

Fourier transform

①

$x(t)$ periodic with period T

Let's say the signals we use today satisfy the Dirichlet conditions

and is integrable

$$so \quad x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j k \omega_0 t} \quad \omega_0 = \frac{2\pi}{T}$$

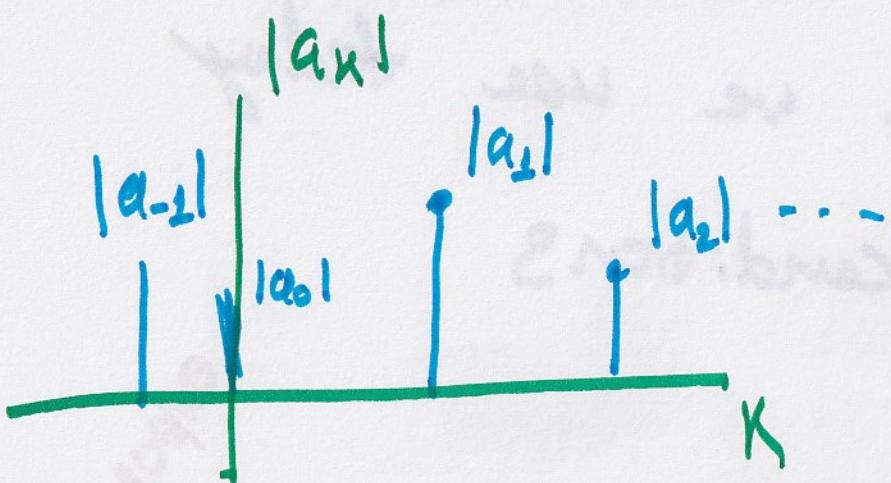
$$a_k = \frac{1}{T} \int_T x(t) e^{-j k \omega_0 t} dt$$

Exponential Fourier Series

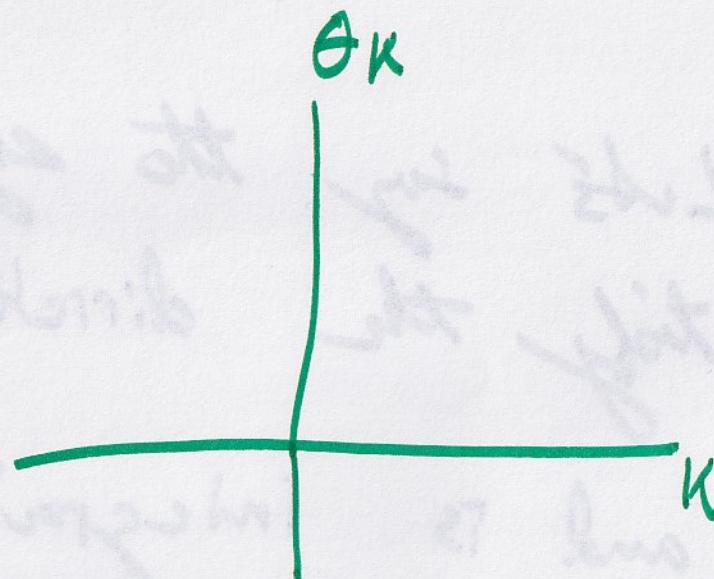
(2)

Fourier Spectrum

$$a_k = |a_k| e^{j\theta_k}$$



Amplitude Spectrum



Phase Spectrum

$a = 1+j \rightarrow |a| = \sqrt{1^2+1^2} = \sqrt{2} \rightarrow \cancel{\theta = 45^\circ}$

$$\Theta = \arctan\left(\frac{1}{1}\right) = \frac{\pi}{4}$$

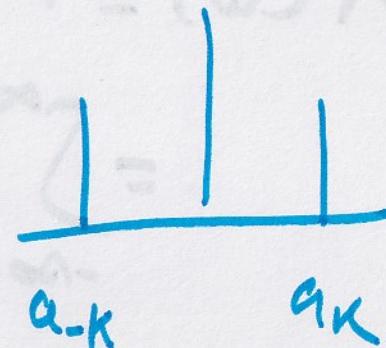
$$a = -5 \rightarrow |a| = 5$$

$$\Theta = \arctan\left(\frac{0}{-5}\right) = \pi$$

(3)

