2.1 Linear Equations:

Method of Integrating Factors

$$\frac{dy}{dt} = f(t, y) \qquad (1)$$

f is a given function of two variables

$y = \phi(t)$ is a differentiable function that satisfy the equation $\forall t \in I$ called a solution

→ does y exist?

— develope the methods to get y

There is no general method for an arbitrary f

(2.1) Linear equations

(2.2) Separable equations

(2.3)

If f of (1) depends linearly on the dependent variable y, Eq (1) is a 1st order linear equation

$$\frac{dy}{dt} = -ay + b \qquad (2)$$

$a, b \in \mathbb{R}$ (falling object in the atmosphere)

The General <u>first order linear equation</u>

$$\frac{dy}{dt} + p(t)y = g(t) \qquad (3)$$

p, g are functions of the independent variable t

Another way of writing it

$$P(t)\frac{dy}{dt} + Q(t)y = G(t) \qquad (4)$$

$$P(t) \neq 0$$

$$\rightarrow \frac{dy}{dt} + \frac{Q(t)}{P(t)}y = \frac{G(t)}{P(t)}$$

Ex)

$$(4+t^2)\frac{dy}{dt} + 2ty = 4t$$

$$\frac{d}{dx}\left[f(x)g(x)\right] = f'(x)g(x) + f(x)g'(x)$$

This is product rule

it's understood that ~~it~~ $y = f(t)$

we can ~~see~~ $4+t^2 = g(t)$

$$\frac{d}{dt}\Big[\underbrace{(4+t^2)}_{g(t)}\underbrace{y}_{f(t)}\Big] = (4+t^2)\frac{dy}{dt} + 2ty = 4t$$

So we can write $\frac{d}{dt}\left[(4+t^2)y\right] = 4t$

$$\rightarrow (4+t^2)y = 2t^2 + c \qquad (7)$$

$$\rightarrow y = \frac{2t^2}{4+t^2} + \frac{c}{4+t^2} \qquad (8)$$

This is the general solution

Equations are not this easy to solve

Leibniz figured something out

— Multiply the diff eq by $\mu(t)$
then the equation is <u>converted</u> into
one that is differentiable

$\mu(t)$ – integrating factor

~~Ex) 2~~

Ex 2) $$\frac{dy}{dt} + \frac{1}{2} y = \frac{1}{2} e^{t/3} \quad (9)$$

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$$\rightarrow \quad \underbrace{\mu(t)\frac{dy}{dt} + \frac{1}{2}\mu(t) y}_{} = \frac{1}{2}\mu(t) e^{t/3} \quad (10)$$

choose $\mu(t)$ so that
the LHS is $\frac{dy}{dt}\left[\mu(t) y\right]$

$$\rightarrow \quad \frac{dy}{dt}\left[\mu(t) y\right] = \mu(t)\frac{dy}{dt} + \frac{d}{dt}\mu(t)\, y \quad (11)$$

