Meeting #6

Linear Systems

Determinants

<u>Linear Algebra</u>: makes use of number systems or AKA number fields

<u>number field</u>: any set K of objects called 'numbers'

→ 4 arithmetic operations → give elements of K
↓
field axioms

Field Axioms

a) every pair of numbers α, β in K corresponds a unique number, $\alpha + \beta$ in K called the sum in K

we have

1) $\alpha+\beta=\beta+\alpha, \quad \forall \alpha, \beta$ Commutative

2) $(\alpha+\beta)+\gamma=\alpha+(\beta+\gamma), \quad \forall \alpha, \beta, \gamma \in K$
Associative

3) $\exists$ a number $0 \in K$ s.t. $0+\alpha=\alpha \quad \forall \alpha \in K$
Trivial (zero exists

4) for every $\alpha \in K$, $\exists$ $\gamma \in K$
s.t. $\alpha+\gamma=0$ negative element

b To every pair of numbers α and $\beta \in K$
there corresponds a unique number $\alpha \cdot \beta \in K$
called the product of α and β

5) $\alpha\beta=\beta\alpha \quad \forall \alpha, \beta \in K$ multiplication is commutative

6) $(\alpha\beta)\gamma=\alpha(\beta\gamma) \quad \forall \alpha, \beta, \gamma \in K$ multiplication is associative

7) $\exists$ a number $1 \in K$ s.t. $1 \cdot \alpha=\alpha \quad \forall \alpha \in K$
Identity

8) $\forall \alpha \neq 0 \in K$, $\exists$ a number $\gamma \in K$
s.t. $\alpha\gamma = 1$ $\left(\gamma = \frac{1}{\alpha}\right)$ Inverse reciprocal

c) multiplication is distributive over addition

9) $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma \quad \forall \alpha, \beta, \gamma \in K$

Fields

a.) Field of rational Numbers $q \in \mathbb{Z}$

$p \in \mathbb{Z}$

for $p \neq 0$, $q \neq 0$ $\quad \frac{p}{q} \in \mathbb{Q}$

b.) Field of Real Numbers

$$\mathbb{R} = \underbrace{\mathbb{Q} + \text{Irrationals}}_{\text{sequenced!}}$$

4

c) $\mathbb{C}$ $\quad \dfrac{a+ib}{a,b \in \mathbb{R}}$

Lets denote K as an arbitrary field

Problems of the Theory of Systems of Linear Equations

System of Linear Equations

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{k1}x_1 + a_{k2}x_2 + \dots + a_{kn}x_n = b_k$$

$x \in K$

$a \in K$ coefficients

$b \in K$ constants

$c \in K$ solutions

Example

$$2x_1 + 3x_2 = 5$$
$$2x_1 + 3x_2 = 6$$

No Solution

A system that has a solution is called Compatible ! (at least 1 solution)

$$\underbrace{c_1^1 \quad c_2^1 \quad c_3^1}_{\text{one sol.}} \qquad \underbrace{c_1^2 \quad c_2^2 \quad c_3^2}_{\text{Another Sol.}}$$

example

$$2x_1 + 3x_2 = 0$$
$$4x_1 + 6x_2 = 0$$

Compatible
2 sol

$$\underbrace{c_1^1 = 0 \qquad c_2^1 = 0}_{\text{Trivial Solution}} \qquad c_1^2 = 3 \qquad c_2^2 = -2$$

\# A compatible system can have ∞ number of solutions

If a compatible system has 1 solution the system is called determinate

if " has at least 2 solutions the system is called indeterminate

For basic problems

a) To ascertain whether the system (1) is compatible or incompatible

b) If the system is compatible, ascertain whether it is determinate

c) If the system is compatible and determinant, what is its unique solution

d) If the system is compatible and determinant we want to find its unique solutions

1.3 Determinants of Order n

1.31 suppose we are given a square matrix (array of n^2 numbers)

$\rightarrow a_{ij}$ $(i, j = 1, 2 \dots n)$, all elements of a field K

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

This matrix is of order n

The number of rows and columns of the matrix is called its order.

a_{ij} – element of the matrix

i – row, j – column

 to Calculate the determinant . . .

The elements $a_{11}, a_{22}, \ldots, a_{nn}$ form the principal diagonal of the matrix

consider any <u>product</u> of <u>n elements</u> which appear in different rows and different columns of the matrix.

— specifically a product containing just one element from each row and each column.

$$\rightarrow a_{\alpha_1 1}\, a_{\alpha_2 2} \cdots a_{\alpha_n n} \quad (5)$$

For the 1st factor, we can choose an element appearing in the 1st column of the matrix

denote

α_1 – the number of the row which the element belongs

α_2 – the number of the row which the second factor (2n element in (5)) appears

Thus, the indices $\alpha_1, \alpha_2, \cdots, \alpha_n$ are the numbers of the rows in which the factors of the product appear, when we agree to write the column indices in increasing order.

$$\rightarrow \underbrace{a_{\alpha_1 1}}_{\text{indices of an element}} a_{\alpha_2 2} \cdots a_{\alpha_n n}$$

The row indices

The column indices

So the elements $a_{\alpha_1 1} a_{\alpha_2 2} \cdots a_{\alpha_n n}$ appear in different rows of the matrix

→ $\alpha_1, \alpha_2, \cdots, \alpha_n$ are all different and represent some permutation of $1, 2, \cdots, n$

Inversion: by inversion in the sequence $\alpha_1, \alpha_2 \cdots$ we mean an arrangement of 2 indices such that the Larger index comes before the smaller index.

$$N(\alpha_1, \alpha_2, \cdots, \alpha_n) = \text{Total \# of inversions}$$

ex)

$$N(2,1,4,3) = 2$$

2 before 1 and 4 before 3

ex)

$$N(4,3,1,2) = 5$$

4 before 3
4 before 1
4 before 2
3 before 1
3 before 2

So then we can say

$$N(\alpha_1, \alpha_2, \cdots, \alpha_n) = \text{Even}$$

→ put a '+' sign before the Product.

$N(\alpha_1, \alpha_2, \cdots, \alpha_N) = \text{odd}$

$\rightarrow$ put a '$-$' sign before the product.

~~So~~ then

$$(-1)^{N(\alpha_1, \alpha_2, \cdots, \alpha_N)} \left[a_{\alpha_1 1} \, a_{\alpha_2 2} \cdots a_{\alpha_n n} \right]$$

The total number of products of the form (5) that can be made from the elements of a given matrix ~~A~~ of order n is equal to the total number of permutations of the numbers $1, 2, \cdots, n$. $(= n!)$

Definition of the Determinant

Determinant D of the matrix is the algebraic sum of the n! products of the form (5), each preceded by the sign determined by the rule just given

$$D = \sum (-1)^{N(\alpha_1, \alpha_2, \cdots \alpha_n)} a_{\alpha_1 1} a_{\alpha_2 2} \cdots a_{\alpha_n n}$$

(5) are the <u>terms</u> of the determinant D

a_{ij} are <u>elements</u> of D

n is the <u>order</u> of D

$$D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \det \| a_{ij} \|$$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} - a_{21} a_{12}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} a_{22} a_{33} + a_{21} a_{32} a_{13} + a_{31} a_{12} a_{23}$$

$$- a_{31} a_{22} a_{13} - a_{21} a_{12} a_{33}$$

$$- a_{11} a_{32} a_{23}$$

Ex)

$$a_{11} x_1 + a_{12} x_2 = b_1$$

$$a_{21} x_1 + a_{22} x_2 = b_2$$

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \qquad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$