$$X(f) \text{ real} \Rightarrow X(t) = X^*(t)$$

$$a_k = a_{-k}^*$$



recall

$$X(t) = a_0 + \sum_{k=1}^{\infty} 2 \operatorname{Re} \left\{ a_k e^{jk\omega t} \right\}$$

$$= a_0 + \sum_{k=1}^{\infty} 2 \operatorname{Re} \left\{ |a_k| e^{j\theta_k} \dots e^{jk\omega t} \right\}$$

$$= a_0 + \sum_{k=1}^{\infty} 2 |a_k| \cos(k\omega t + \theta_k)$$

Trigonometric
Fourier series

(4)

What if $X(f)$ is aperiodic?

$$X(\omega) = \text{FT}\{x(t)\}$$

$$= \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

(5)

$$\text{FT of } \delta(t) \longrightarrow 1$$

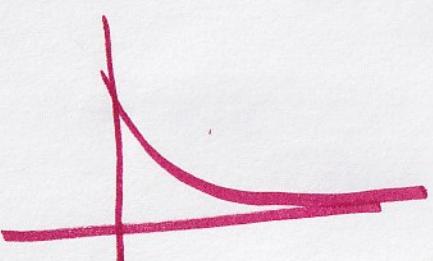
why tho? I will get to that later

$$\text{Ex) FT } e^{-at} u(t) \quad \text{for } \operatorname{Re}\{a\} > 0$$

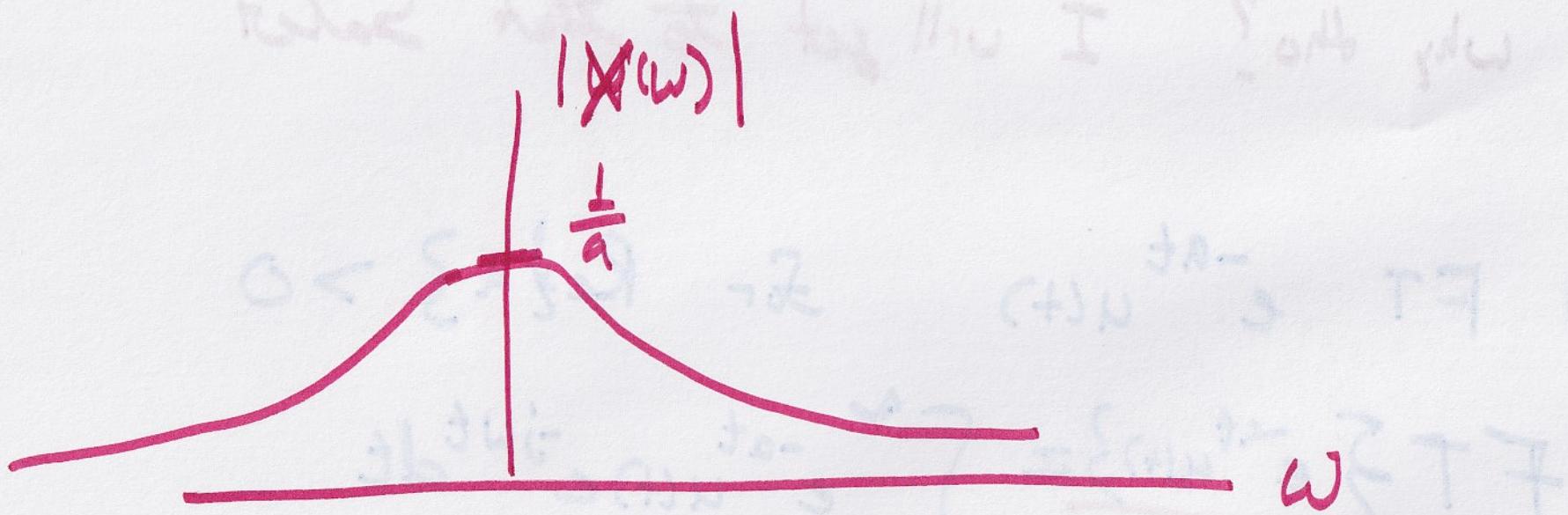
$$\text{FT} \{ e^{-at} u(t) \} = \int_{-\infty}^{\infty} e^{-at} u(t) e^{-j\omega t} dt$$

$$= \int_0^{\infty} e^{-(a+j\omega)t} dt = \frac{-1}{a+j\omega} e^{-(a+j\omega)t} \Big|_0^{\infty}$$

$$= \frac{1}{a+j\omega}$$



$$⑥ |X(\omega)| = \left| \frac{1}{a+j\omega} \right| = \frac{1}{|a+j\omega|} = \frac{1}{\sqrt{a^2+\omega^2}}$$



Amplitude Spectrum!

