Meeting # 5

Solutions to Homogeneous Equations

Example what's the general solution

$$y'' + 5y' + 6y = 0$$

Assume $y = e^{rt}$ r must be a root of the characteristic equation

$$r^2 + 5r + 6 = (r+2)(r+3) = 0$$

$r_1 = -2 \qquad r_2 = -3$

form of the general solution

$$y = c_1 e^{-2t} + c_2 e^{-3t}$$

Example

$$y'' + 5y' + 6y = 0 \qquad y(0) = 2$$
$$y'(0) = 3$$

$$y = c_1 e^{-2t} + c_2 e^{-3t}$$

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$$y(0) = c_1 e^0 + c_2 e^0 = 2, \qquad c_1 + c_2 = 2$$

$$y' = c_1(-2)e^{-2t} + c_2(-3)e^{-3t}$$

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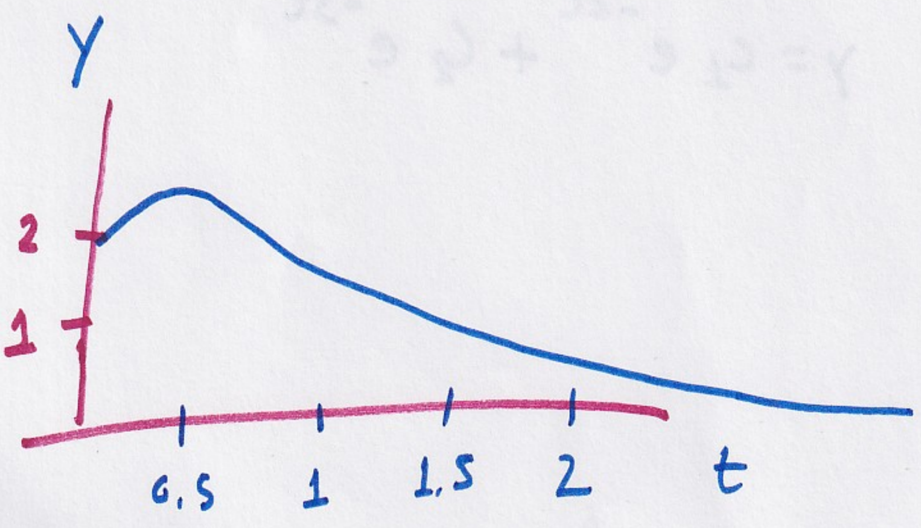
$$y'(0) = -2c_1 + -3c_2 = 3$$

$$\rightarrow \quad c_1 = 9 \qquad c_2 = -7$$

$$y = 9e^{-2t} - 7e^{-3t}$$



Example

$$4y'' - 8y' + 3y = 0 \qquad y(0) = 2$$

$$y'(0) = \frac{1}{2}$$

characteristic equation

$$4r^2 - 8r + 3 = 0$$

$$r = \frac{3}{2} \quad r = \frac{1}{2} \quad \rightarrow y = c_1 e^{\frac{3}{2}t} + c_2 e^{\frac{1}{2}t}$$

Apply Initial Conditions

$$\underline{c_1 + c_2 = 2} \qquad \underline{\frac{3}{2}c_1 + \frac{1}{2}c_2 = \frac{1}{2}}$$

