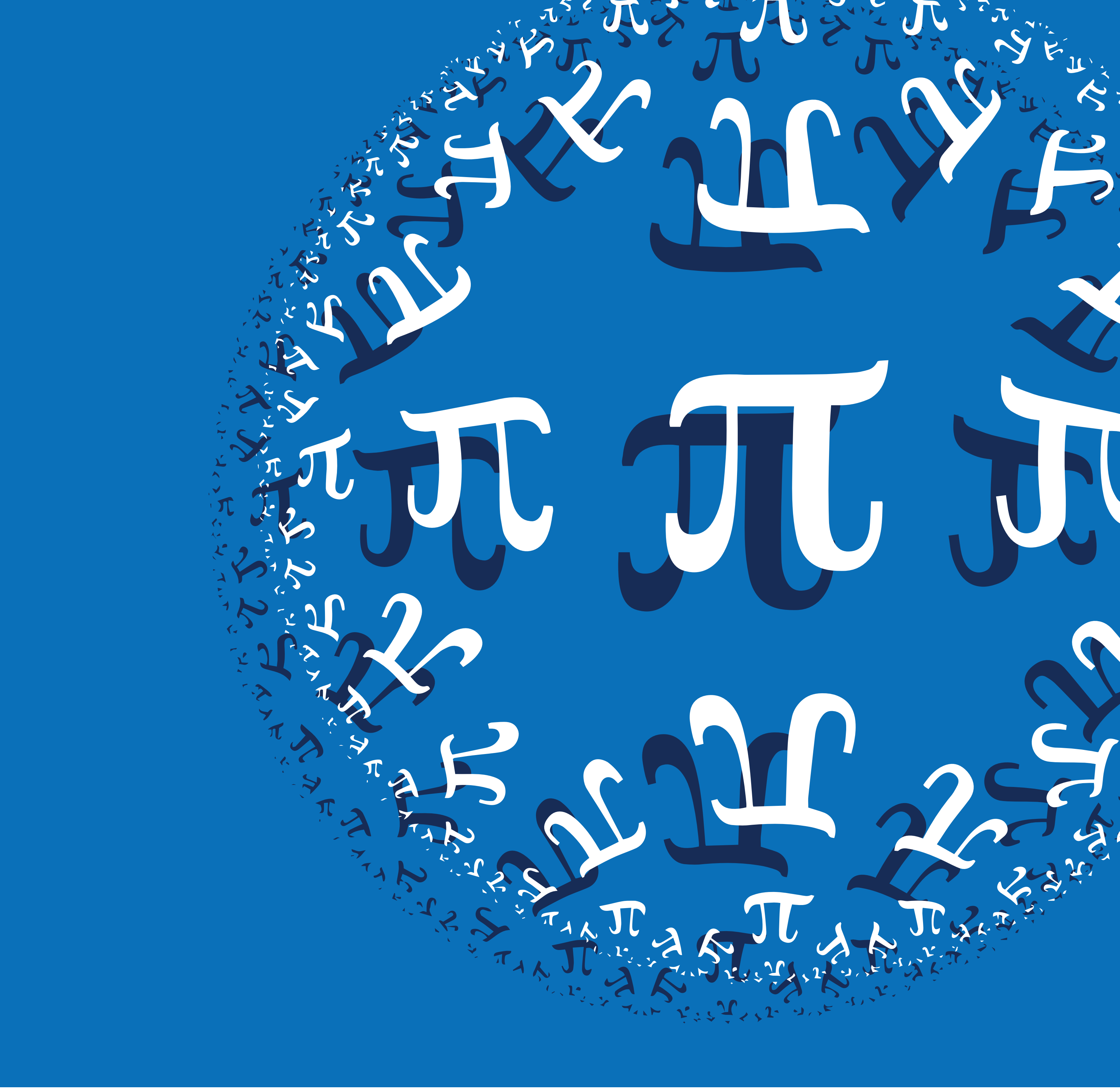

Algebra and Trigonometry

For BYU Math 111 Students

Algebra and Trigonometry

SENIOR CONTRIBUTING AUTHOR

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Notice:

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Brief Contents

Contents

Preface

Welcome to ***Algebra and Trigonometry***, an OpenStax resource. This textbook was written to increase student access to high-quality learning materials, maintaining highest standards of academic rigor at little to no cost.

About OpenStax

OpenStax is a nonprofit based at Rice University, and it's our mission to improve student access to education. Our first openly licensed college textbook was published in 2012, and our library has since scaled to over 20 books for college and AP courses used by hundreds of thousands of students. Our adaptive learning technology, designed to improve learning outcomes through personalized educational paths, is being piloted in college courses throughout the country. Through our partnerships with philanthropic foundations and our alliance with other educational resource organizations, OpenStax is breaking down the most common barriers to learning and empowering students and instructors to succeed.

About OpenStax's Resources

Customization

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Format

You can access this textbook for free in web view or PDF through openstax.org, and for a low cost in print.

About *Algebra and Trigonometry*

Written and reviewed by a team of highly experienced instructors, ***Algebra and Trigonometry*** provides a comprehensive and multi-layered exploration of algebraic principles. The text is suitable for a typical introductory algebra course, and was developed to be used flexibly. While the breadth of topics may go beyond what an instructor would cover, the modular approach and the richness of content ensures that the book meets the needs of a variety of programs.

Algebra and Trigonometry guides and supports students with differing levels of preparation and experience with mathematics. Ideas are presented as clearly as possible, and progress to more complex understandings with considerable reinforcement along the way. A wealth of examples—usually several dozen per chapter—offer detailed, conceptual explanations, in order to build in students a strong, cumulative foundation in the material before asking them to apply what they've learned.

Coverage and Scope

In determining the concepts, skills, and topics to cover, we engaged dozens of highly experienced instructors with a range of student audiences. The resulting scope and sequence proceeds logically while allowing for a significant amount of flexibility in instruction.

Chapters 1 and 2 provide both a review and foundation for study of Functions that begins in Chapter 3. The authors recognize that while some institutions may find this material a prerequisite, other institutions have told us that they have a cohort that need the prerequisite skills built into the course.

- Chapter 1: Prerequisites
- Chapter 2: Equations and Inequalities

Chapters 3-6: The Algebraic Functions

- Chapter 3: Functions
- Chapter 4: Linear Functions
- Chapter 5: Polynomial and Rational Functions
- Chapter 6: Exponential and Logarithm Functions

Chapters 7-10: A Study of Trigonometry

- Chapter 7: The Unit Circle: Sine and Cosine Functions
- Chapter 8: Periodic Functions
- Chapter 9: Trigonometric Identities and Equations
- Chapter 10: Further Applications of Trigonometry

Chapters 11-13: Further Study in Algebra and Trigonometry

- Chapter 11: Systems of Equations and Inequalities
- Chapter 12: Analytic Geometry
- Chapter 13: Sequences, Probability, and Counting Theory

All chapters are broken down into multiple sections, the titles of which can be viewed in the Table of Contents.

Development Overview

OpenStax ***Algebra and Trigonometry*** is the product of a collaborative effort by a group of dedicated authors, editors, and instructors whose collective passion for this project has resulted in a text that is remarkably unified in purpose and voice. Special thanks is due to our Lead Author, Jay Abramson of Arizona State University, who provided the overall vision for the book and oversaw the development of each and every chapter, drawing up the initial blueprint, reading numerous drafts, and assimilating field reviews into actionable revision plans for our authors and editors.

The collective experience of our author team allowed us to pinpoint the subtopics, exceptions, and individual connections that give students the most trouble. The textbook is therefore replete with well-designed features and highlights which help students overcome these barriers. As the students read and practice, they are coached in methods of thinking through problems and internalizing mathematical processes.

Accuracy of the Content

We understand that precision and accuracy are imperatives in mathematics, and undertook a dedicated accuracy program led by experienced faculty.

1. Each chapter's manuscript underwent rounds of review and revision by a panel of active instructors.
2. Then, prior to publication, a separate team of experts checked all text, examples, and graphics for mathematical accuracy; multiple reviewers were assigned to each chapter to minimize the chances of any error escaping notice.
3. A third team of experts was responsible for the accuracy of the Answer Key, dutifully re-working every solution to eradicate any lingering errors. Finally, the editorial team conducted a multi-round post-production review to ensure the integrity of the content in its final form.

Pedagogical Foundations and Features

Learning Objectives

Each chapter is divided into multiple sections (or modules), each of which is organized around a set of learning objectives. The learning objectives are listed explicitly at the beginning of each section and are the focal point of every instructional element.

Narrative Text

Narrative text is used to introduce key concepts, terms, and definitions, to provide real-world context, and to provide transitions between topics and examples. Throughout this book, we rely on a few basic conventions to highlight the most important ideas:

- Key terms are boldfaced, typically when first introduced and/or when formally defined.
- Key concepts and definitions are called out in a blue box for easy reference.

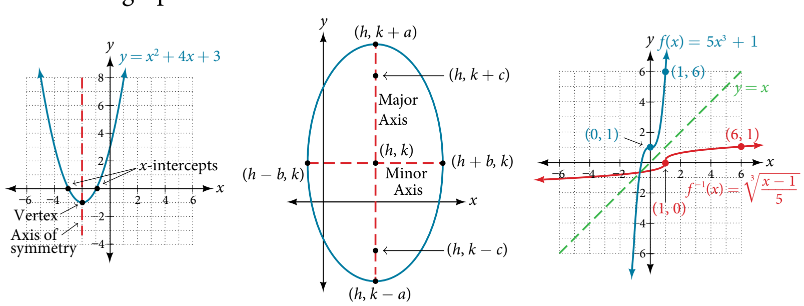
Examples

Each learning objective is supported by one or more worked examples, which demonstrate the problem-solving approaches that students must master. The multiple Examples model different approaches to the same type of problem, or introduce similar problems of increasing complexity.

All Examples follow a simple two- or three-part format. The question clearly lays out a mathematical problem to solve. The Solution walks through the steps, usually providing context for the approach --in other words, why the instructor is solving the problem in a specific manner. Finally, the Analysis (for select examples) reflects on the broader implications of the Solution just shown. Examples are followed by a "Try It," question, as explained below.

Figures

OpenStax ***Algebra and Trigonometry*** contains many figures and illustrations, the vast majority of which are graphs and diagrams. Art throughout the text adheres to a clear, understated style, drawing the eye to the most important information in each figure while minimizing visual distractions. Color contrast is employed with discretion to distinguish between the different functions or features of a graph.



(b)

Supporting Features

Four unobtrusive but important features contribute to and check understanding.

A "*How To*" is a list of steps necessary to solve a certain type of problem. A How To typically precedes an Example that proceeds to demonstrate the steps in action.

A "*Try It*" exercise immediately follows an Example or a set of related Examples, providing the student with an immediate opportunity to solve a similar problem. In the Web View version of the text, students can click an Answer link directly below the question to check their understanding. In the PDF, answers to the Try-It exercises are located in the Answer Key.

A "*Q & A...*" may appear at any point in the narrative, but most often follows an Example. This feature pre-empts misconceptions by posing a commonly asked yes/no question, followed by a detailed answer and explanation.

The "Media" links appear at the conclusion of each section, just prior to the Section Exercises. These are a list of links to online video tutorials that reinforce the concepts and skills introduced in the section.

While we have selected tutorials that closely align to our learning objectives, we did not produce these tutorials, nor were they specifically produced or tailored to accompany Algebra and Trigonometry.

Section Exercises

Each section of every chapter concludes with a well-rounded set of exercises that can be assigned as homework or used selectively for guided practice. With over 6,300 exercises across the 9 chapters, instructors should have plenty to from which to choose[i].

Section Exercises are organized by question type, and generally appear in the following order:

Verbal questions assess conceptual understanding of key terms and concepts.

Algebraic problems require students to apply algebraic manipulations demonstrated in the section.

Graphical problems assess students' ability to interpret or produce a graph.

Numeric problems require the student perform calculations or computations.

Technology problems encourage exploration through use of a graphing utility, either to visualize or verify algebraic results or to solve problems via an alternative to the methods demonstrated in the section.

Extensions pose problems more challenging than the Examples demonstrated in the section. They require students to synthesize multiple learning objectives or apply critical thinking to solve complex problems.

Real-World Applications present realistic problem scenarios from fields such as physics, geology, biology, finance, and the social sciences.

Chapter Review Features

Each chapter concludes with a review of the most important takeaways, as well as additional practice problems that students can use to prepare for exams.

Key Terms provides a formal definition for each bold-faced term in the chapter.

Key Equations presents a compilation of formulas, theorems, and standard-form equations.

Key Concepts summarizes the most important ideas introduced in each section, linking back to the relevant Example(s) in case students need to review.

Chapter Review Exercises include 40-80 practice problems that recall the most important concepts from each section.

Practice Test includes 25-50 problems assessing the most important learning objectives from the chapter. Note that the practice test is not organized by section, and may be more heavily weighted toward cumulative objectives as opposed to the foundational objectives covered in the opening sections.

Additional Resources

Student and Instructor Resources

We've compiled additional resources for both students and instructors, including Getting Started Guides, instructor solution manual, and PowerPoint slides. Instructor resources require a verified instructor account, which can be requested on your openstax.org log-in. Take advantage of these resources to supplement your OpenStax book.

Partner Resources

OpenStax Partners are our allies in the mission to make high-quality learning materials affordable and accessible to students and instructors everywhere. Their tools integrate seamlessly with our OpenStax titles at a low cost. To access the partner resources for your text, visit your book page on openstax.org.

Online Homework

XYZ Homework

XYZ Homework is built using the fastest-growing mathematics cloud platform. XYZ Homework gives instructors access to the Precalculus aligned problems, organized in the Algebra and Trigonometry Course Template. Instructors have access to thousands of additional algorithmically-generated questions for unparalleled course customization. For one low annual price, students can take multiple classes through XYZ Homework. Learn more at www.xyzhomework.com/openstax.

WEBASSIGN

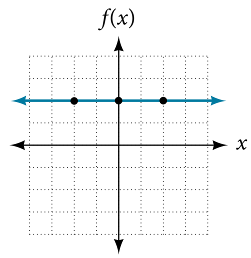
WebAssign is an independent online homework and assessment solution first launched at North Carolina State University in 1997. Today, WebAssign is an employee-owned benefit corporation and participates in the education of over a million students each year. WebAssign empowers faculty to deliver fully customizable assignments and high quality content to their students in an interactive online environment. WebAssign supports Algebra and Trigonometry with hundreds of problems covering every concept in the course, each containing algorithmically-generated values and links directly to the eBook providing a completely integrated online learning experience.

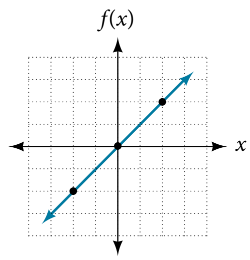
i. 6,367 total exercises. Includes Chapter Reviews and Practice Tests.

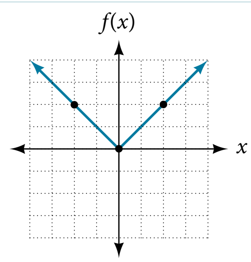
Identifying Basic Toolkit Functions

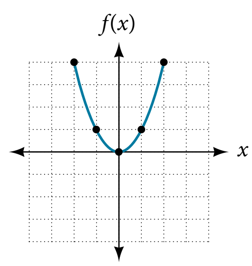
In this text, we will be exploring functions—the shapes of their graphs, their unique characteristics, their algebraic formulas, and how to solve problems with them. When learning to read, we start with the alphabet. When learning to do arithmetic, we start with numbers. When working with functions, it is similarly helpful to have a base set of building-block elements. We call these our "toolkit functions," which form a set of basic named functions for which we know the graph, formula, and special properties. Some of these functions are programmed to individual buttons on many calculators. For these definitions we will use x as the input variable and $y = f(x)$ as the output variable.

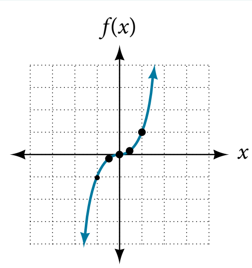
We will see these toolkit functions, combinations of toolkit functions, their graphs, and their transformations frequently throughout this book. It will be very helpful if we can recognize these toolkit functions and their features quickly by name, formula, graph, and basic table properties. The graphs and sample table values are included with each function shown in **Table 13**.

<table>
<tr><th colspan="3">Toolkit Functions</th></tr>
<tr><th>Name</th><th>Function</th><th>Graph</th></tr>
<tr><td>Constant</td><td>$f(x) = c$, where c is a constant</td><td>

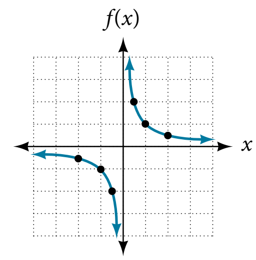
<table><tr><th>x</th><th>f(x)</th></tr><tr><td>–2</td><td>2</td></tr><tr><td>0</td><td>2</td></tr><tr><td>2</td><td>2</td></tr></table></td></tr>
<tr><td>Identity</td><td>$f(x) = x$</td><td>

<table><tr><th>x</th><th>f(x)</th></tr><tr><td>–2</td><td>–2</td></tr><tr><td>0</td><td>0</td></tr><tr><td>2</td><td>2</td></tr></table></td></tr>
<tr><td>Absolute value</td><td>$f(x) = \lvert x \rvert$</td><td>

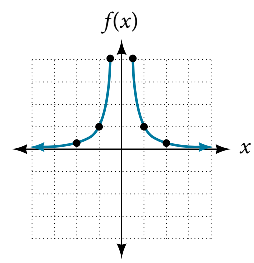
<table><tr><th>x</th><th>f(x)</th></tr><tr><td>–2</td><td>2</td></tr><tr><td>0</td><td>0</td></tr><tr><td>2</td><td>2</td></tr></table></td></tr>
<tr><td>Quadratic</td><td>$f(x) = x^2$</td><td>

<table><tr><th>x</th><th>f(x)</th></tr><tr><td>–2</td><td>4</td></tr><tr><td>–1</td><td>1</td></tr><tr><td>0</td><td>0</td></tr><tr><td>1</td><td>1</td></tr><tr><td>2</td><td>4</td></tr></table></td></tr>
<tr><td>Cubic</td><td>$f(x) = x^3$</td><td>

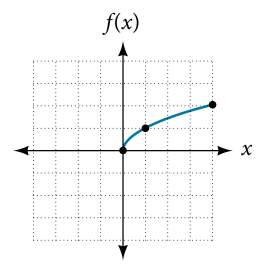
<table><tr><th>x</th><th>f(x)</th></tr><tr><td>–1</td><td>–1</td></tr><tr><td>–0.5</td><td>–0.125</td></tr><tr><td>0</td><td>0</td></tr><tr><td>0.5</td><td>0.125</td></tr><tr><td>1</td><td>1</td></tr></table></td></tr>
</table>

<table>
<tr><td>Reciprocal</td><td>$f(x) = \frac{1}{x}$</td><td>
</td><td>

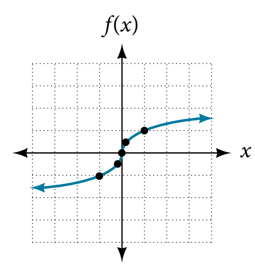
x	$f(x)$
–2	–0.5
–1	–1
–0.5	–2
0.5	2
1	1
2	0.5

</td></tr>
<tr><td>Reciprocal squared</td><td>$f(x) = \frac{1}{x^2}$</td><td>
</td><td>

x	$f(x)$
–2	0.25
–1	1
–0.5	4
0.5	4
1	1
2	0.25

</td></tr>
<tr><td>Square root</td><td>$f(x) = \sqrt{x}$</td><td>
</td><td>

x	$f(x)$
0	0
1	1
4	2

</td></tr>
<tr><td>Cube root</td><td>$f(x) = \sqrt[3]{x}$</td><td>
</td><td>

x	$f(x)$
–1	–1
–0.125	–0.5
0	0
0.125	0.5
1	1

</td></tr>
</table>

Table 13

Access the following online resources for additional instruction and practice with functions.

- Determine if a Relation is a Function (http://openstaxcollege.org/l/relationfunction)
- Vertical Line Test (http://openstaxcollege.org/l/vertlinetest)
- Introduction to Functions (http://openstaxcollege.org/l/introtofunction)
- Vertical Line Test of Graph (http://openstaxcollege.org/l/vertlinegraph)
- One-to-one Functions (http://openstaxcollege.org/l/onetoone)
- Graphs as One-to-one Functions (http://openstaxcollege.org/l/graphonetoone)

LEARNING OBJECTIVES

In this section, you will:

- Graph functions using vertical and horizontal shifts.
- Graph functions using reflections about the *x*-axis and the *y*-axis.
- Determine whether a function is even, odd, or neither from its graph.
- Graph functions using compressions and stretches.
- Combine transformations.

3.5 TRANSFORMATION OF FUNCTIONS

Figure 1 (credit: "Misko"/Flickr)

In Sections 3.5 and 3.7 and the corresponding online HW, you are expected to know about the "Toolkit Functions". They are listed and discussed on pages 173-175. Please look at them and understand them.

We all know that a flat mirror enables us to see an accurate image of ourselves and whatever is behind us. When we tilt the mirror, the images we see may shift horizontally or vertically. But what happens when we bend a flexible mirror? Like a carnival funhouse mirror, it presents us with a distorted image of ourselves, stretched or compressed horizontally or vertically. In a similar way, we can distort or transform mathematical functions to better adapt them to describing objects or processes in the real world. In this section, we will take a look at several kinds of transformations.

Graphing Functions Using Vertical and Horizontal Shifts

Often when given a problem, we try to model the scenario using mathematics in the form of words, tables, graphs, and equations. One method we can employ is to adapt the basic graphs of the toolkit functions to build new models for a given scenario. There are systematic ways to alter functions to construct appropriate models for the problems we are trying to solve.

Identifying Vertical Shifts

One simple kind of transformation involves shifting the entire graph of a function up, down, right, or left. The simplest shift is a **vertical shift**, moving the graph up or down, because this transformation involves adding a positive or negative constant to the function. In other words, we add the same constant to the output value of the function regardless of the input. For a function $g(x) = f(x) + k$, the function $f(x)$ is shifted vertically k units. See **Figure 2** for an example.

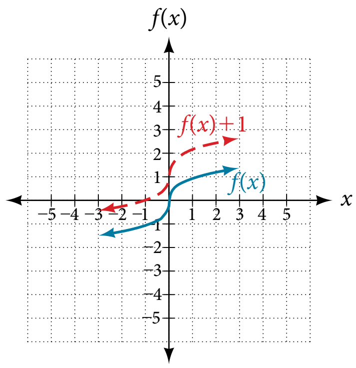


Figure 2 Vertical shift by $k = 1$ of the cube root function $f(x) = \sqrt[3]{x}$.

To help you visualize the concept of a vertical shift, consider that $y = f(x)$. Therefore, $f(x) + k$ is equivalent to $y + k$. Every unit of y is replaced by $y + k$, so the y-value increases or decreases depending on the value of k. The result is a shift upward or downward.

vertical shift

Given a function $f(x)$, a new function $g(x) = f(x) + k$, where k is a constant, is a **vertical shift** of the function $f(x)$. All the output values change by k units. If k is positive, the graph will shift up. If k is negative, the graph will shift down.

Example 1 **Adding a Constant to a Function**

To regulate temperature in a green building, airflow vents near the roof open and close throughout the day. **Figure 3** shows the area of open vents V (in square feet) throughout the day in hours after midnight, t. During the summer, the facilities manager decides to try to better regulate temperature by increasing the amount of open vents by 20 square feet throughout the day and night. Sketch a graph of this new function.

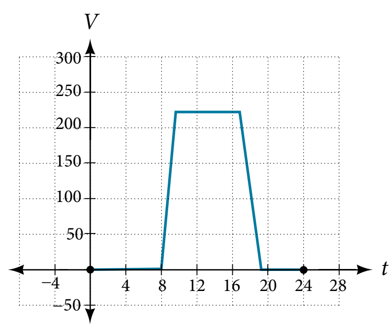


Figure 3

Solution We can sketch a graph of this new function by adding 20 to each of the output values of the original function. This will have the effect of shifting the graph vertically up, as shown in **Figure 4**.

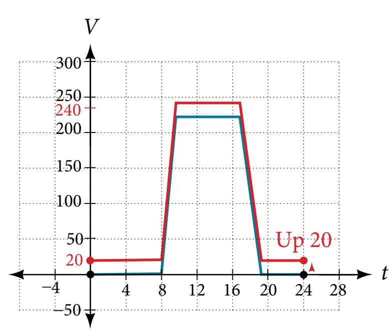


Figure 4

Notice that in **Figure 4**, for each input value, the output value has increased by 20, so if we call the new function $S(t)$, we could write

$$S(t) = V(t) + 20$$

This notation tells us that, for any value of t, $S(t)$ can be found by evaluating the function V at the same input and then adding 20 to the result. This defines S as a transformation of the function V, in this case a vertical shift up 20 units. Notice that, with a vertical shift, the input values stay the same and only the output values change. See **Table 1**.

t	0	8	10	17	19	24
$V(t)$	0	0	220	220	0	0
$S(t)$	20	20	240	240	20	20

Table 1

How To...

Given a tabular function, create a new row to represent a vertical shift.

1. Identify the output row or column.
2. Determine the magnitude of the shift.
3. Add the shift to the value in each output cell. Add a positive value for up or a negative value for down.

Example 2 Shifting a Tabular Function Vertically

A function $f(x)$ is given in **Table 2**. Create a table for the function $g(x) = f(x) - 3$.

x	2	4	6	8
$f(x)$	1	3	7	11

Table 2

Solution The formula $g(x) = f(x) - 3$ tells us that we can find the output values of g by subtracting 3 from the output values of f. For example:

$$\begin{aligned} f(2) &= 1 && \text{Given} \\ g(x) &= f(x) - 3 && \text{Given transformation} \\ g(2) &= f(2) - 3 \\ &= 1 - 3 \\ &= -2 \end{aligned}$$

Subtracting 3 from each $f(x)$ value, we can complete a table of values for $g(x)$ as shown in **Table 3**.

x	2	4	6	8
$f(x)$	1	3	7	11
$g(x)$	-2	0	4	8

Table 3

Analysis *As with the earlier vertical shift, notice the input values stay the same and only the output values change.*

Try It #1

The function $h(t) = -4.9t^2 + 30t$ gives the height h of a ball (in meters) thrown upward from the ground after t seconds. Suppose the ball was instead thrown from the top of a 10-m building. Relate this new height function $b(t)$ to $h(t)$, and then find a formula for $b(t)$.

Identifying Horizontal Shifts

We just saw that the vertical shift is a change to the output, or outside, of the function. We will now look at how changes to input, on the inside of the function, change its graph and meaning. A shift to the input results in a movement of the graph of the function left or right in what is known as a **horizontal shift**, shown in **Figure 5**.

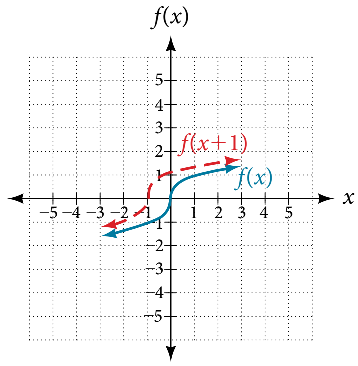


Figure 5 Horizontal shift of the function $f(x) = \sqrt[3]{x}$. Note that $h = +1$ shifts the graph to the left, that is, towards negative values of x.

For example, if $f(x) = x^2$, then $g(x) = (x - 2)^2$ is a new function. Each input is reduced by 2 prior to squaring the function. The result is that the graph is shifted 2 units to the right, because we would need to increase the prior input by 2 units to yield the same output value as given in f.

> ***horizontal shift***
> Given a function f, a new function $g(x) = f(x - h)$, where h is a constant, is a **horizontal shift** of the function f. If h is positive, the graph will shift right. If h is negative, the graph will shift left.

Example 3 Adding a Constant to an Input

Returning to our building airflow example from **Figure 3**, suppose that in autumn the facilities manager decides that the original venting plan starts too late, and wants to begin the entire venting program 2 hours earlier. Sketch a graph of the new function.

Solution We can set $V(t)$ to be the original program and $F(t)$ to be the revised program.

$$V(t) = \text{the original venting plan}$$

$$F(t) = \text{starting 2 hrs sooner}$$

In the new graph, at each time, the airflow is the same as the original function V was 2 hours later. For example, in the original function V, the airflow starts to change at 8 a.m., whereas for the function F, the airflow starts to change at 6 a.m. The comparable function values are $V(8) = F(6)$. See **Figure 6**. Notice also that the vents first opened to 220 ft^2 at 10 a.m. under the original plan, while under the new plan the vents reach 220 ft^2 at 8 a.m., so $V(10) = F(8)$.

In both cases, we see that, because $F(t)$ starts 2 hours sooner, $h = -2$. That means that the same output values are reached when $F(t) = V(t - (-2)) = V(t + 2)$.