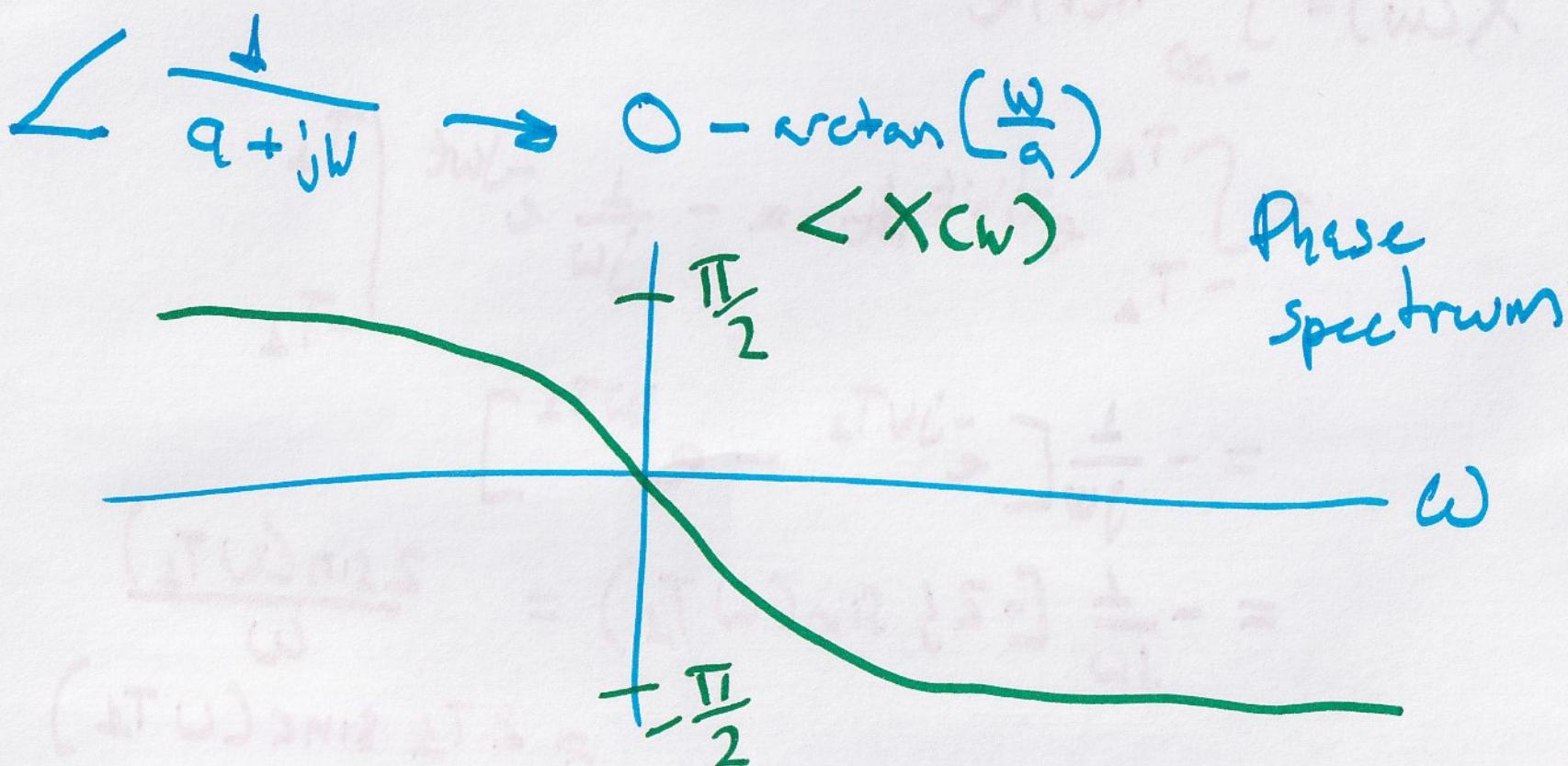
(1st order Low Pass Filter)

