Signals & Systems L 12

①

Properties of the Fourier Transform

① Linearity

$$x_1(t) \xleftrightarrow{F} X_1(\omega)$$

$$x_2(t) \xleftrightarrow{F} X_2(\omega)$$

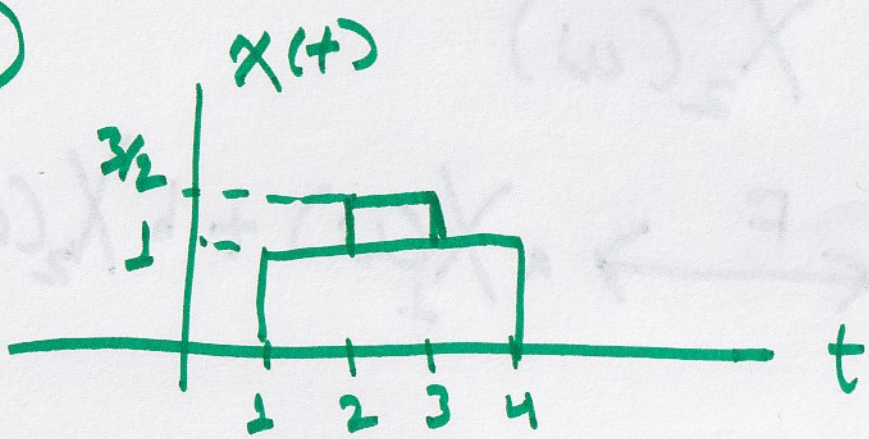
$$a x_1(t) + b x_2(t) \xleftrightarrow{F} a X_1(\omega) + b X_2(\omega)$$

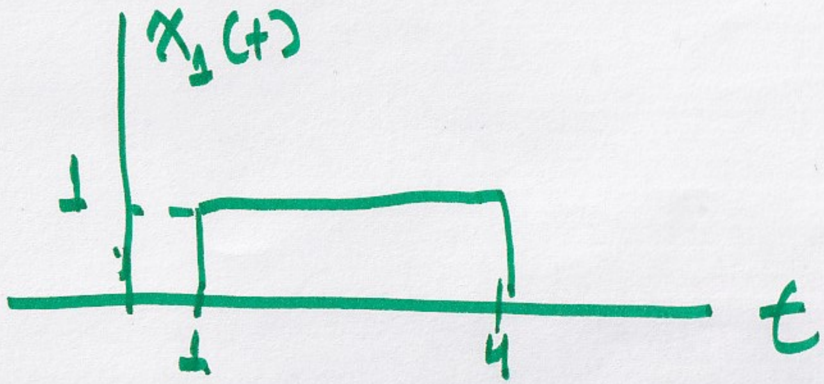
2 Time-Shift

$$x(t) \longleftrightarrow X(\omega)$$

$$x(t-t_0) \longleftrightarrow e^{-j\omega t_0} X(\omega)$$

Ex)





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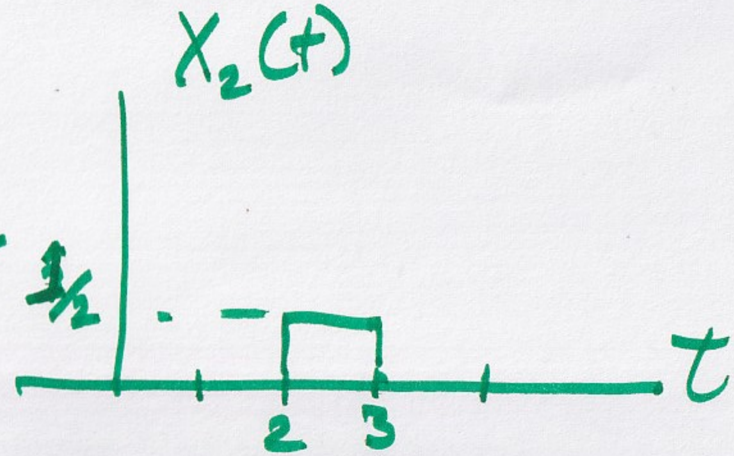
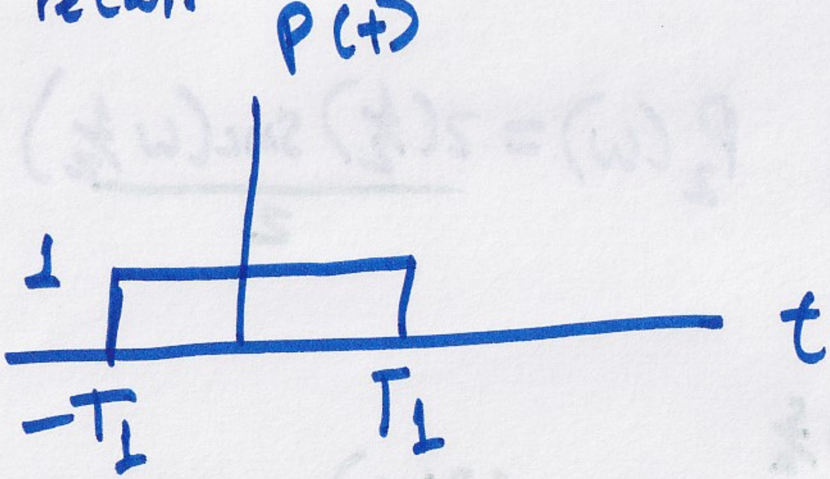


