Meeting #3

Existence & Uniqueness

Theorem 2.4.1

If function p and g are continuous on I (I: $\alpha < t < \beta$) containing the point t_0,

$\exists$ a unique function $y = \phi(t)$ that satisfies the diff eq

$$y' + p(t)y = g(t)$$

for each t in I, and satisfies the IC ~~$y(t)$~~ $y(t_0) = y_0$

Asserts existence and uniqness

- Solution on I where t_0 lies on

(2)

Theorem 2.4.2

Let function f and $\frac{\partial f}{\partial y}$ be continuous in some rectangle $\alpha < t < \beta$, $\gamma < y < \delta$ containing the point (t_0, y_0). Then in some interval $t_0 - h < t < t_0 + h$ contained in $\alpha < t < \beta$ there is a unique solution $y = \phi(t)$ of the initial value problem

$$y' = f(t,y) \qquad y(t_0) = y_0$$

$$= -p(t)y + g(t)$$

and ~~df~~ $\frac{\partial}{\partial y} f(t,y) = -p(t)y$

$$2.4.2 \rightarrow 2.4.1$$

2.4.1 – Linear Diff EQ

2.4.2 – non Linear Diff EQ

→ goes to 2.4.1 if we have a Linear equation

Example 1) Goal: use the Theorems to Assert a solution.

Use Theorem 2.4.1 to find an interval in which the IVP

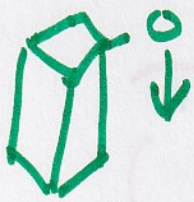
$$ty' + 2y = 4t^2 \quad \leftarrow \text{fix this}$$

$$y(1) = 2 \quad \leftarrow \text{Plug this}$$

→ plug n chug get 4

has a unique solution.

Scenario 1 (from falling object Diff EQ)

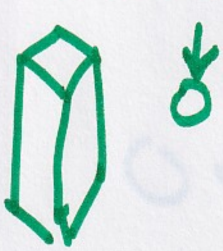


ICs

$V_0 = 0$

$a = g$

Scenario 2, coin already falling



ICs ?

We have a nonzero initial velocity

what is acceleration ?

why ask this question ?

If we have ~~accele~~ acceleration, ~~accelerate~~

and $a \neq 0$, we may state that we are not at terminal velocity!

if $a > 0$ is ~~$V_0 < V_T$~~ $> V_T$?

$V_0 <$ $> V_T$?

if $a < 0$

$V_0 > V_T$

$$ty' + 2y = 4t^2 \longrightarrow y' + \underbrace{\frac{2}{t}}_{p(t)} y = \underbrace{\frac{4t^2}{t}}_{g(t)}$$

g is continuous $\forall\, t$

p is continuous only for $t < 0$ or $t > 0$

What's our interval?

— we have ICs. $t_0 = 1$, $y_0 = 2$

$\rightarrow$ $t > 0$ is our interval

because g $\forall t$

~~f~~ y is cont $\underline{t > 0}$

Diff Eq is Linear

has the form $y' + p(t)y = g(t)$

$\rightarrow \exists$ a Solution and it's unique

due to the ICs

$$\frac{\frac{d}{dt}\mu(t)}{\mu(t)} = \frac{\frac{2}{t}\mu(t)}{\mu(t)}$$

$$\frac{f'}{f} = \frac{d}{dt}\ln|f|$$

$$\frac{d}{dt}\ln|\mu(t)| = \frac{2}{t}$$

$$\ln|\mu(t)| = 2\ln|t|$$

$$\mu(t) = e^{2\ln|t|} = t^2$$

$$t^2\left[y' + \frac{2}{t}y = 4t\right] \rightarrow \frac{d}{dt}t^2 y = 4t(t^2)$$

$$t^2 y' + 2ty = 4t(t^2)$$

⑥

$t^2 y = t^4 + C \qquad y(1) = 2$

$y = t^2 + \frac{C}{t^2}$

$2 = (1)^2 + \frac{C}{(1)^2} = \quad C = 1$

$$\boxed{y = t^2 + \frac{1}{t^2} \qquad 0 < t < \infty}$$

we know a solution exists because 2.4.1 said so!

