Meeting # 4

Homogeneous Equations

$$\frac{d^2y}{dt^2} = f(t, y, \frac{dy}{dt}) \qquad (1)$$

They are linear if it has the form

$$\frac{d^2y}{dt^2} = f(t, y, \frac{dy}{dt}) = g(t) - p(t)\frac{dy}{dt} - q(t)y$$

f is Linear in y and $\frac{dy}{dt}$

$g(t)$, $p(t)$, $q(t)$ are functions of the independent variable (t in this case)

$$y'' + p(t)y' + q(t)y = g(t)$$

$$P(t)y'' + Q(t)y' + R(t)y = G(t)$$

$$p(t) = \frac{Q}{P} \quad q = \frac{R}{P} \quad g = \frac{G}{P}$$

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$$y'' + p(t)\,y' + q(t)\,y = g(t)$$

Continuous

If a 2nd order diff Eq is not of this form, we say its non Linear

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Initial Value Problem IVP

- 2 ICs

$y(t_0) = y_0 \quad , \quad y'(t_0) = y_0'$

Whats going on?

ICs
- we have a point (t_0, y_0)
  - solutions must pass through this point
- we have the slope* of the graph at that point

- 2 ICs because we have 2 integrations with each Integration Introduces an arbitrary constant

if $g(t) = 0 \ \forall t$, diff eq is homogeneous!

$$y'' + p(t)\,y' + q(t)\,y = 0$$

For this chapter (Boyce DiPrima) P, Q, R are constants

$\longrightarrow$ So we write

$$ay'' + by' + cy = 0$$

Example 1)

$$y'' - y = 0 \qquad (9)$$

$$y(0) = 2 \qquad y'(0) = -1$$

what is

$$a = 1, \quad b = 0, \quad c = -1$$

what are the possible solutions?

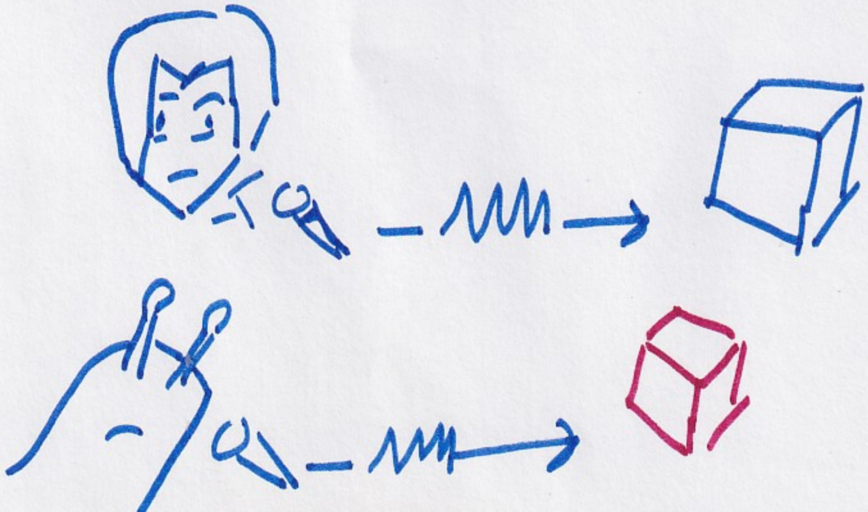
$$y_1(t) = e^{t}, \quad y_2(t) = e^{-t}$$

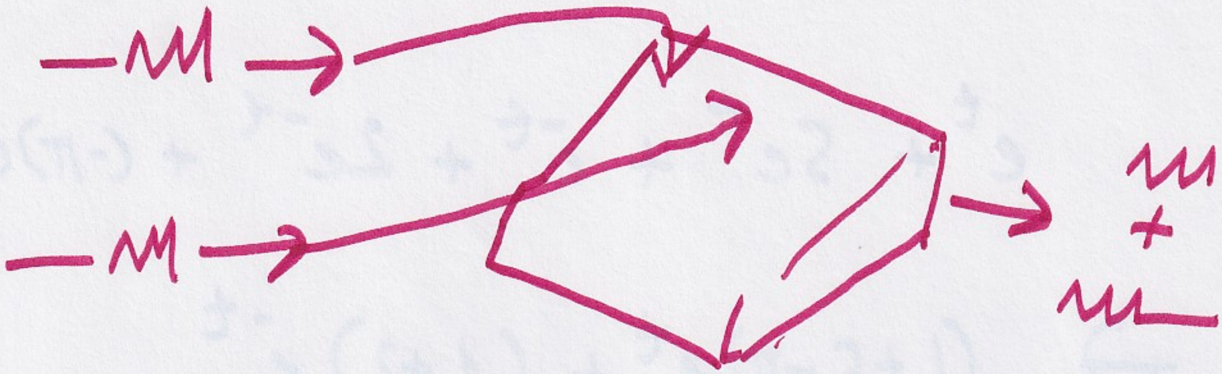
if we introduce coefficients

$$y_3(t) = 2e^{t}, \quad y_4(t) = 5e^{-t}$$

Because, we are dealing with Linearity, we can use Superposition.