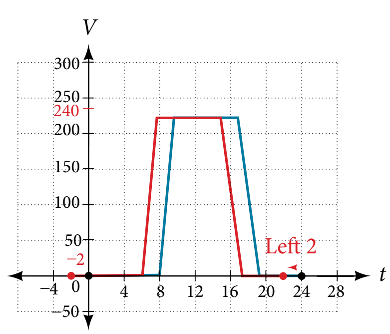


Figure 6

Analysis *Note that $V(t + 2)$ has the effect of shifting the graph to the* left.

Horizontal changes or "inside changes" affect the domain of a function (the input) instead of the range and often seem counterintuitive. The new function F(t) uses the same outputs as V(t), but matches those outputs to inputs 2 hours earlier than those of V(t). Said another way, we must add 2 hours to the input of V to find the corresponding output for $F : F(t) = V(t + 2)$.

How To...

Given a tabular function, create a new row to represent a horizontal shift.

1. Identify the input row or column.
2. Determine the magnitude of the shift.
3. Add the shift to the value in each input cell.

Example 4 Shifting a Tabular Function Horizontally

A function $f(x)$ is given in **Table 4**. Create a table for the function $g(x) = f(x - 3)$.

x	2	4	6	8
$f(x)$	1	3	7	11

Table 4

Solution The formula $g(x) = f(x - 3)$ tells us that the output values of g are the same as the output value of f when the input value is 3 less than the original value. For example, we know that $f(2) = 1$. To get the same output from the function g, we will need an input value that is 3 *larger*. We input a value that is 3 larger for $g(x)$ because the function takes 3 away before evaluating the function f.

$$\begin{aligned} g(5) &= f(5 - 3) \\ &= f(2) \\ &= 1 \end{aligned}$$

We continue with the other values to create **Table 5**.

x	5	7	9	11
$x - 3$	2	4	6	8
$f(x - 3)$	1	3	7	11
$g(x)$	1	3	7	11

Table 5

The result is that the function $g(x)$ has been shifted to the right by 3. Notice the output values for $g(x)$ remain the same as the output values for $f(x)$, but the corresponding input values, x, have shifted to the right by 3. Specifically, 2 shifted to 5, 4 shifted to 7, 6 shifted to 9, and 8 shifted to 11.

Analysis ***Figure 7*** *represents both of the functions. We can see the horizontal shift in each point.*

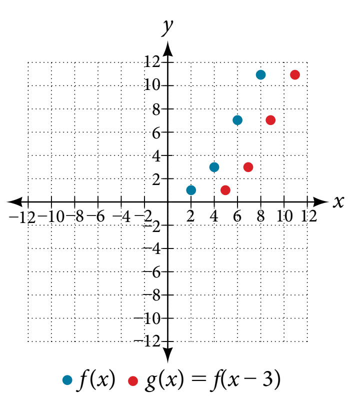


Figure 7

Example 5 **Identifying a Horizontal Shift of a Toolkit Function**

Figure 8 represents a transformation of the toolkit function $f(x) = x^2$. Relate this new function $g(x)$ to $f(x)$, and then find a formula for $g(x)$.

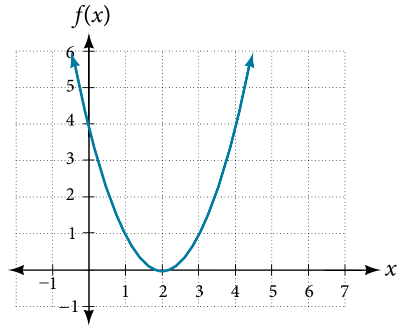


Figure 8

Solution Notice that the graph is identical in shape to the $f(x) = x^2$ function, but the x-values are shifted to the right 2 units. The vertex used to be at (0,0), but now the vertex is at (2,0). The graph is the basic quadratic function shifted 2 units to the right, so

$$g(x) = f(x - 2)$$

Notice how we must input the value $x = 2$ to get the output value $y = 0$; the x-values must be 2 units larger because of the shift to the right by 2 units. We can then use the definition of the $f(x)$ function to write a formula for $g(x)$ by evaluating $f(x - 2)$.

$$f(x) = x^2$$
$$g(x) = f(x - 2)$$
$$g(x) = f(x - 2) = (x - 2)^2$$

Analysis *To determine whether the shift is $+2$ or -2, consider a single reference point on the graph. For a quadratic, looking at the vertex point is convenient. In the original function, $f(0) = 0$. In our shifted function, $g(2) = 0$. To obtain the output value of 0 from the function f, we need to decide whether a plus or a minus sign will work to satisfy $g(2) = f(x - 2) = f(0) = 0$. For this to work, we will need to subtract 2 units from our input values.*

Example 6 **Interpreting Horizontal versus Vertical Shifts**

The function $G(m)$ gives the number of gallons of gas required to drive m miles. Interpret $G(m) + 10$ and $G(m + 10)$.

Solution $G(m) + 10$ can be interpreted as adding 10 to the output, gallons. This is the gas required to drive m miles, plus another 10 gallons of gas. The graph would indicate a vertical shift.

$G(m + 10)$ can be interpreted as adding 10 to the input, miles. So this is the number of gallons of gas required to drive 10 miles more than m miles. The graph would indicate a horizontal shift.

Try It #2

Given the function $f(x) = \sqrt{x}$, graph the original function $f(x)$ and the transformation $g(x) = f(x + 2)$ on the same axes. Is this a horizontal or a vertical shift? Which way is the graph shifted and by how many units?

Combining Vertical and Horizontal Shifts

Now that we have two transformations, we can combine them. Vertical shifts are outside changes that affect the output (y-) axis values and shift the function up or down. Horizontal shifts are inside changes that affect the input (x-) axis values and shift the function left or right. Combining the two types of shifts will cause the graph of a function to shift up or down *and* right or left.

How To...

Given a function and both a vertical and a horizontal shift, sketch the graph.

1. Identify the vertical and horizontal shifts from the formula.
2. The vertical shift results from a constant added to the output. Move the graph up for a positive constant and down for a negative constant.
3. The horizontal shift results from a constant added to the input. Move the graph left for a positive constant and right for a negative constant.
4. Apply the shifts to the graph in either order.

Example 7 Graphing Combined Vertical and Horizontal Shifts

Given $f(x) = |x|$, sketch a graph of $h(x) = f(x + 1) - 3$.

Solution The function f is our toolkit absolute value function. We know that this graph has a V shape, with the point at the origin. The graph of h has transformed f in two ways: $f(x + 1)$ is a change on the inside of the function, giving a horizontal shift left by 1, and the subtraction by 3 in $f(x + 1) - 3$ is a change to the outside of the function, giving a vertical shift down by 3. The transformation of the graph is illustrated in **Figure 9**.

Let us follow one point of the graph of $f(x) = |x|$.

- The point (0, 0) is transformed first by shifting left 1 unit: $(0, 0) \to (-1, 0)$
- The point (−1, 0) is transformed next by shifting down 3 units: $(-1, 0) \to (-1, -3)$

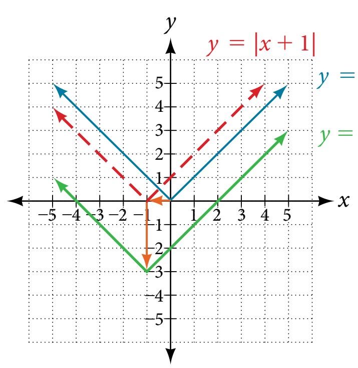


Figure 9

Figure 10 shows the graph of h.

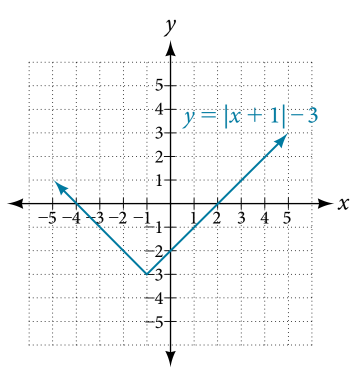


Figure 10

Try It #3

Given $f(x) = |x|$, sketch a graph of $h(x) = f(x - 2) + 4$.

Example 8 **Identifying Combined Vertical and Horizontal Shifts**

Write a formula for the graph shown in **Figure 11**, which is a transformation of the toolkit square root function.

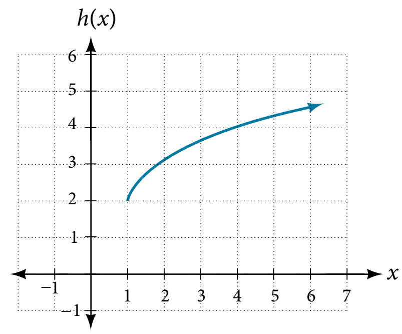


Figure 11

Solution The graph of the toolkit function starts at the origin, so this graph has been shifted 1 to the right and up 2. In function notation, we could write that as

$$h(x) = f(x - 1) + 2$$

Using the formula for the square root function, we can write

$$h(x) = \sqrt{x - 1} + 2$$

Analysis *Note that this transformation has changed the domain and range of the function. This new graph has domain* $[1, \infty)$ *and range* $[2, \infty)$.

Try It #4

Write a formula for a transformation of the toolkit reciprocal function $f(x) = \dfrac{1}{x}$ that shifts the function's graph one unit to the right and one unit up.

Graphing Functions Using Reflections about the Axes

Another transformation that can be applied to a function is a reflection over the x- or y-axis. A **vertical reflection** reflects a graph vertically across the x-axis, while a **horizontal reflection** reflects a graph horizontally across the y-axis. The reflections are shown in **Figure 12**.

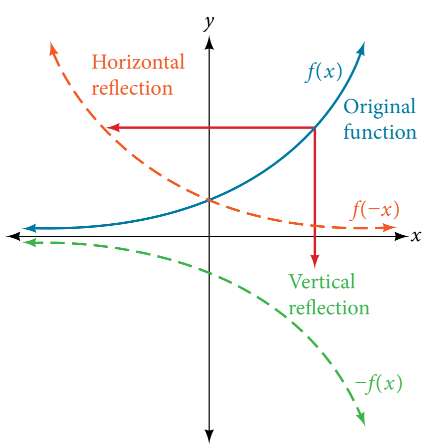


Figure 12 Vertical and horizontal reflections of a function.

Notice that the vertical reflection produces a new graph that is a mirror image of the base or original graph about the x-axis. The horizontal reflection produces a new graph that is a mirror image of the base or original graph about the y-axis.

reflections

Given a function $f(x)$, a new function $g(x) = -f(x)$ is a **vertical reflection** of the function $f(x)$, sometimes called a reflection about (or over, or through) the x-axis.

Given a function $f(x)$, a new function $g(x) = f(-x)$ is a **horizontal reflection** of the function $f(x)$, sometimes called a reflection about the y-axis.

How To...

Given a function, reflect the graph both vertically and horizontally.

1. Multiply all outputs by -1 for a vertical reflection. The new graph is a reflection of the original graph about the x-axis.
2. Multiply all inputs by -1 for a horizontal reflection. The new graph is a reflection of the original graph about the y-axis.

Example 9 Reflecting a Graph Horizontally and Vertically

Reflect the graph of $s(t) = \sqrt{t}$ **a.** vertically and **b.** horizontally.

Solution

a. Reflecting the graph vertically means that each output value will be reflected over the horizontal t-axis as shown in **Figure 13**.

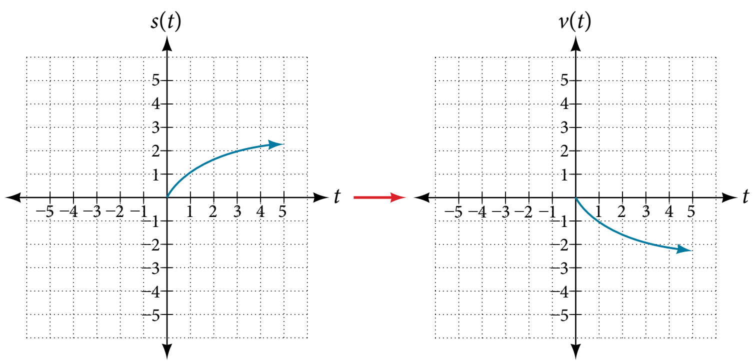


Figure 13 Vertical reflection of the square root function

Because each output value is the opposite of the original output value, we can write

$$V(t) = -s(t) \text{ or } V(t) = -\sqrt{t}$$

Notice that this is an outside change, or vertical shift, that affects the output $s(t)$ values, so the negative sign belongs outside of the function.

b. Reflecting horizontally means that each input value will be reflected over the vertical axis as shown in **Figure 14**.

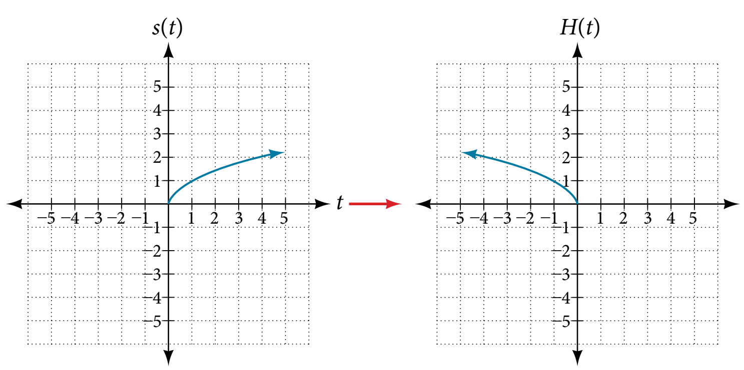


Figure 14 Horizontal reflection of the square root function

Because each input value is the opposite of the original input value, we can write

$$H(t) = s(-t) \text{ or } H(t) = \sqrt{-t}$$

Notice that this is an inside change or horizontal change that affects the input values, so the negative sign is on the inside of the function.

Note that these transformations can affect the domain and range of the functions. While the original square root function has domain $[0, \infty)$ and range $[0, \infty)$, the vertical reflection gives the $V(t)$ function the range $(-\infty, 0]$ and the horizontal reflection gives the $H(t)$ function the domain $(-\infty, 0]$.

Try It #5

Reflect the graph of $f(x) = |x - 1|$ **a.** vertically and **b.** horizontally.

Example 10 Reflecting a Tabular Function Horizontally and Vertically

A function $f(x)$ is given as **Table 6**. Create a table for the functions below.

a. $g(x) = -f(x)$ **b.** $h(x) = f(-x)$

x	2	4	6	8
$f(x)$	1	3	7	11

Table 6

Solution

a. For $g(x)$, the negative sign outside the function indicates a vertical reflection, so the x-values stay the same and each output value will be the opposite of the original output value. See **Table 7**.

x	2	4	6	8
$g(x)$	–1	–3	–7	–11

Table 7

b. For $h(x)$, the negative sign inside the function indicates a horizontal reflection, so each input value will be the opposite of the original input value and the $h(x)$ values stay the same as the $f(x)$ values. See **Table 8**.

x	–2	–4	–6	–8
$h(x)$	1	3	7	11

Table 8

Try It #6

A function $f(x)$ is given as **Table 9**. Create a table for the functions below.

x	–2	0	2	4
$f(x)$	5	10	15	20

Table 9

a. $g(x) = -f(x)$

b. $h(x) = f(-x)$

Example 11 Applying a Learning Model Equation

A common model for learning has an equation similar to $k(t) = -2^{-t} + 1$, where k is the percentage of mastery that can be achieved after t practice sessions. This is a transformation of the function $f(t) = 2^t$ shown in **Figure 15**. Sketch a graph of $k(t)$.

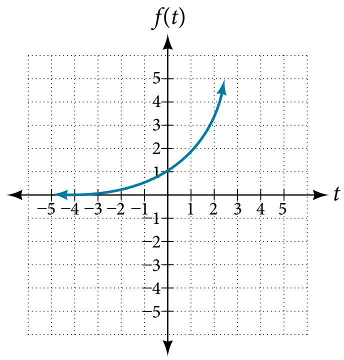


Figure 15

Solution This equation combines three transformations into one equation.

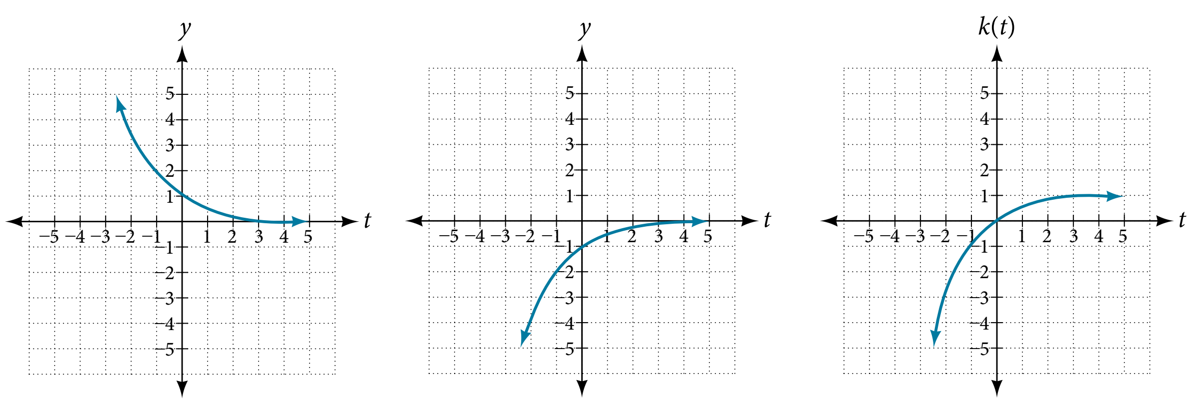
- A horizontal reflection: $f(-t) = 2^{-t}$
- A vertical reflection: $-f(-t) = -2^{-t}$
- A vertical shift: $-f(-t) + 1 = -2^{-t} + 1$

We can sketch a graph by applying these transformations one at a time to the original function. Let us follow two points through each of the three transformations. We will choose the points (0, 1) and (1, 2).

1. First, we apply a horizontal reflection: (0, 1) (−1, 2).
2. Then, we apply a vertical reflection: (0, −1) $\boxed{(-1, -2).}$
3. Finally, we apply a vertical shift: (0, 0) $\boxed{(-1, -1).}$

This means that the original points, (0,1) and (1,2) become (0,0) and ~~(1,1)~~ $\boxed{(-1,-1)}$ after we apply the transformations.

In **Figure 16**, the first graph results from a horizontal reflection. The second results from a vertical reflection. The third results from a vertical shift up 1 unit.



(a) (b) (c)

Figure 16

Analysis *As a model for learning, this function would be limited to a domain of* $t \geq 0$, *with corresponding range* $[0, 1)$.

Try It #7

Given the toolkit function $f(x) = x^2$, graph $g(x) = -f(x)$ and $h(x) = f(-x)$. Take note of any surprising behavior for these functions.

Determining Even and Odd Functions

Some functions exhibit symmetry so that reflections result in the original graph. For example, horizontally reflecting the toolkit functions $f(x) = x^2$ or $f(x) = |x|$ will result in the original graph. We say that these types of graphs are symmetric about the y-axis. A function whose graph is symmetric about the y-axis is called an **even function**.

If the graphs of $f(x) = x^3$ or $f(x) = \frac{1}{x}$ were reflected over *both* axes, the result would be the original graph, as shown in **Figure 17**.

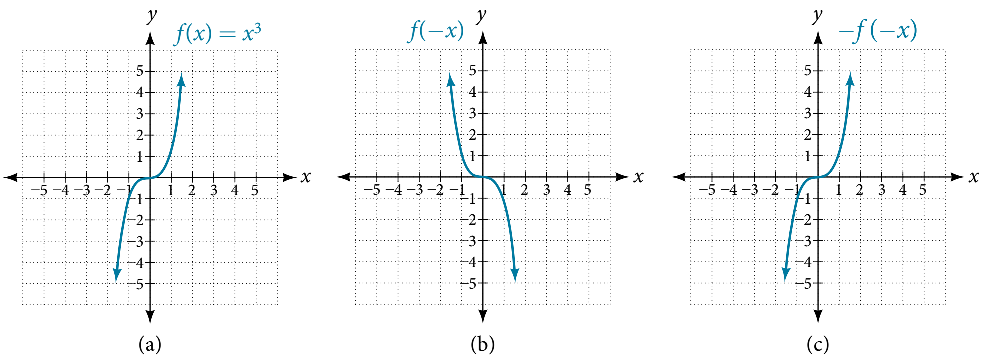


Figure 17 (a) The cubic toolkit function (b) Horizontal reflection of the cubic toolkit function (c) Horizontal and vertical reflections reproduce the original cubic function.

We say that these graphs are symmetric about the origin. A function with a graph that is symmetric about the origin is called an **odd function**.

Note: A function can be neither even nor odd if it does not exhibit either symmetry. For example, $f(x) = 2^x$ is neither even nor odd. Also, the only function that is both even and odd is the constant function $f(x) = 0$.

> ***even and odd functions***
>
> A function is called an **even function** if for every input x: $f(x) = f(-x)$
>
> The graph of an even function is symmetric about the y-axis.
>
> A function is called an **odd function** if for every input x: $f(x) = -f(-x)$
>
> The graph of an odd function is symmetric about the origin.

How To...

Given the formula for a function, determine if the function is even, odd, or neither.

1. Determine whether the function satisfies $f(x) = f(-x)$. If it does, it is even.
2. Determine whether the function satisfies $f(x) = -f(-x)$. If it does, it is odd.
3. If the function does not satisfy either rule, it is neither even nor odd.

Example 12 Determining whether a Function Is Even, Odd, or Neither

Is the function $f(x) = x^3 + 2x$ even, odd, or neither?

Solution Without looking at a graph, we can determine whether the function is even or odd by finding formulas for the reflections and determining if they return us to the original function. Let's begin with the rule for even functions.

$$f(-x) = (-x)^3 + 2(-x) = -x^3 - 2x$$

This does not return us to the original function, so this function is not even. We can now test the rule for odd functions.

$$-f(-x) = -(-x^3 - 2x) = x^3 + 2x$$

Because $-f(-x) = f(x)$, this is an odd function.

Analysis *Consider the graph of f in* ***Figure 18****. Notice that the graph is symmetric about the origin. For every point (x, y) on the graph, the corresponding point* $(-x, -y)$ *is also on the graph. For example,* $(1, 3)$ *is on the graph of f, and the corresponding point* $(-1, -3)$ *is also on the graph.*

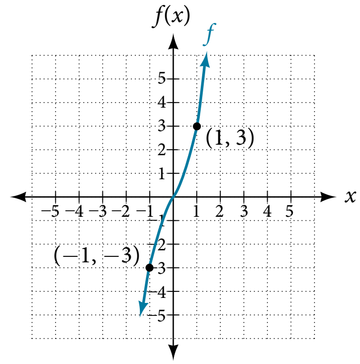


Figure 18

Try It #8

Is the function $f(s) = s^4 + 3s^2 + 7$ even, odd, or neither?

Graphing Functions Using Stretches and Compressions

Adding a constant to the inputs or outputs of a function changed the position of a graph with respect to the axes, but it did not affect the shape of a graph. We now explore the effects of multiplying the inputs or outputs by some quantity.

We can transform the inside (input values) of a function or we can transform the outside (output values) of a function. Each change has a specific effect that can be seen graphically.

Vertical Stretches and Compressions

When we multiply a function by a positive constant, we get a function whose graph is stretched or compressed vertically in relation to the graph of the original function. If the constant is greater than 1, we get a **vertical stretch**; if the constant is between 0 and 1, we get a **vertical compression**. **Figure 19** shows a function multiplied by constant factors 2 and 0.5 and the resulting vertical stretch and compression.

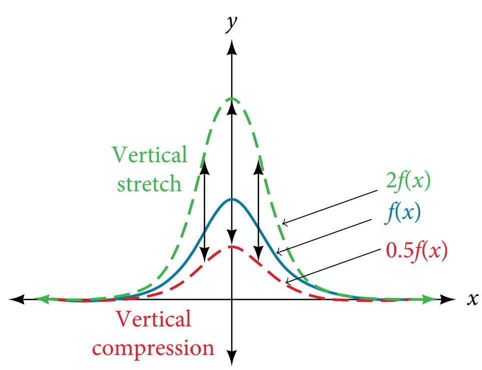


Figure 19 Vertical stretch and compression

vertical stretches and compressions

Given a function $f(x)$, a new function $g(x) = af(x)$, where a is a constant, is a **vertical stretch** or **vertical compression** of the function $f(x)$.

- If $a > 1$, then the graph will be stretched.
- If $0 < a < 1$, then the graph will be compressed.
- If $a < 0$, then there will be combination of a vertical stretch or compression with a vertical reflection.

How To…

Given a function, graph its vertical stretch.

1. Identify the value of a.
2. Multiply all range values by a.
3. If $a > 1$, the graph is stretched by a factor of a.
   If $0 < a < 1$, the graph is compressed by a factor of a.
   If $a < 0$, the graph is either stretched or compressed and also reflected about the x-axis.

Example 13 **Graphing a Vertical Stretch**

A function $P(t)$ models the population of fruit flies. The graph is shown in **Figure 20**.

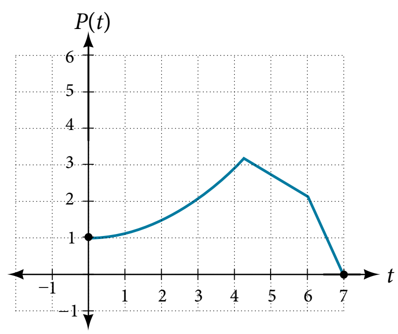


Figure 20

A scientist is comparing this population to another population, Q, whose growth follows the same pattern, but is twice as large. Sketch a graph of this population.

Solution Because the population is always twice as large, the new population's output values are always twice the original function's output values. Graphically, this is shown in **Figure 21**.

If we choose four reference points, (0, 1), (3, 3), (6, 2) and (7, 0) we will multiply all of the outputs by 2.

The following shows where the new points for the new graph will be located.

$$(0, 1) \rightarrow (0, 2)$$
$$(3, 3) \rightarrow (3, 6)$$
$$(6, 2) \rightarrow (6, 4)$$
$$(7, 0) \rightarrow (7, 0)$$

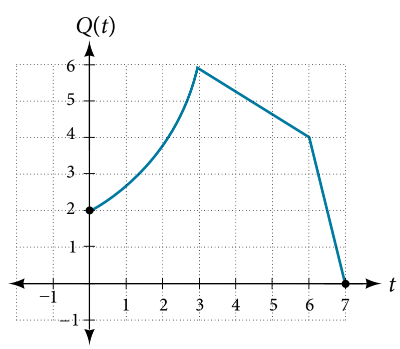


Figure 21

Symbolically, the relationship is written as

$$Q(t) = 2P(t)$$

This means that for any input t, the value of the function Q is twice the value of the function P. Notice that the effect on the graph is a vertical stretching of the graph, where every point doubles its distance from the horizontal axis. The input values, t, stay the same while the output values are twice as large as before.

How To...

Given a tabular function and assuming that the transformation is a vertical stretch or compression, create a table for a vertical compression.

1. Determine the value of a.
2. Multiply all of the output values by a.

Example 14 Finding a Vertical Compression of a Tabular Function

A function f is given as **Table 10**. Create a table for the function $g(x) = \frac{1}{2}f(x)$.

x	2	4	6	8
$f(x)$	1	3	7	11

Table 10

Solution The formula $g(x) = \frac{1}{2}f(x)$ tells us that the output values of g are half of the output values of f with the same inputs. For example, we know that $f(4) = 3$. Then

$$g(4) = \frac{1}{2}f(4) = \frac{1}{2}(3) = \frac{3}{2}$$

We do the same for the other values to produce **Table 11**.

x	2	4	6	8
$g(x)$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{7}{2}$	$\frac{11}{2}$

Table 11

Analysis *The result is that the function $g(x)$ has been compressed vertically by $\frac{1}{2}$. Each output value is divided in half, so the graph is half the original height.*

Try It #9

A function f is given as **Table 12**. Create a table for the function $g(x) = \frac{3}{4}f(x)$.

x	2	4	6	8
$f(x)$	12	16	20	0

Table 12

Example 15 Recognizing a Vertical Stretch

The graph in **Figure 22** is a transformation of the toolkit function $f(x) = x^3$. Relate this new function $g(x)$ to $f(x)$, and then find a formula for $g(x)$.

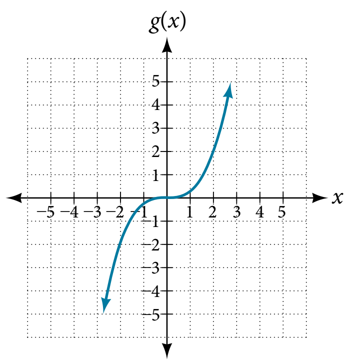


Figure 22

Solution When trying to determine a vertical stretch or shift, it is helpful to look for a point on the graph that is relatively clear. In this graph, it appears that $g(2) = 2$. With the basic cubic function at the same input, $f(2) = 2^3 = 8$. Based on that, it appears that the outputs of g are $\frac{1}{4}$ the outputs of the function f because $g(2) = \frac{1}{4}f(2)$. From this we can fairly safely conclude that $g(x) = \frac{1}{4}f(x)$.

We can write a formula for g by using the definition of the function f.

$$g(x) = \frac{1}{4}f(x) = \frac{1}{4}x^3$$

Try It #10

Write the formula for the function that we get when we stretch the identity toolkit function by a factor of 3, and then shift it down by 2 units.