$y(-1) = 2$ ICs

$2 = (-1)^2 + \frac{C}{(-1)^2} \quad C = 1$

$$\boxed{y = t^2 + \frac{1}{t^2} \qquad \cancel{-\infty > t > 0} \qquad -\infty < t < 0}$$

Example 2)

Apply Theorem 2.4.2 to the Initial value problem

$$\frac{dy}{dx} = \frac{3x^2+4x+2}{2(y-1)}, \qquad y(0) = -1$$

We can see that this is non Linear. so we can't use 2.4.1 → so use 2.4.2

The Idea here is that we need f and $\frac{\partial}{\partial y} f$ to be continuous in the space!

$$y' = f(t, y) \qquad y(t_0) = y_0$$

$$f(x,y) = \frac{3x^2+4x+2}{2(y-1)} \qquad \frac{\partial}{\partial y} f(x,y) = -\frac{3x^2+4x+2}{2(y-1)^2}$$

→ $f(x,y)$ & $\frac{\partial}{\partial y} f(x,y)$ are continuous everywhere on the Line $y = 1$

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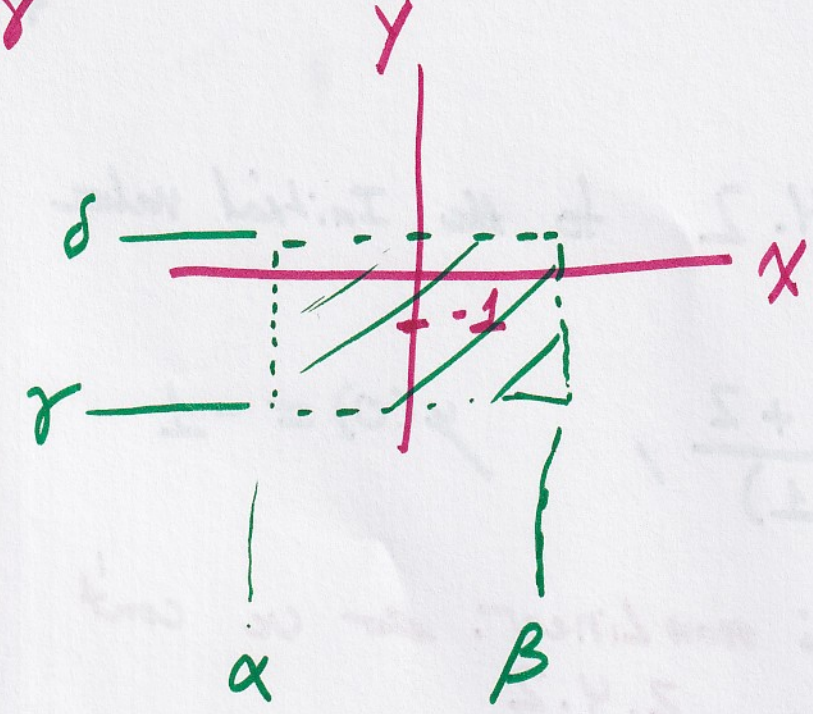

We have the Initial point (0, 1) which comes from the Initial Condition

Theorem 2.4.2 garuntees we will have a unique solution

- we can draw this rectangle infinitely far in both directions (positive X ~~or~~ and negative X)
  - But does that mean we have a solution for all X?

2.4.2 garuntees a solution, doesn't garuntee a solution for all X.

