Meeting # 8

Complex Roots of the Homogeneous Equation

we continue with the equation

$$ay'' + by' + cy = 0 \quad (1)$$

$a, b, c \in \mathbb{R}$

In section 3.1 we found that if we seek ~~solution~~ solutions of the form $y = e^{rt}$, then r must be a root of the characteristic equation

$$ar^2 + br + c = 0 \quad (2)$$

From section 3.1, if the roots r_1 and r_2 are real and different, the general solution is

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t} \quad (3)$$

suppose now that $b^2 - 4ac$ is negative (discriminate)

Then the roots of (2) are complex conjugates

$$r_1 = \lambda + i\mu \quad r_2 = \lambda - i\mu \quad (4)$$

$\lambda, \mu \in \mathbb{R}$

so the corresponding expressions

something like

(5) $\quad y_1(t) = e^{(\lambda + i\mu)t}, \quad y_2(t) = e^{(\lambda - i\mu)t}$

what is meant by these? (5)?

ex) $\lambda = -1, \mu = 2, t = 3$

$\rightarrow y_1(3) = e^{-3+6i}$

Euler's Formula

we know

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}, \quad -\infty < t < \infty$$

$$\cancel{= \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} + i \sum_{n=1}^{\infty} \frac{(-1)^{n-1} t^{2n-1}}{(2n-1)!}}$$

for $t \to it$

$$e^{it} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \quad (8)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} + i \sum_{n=1}^{\infty} \frac{(-1)^{n-1} t^{2n-1}}{(2n-1)!}$$

$i^2 = -1$, $i^3 = -i$, $i^4 = 1$

Taylor series

$\cos(t)$ $\quad i\sin(t)$

This is Euler's Formula

$$e^{it} = \cos t + i\sin t \quad (9)$$

$$e^{-it} = \cos(t) - i\sin(t)$$

$$\sin(-t) = -\sin(t)$$

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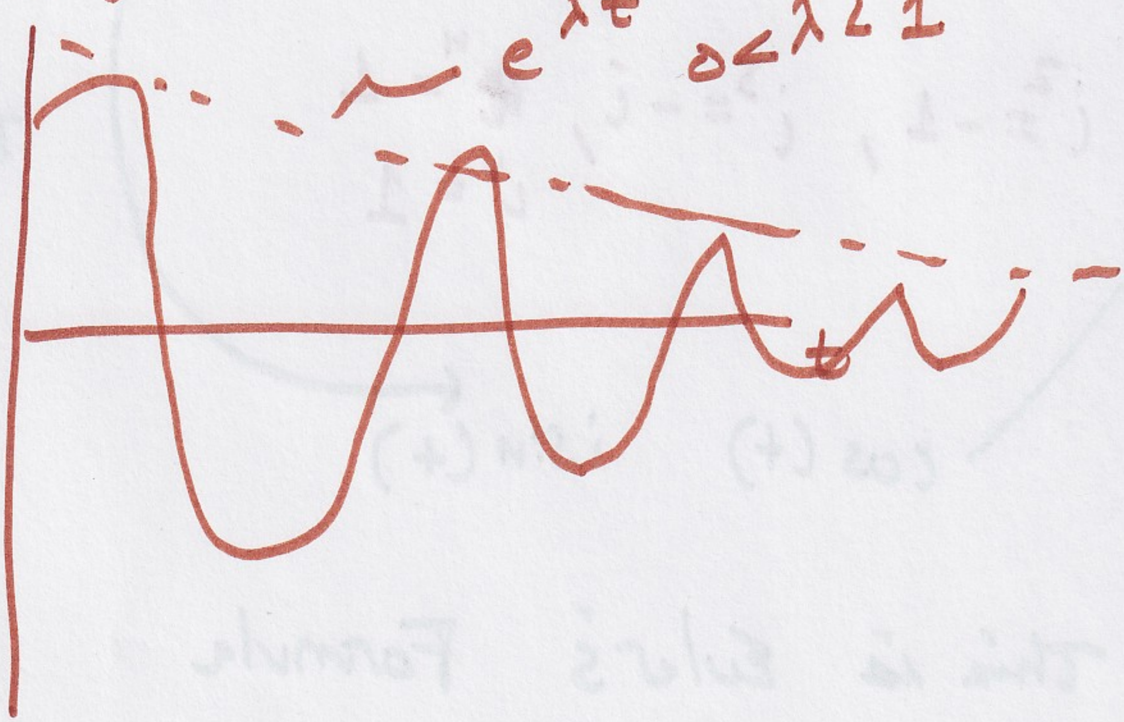
$$e^{i\mu t} = \cos(\mu t) + i\sin(\mu t)$$

