Meeting # 9

Repeated Roots and Reduction of Order

we have the equation

$$ay'' + by' + cy = 0 \qquad (1)$$

And the roots of the characteristic equation

$$ar^2 + br + c = 0 \qquad (2)$$

we had either roots that were

real and Different

or

complex conjugates

Now we consider roots that are equal

$$r_1 = r_2$$

(The discriminant is zero)

from the quadratic formula

$$r_1 = r_2 = -\frac{b}{2a}$$

we can see a problem,

are we going to have the same solution?

Example 1

Solve

$$y'' + 4y' + 4y = 0$$

the characteristic equation is

$$r^2 + 4r + 4 = (r+2)^2 = 0$$

so $r_1 = r_2 = -2$

well, one solution is $y_1 = e^{-2t}$

→ To find the general solution, we need the second solution

Method that originated by D'Alembert

— $y_1(t)$ is a solution of (1)

$\rightarrow c\,y_1(t) \;\; \forall c \in \mathbb{R}$

we generalize this observation by replacing c with a function $v(t)$. Then we determine $v(t)$ so the product $v(t)y_1(t)$ is also a solution of (1)

To do this, we substitute

$$y = v(t)\,y_1(t) = v(t)\,e^{-2t}$$

we have

$$y' = v'(t)\,e^{-2t} - 2\,v(t)\,e^{-2t}$$

and

$$y'' = v''(t)\,e^{-2t} - 4v'(t)\,e^{-2t} + 4\,v(t)\,e^{-2t}$$

with these plug into

$$y'' + 4y' + 4y = 0$$

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we have

$$[v''(t) - 4v'(t) + 4v(t) + 4v'(t) - 8v(t) + 4v(t)]$$

$$(\quad) \times e^{-2t} = 0$$

From this, we find

$$v''(t) = 0$$

$$\cancel{v''(t) = 0}$$

Therefore

$$v'(t) = C_1 , \quad v(t) = C_1 t + C_2$$

Now that we have $v(t)$

$$\rightarrow y = v(t)\, y_1(t)$$

$$= C_1 t e^{-2t} + C_2 e^{-2t}$$

first solution

second solution

we can ~~verify~~ verify that these two solutions form a fundamental set by calculating their Wronskian

$$W(y_1, y_2)(t) = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & (1-2t)e^{-2t} \end{vmatrix}$$

$$= e^{-4t} - 2te^{-4t} + 2te^{-4t} = e^{-4t}$$

Therefore we may say

$$y_1(t) = e^{-2t} \qquad y_2(t) = te^{-2t}$$

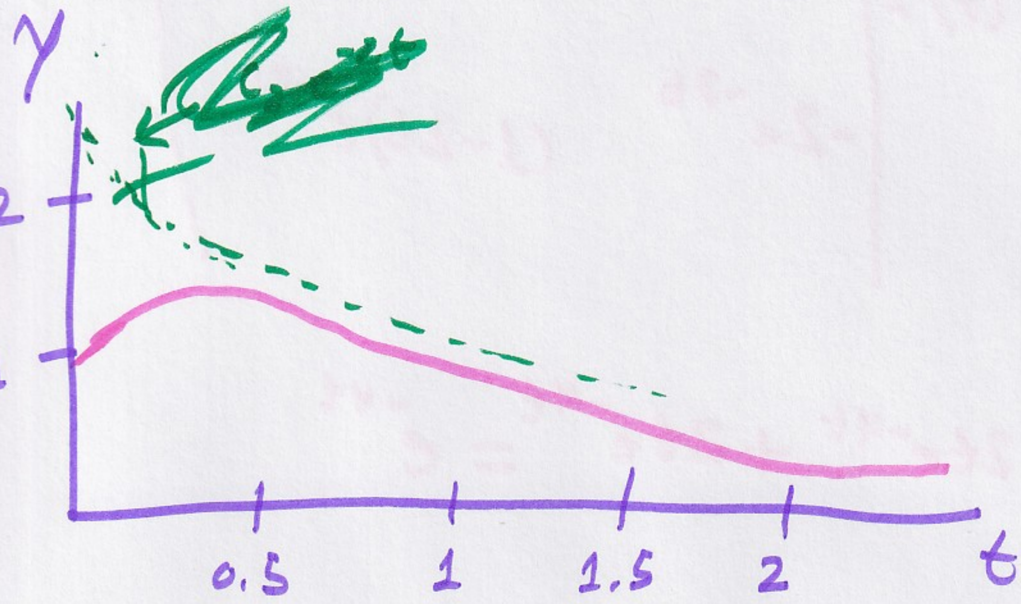
~~These~~ form a fundamental set of solutions for (5)

$$y = c_1 t e^{-2t} + c_2 e^{-2t}$$

is the general solution

$y_1, y_2 \to 0$ as $t \to \infty$

All solutions of (5) behave this way



The procedure used in Example 1 can be extended to a general equation whose characteristic equation has repeated roots. When the coefficients in ~~(2)~~ (1) satisfy $b^2-4ac=0$,

