MATH Analysis Winter 2024 (Spring)

Meeting 2

Summary of Meeting 1

- The Area of successive inscribed polygons form the area of the circumscribed circle, as the number of sides tends to infinity.
- The perimeter of successive inscribed polygons form the area of the circle circumscribing the polygons, as the number of sides tends to infinity.
- Lingering question: The statement $\frac{\pi}{4} = 1$ is not correct, but why? (See prompt 1-7)

(2)

Today's Prompts

1) There are various ways to represent sequences. For example, consider the sequence of numbers.

$$a_n = (1, 4, 9, 16, 25, \ldots) = (n^2)_{n=1}^{\infty}$$

a) Fill in the table.

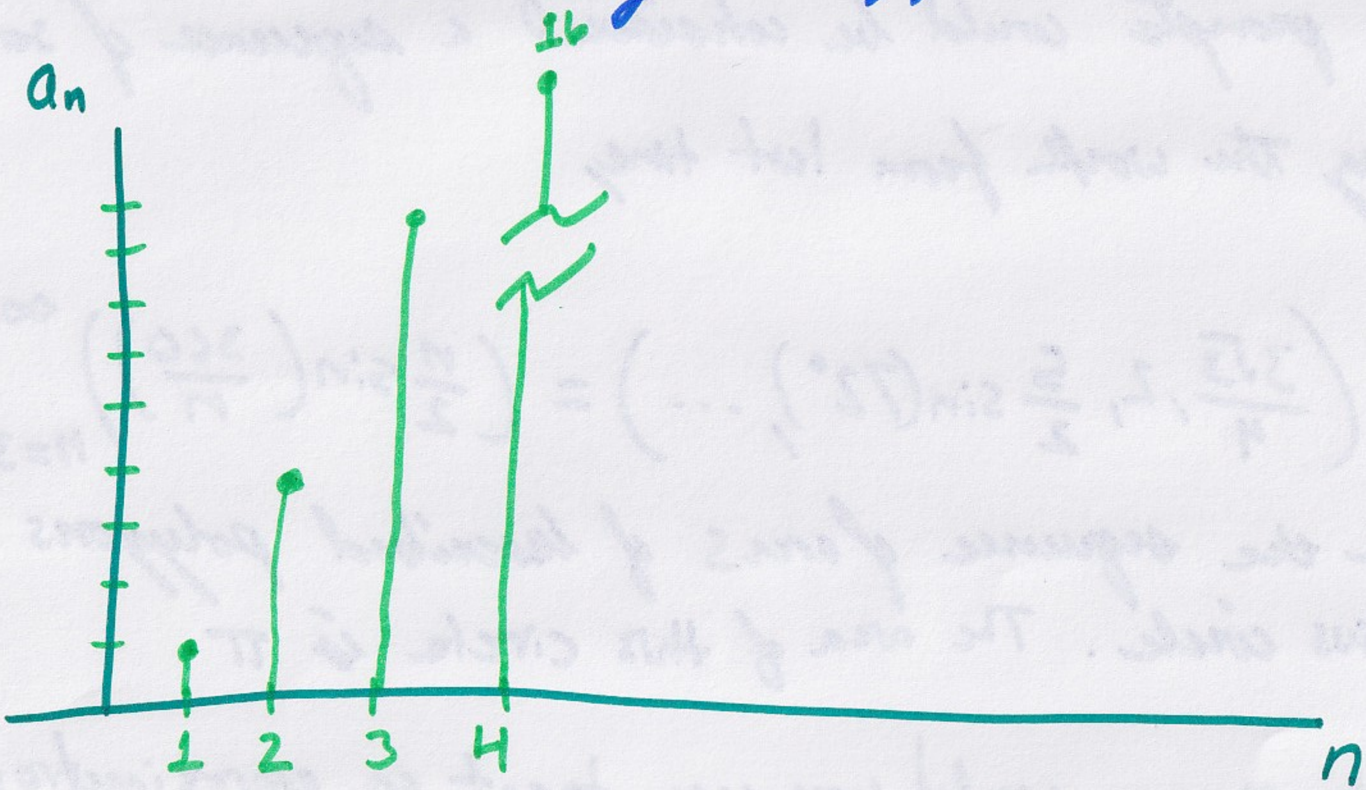
n	1	2	3	4	45	200
a_n	1	4	9	16	↑	40000

2025

$$\begin{array}{r} {}^{2}\\ 45 \\ \times\ 45 \\ \hline 225 \\ 1800 \\ \hline 2025 \end{array}$$

$$\sqrt{4 \cdot 10000} = 2\sqrt{4 \cdot 2500}$$
$$= 2 \cdot 2\sqrt{25 \cdot 100} = 2 \cdot 2 \cdot 5\sqrt{100}$$
$$= 2 \cdot 2 \cdot 5 \cdot 10 = 200$$

b) Make a plot of the table in part (a), "without using technology"



