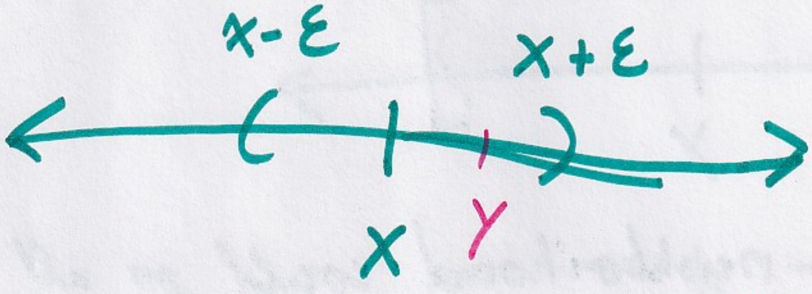
Meeting #8 @MRemark

Notes

A Topology Lesson

Summary of #7

Limit point (accumulation point)



$x, y \in S$

all neighborhoods will contain the point we are concerned with and some other point, we cant garuntee what those points are (we don't know what ε is ahead of time) No matter how small ε is, There is going to another point in the neighborhood

② If we increase the dimension, we increase the dimension of the neighborhood as well.

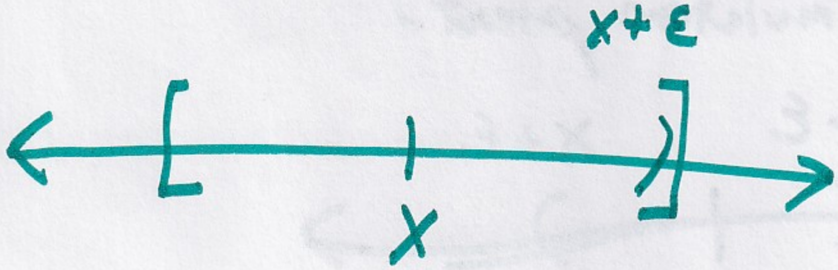
"we could ~~extend~~ extend this to any set"

→ Not limited to the real numbers

---

since the ε-neighborhoods are open,

say that our set S is closed in some manner,



our ε-neighborhood could go all the way out to the edge but will never actually include our boundary points

---

you can also define a closed set whose complement is open

1)

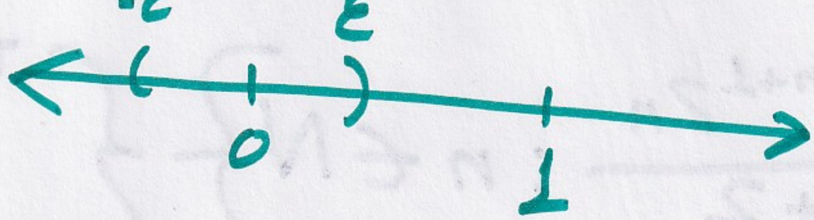
No matter how small ε is, There's always another element in S that's around x that is not x itself (say y)

(a) we are Trying to figure out what the point is
No matter how close we get, there's always another Point ~~[illegible]~~ in that set.

We see $\frac{1}{n^3}$ so $\frac{1}{n^3} \to 0$

→ make a neighborhood around zero
Then no matter how far down the List we go, there's always going to be some other element in our set which is in that neighborhood around zero.

2:25



We don't care that $-\varepsilon$ goes outside of our sequence

(4)

we already know the limit of the sequence is. We want to formally define it.

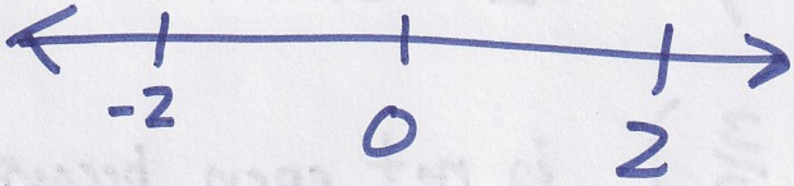
The limit point Idea is talking about the Gradualness of the sequence (Eventualness)

If ~~you~~ you make an interval around zero, I can always capture the tail of the sequence inside any ε I wish.

So zero would be an accumulation point. We have this tail converging to 0 and we are trying to capture that notion.