Horizontal Stretches and Compressions

Now we consider changes to the inside of a function. When we multiply a function's input by a positive constant, we get a function whose graph is stretched or compressed horizontally in relation to the graph of the original function. If the constant is between 0 and 1, we get a **horizontal stretch**; if the constant is greater than 1, we get a **horizontal compression** of the function.

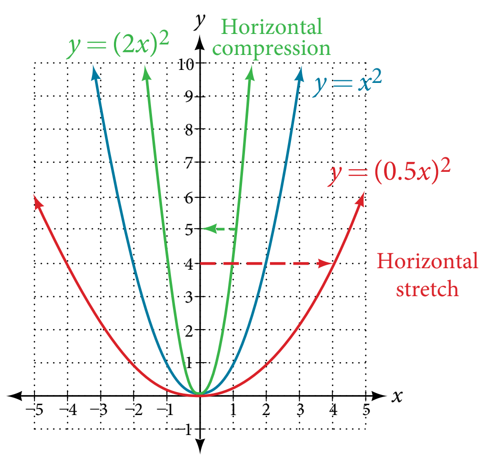


Figure 23

Given a function $y = f(x)$, the form $y = f(bx)$ results in a horizontal stretch or compression. Consider the function $y = x^2$. Observe **Figure 23**. The graph of $y = (0.5x)^2$ is a horizontal stretch of the graph of the function $y = x^2$ by a factor of 2. The graph of $y = (2x)^2$ is a horizontal compression of the graph of the function $y = x^2$ by a factor of 2.

horizontal stretches and compressions

Given a function $f(x)$, a new function $g(x) = f(bx)$, where b is a constant, is a **horizontal stretch** or **horizontal compression** of the function $f(x)$.

- If $b > 1$, then the graph will be compressed by $\frac{1}{b}$.
- If $0 < b < 1$, then the graph will be stretched by $\frac{1}{b}$.
- If $b < 0$, then there will be combination of a horizontal stretch or compression with a horizontal reflection.

How To…

Given a description of a function, sketch a horizontal compression or stretch.

1. Write a formula to represent the function.
2. Set $g(x) = f(bx)$ where $b > 1$ for a compression or $0 < b < 1$ for a stretch.

Example 16 Graphing a Horizontal Compression

Suppose a scientist is comparing a population of fruit flies to a population that progresses through its lifespan twice as fast as the original population. In other words, this new population, R, will progress in 1 hour the same amount as the original population does in 2 hours, and in 2 hours, it will progress as much as the original population does in 4 hours. Sketch a graph of this population.

Solution Symbolically, we could write

$$R(1) = P(2),$$
$$R(2) = P(4), \text{ and in general,}$$
$$R(t) = P(2t).$$

See **Figure 24** for a graphical comparison of the original population and the compressed population.

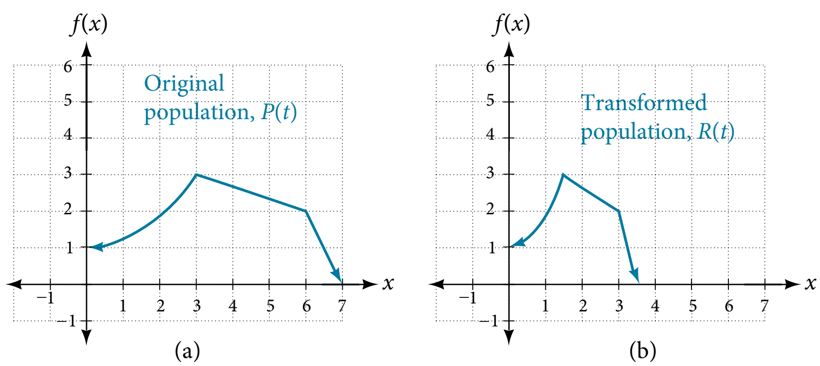


Figure 24 (a) Original population graph (b) Compressed population graph

Analysis *Note that the effect on the graph is a horizontal compression where all input values are half of their original distance from the vertical axis.*

Example 17 Finding a Horizontal Stretch for a Tabular Function

A function $f(x)$ is given as **Table 13**. Create a table for the function $g(x) = f\left(\frac{1}{2}x\right)$.

x	2	4	6	8
$f(x)$	1	3	7	11

Table 13

Solution The formula $g(x) = f\left(\frac{1}{2}x\right)$ tells us that the output values for g are the same as the output values for the function f at an input half the size. Notice that we do not have enough information to determine $g(2)$ because $g(2) = f\left(\frac{1}{2} \cdot 2\right) = f(1)$, and we do not have a value for $f(1)$ in our table. Our input values to g will need to be twice as large to get inputs for f that we can evaluate. For example, we can determine $g(4)$.

$$g(4) = f\left(\frac{1}{2} \cdot 4\right) = f(2) = 1$$

We do the same for the other values to produce **Table 14**.

x	4	8	12	16
$g(x)$	1	3	7	11

Table 14

Figure 25 shows the graphs of both of these sets of points.

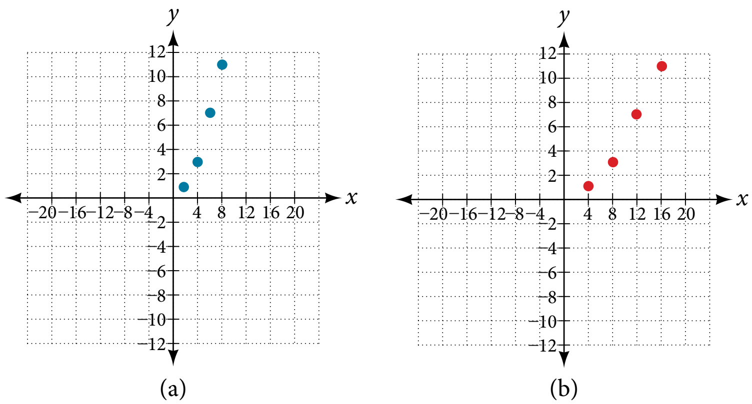


Figure 25

Analysis *Because each input value has been doubled, the result is that the function $g(x)$ has been stretched horizontally by a factor of* 2.

Example 18 Recognizing a Horizontal Compression on a Graph

Relate the function $g(x)$ to $f(x)$ in **Figure 26**.

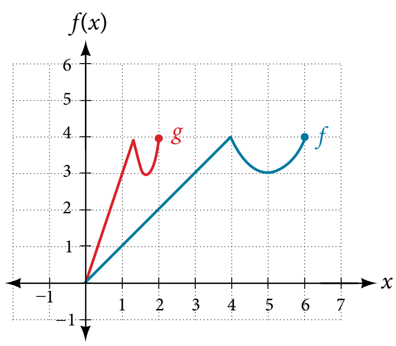


Figure 26

Solution The graph of $g(x)$ looks like the graph of $f(x)$ horizontally compressed. Because $f(x)$ ends at $(6, 4)$ and $g(x)$ ends at $(2, 4)$, we can see that the x-values have been compressed by $\frac{1}{3}$, because $6\left(\frac{1}{3}\right) = 2$. We might also notice that $g(2) = f(6)$ and $g(1) = f(3)$. Either way, we can describe this relationship as $g(x) = f(3x)$. This is a horizontal compression by $\frac{1}{3}$.

Analysis *Notice that the coefficient needed for a horizontal stretch or compression is the reciprocal of the stretch or compression. So to stretch the graph horizontally by a scale factor of* 4*, we need a coefficient of $\frac{1}{4}$ in our function: $f\left(\frac{1}{4}x\right)$. This means that the input values must be four times larger to produce the same result, requiring the input to be larger, causing the horizontal stretching.*

Try It #11

Write a formula for the toolkit square root function horizontally stretched by a factor of 3.

Performing a Sequence of Transformations

When combining transformations, it is very important to consider the order of the transformations. For example, vertically shifting by 3 and then vertically stretching by 2 does not create the same graph as vertically stretching by 2 and then vertically shifting by 3, because when we shift first, both the original function and the shift get stretched, while only the original function gets stretched when we stretch first.

When we see an expression such as $2f(x) + 3$, which transformation should we start with? The answer here follows nicely from the order of operations. Given the output value of $f(x)$, we first multiply by 2, causing the vertical stretch, and then add 3, causing the vertical shift. In other words, multiplication before addition.

Horizontal transformations are a little trickier to think about. When we write $g(x) = f(2x + 3)$, for example, we have to think about how the inputs to the function g relate to the inputs to the function f. Suppose we know $f(7) = 12$. What input to g would produce that output? In other words, what value of x will allow $g(x) = f(2x + 3) = 12$? We would need $2x + 3 = 7$. To solve for x, we would first subtract 3, resulting in a horizontal shift, and then divide by 2, causing a horizontal compression.

This format ends up being very difficult to work with, because it is usually much easier to horizontally stretch a graph before shifting. We can work around this by factoring inside the function.

$$f(bx + p) = f\left(b\left(x + \frac{p}{b}\right)\right)$$

Let's work through an example.

$$f(x) = (2x + 4)^2$$

We can factor out a 2.

$$f(x) = (2(x + 2))^2$$

Now we can more clearly observe a horizontal shift to the left 2 units and a horizontal compression. Factoring in this way allows us to horizontally stretch first and then shift horizontally.

combining transformations

When combining vertical transformations written in the form $af(x) + k$, first vertically stretch by a and then vertically shift by k.

When combining horizontal transformations written in the form $f(bx - h)$, first horizontally shift by h and then horizontally stretch by $\frac{1}{b}$.

When combining horizontal transformations written in the form $f(b(x - h))$, first horizontally stretch by $\frac{1}{b}$ and then horizontally shift by h.

Horizontal and vertical transformations are independent. It does not matter whether horizontal or vertical transformations are performed first.

Example 19 Finding a Triple Transformation of a Tabular Function

Given **Table 15** for the function $f(x)$, create a table of values for the function $g(x) = 2f(3x) + 1$.

x	6	12	18	24
$f(x)$	10	14	15	17

Table 15

Solution There are three steps to this transformation, and we will work from the inside out. Starting with the horizontal transformations, $f(3x)$ is a horizontal compression by $\frac{1}{3}$, which means we multiply each x-value by $\frac{1}{3}$. See **Table 16**.

x	2	4	6	8
$f(3x)$	10	14	15	17

Table 16

Looking now to the vertical transformations, we start with the vertical stretch, which will multiply the output values by 2. We apply this to the previous transformation. See **Table 17**.

x	2	4	6	8
$2f(3x)$	20	28	30	34

Table 17

Finally, we can apply the vertical shift, which will add 1 to all the output values. See **Table 18**.

x	2	4	6	8
$g(x) = 2f(3x) + 1$	21	29	31	35

Table 18

Example 20 Finding a Triple Transformation of a Graph

Use the graph of $f(x)$ in **Figure 27** to sketch a graph of $k(x) = f\left(\frac{1}{2}x + 1\right) - 3$.

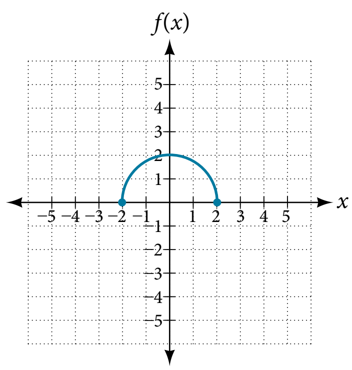


Figure 27

Solution To simplify, let's start by factoring out the inside of the function.

$$f\left(\frac{1}{2}x + 1\right) - 3 = f\left(\frac{1}{2}(x + 2)\right) - 3$$

By factoring the inside, we can first horizontally stretch by 2, as indicated by the $\frac{1}{2}$ on the inside of the function. Remember that twice the size of 0 is still 0, so the point (0, 2) remains at (0, 2) while the point (2, 0) will stretch to (4, 0). See **Figure 28**.

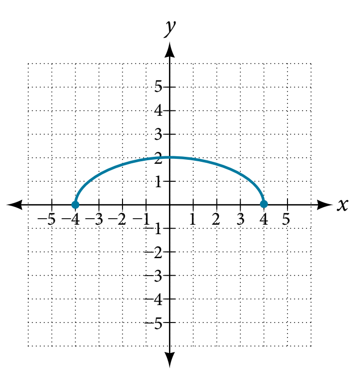


Figure 28

Next, we horizontally shift left by 2 units, as indicated by $x + 2$. See **Figure 29**.

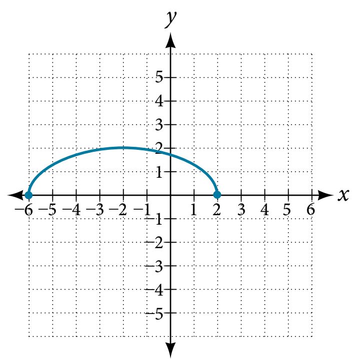


Figure 29

Last, we vertically shift down by 3 to complete our sketch, as indicated by the -3 on the outside of the function. See **Figure 30**.

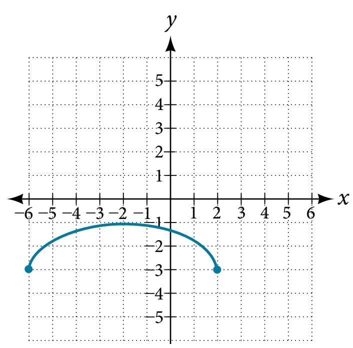


Figure 30

Access this online resource for additional instruction and practice with transformation of functions.

- Function Transformations (http://openstaxcollege.org/l/functrans)

3.5 SECTION EXERCISES

VERBAL

1. When examining the formula of a function that is the result of multiple transformations, how can you tell a horizontal shift from a vertical shift?

2. When examining the formula of a function that is the result of multiple transformations, how can you tell a horizontal stretch from a vertical stretch?

3. When examining the formula of a function that is the result of multiple transformations, how can you tell a horizontal compression from a vertical compression?

4. When examining the formula of a function that is the result of multiple transformations, how can you tell a reflection with respect to the x-axis from a reflection with respect to the y-axis?

5. How can you determine whether a function is odd or even from the formula of the function?

ALGEBRAIC

For the following exercises, write a formula for the function obtained when the graph is shifted as described.

6. $f(x) = \sqrt{x}$ is shifted up 1 unit and to the left 2 units.

7. $f(x) = |x|$ is shifted down 3 units and to the right 1 unit.

8. $f(x) = \dfrac{1}{x}$ is shifted down 4 units and to the right 3 units.

9. $f(x) = \dfrac{1}{x^2}$ is shifted up 2 units and to the left 4 units.

For the following exercises, describe how the graph of the function is a transformation of the graph of the original function f.

10. $y = f(x - 49)$

11. $y = f(x + 43)$

12. $y = f(x + 3)$

13. $y = f(x - 4)$

14. $y = f(x) + 5$

15. $y = f(x) + 8$

16. $y = f(x) - 2$

17. $y = f(x) - 7$

18. $y = f(x - 2) + 3$

19. $y = f(x + 4) - 1$

For the following exercises, determine the interval(*s*) on which the function is increasing and decreasing.

20. $f(x) = 4(x + 1)^2 - 5$

21. $g(x) = 5(x + 3)^2 - 2$

22. $a(x) = \sqrt{-x + 4}$

23. $k(x) = -3\sqrt{x} - 1$

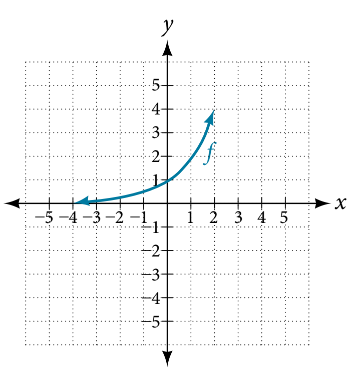


Figure 31

GRAPHICAL

For the following exercises, use the graph of $f(x) = 2^x$ shown in **Figure 31** to sketch a graph of each transformation of $f(x)$.

24. $g(x) = 2^x + 1$

25. $h(x) = 2^x - 3$

26. $w(x) = 2^{x-1}$

For the following exercises, sketch a graph of the function as a transformation of the graph of one of the toolkit functions.

27. $f(t) = (t + 1)^2 - 3$

28. $h(x) = |x - 1| + 4$

29. $k(x) = (x - 2)^3 - 1$

30. $m(t) = 3 + \sqrt{t + 2}$

NUMERIC

31. Tabular representations for the functions f, g, and h are given below. Write $g(x)$ and $h(x)$ as transformations of $f(x)$.

x	−2	−1	0	1	2
$f(x)$	−2	−1	−3	1	2

x	−1	0	1	2	3
$g(x)$	−2	−1	−3	1	2

x	−2	−1	0	1	2
$h(x)$	−1	0	−2	2	3

32. Tabular representations for the functions f, g, and h are given below. Write $g(x)$ and $h(x)$ as transformations of $f(x)$.

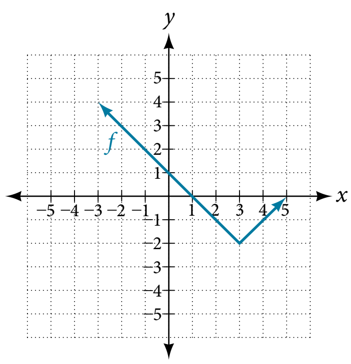
x	−2	−1	0	1	2
$f(x)$	−1	−3	4	2	1

x	−3	−2	−1	0	1
$g(x)$	−1	−3	4	2	1

x	−2	−1	0	1	2
$h(x)$	−2	−4	3	1	0

For the following exercises, write an equation for each graphed function by using transformations of the graphs of one of the toolkit functions.

33.



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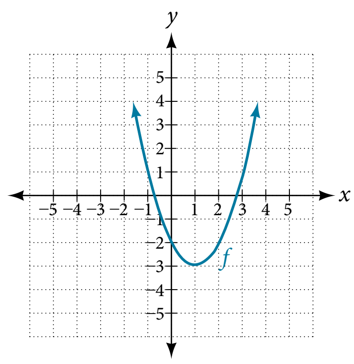

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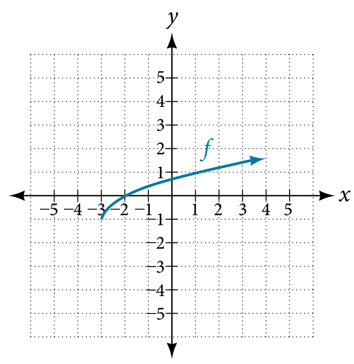

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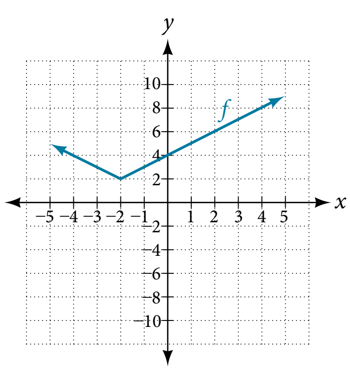

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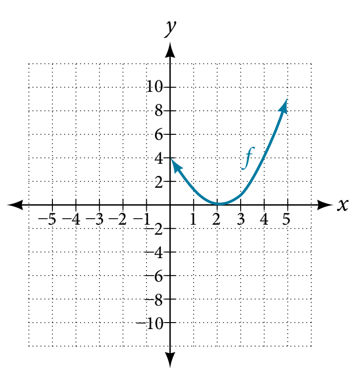

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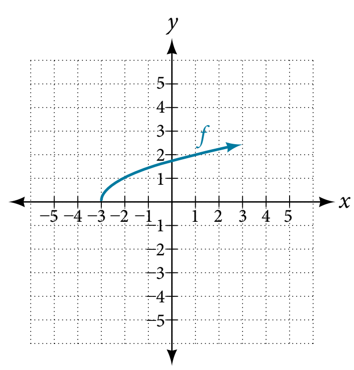

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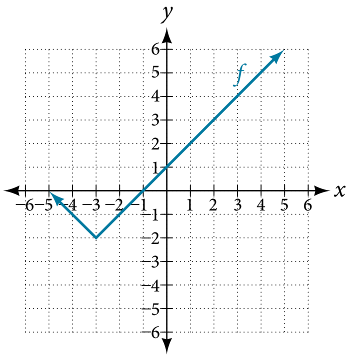

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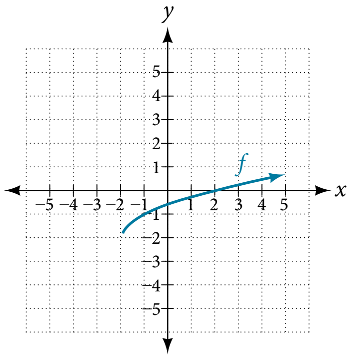
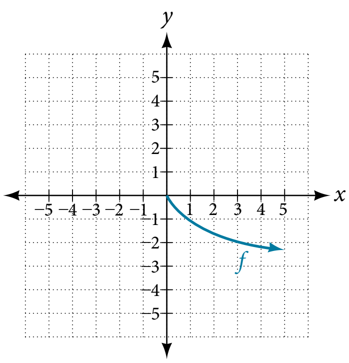


For the following exercises, use the graphs of transformations of the square root function to find a formula for each of the functions.

41.



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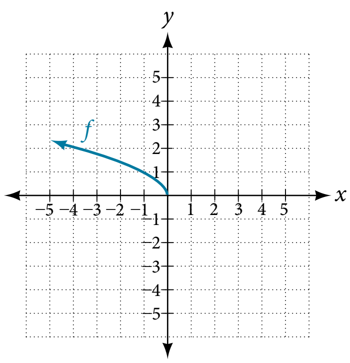
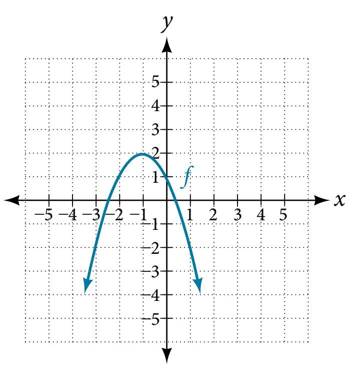


For the following exercises, use the graphs of the transformed toolkit functions to write a formula for each of the resulting functions.

43.



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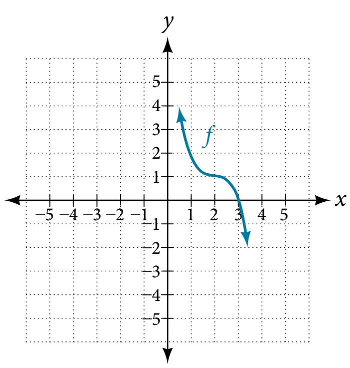

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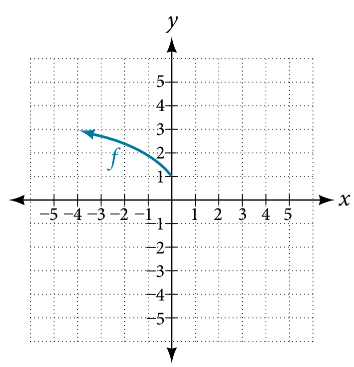

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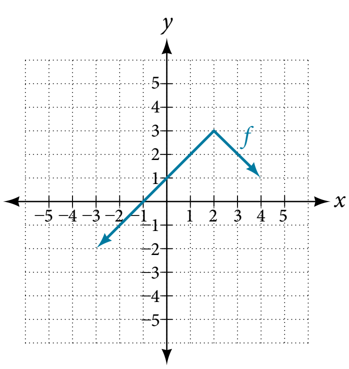


For the following exercises, determine whether the function is odd, even, or neither.

47. $f(x) = 3x^4$

48. $g(x) = \sqrt{x}$

49. $h(x) = \frac{1}{x} + 3x$

50. $f(x) = (x - 2)^2$

51. $g(x) = 2x^4$

52. $h(x) = 2x - x^3$

For the following exercises, describe how the graph of each function is a transformation of the graph of the original function *f*.

53. $g(x) = -f(x)$

54. $g(x) = f(-x)$

55. $g(x) = 4f(x)$

56. $g(x) = 6f(x)$

57. $g(x) = f(5x)$

58. $g(x) = f(2x)$

59. $g(x) = f\left(\frac{1}{3}x\right)$

60. $g(x) = f\left(\frac{1}{5}x\right)$

61. $g(x) = 3f(-x)$

62. $g(x) = -f(3x)$

For the following exercises, write a formula for the function *g* that results when the graph of a given toolkit function is transformed as described.

63. The graph of $f(x) = |x|$ is reflected over the y-axis and horizontally compressed by a factor of $\frac{1}{4}$.

64. The graph of $f(x) = \sqrt{x}$ is reflected over the x-axis and horizontally stretched by a factor of 2.

65. The graph of $f(x) = \frac{1}{x^2}$ is vertically compressed by a factor of $\frac{1}{3}$, then shifted to the left 2 units and down 3 units.

66. The graph of $f(x) = \frac{1}{x}$ is vertically stretched by a factor of 8, then shifted to the right 4 units and up 2 units.

67. The graph of $f(x) = x^2$ is vertically compressed by a factor of $\frac{1}{2}$, then shifted to the right 5 units and up 1 unit.

68. The graph of $f(x) = x^2$ is horizontally stretched by a factor of 3, then shifted to the left 4 units and down 3 units.

For the following exercises, describe how the formula is a transformation of a toolkit function. Then sketch a graph of the transformation.

69. $g(x) = 4(x + 1)^2 - 5$

70. $g(x) = 5(x + 3)^2 - 2$

71. $h(x) = -2|x - 4| + 3$

72. $k(x) = -3\sqrt{x} - 1$

73. $m(x) = \frac{1}{2}x^3$

74. $n(x) = \frac{1}{3}|x - 2|$

75. $p(x) = \left(\frac{1}{3}x\right)^3 - 3$

76. $q(x) = \left(\frac{1}{4}x\right)^3 + 1$

77. $a(x) = \sqrt{-x + 4}$

For the following exercises, use the graph in **Figure 32** to sketch the given transformations.

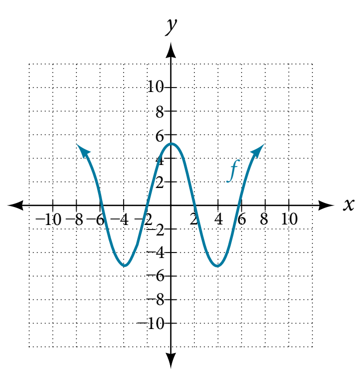


Figure 32

78. $g(x) = f(x) - 2$

79. $g(x) = -f(x)$

80. $g(x) = f(x + 1)$

81. $g(x) = f(x - 2)$

LEARNING OBJECTIVES

In this section, you will:

- Verify inverse functions.
- Determine the domain and range of an inverse function, and restrict the domain of a function to make it one-to-one.
- Find or evaluate the inverse of a function.
- Use the graph of a one-to-one function to graph its inverse function on the same axes.

3.7 INVERSE FUNCTIONS

A reversible heat pump is a climate-control system that is an air conditioner and a heater in a single device. Operated in one direction, it pumps heat out of a house to provide cooling. Operating in reverse, it pumps heat into the building from the outside, even in cool weather, to provide heating. As a heater, a heat pump is several times more efficient than conventional electrical resistance heating.

If some physical machines can run in two directions, we might ask whether some of the function "machines" we have been studying can also run backwards. **Figure 1** provides a visual representation of this question. In this section, we will consider the reverse nature of functions.

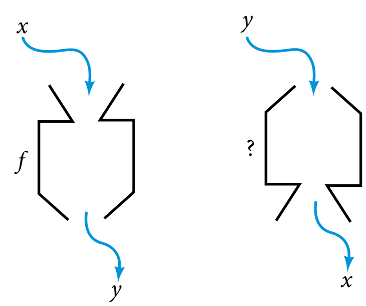


Figure 1 Can a function "machine" operate in reverse?

Verifying That Two Functions Are Inverse Functions

Suppose a fashion designer traveling to Milan for a fashion show wants to know what the temperature will be. He is not familiar with the Celsius scale. To get an idea of how temperature measurements are related, he asks his assistant, Betty, to convert 75 degrees Fahrenheit to degrees Celsius. She finds the formula

$$C = \frac{5}{9}(F - 32)$$

and substitutes 75 for F to calculate

$$\frac{5}{9}(75 - 32) \approx 24°\text{C}.$$

Knowing that a comfortable 75 degrees Fahrenheit is about 24 degrees Celsius, he sends his assistant the week's weather forecast from **Figure 2** for Milan, and asks her to convert all of the temperatures to degrees Fahrenheit.

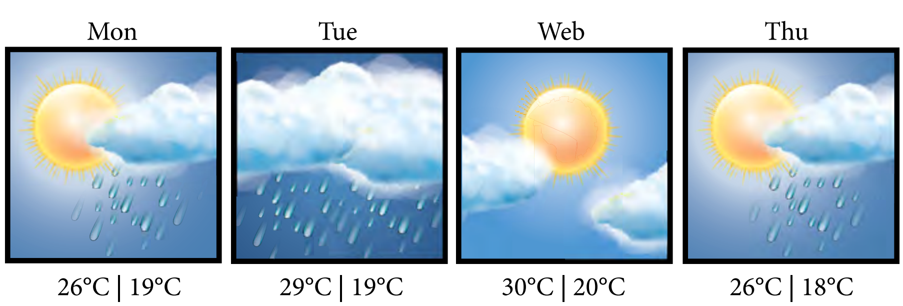


Figure 2

At first, Betty considers using the formula she has already found to complete the conversions. After all, she knows her algebra, and can easily solve the equation for F after substituting a value for C. For example, to convert 26 degrees Celsius, she could write

$$26 = \frac{5}{9}(F - 32)$$
$$26 \cdot \frac{9}{5} = F - 32$$
$$F = 26 \cdot \frac{9}{5} + 32 \approx 79$$

After considering this option for a moment, however, she realizes that solving the equation for each of the temperatures will be awfully tedious. She realizes that since evaluation is easier than solving, it would be much more convenient to have a different formula, one that takes the Celsius temperature and outputs the Fahrenheit temperature.

