Meeting 14

Tangents and Secants

Last time: The Extreme Value Theorem

1.)

Proof of the boundedness Theorem

<u>Statement:</u> if $f(x)$ is continuous on $[a,b]$, Then it is bounded on $[a,b]$

suppose f is not bounded above on the Interval $[a,b]$. Then, for every $n \in \mathbb{N}$, $\exists\ x_n \in [a,b]$ s.t. $f(x_n) > n$. This defines a sequence (x_n), $n \in \mathbb{N}$. Because $[a,b]$ is bounded, The Bolzano-Weierstrass Theorem implies that $\exists$ a convergent subsequence (x_{n_k}), $k \in \mathbb{N}$, of (x_n). Denote ~~this~~ Limit by x. As $[a,b]$ is closed, it contains x.

x_{n_k} must be in $[a,b]$

Because f is continuous at x, we know that $f(x_{n_k})$ converges to the real number $f(x)$ (as f is sequentially continuous at x)

Dedekind-completness → Least uppr bound

<u>Idea</u>

1) By way of ~~contratiction~~ contradiction, assume unbounded

2) construct a sequence (x_n) s.t.

$f(x_n) > n \;\forall\, n \in \mathbb{N}$

3) BW says $\exists$ convergent sequence $x_{n_k} \to L$

so we expect $f(x_{n_k}) \to f(L)$

4) But $f(x_{n_k}) > n_k \geq k \;\forall k$

~~which~~ which gives a contradiction

EVT

1.) Since f is bounded, $\sup f = M$ exists

2.) construct $M - \frac{1}{n}$ $\forall n \in \mathbb{N}$

3.) Since f is continuous $\exists$ $d_n \in [a,b]$ Satisfying $M - \frac{1}{n} < f(d_n)$

4.) This defines a sequence (d_n) with property $M - \frac{1}{n} < f(d_n) \leq M$

$\forall n \in \mathbb{N}$

5.) (d_n) is closed and bounded so $\exists$ (d_{n_k}) that is convergent to $d \in [a,b]$

If all subsequences are monotonic and converges, ~~that~~ The sequence converges

6.) f being continuous implies

$(f(d_{n_k})) \to f(L)$

7.) since $(f(d_n))$ is eventually monotonic we know $(f(d_n)) \to M$

4

8) We conclude the two sequences must be the same: $f(L) = M$

#2)

a)

b) $\cancel{\lim_{h\to 0}}$ $\frac{d}{dx} x^2 = 2x$

$(x+h)(x+h)$

$x^2 + 2xh + h^2$

$$\lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$$

$$\lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$

$$\lim_{h\to 0} \frac{2xh + h^2}{h}$$

$$\lim_{h\to 0} 2x + h = 2x$$

c) $f(x) = x^{\tau}$

$f(x+h) = (x+h)^{\tau}$

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^{\tau} - x^{\tau}}{h}$$

$$\cancel{\lim_{h\to 0} \frac{\sum \binom{\tau}{k} x^{\tau} h^{\tau-k} - x^{\tau}}{h}}$$

$$= \lim_{h\to 0}$$

$$\lim_{h\to 0} \frac{\sum_{k=0}^{\tau} \binom{\tau}{k} x^{\tau-k} h^{k} - x^{\tau}}{h}$$

$$= \lim_{h\to 0} \frac{\sum_{k=1}^{\tau} \binom{\tau}{k} x^{\tau-k} h^{k} + x^{\tau} h^{0} - x^{\tau}}{h}$$

$$= \lim_{h\to 0} \sum_{k=1}^{\tau} \binom{\tau}{k} x^{\tau-k} h^{k-1}$$

$$= \binom{\tau}{1} x^{\tau-1} h^{1-1} = \tau x^{\tau-1}$$

(6)

$$\sum_{k=1}^{r} \binom{r}{k} x^{r-k} h^{k-1}$$

$$\binom{r}{1} x^{r-1} h^{0} + \binom{r}{2} x^{r-2} h^{1} + \binom{r}{3} x^{r-3} h^{2} + \dots + \binom{r}{r} x^{0} h^{r-1}$$

#3

function of x

vs

slope of a tangent Line

derivative is a moving tangent

Like $\frac{d}{dx} \sin x = \cos x$

$\sin(x+\Delta x)$

$= \sin(x)\cos(\Delta x) + \sin(\Delta x)\cos(x)$

$$\lim_{\Delta x \to 0} \frac{\sin(x)\cos(\Delta x) + \sin(\Delta x)\cos(x) - \sin(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sin(x)\cos(\Delta x) + \sin(\Delta x)\cos(x) - \sin(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \sin(x)\left[\frac{\cos(\Delta x)-1}{\Delta x}\right] + \cancel{\frac{\sin(\Delta x)\cos(x)}{\Delta x}}$$

Small angle approx

$= \cos(x)$

4

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a)$$

$$\lim_{h \to 0} \frac{f(u+h) - f(u)}{(u+h) - h} \leftarrow f(u) \text{ change of variables}$$

$h = x - u$

$0 = x - u$

$u = x$

but $u = a$

$x \to a$

is $u+h=x$ and $a = h$

$$\to \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

#5

Consider $f(x) = |x|$

f is continuous but we can show

$$\lim_{h \to 0^+} \frac{|x+h| - |x|}{h} = 1$$

$$\lim_{h \to 0^-} \frac{|x+h| - |x|}{h} = -1$$

So $\lim_{h \to 0} f(x)$ Does not Exist

#6

Continuous (at $x=a$) : $\lim_{x \to a} f(x) = \cancel{f(a)}\ f(a)$

Differentiable (at $x=a$) : $\lim_{x \to a} \frac{f(x)-f(a)}{x-a}$

If I can show

$$\lim_{x \to a} f(x) - f(a)$$

The $f(x)$ is continuous

Differentiability is stronger that continuity

Class C1 v C0

continuous function

$$\lim_{x \to a} f(x) = f(a)$$

$$\lim_{x \to a} f(x) - f(a) = 0$$

$$\lim_{x \to a} f(x) - f(a) = \lim_{x \to 0} \frac{f(x) - f(a)}{x - a} \frac{x - a}{1}$$

$$= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \lim_{x \to a} x - a$$

$$= f'(a) \cdot 0$$

$$= 0$$

So differentiable and continuous