Meeting # 10

Nonhomogeneous Equations; Method of Undetermined Coefficients

We return to the nonhomogeneous equation

$$L[y] = y'' + p(t)y' + q(t)y = g(t) \quad (1)$$

p, q, g are given continuous function on the open Interval I

$$L[y] = y'' + p(t)y' + q(t)y = 0 \quad (2)$$

homogeneous version

Theorem 3.5.1

If Y_1 and Y_2 are two solutions of the nonhomogeneous equation (1), then their differen $Y_1 - Y_2$ is a solution of the corresponding homogeneous equation (2). If, in addition, y_1 and y_2 are a fundamental set of solutions of (2) then,

$$Y_1(t) - Y_2(t) = c_1 y_1(t) + c_2 y_2(t) \quad (3)$$

where c_1, c_2 are certain constants

To prove this result, note that Y_1 and Y_2 satisfy the equations

$$L[Y_1](t) = g(t) \qquad L[Y_2](t) = g(t) \quad (4)$$

If we subtract the second from the first,

$$L[Y_1](t) - L[Y_2](t) = g(t) - g(t) = 0 \quad (5)$$

However

$$L[Y_1] - L[Y_2] = L[Y_1 - Y_2]$$

$$\rightarrow L[Y_1 - Y_2](t) = 0 \quad (6)$$

(6) states that $Y_1 - Y_2$ is a solution of (2)

by Theorem 3.2.4 all solutions of (2) can be expressed as linear combinations of a fundamental set of solutions it follows that the solution $Y_1 - Y_2$ can be so written.

Hence Eq (3) holds and the proof is complete

Theorem 3.5.2

The general solution of the nonhomogeneous equation (1) can be written in the form

$$y = \phi(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t) \quad (7)$$

where y_1 and y_2 are a fundamental set of solutions of the corresponding homogeneous equation (2), c_1 and c_2 are arbitrary constants, and Y is some specific solution of the nonhomogeneous equation (1).

The proof of Theorem 3.5.2 follows quickly from the preceding theorem. Note that Eq (3) holds if we identify Y_1 with an arbitrary solution ϕ of Eq (1) and Y_2 with specific solution Y. from Eq (3) we thereby obtain

$$\phi(t) - Y(t) = c_1 y_1(t) + c_2 y_2(t) \quad (8)$$

which is equivalent to Eq (7) since ϕ is an arbitrary solution of Eq (1), the expression on the right side Eq (7) includes all solutions of Eq (1); thus it is natural to call it the general solution of Eq (1)

→ Theorem 3.5.2 states that to solve the nonhomogeneous equation (1), we must do three things:

1.) find the general solution $c_1 y_1(t) + c_2 y_2(t)$ of the corresponding homogeneous equation. This solution is frequently called the complementary solution and may be denoted by $y_c(t)$

2.) Find some single solution $Y(t)$ of the nonhomogeneous equation. Often this solution is referred to as a particular solution

3.) Form the sum of the functions found in steps 1 and 2

To develop methods to find the particular solution.

Method of Undetermined Coefficients

- We must make an assumption of $Y(t)$
  - coefficients are unspecified

- If we substitute the assumed expression into eq(1) and attempt to determine the coefficients so as to satisfy that equation
  - If we are successful, we found a solution to (1)
    - We can use our findings to get the ~~...~~ particular solution $Y(t)$

- If we cannot determine the coefficients, then there is no solution of the form we assumed.
  - So modify the assumption and try again.

<u>Weakness</u>: basically works if we have the correct form of the particular solution in advanced.

<u>Consequently</u>: Limitations are
- homogeneous eq has constant coefficients
- non homogeneous term is restricted to a small class of functions

6.

