Meeting # 7

Fundamental sets of solutions

We have a system

$$c_1 y_1(t_0) + c_2 y_2(t_0) = y_0 \qquad (8)$$

$$c_1 y_1'(t_0) + c_2 y_2'(t_0) = y_0'$$

if $W = 0$

(8) has no solution, unless y_0 and y_0' have values that make the numerators

in

$$c_1 = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y_0' & y_2'(t_0) \end{vmatrix}}{W} \qquad c_2 = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y_1'(t_0) & y_0' \end{vmatrix}}{W} \qquad (11)$$

~~If $W=0$, there are many~~

equal zero

if $W = 0$, there are many ICs that cannot be satisfied no matter how c_1, c_2 are chosen

The Wronskian determinant depends on solutions y_1 and y_2 evaluated at the point t_0

<u>Theorem 3.2.3</u> say y_1 and y_2 are two solutions of

$$L[y] = y'' + p(t)y' + q(t)y = 0 \qquad (2)$$

and the initial conditions

$$y(t_0) = y_0 \qquad y'(t_0) = y_0' \qquad (3)$$

Then it's always possible to choose <u>constants</u> c_1, c_2 so that

$$y = c_1 y_1(t) + c_2 y_2(t)$$

that satisfies (2) and ICs (3) if and only if...

... the Wronskian

$$W = y_1 y_2' - y_1' y_2$$

is not zero at t_0

Example 3

In example 2 of section 3.1 we found $y_1(t) = e^{-2t}$, $y_2(t) = e^{-3t}$ are solutions of the of the differential equation

$$y'' + 5y' + 6y = 0$$

Find the Wronskian of y_1 and y_2

$$W = \begin{vmatrix} e^{-2t} & e^{-3t} \\ -2e^{-2t} & -3e^{-3t} \end{vmatrix} = -e^{-5t}$$

Since W is nonzero $\forall t$, y_1 and y_2 can be use to construct ~~solutions~~ of the given diff EQ and ICs

Theorem 3.2.4

Suppose Y_1 and Y_2 are two solutions of the diff eq

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

Then the family of solutions

$$y = c_1 Y_1(t) + c_2 Y_2(t)$$

with arbitrary coefficients c_1 and c_2 include every solution of (2) if and only if there is a point t_0 where the Wronskian of Y_1 and Y_2 is not zero.

Let ϕ be any solution of (2)

to prove the theorem, we must determine if ϕ is included in the linear combination

$c_1 Y_1 + c_2 Y_2$

Are there values of c_1, c_2 that make linear combinations the same as ϕ

Let t_0 be a point where the Wronskian of y_1 and y_2 is nonzero

– Then we want to evaluate ϕ and ϕ' at this point and call these values y_0 and y_0' respectively

Thus

$$y_0 = \phi(t_0) \qquad y_0' = \phi'(t_0)$$

The function ϕ is certainly a solution of this Initial Value problem

$$(12) \quad y'' + p(t)y' + q(t)y = 0 \qquad \begin{array}{l} y(t_0) = y_0 \\ y'(t_0) = y_0' \end{array}$$

We are assuming $W(y_1, y_2)(t_0)$ is nonzero it is possible by 3.2.3 to choose c_1, c_2

s.t. $y = c_1 y_1(t) + c_2 y_2(t)$ is a sol of (12)

The proper values of C_1, C_2 are given by

$$C_1 = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y_0' & y_2'(t_0) \end{vmatrix}}{W(y_1, y_2)(t_0)} \qquad C_2 = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y_1'(t_0) & y_0' \end{vmatrix}}{W(y_1, y_2)(t_0)}$$

Due to 3.2.1 (Uniqueness part) guarantees that these 2 solutions of the Initial Value Problem are actually the same function.

Thus, for the proper choice of C_1 and C_2

$$\phi(t) = C_1 y_1(t) + C_2 y_2(t)$$

Therefor, ϕ is included in the family of functions

$$C_1 y_1 + C_2 y_2$$

Finally since ϕ is an arbitrary solution of (2)

$\rightarrow$ it follows that every solution of this equation is included in the family

~~the~~ Suppose that there is no point t_0 where the Wronskian is nonzero

So $W(Y_1, Y_2)(t_0) = 0$ no matter what t_0 is chosen.

So by 3.2.3 there are values of Y_0 and Y_0' s.t. the system (8)

$$\begin{aligned} c_1 Y_1(t_0) + c_2 Y_2(t_0) &= Y_0 \\ c_1 Y_1'(t_0) + c_2 Y_2'(t_0) &= Y_0' \end{aligned} \qquad (8)$$

has no solution

So Let's select such values

— t_0, Y_0, Y_0' which yield $W(Y_1, Y_2)(t_0) = 0$

Then we select a solution $\phi(t)$ of (2)

And also $\phi(t)$ satisfies ~~the~~ ICs

$Y(t_0) = Y_0 \qquad Y'(t_0) = Y_0' \qquad (3)$

We can see that a ~~solution~~ solution $\Phi(t)$ is garunteed to exist by Theorem 3.2.1

(we know 3.2.1 does garuntee a solution)

but the solution $\Phi(t)$ is not included in the family $y = c_1 Y_1 + c_2 Y_2$

$\Rightarrow$ Thus this linear combination does not include all solutions of (2)

if $W(Y_1, Y_2) = 0$

This completes the proof.

remember that

Theorem 3.2.3

basically says

$$y = c_1 y_1(t) + c_2 y_2(t)$$

is a solution to (2) and ICs (3)

if Wronskian $W = y_1 y_2' - y_1' y_2$

is nonzero at t_0

and Theorem 3.2.4

says $y = c_1 y_1(t) + c_2 y_2(t)$

is a family of solution and includes all solutions to (2) if and only if $W(y_1, y_2)(t_0) \neq 0$