④

2) In meeting 1 (cf. prompts 1-1 to 1-5), each of these prompts could be considered a sequence of sorts. Using the work from last time,

$$\left(\frac{3\sqrt{3}}{4}, 2, \frac{5}{2}\sin(72^\circ), \ldots\right) = \left(\frac{n}{2}\sin\left(\frac{360}{n}\right)\right)_{n=3}^{\infty}$$

represents the sequence of areas of inscribed polygons inside a unit-radius circle. The area of this circle is π

a) What n-gon could you use to get an approximation to π within 1?

$n = 5$

$$\left.\frac{n}{2}\sin\left(\frac{360^\circ}{n}\right)\right|_{n=5} \approx 2.37$$

b) What n-gon could you use to get an approximation to π within 0.1?

$n = 15$

$$\frac{n}{2}\sin\left(\frac{2\pi}{n}\right)\bigg|_{n=15} \simeq 3.037$$

So $$\left|\pi - \frac{15}{2}\sin\left(\frac{2\pi}{15}\right)\right| \simeq 0.091$$

c) What n-gon could you use to get an approximation to π within 0.01?

$n = 50$

$$\frac{n}{2}\sin\left(\frac{2\pi}{n}\right)\bigg|_{n=50} \simeq 3.133$$

So $$\left|\pi - \frac{50}{2}\sin\left(\frac{2\pi}{50}\right)\right| = 8.26 \times 10^{-3}$$

(1)

d) Describe an algorithm that you could use to get an approximation of π within some K.

for $n \geq 3$

While $\left|\pi - \frac{n}{2}\sin\left(\frac{2\pi}{n}\right)\right| > K$

$n{+}{+}$

3.) Here's the sequence of perimeters from prompt 1-4. Do the same thing as prompt 2-2, except remember that the circumference of a unit-radius circle is 2π.

$$P_n = (3\sqrt{3}, 4\sqrt{2}, \ldots) = \left(2n\sin\left(\frac{\pi}{n}\right)\right)_{n=3}^{\infty}$$

a) what n-gon could you use to get an approximation to 2π within 1? $n=4$

$$2n\sin(\pi/n)\Big|_{n=4} \simeq 5.66$$

$$|2\pi - 2(4)\sin(\pi/4)| \simeq 0.626$$

b) what n-gon could you use to get an approximation to 2π within 0.1? $n=11$

$$2n\sin(\pi/n)\Big|_{n=11} \simeq 6.198$$

$$\left|2\pi - 2(11)\sin\left(\frac{\pi}{11}\right)\right| \simeq 0.085$$

⑧ c.) " within 0.01?

$n = 33$

$$2n\sin\left(\frac{\pi}{n}\right)\Big|_{n=33} \simeq \cancel{0.00948}\ 6.27$$

$$\left|2\pi - 2n\sin\left(\frac{\pi}{n}\right)\right| \simeq 0.00948$$

d) Describe an algorithm that you could use to get an approximation of 2π within some K.

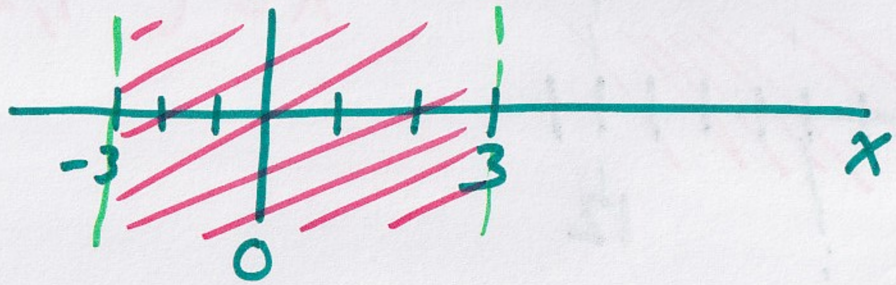
for ~~$n \geq 0$~~ $n \geq 3$

While $\left|2\pi - 2n\sin\left(\frac{\pi}{n}\right)\right| > K$

n++

4.) On a number line, draw a picture of the solutions to the following inequalities.