$\rightarrow$ do $g(t)$ = polynomials, exponentials, sin + cos

Example 1

Find a particular solution of

$$y'' - 3y' - 4y = 3e^{2t} \qquad (9)$$

we seek a function Y such that

$$Y''(t) - 3Y'(t) - 4Y(t) = 3e^{2t}$$

The exponent reproduces itself through differentiation

$\rightarrow$ pick $Y(t) = Ae^{2t}$

(particular solution is some multiple of e^{2t}

'A' is a coefficient to be determined,

$\longrightarrow Y'(t) = 2Ae^{2t} \quad Y''(t) = 4Ae^{2t}$

plug in!

$$(4A - 6A - 4A)e^{2t} = 3e^{2t}$$

$$-6Ae^{2t} = 3e^{2t}, \quad A = -\tfrac{1}{2}$$

$$Y(t) = -\tfrac{1}{2}e^{2t}$$

And we see

$$r^2 - 3r - 4 = 0$$

$$(r+1)(r-4) \rightarrow y_1(t) = c_1 e^{-t}, \quad y_2(t) = c_2 e^{4t}$$

check!

$$y_1' = -c_1 e^{-t}, \quad y_1'' = c_1 e^{-t}$$

$$\rightarrow c_1 e^{-t} + 3c_1 e^{-t} - 4c_1 e^{-t} = 0 \quad \checkmark$$

$$y_2' = 4c_2 e^{4t} \quad y_2'' = 16 c_2 e^{4t}$$

$$\rightarrow 16 c_2 e^{4t} - 12 c_2 e^{4t} - 4c_2 e^{4t} = 0 \quad \checkmark$$

The General Solution is

$$\phi(t) = c_1 e^{-t} + c_2 e^{4t} + \left(-\tfrac{1}{2}\right) e^{2t}$$

Example 2

find the particular solution of

$$y'' - 3y' - 4y = 2\sin t \qquad (11)$$

Following from example 1, Let's assume $Y(t) = A\sin t$

substitute $Y(t)$ into (11)

$$-A\sin(t) - 3A\cos(t) - 4A\sin(t) = 2\sin(t)$$

$$(2+5A)\sin(t) + 3A\cos t = 0 \qquad (12)$$

we want this to hold for all t,

Look at $t=0$, $t=\pi/2$ (critical points)

$t=0$, $3A=0$
$t=\pi/2$, $2+5A=0$ } contradiction! → No choice for A

we need an expression $\forall t$

→ Our Assumption is inadequate

if we Look at (12), it suggests we modify our original assumption to include a cosine

$$Y(t) = A\sin(t) + B\cos(t)$$

we need to know A and B

$$Y'(t) = A\cos t - B\sin t \qquad Y''(t) = -A\sin t - B\cos t$$

By substituting these expressions for Y, Y', Y'' in (11)

(13) $\quad (-A+3B-4A)\sin t + (-B-3A-4B)\cos t = 2\sin t$

From the Analysis of coefficients,

$$-5A+3B=2, \quad -3A-5B=0$$

$$A=\frac{-5}{17} \qquad B=\frac{3}{17}$$

so $Y(t)=-\frac{5}{17}\sin t+\frac{3}{17}\cos t$

For a polynomial

$$y''-3y'-4y=4t^2-1 \qquad (14)$$

we make an Assumption $Y(t)=At^2+Bt+C$

$$\left.\begin{aligned} Y'(t) &= 2At+B \\ Y''(t) &= 2A \end{aligned}\right\}$$ Plug into (14) → Find Coefficients