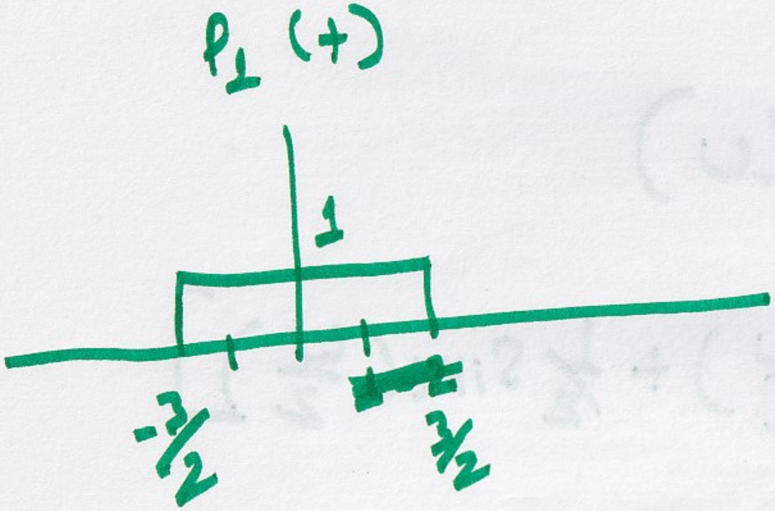
recall

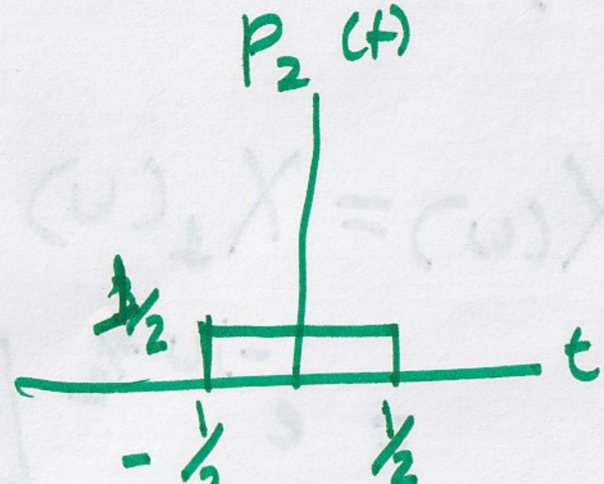


$$p(t) \xrightarrow{FT} P(\omega) = 2T_1 \operatorname{sinc}(\omega T_1)$$

we may write,

$$x_1(t) = p_1\left(t - \tfrac{5}{2}\right) \qquad x_2(t) = p_2\left(t - \tfrac{5}{2}\right)$$





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So we may write

$$P_1(\omega) = 2\left(\tfrac{3}{2}\right)\operatorname{sinc}\left(\omega \tfrac{3}{2}\right), \quad P_2(\omega) = \frac{2\left(\tfrac{1}{2}\right)\operatorname{sinc}\left(\omega \tfrac{1}{2}\right)}{2}$$

So then

$$X_1(\omega) = e^{-j\omega 5/2} P_1(\omega) = e^{-j\omega 5/2}\, 3\operatorname{sinc}\left(\tfrac{3\omega}{2}\right)$$

$$X_2(\omega) = e^{-j\omega 5/2} P_2(\omega) = \frac{e^{-j\omega 5/2}}{2}\operatorname{sinc}\left(\tfrac{\omega}{2}\right)$$

So $$X(\omega) = X_1(\omega) + X_2(\omega)$$

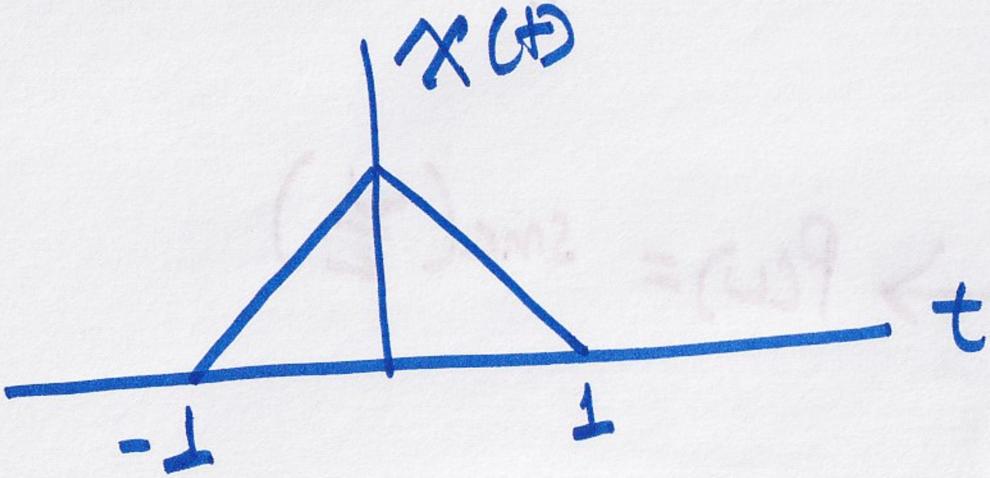
$$= e^{-j\omega 5/2}\left[3\operatorname{sinc}\left(\tfrac{3\omega}{2}\right) + \tfrac{1}{2}\operatorname{sinc}\left(\tfrac{\omega}{2}\right)\right]$$

