MATH 4200

MATHEMATICAL ANALYSIS I

These are my notes of Prof. Gregor Kovačič's Notes. (And class)

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INTEGERS ($\mathbb{Z}$)

$$\underbrace{\ldots -3, -2, -1}_{\text{Negative}}, 0, \underbrace{1, 2, 3, \ldots}_{\text{Positive}}$$

$$\underbrace{0, 1, 2, 3, \ldots}_{\text{Non-Negative}}$$

There are Infinitely Many Integers;

A Countable Infinity

Arithmetic Roles (Axioms)

1.) Commutativity: $n+m=m+n$

2.) Associativity: $[n+m]+p=n+[m+p]$

3.) $0+m=m+0=m$ (Zero)

4.) Negative: $m+(-m)=(-m)+m=0$

$-m$ = Negative m

Subtraction: $n+(-m)\equiv n-m$

1.) – 4.) Make the Integers a Commutative Group.

SHortHand: Multiplication: $mn=\underbrace{n+n+\ldots n}_{m}$

Induction Principle

Theorem: If some Property (E.g., a Formula) holds for $n=1$, And we can show That it holds for $n+1$ whenever it holds for n, Then it holds for all Positive Integers

Proof. It holds for $n=1$, Therefore For $1+1=2$, $2+1=3$, 4, 5, ETC.

Examples: 1.) $1+2+\ldots+n=?$

$1+2+\ldots+n=S$ Let's add these

$\cancel{n+n-1}$

$n+n-1+\ldots+1=S$

Looking at the second S,

$$n + n-1 + n-2 \cdots \underbrace{n-(n-1)}_{\text{This } = 1} = S$$

re-align and add

$$\begin{array}{r} 1 + 2 + 3 + \cdots \quad n-2 + n-1 + n \\ n + n-1 + n-2 + n-3 + \cdots n-(n-2) + n-(n-1) \\ + \quad \\ \hline n^2 + n \end{array}$$

we have n n's plus one more n

So we have $2S = n(n+1)$

$$S = \frac{n(n+1)}{2}$$

Prove this by Induction:

$n=1$: $1 = \frac{1\cdot(1+1)}{2} = 1$ ✓

$n \to n+1$: Let $1+2+\cdots+n = \frac{n(n+1)}{2}$

What is $1+2+\cdots+n+(n+1)$?

we can write

$1+2+\cdots+n+(n+1) = (1+2+\cdots+n)+(n+1)$

$= \frac{n(n+1)}{2} + (n+1) = (n+1)(\frac{n}{2}+1) = \frac{(n+1)(n+2)}{2}$

By Hypothesis

$= \frac{(n+1)((n+1)+1)}{2}$

So $1+2+\cdots n = \frac{n(n+1)}{2}$ $n \to n+1 \xrightarrow{\text{So}} \frac{(n+1)(n+1+1)}{2}$

2.) Show $1^2 + 2^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

$\underline{n=1}$: $1^2 = 1 = \dfrac{1 \cdot 2 \cdot 3}{6}$ ✓

$\underline{n \longrightarrow n+1}$: $1^2 + 2^2 + \cdots + n^2 + (n+1)^2 =$

$$= \frac{n(n+1)(2n+1)}{6} + (n+1)^2$$

$$= (n+1)\left[\frac{n(2n+1)}{6} + (n+1)\right]$$

$$= (n+1)\,\frac{n(2n+1) + 6(n+1)}{6}$$

$$= \frac{n+1}{6}\left[2n^2 + 7n + 6\right]$$

Note, $(n+2)(2(n+1)+1) = (n+2)(2n+3)$
$= 2n^2 + 3n + 4n + 6 = 2n^2 + 7n + 6$

So we can write,

$$\frac{(n+1)((n+1)+1)(2(n+1)+1)}{6}$$

Binomial Formula:

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots$$

$$\cdots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$$

$$= \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

(a, b - any Real or Even Complex Numbers, To be Discussed a bit Later)

