Fourier transforms

$x(t)$ periodic with period T

Let's say the signals we use ~~today~~ satisfy the dirichlet conditions

and is integrable

So $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$ $\omega_0 = \frac{2\pi}{T}$

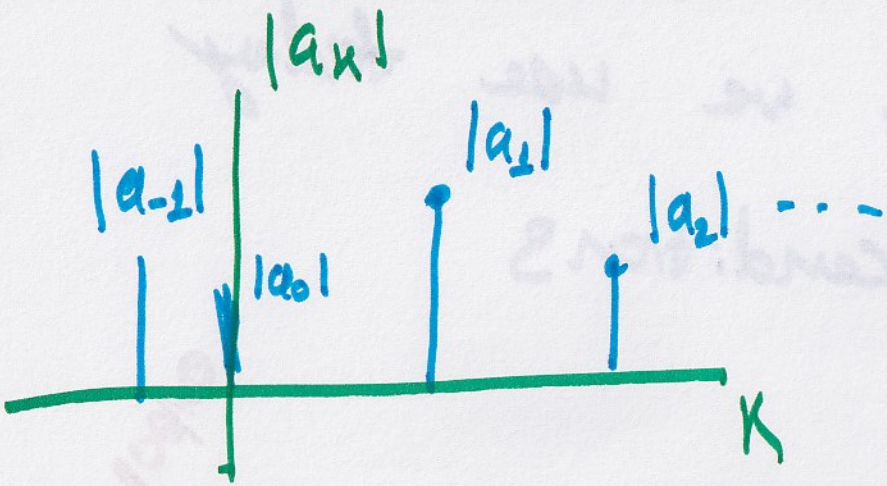
$a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt$

exponential fourier series

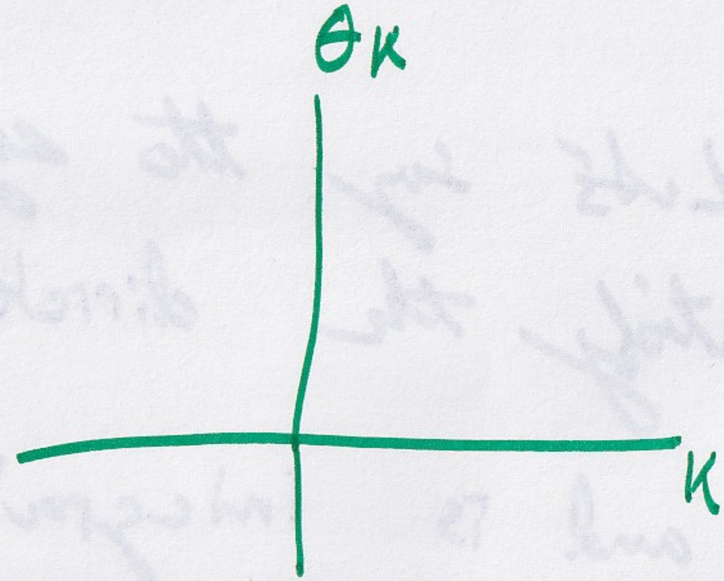
②

Fourier Spectrum

$$a_k = |a_k| e^{j\theta_k}$$



Amplitude Spectrum



phase spectrum

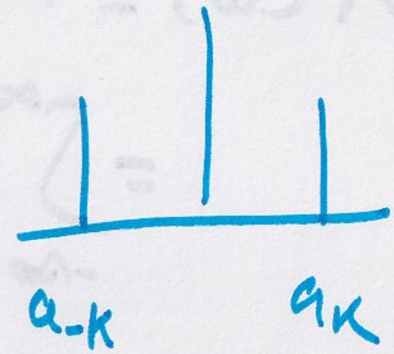
Ex

$\underline{a} = 1 + j \rightarrow |a| = \sqrt{1^2+1^2} = \sqrt{2} \rightarrow$ ~~theta~~ $\theta = \arctan\left(\frac{1}{1}\right) = \pi/4$