<u>Think</u> Lobster Wiggles





Consequently we may say

$c_1 y_1(t)$ & $c_2 y_2(t)$ satisfy

the Diff EQ $\forall\, c_1, c_2 \in \mathbb{R}$

Therefore we can write

$$y = c_1 y_1(t) + c_2 y_2(t) \qquad (11)$$

solution to the differential Eq

$$y' = c_1 e^t - c_2 e^{-t}$$

$$y'' = c_1 e^t + c_2 e^{-t}$$

$$e^t + 5e^t + e^{-t} + 2e^{-t} + (-\pi)e^t$$

$$\rightarrow \underbrace{(1+5-\pi)}_{c_1} e^t + \underbrace{(1+2)}_{c_2} e^{-t}$$

Linear Combination

→ Infinite Solutions

— Because c_1, c_2 are arbitrary

we want to find the solutions that passes through the point $(0, -2)$ and has the slope -1.

$$y = c_1 e^{t} + c_2 e^{-t}$$

ICs

(0, 2)

$y'(0) = -1$

$t=0,\ y=2$

$$\rightarrow \quad 2 = c_1 + c_2 \qquad (1)$$

$t=0,\ y'=-1$

$$\rightarrow \quad y' = c_1 e^{t} - c_2 e^{-t}$$

$$-1 = c_1 - c_2 \qquad (2)$$

2 eqns, 2 unknowns

$$c_1 = \tfrac{1}{2} \qquad c_2 = \tfrac{3}{2}$$

~~Some~~ conclusions

$$ay'' + by' + cy = 0$$

a, b, c are arbitrary (real) constants.

In the ~~examples~~ example, the solutions were exponential functions.

— Once we identified two solutions, we were able to use a linear combination of ~~them~~ them to satisfy the ICs and ~~the~~ Differential Equations

For any value of a, b, c and ICs for y and y', we can solve ~~the~~ diff Eq

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Start with $y = e^{rt}$

r is a parameter to be determined

$\longrightarrow y' = re^{rt}$

$y'' = r^2 e^{rt}$

So for

$$ay'' + by' + cy = 0$$

we may substitute for y'', y', y

$$ar^2e^{rt} + bre^{rt} + ce^{rt} = 0$$

$$(ar^2 + br + c)e^{rt} = 0$$

$\rightarrow e^{rt} \neq 0$

$$\underline{ar^2 + br + c}$$

Characteristic equation for the differential Equation

$\rightarrow$ if r is a root of the polynomial, $y = e^{rt}$ is a solution of the differential Equation.

$a, b, c \in \mathbb{R}$

So the roots are
- real and different
- real and repeated
- complex conjugates

Real and different

if the ~~roots~~ are $r_1, r_2 \in \mathbb{R}$ and $r_1 \neq r_2$

— we have two ~~solutions~~

$$y_1(t) = e^{r_1 t}, \quad y_2(t) = e^{r_2 t}$$

we may write

$$y = c_1 y_1(t) + c_2 y_2(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$
$$= \phi(t)$$

This is a ~~solution~~ to the differential equation

To verify

$$y' = c_1 r_1 e^{r_1 t} + c_2 r_2 e^{r_2 t}$$

$$y'' = c_1 r_1^2 e^{r_1 t} + c_2 r_2^2 e^{r_2 t}$$

So then, $ay'' + by' + cy = 0$

$$a[c_1 r_1^2 e^{r_1 t} + c_2 r_2^2 e^{r_2 t}]$$
$$+ b[c_1 r_1 e^{r_1 t} + c_2 r_2 e^{r_2 t}]$$
$$+ c[c_1 e^{r_1 t} + c_2 e^{r_2 t}] = 0$$

$$= c_1[ar_1^2 + br_1 + c]e^{r_1 t}$$
$$+ c_2[ar_2^2 + br_2 + c]e^{r_2 t} = 0$$

since r_1, r_2 are roots,

$$ar_1^2 + br_1 + c = 0$$

$$ar_2^2 + br_2 + c = 0$$

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For Initial Conditions

$$y(t_0) = y_0 \quad y'(t_0) = y_0'$$

For $t = t_0$ $y = y_0$

$$c_1 e^{r_1 t_0} + c_2 e^{r_2 t_0} = y_0$$

for $t = t_0$ $y' = y_0'$

$$c_1 r_1 e^{r_1 t_0} + c_2 r_2 e^{r_2 t_0} = y_0'$$

we have

$$c_1 = \frac{y_0' - y_0 r_2}{r_1 - r_2} e^{-r_1 t_0}$$

$$c_2 = \frac{y_0 r_1 - y_0'}{r_1 - r_2} e^{-r_2 t_0}$$

we know $r_1 \neq r_2$, so these make sense

regardless of t_0, y_0, y_0'

we can get c_1, c_2 that satisfies

The ICs