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} = \frac{n!}{k!\,(n-k)!}$$

→ This is the Binomial Coefficient "n choose k" for $K = 0, 1, 2, \ldots, n$

with $K! = 1 \cdot 2 \cdot \ldots \cdot K$, $0! = 1$

for $\underline{n = 1}$: $a+b = \binom{1}{0}a + \binom{1}{1}b$

$$= \frac{1!}{0!\,1!}a + \frac{1!}{1!\,0!}b = a+b \quad \checkmark$$

So what about

$\underline{n \to n+1}$: $(a+b)^{n+1}$

$$(a+b)^{n+1} = (a+b)(a+b)^n =$$

$$= (a+b)\left[\binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \cdots + \binom{n}{n}b^n\right]$$

$$= \binom{n}{0}a^{n+1} + \left[\binom{n}{0} + \binom{n}{1}\right]a^n b + \left[\binom{n}{1} + \binom{n}{2}\right]a^{n-1}b^2$$

$$+ \cdots + \left[\binom{n}{n-1} + \binom{n}{n}\right]ab^n + \binom{n}{n}b^{n+1}$$

Therefore, we must show

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

We deduce this from inside the brackets "[...]"

$$\binom{n}{k} + \binom{n}{k+1} = \frac{n!}{k!\,(n-k)!} + \frac{n!}{(k+1)!\,(n-(k+1))!}$$

$$= \frac{n!}{k!(n-k-1)!}\left[\frac{1}{n-k} + \frac{1}{k+1}\right]$$

$$\frac{1}{(n-k)!} = \frac{1}{(n-k-1)!} * \frac{1}{n-k}$$

$$= \frac{n!}{k!(n-k-1)!}\,\frac{k+1+n-k}{(n-k)(k+1)}$$

$$= \frac{n!\,(n+1)}{(k+1)!\,(n-k)!} = \frac{(n+1)!}{(k+1)!\,((n+1)-(k+1))!}$$

$$= \binom{n+1}{k+1}$$

Note $\binom{n}{0} = \binom{n+1}{0} = \binom{n}{n} = \binom{n+1}{n+1} = 1$

So we have

$$(a+b)^{n+1} = \binom{n+1}{0} a^{n+1} + \binom{n+1}{1} a^n b + \cdots$$

$$\cdots + \binom{n+1}{n} a b^n + \binom{n+1}{n+1} b^{n+1}$$

$\Rightarrow$ Pascal's Triangle

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

n rows

$\downarrow n$

$\xrightarrow{\quad} k$

The entry in the nth row and the Kth column of Pascal's triangle is denoted $\binom{n}{k}$

We can see that it clearly follows from

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}, \quad \binom{n}{0} = \binom{n+1}{0} = \binom{n}{n} = \binom{n+1}{n+1} = 1$$

Rational Numbers ($\mathbb{Q}$)

Fractions: $\frac{p}{q}$, $q \neq 0$

$\frac{m}{n} = \frac{p}{q}$ Precisely when $np = mq$

Representation $\frac{p}{q}$ is made unique by cancelling common Factors

Rules of Arithmetic (Axioms)

I. Addition (Like Integers)

1.) $\alpha + \beta = \beta + \alpha$ (commutativity)

2.) $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$ (Associativity)

3.) $\alpha + 0 = 0 + \alpha = \alpha$ (Zero)

4.) $\alpha + (-\alpha) = (-\alpha) + \alpha = 0$ (Negative)

(Subtraction: $\alpha - \beta \equiv \alpha + (-\beta)$)

II Multiplication (Exclude 0)

1.) $\alpha\beta = \beta\alpha$

2.) $\alpha(\beta\gamma) = (\alpha\beta)\gamma$

3.) $\alpha \cdot 1 = 1 \cdot \alpha = \alpha$ (Identity)

4.) $\alpha\left(\frac{1}{\alpha}\right) = \left(\frac{1}{\alpha}\right)\alpha = 1$ (Inverse, Reciprocal)