(7)

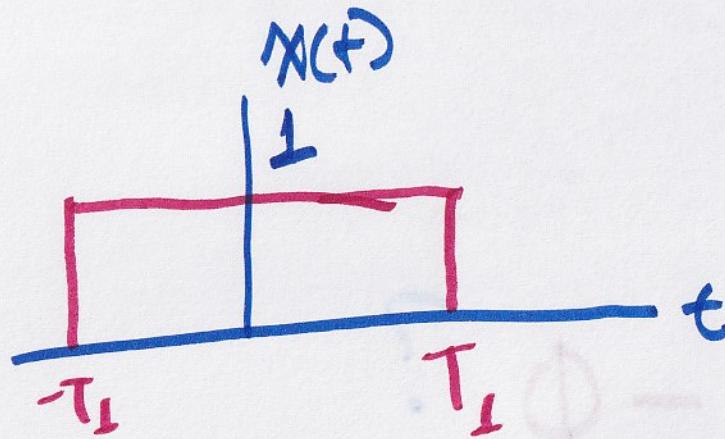
Angle of $\chi(\omega)$?Hence on EE trick

$$\frac{|a_k| \angle \theta}{|b_k| \angle \phi} = |c_k| \angle \underline{\theta - \phi} ?$$

Polar / Phasor form



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$$X(t) = \begin{cases} C & -T_1 \leq t \leq T_1 \\ 0 & \text{else} \end{cases}$$

$$C = 1$$

so

$$X(\omega) = \int_{-\infty}^{\infty} X(t) e^{-j\omega t} dt$$

$$= \int_{-T_1}^{T_1} e^{-j\omega t} dt = -\frac{1}{j\omega} e^{-j\omega t} \Big|_{-T_1}^{T_1}$$

$$= -\frac{1}{j\omega} [e^{-j\omega T_1} - e^{j\omega T_1}]$$

$$= -\frac{1}{j\omega} [-2j \sin(\omega T_1)] = \frac{2 \sin(\omega T_1)}{\omega}$$

$$\approx 2T_1 \operatorname{sinc}(U T_1)$$

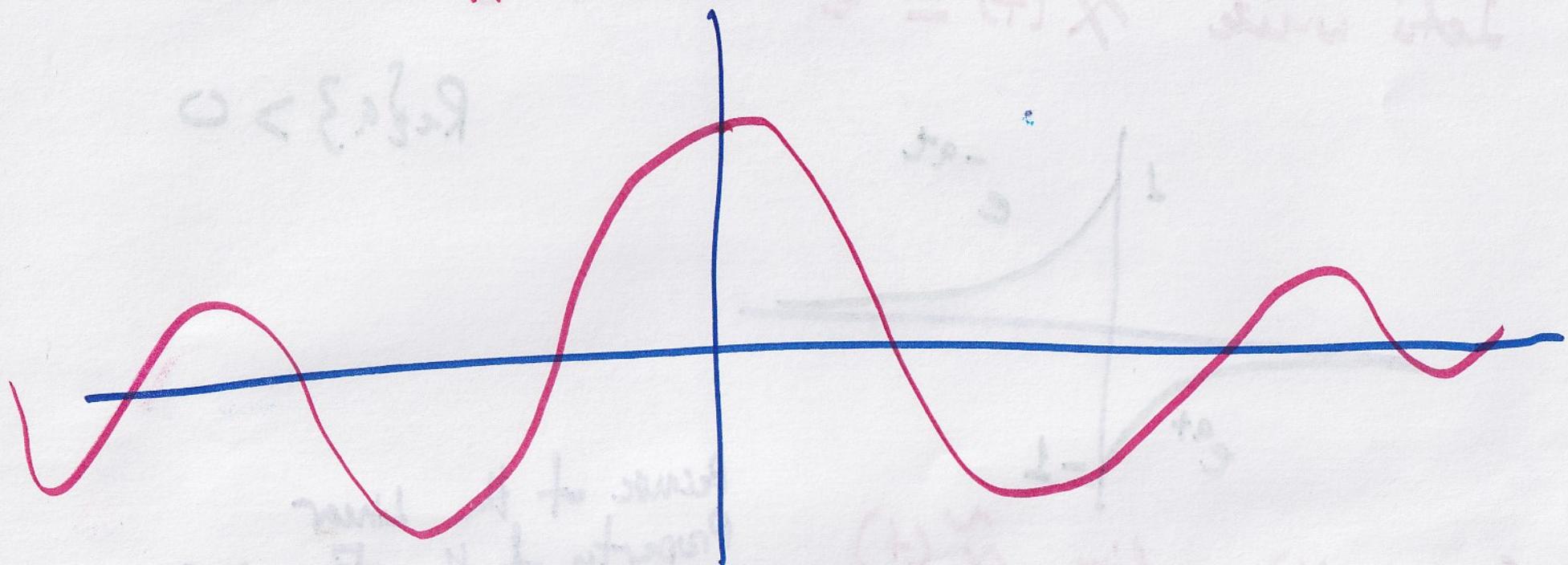
$$\frac{2 \sin(\omega T_L)}{\omega} = \frac{2 T_L \sin(\omega T_L)}{\omega T_L}$$