The formula for which Betty is searching corresponds to the idea of an **inverse function**, which is a function for which the input of the original function becomes the output of the inverse function and the output of the original function becomes the input of the inverse function.

Given a function $f(x)$, we represent its inverse as $f^{-1}(x)$, read as "f inverse of x." The raised -1 is part of the notation. It is not an exponent; it does not imply a power of -1. In other words, $f^{-1}(x)$ does *not* mean $\frac{1}{f(x)}$ because $\frac{1}{f(x)}$ is the reciprocal of f and not the inverse.

The "exponent-like" notation comes from an analogy between function composition and multiplication: just as $a^{-1} a = 1$ (1 is the identity element for multiplication) for any nonzero number a, so $f^{-1} \circ f$ equals the identity function, that is,

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(y) = x$$

This holds for all x in the domain of f. Informally, this means that inverse functions "undo" each other. However, just as zero does not have a reciprocal, some functions do not have inverses.

Given a function $f(x)$, we can verify whether some other function $g(x)$ is the inverse of $f(x)$ by checking whether either $g(f(x)) = x$ or $f(g(x)) = x$ is true. We can test whichever equation is more convenient to work with because they are logically equivalent (that is, if one is true, then so is the other.)

The book assumes that you know this, but just in case: A *one-to-one* function is a function $y = f(x)$ such that, for every number b, there is never more than one number a such that $b = f(a)$. You can recognize a one-to-one function from its graph. If every horizontal line intersects the graph at either 1 point or at 0 points, then the function is one-to-one. For example, $f(x) = \sqrt{x}$ and $f(x) = 1/x$ are one-to-one functions. $f(x) = x^2$ is not one-to-one because $2^2 = (-2)^2$, i.e., $f(2) = f(-2)$. On p. 245 of this book, the functions graphed in problems 44,45 are one-to-one, but the functions graphed in problems 43,46 are not one-to-one.

For example, $y = 4x$ and $y = \frac{1}{4}x$ are inverse functions.

$$(f^{-1} \circ f)(x) = f^{-1}(4x) = \frac{1}{4}(4x) = x$$

and

$$(f \circ f^{-1})(x) = f\left(\frac{1}{4}x\right) = 4\left(\frac{1}{4}x\right) = x$$

A few coordinate pairs from the graph of the function $y = 4x$ are $(-2, -8)$, $(0, 0)$, and $(2, 8)$. A few coordinate pairs from the graph of the function $y = \frac{1}{4}x$ are $(-8, -2)$, $(0, 0)$, and $(8, 2)$. If we interchange the input and output of each coordinate pair of a function, the interchanged coordinate pairs would appear on the graph of the inverse function.

inverse function

For any one-to-one function $f(x) = y$, a function $f^{-1}(x)$ is an **inverse function** of f if $f^{-1}(y) = x$. This can also be written as $f^{-1}(f(x)) = x$ for all x in the domain of f. It also follows that $f(f^{-1}(x)) = x$ for all x in the domain of f^{-1} if f^{-1} is the inverse of f.

The notation f^{-1} is read "f inverse." Like any other function, we can use any variable name as the input for f^{-1}, so we will often write $f^{-1}(x)$, which we read as "f inverse of x." Keep in mind that

$$f^{-1}(x) \neq \frac{1}{f(x)}$$

and not all functions have inverses.

Example 1 Identifying an Inverse Function for a Given Input-Output Pair

If for a particular one-to-one function $f(2) = 4$ and $f(5) = 12$, what are the corresponding input and output values for the inverse function?

Solution The inverse function reverses the input and output quantities, so if

$$f(2) = 4, \text{ then } f^{-1}(4) = 2;$$
$$f(5) = 12, \text{ then } f^{-1}(12) = 5.$$

Alternatively, if we want to name the inverse function g, then $g(4) = 2$ and $g(12) = 5$.

Analysis *Notice that if we show the coordinate pairs in a table form, the input and output are clearly reversed. See* ***Table 1***.

$(x, f(x))$	$(x, g(x))$
(2, 4)	(4, 2)
(5, 12)	(12, 5)

Table 1

Try It #1

Given that $h^{-1}(6) = 2$, what are the corresponding input and output values of the original function h?

How To…

Given two functions $f(x)$ and $g(x)$, test whether the functions are inverses of each other.

1. Determine whether $f(g(x)) = x$ or $g(f(x)) = x$.
2. If either statement is true, then both are true, and $g = f^{-1}$ and $f = g^{-1}$. If either statement is false, then both are false, and $g \neq f^{-1}$ and $f \neq g^{-1}$.

Example 2 Testing Inverse Relationships Algebraically

If $f(x) = \dfrac{1}{x+2}$ and $g(x) = \dfrac{1}{x} - 2$, is $g = f^{-1}$?

Solution

$$\begin{aligned} g(f(x)) &= \frac{1}{\left(\frac{1}{x+2}\right)} - 2 \\ &= x + 2 - 2 \\ &= x \end{aligned}$$

so

$$g = f^{-1} \text{ and } f = g^{-1}$$

This is enough to answer yes to the question, but we can also verify the other formula.

$$\begin{aligned} f(g(x)) &= \frac{1}{\frac{1}{x} - 2 + 2} \\ &= \frac{1}{\frac{1}{x}} \\ &= x \end{aligned}$$

Analysis *Notice the inverse operations are in reverse order of the operations from the original function.*

Try It #2

If $f(x) = x^3 - 4$ and $g(x) = \sqrt[3]{x - 4}$, is $g = f^{-1}$?

Example 3 Determining Inverse Relationships for Power Functions

If $f(x) = x^3$ (the cube function) and $g(x) = \frac{1}{3}x$, is $g = f^{-1}$?

Solution $f(g(x)) = \frac{x^3}{27} \neq x$

No, the functions are not inverses.

Analysis *The correct inverse to the cube is, of course, the cube root* $\sqrt[3]{x} = x^{1/3}$ *that is, the one-third is an exponent, not a multiplier.*

Try It #3

If $f(x) = (x - 1)^3$ and $g(x) = \sqrt[3]{x} + 1$, is $g = f^{-1}$?

The definition of a function $f(x)$ is not complete unless the domain has been specified. However, when the domain consists of all input values x for which $f(x)$ can be computed, the domain is usually not explicitly mentioned. For example, in Example 2 above the authors wrote "f(x) = 1/(x+2)" . The domain is implicit; it is all values of x for which x+2 is non-zero, i.e., $(-\infty, -2) \cup (-2, \infty)$. In Example 9 below the authors will write "$f(x) = 2 + \sqrt{x-4}$". The domain is all values of x for which x–4 is non-negative, i.e., $[4, \infty)$.

Finding Domain and Range of Inverse Functions

The outputs of the function f are the inputs to f^{-1}, so the range of f is also the domain of f^{-1}. Likewise, because the inputs to f are the outputs of f^{-1}, the domain of f is the range of f^{-1}. We can visualize the situation as in **Figure 3**.

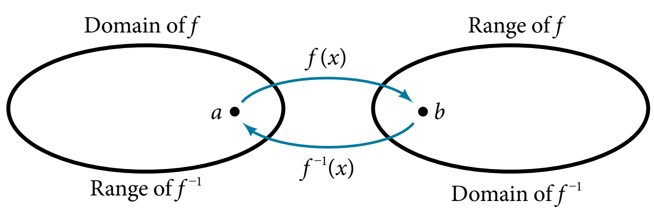


Figure 3 Domain and range of a function and its inverse

When a function has no inverse function, it is possible to create a new function where that new function on a limited domain does have an inverse function. For example, the inverse of $f(x) = \sqrt{x}$ is $f^{-1}(x) = x^2$, because a square "undoes" a square root; but the square is only the inverse of the square root on the domain $[0, \infty)$, since that is the range of $f(x) = \sqrt{x}$.

We can look at this problem from the other side, starting with the square (toolkit quadratic) function $f(x) = x^2$. If we want to construct an inverse to this function, we run into a problem, because for every given output of the quadratic function, there are two corresponding inputs (except when the input is 0). For example, the output 9 from the quadratic function corresponds to the inputs 3 and -3. But an output from a function is an input to its inverse; if this inverse input corresponds to more than one inverse output (input of the original function), then the "inverse" is not a function at all! To put it differently, the quadratic function is not a one-to-one function; it fails the horizontal line test, so it does not have an inverse function. In order for a function to have an inverse, it must be a one-to-one function.

In many cases, if a function is not one-to-one, we can still restrict the function to a part of its domain on which it is one-to-one. For example, we can make a restricted version of the square function $f(x) = x^2$ with its domain limited to $[0, \infty)$, which is a one-to-one function (it passes the horizontal line test) and which has an inverse (the square-root function).

If $f(x) = (x - 1)^2$ on $[1, \infty)$, then the inverse function is $f^{-1}(x) = \sqrt{x} + 1$.

- The domain of f = range of $f^{-1} = [1, \infty)$.
- The domain of f^{-1} = range of $f = [0, \infty)$.

Q & A…

Is it possible for a function to have more than one inverse?

No. If two supposedly different functions, say, g and h, both meet the definition of being inverses of another function f, then you can prove that $g = h$. We have just seen that some functions only have inverses if we restrict the domain of the original function. In these cases, there may be more than one way to restrict the domain, leading to different inverses. However, on any one domain, the original function still has only one unique inverse.

domain and range of inverse functions

The range of a function $f(x)$ is the domain of the inverse function $f^{-1}(x)$. The domain of $f(x)$ is the range of $f^{-1}(x)$.

How To…

Given a function, find the domain and range of its inverse.

1. If the function is one-to-one, write the range of the original function as the domain of the inverse, and write the domain of the original function as the range of the inverse.
2. If the domain of the original function needs to be restricted to make it one-to-one, then this restricted domain becomes the range of the inverse function.

Example 4 Finding the Inverses of Toolkit Functions

Identify which of the toolkit functions besides the quadratic function are not one-to-one, and find a restricted domain on which each function is one-to-one, if any. The toolkit functions are reviewed in **Table 2**. We restrict the domain in such a fashion that the function assumes all y-values exactly once.

Constant	Identity	Quadratic	Cubic	Reciprocal
$f(x) = c$	$f(x) = x$	$f(x) = x^2$	$f(x) = x^3$	$f(x) = \frac{1}{x}$

Reciprocal squared	Cube root	Square root	Absolute value	
$f(x) = \frac{1}{x^2}$	$f(x) = \sqrt[3]{x}$	$f(x) = \sqrt{x}$	$f(x) = \lvert x \rvert$	

Table 2

Solution The constant function is not one-to-one, and there is no domain (except a single point) on which it could be one-to-one, so the constant function has no meaningful inverse.

The absolute value function can be restricted to the domain $[0, \infty)$, where it is equal to the identity function.

The reciprocal-squared function can be restricted to the domain $(0, \infty)$.

Analysis *We can see that these functions (if unrestricted) are not one-to-one by looking at their graphs, shown in* ***Figure 4****. They both would fail the horizontal line test. However, if a function is restricted to a certain domain so that it passes the horizontal line test, then in that restricted domain, it can have an inverse.*

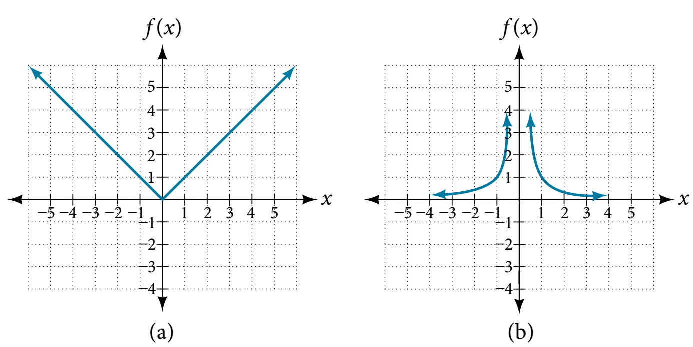


Figure 4 (a) Absolute value (b) Reciprocal squared

Try It #4

The domain of function f is $(1, \infty)$ and the range of function f is $(-\infty, -2)$. Find the domain and range of the inverse function.

Example 4.5 Consider the function $y = (x-3)^2 - 7$. The function is not one-to-one. However, we can create a related one-to-one function by restricting the domain. The function $y = (x-3)^2 - 7, x \leq 3$ is a one-to-one function, as the algebra below indicates.

$y = (x-3)^2 - 7 \;\Leftrightarrow\; y + 7 = (x-3)^2 \;\Leftrightarrow\; \sqrt{y+7} = |x-3|$. But $x \leq 3 \Leftrightarrow x - 3 \leq 0$,
so this is equivalent to $\sqrt{y+7} = 3 - x \Leftrightarrow x = 3 - \sqrt{y+7}$.

This algebra shows that $f(x)$ is one-to-one because for every fixed value of y, either there is a unique x satisfying $y = f(x)$ (namely, $x = 3 - \sqrt{y+7}$), or else there is no value of x such that $y = f(x)$ (if, for example, $y = -11$). We also discovered that $f^{-1}(x) = 3 - \sqrt{x+7}$. Usually we would not state it explicitly, but the domain of $f^{-1}(x)$ is $[-7, \infty)$.

The graph of $y = (x-3)^2 - 7$ is a parabola with vertex $(3, -7)$ which opens upwards. The graph of $y = (x-3)^2 - 7, x \leq 3$ is the left branch of the parabola. It starts at the vertex $(3, -7)$ and extends to the left and upwards.

Finding and Evaluating Inverse Functions

Once we have a one-to-one function, we can evaluate its inverse at specific inverse function inputs or construct a complete representation of the inverse function in many cases.

Inverting Tabular Functions

Suppose we want to find the inverse of a function represented in table form. Remember that the domain of a function is the range of the inverse and the range of the function is the domain of the inverse. So we need to interchange the domain and range.

Each row (or column) of inputs becomes the row (or column) of outputs for the inverse function. Similarly, each row (or column) of outputs becomes the row (or column) of inputs for the inverse function.

Example 5 **Interpreting the Inverse of a Tabular Function**

A function $f(t)$ is given in **Table 3**, showing distance in miles that a car has traveled in t minutes. Find and interpret $f^{-1}(70)$.

t (minutes)	30	50	70	90
$f(t)$ (miles)	20	40	60	70

Table 3

Solution The inverse function takes an output of f and returns an input for f. So in the expression $f^{-1}(70)$, 70 is an output value of the original function, representing 70 miles. The inverse will return the corresponding input of the original function f, 90 minutes, so $f^{-1}(70) = 90$. The interpretation of this is that, to drive 70 miles, it took 90 minutes.

Alternatively, recall that the definition of the inverse was that if $f(a) = b$, then $f^{-1}(b) = a$. By this definition, if we are given $f^{-1}(70) = a$, then we are looking for a value a so that $f(a) = 70$. In this case, we are looking for a t so that $f(t) = 70$, which is when $t = 90$.

Try It #5

Using **Table 4**, find and interpret **a.** $f(60)$, and **b.** $f^{-1}(60)$.

t (minutes)	30	50	60	70	90
$f(t)$ (miles)	20	40	50	60	70

Table 4

Evaluating the Inverse of a Function, Given a Graph of the Original Function

We saw in **Functions and Function Notation** that the domain of a function can be read by observing the horizontal extent of its graph. We find the domain of the inverse function by observing the *vertical* extent of the graph of the original function, because this corresponds to the horizontal extent of the inverse function. Similarly, we find the range of the inverse function by observing the *horizontal* extent of the graph of the original function, as this is the vertical extent of the inverse function. If we want to evaluate an inverse function, we find its input within its domain, which is all or part of the vertical axis of the original function's graph.

How To...

Given the graph of a function, evaluate its inverse at specific points.

1. Find the desired input on the y-axis of the given graph.
2. Read the inverse function's output from the x-axis of the given graph.

Example 6 Evaluating a Function and Its Inverse from a Graph at Specific Points

A function $g(x)$ is given in **Figure 5**. Find $g(3)$ and $g^{-1}(3)$.

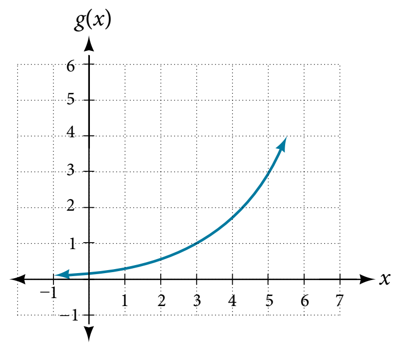


Figure 5

Solution To evaluate $g(3)$, we find 3 on the x-axis and find the corresponding output value on the y-axis. The point $(3, 1)$ tells us that $g(3) = 1$.

To evaluate $g^{-1}(3)$, recall that by definition $g^{-1}(3)$ means the value of x for which $g(x) = 3$. By looking for the output value 3 on the vertical axis, we find the point $(5, 3)$ on the graph, which means $g(5) = 3$, so by definition, $g^{-1}(3) = 5$. See **Figure 6**.

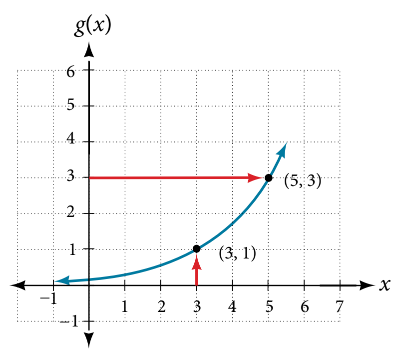


Figure 6

Try It #6

Using the graph in **Figure 6**, **a.** find $g^{-1}(1)$, and **b.** estimate $g^{-1}(4)$.

Finding Inverses of Functions Represented by Formulas

Sometimes we will need to know an inverse function for all elements of its domain, not just a few. If the original function is given as a formula—for example, y as a function of x—we can often find the inverse function by solving to obtain x as a function of y.

How To…

Given a function represented by a formula, find the inverse.

1. Make sure f is a one-to-one function.
2. Solve for x.
3. Interchange x and y.

Example 7 Inverting the Fahrenheit-to-Celsius Function

Find a formula for the inverse function that gives Fahrenheit temperature as a function of Celsius temperature.

$$C = \frac{5}{9}(F - 32)$$

Solution

$$C = \frac{5}{9}(F - 32)$$
$$C \cdot \frac{9}{5} = F - 32$$
$$F = \frac{9}{5}C + 32$$

By solving in general, we have uncovered the inverse function. If

$$C = h(F) = \frac{5}{9}(F - 32),$$

then

$$F = h^{-1}(C) = \frac{9}{5}C + 32.$$

In this case, we introduced a function h to represent the conversion because the input and output variables are descriptive, and writing C^{-1} could get confusing.

Try It #7

Solve for x in terms of y given $y = \frac{1}{3}(x - 5)$

Example 8 Solving to Find an Inverse Function

Find the inverse of the function $f(x) = \frac{2}{x-3} + 4$.

Solution

$$y = \frac{2}{x-3} + 4 \quad \text{Set up an equation.}$$
$$y - 4 = \frac{2}{x-3} \quad \text{Subtract 4 from both sides.}$$
$$x - 3 = \frac{2}{y-4} \quad \text{Multiply both sides by } x-3 \text{ and divide by } y-4.$$
$$x = \frac{2}{y-4} + 3 \quad \text{Add 3 to both sides.}$$

So $f^{-1}(y) = \frac{2}{y-4} + 3$ or $f^{-1}(x) = \frac{2}{x-4} + 3$.

Analysis *The domain and range of f exclude the values 3 and 4, respectively. f and f^{-1} are equal at two points but are not the same function, as we can see by creating **Table 5**.*

x	1	2	5	$f^{-1}(y)$
$f(x)$	3	2	5	y

Table 5

Example 9 Solving to Find an Inverse with Radicals

Find the inverse of the function $f(x) = 2 + \sqrt{x-4}$.

Solution

$$y = 2 + \sqrt{x-4}$$
$$(y-2)^2 = x - 4$$
$$x = (y-2)^2 + 4$$

So $f^{-1}(x) = (x-2)^2 + 4$.

The domain of f is $[4, \infty)$. Notice that the range of f is $[2, \infty)$, so this means that the domain of the inverse function f^{-1} is also $[2, \infty)$.

Analysis *The formula we found for $f^{-1}(x)$ looks like it would be valid for all real x. However, f^{-1} itself must have an inverse (namely, f) so we have to restrict the domain of f^{-1} to $[2, \infty)$ in order to make f^{-1} a one-to-one function. This domain of f^{-1} is exactly the range of f.*

Try It #8

What is the inverse of the function $f(x) = 2 - \sqrt{x}$? State the domains of both the function and the inverse function.

Finding Inverse Functions and Their Graphs

Now that we can find the inverse of a function, we will explore the graphs of functions and their inverses. Let us return to the quadratic function $f(x) = x^2$ restricted to the domain $[0, \infty)$, on which this function is one-to-one, and graph it as in **Figure 7**.

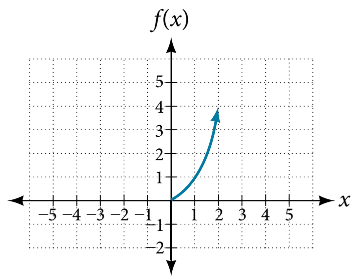


Figure 7 Quadratic function with domain restricted to $[0, \infty)$.

Restricting the domain to $[0, \infty)$ makes the function one-to-one (it will obviously pass the horizontal line test), so it has an inverse on this restricted domain.

We already know that the inverse of the toolkit quadratic function is the square root function, that is, $f^{-1}(x) = \sqrt{x}$. What happens if we graph both f and f^{-1} on the same set of axes, using the x-axis for the input to both f and f^{-1}?

We notice a distinct relationship: The graph of $f^{-1}(x)$ is the graph of $f(x)$ reflected about the diagonal line $y = x$, which we will call the identity line, shown in **Figure 8**.

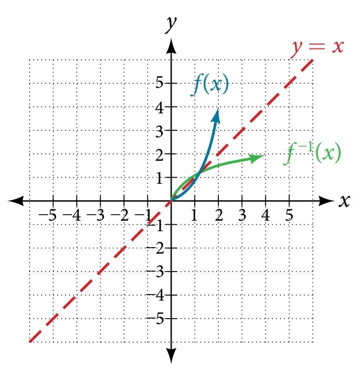


Figure 8 Square and square-root functions on the non-negative domain

This relationship will be observed for all one-to-one functions, because it is a result of the function and its inverse swapping inputs and outputs. This is equivalent to interchanging the roles of the vertical and horizontal axes.

Example 10 **Finding the Inverse of a Function Using Reflection about the Identity Line**

Given the graph of $f(x)$ in **Figure 9**, sketch a graph of $f^{-1}(x)$.

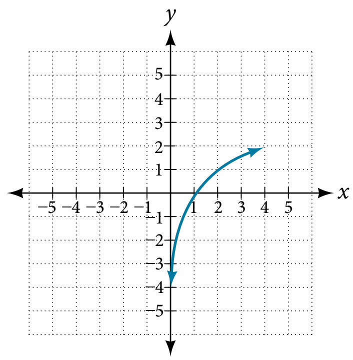


Figure 9

Solution This is a one-to-one function, so we will be able to sketch an inverse. Note that the graph shown has an apparent domain of $(0, \infty)$ and range of $(-\infty, \infty)$, so the inverse will have a domain of $(-\infty, \infty)$ and range of $(0, \infty)$.

If we reflect this graph over the line $y = x$, the point $(1, 0)$ reflects to $(0, 1)$ and the point $(4, 2)$ reflects to $(2, 4)$. Sketching the inverse on the same axes as the original graph gives **Figure 10**.

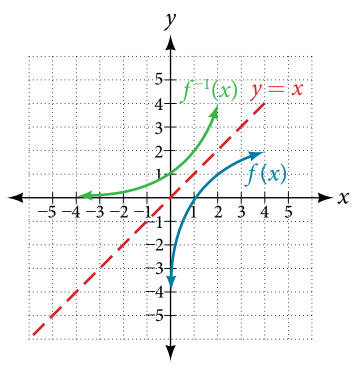


Figure 10 The function and its inverse, showing reflection about the identity line

Try It #9

Draw graphs of the functions f and f^{-1} from **Example 8**.

Q & A...

Is there any function that is equal to its own inverse?

Yes. If $f = f^{-1}$, then $f(f(x)) = x$, and we can think of several functions that have this property. The identity function does, and so does the reciprocal function, because

$$\frac{1}{\frac{1}{x}} = x$$

Any function $f(x) = c - x$, where c is a constant, is also equal to its own inverse.

Access these online resources for additional instruction and practice with inverse functions.

- Inverse Functions (http://openstaxcollege.org/l/inversefunction)
- One-to-one Functions (http://openstaxcollege.org/l/onetoone)
- Inverse Function Values Using Graph (http://openstaxcollege.org/l/inversfuncgraph)
- Restricting the Domain and Finding the Inverse (http://openstaxcollege.org/l/restrictdomain)

3.7 SECTION EXERCISES

VERBAL

1. Describe why the horizontal line test is an effective way to determine whether a function is one-to-one?

2. Why do we restrict the domain of the function $f(x) = x^2$ to find the function's inverse?

3. Can a function be its own inverse? Explain.

4. Are one-to-one functions either always increasing or always decreasing? Why or why not?

5. How do you find the inverse of a function algebraically?

ALGEBRAIC

6. Show that the function $f(x) = a - x$ is its own inverse for all real numbers a.

For the following exercises, find $f^{-1}(x)$ for each function.

7. $f(x) = x + 3$

8. $f(x) = x + 5$

9. $f(x) = 2 - x$

10. $f(x) = 3 - x$

11. $f(x) = \dfrac{x}{x+2}$

12. $f(x) = \dfrac{2x+3}{5x+4}$

For the following exercises, find a domain on which each function f is one-to-one and non-decreasing. Write the domain in interval notation. Then find the inverse of f restricted to that domain.

13. $f(x) = (x + 7)^2$

14. $f(x) = (x - 6)^2$

15. $f(x) = x^2 - 5$

16. Given $f(x) = \dfrac{x}{2+x}$ and $g(x) = \dfrac{2x}{1-x}$:

a. Find $f(g(x))$ and $g(f(x))$.

b. What does the answer tell us about the relationship between $f(x)$ and $g(x)$?

For the following exercises, use function composition to verify that $f(x)$ and $g(x)$ are inverse functions.

17. $f(x) = \sqrt[3]{x-1}$ and $g(x) = x^3 + 1$

18. $f(x) = -3x + 5$ and $g(x) = \dfrac{x-5}{-3}$

GRAPHICAL

For the following exercises, use a graphing utility to determine whether each function is one-to-one.

19. $f(x) = \sqrt{x}$

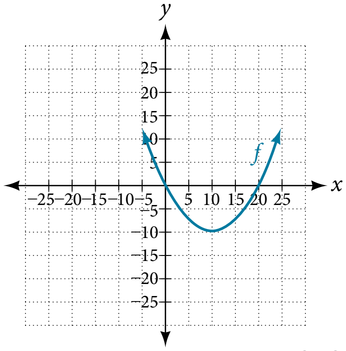
20. $f(x) = \sqrt[3]{3x+1}$

21. $f(x) = -5x + 1$

22. $f(x) = x^3 - 27$

For the following exercises, determine whether the graph represents a one-to-one function.

23.



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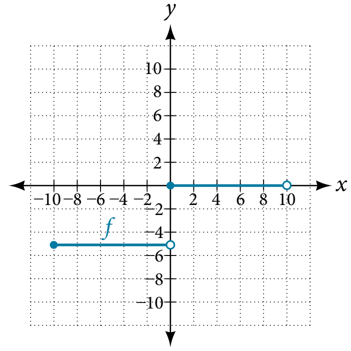


For the following exercises, use the graph of f shown in **Figure 11**.

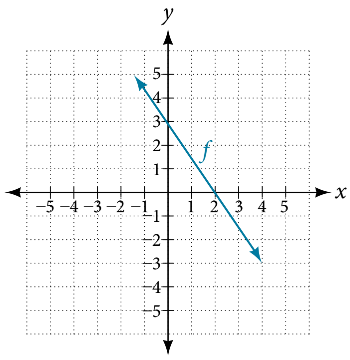


Figure 11

25. Find $f(0)$.

26. Solve $f(x) = 0$.

27. Find $f^{-1}(0)$.

28. Solve $f^{-1}(x) = 0$.

For the following exercises, use the graph of the one-to-one function shown in **Figure 12**.