Division: $\frac{\alpha}{\beta} = \alpha \cdot \left(\frac{1}{\beta}\right)$

$0 \cdot \alpha = 0 \Rightarrow 0$ Has no Inverse

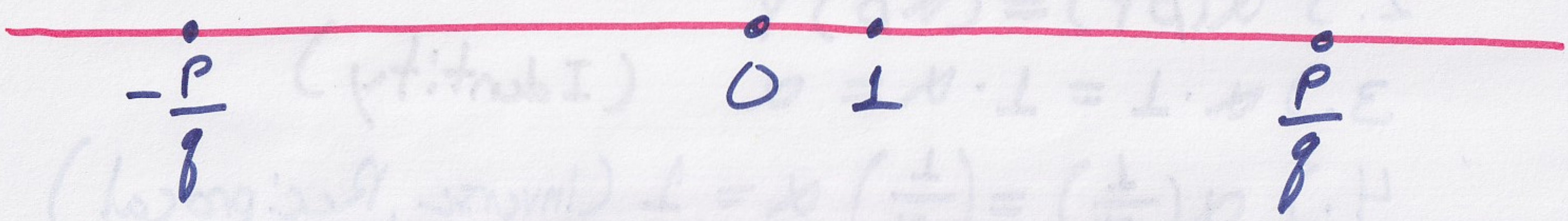
III. Distributive Law

1.) $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$

Integers can be thought of as rationals of the form

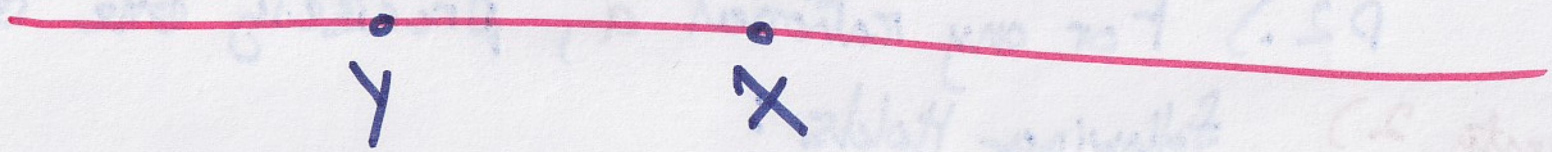
$$n = \frac{n}{1}$$

We can represent rationals as points on a straight Line:



This induces Ordering Among Rationals:

$x > y$ (Greater Than), if the point representing x is to the right of the point representing y.



x is to the right of y

How to Introduce order Logically?

We Observe that Positive Rationals ($\frac{p}{q} > 0$) Satisfy the following:

IV Positive Numbers

P1.) if $a>0$ and ~~$b>0$~~ $b>0$, Then

(Property 1) $a+b>0$ and $ab>0$

P2.) For any rational a, precisely one of the

(Property 2) following Holds:

$a>0$, $\alpha=0$, or $-a>0$.

Now, we define: $a>b$ If $a-b>0$.

Simple Consequences

1.) Addition of Inequalities:

IF $a > b$ and ~~c>d~~ $c > d$ then $a+c > b+d$

proof. $a+c-(b+d) = (a-b)+(c-d) > 0$ (by P1.)

P1.) $a>0$, $b>0 \rightarrow a+b>0$, $ab>0$

we are saying "$a=(a-b)$" and "$b=(c-d)$"

so by P1 $\boxed{(a-b)+(c-d)>0}$

"$a>b \rightarrow a-b>0$"

2.) Multiplication by a Positive Number:

If $a>b$ and $c>0$, Then $ac>bc$

Proof. $ac-bc = (a-b)c > 0$ (by P1.)

from P1, we are saying "$a = ac$", "$b=-bc$"

basically we are saying the inequality $ac>bc$ is true if P1 is true. P1 is true because of the number line

Definition $x < y$ (Smaller, Less than) if $y > x$

we basically showed that, $x < y$ is true (we are saying the concept of 'Less than' exists) because 'greater than' is true for the reverse.

3.) Multiplication by a Negative Number:

If $a > b$ and $c < 0$, Then $ac < bc$

proof $(-c) > 0$ (by P2), ~~So~~ $(-c)(a-b) = -ca + cb > 0$

~~So~~ $bc > ac$ P2) '$-a > 0$'

4.) IF $a > b > 0$ and $c > d > 0$, Then $ac > bd > 0$

5.) INEquality is Transitive:

if $a > b$ and $b > c$, Then $a > c$

proof $a - c = (a-b) + (b-c) > 0$

Basically, we are saying P1 applies if we say
$a > b \rightarrow a - b > 0$, ~~they~~ They are both positive numbers on the number line