To solve the differential equation, we do separation of variables recall, the equation must be of the form (9)

$$\frac{dy}{dx} = f(x,y) \rightarrow M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

$$M(x,y) = -f(x,y) \qquad N(x,y) = 1$$

if M only has x, if N only has y

$$\rightarrow M(x) + N(y)\frac{dy}{dx} = 0$$

$$\rightarrow M(x)\,dx + N(y)\,dy = 0$$

so we may write

$$2(y-1)\,dy = (3x^2 + 4x + 2)\,dx$$

<u>Note</u>! $\quad H_1'(x) = M(x) \qquad H_2'(y) = N(y)$

$$\rightarrow H_1'(x) + \underbrace{H_2'(y)\frac{dy}{dx}}_{\frac{d}{dx}H_2(y)} = 0$$

$$\rightarrow \frac{d}{dx}H_1(x) + \frac{d}{dx}H_2(y) = 0$$

$$\frac{d}{dx}[H_1(x) + H_2(y)] = 0$$

(10) So then, $H_1(x) + H_2(y) = C$

Therefore It can be stated

$$2(y-1)dy = (3x^2+4x+2)dx$$

becomes

$$y^2-2y = x^3+2x^2+2x+C$$

for $(0,-1)$ we see that

$\cancel{0^2-2(0)}$

$$(-1)^2-2(-1) = 0^3+2(0)^2+2(0)+C$$

$$3 = C$$

To get y in terms of x, use the quadratic equation.

$$y = 1 \pm \sqrt{x^3+2x^2+2x+4}$$

$$\rightarrow y = 1 - \sqrt{x^3+2x^2+2x+4}$$

if $x < -2$, we get a complex number

$\rightarrow$ choose $x > -2$

~~Example 5~~

So for $y(0)=1$, the Initial point now lies on the line $y=1$.

- Because of this, we can't draw a box around the point $(0,1)$ where f, $\frac{\partial f}{\partial y}$ are continuous

So Theorem 2.4.2 doesn't say there is no solution

$$\rightarrow y^2-2y = x^3+2x^2+2x+c$$

for $x=0$, $y=1$, $c=-1$

so when we use the quadratic equation we get

$$y = 1 \pm \sqrt{x^3+2x^2+2x}$$

$\rightarrow$ for $x>0$, we have 2 solutions that satisfy the differential Equation with $y(0)=1$

$\rightarrow$ Non unique solution

Example 3

consider the Initial value problem

$$y' = y^{1/3}, \quad y(0) = 0 \qquad t \geq 0$$

Apply Theorem 2.4.2 to this initial value problem and then solve the problem.

$f(t,y) = y^{1/3}$ is continuous everywhere

and we can see that $\frac{\partial f}{\partial y}$ doesn't exist

when $y=0$. (Not continuous at $y=0$)

→ The continuity of $f(t,y)$ ensures the existence of solutions, but we don't have uniqueness here!

To solve, $\frac{dy}{dt} = y^{1/3} \rightarrow y^{-1/3} dy = dt$

$$\rightarrow \frac{3}{2} y^{2/3} = t + C$$

$y = \left[\frac{2}{3}(t+c)\right]^{3/2}$

for $y(0)=0$ $\quad c=0$

$y = \phi(t) = \left(\frac{2}{3}t\right)^{3/2}, \quad t \geq 0$

$\longrightarrow$ This satisfies the differential equation

but $y = \rho(t) = -\left(\frac{2}{3}t\right)^{3/2}, \quad t \geq 0$

also satisfies the differential equation

Also $y = \psi(t) = 0, \quad t \geq 0$ is a solution too!

~~The~~ These ~~solutions~~ do not contradict the existence and uniqueness ~~theorem~~ theorems

$\longrightarrow$ The theorem is not applicable if the initial point (t_0, y_0) lies on the t-axis!

(14) Look!

if we have $y(1) = 1$ for $y = [\frac{2}{3}(t + c)]^{3/2}$

$c = \frac{1}{2}$

$\rightarrow y = [\frac{2}{3}(t + \frac{1}{2})]^{3/2}$ is a unique ~~solution~~ solution!

if we have $y(1) = -1$

$\rightarrow y = \cancel{-[\frac{2}{3}(t+\frac{1}{2})]^{3/2}}$ is a unique

$-[\frac{2}{3}(t+\frac{1}{2})]^{3/2}$ Solution!