Meeting 21 Notes

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Sums and differences

Part II

Summary of Meeting #20

- Derivatives and Integrals are related by the Fundamental Theorem of Calculus

C^0 function, f is continuous

C^1 function, f has a continuous derivative

An Integral Property: for an integrable function $g(x)$ and $\alpha, \beta, \gamma \in \mathbb{R}$

$$\int_\alpha^\gamma g(x) - \int_\alpha^\beta g(x)\,dx = \int_\beta^\gamma g(x)\,dx$$

\# 1)

a) i) $\left|\overline{\int_\alpha^\beta} f(x)dx - \underline{\int_\alpha^\beta} f(x)dx\right| < \varepsilon$

$[\alpha, \beta]$ is compact

for $\varepsilon > 0$ and $[\alpha, \beta]$ a compact ~~set~~ we say $f(x)$ is integrable when

~~lim~~ $\left|\overline{\int_\alpha^\beta} f(x)dx - \underline{\int_\alpha^\beta} f(x)dx\right| < \varepsilon$

There is a mesh size $< \delta$ s.t.

ii) $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

for a continuous function, f,

$\underline{|x+h-x|} < \delta \Longrightarrow |f(x+h) - f(x)| < \varepsilon$

This distance is approaching zero

$$f'(x) = \underbrace{\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}_{L}$$

$$|x-h| < \delta \longrightarrow \left|\frac{f(x) - f(h)}{x-h} - L\right| < \varepsilon$$

$$f'(x) = \frac{f(x) - f(h)}{x - h}$$

iii A function ~~is~~ continuous at $x=a$ if for any $\varepsilon > 0$ we can find $\delta > 0$ s.t. $|x-a| < \delta$ implies $|f(x) - f(a)| < \varepsilon$

$$\lim_{x \to a} f(x) = f(a)$$

b) i) $f(x) = \cancel{\lceil x \rceil}$ on $[0,1]$

$\lceil x \rceil$ ← ceiling function

Heaviside function

ii) Trick question!

$f(x) = \frac{1}{x}$ $\cancel{(0,\infty)}$, $[a,b]$

can work if we are allowed to not use a compact set. We must use a compact set

A continuous function on a compact set is always integrable

iii] ~~A continuous function f is also differentiable~~

A differentiable function f is also continuous

iv] ~~A continuous function f is also differentiable~~

$\Rightarrow f(x) = |x|$

is a contradiction to this

#2

$$H(x) = \int_a^x \eta(t)\,dt + C$$

$$\frac{dH}{dx} = \eta(x) + 0$$ by the fund. Thm of calculus.

if

$$H(x) = \int_a^x \eta(t)\,dt + N(x)$$

so $$\frac{dH}{dx} = \cancel{\int_a^x \eta(t)\,d}$$

$$\eta(x) + N'(x)$$

but $\eta(x) = \frac{dH}{dx}$

so $\eta(x) = \eta(x) + N'(x)$

$$\Rightarrow N' = 0 \Rightarrow N = C$$

we have '+C' because we don't know ~~if~~ the original function had a constant or something

Suppose $H(x) - \int_c^x \eta(t)\,dt = f(x)$

Then $H'(x) - \eta(x) = f'(x)$

$$\underbrace{H'(x)}_{\eta(x)}$$

$\Rightarrow$ ~~$0 = f'(x)$~~ $\quad 0 = f'(x)$

$\rightarrow f(x) = c \qquad c \in \mathbb{R}$

very specific c, $(H(\alpha))$

#3

Let $\eta(x) = \frac{dH}{dx}$, where $\eta : [\alpha, \beta] \to \mathbb{R}$ is continuous, Then

$$\int_\alpha^\beta \eta(x)\,dx = H(\beta) - H(\alpha)$$

$$\rightarrow \quad -\int_\alpha^\beta \eta(x)\,dx = -H(\beta) + H(\alpha)$$

$$H(\beta) - \int_\alpha^\beta \eta(x)\,dx = H(\alpha)$$

① $H(x) - C = \int_{\alpha}^{x} \eta(t)\,dt$

Let $x = \beta$

② $H(\alpha) - C = \int_{\alpha}^{\alpha} \eta(x)\,dx = 0$

$H(\alpha) = C$

③ $H(B) - C = \int_{\alpha}^{\beta} \eta(x)\,dx$

④ $H(\beta) - H(\alpha) = \int_{\alpha}^{B} \eta(x)\,dx$

4

The Integral of derivative is tru

Last time was derivative of Integral

Now we can use fundamental Theorem of Calculus

$h(y) \quad K \in \mathbb{R}$

$$\int_{5}^{x^3} h(y)\,dy = x^4 + K$$

8

$$h(x^3)\cdot 3x^2 = 4x^3$$

$$h(x^3) = \frac{4}{3}x$$

$$h(x) = \frac{4}{3}x^{1/3}$$

$$h(y) = \frac{4}{3}y^{1/3}$$

$$\rightarrow \frac{\frac{4}{3}y^{4/3}}{4/3}\Big|_5^{x^3} = (x^3)^{4/3} - (5)^{4/3}$$

$$= x^4 + K$$

is this a unique solution?

#5) $h(x) = \alpha(x)\beta(x)$

$$h'(x) = \alpha'(x)\beta(x) + \beta'(x)\alpha(x)$$

$$\int_y^z h'(x)\,dx = \int_y^z \alpha'(x)\beta(x)\,dx + \int_y^z \beta'(x)\alpha(x)\,dx$$

$$\int_y^z h'(x)\,dx - \int_y^z \beta(x)\,\alpha'(x)\,dx$$

$$= \int_y^z \alpha(x)\,\beta'(x)\,dx$$

$$\int_y^z \alpha(x)\,\beta(x)\,dx$$

#6) $u : [\alpha, \beta] \to \mathbb{R}$

C^1 $z(u)$ is continuous Then,

$$\int_{u(\alpha)}^{u(\beta)} z(u)\,du = \int_\alpha^\beta z(u)\,u'(x)\,dx$$

$$\int_{u(\alpha)}^{u(x)} z(u)\,du \qquad \int_\alpha^x z(u)\,\frac{du}{dx}\,dx$$

Let $\beta = x$

$h(x) = f(g(x))$

$h'(x) = f'(g(x)) \cdot g'(x)$

$$\int_{\alpha}^{\beta} h'(x)\,dx = \underline{\int_{\alpha}^{\beta} f'(g(x))\, g'(x)}\, dx$$

$$f(g(x))\Big|_{\alpha}^{\beta} = {}''$$

$$f(g(\beta)) - f(g(\alpha)) = \int_{g(\alpha)}^{g(\beta)} f'(x)\,dx$$

$$= \int_{\alpha}^{\beta} f'(g(x))\, g'(x)\, dx$$

Let $f' = z$, $\cancel{g}$ $g' = u'$

$$\int z(u)\,du$$