DEF $\geq$ Means $>$ OR $=$

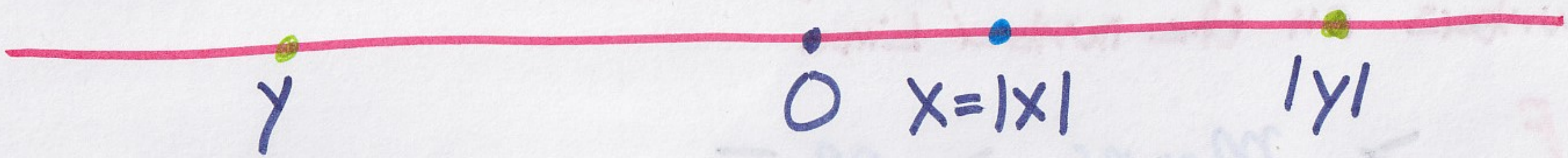
$\leq$ Means $<$ OR $=$

E.G. $a \geq b$ IF $a - b \geq 0$

All the above consequences hold for $\geq$, too.

We can replace $\geq \longrightarrow >$, vice versa
for all of the above consequences

Absolute Value



Def:

$$|0|=0, \quad |x| = \begin{cases} x, & x>0 \\ -x, & x<0 \end{cases}$$

This is the (Non-Negative) Distance from x to the Origin, 0

Proposition Let $a, b \geq 0$, and $a > b$. Then $a^2 > b^2$, and, In general, $a^n > b^n$ for any positive Integer n

Proof ~~If a>b~~

$$a^2 - b^2 = (a+b)(a-b) > 0$$

And $a^n - b^n = (a^{n-1} + a^{n-2}b + \dots + ab^{n-2} + b^{n-1})(a-b) > 0$

because $a^k, b^l, a^i b^j \underset{(=)}{>} 0$ by P1

Proposition If $a, b > 0$ and $a^2 > b^2$, Then $a > b$

Proof. $a - b = \frac{1}{a+b}(a^2 - b^2)$

We see that $(a^2 - b^2) = (a+b)(a-b)$

so $\frac{1}{a+b}(a^2-b^2) = \frac{(a+b)}{a+b}(a-b) = a-b$

If $\frac{1}{a+b} \leq 0$ It would contradict P1.

because P1 says a and b are greater than zero

(Likewise: $a^n > b^n \Rightarrow a > b$)

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How Many (positive) Rationals are There?

As Many as (Positive) Integers.

$$\begin{array}{cccccc}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\[1em]
\frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \frac{2}{4} & \frac{2}{5} & \cdots \\[1em]
\frac{3}{1} & \frac{3}{2} & \frac{3}{3} & \frac{3}{4} & \frac{3}{5} & \cdots \\[1em]
\frac{4}{1} & \frac{4}{2} & & & &
\end{array}$$

The numerators go up or down and the denominators too

We have a (countable) Infinity of rational numbers

But we have "Gaps" Amoung Rational numbers

Proposition There is no Rational l s.t. $l^2 = 2$

(i.e., $\sqrt{2}$ is not rational)

Proof. Let $l^2 = \frac{p^2}{q^2} = 2 \Longrightarrow p^2 = 2q^2$

We can assume that p and q have no common factors.
We see that p^2 is even $\Longrightarrow$ p is even, $p = 2p'$.
$\Rightarrow 4p'^2 = 2q^2 \Longrightarrow q^2 = 2p'^2$.

Therefore q^2 is even

b/c p^2 is even
$\rightarrow p$ is even
$p = 2p' \rightarrow \frac{p}{2} = p'$
$p'^2 = \frac{p^2}{4}$

$\Rightarrow q$ is even

$\Rightarrow p$ and q have a common factor, 2, which is a contradiction

by definition $\frac{p}{q}$ is a rational ~~number~~ number
not $\frac{p}{2p}$, so ~~we~~ we have a contradiction

So what are we going to do about this?

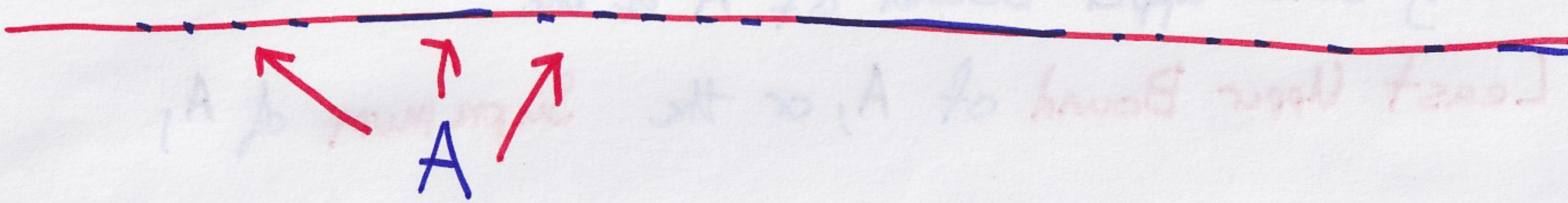
⇒ We fill all such "Gaps" on the Number Axis: REAL NUMBERS ($\mathbb{R}$)