from $y(0)=2$ from $y'(0) = \frac{1}{2}$

$$c_1 = -\frac{1}{2} \quad c_2 = \frac{5}{2}$$

$$\rightarrow y = -\frac{1}{2} e^{\frac{3}{2}t} + \frac{5}{2} e^{\frac{1}{2}t}$$

This term grows faster

cuz $\frac{3}{2} > \frac{1}{2}$ for exponential

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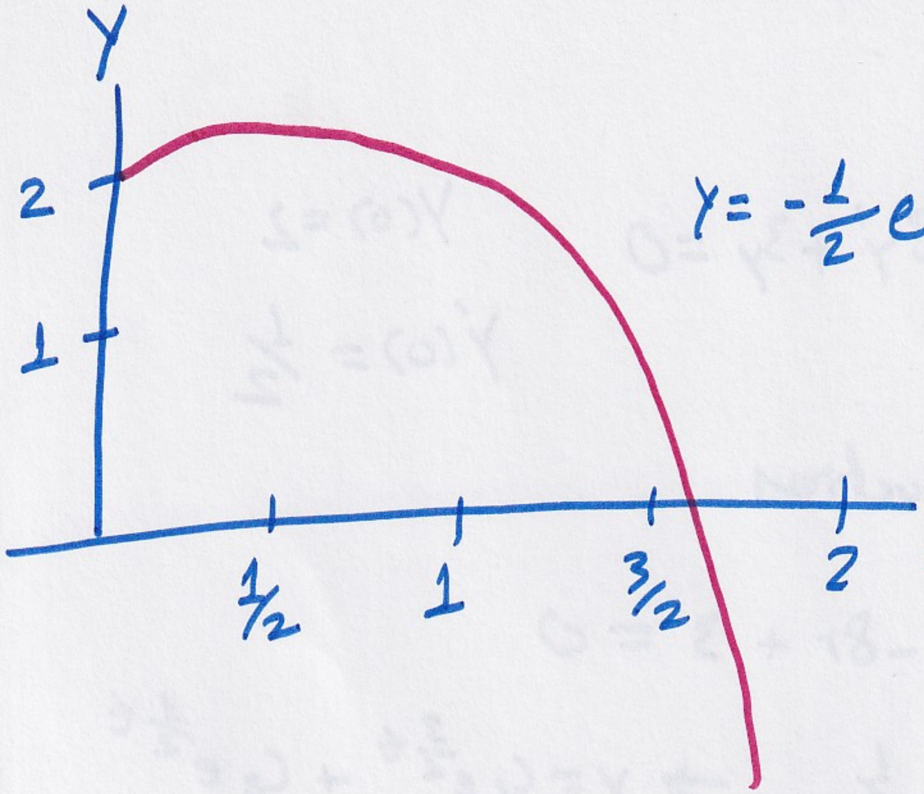

$$y = -\frac{1}{2}e^{\frac{3}{2}t} + \frac{5}{2}e^{\frac{1}{2}t}$$

To find the maximum of $y = 9e^{-2t} - 7e^{-3t}$

$$y' = -18e^{-2t} + 21e^{-3t}$$

Set $y' = 0 \rightarrow t_m = \ln\left(\frac{7}{6}\right) \cong 0.15415$

So to get $y_m = y(t_m) = \frac{108}{49} \cong 2.204$

Part 2 the Wronskian

$$ay'' + by' + cy = 0$$

a, b, c are constants

To see another way, Let's Introduce an Operator.

Let p, q be continuous on an open Interval

$I \rightarrow \alpha < t < \beta$

for any function ϕ that is $\underbrace{\text{twice differentiable}}_{C^2 \text{ on } I}$ on I, we define the operator

$$\underbrace{L[\phi]} = \phi'' + p\phi' + q\phi$$

is a function on I the value of $L[\phi]$ at point t is

$$L[\phi](t) = \phi''(t) + p(t)\phi'(t) + q(t)\phi(t)$$

Let $p(t) = t^2$, $q(t) = 1 + t$,

$$\phi(t) = \sin 3t$$

$$L[\phi](t) = (\sin 3t)'' + t^2 \cdot (\sin 3t)' + (1+t)\sin 3t$$
$$= -9\sin 3t + 3t^2 \cos 3t + (1+t)\sin 3t$$

The operator L is often written as

$L = D^2 + pD + q$, where D is the

Derivative operator

We'll study $\underbrace{L[\phi](t) = 0}_{\text{2nd order linear homogeneous equation}}$

Let $y = \phi(t)$

$\rightarrow L[y] = y'' + p(t)y' + q(t)y = 0$

We have Initial Conditions

$$y(t_0) = y_0 \quad y'(t_0) = y_0'$$

$$t_0 \in I \quad y_0, y_0' \in \mathbb{R}$$

Theorem 3.2.1 Existence and Uniqueness Theorem

Consider the Initial value problem

$$y'' + p(t)y' + q(t)y = g(t)$$

$$y(t_0) = y_0 \quad y'(t_0) = y_0'$$

Where p, q, and g are continuous on an open interval I ($t_0 \in I$)

Then there is exactly one solution $y = \phi(t)$ of this problem. That solution exist througout I (for $t_0 \in I$)

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1) The IVP has a solution
- a solution exists

2) The IVP has only 1 solution
- The solution is unique

3) solution ϕ is defined throughout the Interval I where p, q, g are C^2 on I

Example

$$y'' - y = 0 \quad y(0) = 2 \quad y'(0) = -1$$

The solution is $y = \frac{1}{2}e^{t} + \frac{3}{2}e^{-t}$

1.) a solution exists

2.) The solution is twice differentiable on I
- because p, q, g are continous on I
$\rightarrow C^2$ as well, $\longrightarrow$ $y = \frac{1}{2}e^{t} + \frac{3}{2}e^{-t}$ is the only solution