$a = -5 \rightarrow |a| = 5 \qquad \theta = \arctan\left(\frac{0}{-5}\right) = \pi$

$x(t)$ real $\longrightarrow x(t) = x^*(t)$

$$a_k = a_{-k}^*$$



recall

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2\,\mathrm{Re}\left\{ a_k e^{jk\omega_0 t} \right\}$$

$$= a_0 + \sum_{k=1}^{\infty} 2\,\mathrm{Re}\left\{ |a_k| e^{j\theta_k} \cdots e^{jk\omega_0 t} \right\}$$

$$= a_0 + \sum_{k=1}^{\infty} 2|a_k| \cos(k\omega_0 t + \theta_k)$$

Trigonometric Fourier Series

④

what if $x(t)$ is aperiodic?

$$X(\omega) = FT\{x(t)\}$$

$$= \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

FT of $\delta(t) \longrightarrow 1$

why tho? I will get to that later

Ex) FT $e^{-at}u(t)$ for $\mathrm{Re}\{a\} > 0$

$$FT\{e^{-at}u(t)\} = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}dt$$

$$= \int_0^{\infty} e^{-(a+j\omega)t}dt = \frac{-1}{a+j\omega} e^{-(a+j\omega)t}\Big|_0^{\infty}$$

$$= \frac{1}{a+j\omega}$$

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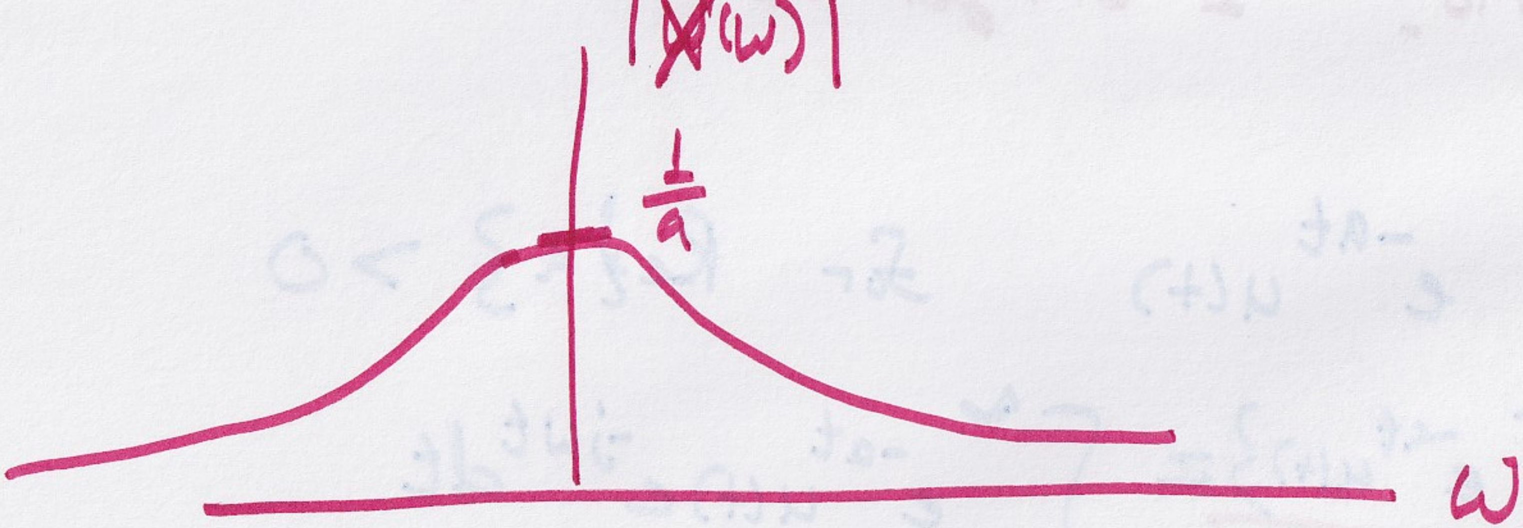
$$|X(\omega)| = \left|\frac{1}{a+j\omega}\right| = \frac{1}{|a+j\omega|} = \frac{1}{\sqrt{a^2+\omega^2}}$$



Amplitude Spectrum!