from this example, we must choose

$$\frac{d}{dt}\mu(t) = \frac{1}{2}\mu(t)$$

Note $\frac{1}{f} f' = \frac{d}{dx} \ln|f|$

so $$\frac{\frac{d}{dt}\mu(t)}{\mu(t)} = \frac{1}{2} = \frac{d}{dt}\ln|\mu(t)|$$

$$\rightarrow \ln|\mu(t)| = \frac{1}{2}t + C$$

$$e^{\ln|\mu(t)|} = e^{\frac{1}{2}t + C}$$

$$\rightarrow \mu(t) = e^{C}e^{\frac{1}{2}t} \qquad (14)$$

Let $e^C = 1$

$$e^{t/2}\frac{dy}{dt} + \frac{1}{2}e^{t/2}y = \frac{1}{2}e^{t/3}e^{t/2}$$

$\mu(t)$, so product rule

$$\frac{d}{dt}(e^{t/2}y) = \frac{1}{2}e^{5t/6} \qquad (16)$$

$$\frac{1}{2}\,\frac{6}{5}$$

$$\rightarrow e^{t/2}y = \frac{1}{2}\,\frac{6}{5}e^{5t/6} + C \qquad (17)$$

$$y = \frac{\frac{1}{2}\frac{6}{5}e^{5t/6} + C}{e^{t/2}}$$

6

$$y = \frac{3}{5}e^{t/3} + Ce^{-t/2} \qquad (18)$$

for a solution thru $(0,1)$

$t = 0$, $y = 1$

$\rightarrow 1 = \frac{3}{5} + C$ $\quad C = 2/5$ $\quad$ ur done!

OK, go over general solution

then proceed in the book

$(0,4)$

$4 = \frac{3}{5} + C \rightarrow \frac{20}{5} - \frac{3}{5} = \frac{17}{5} = C$

2.2 Separable Equations

The general 1st order equation is

$$\frac{dy}{dx} = f(x,y) \qquad (2)$$

subclass of 1st order equations that can be solved by direct integration. Rewrite (2) as

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \qquad (3)$$

It's always possible to do this by setting

$$M(x,y) = -f(x,y) \qquad N(x,y) = 1$$

if ~~$M(x,y) = f(x,y)$~~

M is a function of x only and N is a function of y only. Eq (3) becomes

$$M(x) + N(y)\frac{dy}{dx} = 0 \qquad (4)$$

(2) Such an equation is said to be <u>separable</u> if written

$$M(x)dx + N(y)dy = 0 \qquad (5)$$

Ex 1) $\dfrac{dy}{dx} = \dfrac{x^2}{1-y^2}$ (6)

$\rightarrow$ $-x^2 + (1-y^2)\dfrac{dy}{dx} = 0$ (7)

chain rule $\dfrac{d}{dx} f(y) = f'(y)\dfrac{dy}{dx}$

$$\frac{d}{dx}\left[y - \frac{y^3}{3}\right] = (1-y^2)\frac{dy}{dx}$$

we know $-x^2 = \dfrac{d}{dx}\left(-\dfrac{x^3}{3}\right)$

$$\rightarrow \frac{d}{dx}\left(-\frac{x^3}{3}\right) + \frac{d}{dx}\left(y - \frac{y^3}{3}\right) = 0$$

$$\frac{d}{dx}\left(-\frac{x^3}{3} + y - \frac{y^3}{3}\right) = 0$$

$$\rightarrow -\frac{x^3}{3} + y - \frac{y^3}{3} = C$$

$$H_1'(x) = M(x) \qquad H_2'(y) = N(y)$$

$$(4) \quad M(x) + N(y)\frac{dy}{dx} = 0$$

$$H_1'(x) + H_2'(y)\frac{dy}{dx} = 0$$

y is regarded as a function of x

$$H_2'(y)\frac{dy}{dx} = \left(\frac{d}{dy} H(y)\right)\left(\frac{dy}{dx}\right) = \frac{d}{dx} H_2(y)$$

$$\frac{d}{dx} H_1(x) + \frac{d}{dx} H_2(y) = 0$$

$$\frac{d}{dx}\left[H_1(x) + H_2(y)\right] = 0$$

$$H_1(x) + H_2(y) = C$$

ICs $\quad y(x_0) = y_0$

$$C = H_1(x_0) + H_2(y_0)$$

4

Ex 2) $\dfrac{dy}{dx} = \dfrac{3x^2 + 4x + 2}{2(y-1)}$ $y(0) = -1$

$$2(y-1)\frac{dy}{dx} - (3x^2 + 4x + 2) = 0$$

$$\frac{d}{dx}[\cancel{4}y^2 - 2y] - \frac{d}{dx}(x^3 + 2x^2 + 2x) = 0$$

$$y^2 - 2y - (x^3 + 2x^2 + 2x) = c$$

$$2(y-1)\,dy = (3x^2 + 4x + 2)\,dx$$