$y' = f(t, y)$ $f(t, y) = \underline{\qquad\qquad}$

→ proving uniqueness is hard, but 3.2.1 says it's unique. Boom^5

Example 1) find the longest interval in which the solution ~~of~~ of the IVP

$$(t^2 - 3t)y'' + ty' - (t+3)y = 0$$

$$y(1) = 2 \qquad y'(1) = 1$$

is certain to exist.

$$p(t) = \frac{t}{t^2 - 3t} = \frac{1}{t^1 - 3} \qquad q(t) = \frac{-(t+3)}{t^2 - 3t} = \frac{-(t+3)}{t(t-3)}$$

$t = 0 \rightarrow q(t)$ undefined

$t = 3 \rightarrow p, q$ undefined

3.2.1 ~~says~~ p, q, g have to be C^2 on I

at $t=0$, $t=3$ its not C^2

we have $y(1) = 2$, $y'(1) = 1$

This $t=1$, lies on $0 < t < 3$

So $I: 0 < t < 3$ (open Interval)

Example 2 Find the unique solution of the IVP

$$y'' + p(t)y' + q(t)y = 0$$

$$y(t_0) = 0, \quad y'(t_0) = 0$$

where p, q are continuous on I containing t_0

$$y = \phi(t) \equiv 0 \quad \forall t \in I$$

by 3.2.1 This is the only solution.

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

Let us assume y_1, y_2 are two solutions

So we may write

$$\rightarrow \quad L[y_1] = y_1'' + py_1' + qy_1 = 0$$

same for y_2 and

$L[y_1 + y_2]$ works too!

we can generate more solutions by forming Linear combinations of y_1 and y_2

<u>Theorem 3.2.2</u> (Principle of Superposition)

If y_1 and y_2 are solutions of the diff EQ

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

Then the Linear combination of $c_1 y_1 + c_2 y_2$ is also a solution for any of the values of c_1, c_2

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Proof Let $y = c_1 y_1 + c_2 y_2$

$$\rightarrow L[c_1 y_1 + c_2 y_2] = [c_1 y_1 + c_2 y_2]'' + p[c_1 y_1 + c_2 y_2]'$$

$$+ q[c_1 y_1 + c_2 y_2]$$

$$= c_1 y_1'' + c_2 y_2'' + c_1 p y_1' + c_2 p y_2'$$

$$+ c_1 q y_1 + c_2 q y_2$$

$$= c_1 [y_1'' + p y_1' + q y_1]$$

$$+ c_2 [y_2'' + p y_2' + q y_2]$$

$$= c_1 L[y_1] + c_2 L[y_2]$$

$$L[c_1 y_1 + c_2 y_2] = c_1 L[y_1] + c_2 L[y_2]$$

Since $L[y_1] = 0$, $L[y_2] = 0$, it follows $L[c_1 y_1 + c_2 y_2] = 0$

regardless of c_1, c_2, $y = c_1 y_1 + c_2 y_2$ satisfies the diff EQ

3.2.2 says

if we begin with only 2 solutions of

$L[y]$

we can construct an infinite family of solutions

$$y = c_1 y_1 + c_2 y_2 + \dots c_n y_n$$

We Now ask ourselves,

do all solutions of $L[y]$ are

included in y?

or are they solutions of a different form

Can $c_1, c_2 \dots c_n$ be chosen so as to satisfy the Initial conditions?

for c_1, c_2

$$\left.\begin{aligned} c_1 Y_1(t_0) + c_2 Y_2(t_0) &= Y_0 \\ c_1 Y_1'(t_0) + c_2 Y_2'(t_0) &= Y_0' \end{aligned}\right\} \text{System 1}$$

The determinant of the coefficients of the system

$$W = \begin{vmatrix} Y_1(t_0) & Y_2(t_0) \\ Y_1'(t_0) & Y_2'(t_0) \end{vmatrix} = Y_1(t_0) Y_2'(t_0) - Y_1'(t_0) Y_2(t_0)$$

If $W \neq 0$, then System 1 has a unique solution (c_1, c_2) regardless of the values of Y_0 ~~t_0~~ and Y_0'

$$c_1 = \frac{Y_0 Y_2'(t_0) - Y_0' Y_2(t_0)}{W}$$

$$c_2 = \frac{-y_0 y_1'(t_0) + y_0' y_1(t_0)}{W}$$

$$c_1 = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y_0' & y_2'(t_0) \end{vmatrix}}{W} \qquad c_2 = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y_1'(t_0) & y_0' \end{vmatrix}}{W}$$

$\rightarrow$ c_1, c_2 will satisfy ICs and the diff EQ ($L[y]$)