(1st order Low Pass Filter)

Angle of $X(\omega)$?

Here's an <u>EE trick</u>

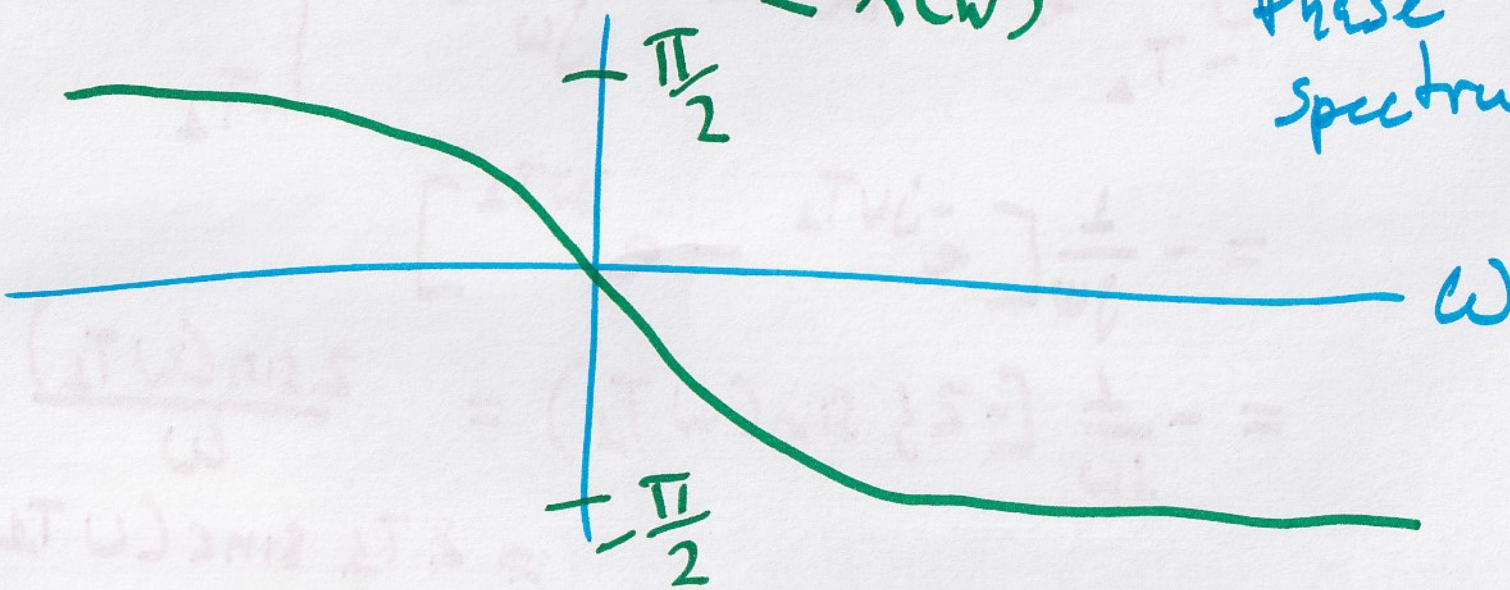
$$\frac{|a_k|\angle\theta}{|b_k|\angle\phi} = |c_k|\angle\underline{\theta-\phi}\ ?$$

Polar / Phasor form

$$\angle\frac{1}{a+j\omega} \rightarrow 0 - \arctan\left(\frac{\omega}{a}\right)$$