$$y_1(t)=e^{-bt/2a}$$

is a solution. To find a second solution, we assume ~~that~~

$$y=v(t)\,y_1(t)=v(t)\,e^{-bt/2a} \qquad (13)$$

went from c_1 to $v(t)$ a generalization

we substitute for y in eq (1) to determine $v(t)$. We have

$$y'=v'(t)\,e^{-bt/2a}-\frac{b}{2a}v(t)\,e^{-bt/2a} \qquad (14)$$

$$y''=v''(t)\,e^{-bt/2a}-\frac{b}{a}v'(t)\,e^{-bt/2a}+\frac{b^2}{4a^2}v(t)\,e^{-bt/2a} \qquad (15)$$

we substitute into eq (1) $(ay''+by'+cy=0)$

$$\left[a\left(v''(t)-\frac{b}{a}v'(t)+\frac{b^2}{4a^2}v(t)\right)+b\left(v'(t)-\frac{b}{2a}v(t)\right)+cv(t)\right]e^{-bt/2a}=0$$

(16)

cancel out $e^{-bt/2a}$ and rearrange the terms.

$$a\ddot{v}(t) + (-b+b)\dot{v}(t) + \underline{\left(\frac{b^2}{4a} - \frac{b^2}{2a} + c\right)} v(t) = 0 \quad (17)$$

$$\downarrow$$

$c - \frac{b^2}{4a} \rightarrow$ we know $b^2 - 4ac = 0$

$\rightarrow \frac{b^2}{4a} = \frac{4ac}{4a}$

coefficient is zero

(17) becomes $\ddot{v}(t) = 0$

so we can write $v(t) = c_1 + c_2 t$

so from (13), we have

$$y = c_1 e^{-bt/2a} + c_2 t e^{-bt/2a} \quad (18)$$

Thus y is a linear combination of the two solutions

$$y_1(t) = e^{-bt/2a} \quad y_2(t) = te^{-bt/2a} \quad (19$$

The Wronskian of these two solutions is

$$W(y_1, y_2)(t) = \begin{vmatrix} e^{-bt/2a} & te^{-bt/2a} \\ -\frac{b}{2a}e^{-bt/2a} & \left(1 - \frac{bt}{2a}\right)e^{-bt/2a} \end{vmatrix} = e^{-bt/a} \qquad (20)$$

Since $W(y_1, y_2)(t)$ is never zero, the solutions y_1 and y_2 given by (19) are fundamental set of solutions

(18) is the general solution of (1)

when the roots of the characteristic equation are equal.