a) $|x| < 3$

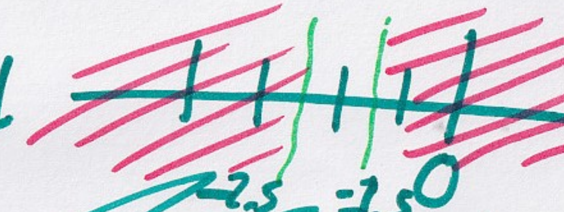


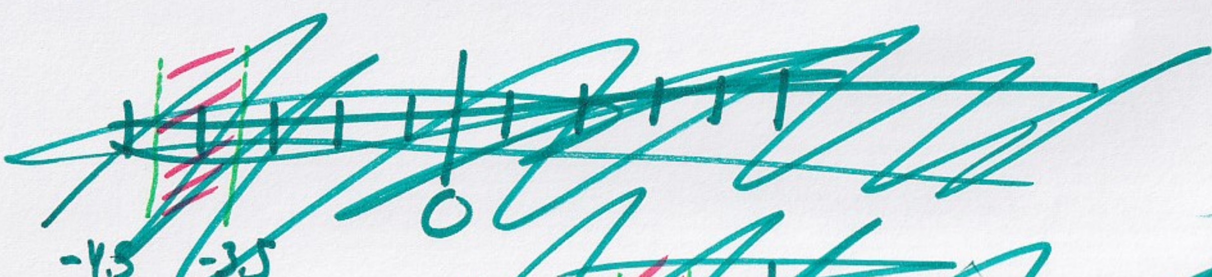
$x \in (-3, 3)$

b) $|x-3| < 5$

$x \in (-2, 8)$

c) $|2x+4| > 1 \rightarrow |2(x+2)| > 1$





~~$x \in (-2.5, -1.5)$~~

$x < -2.5$

$x > -1.5$

1c

d) $|4-x| < 8$

$-x+4 = -1(x-4)$

$|-1(x-4)| < 8$

$x \in (-4, 12)$

5.)

5. Consider the following sequences.

$$\alpha_n = (1, -2, 3, -4, 5, -6, \cdots)$$

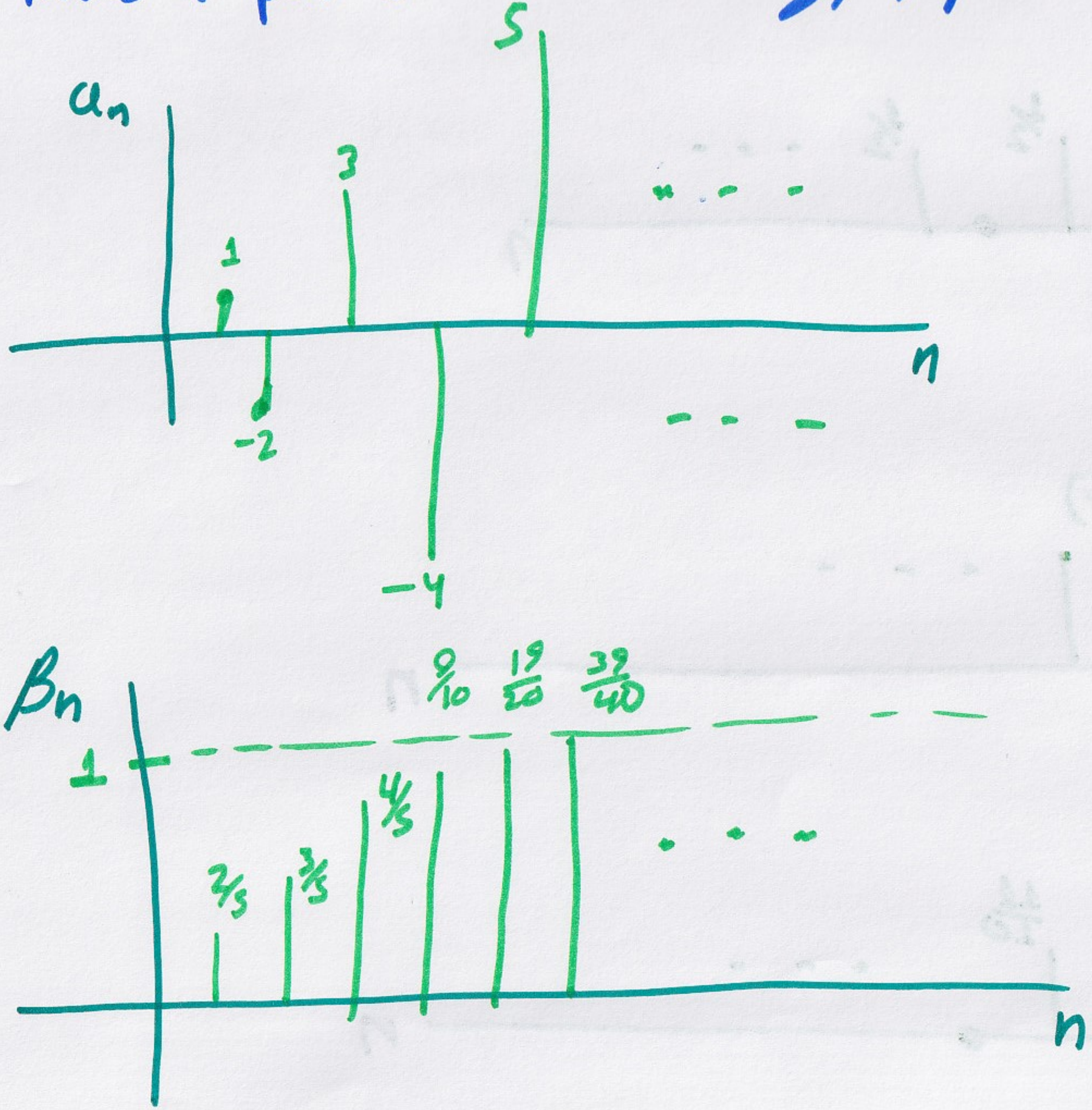
$$\beta_n = \left(\frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{9}{10}, \frac{19}{20}, \frac{39}{40}, \cdots\right)$$