3 Time-Differentiation

$$\frac{d}{dt}x(t) \longleftrightarrow j\omega X(\omega)$$

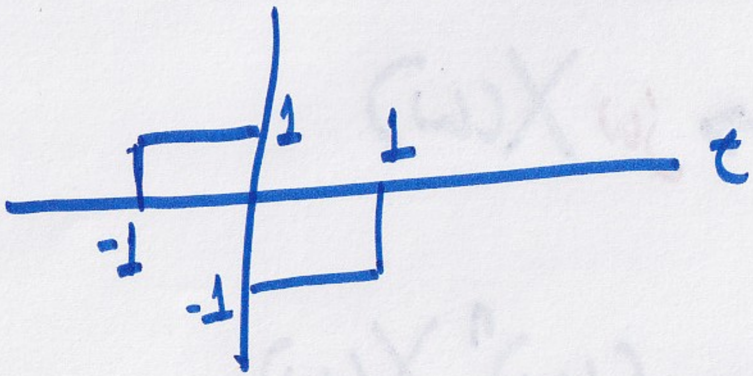
$$\frac{d^n x(t)}{dt^n} \longleftrightarrow (j\omega)^n X(\omega)$$

Ex) Find the FT of



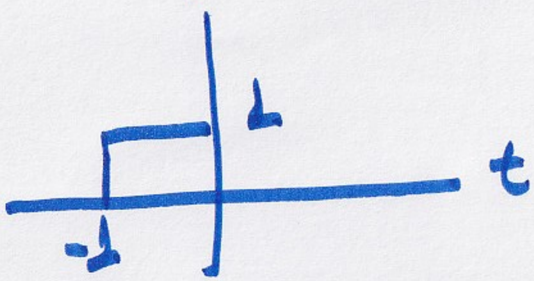
(b)

Let $\tilde{x}(t) = \frac{d}{dt} x(t)$

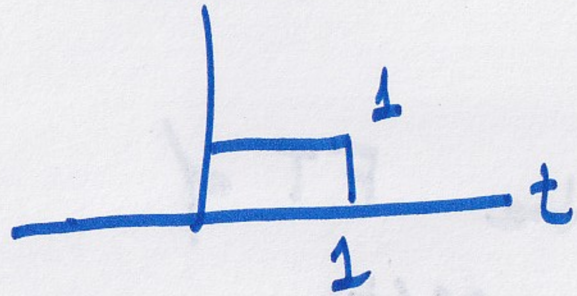


So

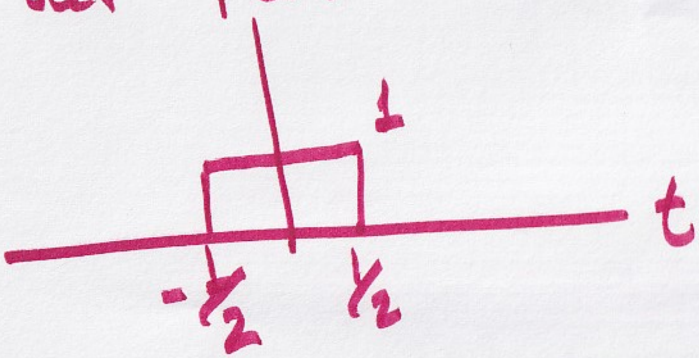
$\tilde{x}_1(t)$



$\tilde{x}_2(t)$



Let $p(t)$



$\longrightarrow P(\omega) = \text{sinc}\left(\frac{\omega}{2}\right)$

So we may write

$$\tilde{x}_1(t) = p(t+\tfrac{1}{2}), \quad \tilde{x}_2(t) = -p(t-\tfrac{1}{2})$$

$$\tilde{X}_1(\omega) = e^{j\omega/2}\,\mathrm{sinc}(\tfrac{\omega}{2}), \quad \tilde{X}_2(\omega) = -e^{-j\omega/2}\,\mathrm{sinc}(\omega/2)$$

So $$\tilde{X}(\omega) = \tilde{X}_1(\omega) + \tilde{X}_2(\omega) = \mathrm{sinc}(\tfrac{\omega}{2})\Big[\underbrace{e^{j\omega/2} - e^{-j\omega/2}}_{2j\sin(\omega/2)}\Big]$$

$$\tilde{x}(t) = \frac{d}{dt}x(t) \xleftrightarrow{FT} \tilde{X}(\omega) = j\omega X(\omega)$$

$$\rightarrow X(\omega) = \frac{\tilde{X}(\omega)}{j\omega} = \frac{\mathrm{sinc}(\omega/2)\cdot 2j\sin(\omega/2)}{j\omega}$$

$$= \mathrm{sinc}^2(\omega/2)$$

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4 Convolution

$$x(t) * y(t) \longleftrightarrow X(\omega)\,Y(\omega)$$

5 Integral

$$\int_{-\infty}^{t} x(\tau)\,d\tau \longleftrightarrow \left(\frac{1}{j\omega} + \pi\delta(\omega)\right) X(\omega)$$

