta$

$\lim_{z\to 0} f$

$= \lim_{r\to 0} \frac{r^2\cos^2\theta}{r^2} = \cos^2\theta$

Limit depends on θ

So Limit does not exist

cuz $\lim_{z\to 0} f = \lim_{r\to 0} \frac{r^2\cos^2\theta}{r^2} + 2i$

$= \underbrace{\cos^2\theta}_{\text{dependent on } \theta} + 2i$

$$\lim_{x,y \to 0,0} \frac{x^2}{x^2+y^2} + 2i$$

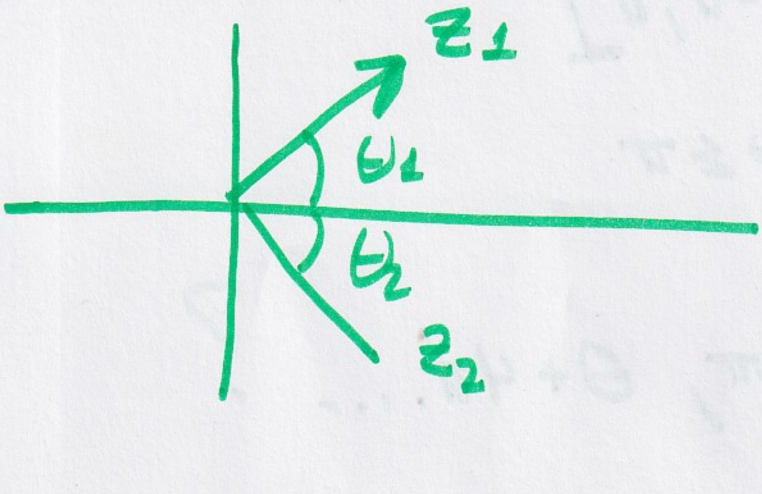
1. fix $x=0$, $y \to 0$

$$\lim_{y \to 0} \frac{0^2}{0^2+y^2} + 2i = 2i$$

2. fix $y=0$, $x \to 0$

$$\lim_{x \to 0} \frac{x^2}{x^2+0^2} + 2i = 1 + 2i$$

Since these limits are not the same, overall limit Does not Exist



If we approach along different rays we get different limits

4.) (a) Let's define the Argument of a complex number to be ~~the~~ function ~~Arg(z): X~~

$\mathrm{Arg}(z): X \to Y$, where $Y = (-\pi, \pi]$.

Explain why the restriction on Y must be present. Why can we not allow $Y = \mathbb{R}$, for example.

$\mathrm{Arg}(z)$ $\qquad e^{i\theta} = e^{i[\theta + 2\pi]}$

principal argument

$e^{i\theta} \longrightarrow$ if $y = [-\pi, \pi]$

$-\pi < \theta \leq \pi$

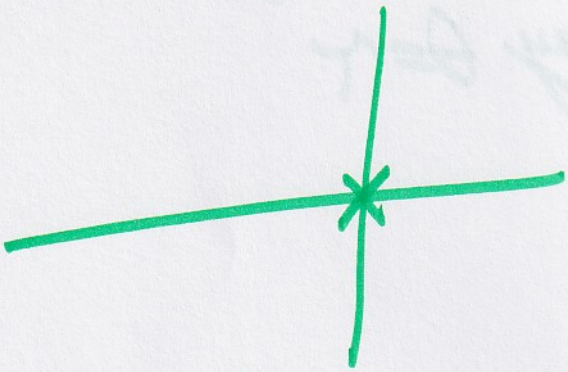
$e^{i\theta} \longrightarrow \theta,\ \theta + 2\pi,\ \theta + 4\pi \ldots$?

* we need to be able to map to the entire unit circle in non-ambiguous way

(b) Describe ~~the~~ domain X of our function

What is $\text{Arg}(0)$?

→ not defined



$X = \mathbb{C} - \{0\}$

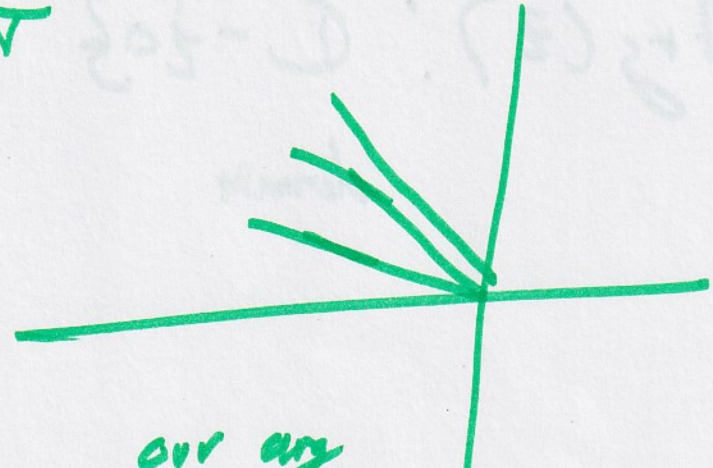
remove that point at 0,0

$$\text{Arg}(z) : \underbrace{\mathbb{C} - \{0\}}_{\text{domain}} \longrightarrow \underbrace{(-\pi, \pi]}_{\text{range}}$$