So we need to Demand that they satisfy the Arithmetic Rules I, II, III, and the Ordering Rules IV

And, in addition, they must satisfy a "completeness" Property

Before we explain this, we need to go on a

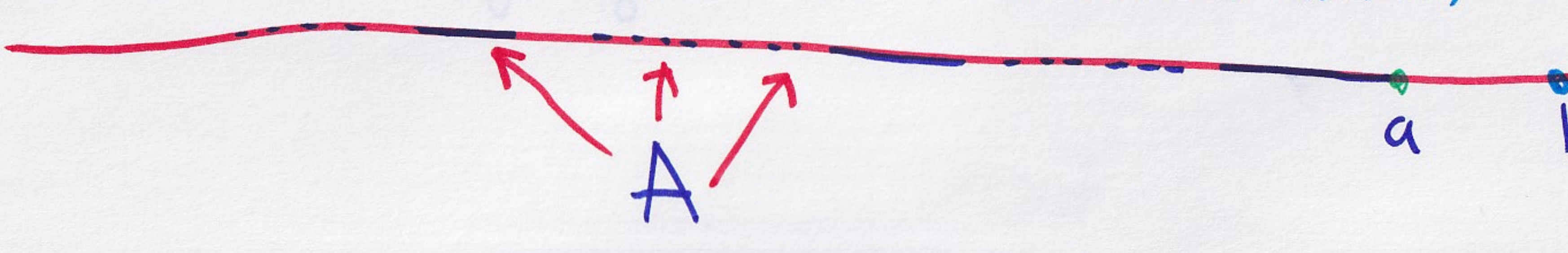
Digression: Let A be a set (collection) of rational (or Real) numbers.



x Belonging to A is denoted as $x \in A$.

(x is an Element of A)

Definition: If every $x \in A$ satisfies $x < b$, then b is an Upper Bound of A. (And A is Bounded above.)



Clearly, if $c > b$, Then c is also an upper bound of A.

Definition The Upper bound of A, Say a, s.t. $a \leq b$ for any other upper bound of A is the **Least Upper Bound** of A, or the **Supremum** of A,

$$a = \sup A$$

V Least Upper Bound Property:

Any set of Real numbers, A, which is Bounded Above, has the Least Upper Bound.

(IT is clearly Unique)

Consequences:

1.) $\sqrt{a}$ Exists for all $a \geq 0$.

Proof (Sketch) $\sqrt{a} = \sup$ of the set A of all Rationals $\frac{p}{q}$ s.t. $\frac{p^2}{q^2} \leq a$ $\quad a = \sup A^2$

Analogons Properties: B - Bounded Below, Lower Bound, Greatest Lower Bound = Infimum

Clearly: $\sup(-A) = -\inf A$

2.) Archimedean Property: For any $x \in \mathbb{R}$ There is an Integer, $n \in \mathbb{Z}$, s.t. $n > x$

Proof Let this be wrong for some $x \in \mathbb{R}$. Then $n \leq x$ for all $n \in \mathbb{Z}$ $\Rightarrow$ $\mathbb{Z}$ is bounded above

Well if $n \leq x$ for all $n \in \mathbb{Z}$ and $x \in \mathbb{R}$,
we have to have a supremum

$\Rightarrow \exists$ $a = \sup \mathbb{Z}$. For $n \in \mathbb{Z}$ also
$n+1 \in \mathbb{Z}$, so $n+1 \leq a$. But then $n \leq a-1$
$\Rightarrow$ $a-1$ is an Upper Bound of $\mathbb{Z}$ $\Rightarrow$ $a \neq \sup \mathbb{Z}$
$\Rightarrow$ Contradiction

We know $n \in \mathbb{Z}$ and $n+1 \in \mathbb{Z}$ but x is $n \leq x$
So we know $\exists$ $a = \sup \mathbb{Z}$, we are saying there $\exists$ a
~~sup~~ sup of $\mathbb{Z}$ but as $n+1 \in \mathbb{Z}$, there isn't
a sup, so we have a contradiction

~~4.) For any $x \in \mathbb{R}$, $\exists$ $n \in \mathbb{Z}$ s.t. $n \leq x < n+1$~~

3.) For any $\epsilon > 0$, $\exists$ an integer $n \in \mathbb{Z}$, $n > 0$,
s.t. $0 < \frac{1}{n} < \epsilon$.