Example 2

Find the solution of the initial value problem

$$y'' - y' + 0.25y = 0 \qquad y(0) = 2,$$

The characteristic equation is

$$r^2 - r + 0.25 = 0$$

so the roots are $r_1 = r_2 = \frac{1}{2}$

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The general solution is

$$y = c_1 e^{t/2} + c_2 t e^{t/2}$$

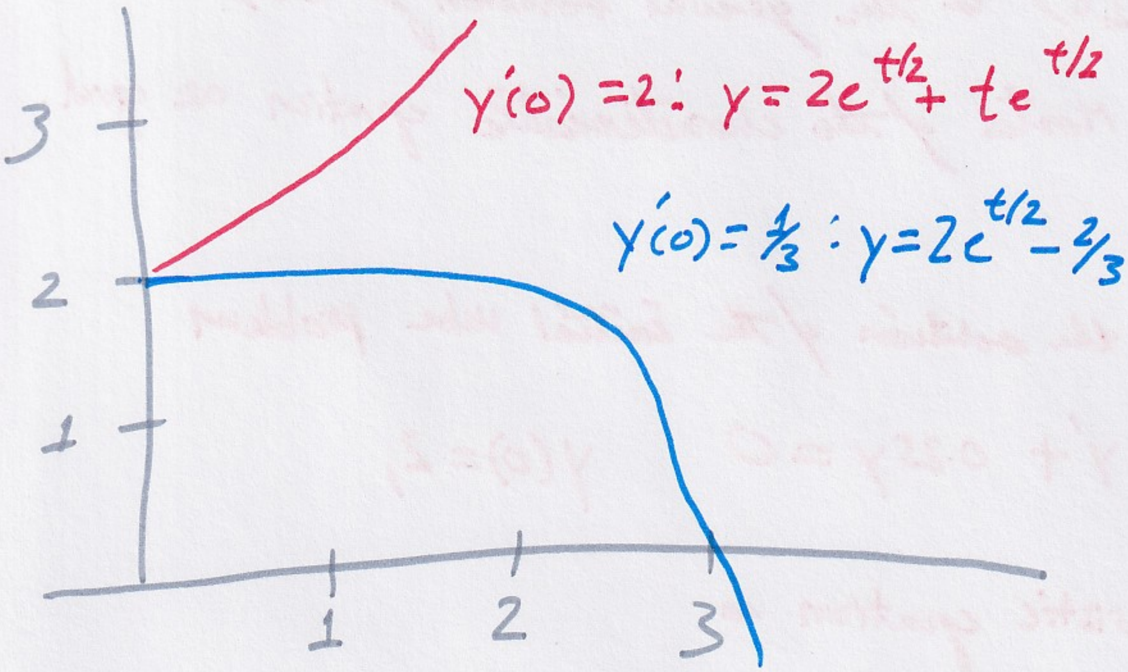
And the ICs

$$y(0) = c_1 = 2$$

$$y'(0) = \tfrac{1}{2} c_1 + c_2 = \tfrac{1}{3}, \quad c_2 = -\tfrac{2}{3}$$

The solution is

$$y = 2e^{t/2} - \tfrac{2}{3} t e^{t/2}$$



There's a critical initial slope between $\frac{1}{3}$ and 2 that separates those that grow positively and those that grow negatively

Summary

$$ay'' + by' + cy = 0 \qquad (1)$$

r_1 and r_2 are the roots of the corresponding characteristic equation

$$ar^2 + br + c = 0 \qquad (2)$$

r_1 and r_2 are real but not equal, the general solution

$$\longrightarrow y = c_1 e^{r_1 t} + c_2 e^{r_2 t} \qquad (24)$$

if r_1 and r_2 are complex conjugates $\lambda \pm i\mu$, then the general solution is

$$y = c_1 e^{\lambda t} \cos \mu t + c_2 e^{\lambda t} \sin \mu t \quad (25)$$

If $r_1 = r_2$, the general solution is

$$y = c_1 e^{r_1 t} + c_2 t e^{r_1 t} \qquad (26)$$

Reduction of Order

The procedure used in this section for equations with constant coefficients is more generally applicable.

Suppose that we know one solution $y_1(t)$,

$$y_1(t) \neq 0 \quad \forall t \in \mathbb{R}$$

$$y'' + p(t)\,y' + q(t)\,y = 0 \qquad (27)$$

To find a second solution, let

$$y = v(t)\,y_1(t) \qquad (28)$$

$$y' = v'(t)\,y_1(t) + v(t)\,y_1'(t)$$

$$y'' = v''(t)\,y_1(t) + 2v'(t)\,y_1'(t) + v(t)\,y_1''(t)$$

substituting for y, y', and y'' in (27) and collecting terms, we find that

$$y_1 v'' + (2y_1' + p y_1)v' + (y_1'' + p y_1' + q y_1)v = 0 \quad (29)$$

Since y_1 is a solution of (27), the coefficient of v in (29) is zero

$$\longrightarrow y_1 v'' + (2y_1' + p y_1)v' = 0 \quad (30)$$

Note that (30) is actually a first order equation for the function v'.

we can solve it as a first order Linear equation or as a separable equation.

when we get v', ~~the~~ v is obtained by Integration

Finally y is obtained from $y = v(t)\, y_1(t)$ (28)

This procedure is called the method of reduction of order

— set up a diff eq for v' (1st order)

— Avoid 2nd order for y

Example 3

Given that $y_1(t) = t^{-1}$ is a solution of

$$2t^2 y'' + 3ty' - y = 0, \quad t > 0 \qquad (31)$$

find a fundamental set of solutions

→ set $y = v(t)\, t^{-1}$; then

$$y' = v' t^{-1} - v t^{-2}, \quad y'' = v'' t^{-1} - 2v' t^{-2} + 2v t^{-3}$$

Substituting for y, y' and y'' in (31) and collecting terms, we get

$$2t^2(v''t^{-1}-2v't^{-2}+2vt^{-3})+3t(v't^{-1}-vt^{-2})$$
$$-vt^{-1}$$

$$=2tv''+(-4+3)v'+(4t^{-1}-3t^{-1}-t^{-1})v$$
$$=2tv''-v'=0 \quad (32)$$

The coefficient of v should be zero, this should be a verification of your algebra

Let $w=v' \rightarrow 2tw'-w=0$

now use separation of variables

$$w(t)=v'(t)=ct^{1/2}$$

$$\rightarrow v(t)=\tfrac{2}{3}ct^{3/2}+K \quad \text{(use integration)}$$

It follows that

$$y=v(t)t^{-1}=\tfrac{2}{3}ct^{1/2}+Kt^{-1} \quad (33)$$

c, K are arbitrary constants

$y_2(t) = t^{1/2}$ this is the new solution we wanted!

$$W(y_1, y_2)(t) = \frac{3}{2} t^{-3/2} \neq 0 \quad \text{for } t > 0$$

→ y_1 and y_2 form a fundamental set of solutions of (31) for $t > 0$