(c) Explain why $Arg(z)$ is not continuous along ~~the~~ the ray $Re\{z\} < 0$. Due to this, we create an artificial barrier along this ray that we agree not to cross, and this forms the principal branch of the argument function. We note that other branches can be selected using the map $arg_\tau(z): X \to [\tau, \tau + 2\pi]$, causing the branch cut to land along the ray $\theta = \tau$

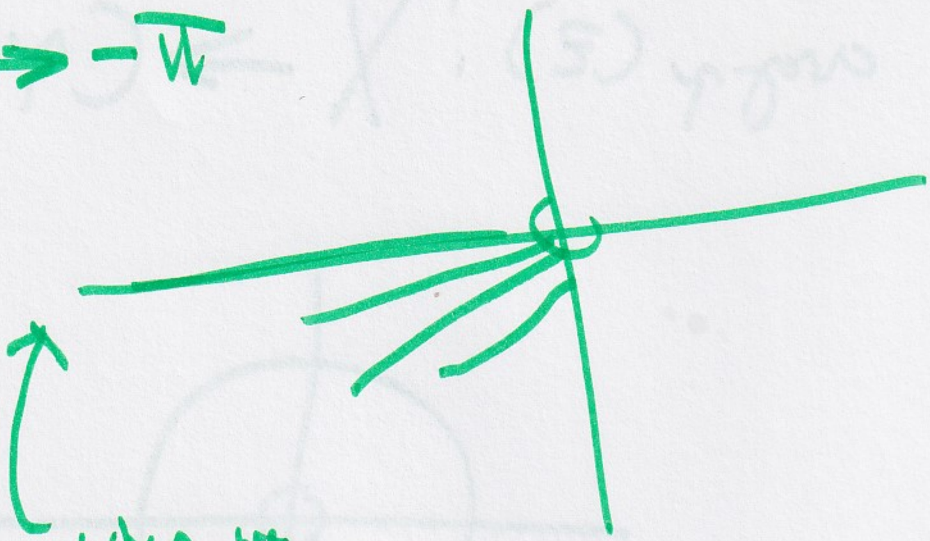
(1) As we approach from above

$Arg(z) \longrightarrow \pi$



(2) As we approach from below

$Arg(z) \longrightarrow -\pi$



when we run ~~the~~ into this piece, we come across issues

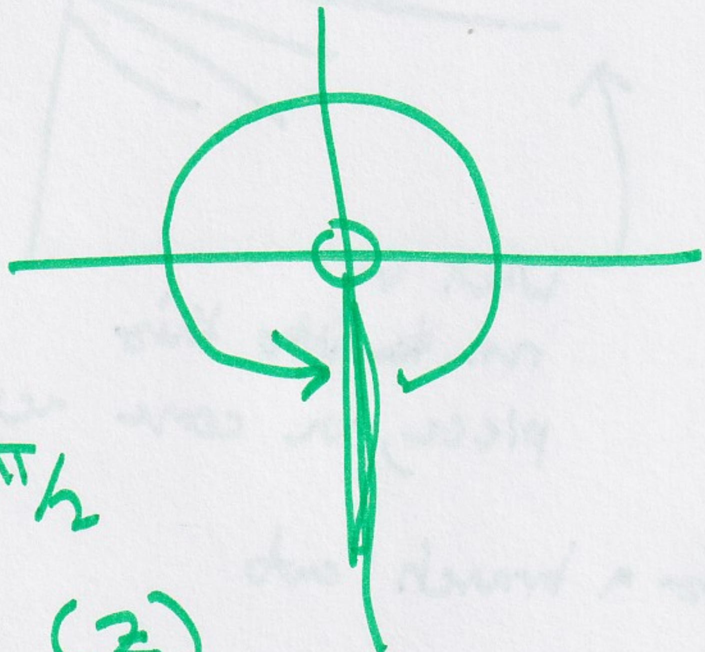
we call this a branch cut

The Limit doesn't Exist along the array

so $Arg(z)$ is not continuous on $Re(z) < 0$

18 zero is not included in the domain

$$\arg_\tau(z) : X \to (\tau, \tau + 2\pi]$$



$\tau = -\pi/2$

$\arg_{-\pi/2}(z)$

$-\frac{\pi}{2} < \theta \le \frac{3\pi}{2}$

Branch cut is a place of discontinuity