what if

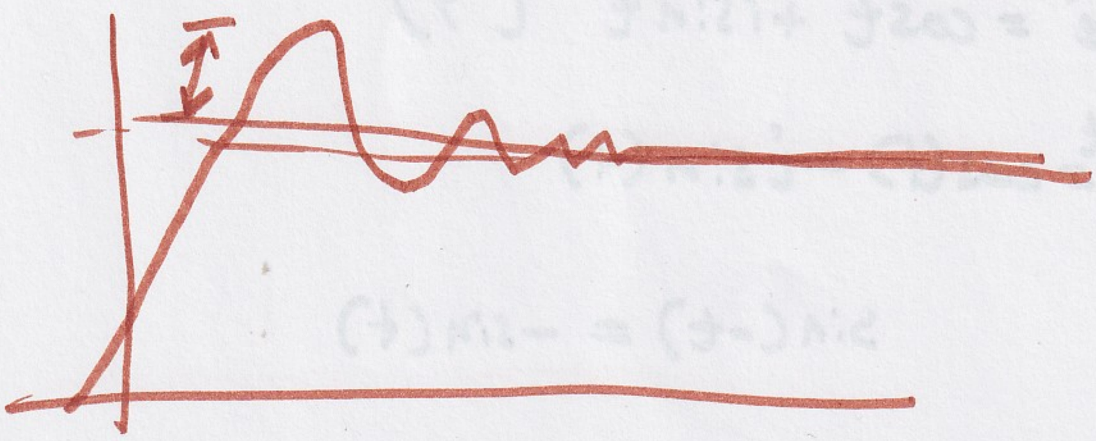
$$e^{(\lambda + i\mu)t} = e^{\lambda t} e^{i\mu t}$$

$$= e^{\lambda t}\left[\cos(\mu t) + i\sin\mu t\right]$$

$Re\{f(t)\}$



Equi



Example 1)

Find the general solution of the differential equation

$$y'' + y' + 9.25y = 0, \qquad (15)$$

Also find the solution that satisfies the initial conditions

$$y(0) = 2, \quad y'(0) = 8 \qquad (16)$$

and draw its graph

The characteristic equation for (15)

$$r^2 + r + 9.25 = 0$$

so its root are

$$r_1 = -\tfrac{1}{2} + 3i, \qquad r_2 = -\tfrac{1}{2} - 3i$$

So the solutions are

$$y_1(t) = e^{(-\frac{1}{2}+3i)t}$$

$$y_2(t) = e^{(-\frac{1}{2}-3i)t}$$

you can verify that the Wronskian

$$W(y_1, y_2)(t) = -6ie^{-t}$$

which is non zero, so the general solution of Eq (15) can be expressed as a linear combination of $y_1(t)$ and $y_2(t)$ with arbitrary coefficients

However, the initial value problem (15), (16) has only real coefficients, and it is often desirable to express ~~the~~ solution of such a problem in terms of real-valued functions.