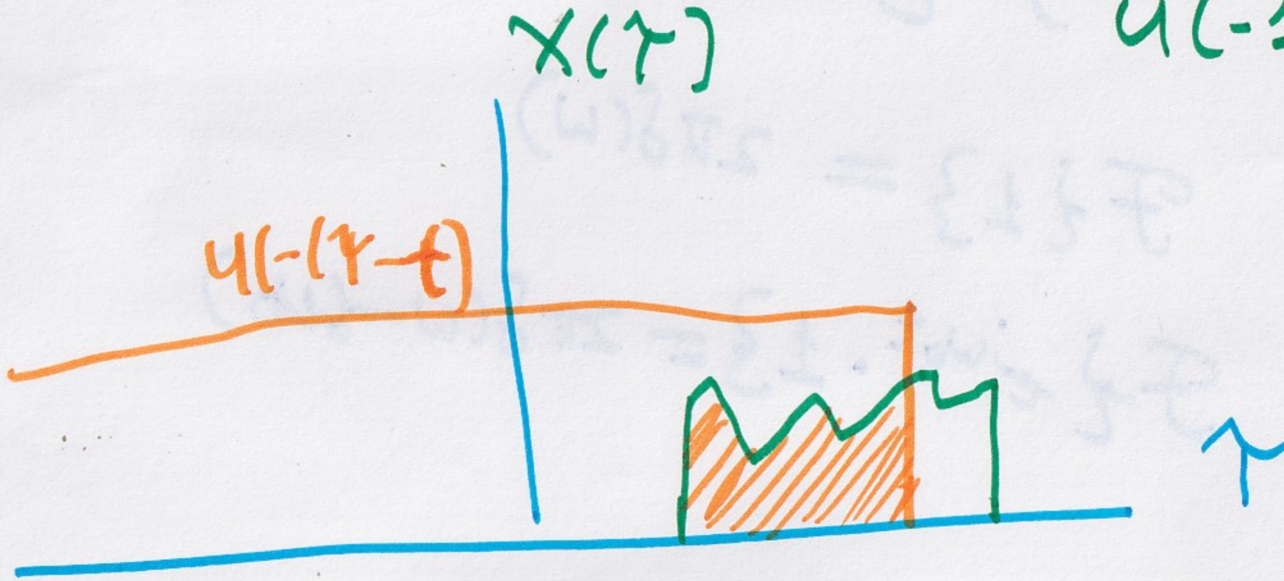
So This means

$$x(t) * u(t) \longleftrightarrow X(\omega)\,U(\omega)$$

$$x(t) * u(t) = \int_{-\infty}^{\infty} x(\tau)\, u(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{t} x(\tau)\, d\tau \quad \text{for } \tau \le t$$

$u(-\tau + t)$
$u(-1(\tau - t))$



6 Frequency shift

$$e^{j\omega_0 t} x(t) \longleftrightarrow X(\omega-\omega_0)$$

Ex) $\mathcal{F}\{e^{j\omega_0 t}\}$

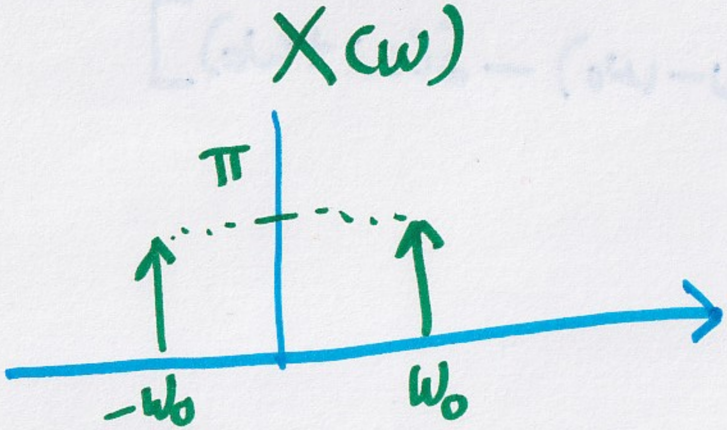
$$\mathcal{F}\{1\} = 2\pi\delta(\omega)$$

$$\mathcal{F}\{e^{j\omega_0 t}\cdot 1\} = 2\pi\delta(\omega-\omega_0)$$

Ex) $\mathcal{F}\{\cos(\omega_0 t)\}$

$$\cos(\omega_0 t) = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$$

$$\mathcal{F}\{\cos(\omega_0 t)\} = \pi\left[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)\right]$$