Proof by 2.), $\exists\, n \in \mathbb{Z}$ $(n>0)$ s.t. $n > \frac{1}{\varepsilon}$.
multiply by ε and $\frac{1}{n}$. Both are >0, so the inequality is preserved $\Rightarrow \varepsilon > \frac{1}{n} > 0$.

4.) For any $x \in \mathbb{R}$, There is $n \in \mathbb{Z}$ s.t.
$n \leq x < n+1$

Proof choose $N \in \mathbb{Z}$, $N > |x|$, i.e, $-N < x < N$

$\Rightarrow$ There are finitely Many Integers
$-N, -N+1, \ldots, -1, 0, 1, \ldots, N-1, N$

We are in a range $(-N<x<N)$ So it's finite

Pick n to be the greatest of these that is $\leq x$

5.) For any $x \in \mathbb{R}$ and $N \in \mathbb{Z}$, $N>0$, There exists an $n \in \mathbb{Z}$ s.t.

$$\frac{n}{N} \leq x < \frac{n+1}{N}$$

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Proof Apply 4.) to Nx, so that $n \le Nx < n+1$

$$\frac{n}{N} \le x < \frac{n+1}{N}$$

$$n \le xN < n+1$$

6.) **Approximation by Rational Numbers:**

If $x, \varepsilon \in \mathbb{R}$, $\varepsilon > 0$, There Exists a Rational $r \in \mathbb{Q}$ s.t. $|x-r| < \varepsilon$

Proof by 3.), we find $N \in \mathbb{Z}$, $N > 0$ s.t. $\frac{1}{N} < \varepsilon$.

by 5.), Find an Integer n s.t. $\frac{n}{N} \le x < \frac{n+1}{N}$.

Then $0 \le x - \frac{n}{N} < \frac{n+1}{N} - \frac{n}{N} = \frac{1}{N} < \varepsilon$,

so $|x - \frac{n}{N}| < \varepsilon$.

3.) $\varepsilon > 0$, $\exists\ n \in \mathbb{Z}$, $n > 0$, s.t. $0 < \frac{1}{n} < \varepsilon$

5.) $x \in \mathbb{R}$ and $N \in \mathbb{Z}$, $N > 0$, $\exists$ an $n \in \mathbb{Z}$ s.t. $\frac{n}{N} \le x < \frac{n+1}{N}$

Decimal Representation of Real Numbers

Let $a_1, a_2, \ldots, a_n$ be any integers chosen from among $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$

$a_0 . a_1 a_2 \cdots a_n$ Denotes the rational number

$$r = a_0 + \frac{a_1}{10} + \frac{a_2}{100} + \cdots + \frac{a_n}{10^n} \quad (a_0 \in \mathbb{Z})$$

If $m, n \in \mathbb{Z}$, $n > m$, then

$$a_0 . a_1 \cdots a_m \leq a_0 . a_1 \cdots a_n$$

$$a_0 . a_1 \cdots a_m + \frac{a_{m+1}}{10^{m+1}} + \cdots + \frac{a_n}{10} \leq \underbrace{a_0 . a_1 \cdots a_m + \frac{9}{10^{m+1}} + \cdots + \frac{9}{10^n} + \frac{1}{10^n}}_{a_0 . a_1 \cdots a_m + \frac{1}{10^m}}$$

Infinite Decimal

$$a_0.a_1a_2a_3\cdots = \sup\{a_0.a_1a_2a_3\cdots a_n \mid n \in \mathbb{Z}, n>0\}$$

For any $n \in \mathbb{Z}$, $n>0$

$$a_0.a_1\cdots a_n \leq a_0.a_1a_2a_3\cdots \leq a_0.a_1a_2\cdots a_n + \frac{1}{10^n}$$