(9)

$\operatorname{sinc}(\omega T_L)$

$2 T_L \frac{\cos(\omega T_L)}{T_L}$

$\frac{\sin(x)}{x} = \operatorname{sinc}(x)$



$$\operatorname{sinc}(\omega T_L) \rightarrow \frac{1}{T_L}$$

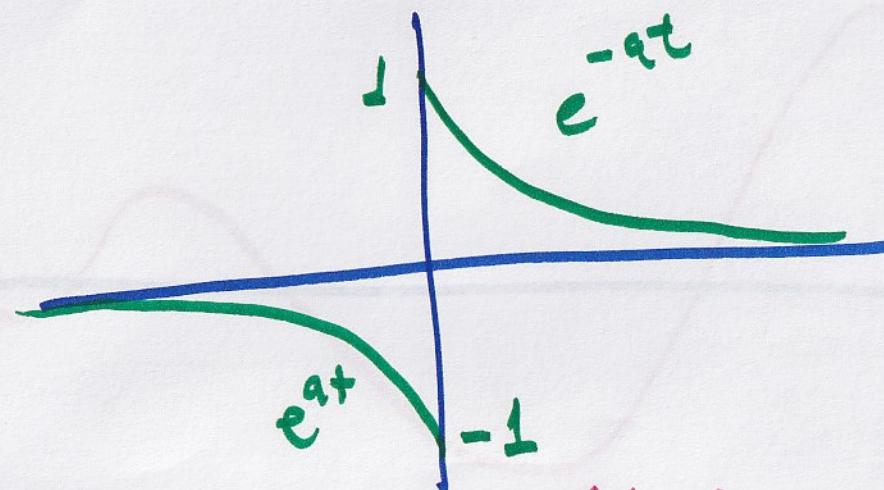
$\omega \rightarrow 0$

(10)

Ex FT of $\text{sgn}(t) = \begin{cases} -1 & t \leq 0 \\ 1 & t > 0 \end{cases}$

(signum)

Let's write $\tilde{x}(t) = e^{-at}u(t) + e^{at}u(-t)$



$$\therefore x(t) = \lim_{a \rightarrow 0} \tilde{x}(t)$$

Because of the Linear
Property of the Fourier
Transform

$$\rightarrow X(\omega) = \lim_{a \rightarrow 0} \tilde{X}(\omega)$$

(11)

s₀

$$\tilde{x}(\omega) = \text{FT}\{\tilde{x}(+)\}$$

$$= \int_{-\infty}^0 e^{at} e^{-j\omega t} dt + \int_0^\infty e^{-at} e^{-j\omega t} dt$$

$$= -\frac{1}{a-j\omega} + \frac{1}{a+j\omega}$$

$$a \rightarrow 0 \quad X(\omega) = -\frac{1}{j\omega} + \frac{1}{j\omega} = \frac{2}{j\omega}$$

s₀

$$\text{FT}\{\text{sgn}(t)\} = \frac{2}{j\omega}$$

(12) What is the Inverse Fourier Transform of $2\pi\delta(\omega)$?