$$\gamma_n = \left(0, \frac{1}{2}, 1, \frac{1}{2}, 0, \frac{1}{2}, \cdots\right)$$

$$\xi_n = \left(3\tan\left(\frac{180^\circ}{3}\right), 4\tan\left(\frac{180^\circ}{4}\right), 5\tan\left(\frac{180^\circ}{5}\right), 6\tan\left(\frac{180^\circ}{6}\right), \cdots\right)$$

$$\eta_n = \left(\frac{5}{2}, -1, \frac{3}{2}, -\frac{1}{4}, \frac{11}{10}, 0, \cdots\right)$$

a) make a plot ("without technology") of each sequence.



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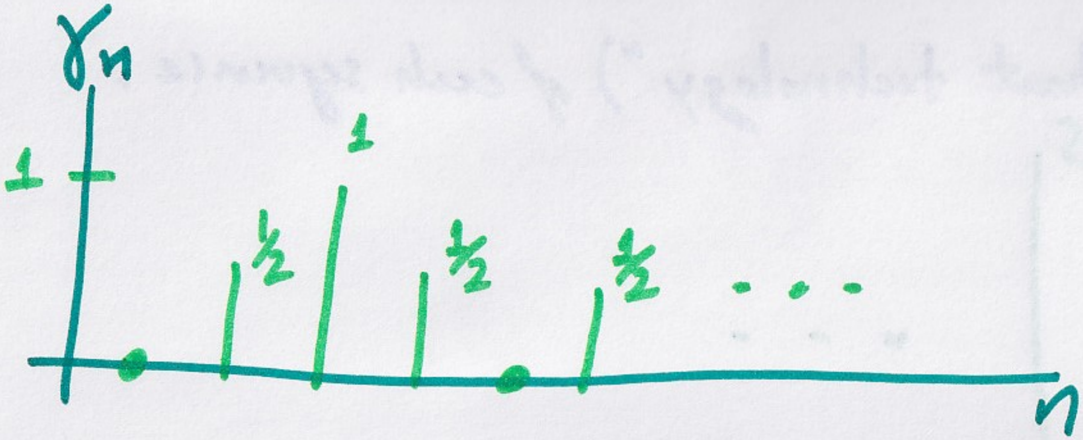
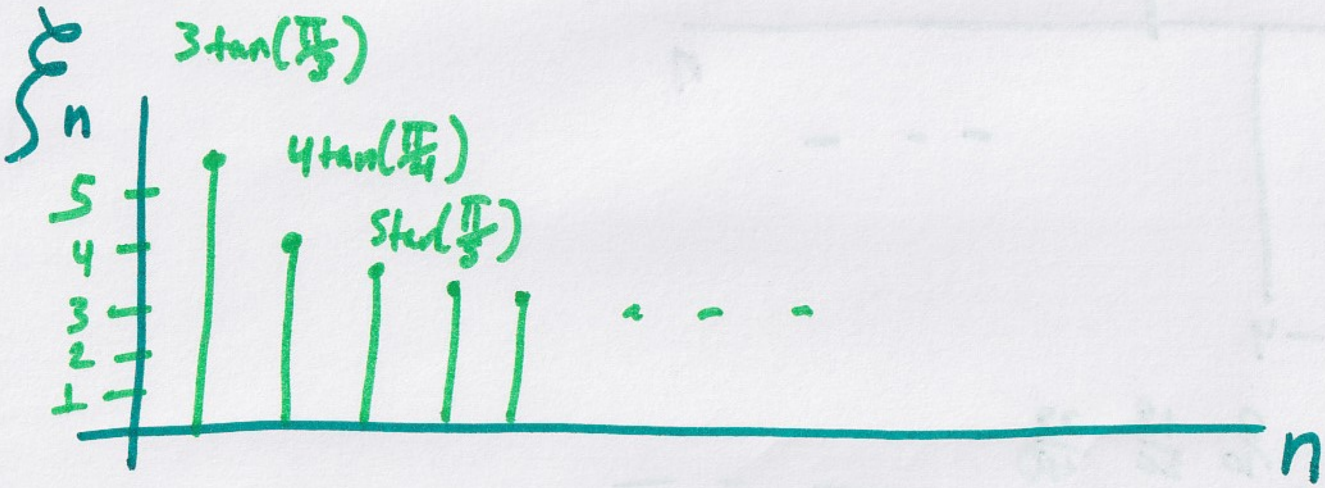
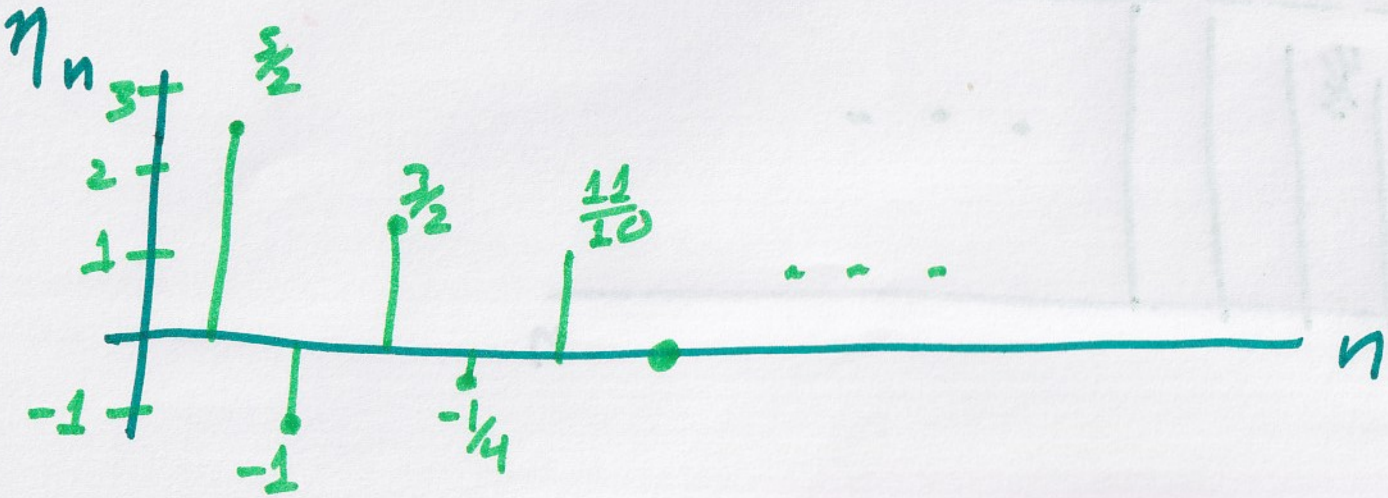


γn
1
1/2
1
1/2
1/2
n



ξn
3tan(π/3)
4tan(π/4)
5tan(π/5)
5
4
3
2
1
n



ηn
5/2
3
2
1
-1
3/2
11/10
-1
-1/4
n

b) What is happening in these sequences "in the long run"? 13

α_n, the magnitude will increase to a very large number.

β_n, ~~the higher entries~~ as $n \to \infty$, the entries will get closer to $\underline{1}$

γ_n, repeats infinitely

ξ_n, as $n \to \infty$, the entries will get closer to π

η_n, as $n \to \infty$ looks like it converges to some number, in perhaps a discrete sampling of a attenuating sinusoid.

(14)

c) Is there a term of the sequence that you can get within $K=0.1$ of the "Long term behavior"?

α_n will increase for ever.

β_n, at $n=4$ we are exactly 0.1 away from the "Long term behavior". So then at $n=5$ we are within $K=0.1$

γ_n, We'll say there's no term

ξ_n, at $n=11$, $n\tan(\frac{\pi}{n})\Big|_{n=11} \approx 3.23$

and $|\pi - n\tan(\frac{\pi}{n})| \simeq 0.0883$

η_n may converge to some number after all (Likely)

η_n, I don't see a discernable sequence, so I don't know if there is a Long term behavior

d) How many other terms meets the requirement in part (c)?