We need Theorem 3.2.6

The real and the Imaginary parts of a complex-valued sol of (15) are solutions of (15)

Thus, starting from either $y_1(t)$ or $y_2(t)$

we obtain

$$u(t) = e^{-t/2}\cos 3t \qquad v(t) = e^{-t/2}\sin 3t$$

as real-valued solutions of (15)

So $W(u,v)(t) = 3e^{-t}$ (non zero)

so $u(t)$ $v(t)$ form a fundamental set of solutions

The general solution can be written as

$$y = C_1 u(t) + C_2 V(t) = e^{-t/2}(C_1 \cos 3t + C_2 \sin 3t)$$

To satisfy ICs (16)

from $\begin{matrix} t=0 \\ y=2 \end{matrix} \longrightarrow C_1 = 2$

$\cancel{y}$

$y'(0) = 8 \rightarrow C_2 = 3$

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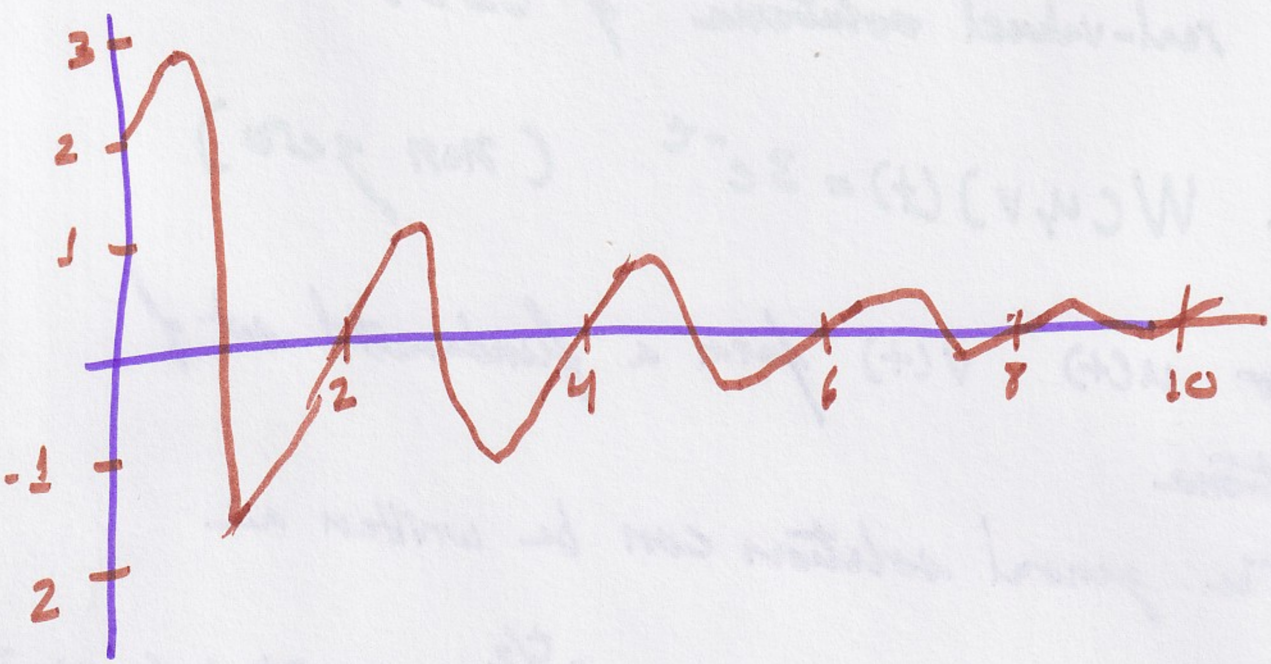
so the solution of the Initial Value Problem

$$y = e^{-t/2}(2\cos 3t + 3\sin 3t) \quad (21)$$



— decaying oscillations

- sine and cosine illustrate the oscillations factor

- the negative exponent ~~cause~~ causes the sinusoids to decay

Complex Roots: The General Case

$y_1(t), y_2(t)$ are solutions of (1)

when the roots of the characteristic equation (2) are complex numbers $\lambda \pm j\mu$

y_1, y_2 are complex-valued functions

In the General solution, we prefer to have real-valued solutions (The Diff EQ has real coefficients)