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} x(\omega) e^{+j\omega t} d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi\delta(\omega) \cdot e^{+j\omega t} d\omega = 1$$

This means $\text{FT}\{1\} = 2\pi\delta(\omega)$!!!

(13)

$$\delta(t) \xrightarrow{FT} 1$$

$$1 \xrightarrow{FT} 2\pi\delta(\omega)$$

dirac delta (Impulse Function)

$$\int_{-\infty}^{\infty} e^{-j\omega t} dt = \underbrace{\frac{-1}{j\omega} \left[e^{-j\omega t} \right]_{-\infty}^{\infty}}_{\text{Blows Up?}}$$

No! $\rightarrow 2\pi \delta(\omega)$ ~~$\sum_{n=1}^{\infty}$~~

Friday Stream:

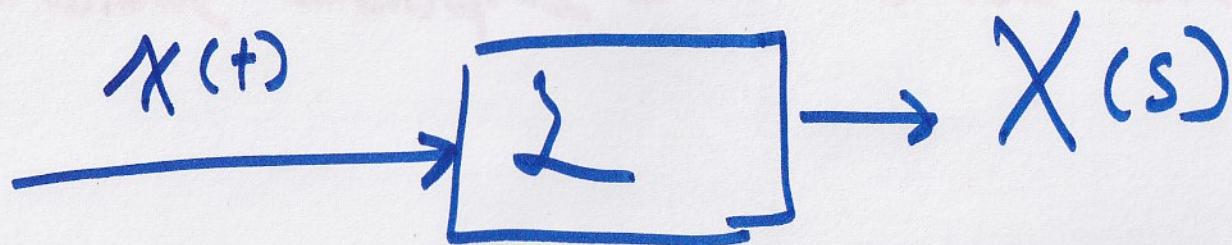
How you'll die on every planet

(14)

FT of $e^{j\omega_0 t}$?

$$e^{j\omega_0 t} \times 1 \xrightarrow{\text{FT}} 2\pi\delta(\omega - \omega_0)$$

Y tho?



$$\mathcal{L}\{X(t)e^{s_0 t}\} = X(s-s_0)$$

Let $s \rightarrow j\omega$, we get a lot
just from that

$$x(+)$$

$$\longrightarrow X(s)$$

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$$e^{s_0 t} x(+)$$

$$\longrightarrow X(s - s_0)$$

(from Laplace)

$$1$$

$$\longrightarrow$$

$$2\pi \delta(\omega)$$

$$e^{j\omega_0 t} \cdot 1$$

$$\longrightarrow$$

$$2\pi \delta(\omega - \omega_0)$$

FT { }

FT { }

(16)

FT of $\cos(\omega_0 t)$

$$\cos(\omega_0 t) = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$$

$$\mathcal{F}\{\cos(\omega_0 t)\} = \frac{1}{2} \mathcal{F}\{e^{j\omega_0 t}\} + \frac{1}{2} \mathcal{F}\{e^{-j\omega_0 t}\}$$

$$= \frac{1}{2} (2\pi\delta(\nu - \omega_0)) + \frac{1}{2} (2\pi\delta(\nu + \omega_0))$$

$$= \pi [\delta(\nu - \omega_0) + \delta(\nu + \omega_0)]$$

FT of $u(t)$?

$$u(t) = \frac{1}{2} (\sin(t) + 1)$$

$$\begin{aligned}\mathcal{F}\{u(t)\} &= \mathcal{F}\left\{\frac{\sin(t)}{2} + \frac{1}{2}\right\} \\ &= \frac{1}{2} \left[\frac{2}{j\omega} + 2\pi\delta(\omega) \right]\end{aligned}$$

(18)

FT of Periodic Signals

$x(t)$ periodic with period T
(Dirichlet cond.)

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$\mathcal{F} \left\{ \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \right\} = \sum_{k=-\infty}^{\infty} a_k \underbrace{\mathcal{F} \left\{ e^{jk\omega_0 t} \right\}}_{= 2\pi a_k \delta(\omega - k\omega_0)}$$

(19)