$\Rightarrow$ Sup Exists

Non-uniqueness $\quad 1.000\cdots = 0.9999\cdots$

I suppose $1.000\cdots$ is the sup of something that is ~~equiv~~ equivalent of $0.9999\cdots$

$$1.000\cdots = \sup\{\quad\}$$

Proposition: Real Numbers are Uncountably Infinite

proof Suppose they are countable, so enumerate them in some fashion.
It is enough to enumerate just those with $0 \leq x \leq 1$
(because if you can't count them, you can count all the Reals even Less.)

$$0.a_1^{(1)}\,a_2^{(1)}\,a_3^{(1)}\cdots$$
$$0.a_1^{(2)}\,a_2^{(2)}\,a_3^{(2)}\cdots$$
$$0.a_1^{(3)}\,a_2^{(3)}\,a_3^{(3)}\cdots$$

Focus on $a_j^{(j)}$. Construct a new number b:
If $a_j^{(j)} = 5$, Let $b_j = 6$. If $a_j^{(j)} \neq 5$, Let $b_j = 5 \Rightarrow b \neq a^{(j)}$ for any (j),
so there is a real number, $0 \leq b \leq 1$
That's different from any $a^{(j)}$. But, by assumption,
$a^{(j)}$'s represent all Real numbers between 0 and 1
$\Rightarrow$ contradiction $\Rightarrow$ You cannot count Reals

$a^{(j)}$ represent all the Real numbers between 0 and 1
but as b is a new number, we showed it also represents all the real numbers between 0 and 1. ($\leq$, $\geq$ are implied) which produces a certain contradiction that shows we cannot count the real numbers.

The Explicit construction of Reals will come Later

Powers: Let $a \in \mathbb{R}$. Then $a^n = \underbrace{a \cdot a \cdots a}_{n}$

Rules: 1.) $a^{n+m} = a^n a^m$ 2.) $(a^n)^m = a^{nm}$
3.) $a^{2n} > 0$, $a \neq 0$

Proof of 3.): if $a>0$, clear, we got it. If $a<0$ Then

$$a \cdot a > a \cdot 0 \Rightarrow a^2 > 0 \Rightarrow a^{2n} = (a^2)^n > 0$$

N-th RooT of a: $\sqrt[n]{a} = x$ s.t. $a = x^n$

Proposition: if $a \geq 0$, $\sqrt[n]{a}$ Exists

Proof $\sqrt[n]{a} = \sup\{\frac{p}{q} \mid (\frac{p}{q})^n \leq a\}$

This is showing a ~~to~~ be bounded and a lot more.

Rational Powers: $a^{\frac{p}{q}} = \sqrt[q]{a^p}$

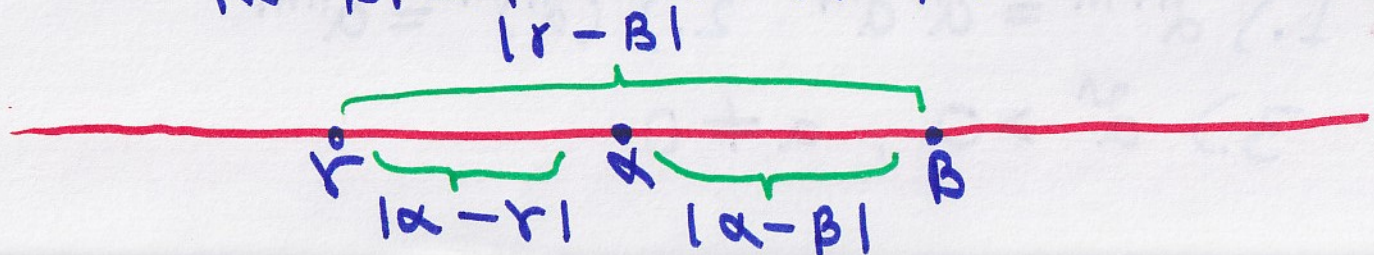
Rules 1.) and 2.) Above hold for rational Powers