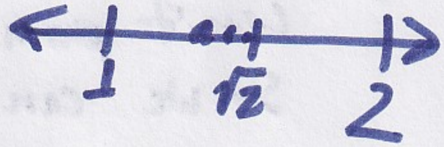
(b) $\left\{ \dfrac{(-1)^{n+1}2n}{n+2} : n \in N \right\} \begin{cases} +2 \\ -2 \end{cases}$

as we gonna find ~~infinitly~~ infinitely many points approaching this value



for 2, -2 we are going to find an infinite subsequence which have a convergent value of 2 or -2

(c) $\rightarrow \sqrt{2}$



(d) $\rightarrow$ None

∞ is not defined as a number, it's unbounded.

$$\left\lfloor \frac{10^n \sqrt{2}}{10^n} \right\rfloor$$

⑥

#2)

a) $[-31, -0.3]$ closed

b) $[1, \frac{3}{2})$ is not open because any ε-neighborhood of 1 is not contained inside the interval $[1, 3/2]$

c) $[5, \infty)$ we have no defined limit point on the right, so we can't contain it.

→ ~~both~~ ~~neither~~ → closed

(infinity is not a limit point)

d) $(-\infty, \infty)$

can infinity be a limit point? its not in S, so we just have 1 limit point

ε-neighborhoods are still contained in here

we can pick a number without bound, and I can find a ~~number~~ ε-neighborhood within it.

Pick a # $< \infty$ and I can find something bigger and something smaller inside that interval as long as I am bigger than 5

d) So is (d) both open and closed?

Yes! open & closed

You can draw an ε-neighborhood around its Limits points and all of those are contained in the set, so its open

~~Also~~ Also
all of these Limit points are in the set itself so it's also closed.

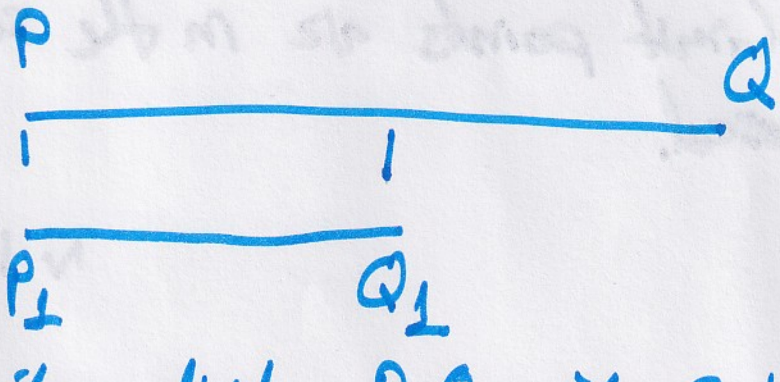
~43

<u>Thm</u>: If a set S contains infinitely many points, and is entirely situated in an interval $[\alpha, \beta]$, then at least one point of the interval is a point of accumulation of S.

In this proof, $[\alpha, \beta]$ is considered to be the segment PQ

<u>Proof</u>:

if we divide PQ into 2 equal parts, at least one of them must contain infinitely many points of S. We select the one which does, or, if both do, we select the ~~left~~ Left half (Left-hand half); and we denote the selected half by P_1Q_1. If P_1Q_1 is the Left-hand half, P_1 is the same point as P.



Similarly, if we divide P_1Q_1 into 2 halves, at least one of them must contain infinitely many points of S. We select the half P_2Q_2 which does so, or, if both do, we select the Left-hand half.

Proceeding in this way, we can define a sequence of intervals

$$(PQ, P_1Q_1, P_2Q_2, P_3Q_3. \cdots)$$

each of which is a half of its predecessors and each of which contains infinitely many points of S

The points $P, P_1, P_2, \ldots$ progress steadily from left to right, and so P_n tends to a limiting position T. Similarly Q_n tends to a limiting position T'. But TT' is plainly less than P_nQ_n, whatever the value of n; and P_nQ_n, being equal to $\frac{PQ}{2^n}$, tends to zero. (It's equal because we are halving PQ every iteration)

Hence T' coincides with T, and P_n and Q_n both tend to T.