$\angle X(\omega)$

Phase spectrum



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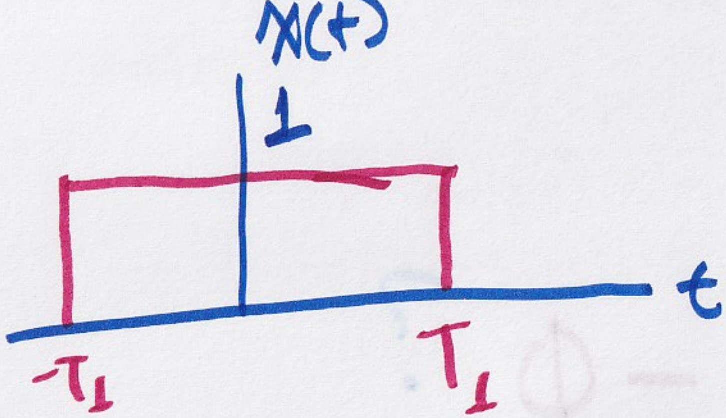


$$x(t)=\begin{cases} c & -T_1 \le t \le T_1 \\ 0 & \text{else} \end{cases}$$

$$c = 1$$

so

$$X(\omega)=\int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

$$=\int_{-T_1}^{T_1} e^{-j\omega t}\,dt = -\frac{1}{j\omega}e^{-j\omega t}\Big|_{-T_1}^{T_1}$$

$$=-\frac{1}{j\omega}\left[e^{-j\omega T_1}-e^{j\omega T_1}\right]$$

$$=-\frac{1}{j\omega}\left[-2j\sin(\omega T_1)\right)=\frac{2\sin(\omega T_1)}{\omega}$$