Important Inequalities

Triangle Inequality: $|a+b| \leq |a| + |b|$

Equivalent: if $a = \alpha - \gamma$, $b = \gamma - \beta$ Then

$$|\alpha - \beta| \leq |\alpha - \gamma| + |\gamma - \beta|$$



Proof Distinguish $a+b \geq 0$ and $a+b < 0$

if $a+b \geq 0$ Then $|a+b| = a+b \leq |a| + |b|$

(because $a \leq |a|$, $b \leq |b|$)

if $a+b<0$ Then $|a+b| = -(a+b) = -a + (-b) \leq |a| + |b|$

This is how the inequality work (and absolute value)

Corollary $|a+b+c| \leq |a|+|b|+|c|$

because $|a+b+c| \leq |a+b| + |c| \leq |a| + |b| + |c|$

Likewise, $|a_1 + \cdots + a_n| \leq |a_1| + \cdots + |a_n|$

Estimate $|a+b|$ **From Below**:

$$|a| = |(a+b) + (-b)| \leq \underbrace{|a+b| + |-b|}_{|a+b| + |b|}$$

$\Rightarrow |a+b| \geq |a| - |b|$

If you exchange a and b, IT makes no Difference

$\Rightarrow |a+b| \geq |b| - |a| \Rightarrow |a+b| \geq \big||a| - |b|\big|$

If you take $-b$ instead of b:

$$\boxed{|a-b| \geq \big||a|-|b|\big|}$$

from $|a+b| \geq \big||a|-|b|\big|$

This is supposed to be the largest side

Cauchy-Schwartz Inequality

Let $a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{R}$ Then

$$(a_1b_1 + \dots + a_nb_n)^2 \leq (a_1^2 + \dots + a_n^2)(b_1^2 + \dots + b_n^2)$$

$$\left(\sum_{i=1}^{n} a_ib_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$$

$\bar{a}\cdot\bar{b} \leq |\bar{a}|\,|\bar{b}|$

Proof Let $A = a_1^2 + \dots + a_n^2$, $B = a_1b_1 + \dots + a_nb_n$

$C = b_1^2 + \dots + b_n^2$

The Inequality is $B^2 \leq AC$. To Prove it, observe that for any $t \in \mathbb{R}$,

$$0 \leq \underbrace{(a_1 + tb_1)^2 + \dots + (a_n + tb_n)^2}$$

$$a_1^2 + 2ta_1b_1 + t^2b_1^2 + \cdots + a_n^2 + 2ta_nb_n + t^2b_n^2$$
$$= a_1^2 + \cdots + a_n^2 + 2t(a_1b_1 + \cdots + a_nb_n) + t^2(b_1^2 + \cdots + b_n^2)$$
$$= A + 2Bt + Ct^2$$

You can assume $C > 0$ (If not, $B = 0$, so $0 \leq 0$.)

$\Rightarrow$ The discriminant of the Quadratic expression for t must be ≤ 0

$$\Rightarrow B^2 - AC \leq 0 \Rightarrow AC \geq B^2$$

B, C, A ~~have~~ are composed of related variables so one grows/decrease with the others, so $4B^2 - 4AC$ must be negative or the initial statement is false

Alternative proof: $0 \le \underbrace{A + 2Bt + Ct^2}$

$\cancel{C\left[\left(t+\frac{B}{C}\right)^2 + \left(\frac{A}{C} - \frac{B^2}{C^2}\right)\right]}$

originally the equation was $C\left[\left(t+\frac{B}{C}\right)^2 + \left(A - \frac{B^2}{C}\right)\right]$

The minimum of this expression is attained

by $t = -\frac{B}{C} \Rightarrow 0 \le AC - B^2$

$$A + 2Bt + Ct^2 \Big|_{t=-\frac{B}{C}} = A - \frac{B^2}{C}$$

you can see the inequality from this

Ex.) Let $a_1 = \sqrt{x}$, $b_1 = \sqrt{y}$, $a_2 = \sqrt{y}$, $b_2 = \sqrt{x}$

$\Rightarrow (2\sqrt{xy})^2 \le (x+y)^2$

$$\sqrt{xy} \le \frac{x+y}{2}$$

$\cancel{a_1 b_1 \le (}$

$a_1 b_1 + a_2 b_2 \le (a_1^2 + a_2^2)(b_1^2 + b_2^2)$

$\sqrt{x}\sqrt{y} + \sqrt{x}\sqrt{y} \le (x+y)(y+x)$

$2\sqrt{x}\sqrt{y} \le (x+y)^2$

$\Rightarrow$ Geometric mean is Less than Arithmetic mean