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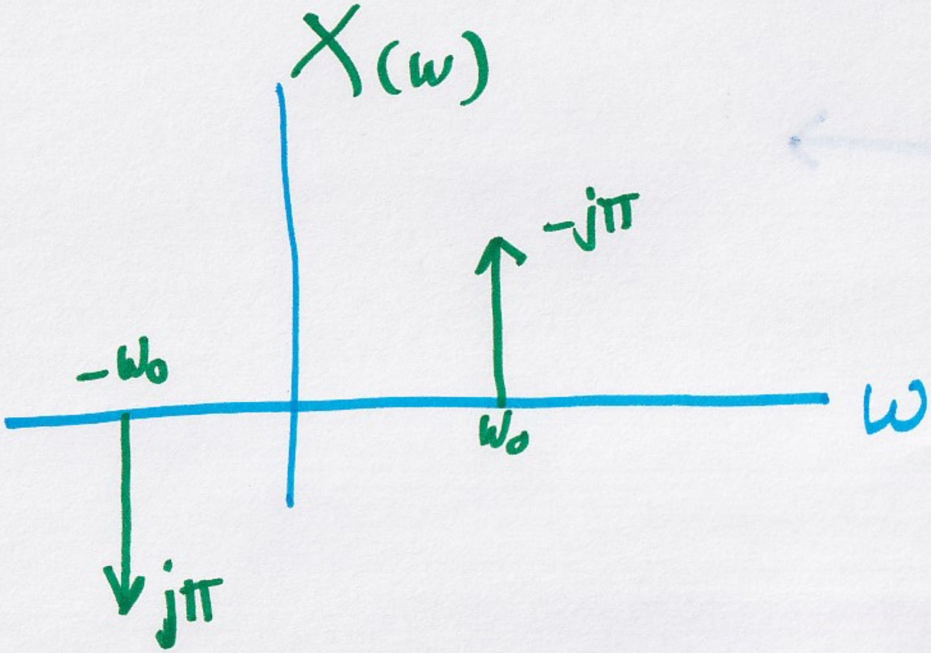
FT of $\sin \omega_0 t$?

$$\sin \omega_0 t = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}$$

$$\mathcal{F}\{\sin \omega_0 t\} = \frac{1}{2j} 2\pi \left[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)\right]$$

$$= -j\pi \left[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)\right]$$



7 Fourier Transforms of periodic Signals

$x(t)$ with period T

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T}$$

so

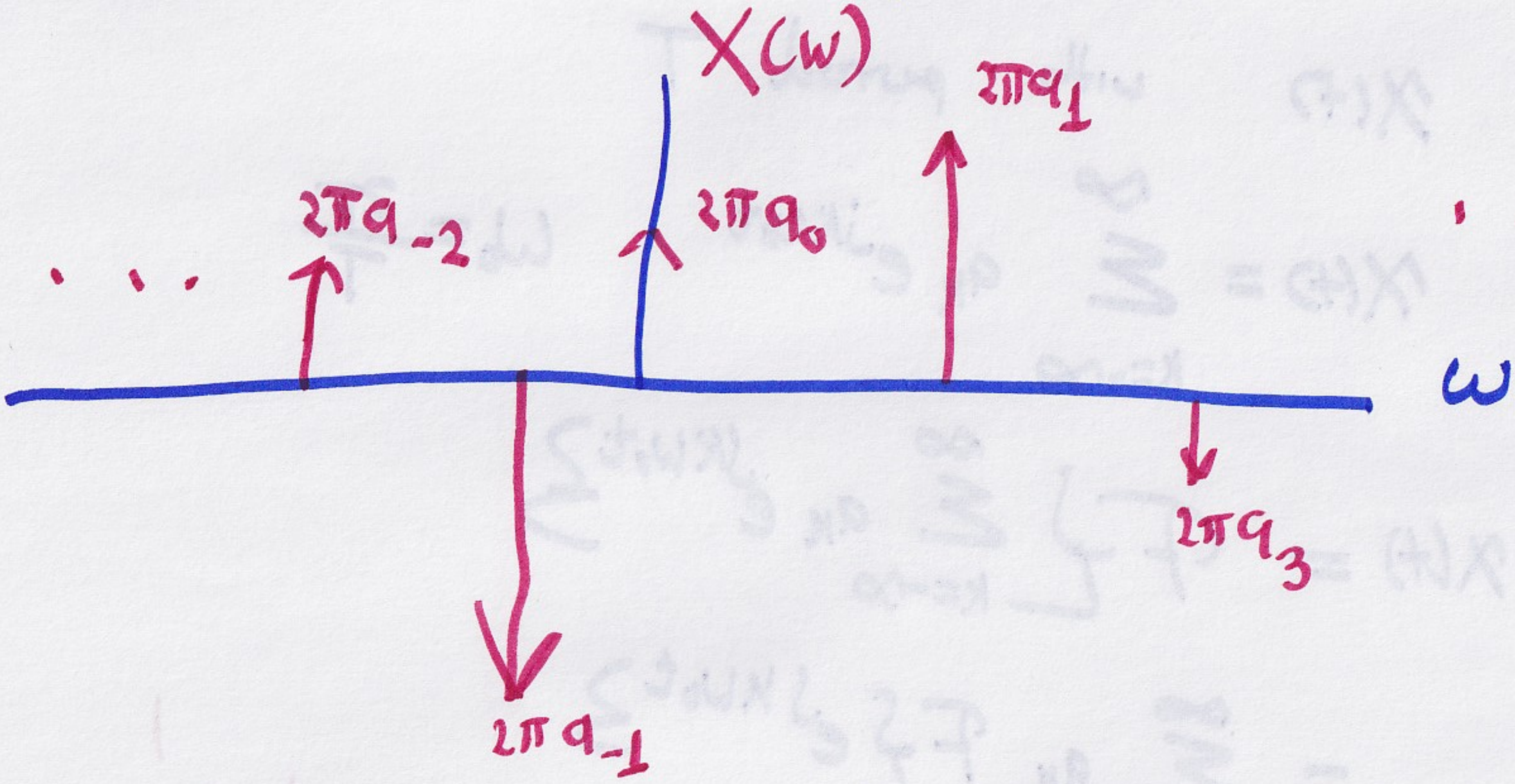
$$\mathcal{F}\{x(t)\} = \mathcal{F}\left\{\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\right\}$$

$$= \sum_{k=-\infty}^{\infty} a_k \underbrace{\mathcal{F}\{e^{jk\omega_0 t}\}}_{2\pi\,\delta(\omega - k\omega_0)}$$