And we just proved 3.2.4

So if Theorem 3.2.1 applies,

we may say

$$y = c_1 y_1(t) + c_2 y_2(t)$$

with arbitrary coefficients is the General Solution of (2)

and y_1, y_2 are Fundamental set of Solutions of (2)

if and only if their Wronskian is nonzero

To restate 3.2.4

to find the general solution (all solutions) of (2), we need only to find two solutions of the given equation whose Wronskian is nonzero

Example 4)

Suppose that $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$ are two solutions of an equation of the form (2). Show that they form a fundamental set of solutions if

$r_1 \neq r_2$

We calculate the Wronskian of y_1 and y_2:

$$W = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix} = (r_2 - r_1)\, e^{(r_1 + r_2)t}$$

~~Since~~ Since the exponential function is never ~~zero~~ zero, and since we are assuming that $r_2 - r_1 \neq 0$, it follows that W is nonzero for every value of t. Consequently, y_1 and y_2 form a fundamental set of solutions.

Example 5

Show that $y_1(t) = t^{1/2}$ and $y_2(t) = t^{-1}$ form a fundamental set of solutions of

$$2t^2y'' + 3ty' - y = 0, \qquad t > 0 \qquad (14)$$

We will show how to solve (14) later. We want to verify by direct substitution that y_1 and y_2 are solutions of ~~the~~ differential equation.

Since $y_1'(t) = \frac{1}{2}t^{-1/2}$

$y_1''(t) = -\frac{1}{4}t^{-3/2}$

we are putting y_1 into the diff eq

we have

$$2t^2\left(-\tfrac{1}{4}t^{-3/2}\right) + 3t\left(-t^{-1}\right) - t^{-1} = \cancel{(4-3-1)}t^{-1} = 0$$

$$= \left(-\tfrac{1}{2} + \tfrac{3}{2} - 1\right)t^{1/2}$$

$$= 0$$

Similarly, $y_2'(t) = -t^{-2}$ and $y_2''(t) = 2t^{-3}$, so

$$2t^2(2t^{-3}) + 3t(-t^{-2}) - t^{-1} = (4-3-1)t^{-1} = 0$$

Next we calculate the Wronskian W of y_1 and y_2:

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} t^{1/2} & t^{-1} \\ \frac{1}{2}t^{-1/2} & -t^{-2} \end{vmatrix} = -\frac{3}{2}t^{-3/2}$$

Since $W \neq 0$ for $t>0$, we conclude that y_1 and y_2 form a fundamental set of solutions there.

Sometimes it's hard to get / find a solution to the differential equation we have an alternative.

<u>Theorem 3.2.5</u> consider the differential equation

$$L[y] = y'' + p(t)y' + q(t)y = 0 \quad (2)$$

whose coefficients p and q are continuous on some open interval I. Choose some point t_0 in I. Let y_1 be the solution of Eq (2) that also satisfies the initial ~~and~~ conditions

$$y(t_0) = 1, \quad y'(t_0) = 0$$

and let y_2 be the solution of Eq (2) that satisfies the initial conditions

$$y(t_0) = 0, \quad y'(t_0) = 1$$

Then y_1 and y_2 form a fundamental set of solutions of Eq (2)

First observe that the existence of the functions y_1 and y_2 is ensured by the existence part of Theorem 3.2.1

To show that ~~the~~ they form a fundamental set of solutions, we need only calculate their Wronskian at t_0:

$$W(y_1, y_2)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

Since their Wronskian is not zero at the point t_0, the functions y_1 and y_2 do form a fundamental set of solutions, thus completing the proof of Theorem 3.2.5

Note: we used Theorem ~~3~~ 32.1

Example 6)

Find the fundamental set of solutions y_1 and y_2 specified by Theorem 3.2.5 for the differential equation

$$y'' - y = 0 \qquad (16)$$

using the initial point $t_0 = 0$

recall in section 3.1, we noted the two solutions of (16) $y_1(t) = e^t$ and $y_2(t) = e^{-t}$.

The Wronskian of these solutions is

$$W(y_1, y_2)(t) = -2 \neq 0$$

so they form a set of solutions. However, they are not fundamental solutions indicated 3.2.5

— Because they do not satisfy the ICs mentioned in the theorem at point $t = 0$

To find the fundamental solutions specified by the theorem (3.2.5) we need to find the solutions satisfying the ~~ICs~~ proper ICs

Let us denote by $Y_3(t)$ the solution of (16)

that satisfy the ICs

$$y(0) = 1 \quad y'(0) = 0 \qquad (17)$$

The general solution of (16) is

$$y = c_1 e^{t} + c_2 e^{-t}$$

To satisfy the ICs we have

$$c_1 = \frac{1}{2} \qquad c_2 = \frac{1}{2}$$

$$\rightarrow Y_3(t) = \frac{1}{2} e^{t} + \frac{1}{2} e^{-t} = \cosh t$$

Similarly, if $Y_4(t)$ satisfies the ICs

$$y(0) = 0 \quad y'(0) = 1$$

$$\rightarrow Y_4(t) = \frac{1}{2} e^{t} - \frac{1}{2} e^{-t} = \sinh t$$

And the Wronskian

$$W(Y_3, Y_4)(t) = \cosh^2 t - \sinh^2 t = 1$$

These functions form a fundamental set of solutions as stated by Theorem 3.2.5.

So the general solution of (16)

can be written as

$$y = k_1 \cosh t + k_2 \sinh t \qquad (20)$$

k_1, k_2 are arbitrary constants
(Not the same as c_1 or c_2)

One purpose of this illustration is to make it clear that a given differential Equation has ~~more~~ more than one fundamental set of solutions

— it has infinitely many

— choose the set that is most convenient