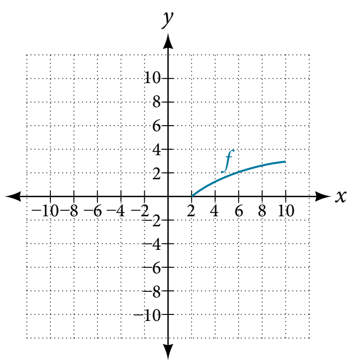


Figure 12

29. Sketch the graph of f^{-1}.

30. Find $f(6)$ and $f^{-1}(2)$.

31. If the complete graph of f is shown, find the domain of f.

32. If the complete graph of f is shown, find the range of f.

NUMERIC

For the following exercises, evaluate or solve, assuming that the function f is one-to-one.

33. If $f(6) = 7$, find $f^{-1}(7)$.

34. If $f(3) = 2$, find $f^{-1}(2)$.

35. If $f^{-1}(-4) = -8$, find $f(-8)$.

36. If $f^{-1}(-2) = -1$, find $f(-1)$.

For the following exercises, use the values listed in **Table 6** to evaluate or solve.

x	0	1	2	3	4	5	6	7	8	9
$f(x)$	8	0	7	4	2	6	5	3	9	1

Table 6

37. Find $f(1)$.

38. Solve $f(x) = 3$.

39. Find $f^{-1}(0)$.

40. Solve $f^{-1}(x) = 7$.

41. Use the tabular representation of f in **Table 7** to create a table for $f^{-1}(x)$.

x	3	6	9	13	14
$f(x)$	1	4	7	12	16

Table 7

TECHNOLOGY

For the following exercises, find the inverse function. Then, graph the function and its inverse.

42. $f(x) = \dfrac{3}{x - 2}$

43. $f(x) = x^3 - 1$

44. Find the inverse function of $f(x) = \dfrac{1}{x - 1}$. Use a graphing utility to find its domain and range. Write the domain and range in interval notation.

REAL-WORLD APPLICATIONS

45. To convert from x degrees Celsius to y degrees Fahrenheit, we use the formula $f(x) = \dfrac{9}{5}x + 32$. Find the inverse function, if it exists, and explain its meaning.

46. The circumference C of a circle is a function of its radius given by $C(r) = 2\pi r$. Express the radius of a circle as a function of its circumference. Call this function $r(C)$. Find $r(36\pi)$ and interpret its meaning.

47. A car travels at a constant speed of 50 miles per hour. The distance the car travels in miles is a function of time, t, in hours given by $d(t) = 50t$. Find the inverse function by expressing the time of travel in terms of the distance traveled. Call this function $t(d)$. Find $t(180)$ and interpret its meaning.

CHAPTER 3 REVIEW

Key Terms

absolute maximum the greatest value of a function over an interval

absolute minimum the lowest value of a function over an interval

average rate of change the difference in the output values of a function found for two values of the input divided by the difference between the inputs

composite function the new function formed by function composition, when the output of one function is used as the input of another

decreasing function a function is decreasing in some open interval if $f(b) < f(a)$ for any two input values a and b in the given interval where $b > a$

dependent variable an output variable

domain the set of all possible input values for a relation

even function a function whose graph is unchanged by horizontal reflection, $f(x) = f(-x)$, and is symmetric about the y-axis

function a relation in which each input value yields a unique output value

horizontal compression a transformation that compresses a function's graph horizontally, by multiplying the input by a constant $b > 1$

horizontal line test a method of testing whether a function is one-to-one by determining whether any horizontal line intersects the graph more than once

horizontal reflection a transformation that reflects a function's graph across the y-axis by multiplying the input by -1

horizontal shift a transformation that shifts a function's graph left or right by adding a positive or negative constant to the input

horizontal stretch a transformation that stretches a function's graph horizontally by multiplying the input by a constant $0 < b < 1$

increasing function a function is increasing in some open interval if $f(b) > f(a)$ for any two input values a and b in the given interval where $b > a$

independent variable an input variable

input each object or value in a domain that relates to another object or value by a relationship known as a function

interval notation a method of describing a set that includes all numbers between a lower limit and an upper limit; the lower and upper values are listed between brackets or parentheses, a square bracket indicating inclusion in the set, and a parenthesis indicating exclusion

inverse function for any one-to-one function $f(x)$, the inverse is a function $f^{-1}(x)$ such that $f^{-1}(f(x)) = x$ for all x in the domain of f; this also implies that $f(f^{-1}(x)) = x$ for all x in the domain of f^{-1}

local extrema collectively, all of a function's local maxima and minima

local maximum a value of the input where a function changes from increasing to decreasing as the input value increases.

local minimum a value of the input where a function changes from decreasing to increasing as the input value increases.

odd function a function whose graph is unchanged by combined horizontal and vertical reflection, $f(x) = -f(-x)$, and is symmetric about the origin

one-to-one function a function for which each value of the output is associated with a unique input value

output each object or value in the range that is produced when an input value is entered into a function

piecewise function a function in which more than one formula is used to define the output

range the set of output values that result from the input values in a relation

rate of change the change of an output quantity relative to the change of the input quantity

relation a set of ordered pairs

set-builder notation a method of describing a set by a rule that all of its members obey; it takes the form $\{x \mid \text{statement about } x\}$

vertical compression a function transformation that compresses the function's graph vertically by multiplying the output by a constant $0 < a < 1$

vertical line test a method of testing whether a graph represents a function by determining whether a vertical line intersects the graph no more than once

vertical reflection a transformation that reflects a function's graph across the x-axis by multiplying the output by -1

vertical shift a transformation that shifts a function's graph up or down by adding a positive or negative constant to the output

vertical stretch a transformation that stretches a function's graph vertically by multiplying the output by a constant $a > 1$

Key Equations

Constant function	$f(x) = c$, where c is a constant
Identity function	$f(x) = x$
Absolute value function	$f(x) = \lvert x \rvert$
Quadratic function	$f(x) = x^2$
Cubic function	$f(x) = x^3$
Reciprocal function	$f(x) = \frac{1}{x}$
Reciprocal squared function	$f(x) = \frac{1}{x^2}$
Square root function	$f(x) = \sqrt{x}$
Cube root function	$f(x) = \sqrt[3]{x}$
Average rate of change	$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$
Composite function	$(f \circ g)(x) = f(g(x))$
Vertical shift	$g(x) = f(x) + k$ (up for $k > 0$)
Horizontal shift	$g(x) = f(x - h)$ (right for $h > 0$)
Vertical reflection	$g(x) = -f(x)$
Horizontal reflection	$g(x) = f(-x)$
Vertical stretch	$g(x) = af(x)$ $(a > 0)$
Vertical compression	$g(x) = af(x)$ $(0 < a < 1)$
Horizontal stretch	$g(x) = f(bx)$ $(0 < b < 1)$
Horizontal compression	$g(x) = f(bx)$ $(b > 1)$

Key Concepts

3.1 Functions and Function Notation

- A relation is a set of ordered pairs. A function is a specific type of relation in which each domain value, or input, leads to exactly one range value, or output. See **Example 1** and **Example 2**.
- Function notation is a shorthand method for relating the input to the output in the form $y = f(x)$. See **Example 3** and **Example 4**.
- In tabular form, a function can be represented by rows or columns that relate to input and output values. See **Example 5**.
- To evaluate a function, we determine an output value for a corresponding input value. Algebraic forms of a function can be evaluated by replacing the input variable with a given value. See **Example 6** and **Example 7**.
- To solve for a specific function value, we determine the input values that yield the specific output value. See **Example 8**.
- An algebraic form of a function can be written from an equation. See **Example 9** and **Example 10**.
- Input and output values of a function can be identified from a table. See **Example 11**.
- Relating input values to output values on a graph is another way to evaluate a function. See **Example 12**.
- A function is one-to-one if each output value corresponds to only one input value. See **Example 13**.
- A graph represents a function if any vertical line drawn on the graph intersects the graph at no more than one point. See **Example 14**.
- The graph of a one-to-one function passes the horizontal line test. See **Example 15**.

3.2 Domain and Range

- The domain of a function includes all real input values that would not cause us to attempt an undefined mathematical operation, such as dividing by zero or taking the square root of a negative number.
- The domain of a function can be determined by listing the input values of a set of ordered pairs. See **Example 1.**
- The domain of a function can also be determined by identifying the input values of a function written as an equation. See **Example 2**, **Example 3**, and **Example 4**.
- Interval values represented on a number line can be described using inequality notation, set-builder notation, and interval notation. See **Example 5**.
- For many functions, the domain and range can be determined from a graph. See **Example 6** and **Example 7**.
- An understanding of toolkit functions can be used to find the domain and range of related functions. See **Example 8**, **Example 9**, and **Example 10**.
- A piecewise function is described by more than one formula. See **Example 11** and **Example 12**.
- A piecewise function can be graphed using each algebraic formula on its assigned subdomain. See **Example 13**.

3.3 Rates of Change and Behavior of Graphs

- A rate of change relates a change in an output quantity to a change in an input quantity. The average rate of change is determined using only the beginning and ending data. See **Example 1**.
- Identifying points that mark the interval on a graph can be used to find the average rate of change. See **Example 2**.
- Comparing pairs of input and output values in a table can also be used to find the average rate of change. See **Example 3**.
- An average rate of change can also be computed by determining the function values at the endpoints of an interval described by a formula. See **Example 4** and **Example 5**.
- The average rate of change can sometimes be determined as an expression. See **Example 6**.
- A function is increasing where its rate of change is positive and decreasing where its rate of change is negative. See **Example 7**.
- A local maximum is where a function changes from increasing to decreasing and has an output value larger (more positive or less negative) than output values at neighboring input values.

- A local minimum is where the function changes from decreasing to increasing (as the input increases) and has an output value smaller (more negative or less positive) than output values at neighboring input values.
- Minima and maxima are also called extrema.
- We can find local extrema from a graph. See **Example 8** and **Example 9**.
- The highest and lowest points on a graph indicate the maxima and minima. See **Example 10**.

3.4 Composition of Functions

- We can perform algebraic operations on functions. See **Example 1**.
- When functions are combined, the output of the first (inner) function becomes the input of the second (outer) function.
- The function produced by combining two functions is a composite function. See **Example 2** and **Example 3**.
- The order of function composition must be considered when interpreting the meaning of composite functions. See **Example 4**.
- A composite function can be evaluated by evaluating the inner function using the given input value and then evaluating the outer function taking as its input the output of the inner function.
- A composite function can be evaluated from a table. See **Example 5**.
- A composite function can be evaluated from a graph. See **Example 6**.
- A composite function can be evaluated from a formula. See **Example 7**.
- The domain of a composite function consists of those inputs in the domain of the inner function that correspond to outputs of the inner function that are in the domain of the outer function. See **Example 8** and **Example 9**.
- Just as functions can be combined to form a composite function, composite functions can be decomposed into simpler functions.
- Functions can often be decomposed in more than one way. See **Example 10**.

3.5 Transformation of Functions

- A function can be shifted vertically by adding a constant to the output. See **Example 1** and **Example 2**.
- A function can be shifted horizontally by adding a constant to the input. See **Example 3**, **Example 4**, and **Example 5**.
- Relating the shift to the context of a problem makes it possible to compare and interpret vertical and horizontal shifts. See **Example 6**.
- Vertical and horizontal shifts are often combined. See **Example 7** and **Example 8**.
- A vertical reflection reflects a graph about the x-axis. A graph can be reflected vertically by multiplying the output by -1.
- A horizontal reflection reflects a graph about the y-axis. A graph can be reflected horizontally by multiplying the input by -1.
- A graph can be reflected both vertically and horizontally. The order in which the reflections are applied does not affect the final graph. See **Example 9**.
- A function presented in tabular form can also be reflected by multiplying the values in the input and output rows or columns accordingly. See **Example 10**.
- A function presented as an equation can be reflected by applying transformations one at a time. See **Example 11**.
- Even functions are symmetric about the y-axis, whereas odd functions are symmetric about the origin.
- Even functions satisfy the condition $f(x) = f(-x)$.
- Odd functions satisfy the condition $f(x) = -f(-x)$.
- A function can be odd, even, or neither. See **Example 12**.
- A function can be compressed or stretched vertically by multiplying the output by a constant. See **Example 13**, **Example 14**, and **Example 15**.
- A function can be compressed or stretched horizontally by multiplying the input by a constant. See **Example 16**, **Example 17**, and **Example 18**.

- The order in which different transformations are applied does affect the final function. Both vertical and horizontal transformations must be applied in the order given. However, a vertical transformation may be combined with a horizontal transformation in any order. See **Example 19** and **Example 20**.

3.6 Absolute Value Functions

- Applied problems, such as ranges of possible values, can also be solved using the absolute value function. See **Example 1**.
- The graph of the absolute value function resembles a letter V. It has a corner point at which the graph changes direction. See **Example 2**.
- In an absolute value equation, an unknown variable is the input of an absolute value function.
- If the absolute value of an expression is set equal to a positive number, expect two solutions for the unknown variable. See **Example 3**.

3.7 Inverse Functions

- If $g(x)$ is the inverse of $f(x)$, then $g(f(x)) = f(g(x)) = x$. See **Example 1**, **Example 2**, and **Example 3**.
- Each of the toolkit functions has an inverse. See **Example 4**.
- For a function to have an inverse, it must be one-to-one (pass the horizontal line test).
- A function that is not one-to-one over its entire domain may be one-to-one on part of its domain.
- For a tabular function, exchange the input and output rows to obtain the inverse. See **Example 5**.
- The inverse of a function can be determined at specific points on its graph. See **Example 6**.
- To find the inverse of a formula, solve the equation $y = f(x)$ for x as a function of y. Then exchange the labels x and y. See **Example 7**, **Example 8**, and **Example 9**.
- The graph of an inverse function is the reflection of the graph of the original function across the line $y = x$. See **Example 10**.

CHAPTER 3 REVIEW EXERCISES

FUNCTIONS AND FUNCTION NOTATION

For the following exercises, determine whether the relation is a function.

1. $\{(a, b), (c, d), (e, d)\}$

2. $\{(5, 2), (6, 1), (6, 2), (4, 8)\}$

3. $y^2 + 4 = x$, for x the independent variable and y the dependent variable

4. Is the graph in **Figure 1** a function?

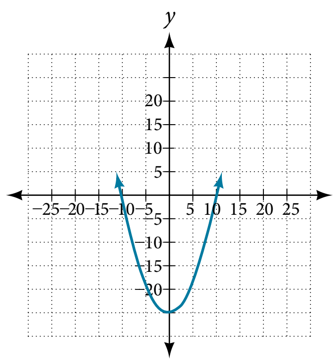


Figure 1

For the following exercises, evaluate the function at the indicated values: $f(-3); f(2); f(-a); -f(a); f(a+h)$.

5. $f(x) = -2x^2 + 3x$

6. $f(x) = 2|3x - 1|$

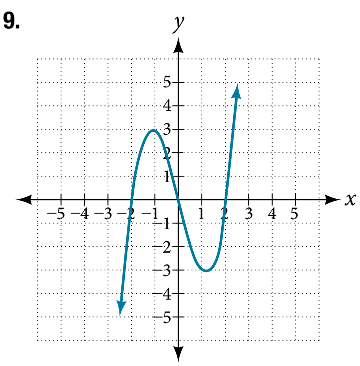
For the following exercises, determine whether the functions are one-to-one.

7. $f(x) = -3x + 5$

8. $f(x) = |x - 3|$

For the following exercises, use the vertical line test to determine if the relation whose graph is provided is a function.

9.



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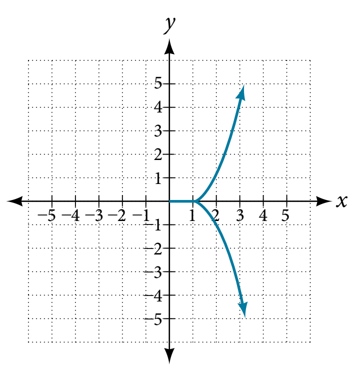

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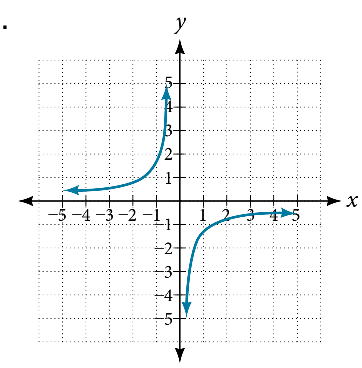


For the following exercises, graph the functions.

12. $f(x) = |x + 1|$

13. $f(x) = x^2 - 2$

For the following exercises, use **Figure 2** to approximate the values.

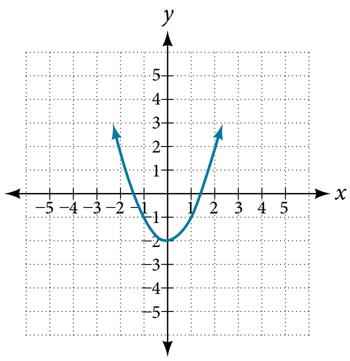


Figure 2

14. $f(2)$

15. $f(-2)$

16. If $f(x) = -2$, then solve for x.

17. If $f(x) = 1$, then solve for x.

For the following exercises, use the function $h(t) = -16t^2 + 80t$ to find the values.

18. $\dfrac{h(2) - h(1)}{2 - 1}$

19. $\dfrac{h(a) - h(1)}{a - 1}$

DOMAIN AND RANGE

For the following exercises, find the domain of each function, expressing answers using interval notation.

20. $f(x) = \dfrac{2}{3x + 2}$

21. $f(x) = \dfrac{x - 3}{x^2 - 4x - 12}$

22. $f(x) = \dfrac{\sqrt{x - 6}}{\sqrt{x - 4}}$

23. Graph this piecewise function: $f(x) = \begin{cases} x + 1 & x < -2 \\ -2x - 3 & x \geq -2 \end{cases}$

RATES OF CHANGE AND BEHAVIOR OF GRAPHS

For the following exercises, find the average rate of change of the functions from $x = 1$ to $x = 2$.

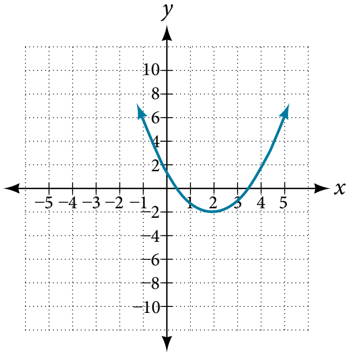
24. $f(x) = 4x - 3$

25. $f(x) = 10x^2 + x$

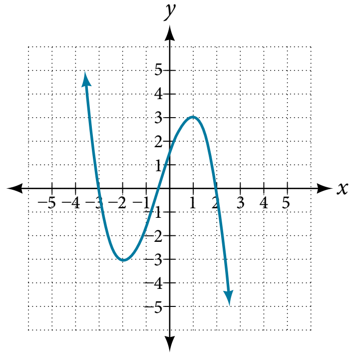
26. $f(x) = -\dfrac{2}{x^2}$

For the following exercises, use the graphs to determine the intervals on which the functions are increasing, decreasing, or constant.

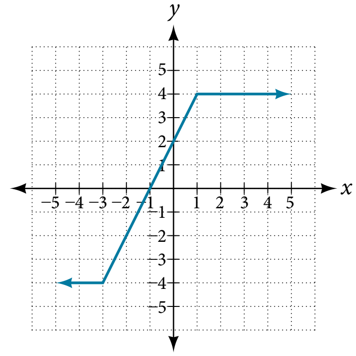
27.



28.



29.



30. Find the local minimum of the function graphed in **Exercise 27**.

31. Find the local extrema for the function graphed in **Exercise 28**.

32. For the graph in **Figure 3**, the domain of the function is $[-3, 3]$. The range is $[-10, 10]$. Find the absolute minimum of the function on this interval.

33. Find the absolute maximum of the function graphed in **Figure 3**.

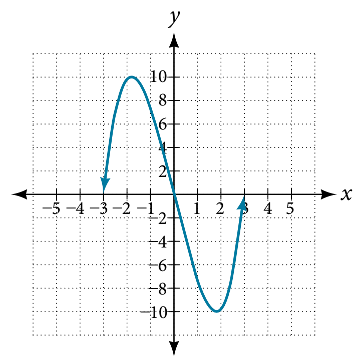


Figure 3

COMPOSITION OF FUNCTIONS

For the following exercises, find $(f \circ g)(x)$ and $(g \circ f)(x)$ for each pair of functions.

34. $f(x) = 4 - x,\ g(x) = -4x$

35. $f(x) = 3x + 2,\ g(x) = 5 - 6x$

36. $f(x) = x^2 + 2x,\ g(x) = 5x + 1$

37. $f(x) = \sqrt{x + 2},\ g(x) = \dfrac{1}{x}$

38. $f(x) = \dfrac{x + 3}{2},\ g(x) = \sqrt{1 - x}$

For the following exercises, find $(f \circ g)$ and the domain for $(f \circ g)(x)$ for each pair of functions.

39. $f(x) = \dfrac{x + 1}{x + 4},\ g(x) = \dfrac{1}{x}$

40. $f(x) = \dfrac{1}{x + 3},\ g(x) = \dfrac{1}{x - 9}$

41. $f(x) = \dfrac{1}{x},\ g(x) = \sqrt{x}$

42. $f(x) = \dfrac{1}{x^2 - 1},\ g(x) = \sqrt{x + 1}$

For the following exercises, express each function H as a composition of two functions f and g where $H(x) = (f \circ g)(x)$.

43. $H(x) = \sqrt{\dfrac{2x - 1}{3x + 4}}$

44. $H(x) = \dfrac{1}{(3x^2 - 4)^{-3}}$

TRANSFORMATION OF FUNCTIONS

For the following exercises, sketch a graph of the given function.

45. $f(x) = (x - 3)^2$

46. $f(x) = (x + 4)^3$

47. $f(x) = \sqrt{x} + 5$

48. $f(x) = -x^3$

49. $f(x) = \sqrt[3]{-x}$

50. $f(x) = 5\sqrt{-x} - 4$

51. $f(x) = 4[|x - 2| - 6]$

52. $f(x) = -(x + 2)^2 - 1$

For the following exercises, sketch the graph of the function g if the graph of the function f is shown in **Figure 4**.

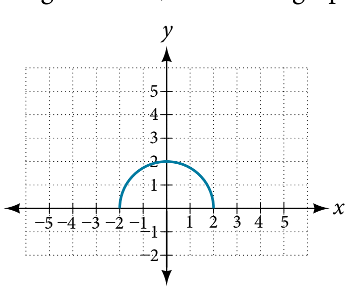


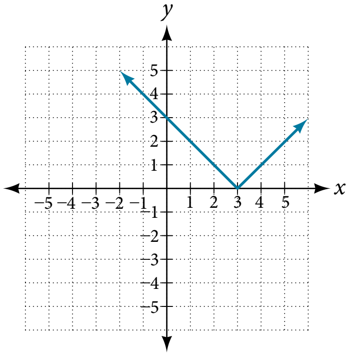
Figure 4

53. $g(x) = f(x - 1)$

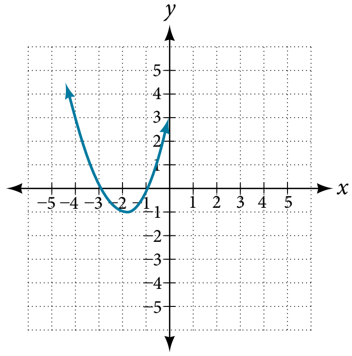
54. $g(x) = 3f(x)$

For the following exercises, write the equation for the standard function represented by each of the graphs below.

55.



56.



For the following exercises, determine whether each function below is even, odd, or neither.

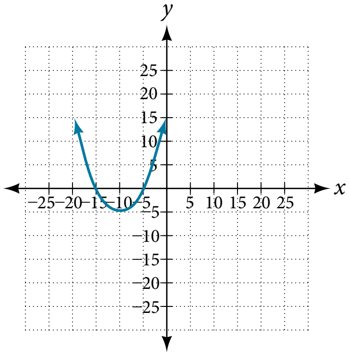
57. $f(x) = 3x^4$

58. $g(x) = \sqrt{x}$

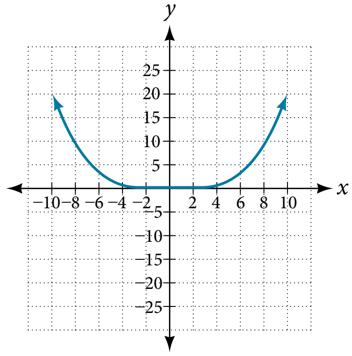
59. $h(x) = \dfrac{1}{x} + 3x$

For the following exercises, analyze the graph and determine whether the graphed function is even, odd, or neither.

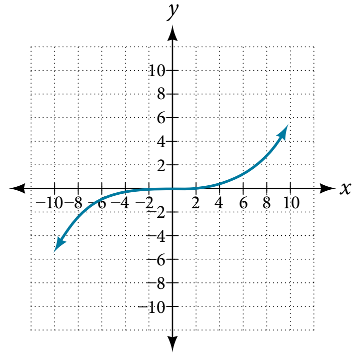
60.



61.



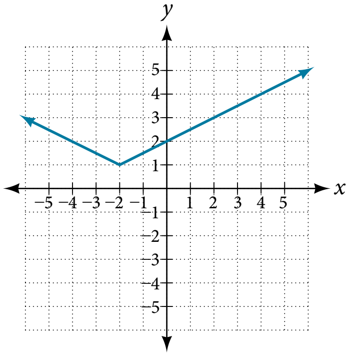
62.



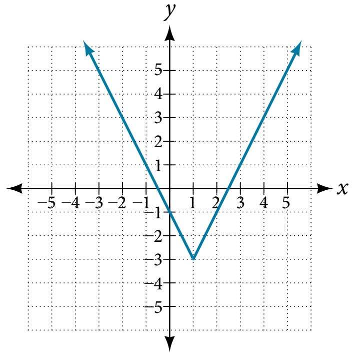
ABSOLUTE VALUE FUNCTIONS

For the following exercises, write an equation for the transformation of $f(x) = |x|$.

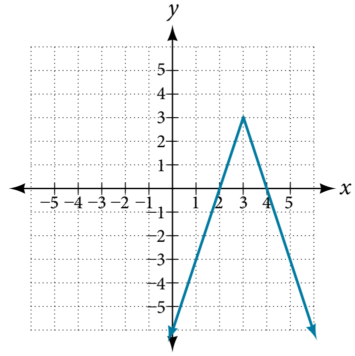
63.



64.



65.



For the following exercises, graph the absolute value function.

66. $f(x) = |x - 5|$

67. $f(x) = -|x - 3|$

68. $f(x) = |2x - 4|$

INVERSE FUNCTIONS

For the following exercises, find $f^{-1}(x)$ for each function.

69. $f(x) = 9 + 10x$

70. $f(x) = \dfrac{x}{x+2}$

For the following exercise, find a domain on which the function f is one-to-one and non-decreasing. Write the domain in interval notation. Then find the inverse of f restricted to that domain.

71. $f(x) = x^2 + 1$

72. Given $f(x) = x^3 - 5$ and $g(x) = \sqrt[3]{x+5}$:

a. Find $f(g(x))$ and $g(f(x))$.

b. What does the answer tell us about the relationship between $f(x)$ and $g(x)$?

For the following exercises, use a graphing utility to determine whether each function is one-to-one.

73. $f(x) = \dfrac{1}{x}$

74. $f(x) = -3x^2 + x$

75. If $f(5) = 2$, find $f^{-1}(2)$.

76. If $f(1) = 4$, find $f^{-1}(4)$.

CHAPTER 3 PRACTICE TEST

For the following exercises, determine whether each of the following relations is a function.

1. $y = 2x + 8$

2. $\{(2, 1), (3, 2), (-1, 1), (0, -2)\}$

For the following exercises, evaluate the function $f(x) = -3x^2 + 2x$ at the given input.

3. $f(-2)$

4. $f(a)$

5. Show that the function $f(x) = -2(x - 1)^2 + 3$ is not one-to-one.

6. Write the domain of the function $f(x) = \sqrt{3 - x}$ in interval notation.

7. Given $f(x) = 2x^2 - 5x$, find $f(a + 1) - f(1)$.

8. Graph the function $f(x) = \begin{cases} x + 1 \text{ if } -2 < x < 3 \\ -x \quad \text{if} \quad x \geq 3 \end{cases}$

9. Find the average rate of change of the function $f(x) = 3 - 2x^2 + x$ by finding $\dfrac{f(b) - f(a)}{b - a}$.

For the following exercises, use the functions $f(x) = 3 - 2x^2 + x$ and $g(x) = \sqrt{x}$ to find the composite functions.

10. $(g \circ f)(x)$

11. $(g \circ f)(1)$

12. Express $H(x) = \sqrt[3]{5x^2 - 3x}$ as a composition of two functions, f and g, where $(f \circ g)(x) = H(x)$.

For the following exercises, graph the functions by translating, stretching, and/or compressing a toolkit function.

13. $f(x) = \sqrt{x + 6} - 1$

14. $f(x) = \dfrac{1}{x + 2} - 1$

For the following exercises, determine whether the functions are even, odd, or neither.

15. $f(x) = -\dfrac{5}{x^2} + 9x^6$

16. $f(x) = -\dfrac{5}{x^3} + 9x^5$

17. $f(x) = \dfrac{1}{x}$

18. Graph the absolute value function $f(x) = -2|x - 1| + 3$.

For the following exercises, find the inverse of the function.

19. $f(x) = 3x - 5$

20. $f(x) = \dfrac{4}{x + 7}$

For the following exercises, use the graph of g shown in **Figure 1**.