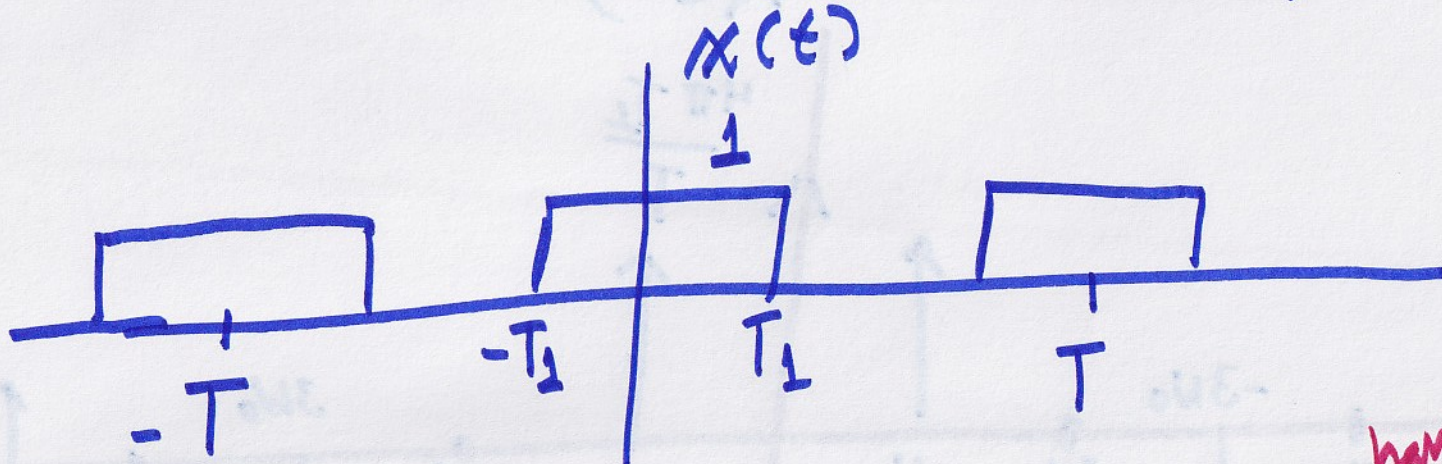
whoa!

(14)

So we have $X(\omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0)$



Ex) what's the Fourier ~~Tran~~ Transform?



$$a_k = \frac{2T_1}{T}\,\mathrm{sinc}(k\omega_0) \quad \text{where} \quad \omega_0 = \frac{2\pi}{T}$$

we have a_k! so plug and chug!

$$X(\omega) = 2\pi \sum_{k=-\infty}^{\infty} \left(\frac{2T_1}{T}\,\mathrm{sinc}(k\omega_0) \right) \delta(\omega - \omega_0 k)$$

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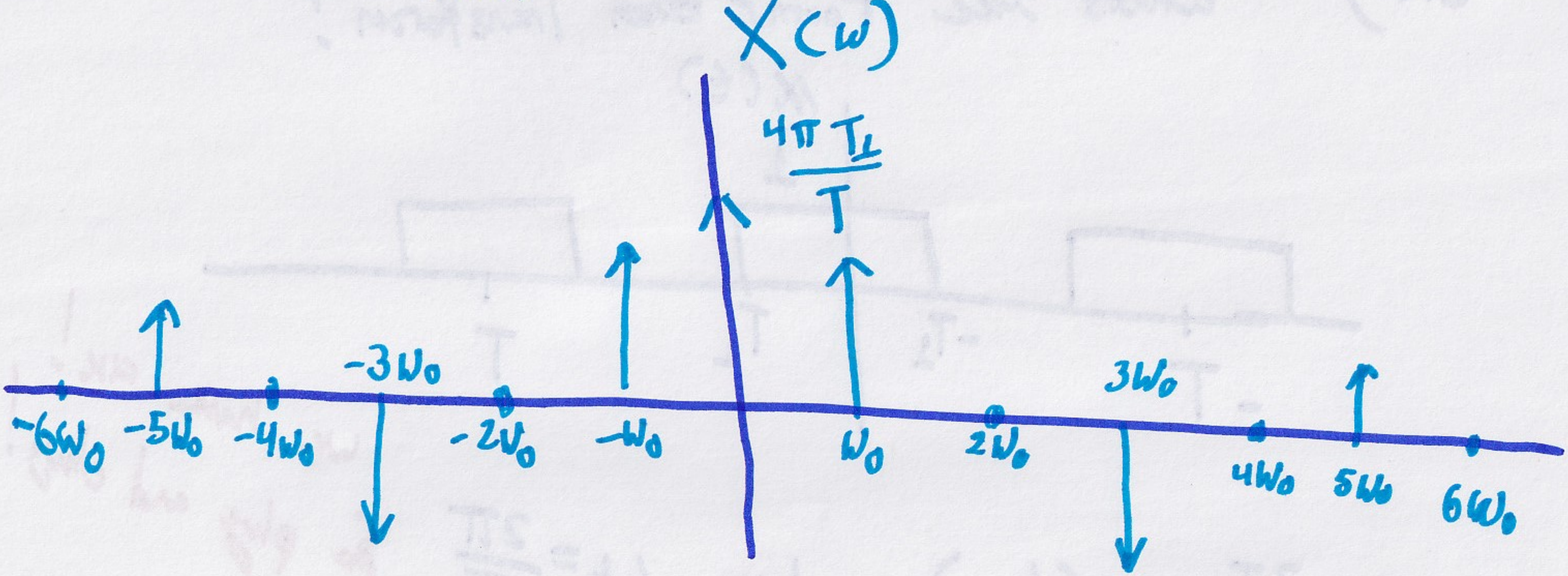
So that would look like