$$=2T_1\,\text{sinc}(\omega T_1)$$

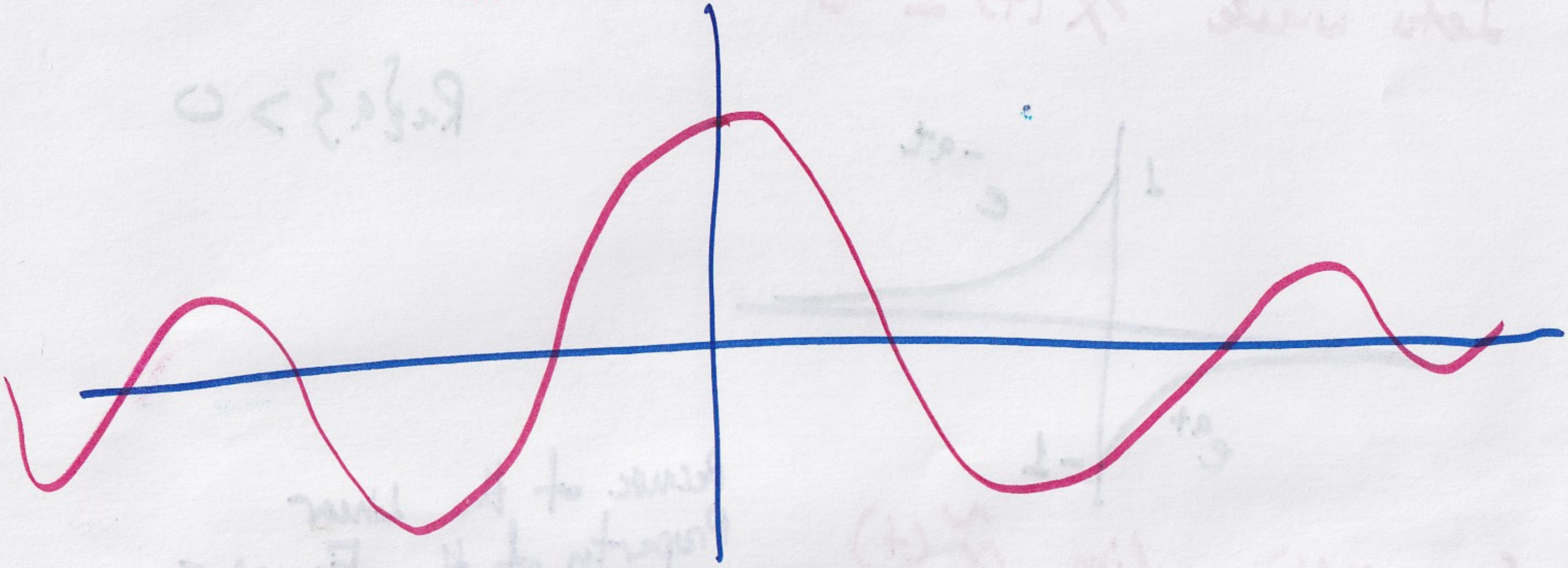
⑨

$$\frac{2\sin(\omega T_1)}{\omega} = \frac{2T_1 \sin(\omega T_1)}{\omega T_1}$$

$\operatorname{sinc}(\omega T_1)$

$2T_1 \quad \frac{\cos(\omega T_1}{T_1})$

$$\frac{\sin(x)}{x} = \operatorname{sinc}(x)$$



$$\underset{\omega \to 0}{\sin(\omega T_1)} \to \frac{1}{T_1}$$

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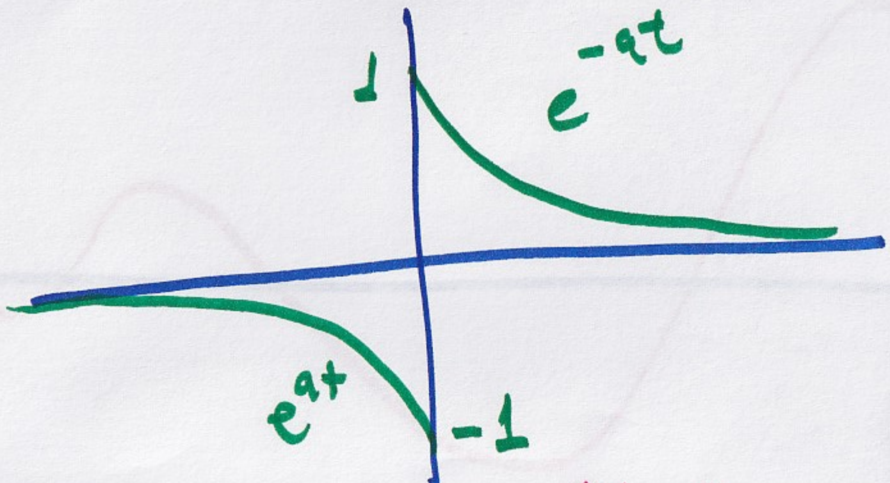
Ex FT of $\operatorname{sgn}(t) = \begin{cases} -1 & t<0 \\ 1, & t>0 \end{cases}$

(signum)

Let's write $\tilde{x}(t) = e^{-at}u(t) + e^{at}u(-t)$

$\mathrm{Re}\{a\} > 0$



So $x(t) = \lim_{a \to 0} \tilde{x}(t)$

Because of the Linear Property of the Fourier Transform

$\longrightarrow X(\omega) = \lim_{a \to 0} \tilde{X}(\omega)$

so

$$\tilde{X}(\omega) = FT\{\tilde{x}(t)\}$$

$$= \int_{-\infty}^{0} e^{at} e^{-j\omega t}\,dt + \int_{0}^{\infty} e^{-at} e^{-j\omega t}\,dt$$

$$= -\frac{1}{a-j\omega} + \frac{1}{a+j\omega}$$

$a \to 0$ $\quad X(\omega) = -\frac{1}{-j\omega} + \frac{1}{j\omega} = \frac{2}{j\omega}$

so

$$FT\{\mathrm{sgn}(t)\} = \frac{2}{j\omega}$$

12 What is ~~the~~ Inverse Fourier Transform of $2\pi\delta(\omega)$?

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{+j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\,\delta(\omega)\cdot e^{+j\omega t}\, d\omega = 1$$

This means $FT\{1\} = 2\pi\,\delta(\omega)$!!!

$$\delta(t) \xrightarrow{FT} 1$$

$$1 \xrightarrow{FT} 2\pi\delta(\omega)$$

dirac delta (Impulse function)

$$\int_{-\infty}^{\infty} e^{-j\omega t}\,dt = \frac{-1}{j\omega}\left[e^{-j\omega t}\right]_{-\infty}^{\infty}$$

Blows Up?

No! $\rightarrow$ $2\pi\delta(\omega)$

Friday stream!

How you'll die on every planet