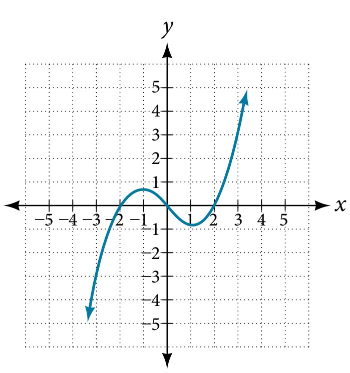


Figure 1

21. On what intervals is the function increasing?

22. On what intervals is the function decreasing?

23. Approximate the local minimum of the function. Express the answer as an ordered pair.

24. Approximate the local maximum of the function. Express the answer as an ordered pair.

For the following exercises, use the graph of the piecewise function shown in **Figure 2**.

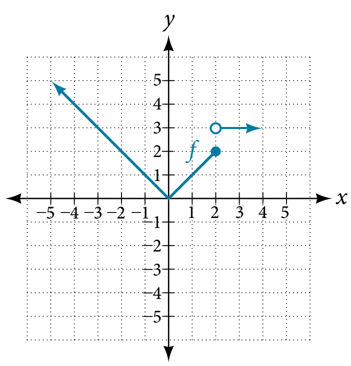


Figure 2

25. Find $f(2)$.

26. Find $f(-2)$.

27. Write an equation for the piecewise function.

For the following exercises, use the values listed in **Table 1**.

x	0	1	2	3	4	5	6	7	8
$F(x)$	1	3	5	7	9	11	13	15	17

Table 1

28. Find $F(6)$.

29. Solve the equation $F(x) = 5$.

30. Is the graph increasing or decreasing on its domain?

31. Is the function represented by the graph one-to-one?

32. Find $F^{-1}(15)$.

33. Given $f(x) = -2x + 11$, find $f^{-1}(x)$.

7

The Unit Circle: Sine and Cosine Functions

Figure 1 The tide rises and falls at regular, predictable intervals. (credit: Andrea Schaffer, Flickr)

CHAPTER OUTLINE

Introduction

Life is dense with phenomena that repeat in regular intervals. Each day, for example, the tides rise and fall in response to the gravitational pull of the moon. Similarly, the progression from day to night occurs as a result of Earth's rotation, and the pattern of the seasons repeats in response to Earth's revolution around the sun. Outside of nature, many stocks that mirror a company's profits are influenced by changes in the economic business cycle.

In mathematics, a function that repeats its values in regular intervals is known as a periodic function. The graphs of such functions show a general shape reflective of a pattern that keeps repeating. This means the graph of the function has the same output at exactly the same place in every cycle. And this translates to all the cycles of the function having exactly the same length. So, if we know all the details of one full cycle of a true periodic function, then we know the state of the function's outputs at all times, future and past. In this chapter, we will investigate various examples of periodic functions.

LEARNING OBJECTIVES

In this section, you will:

- Draw angles in standard position.
- Convert between degrees and radians.
- Find coterminal angles.
- Find the length of a circular arc.
- Use linear and angular speed to describe motion on a circular path.

7.1 ANGLES

A golfer swings to hit a ball over a sand trap and onto the green. An airline pilot maneuvers a plane toward a narrow runway. A dress designer creates the latest fashion. What do they all have in common? They all work with angles, and so do all of us at one time or another. Sometimes we need to measure angles exactly with instruments. Other times we estimate them or judge them by eye. Either way, the proper angle can make the difference between success and failure in many undertakings. In this section, we will examine properties of angles.

Drawing Angles in Standard Position

Properly defining an angle first requires that we define a ray. A **ray** consists of one point on a line and all points extending in one direction from that point. The first point is called the endpoint of the ray. We can refer to a specific ray by stating its endpoint and any other point on it. The ray in **Figure 1** can be named as ray EF, or in symbol form $\overrightarrow{EF}$.

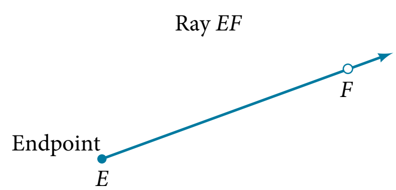


Figure 1

An **angle** is the union of two rays having a common endpoint. The endpoint is called the **vertex** of the angle, and the two rays are the sides of the angle. The angle in **Figure 2** is formed from $\overrightarrow{ED}$ and $\overrightarrow{EF}$. Angles can be named using a point on each ray and the vertex, such as angle *DEF*, or in symbol form $\angle DEF$.

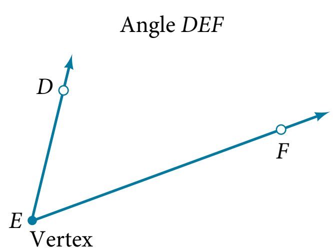


Figure 2

Greek letters are often used as variables for the measure of an angle. **Table 1** is a list of Greek letters commonly used to represent angles, and a sample angle is shown in **Figure 3**.

θ	φ or ϕ	α	β	γ
theta	phi	alpha	beta	gamma

Table 1

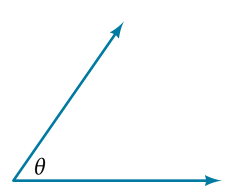


Figure 3 Angle theta, shown as $\angle\theta$

Angle creation is a dynamic process. We start with two rays lying on top of one another. We leave one fixed in place, and rotate the other. The fixed ray is the **initial side,** and the rotated ray is the **terminal side**. In order to identify the different sides, we indicate the rotation with a small arc and arrow close to the vertex as in **Figure 4**.

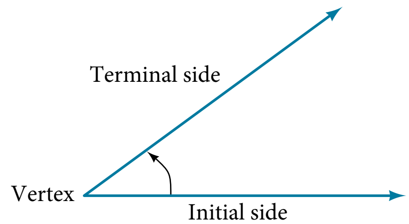


Figure 4

As we discussed at the beginning of the section, there are many applications for angles, but in order to use them correctly, we must be able to measure them. The **measure of an angle** is the amount of rotation from the initial side to the terminal side. Probably the most familiar unit of angle measurement is the degree. One **degree** is $\frac{1}{360}$ of a circular rotation, so a complete circular rotation contains 360 degrees. An angle measured in degrees should always include the unit "degrees" after the number, or include the degree symbol °. For example, 90 degrees = 90°.

To formalize our work, we will begin by drawing angles on an *x-y* coordinate plane. Angles can occur in any position on the coordinate plane, but for the purpose of comparison, the convention is to illustrate them in the same position whenever possible. An angle is in **standard position** if its vertex is located at the origin, and its initial side extends along the positive *x*-axis. See **Figure 5**.

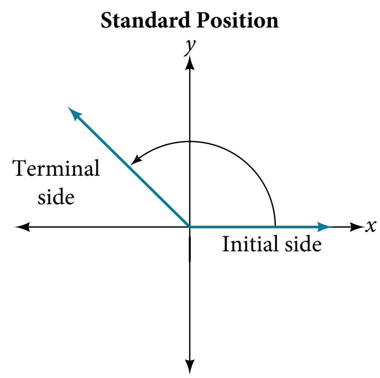


Figure 5

If the angle is measured in a counterclockwise direction from the initial side to the terminal side, the angle is said to be a **positive angle**. If the angle is measured in a clockwise direction, the angle is said to be a **negative angle**.

Drawing an angle in standard position always starts the same way—draw the initial side along the positive *x*-axis. To place the terminal side of the angle, we must calculate the fraction of a full rotation the angle represents. We do that by dividing the angle measure in degrees by 360°. For example, to draw a 90° angle, we calculate that $\frac{90°}{360°} = \frac{1}{4}$. So, the terminal side will be one-fourth of the way around the circle, moving counterclockwise from the positive *x*-axis. To draw a 360° angle, we calculate that $\frac{360°}{360°} = 1$. So the terminal side will be 1 complete rotation around the circle, moving counterclockwise from the positive *x*-axis. In this case, the initial side and the terminal side overlap. See **Figure 6**.

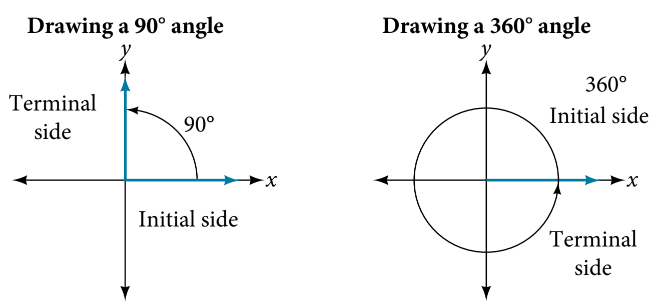


Figure 6

Since we define an angle in standard position by its terminal side, we have a special type of angle whose terminal side lies on an axis, a **quadrantal angle**. This type of angle can have a measure of 0°, 90°, 180°, 270° or 360°. See **Figure 7**.

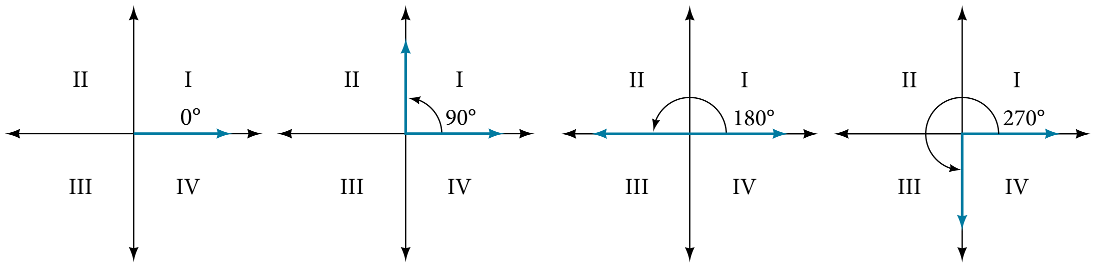


Figure 7 Quadrantal angles have a terminal side that lies along an axis. Examples are shown.

quadrantal angles

An angle is a **quadrantal angle** if its terminal side lies on an axis, including 0°, 90°, 180°, 270°, or 360°.

How To…

Given an angle measure in degrees, draw the angle in standard position.

1. Express the angle measure as a fraction of 360°.
2. Reduce the fraction to simplest form.
3. Draw an angle that contains that same fraction of the circle, beginning on the positive x-axis and moving counterclockwise for positive angles and clockwise for negative angles.

Example 1 Drawing an Angle in Standard Position Measured in Degrees

a. Sketch an angle of 30° in standard position.

b. Sketch an angle of $-135°$ in standard position.

Solution

a. Divide the angle measure by 360°.

$$\frac{30°}{360°} = \frac{1}{12}$$

To rewrite the fraction in a more familiar fraction, we can recognize that

$$\frac{1}{12} = \frac{1}{3}\left(\frac{1}{4}\right)$$

One-twelfth equals one-third of a quarter, so by dividing a quarter rotation into thirds, we can sketch a line at 30° as in **Figure 8**.

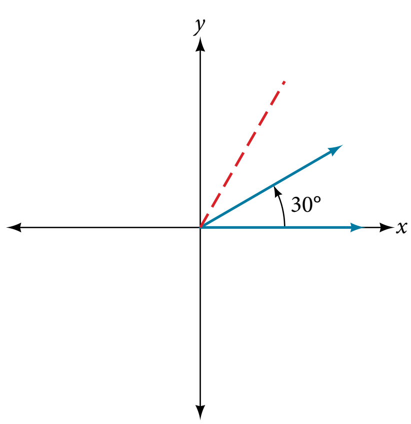


Figure 8

b. Divide the angle measure by 360°.

$$\frac{-135°}{360°} = -\frac{3}{8}$$

In this case, we can recognize that

$$-\frac{3}{8} = -\frac{3}{2}\left(\frac{1}{4}\right)$$

Negative three-eighths is one and one-half times a quarter, so we place a line by moving clockwise one full quarter and one-half of another quarter, as in **Figure 9**.

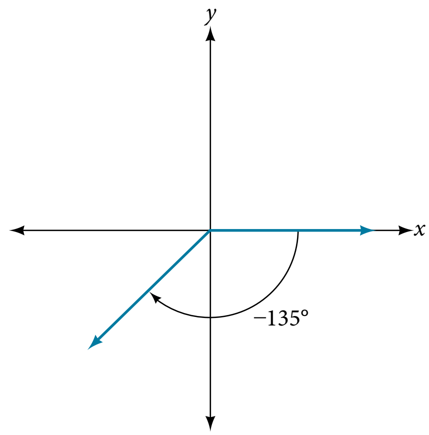


Figure 9

Try It #1

Show an angle of 240° on a circle in standard position.

Converting Between Degrees and Radians

Dividing a circle into 360 parts is an arbitrary choice, although it creates the familiar degree measurement. We may choose other ways to divide a circle. To find another unit, think of the process of drawing a circle. Imagine that you stop before the circle is completed. The portion that you drew is referred to as an arc. An arc may be a portion of a full circle, a full circle, or more than a full circle, represented by more than one full rotation. The length of the arc around an entire circle is called the circumference of that circle.

The circumference of a circle is $C = 2\pi r$. If we divide both sides of this equation by r, we create the ratio of the circumference to the radius, which is always 2π regardless of the length of the radius. So the circumference of any circle is $2\pi \approx 6.28$ times the length of the radius. That means that if we took a string as long as the radius and used it to measure consecutive lengths around the circumference, there would be room for six full string-lengths and a little more than a quarter of a seventh, as shown in **Figure 10**.

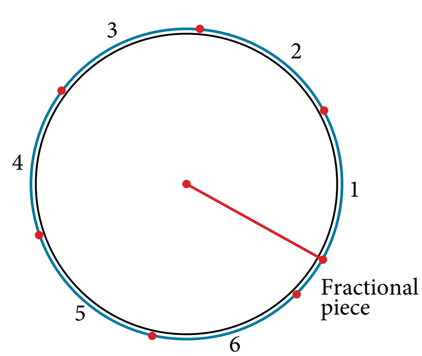


Figure 10

This brings us to our new angle measure. One **radian** is the measure of a central angle of a circle that intercepts an arc equal in length to the radius of that circle. A central angle is an angle formed at the center of a circle by two radii. Because the total circumference equals 2π times the radius, a full circular rotation is 2π radians. So

$$2\pi \text{ radians} = 360°$$

$$\pi \text{ radians} = \frac{360°}{2} = 180°$$

$$1 \text{ radian} = \frac{180°}{\pi} \approx 57.3°$$

See **Figure 11**. Note that when an angle is described without a specific unit, it refers to radian measure. For example, an angle measure of 3 indicates 3 radians. In fact, radian measure is dimensionless, since it is the quotient of a length (circumference) divided by a length (radius) and the length units cancel out.

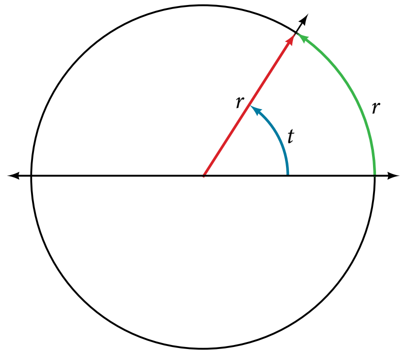


Figure 11 The angle t sweeps out a measure of one radian. Note that the length of the intercepted arc is the same as the length of the radius of the circle.

Relating Arc Lengths to Radius

An **arc length** s is the length of the curve along the arc. Just as the full circumference of a circle always has a constant ratio to the radius, the arc length produced by any given angle also has a constant relation to the radius, regardless of the length of the radius.

This ratio, called the radian measure, is the same regardless of the radius of the circle—it depends only on the angle. This property allows us to define a measure of any angle as the ratio of the arc length s to the radius r. See **Figure 12**.

$$s = r\theta$$

$$\theta = \frac{s}{r}$$

If $s = r$, then $\theta = \frac{r}{r} = 1$ radian.

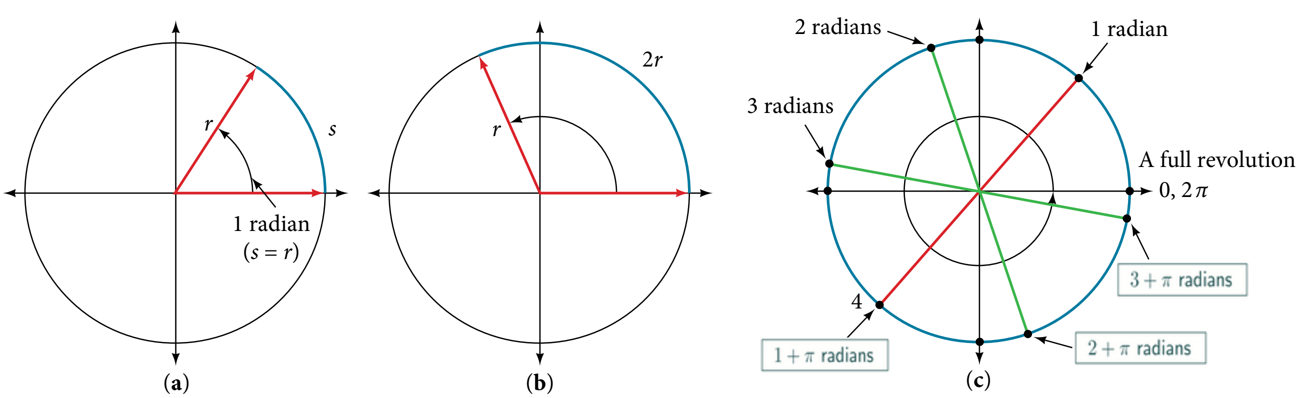


Figure 12 (a) In an angle of 1 radian, the arc length s equals the radius r.
(b) An angle of 2 radians has an arc length $s = 2r$. (c) A full revolution is 2π or about 6.28 radians.

To elaborate on this idea, consider two circles, one with radius 2 and the other with radius 3. Recall the circumference of a circle is $C = 2\pi r$, where r is the radius. The smaller circle then has circumference $2\pi(2) = 4\pi$ and the larger has circumference $2\pi(3) = 6\pi$. Now we draw a 45°angle on the two circles, as in **Figure 13**.

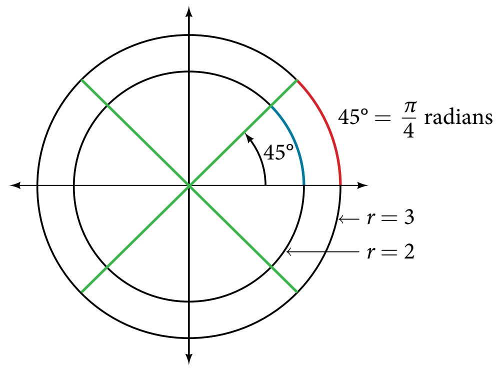


Figure 13 A 45° angle contains one-eighth of the circumference of a circle, regardless of the radius.

Notice what happens if we find the ratio of the arc length divided by the radius of the circle.

$$\text{Smaller circle: } \frac{\frac{1}{2}\pi}{2} = \frac{1}{4}\pi$$

$$\text{Larger circle: } \frac{\frac{3}{4}\pi}{3} = \frac{1}{4}\pi$$

Since both ratios are $\frac{1}{4}\pi$, the angle measures of both circles are the same, even though the arc length and radius differ.

> ***radians***
>
> One radian is the measure of the central angle of a circle such that the length of the arc between the initial side and the terminal side is equal to the radius of the circle. A full revolution (360°) equals 2π radians. A half revolution (180°) is equivalent to π radians.
>
> The **radian measure** of an angle is the ratio of the length of the arc subtended by the angle to the radius of the circle. In other words, if s is the length of an arc of a circle, and r is the radius of the circle, then the central angle containing that arc measures $\frac{s}{r}$ radians. In a circle of radius 1, the radian measure corresponds to the length of the arc.

Q & A...

A measure of 1 radian looks to be about 60°. Is that correct?

Yes. It is approximately 57.3°. Because 2π radians equals 360°, 1 radian equals $\frac{360°}{2\pi} \approx 57.3°$.

Using Radians

Because radian measure is the ratio of two lengths, it is a unitless measure. For example, in **Figure 11**, suppose the radius was 2 inches and the distance along the arc was also 2 inches. When we calculate the radian measure of the angle, the "inches" cancel, and we have a result without units. Therefore, it is not necessary to write the label "radians" after a radian measure, and if we see an angle that is not labeled with "degrees" or the degree symbol, we can assume that it is a radian measure.

Considering the most basic case, the unit circle (a circle with radius 1), we know that 1 rotation equals 360 degrees, 360°. We can also track one rotation around a circle by finding the circumference, $C = 2\pi r$, and for the unit circle $C = 2\pi$. These two different ways to rotate around a circle give us a way to convert from degrees to radians.

$$1 \text{ rotation} = 360° = 2\pi \text{ radians}$$

$$\frac{1}{2} \text{rotation} = 180° = \pi \text{ radians}$$

$$\frac{1}{4} \text{rotation} = 90° = \frac{\pi}{2} \text{ radians}$$

Identifying Special Angles Measured in Radians

In addition to knowing the measurements in degrees and radians of a quarter revolution, a half revolution, and a full revolution, there are other frequently encountered angles in one revolution of a circle with which we should be familiar. It is common to encounter multiples of 30, 45, 60, and 90 degrees. These values are shown in **Figure 14**. Memorizing these angles will be very useful as we study the properties associated with angles.

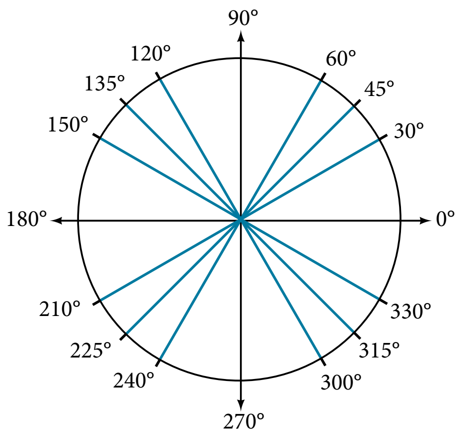


Figure 14 Commonly encountered angles measured in degrees

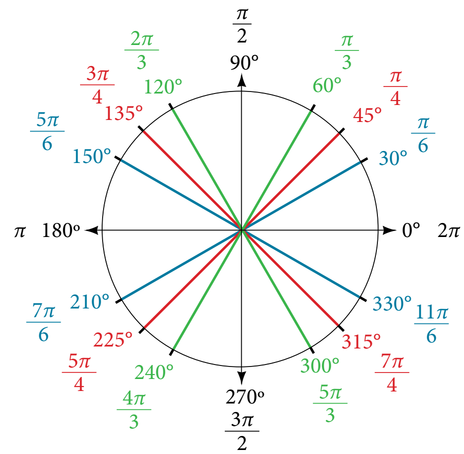


Figure 15 Commonly encountered angles measured in radians

Now, we can list the corresponding radian values for the common measures of a circle corresponding to those listed in **Figure 14**, which are shown in **Figure 15**. Be sure you can verify each of these measures.

Example 2 Finding a Radian Measure

Find the radian measure of one-third of a full rotation.

Solution For any circle, the arc length along such a rotation would be one-third of the circumference. We know that

$$1 \text{ rotation} = 2\pi r$$

So,

$$\begin{aligned} s &= \frac{1}{3}(2\pi r) \\ &= \frac{2\pi r}{3} \end{aligned}$$

The radian measure would be the arc length divided by the radius.

$$\begin{aligned} \text{radian measure} &= \frac{\frac{2\pi r}{3}}{r} \\ &= \frac{2\pi r}{3r} \\ &= \frac{2\pi}{3} \end{aligned}$$

Try It #2

Find the radian measure of three-fourths of a full rotation.

Converting Between Radians and Degrees

Because degrees and radians both measure angles, we need to be able to convert between them. We can easily do so using a proportion where θ is the measure of the angle in degrees and θ_R is the measure of the angle in radians.

$$\frac{\theta}{180} = \frac{\theta_R}{\pi}$$

This proportion shows that the measure of angle θ in degrees divided by 180 equals the measure of angle θ in radians divided by π. Or, phrased another way, degrees is to 180 as radians is to π.

$$\frac{\text{Degrees}}{180} = \frac{\text{Radians}}{\pi}$$

converting between radians and degrees

To convert between degrees and radians, use the proportion

$$\frac{\theta}{180} = \frac{\theta_R}{\pi}$$

Example 3 Converting Radians to Degrees

Convert each radian measure to degrees.

a. $\frac{\pi}{6}$ **b.** 3

Solution Because we are given radians and we want degrees, we should set up a proportion and solve it.

a. We use the proportion, substituting the given information.

$$\frac{\theta}{180} = \frac{\theta_R}{\pi}$$

$$\frac{\theta}{180} = \frac{\frac{\pi}{6}}{\pi}$$

$$\theta = \frac{180}{6}$$

$$\theta = 30°$$

b. We use the proportion, substituting the given information.

$$\frac{\theta}{180} = \frac{\theta_R}{\pi}$$

$$\frac{\theta}{180} = \frac{3}{\pi}$$

$$\theta = \frac{3(180)}{\pi}$$

$$\theta \approx 172°$$

Try It #3

Convert $-\frac{3\pi}{4}$ radians to degrees.

Example 4 Converting Degrees to Radians

Convert 15 degrees to radians.

Solution In this example, we start with degrees and want radians, so we again set up a proportion and solve it, but we substitute the given information into a different part of the proportion.

$$\frac{\theta}{180} = \frac{\theta_R}{\pi}$$

$$\frac{15}{180} = \frac{\theta_R}{\pi}$$

$$\frac{15\pi}{180} = \theta_R$$

$$\frac{\pi}{12} = \theta_R$$

Analysis *Another way to think about this problem is by remembering that* $30° = \frac{\pi}{6}$. *Because* $15° = \frac{1}{2}(30°)$, *we can find that* $\frac{1}{2}\left(\frac{\pi}{6}\right)$ *is* $\frac{\pi}{12}$.

Try It #4

Convert 126° to radians.

Finding Coterminal Angles

Converting between degrees and radians can make working with angles easier in some applications. For other applications, we may need another type of conversion. Negative angles and angles greater than a full revolution are more awkward to work with than those in the range of 0° to 360°, or 0 to 2π. It would be convenient to replace those out-of-range angles with a corresponding angle within the range of a single revolution.

It is possible for more than one angle to have the same terminal side. Look at **Figure 16**. The angle of 140° is a positive angle, measured counterclockwise. The angle of $-220°$ is a negative angle, measured clockwise. But both angles have the same terminal side. If two angles in standard position have the same terminal side, they are coterminal angles. Every angle greater than 360° or less than 0° is coterminal with an angle between 0° and 360°, and it is often more convenient to find the coterminal angle within the range of 0° to 360° than to work with an angle that is outside that range.

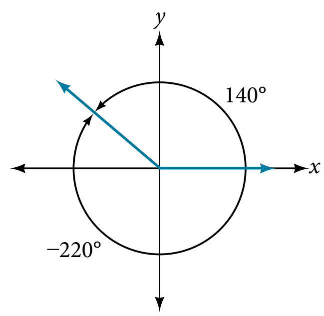


Figure 16 An angle of 140° and an angle of −220° are coterminal angles.

An *actue* angle is an angle whose measure θ satisfies $0° < \theta < 90°$ or, in radians, $0 < \theta < \pi/2$.

Any angle has infinitely many coterminal angles because each time we add 360° to that angle—or subtract 360° from it—the resulting value has a terminal side in the same location. For example, 100° and 460° are coterminal for this reason, as is $-260°$.

An angle's reference angle is the measure of the smallest, positive, acute angle t formed by the terminal side of the angle t and the horizontal axis. Thus positive reference angles have terminal sides that lie in the first quadrant and can be used as models for angles in other quadrants. See **Figure 17** for examples of reference angles for angles in different quadrants.

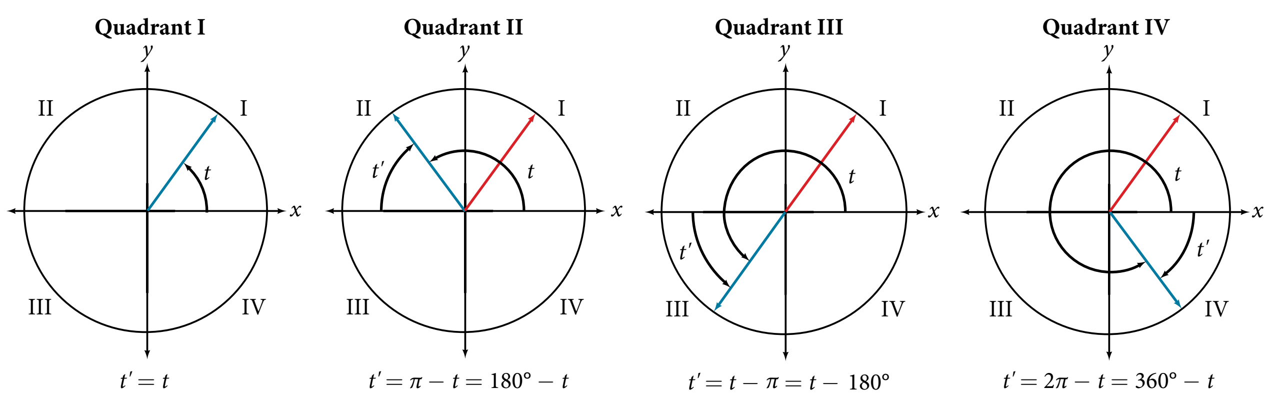


Figure 17

coterminal and reference angles

Coterminal angles are two angles in standard position that have the same terminal side.