Use Theorem 3.2.6

to find a fundamental set of real-valued solutions

- by choosing the real and Imaginary parts of either $y_1(t)$ or $y_2(t)$

→ so we obtain solutions

$u(t) = e^{\lambda t}\cos\mu t \qquad v(t) = e^{\lambda t}\sin\mu t$

we can show that the Wronskian is

$$W(u, v)(t) = \mu e^{2\lambda t}$$

So as long as $\mu \neq 0$, the Wronskian is not zero so u and v form a fundamental set of solutions

if $\mu = 0$, roots are real

So with $\mu \neq 0$, the general solution is

$$y = c_1 e^{\lambda t} \cos(\mu t) + c_2 e^{\lambda t} \sin(\mu t)$$

Example 2

Find the solution of the Initial Value Problem

$$16y'' - 8y' + 145y = 0, \quad y(0) = -2, \quad y'(0) = 1$$

The characteristic equation is

$$16r^2 - 8r + 145 = 0$$

The roots, $r = \frac{1}{4} \pm 3i$

so we can simply write, ..

$$y = c_1 e^{t/4} \cos(3t) + c_2 e^{t/4} \sin(3t)$$

(Thanks to the previous Analysis)

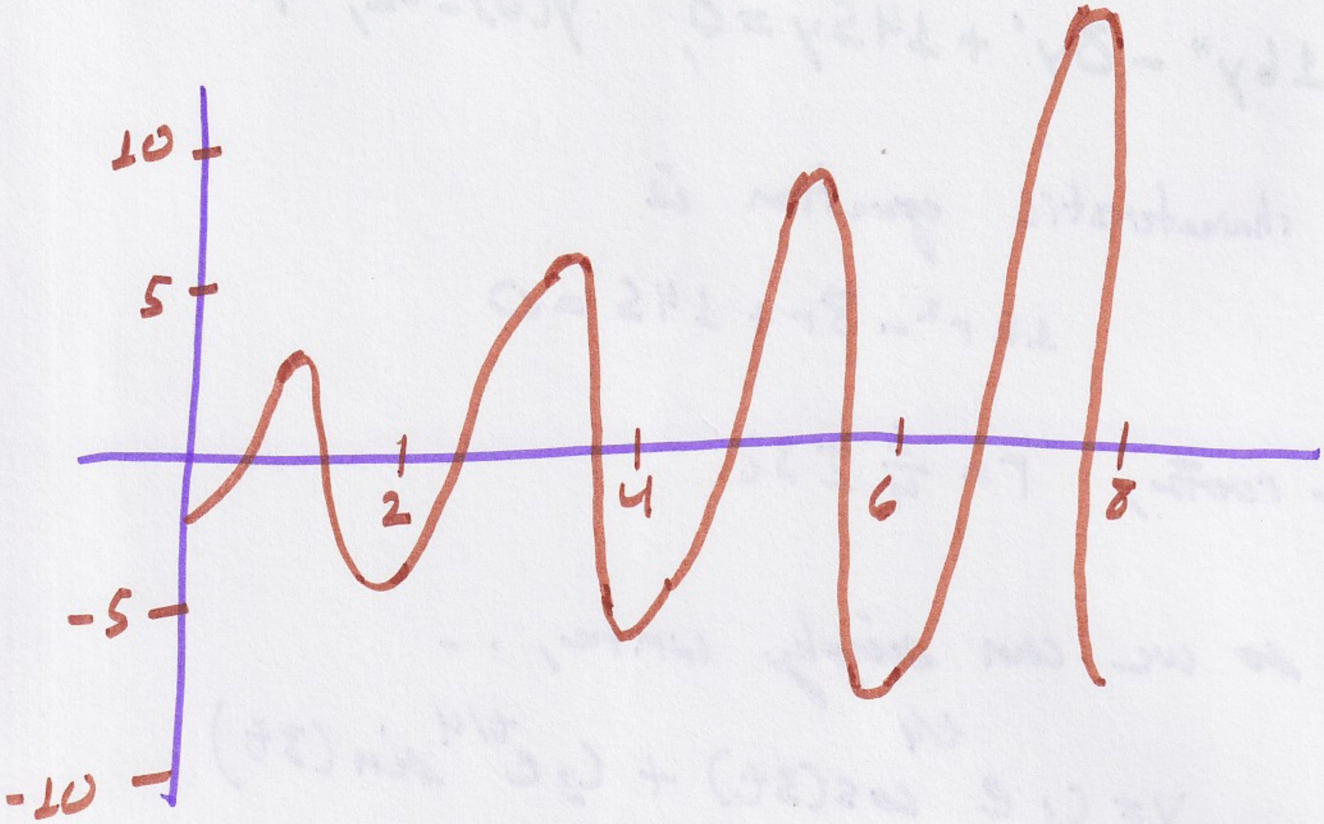
$$y(0) = -2 = c_1$$

And

$$y'(0) = \frac{1}{4} c_1 + 3c_2 = 1, \quad c_2 = \frac{1}{2}$$

So then

$$y = -2e^{t/4}\cos 3t + \tfrac{1}{2}e^{t/4}\sin 3t$$



The solution is a growing oscillation

- The sinusoids form the oscillatory part
- The exponential part is positive so the magnitude increases over time.

Example 3

Find the general solution of

$$y'' + 9y = 0 \qquad (28)$$

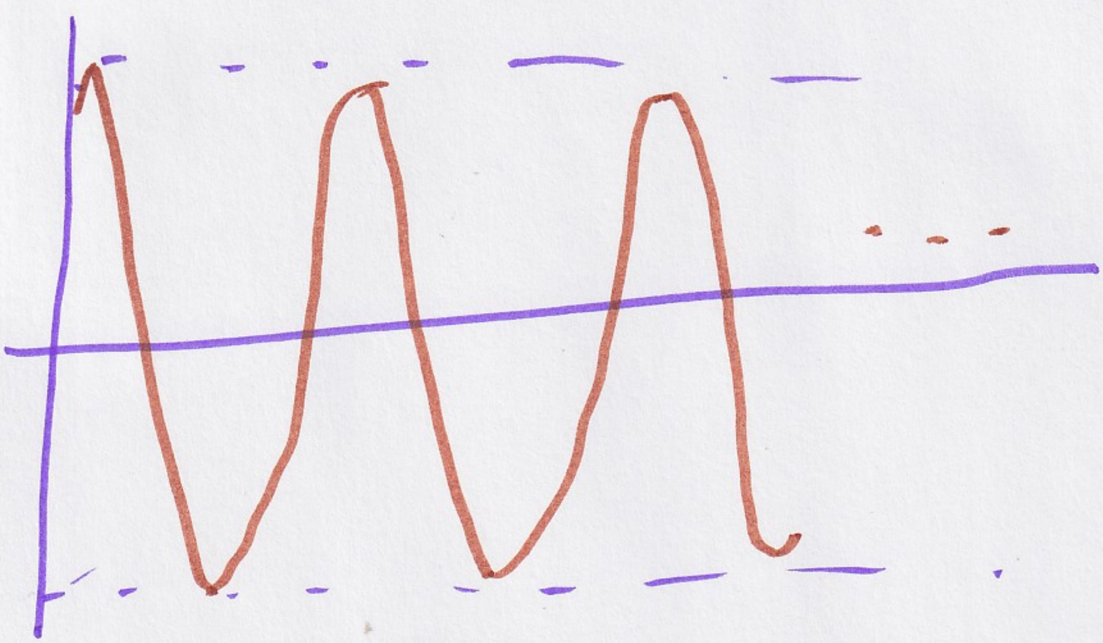
The ~~characteristic~~ characteristic equation is $r^2 + 9 = 0$

so the roots are $r = \pm 3i$

$\to \lambda = 0, \quad \mu = 3$

The general solution is

$$y = c_1 \cos 3t + c_2 \sin 3t$$



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The real part of the roots are zero so there is no exponential factor - sinusoids have constant amplitude