Capturing instances of the sinc (the a_k)

8 Time Reversal

$$x(-t) \longleftrightarrow X(-\omega)$$

9 Time Scaling

$$x(at) \longleftrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

Ex) FT of $e^{-a|t|}$ $\quad a>0$

$$x(t) = e^{-a|t|} = e^{-at}u(t) + e^{at}u(-t)$$

Let $x_1(t) = e^{-at}u(t)$

$$x_2(t) = e^{at}u(-t) = x_1(-t)$$

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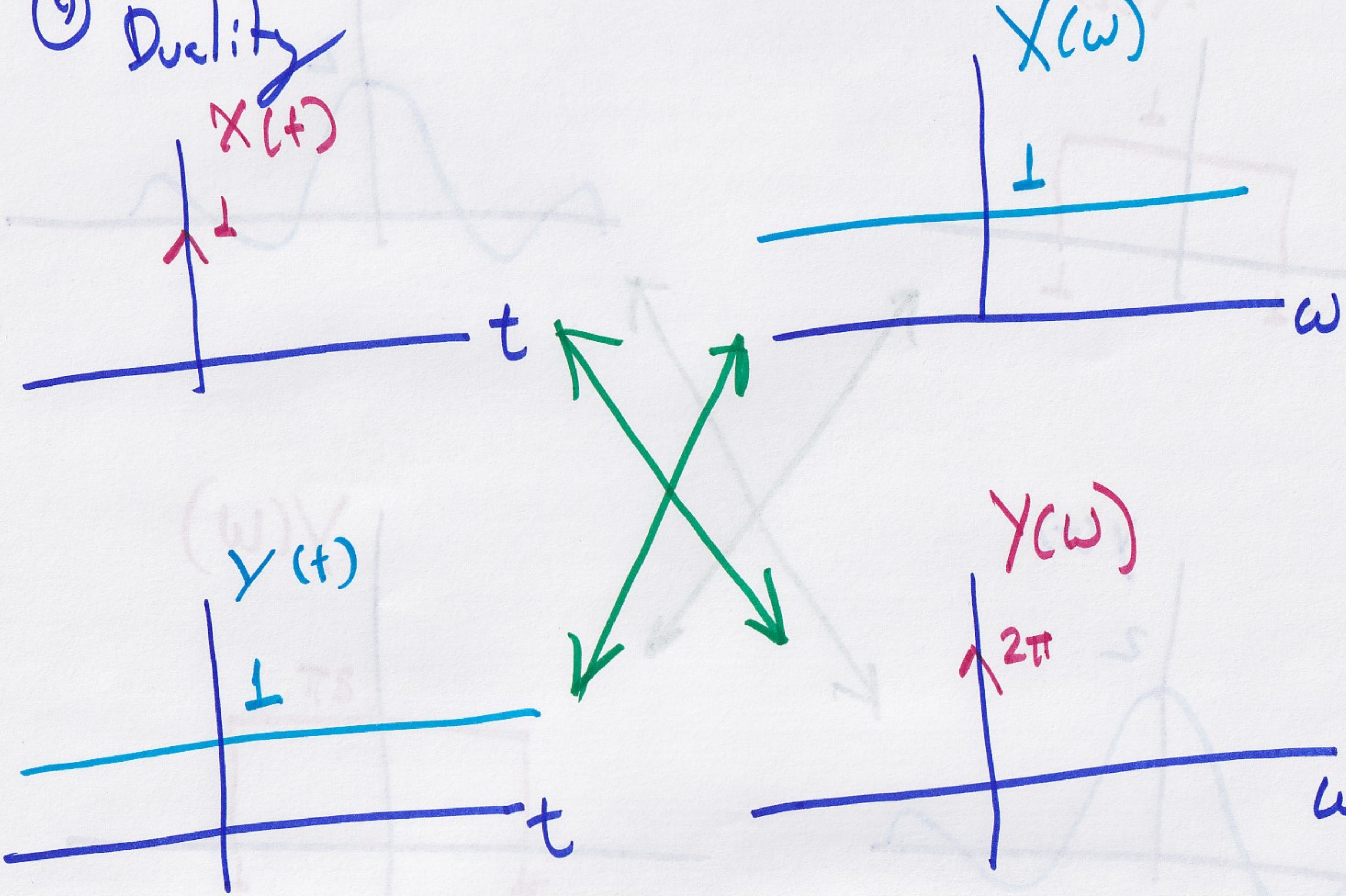
So $X_1(\omega) = \dfrac{1}{a+j\omega}$