Example 3

Find Particular solution of

$$y''-3y'-4y=-8e^t\cos 2t \qquad (15)$$

for this case we assume $Y(t)$ is the product of e^t and a Linear combination of $\cos 2t$ and $\sin 2t$

$$\rightarrow Y(t) = Ae^t\cos 2t + Be^t\sin 2t$$

$$Y'(t) = (A+2B)e^t\cos(2t) + (-2A+B)e^t\sin 2t$$

$$Y''(t) = (-3A+4B)e^t\cos 2t + (-4A-3B)e^t\sin 2t$$

By substituting these expressions in (15)

$$\rightarrow 10A + 2B = 8 \qquad 2A - 10B = 0$$

(sin term is 0)

$$A = \frac{10}{13} \qquad B = \frac{2}{13}$$

$$\rightarrow Y(t) = \frac{10}{13}e^t\cos 2t + \frac{2}{13}e^t\sin 2t$$

suppose that $g(t)$ is the sum of

$$g(t) = g_1(t) + g_2(t)$$

suppose Y_1 and Y_2 are solutions of

$$ay'' + by' + cy = g_1(t) \quad (16)$$

and

$$ay'' + by' + cy = g_2(t) \quad (17)$$

Then $Y_1 + Y_2$ is a solution of the equation

$$ay'' + by' + cy = g(t) \quad (18)$$

<u>Example</u> 4

Find a particular solution of

(19) $$y'' - 3y' - 4y = 3e^{2t} + 2\sin t - 8e^{t}\cos 2t$$

→ use Superposition

recall

$y'' - 3y' - 4y = 3e^{2t}$ $\quad Y_1(t) = Ae^{2t} = -\frac{1}{2}e^{2t}$

$y'' - 3y' - 4y = 2\sin t$ $\quad Y_2(t) = A\sin t + B\cos t = -\frac{5}{17}\sin t + \frac{3}{17}\cos t$

$y'' - 3y' - 4y = -8e^{t}\cos 2t$ $\quad Y_3(t) = Ae^{t}\cos 2t + Be^{t}\sin 2t$

$$Y_3(t) = \frac{10}{13}e^{t}\cos 2t + \frac{2}{13}e^{t}\sin 2t$$

$$Y(t) = Y_1(t) + Y_2(t) + Y_3(t)$$

Example 5

Find the particular solution of

$$y'' - 3y' - 4y = 2e^{-t} \qquad (20)$$

From Example 1, we can assume

$$y(t) = A e^{-t}$$

Sub into (20)

$$\rightarrow \qquad \underbrace{(A + 3A - 4A)}_{\text{This is } 0} e^{-t} = 2e^{-t} \qquad (21)$$

No choice of A that satisfy the equation

Why not? choose Another Assumption?

Try solving the homogeneous equation

$$y'' - 3y' - 4y = 0 \qquad (22)$$

A fundamental set of solutions of (22) is

$$y_1(t) = e^{-t} \quad y_2(t) = e^{4t}$$

So our Particular solution is actually a solution of the homogeneous equation.

— So it cannot be a solution of the nonhomogeneous equation.

Options

— Use another method
  — use the result to guide our assumption
  — Try Problems 29, 35

— Seek a simpler equation where we find a similar problem, and use the solution to suggest how we should proceed further

1st order analogous eqn,

$$y' + y = 2e^{-t} \quad (23)$$

a particular solution for (23) of the form Ae^{-t} fail because e^{-t} is a solution of $y'+y=0$

$\rightarrow$ use $\mu(t) = e^{t}$

we obtain

$$y = \underset{\text{Particular sol}}{2te^{-t}} + \underset{\text{homogeneous sol}}{ce^{-t}} \qquad (24)$$

So now, Let's use this as a clue and try

$$y(t) = Ate^{-t} \quad \text{for} \quad (20)$$

$$\rightarrow y'(t) = Ae^{-t} - Ate^{-t} \qquad y''(t) = -2Ae^{-t} + Ate^{-t}$$

Now we substitute into (20)

$$\rightarrow (-2A - 3A)e^{-t} + (A + 3A - 4A)te^{-t} = 2e^{-t}$$

$$\rightarrow -5A = 2, \quad A = -\frac{2}{5}$$

$$\rightarrow y(t) = -\frac{2}{5}te^{-t}$$

So if Initial Assumption $y(t)$ is the same as one of the solutions to the homogeneous version,

$\rightarrow$ multiply by 't' or 't^2'