Then T is point of accumulation of S. Suppose ξ is its coordinate, and consider any interval of the type $[\xi - \varepsilon, \xi + \varepsilon]$. If n is sufficiently large, P_nQ_n will lie entirely inside this interval.

10

Hence $[\xi - \varepsilon, \xi + \varepsilon]$ contains ~~a fite~~ infinitely many points of S

#4) What are the accumulation points of $\mathbb{Q}$, the rational numbers? What about $\mathbb{Q}^+$, the positive rational numbers?

$$\mathbb{Q} = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z},\ b \neq 0 \right\}$$

$\mathbb{Q}$: All $\mathbb{R}$

$\mathbb{Q}^+$: $\mathbb{R}^+ \cup \{0\}$

Accumulation point: Given a set S, a point $x \in S$ that, when creating any ε-neighborhood around x, contains a distinct $y \in S$, ~~with~~ $y \neq x$

1:08

Well, we do have an ∞ amount of rational numbers.

When we talk about ~~the~~ accumulation points of $\mathbb{Q}$, it sounds like every number in $\mathbb{Q}$

But Look at 1c,

It has an accumulation point $\sqrt{2}$

So how does $\sqrt{2}$ play into this?
(How does 1c play into this?)

$\sqrt{2}$ cannot be an accumulation point because it doesn't belong to the set $\mathbb{Q}$

"But its truncated versions are rational numbers"

We know real numbers exist, but it is conceivable to get an irrational number out of a sequence of rational numbers

??? what does this mean?

Anders: limit points are not required to be ~~but~~ in the set

→ if that happens (for all limit points) then we have a closed set

You can take a limit point in $\mathbb{R}$, if the real numbers are all the limit points of all the rational numbers

If we take all the points that are sequences in $\mathbb{Q}$, what are the accumulation points of those sequences in $\mathbb{Q}$?

These limit points can be irrational or just real numbers.

___ ___ ___ ___ ___ ___

Gaps in $\mathbb{Q}$?

Rational numbers aren't complete
"There are no holes in the real numbers"

Anders: $\sqrt{2}$ is a limit point of $\mathbb{Q}$ since no matter how small, all ε-neighborhoods centered at $\sqrt{2}$ will have rational numbers

"We want to define the real numbers as the closure of the rationals"

1:09

Another way to look at this,

$\left(\quad \frac{1}{2} \quad \right)$ $\qquad$ $\left(\quad \sqrt{2} \quad \right)$

$\frac{1}{2}-\varepsilon \qquad \frac{1}{2}+\varepsilon \qquad \sqrt{2}-\varepsilon \qquad \sqrt{2}+\varepsilon$

$\mathbb{Q}$? $\qquad$ $\mathbb{Q}$?

We can make an ε-neighborhood around $\frac{1}{2}$ or $\sqrt{2}$ and we can find a rational number inside the ε-neighborhood

(14) If we can do this for any ε it implies that all rational and irrational numbers are accumulation.

5) True or false: Cauchy sequences converge in $\mathbb{Q}$

$\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ s.t.

$$N > a, b \rightarrow |X_a - X_b| < \varepsilon$$

In a cauchy-sequence, eventually we will have this phenomenon

If they converge in the real numbers then we should 'theoretically' be able to ~~converge~~ convince ourselves they converge in $\mathbb{Q}$

"We know they converge in $\mathbb{R}$ but if we constrict ourselves to only rational numbers, we know that the limit points for the rational numbers are actually real numbers, so they should converge in $\mathbb{Q}$ as well."

but $(1, 1.4, 1.41, 1.414, 1.4142 \ldots \ldots$

is a Cauchy sequence.

$\rightarrow$ Not in $\mathbb{Q}$, False

If I restrict this sequence to be only rational numbers, and we know it's cauchy, it's going to have a limit. (we can show it converges)

Where we want the limit to be, may not be in $\mathbb{Q}$.

So if we restrict our sequence in $\mathbb{Q}$, does the limit have to be in $\mathbb{Q}$?

"If cauchy sequences converge in the space, then the space is complete. So $\mathbb{Q}$ is not complete because not all cauchy sequences converge in

(16) Q.

$\mathbb{R}$ is called complete