β_n, $\cancel{\ }$, ξ_n ~~met~~ meet the requirements.

And η_n

e) Try the same thing, but for $k = 0.0001$

for β_n, at $n = 11$ we have $1 - \frac{1}{1280}$

which is within $k = 0.0001$ (1×10^{-4})

ξ_n, for $n = 325$ we have

$$(325)\tan\left(\frac{\pi}{325}\right) \simeq 3.1417$$

and $\left|\pi - 325\tan\left(\frac{\pi}{325}\right)\right| \simeq 0.978 \times 10^{-4}$

η_n may converge but I don't know what n would yield a $k = 1 \times 10^{-4}$.

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f) How small can K be...?

as Long as $K > 0$, it can be as small as we need it.

6.) Sort the sequences in prompt 5 into categories based on their behavior. How is the behavior in these sequences different? Try to define the categories, if possible.

α_n, is unbounded and oscillatory (sign-wise), and doesn't converge

β_n, is bounded and 'converges'

γ_n, is a repeating pattern

ξ_n, is bounded and converges to π

η_n, doesn't seem to have a discernable pattern.

X - it does have a pattern like a decaying sinusoid with a dc component, it converges to some number

The sequence entries to η_n may take the form

$a * f(b(x-c)) + d$ (n Simplex Pachinko's function transformation form)

could be $a\sin(\frac{\pi}{2}n) + \frac{1}{2}$, $a = 2^{-n}$ or some decaying exponent

So then η_n converges

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7.) consider the sequence of functions

$$(M_1(x), M_2(x), M_3(x), \ldots)$$

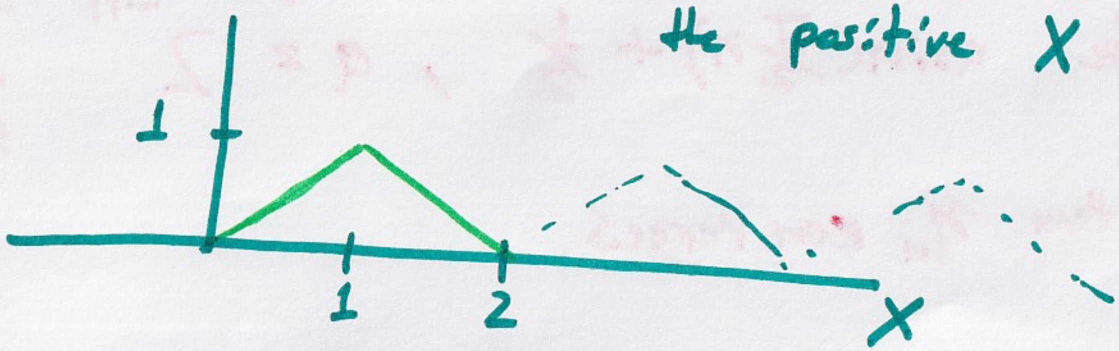
$$M_1(x) = \begin{cases} x & x \in [0, \frac{1}{n^2}] \\ \frac{2}{n^2} - x & x \in (\frac{1}{n^2}, \frac{2}{n^2}] \end{cases} \quad \text{for } n = 1$$

Assume that $M_n(x + \frac{2}{n^2}) = M_n(x)$

a) what does $M_n(x + \frac{2}{n^2}) = M_n(x)$ mean?