An angle's **reference angle** is the size of the smallest acute angle, t', formed by the terminal side of the angle t and the horizontal axis.

How To...

Given an angle greater than 360°, find a coterminal angle between 0° and 360°.

1. Subtract 360° from the given angle.
2. If the result is still greater than 360°, subtract 360° again till the result is between 0° and 360°.
3. The resulting angle is coterminal with the original angle.

Example 5 Finding an Angle Coterminal with an Angle of Measure Greater Than 360°

Find the least positive angle θ that is coterminal with an angle measuring 800°, where $0° \leq \theta < 360°$.

Solution An angle with measure 800° is coterminal with an angle with measure $800 - 360 = 440°$, but 440° is still greater than 360°, so we subtract 360° again to find another coterminal angle: $440 - 360 = 80°$.

The angle $\theta = 80°$ is coterminal with 800°. To put it another way, 800° equals 80° plus two full rotations, as shown in **Figure 18**.

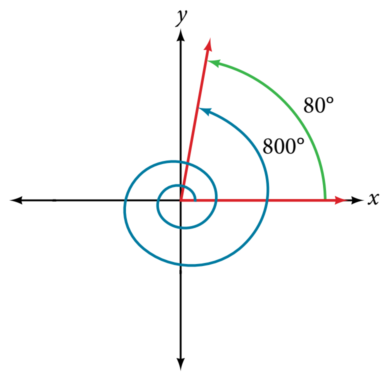


Figure 18

Try It #5

Find an angle α that is coterminal with an angle measuring 870°, where $0° \leq \alpha < 360°$.

How To...

Given an angle with measure less than 0°, find a coterminal angle having a measure between 0° and 360°.

1. Add 360° to the given angle.
2. If the result is still less than 0°, add 360° again until the result is between 0° and 360°.
3. The resulting angle is coterminal with the original angle.

Example 6 Finding an Angle Coterminal with an Angle Measuring Less Than 0°

Show the angle with measure $-45°$ on a circle and find a positive coterminal angle α such that $0° \leq \alpha < 360°$.

Solution Since 45° is half of 90°, we can start at the positive horizontal axis and measure clockwise half of a 90° angle.

Because we can find coterminal angles by adding or subtracting a full rotation of 360°, we can find a positive coterminal angle here by adding 360°:

$$-45° + 360° = 315°$$

We can then show the angle on a circle, as in **Figure 19**.

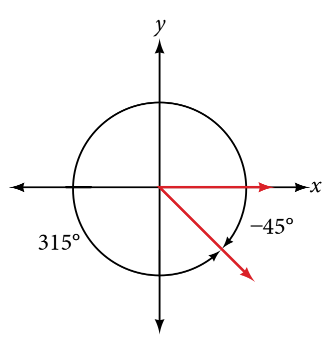


Figure 19

Try It #6

Find an angle β that is coterminal with an angle measuring $-300°$ such that $0° \leq \beta < 360°$.

Finding Coterminal Angles Measured in Radians

We can find coterminal angles measured in radians in much the same way as we have found them using degrees. In both cases, we find coterminal angles by adding or subtracting one or more full rotations.

How To...

Given an angle greater than 2π, find a coterminal angle between 0 and 2π.

1. Subtract 2π from the given angle.
2. If the result is still greater than 2π, subtract 2π again until the result is between 0 and 2π.
3. The resulting angle is coterminal with the original angle.

Example 7 Finding Coterminal Angles Using Radians

Find an angle β that is coterminal with $\frac{19\pi}{4}$, where $0 \leq \beta < 2\pi$.

Solution When working in degrees, we found coterminal angles by adding or subtracting 360 degrees, a full rotation. Likewise, in radians, we can find coterminal angles by adding or subtracting full rotations of 2π radians:

$$\begin{aligned}\frac{19\pi}{4} - 2\pi &= \frac{19\pi}{4} - \frac{8\pi}{4}\\ &= \frac{11\pi}{4}\end{aligned}$$

The angle $\frac{11\pi}{4}$ is coterminal, but not less than 2π, so we subtract another rotation:

$$\begin{aligned}\frac{11\pi}{4} - 2\pi &= \frac{11\pi}{4} - \frac{8\pi}{4}\\ &= \frac{3\pi}{4}\end{aligned}$$

The angle $\frac{3\pi}{4}$ is coterminal with $\frac{19\pi}{4}$, as shown in **Figure 20**.

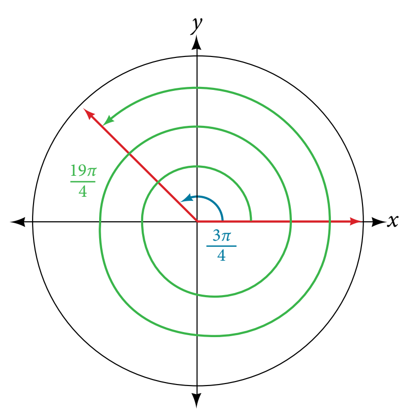


Figure 20

Try It #7

Find an angle of measure θ that is coterminal with an angle of measure $-\frac{17\pi}{6}$ where $0 \leq \theta < 2\pi$.

Determining the Length of an Arc

Recall that the radian measure θ of an angle was defined as the ratio of the arc length s of a circular arc to the radius r of the circle, $\theta = \frac{s}{r}$. From this relationship, we can find arc length along a circle, given an angle.

arc length on a circle

In a circle of radius r, the length of an arc s subtended by an angle with measure θ in radians, shown in **Figure 21**, is

$$s = r\theta$$

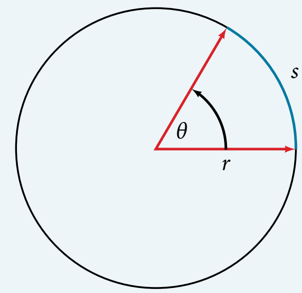


Figure 21

How To…

Given a circle of radius r, calculate the length s of the arc subtended by a given angle of measure θ.

1. If necessary, convert θ to radians.
2. Multiply the radius r by the radian measure of $\theta : s = r\theta$.

Example 8 Finding the Length of an Arc

Assume the orbit of Mercury around the sun is a perfect circle. Mercury is approximately 36 million miles from the sun.

a. In one Earth day, Mercury completes 0.0114 of its total revolution. How many miles does it travel in one day?

b. Use your answer from part (a) to determine the radian measure for Mercury's movement in one Earth day.

Solution

a. Let's begin by finding the circumference of Mercury's orbit.

$$\begin{aligned} C &= 2\pi r \\ &= 2\pi(36 \text{ million miles}) \\ &\approx 226 \text{ million miles} \end{aligned}$$

Since Mercury completes 0.0114 of its total revolution in one Earth day, we can now find the distance traveled:

$$(0.0114)226 \text{ million miles} = 2.58 \text{ million miles}$$

b. Now, we convert to radians:

$$\begin{aligned} \text{radian} &= \frac{\text{arclength}}{\text{radius}} \\ &= \frac{2.58 \text{ million miles}}{36 \text{ million miles}} \\ &= 0.0717 \end{aligned}$$

Try It #8

Find the arc length along a circle of radius 10 units subtended by an angle of 215°.

Finding the Area of a Sector of a Circle

In addition to arc length, we can also use angles to find the area of a sector of a circle. A sector is a region of a circle bounded by two radii and the intercepted arc, like a slice of pizza or pie. Recall that the area of a circle with radius r can be found using the formula $A = \pi r^2$. If the two radii form an angle of θ, measured in radians, then $\frac{\theta}{2\pi}$ is the ratio of the angle measure to the measure of a full rotation and is also, therefore, the ratio of the area of the sector to the area of the circle. Thus, the **area of a sector** is the fraction $\frac{\theta}{2\pi}$ multiplied by the entire area. (Always remember that this formula only applies if θ is in radians.)

$$\begin{aligned} \text{Area of sector} &= \left(\frac{\theta}{2\pi}\right)\pi r^2 \\ &= \frac{\theta\pi r^2}{2\pi} \\ &= \frac{1}{2}\theta r^2 \end{aligned}$$

area of a sector

The **area of a sector** of a circle with radius r subtended by an angle θ, measured in radians, is

$$A = \frac{1}{2}\theta r^2$$

See **Figure 22**.

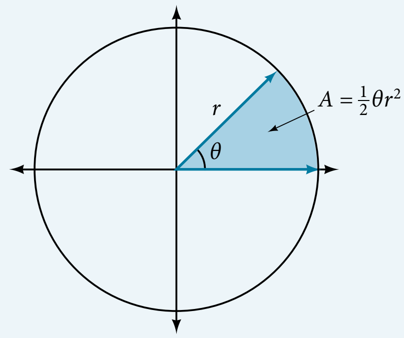


Figure 22 The area of the sector equals half the square of the radius times the central angle measured in radians.

How To...

Given a circle of radius r, find the area of a sector defined by a given angle θ.

1. If necessary, convert θ to radians.
2. Multiply half the radian measure of θ by the square of the radius r: $A = \frac{1}{2}\theta r^2$.

Example 9 Finding the Area of a Sector

An automatic lawn sprinkler sprays a distance of 20 feet while rotating 30 degrees, as shown in **Figure 23**. What is the area of the sector of grass the sprinkler waters?

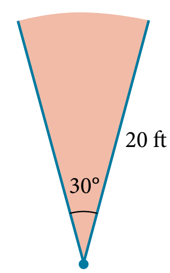


Figure 23 The sprinkler sprays 20 ft within an arc of 30°.

Solution First, we need to convert the angle measure into radians. Because 30 degrees is one of our special angles, we already know the equivalent radian measure, but we can also convert:

$$\begin{aligned} 30 \text{ degrees} &= 30 \cdot \frac{\pi}{180} \\ &= \frac{\pi}{6} \text{ radians} \end{aligned}$$

The area of the sector is then

$$\begin{aligned} \text{Area} &= \frac{1}{2}\left(\frac{\pi}{6}\right)(20)^2 \\ &\approx 104.72 \end{aligned}$$

So the area is about 104.72 ft^2.

Try It #9

In central pivot irrigation, a large irrigation pipe on wheels rotates around a center point. A farmer has a central pivot system with a radius of 400 meters. If water restrictions only allow her to water 150 thousand square meters a day, what angle should she set the system to cover? Write the answer in radian measure to two decimal places.

Use Linear and Angular Speed to Describe Motion on a Circular Path

In addition to finding the area of a sector, we can use angles to describe the speed of a moving object. An object traveling in a circular path has two types of speed. **Linear speed** is speed along a straight path and can be determined by the distance it moves along (its displacement) in a given time interval. For instance, if a wheel with radius 5 inches rotates once a second, a point on the edge of the wheel moves a distance equal to the circumference, or 10π inches, every second. So the linear speed of the point is 10π in./s. The equation for linear speed is as follows where v is linear speed, s is displacement, and t is time.

$$v = \frac{s}{t}$$

Angular speed results from circular motion and can be determined by the angle through which a point rotates in a given time interval. In other words, **angular speed** is angular rotation per unit time. So, for instance, if a gear makes a full rotation every 4 seconds, we can calculate its angular speed as $\frac{360 \text{ degrees}}{4 \text{ seconds}} = 90$ degrees per second. Angular speed can be given in radians per second, rotations per minute, or degrees per hour for example. The equation for angular speed is as follows, where ω (read as omega) is angular speed, θ is the angle traversed, and t is time.

$$\omega = \frac{\theta}{t}$$

Combining the definition of angular speed with the arc length equation, $s = r\theta$, we can find a relationship between angular and linear speeds. The angular speed equation can be solved for θ, giving $\theta = \omega t$. Substituting this into the arc length equation gives:

$$\begin{aligned} s &= r\theta \\ &= r\omega t \end{aligned}$$

Substituting this into the linear speed equation gives:

$$\begin{aligned} v &= \frac{s}{t} \\ &= \frac{r\omega t}{t} \\ &= r\omega \end{aligned}$$

angular and linear speed

As a point moves along a circle of radius r, its **angular speed**, ω, is the angular rotation θ per unit time, t.

$$\omega = \frac{\theta}{t}$$

The **linear speed**, v, of the point can be found as the distance traveled, arc length s, per unit time, t.

$$v = \frac{s}{t}$$

When the angular speed is measured in radians per unit time, linear speed and angular speed are related by the equation

$$v = r\omega$$

This equation states that the angular speed in radians, ω, representing the amount of rotation occurring in a unit of time, can be multiplied by the radius r to calculate the total arc length traveled in a unit of time, which is the definition of linear speed.

How To...

Given the amount of angle rotation and the time elapsed, calculate the angular speed.

1. If necessary, convert the angle measure to radians.
2. Divide the angle in radians by the number of time units elapsed: $\omega = \frac{\theta}{t}$.
3. The resulting speed will be in radians per time unit.

Example 10 Finding Angular Speed

A water wheel, shown in **Figure 24**, completes 1 rotation every 5 seconds. Find the angular speed in radians per second.

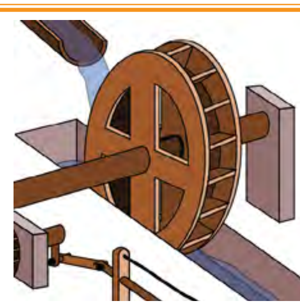


Figure 24

Solution The wheel completes 1 rotation, or passes through an angle of 2π radians in 5 seconds, so the angular speed would be $\omega = \frac{2\pi}{5} \approx 1.257$ radians per second.

Try It #10

An old vinyl record is played on a turntable rotating clockwise at a rate of 45 rotations per minute. Find the angular speed in radians per second.

How To…

Given the radius of a circle, an angle of rotation, and a length of elapsed time, determine the linear speed.

1. Convert the total rotation to radians if necessary.
2. Divide the total rotation in radians by the elapsed time to find the angular speed: apply $\omega = \dfrac{\theta}{t}.$
3. Multiply the angular speed by the length of the radius to find the linear speed, expressed in terms of the length unit used for the radius and the time unit used for the elapsed time: apply $v = r\omega$.

Example 11 Finding a Linear Speed

A bicycle has wheels 28 inches in diameter. A tachometer determines the wheels are rotating at 180 RPM (revolutions per minute). Find the speed the bicycle is traveling down the road.

Solution Here, we have an angular speed and need to find the corresponding linear speed, since the linear speed of the outside of the tires is the speed at which the bicycle travels down the road.

We begin by converting from rotations per minute to radians per minute. It can be helpful to utilize the units to make this conversion:

$$180\,\frac{\cancel{\text{rotations}}}{\text{minute}}\cdot\frac{2\pi\text{ radians}}{\cancel{\text{rotation}}} = 360\pi\,\frac{\text{radians}}{\text{minute}}$$

Using the formula from above along with the radius of the wheels, we can find the linear speed:

$$\begin{aligned} v &= (14\text{ inches})\left(360\pi\,\frac{\text{radians}}{\text{minute}}\right)\\ &= 5040\pi\,\frac{\text{inches}}{\text{minute}} \end{aligned}$$

Remember that radians are a unitless measure, so it is not necessary to include them.

Finally, we may wish to convert this linear speed into a more familiar measurement, like miles per hour.

$$5040\pi\,\frac{\cancel{\text{inches}}}{\cancel{\text{minute}}}\cdot\frac{1\,\cancel{\text{feet}}}{12\,\cancel{\text{inches}}}\cdot\frac{1\text{ mile}}{5280\,\cancel{\text{feet}}}\cdot\frac{60\,\cancel{\text{minutes}}}{1\text{ hour}} \approx 14.99\text{ miles per hour (mph)}$$

Try It #11

A satellite is rotating around Earth at 0.25 radians per hour at an altitude of 242 km above Earth. If the radius of Earth is 6378 kilometers, find the linear speed of the satellite in kilometers per hour.

Access these online resources for additional instruction and practice with angles, arc length, and areas of sectors.

- Angles in Standard Position (http://openstaxcollege.org/l/standardpos)
- Angle of Rotation (http://openstaxcollege.org/l/angleofrotation)
- Coterminal Angles (http://openstaxcollege.org/l/coterminal)
- Determining Coterminal Angles (http://openstaxcollege.org/l/detcoterm)
- Positive and Negative Coterminal Angles (http://openstaxcollege.org/l/posnegcoterm)
- Radian Measure (http://openstaxcollege.org/l/radianmeas)
- Coterminal Angles in Radians (http://openstaxcollege.org/l/cotermrad)
- Arc Length and Area of a Sector (http://openstaxcollege.org/l/arclength)

7.1 SECTION EXERCISES

VERBAL

1. Draw an angle in standard position. Label the vertex, initial side, and terminal side.

2. Explain why there are an infinite number of angles that are coterminal to a certain angle.

3. State what a positive or negative angle signifies, and explain how to draw each.

4. How does radian measure of an angle compare to the degree measure? Include an explanation of 1 radian in your paragraph.

5. Explain the differences between linear speed and angular speed when describing motion along a circular path.

For the following exercises, draw an angle in standard position with the given measure.

6. $30°$ **7.** $300°$ **8.** $-80°$ **9.** $135°$ **10.** $-150°$ **11.** $\frac{2\pi}{3}$

12. $\frac{7\pi}{4}$ **13.** $\frac{5\pi}{6}$ **14.** $\frac{\pi}{2}$ **15.** $-\frac{\pi}{10}$ **16.** $415°$ **17.** $-120°$

18. $-315°$ **19.** $\frac{22\pi}{3}$ **20.** $-\frac{\pi}{6}$ **21.** $-\frac{4\pi}{3}$

For the following exercises, refer to **Figure 25**. Round to two decimal places.

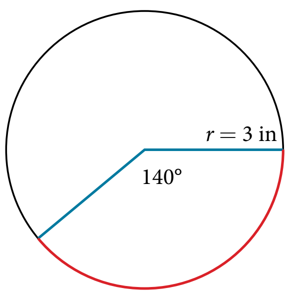


Figure 25

22. Find the arc length.

23. Find the area of the sector.

For the following exercises, refer to **Figure 26**. Round to two decimal places.

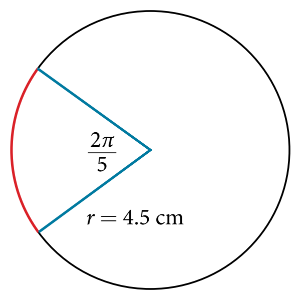


Figure 26

24. Find the arc length.

25. Find the area of the sector.

ALGEBRAIC

For the following exercises, convert angles in radians to degrees.

26. $\frac{3\pi}{4}$ radians **27.** $\frac{\pi}{9}$ radians **28.** $-\frac{5\pi}{4}$ radians **29.** $\frac{\pi}{3}$ radians

30. $-\frac{7\pi}{3}$ radians **31.** $-\frac{5\pi}{12}$ radians **32.** $\frac{11\pi}{6}$ radians

For the following exercises, convert angles in degrees to radians.

33. $90°$ **34.** $100°$ **35.** $-540°$ **36.** $-120°$

37. $180°$ **38.** $-315°$ **39.** $150°$

For the following exercises, use to given information to find the length of a circular arc. Round to two decimal places.

40. Find the length of the arc of a circle of radius 12 inches subtended by a central angle of $\frac{\pi}{4}$ radians.

41. Find the length of the arc of a circle of radius 5.02 miles subtended by the central angle of $\frac{\pi}{3}$.

42. Find the length of the arc of a circle of diameter 14 meters subtended by the central angle of $\frac{5\pi}{6}$.

43. Find the length of the arc of a circle of radius 10 centimeters subtended by the central angle of $50°$.

44. Find the length of the arc of a circle of radius 5 inches subtended by the central angle of $220°$.

45. Find the length of the arc of a circle of diameter 12 meters subtended by the central angle is $63°$.

For the following exercises, use the given information to find the area of the sector. Round to four decimal places.

46. A sector of a circle has a central angle of 45° and a radius 6 cm.

47. A sector of a circle has a central angle of 30° and a radius of 20 cm.

48. A sector of a circle with diameter 10 feet and an angle of $\frac{\pi}{2}$ radians.

49. A sector of a circle with radius of 0.7 inches and an angle of π radians.

For the following exercises, find the angle between 0° and 360° that is coterminal to the given angle.

50. $-40°$

51. $-110°$

52. 700°

53. 1400°

For the following exercises, find the angle between 0 and 2π in radians that is coterminal to the given angle.

54. $-\frac{\pi}{9}$

55. $\frac{10\pi}{3}$

56. $\frac{13\pi}{6}$

57. $\frac{44\pi}{9}$

REAL-WORLD APPLICATIONS

58. A truck with 32-inch diameter wheels is traveling at 60 mi/h. Find the angular speed of the wheels in rad/min. How many revolutions per minute do the wheels make?

59. A bicycle with 24-inch diameter wheels is traveling at 15 mi/h. Find the angular speed of the wheels in rad/min. How many revolutions per minute do the wheels make?

60. A wheel of radius 8 inches is rotating 15°/s. What is the linear speed v, the angular speed in RPM, and the angular speed in rad/s?

61. A wheel of radius 14 inches is rotating 0.5 rad/s. What is the linear speed v, the angular speed in RPM, and the angular speed in deg/s?

62. A CD has diameter of 120 millimeters. When playing audio, the angular speed varies to keep the linear speed constant where the disc is being read. When reading along the outer edge of the disc, the angular speed is about 200 RPM (revolutions per minute). Find the linear speed.

63. When being burned in a writable CD-R drive, the angular speed of a CD varies to keep the linear speed constant where the disc is being written. When writing along the outer edge of the disc, the angular speed of one drive is about 4,800 RPM (revolutions per minute). Find the linear speed if the CD has diameter of 120 millimeters.

64. A person is standing on the equator of Earth (radius 3960 miles). What are his linear and angular speeds?

65. Find the distance along an arc on the surface of Earth that subtends a central angle of 5 minutes $\left(1 \text{ minute} = \frac{1}{60} \text{ degree}\right)$. The radius of Earth is 3,960 mi.

66. Find the distance along an arc on the surface of Earth that subtends a central angle of 7 minutes $\left(1 \text{ minute} = \frac{1}{60} \text{ degree}\right)$. The radius of Earth is 3,960 miles.

67. Consider a clock with an hour hand and minute hand. What is the measure of the angle the minute hand traces in 20 minutes?

EXTENSIONS

68. Two cities have the same longitude. The latitude of city A is 9.00 degrees north and the latitude of city B is 30.00 degree north. Assume the radius of the earth is 3960 miles. Find the distance between the two cities.

69. A city is located at 40 degrees north latitude. Assume the radius of the earth is 3960 miles and the earth rotates once every 24 hours. Find the linear speed of a person who resides in this city.

70. A city is located at 75 degrees north latitude. Assume the radius of the earth is 3960 miles and the earth rotates once every 24 hours. Find the linear speed of a person who resides in this city.

71. Find the linear speed of the moon if the average distance between the earth and moon is 239,000 miles, assuming the orbit of the moon is circular and requires about 28 days. Express answer in miles per hour.

72. A bicycle has wheels 28 inches in diameter. A tachometer determines that the wheels are rotating at 180 RPM (revolutions per minute). Find the speed the bicycle is travelling down the road.

73. A car travels 3 miles. Its tires make 2640 revolutions. What is the radius of a tire in inches?

74. A wheel on a tractor has a 24-inch diameter. How many revolutions does the wheel make if the tractor travels 4 miles?

LEARNING OBJECTIVES

In this section, you will:

- Use right triangles to evaluate trigonometric functions.
- Find function values for $30°\left(\frac{\pi}{6}\right)$, $45°\left(\frac{\pi}{4}\right)$, and $60°\left(\frac{\pi}{3}\right)$.
- Use equal cofunctions of complementary angles.
- Use the definitions of trigonometric functions of any angle.
- Use right-triangle trigonometry to solve applied problems.

7.2 RIGHT TRIANGLE TRIGONOMETRY

Mt. Everest, which straddles the border between China and Nepal, is the tallest mountain in the world. Measuring its height is no easy task and, in fact, the actual measurement has been a source of controversy for hundreds of years. The measurement process involves the use of triangles and a branch of mathematics known as trigonometry. In this section, we will define a new group of functions known as trigonometric functions, and find out how they can be used to measure heights, such as those of the tallest mountains.

Using Right Triangles to Evaluate Trigonometric Functions

Figure 1 shows a right triangle with a vertical side of length y and a horizontal side has length x. Notice that the triangle is inscribed in a circle of radius 1. Such a circle, with a center at the origin and a radius of 1, is known as a **unit circle**.

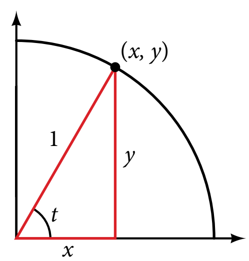


Figure 1

We can define the trigonometric functions in terms an angle t and the lengths of the sides of the triangle. The **adjacent side** is the side closest to the angle, x. (Adjacent means "next to.") The **opposite side** is the side across from the angle, y. The **hypotenuse** is the side of the triangle opposite the right angle, 1. These sides are labeled in **Figure 2**.

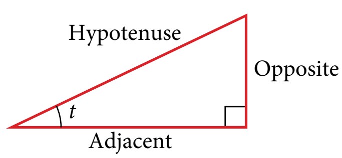


Figure 2 The sides of a right triangle in relation to angle *t*.

Given a right triangle with an acute angle of t, the first three trigonometric functions are listed.

$$\text{Sine} \quad \sin t = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \text{Cosine} \quad \cos t = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \text{Tangent} \quad \tan t = \frac{\text{opposite}}{\text{adjacent}}$$

A common mnemonic for remembering these relationships is SohCahToa, formed from the first letters of "**S**ine is **o**pposite over **h**ypotenuse, **C**osine is **a**djacent over **h**ypotenuse, **T**angent is **o**pposite over **a**djacent."

For the triangle shown in **Figure 1**, we have the following.

$$\sin t = \frac{y}{1} \qquad \cos t = \frac{x}{1} \qquad \tan t = \frac{y}{x}$$

How To...

Given the side lengths of a right triangle and one of the acute angles, find the sine, cosine, and tangent of that angle.

1. Find the sine as the ratio of the opposite side to the hypotenuse.
2. Find the cosine as the ratio of the adjacent side to the hypotenuse.
3. Find the tangent as the ratio of the opposite side to the adjacent side.

Example 1 Evaluating a Trigonometric Function of a Right Triangle

Given the triangle shown in **Figure 3**, find the value of $\cos \alpha$.

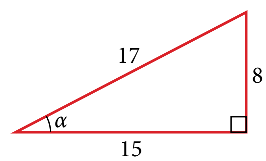


Figure 3

Solution The side adjacent to the angle is 15, and the hypotenuse of the triangle is 17.

$$\cos(\alpha) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$= \frac{15}{17}$$

Try It #1

Given the triangle shown in **Figure 4**, find the value of $\sin t$.

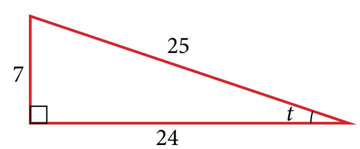


Figure 4

Reciprocal Functions

In addition to sine, cosine, and tangent, there are three more functions. These too are defined in terms of the sides of the triangle.

$$\text{Secant} \quad \sec t = \frac{\text{hypotenuse}}{\text{adjacent}} \qquad \text{Cosecant} \quad \csc t = \frac{\text{hypotenuse}}{\text{opposite}} \qquad \text{Cotangent} \quad \cot t = \frac{\text{adjacent}}{\text{opposite}}$$

Take another look at these definitions. These functions are the reciprocals of the first three functions.

$$\sin t = \frac{1}{\csc t} \qquad \csc t = \frac{1}{\sin t}$$

$$\cos t = \frac{1}{\sec t} \qquad \sec t = \frac{1}{\cos t}$$

$$\tan t = \frac{1}{\cot t} \qquad \cot t = \frac{1}{\tan t}$$

When working with right triangles, keep in mind that the same rules apply regardless of the orientation of the triangle. In fact, we can evaluate the six trigonometric functions of either of the two acute angles in the triangle in **Figure 5**. The side opposite one acute angle is the side adjacent to the other acute angle, and vice versa.

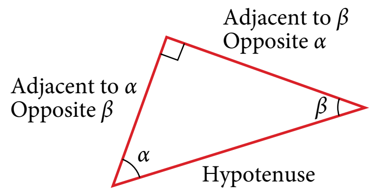


Figure 5 The side adjacent to one angle is opposite the other

Many problems ask for all six trigonometric functions for a given angle in a triangle. A possible strategy to use is to find the sine, cosine, and tangent of the angles first. Then, find the other trigonometric functions easily using the reciprocals.

How To...

Given the side lengths of a right triangle, evaluate the six trigonometric functions of one of the acute angles.