Then $x_1(-t) \xrightarrow{FT} \dfrac{1}{a-j\omega}$

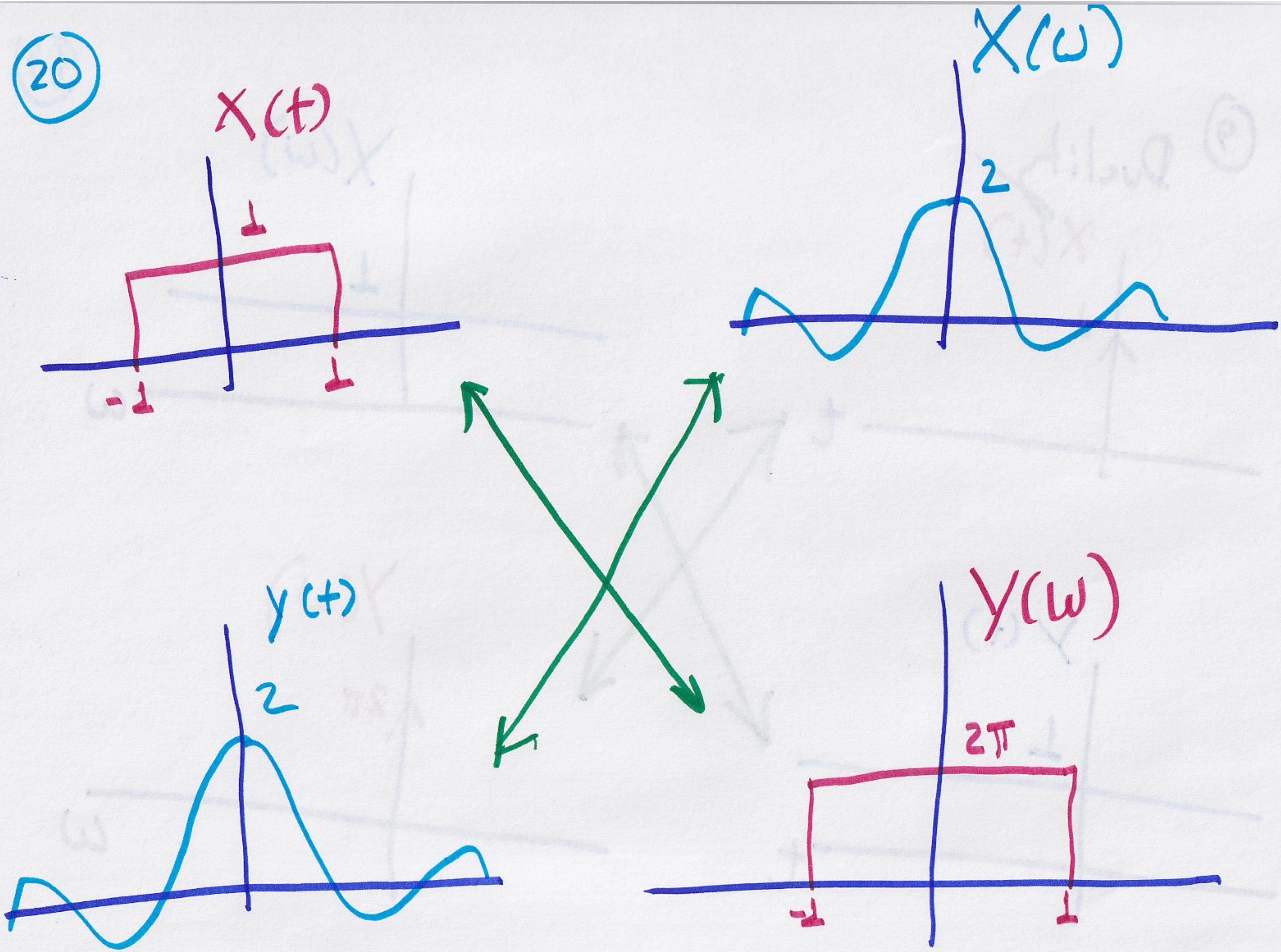
So $X(\omega) = X_1(\omega) + X_1(-\omega)$

$$= \frac{1}{a+j\omega} + \frac{1}{a-j\omega} = \frac{a-j\omega+a+j\omega}{(a+j\omega)(a-j\omega)} = \frac{2a}{a^2+\omega^2}$$

9 Duality



(20)



~~to~~ say it straight up

$$x(t) \longleftrightarrow X(\omega)$$

$$X(t) \longleftrightarrow 2\pi x(-\omega)$$

lower case

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

let $t \to -t$

$$2\pi x(-t) = \int_{-\infty}^{\infty} X(\omega)e^{-j\omega t}d\omega$$

12 $t \longleftrightarrow \omega$

$$2\pi\, x(-\omega) = \int_{-\infty}^{\infty} X(t)\, e^{-j\omega t}\, dt$$

so

$$\Rightarrow \mathcal{F}\{X(t)\} = 2\pi\, x(-\omega)$$

10 ~~Mod~~ Multiplication

$$x(t)\, y(t) \longleftrightarrow \frac{X(\omega) * Y(\omega)}{2\pi}$$