Lets plot M_1

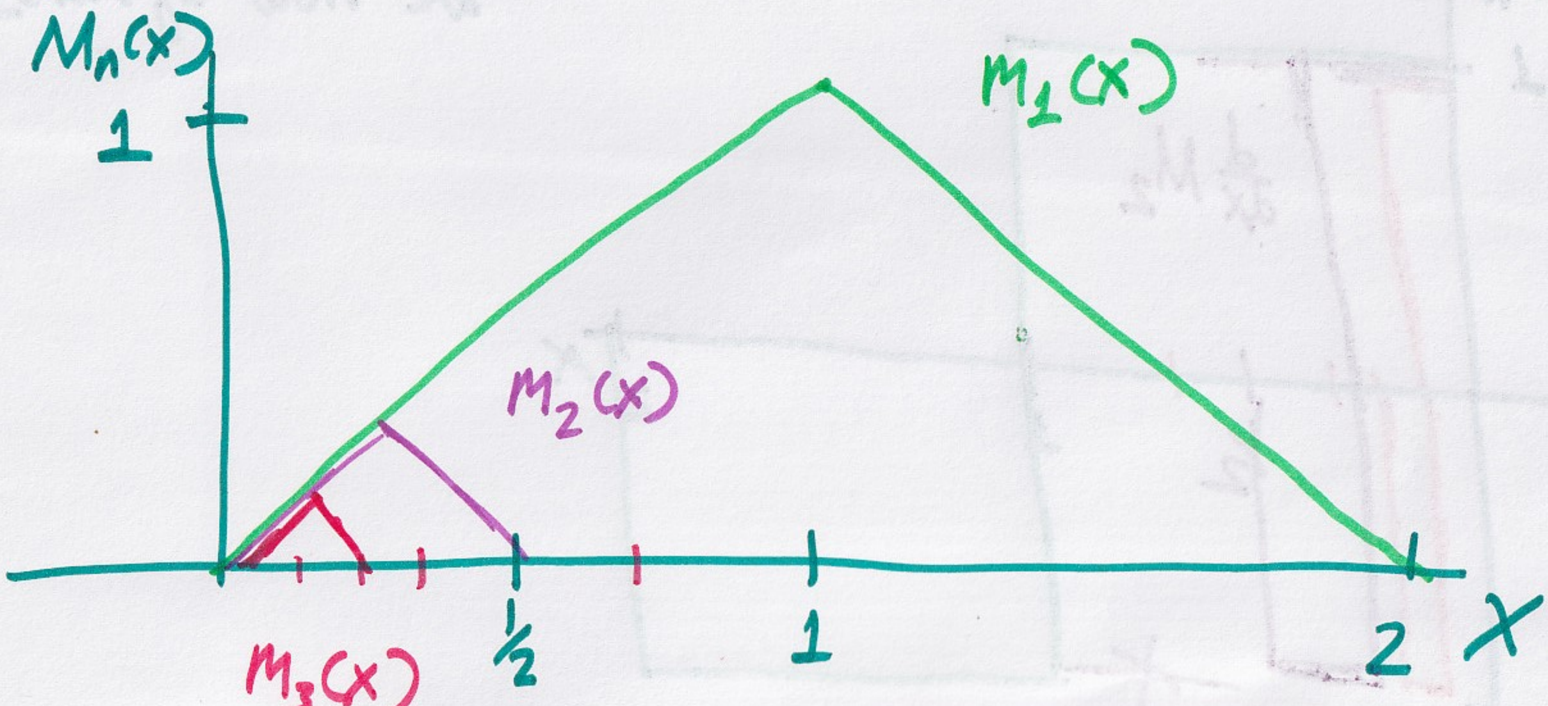
The function is periodic in the positive X axis



b) what is $M_4(x)$?

$$M_4(x) = \begin{cases} x & x \in [0, \frac{1}{16}] \\ \frac{2}{16} - x & x \in (\frac{1}{16}, \frac{2}{16}] \end{cases}$$

c) Graph $M_n(x)$ for various values of n. What's going on? Why?



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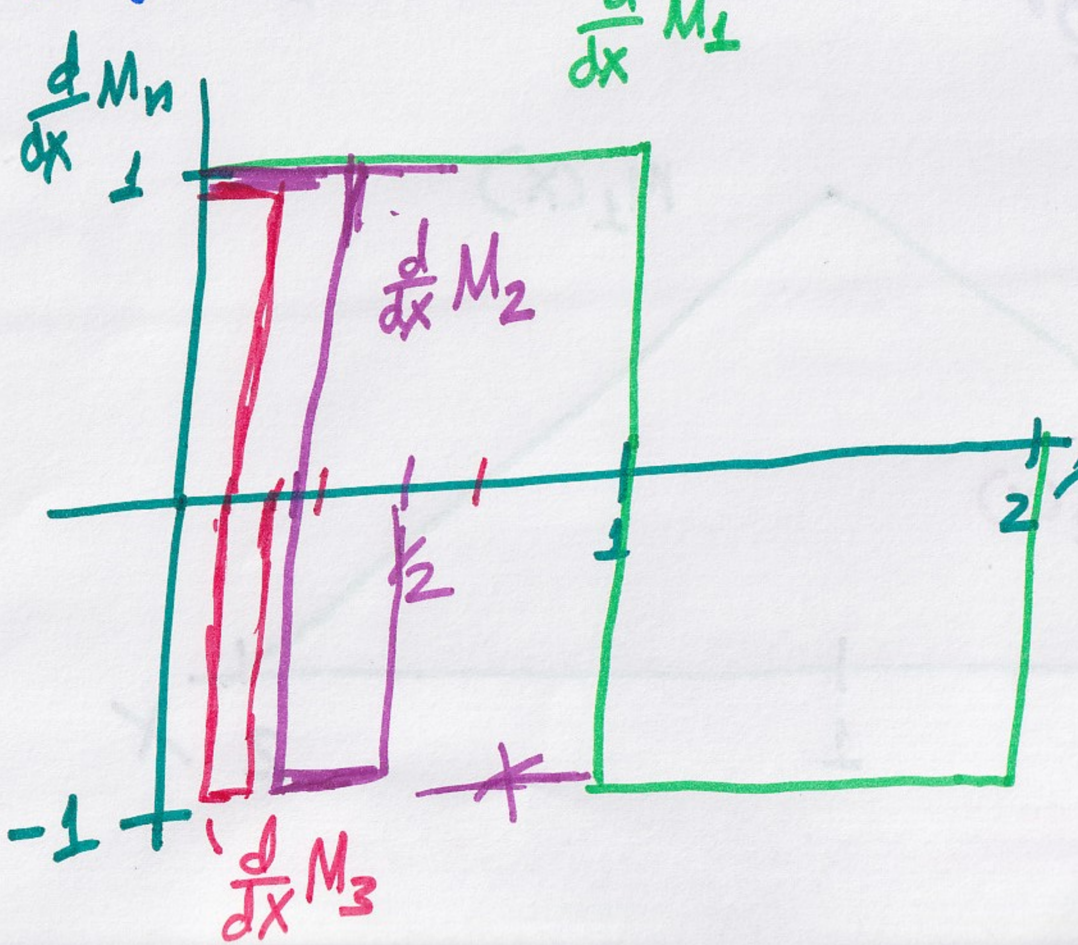
The graph is shrinking by a scale of n^2 because n^2 is in the denominator.

d) Graph $\frac{d}{dx} M_n$ for various values of n. What's going on?



The triangles are now squares.

8.) Consider the following recursively-defined sequence:

$$p_n = \frac{1}{2}\left(p_{n-1} + \frac{4}{p_{n-1}}\right)$$

where $n > 1$ and you ~~chose~~ choose $p_0 > 0$

a) what is the "end behavior" in this sequence?

Say $1 = p_0$

$p_n = (2.5, 2.05, 2.0006, \ldots$

$\cancel{p_n = (\frac{5}{2}, \frac{41}{20}}$ $\frac{41}{20}$

$p_1 = \frac{1}{2}\left(1 + \frac{4}{1}\right)$

$p_3 = \frac{1}{2}\left(\frac{13}{10} + 4\,\frac{10}{13}\right)$

$p_2 = \frac{1}{2}\left(\frac{5}{2} + 4\left(\frac{2}{5}\right)\right) = \cancel{\frac{13}{10}}$

$\cancel{\frac{58}{10}}$

$\cancel{\frac{8}{5}}$ $\cancel{\frac{13}{5}}$ $\frac{25}{10} + \frac{16}{10}$ $\frac{41}{10}$ $\frac{41}{20}$

22.)

for $P_0 = \frac{1}{2}$ $\quad P_n = (4.25, 2.59, 2.06, 2.0011 \ldots$

for $P_0 = 10$

$P_n = (5.2, 2.984, 2.162, 2.0061 \ldots$

for $P_0 = 10^6$ $\quad P_n = (5\times10^5, 2.5\times10^5, 1.25\times10^5, \ldots$

$\ldots, 15.3, 7.8, 4.16, 2.56, 2.06 \ldots)$

(eventually)

eventually we get to ~ 2

b) what happen when you choose a number other than 4? what happens? why?

$$P_n = \frac{1}{2}\left(P_{n-1} + \frac{\cancel{4}^{\,5}}{P_{n-1}}\right), \quad n \to \infty, \; P_n \to 2.236$$

$$2 + \frac{4}{2} = 4$$

$$P_n = \frac{1}{2}\left(P_{n-1} + \frac{3}{P_{n-1}}\right), \quad n \to \infty, \; P_n \to 1.7321$$

other than 4, P_n ~~goes~~ converses to another numbers. 4 or whatever numbers acts as a Locus of convergence.

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The coefficient ($\frac{1}{2}$) also is a Locus of control for convergence. In this case, it garuntees the P_n will not converge to 4 or whatever number we choose.

c) what happens when you change out the $\frac{1}{2}$ for something else? why?

for $\frac{1}{2} \to 1$ $\quad P_0 = 1$

$P_n = (5, 5.8, 6.49, 7.10, \ldots$

for $\frac{1}{2} \to \frac{4}{5}$ $\quad P_0 = 1/2$

$P_n = ($ ~~#~~ $6.8, 5.91, 5.27, 4.82, 4.5$

changing $\frac{1}{2}$ changes the convergence, but can make the sequence not converge "$1 \leq$"

2 The $\frac{1}{2}$ is a coefficient that controls a region of convergense