1. If needed, draw the right triangle and label the angle provided.
2. Identify the angle, the adjacent side, the side opposite the angle, and the hypotenuse of the right triangle.
3. Find the required function:
   - sine as the ratio of the opposite side to the hypotenuse
   - cosine as the ratio of the adjacent side to the hypotenuse
   - tangent as the ratio of the opposite side to the adjacent side
   - secant as the ratio of the hypotenuse to the adjacent side
   - cosecant as the ratio of the hypotenuse to the opposite side
   - cotangent as the ratio of the adjacent side to the opposite side

Example 2 Evaluating Trigonometric Functions of Angles Not in Standard Position

Using the triangle shown in **Figure 6**, evaluate sin α, cos α, tan α, sec α, csc α, and cot α.

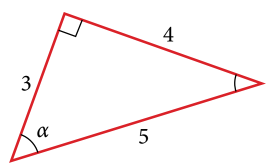


Figure 6

Solution

$$\sin \alpha = \frac{\text{opposite } \alpha}{\text{hypotenuse}} = \frac{4}{5} \qquad \sec \alpha = \frac{\text{hypotenuse}}{\text{adjacent to } \alpha} = \frac{5}{3}$$

$$\cos \alpha = \frac{\text{adjacent to } \alpha}{\text{hypotenuse}} = \frac{3}{5} \qquad \csc \alpha = \frac{\text{hypotenuse}}{\text{opposite } \alpha} = \frac{5}{4}$$

$$\tan \alpha = \frac{\text{opposite } \alpha}{\text{adjacent to } \alpha} = \frac{4}{3} \qquad \cot \alpha = \frac{\text{adjacent to } \alpha}{\text{opposite } \alpha} = \frac{3}{4}$$

Analysis *Another approach would have been to find sine, cosine, and tangent first. Then find their reciprocals to determine the other functions.*

$$\sec \alpha = \frac{1}{\cos \alpha} = \frac{1}{\frac{3}{5}} = \frac{5}{3}$$

$$\csc \alpha = \frac{1}{\sin \alpha} = \frac{1}{\frac{4}{5}} = \frac{5}{4}$$

$$\cot \alpha = \frac{1}{\tan \alpha} = \frac{1}{\frac{4}{3}} = \frac{3}{4}$$

Try It #2

Using the triangle shown in **Figure 7**, evaluate sin t, cos t, tan t, sec t, csc t, and cot t.

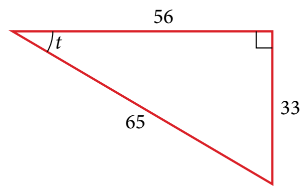


Figure 7

Finding Trigonometric Functions of Special Angles Using Side Lengths

It is helpful to evaluate the trigonometric functions as they relate to the special angles—multiples of 30°, 60°, and 45°. Remember, however, that when dealing with right triangles, we are limited to angles between 0° and 90°.

Suppose we have a 30°, 60°, 90° triangle, which can also be described as a $\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}$ triangle. The sides have lengths in the relation $s, \sqrt{3}s, 2s$. The sides of a 45°, 45°, 90° triangle, which can also be described as a $\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{2}$ triangle, have lengths in the relation $s, s, \sqrt{2}s$. These relations are shown in **Figure 8**.

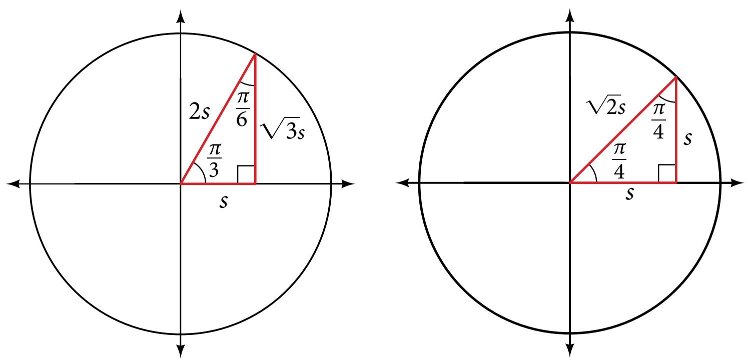


Figure 8 Side lengths of special triangles

We can then use the ratios of the side lengths to evaluate trigonometric functions of special angles.

How To…

Given trigonometric functions of a special angle, evaluate using side lengths.

1. Use the side lengths shown in **Figure 8** for the special angle you wish to evaluate.
2. Use the ratio of side lengths appropriate to the function you wish to evaluate.

Example 3 Evaluating Trigonometric Functions of Special Angles Using Side Lengths

Find the exact value of the trigonometric functions of $\frac{\pi}{3}$, using side lengths.

Solution

$$\sin\left(\frac{\pi}{3}\right) = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{3}s}{2s} = \frac{\sqrt{3}}{2} \qquad \sec\left(\frac{\pi}{3}\right) = \frac{\text{hyp}}{\text{adj}} = \frac{2s}{s} = 2$$

$$\cos\left(\frac{\pi}{3}\right) = \frac{\text{adj}}{\text{hyp}} = \frac{s}{2s} = \frac{1}{2} \qquad \csc\left(\frac{\pi}{3}\right) = \frac{\text{hyp}}{\text{opp}} = \frac{2s}{\sqrt{3}s} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

$$\tan\left(\frac{\pi}{3}\right) = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{3}s}{s} = \sqrt{3} \qquad \cot\left(\frac{\pi}{3}\right) = \frac{\text{adj}}{\text{opp}} = \frac{s}{\sqrt{3}s} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

Try It #3

Find the exact value of the trigonometric functions of $\frac{\pi}{4}$, using side lengths.

Using Equal Cofunction of Complements

If we look more closely at the relationship between the sine and cosine of the special angles, we notice a pattern. In a right triangle with angles of $\frac{\pi}{6}$ and $\frac{\pi}{3}$, we see that the sine of $\frac{\pi}{3}$ namely $\frac{\sqrt{3}}{2}$, is also the cosine of $\frac{\pi}{6}$, while the sine of $\frac{\pi}{6}$, namely $\frac{1}{2}$, is also the cosine of $\frac{\pi}{3}$.

$$\sin\frac{\pi}{3} = \cos\frac{\pi}{6} = \frac{\sqrt{3}s}{2s} = \frac{\sqrt{3}}{2} \qquad \sin\frac{\pi}{6} = \cos\frac{\pi}{3} = \frac{s}{2s} = \frac{1}{2}$$

See **Figure 9.**

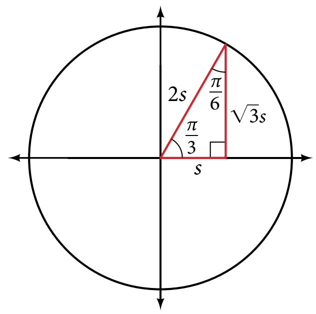


Figure 9 The sine of $\frac{\pi}{3}$ equals the cosine of $\frac{\pi}{6}$ and vice versa

This result should not be surprising because, as we see from **Figure 9**, the side opposite the angle of $\frac{\pi}{3}$ is also the side adjacent to $\frac{\pi}{6}$, so $\sin\left(\frac{\pi}{3}\right)$ and $\cos\left(\frac{\pi}{6}\right)$ are exactly the same ratio of the same two sides, $\sqrt{3}s$ and $2s$. Similarly, $\cos\left(\frac{\pi}{3}\right)$ and $\sin\left(\frac{\pi}{6}\right)$ are also the same ratio using the same two sides, s and $2s$.

The interrelationship between the sines and cosines of $\frac{\pi}{6}$ and $\frac{\pi}{3}$ also holds for the two acute angles in any right triangle, since in every case, the ratio of the same two sides would constitute the sine of one angle and the cosine of the other. Since the three angles of a triangle add to π, and the right angle is $\frac{\pi}{2}$, the remaining two angles must also add up to $\frac{\pi}{2}$. That means that a right triangle can be formed with any two angles that add to $\frac{\pi}{2}$—in other words, any two complementary angles. So we may state a *cofunction identity*: If any two angles are complementary, the sine of one is the cosine of the other, and vice versa. This identity is illustrated in **Figure 10**.

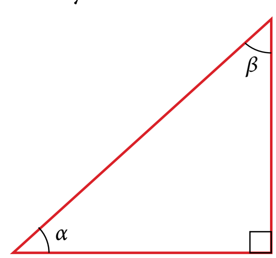


Figure 10 Cofunction identity of sine and cosine of complementary angles

Using this identity, we can state without calculating, for instance, that the sine of $\frac{\pi}{12}$ equals the cosine of $\frac{5\pi}{12}$, and that the sine of $\frac{5\pi}{12}$ equals the cosine of $\frac{\pi}{12}$. We can also state that if, for a certain angle t, $\cos t = \frac{5}{13}$, then $\sin\left(\frac{\pi}{2} - t\right) = \frac{5}{13}$ as well.

cofunction identities

The **cofunction identities** in radians are listed in **Table 1.**

$\sin t = \cos\left(\frac{\pi}{2} - t\right)$	$\sec t = \csc\left(\frac{\pi}{2} - t\right)$	$\tan t = \cot\left(\frac{\pi}{2} - t\right)$
$\cos t = \sin\left(\frac{\pi}{2} - t\right)$	$\csc t = \sec\left(\frac{\pi}{2} - t\right)$	$\cot t = \tan\left(\frac{\pi}{2} - t\right)$

Table 1

How To…

Given the sine and cosine of an angle, find the sine or cosine of its complement.

1. To find the sine of the complementary angle, find the cosine of the original angle.
2. To find the cosine of the complementary angle, find the sine of the original angle.

Example 4 Using Cofunction Identities

If $\sin t = \frac{5}{12}$, find $\cos\left(\frac{\pi}{2} - t\right)$.

Solution According to the cofunction identities for sine and cosine,

$$\sin t = \cos\left(\frac{\pi}{2} - t\right).$$

So

$$\cos\left(\frac{\pi}{2} - t\right) = \frac{5}{12}.$$

Try It #4

If $\csc\left(\frac{\pi}{6}\right) = 2$, find $\sec\left(\frac{\pi}{3}\right)$.

Using Trigonometric Functions

In previous examples, we evaluated the sine and cosine in triangles where we knew all three sides. But the real power of right-triangle trigonometry emerges when we look at triangles in which we know an angle but do not know all the sides.

How To…

Given a right triangle, the length of one side, and the measure of one acute angle, find the remaining sides.

1. For each side, select the trigonometric function that has the unknown side as either the numerator or the denominator. The known side will in turn be the denominator or the numerator.
2. Write an equation setting the function value of the known angle equal to the ratio of the corresponding sides.
3. Using the value of the trigonometric function and the known side length, solve for the missing side length.

Example 5 Finding Missing Side Lengths Using Trigonometric Ratios

Find the unknown sides of the triangle in **Figure 11**.

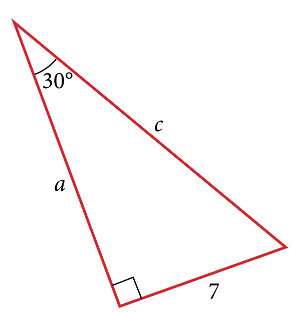


Figure 11

Solution We know the angle and the opposite side, so we can use the tangent to find the adjacent side.

$$\tan(30°) = \frac{7}{a}$$

We rearrange to solve for a.

$$a = \frac{7}{\tan(30°)}$$
$$\approx 12.1$$

We can use the sine to find the hypotenuse.

$$\sin(30°) = \frac{7}{c}$$

Again, we rearrange to solve for c.

$$c = \frac{7}{\sin(30°)}$$
$$\approx 14$$

Try It #5

A right triangle has one angle of $\frac{\pi}{3}$ and a hypotenuse of 20. Find the unknown sides and angle of the triangle.

Using Right Triangle Trigonometry to Solve Applied Problems

Right-triangle trigonometry has many practical applications. For example, the ability to compute the lengths of sides of a triangle makes it possible to find the height of a tall object without climbing to the top or having to extend a tape measure along its height. We do so by measuring a distance from the base of the object to a point on the ground some distance away, where we can look up to the top of the tall object at an angle. The **angle of elevation** of an object above an observer relative to the observer is the angle between the horizontal and the line from the object to the observer's eye. The right triangle this position creates has sides that represent the unknown height, the measured distance from the base, and the angled line of sight from the ground to the top of the object. Knowing the measured distance to the base of the object and the angle of the line of sight, we can use trigonometric functions to calculate the unknown height.

Similarly, we can form a triangle from the top of a tall object by looking downward. The **angle of depression** of an object below an observer relative to the observer is the angle between the horizontal and the line from the object to the observer's eye. See **Figure 12**.

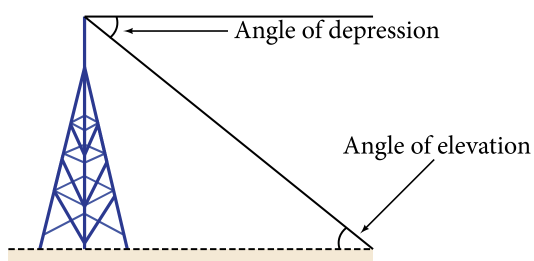


Figure 12

How To...

Given a tall object, measure its height indirectly.

1. Make a sketch of the problem situation to keep track of known and unknown information.
2. Lay out a measured distance from the base of the object to a point where the top of the object is clearly visible.
3. At the other end of the measured distance, look up to the top of the object. Measure the angle the line of sight makes with the horizontal.
4. Write an equation relating the unknown height, the measured distance, and the tangent of the angle of the line of sight.
5. Solve the equation for the unknown height.

Example 6 Measuring a Distance Indirectly

To find the height of a tree, a person walks to a point 30 feet from the base of the tree. She measures an angle of 57° between a line of sight to the top of the tree and the ground, as shown in **Figure 13**. Find the height of the tree.

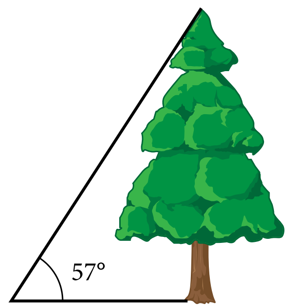


Figure 13

Solution We know that the angle of elevation is 57° and the adjacent side is 30 ft long. The opposite side is the unknown height.

The trigonometric function relating the side opposite to an angle and the side adjacent to the angle is the tangent. So we will state our information in terms of the tangent of 57°, letting h be the unknown height.

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan(57°) = \frac{h}{30} \qquad \text{Solve for } h.$$

$$h = 30\tan(57°) \qquad \text{Multiply.}$$

$$h \approx 46.2 \qquad \text{Use a calculator.}$$

The tree is approximately 46 feet tall.

Try It #6

How long a ladder is needed to reach a windowsill 50 feet above the ground if the ladder rests against the building making an angle of $\frac{5\pi}{12}$ with the ground? Round to the nearest foot.

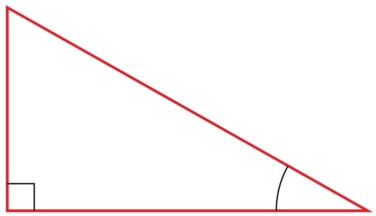
Access these online resources for additional instruction and practice with right triangle trigonometry.

- Finding Trig Functions on Calculator (http://openstaxcollege.org/l/findtrigcal)
- Finding Trig Functions Using a Right Triangle (http://openstaxcollege.org/l/trigrttri)
- Relate Trig Functions to Sides of a Right Triangle (http://openstaxcollege.org/l/reltrigtri)
- Determine Six Trig Functions from a Triangle (http://openstaxcollege.org/l/sixtrigfunc)
- Determine Length of Right Triangle Side (http://openstaxcollege.org/l/rttriside)

7.2 SECTION EXERCISES

VERBAL

1. For the given right triangle, label the adjacent side, opposite side, and hypotenuse for the indicated angle.



2. When a right triangle with a hypotenuse of 1 is placed in the unit circle, which sides of the triangle correspond to the x- and y-coordinates?

3. The tangent of an angle compares which sides of the right triangle?

4. What is the relationship between the two acute angles in a right triangle?

5. Explain the cofunction identity.

ALGEBRAIC

For the following exercises, use cofunctions of complementary angles.

6. $\cos(34°) = \sin(\_\_\_\_\_°)$

7. $\cos\left(\frac{\pi}{3}\right) = \sin(\_\_\_\_\_)$

8. $\csc(21°) = \sec(\_\_\_\_\_°)$

9. $\tan\left(\frac{\pi}{4}\right) = \cot(\_\_\_\_\_)$

For the following exercises, find the lengths of the missing sides if side a is opposite angle A, side b is opposite angle B, and side c is the hypotenuse.

10. $\cos B = \frac{4}{5}, a = 10$

11. $\sin B = \frac{1}{2}, a = 20$

12. $\tan A = \frac{5}{12}, b = 6$

13. $\tan A = 100, b = 100$

14. $\sin B = \frac{1}{\sqrt{3}}, a = 2$

15. $a = 5, \measuredangle A = 60°$

16. $c = 12, \measuredangle A = 45°$

GRAPHICAL

For the following exercises, use **Figure 14** to evaluate each trigonometric function of angle A.

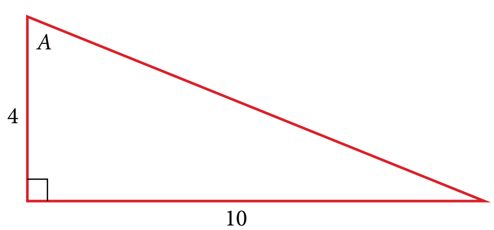


Figure 14

17. $\sin A$

18. $\cos A$

19. $\tan A$

20. $\csc A$

21. $\sec A$

22. $\cot A$

For the following exercises, use **Figure 15** to evaluate each trigonometric function of angle A.

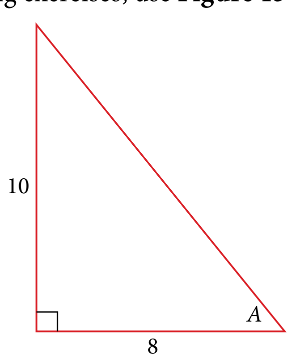


Figure 15

23. $\sin A$

24. $\cos A$

25. $\tan A$

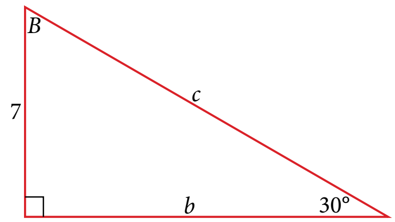
26. $\csc A$

27. $\sec A$

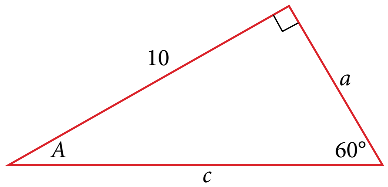
28. $\cot A$

For the following exercises, solve for the unknown sides of the given triangle.

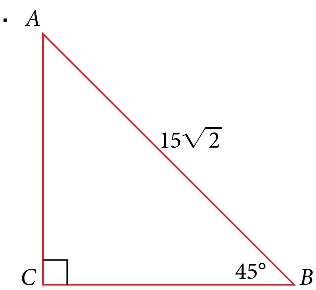
29.



30.



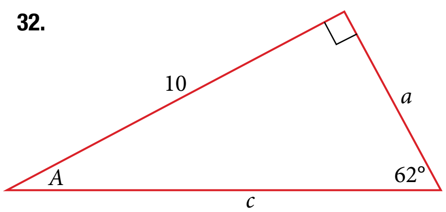
31.



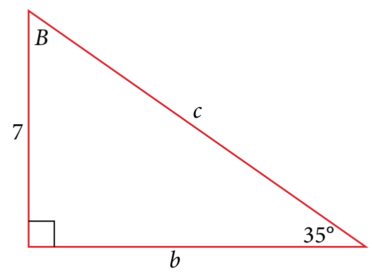
TECHNOLOGY

For the following exercises, use a calculator to find the length of each side to four decimal places.

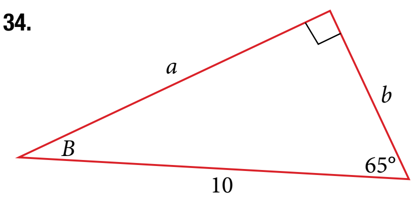
32.



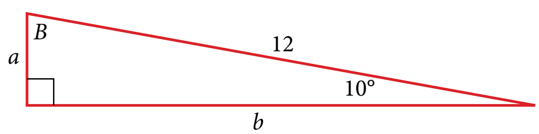
33.



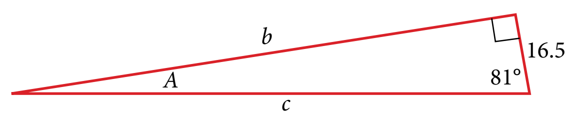
34.



35.



36.



37. $b = 15$, $\measuredangle B = 15°$

38. $c = 200$, $\measuredangle B = 5°$

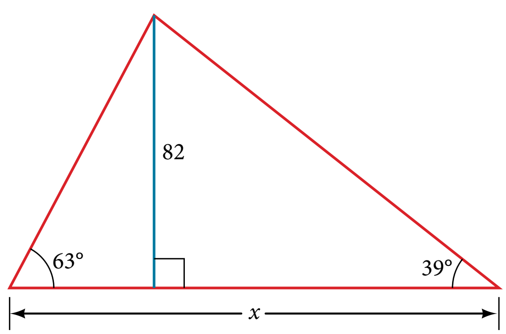
39. $c = 50$, $\measuredangle B = 21°$

40. $a = 30$, $\measuredangle A = 27°$

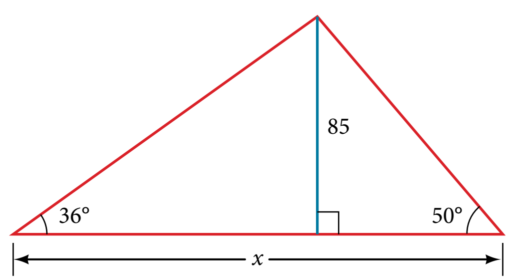
41. $b = 3.5$, $\measuredangle A = 78°$

EXTENSIONS

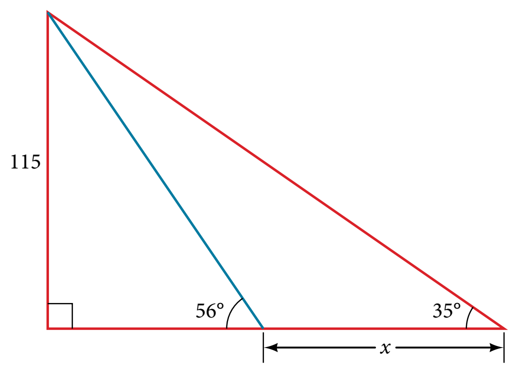
42. Find x.



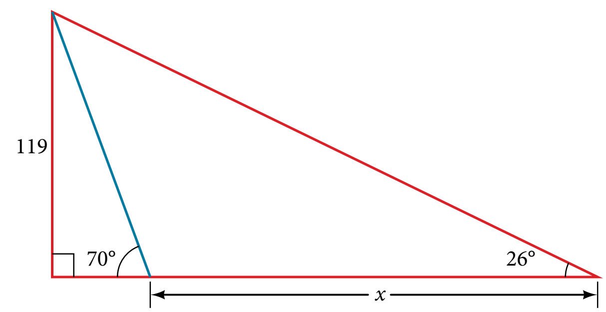
43. Find x.



44. Find *x*.



45. Find *x*.



46. A radio tower is located 400 feet from a building. From a window in the building, a person determines that the angle of elevation to the top of the tower is 36°, and that the angle of depression to the bottom of the tower is 23°. How tall is the tower?

47. A radio tower is located 325 feet from a building. From a window in the building, a person determines that the angle of elevation to the top of the tower is 43°, and that the angle of depression to the bottom of the tower is 31°. How tall is the tower?

48. A 200-foot tall monument is located in the distance. From a window in a building, a person determines that the angle of elevation to the top of the monument is 15°, and that the angle of depression to the bottom of the tower is 2°. How far is the person from the monument?

49. A 400-foot tall monument is located in the distance. From a window in a building, a person determines that the angle of elevation to the top of the monument is 18°, and that the angle of depression to the bottom of the monument is 3°. How far is the person from the monument?

50. There is an antenna on the top of a building. From a location 300 feet from the base of the building, the angle of elevation to the top of the building is measured to be 40°. From the same location, the angle of elevation to the top of the antenna is measured to be 43°. Find the height of the antenna.

51. There is lightning rod on the top of a building. From a location 500 feet from the base of the building, the angle of elevation to the top of the building is measured to be 36°. From the same location, the angle of elevation to the top of the lightning rod is measured to be 38°. Find the height of the lightning rod.

REAL-WORLD APPLICATIONS

52. A 33-ft ladder leans against a building so that the angle between the ground and the ladder is 80°. How high does the ladder reach up the side of the building?

53. A 23-ft ladder leans against a building so that the angle between the ground and the ladder is 80°. How high does the ladder reach up the side of the building?

54. The angle of elevation to the top of a building in New York is found to be 9 degrees from the ground at a distance of 1 mile from the base of the building. Using this information, find the height of the building.

55. The angle of elevation to the top of a building in Seattle is found to be 2 degrees from the ground at a distance of 2 miles from the base of the building. Using this information, find the height of the building.

56. Assuming that a 370-foot tall giant redwood grows vertically, if I walk a certain distance from the tree and measure the angle of elevation to the top of the tree to be 60°, how far from the base of the tree am I?

LEARNING OBJECTIVES

In this section, you will:

- Find function values for the sine and cosine of 30° or $\left(\frac{\pi}{6}\right)$, 45° or $\left(\frac{\pi}{4}\right)$ and 60° or $\left(\frac{\pi}{3}\right)$.
- Identify the domain and range of sine and cosine functions.
- Find reference angles.
- Use reference angles to evaluate trigonometric functions.

7.3 UNIT CIRCLE

Figure 1 The Singapore Flyer is the world's tallest Ferris wheel. (credit: "Vibin JK"/Flickr)

Looking for a thrill? Then consider a ride on the Singapore Flyer, the world's tallest Ferris wheel. Located in Singapore, the Ferris wheel soars to a height of 541 feet—a little more than a tenth of a mile! Described as an observation wheel, riders enjoy spectacular views as they travel from the ground to the peak and down again in a repeating pattern. In this section, we will examine this type of revolving motion around a circle. To do so, we need to define the type of circle first, and then place that circle on a coordinate system. Then we can discuss circular motion in terms of the coordinate pairs.

Finding Trigonometric Functions Using the Unit Circle

We have already defined the trigonometric functions in terms of right triangles. In this section, we will redefine them in terms of the unit circle. Recall that a unit circle is a circle centered at the origin with radius 1, as shown in **Figure 2**. The angle (in radians) that t intercepts forms an arc of length s. Using the formula $s = rt$, and knowing that $r = 1$, we see that for a unit circle, $s = t$.

The x- and y-axes divide the coordinate plane into four quarters called quadrants. We label these quadrants to mimic the direction a positive angle would sweep. The four quadrants are labeled I, II, III, and IV.

For any angle t, we can label the intersection of the terminal side and the unit circle as by its coordinates, (x, y). The coordinates x and y will be the outputs of the trigonometric functions $f(t) = \cos t$ and $f(t) = \sin t$, respectively. This means $x = \cos t$ and $y = \sin t$.

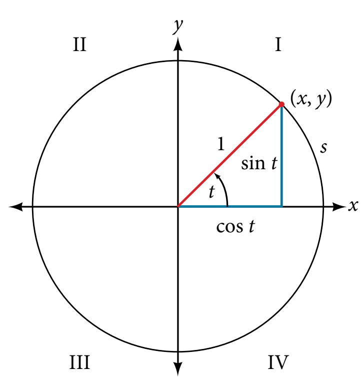


Figure 2 Unit circle where the central angle is t radians

unit circle

A **unit circle** has a center at (0, 0) and radius 1. In a unit circle, the length of the intercepted arc is equal to the radian measure of the central angle t.

Let (x, y) be the endpoint on the unit circle of an arc of arc length s. The (x, y) coordinates of this point can be described as functions of the angle.

Defining Sine and Cosine Functions from the Unit Circle

The sine function relates a real number t to the y-coordinate of the point where the corresponding angle intercepts the unit circle. More precisely, the sine of an angle t equals the y-value of the endpoint on the unit circle of an arc of length t. In **Figure 2**, the sine is equal to y. Like all functions, the **sine function** has an input and an output; its input is the measure of the angle; its output is the y-coordinate of the corresponding point on the unit circle.

The **cosine function** of an angle t equals the x-value of the endpoint on the unit circle of an arc of length t. In **Figure 3**, the cosine is equal to x.

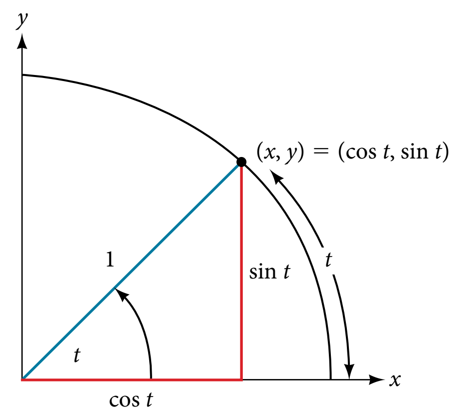


Figure 3

Because it is understood that sine and cosine are functions, we do not always need to write them with parentheses: $\sin t$ is the same as $\sin(t)$ and $\cos t$ is the same as $\cos(t)$. Likewise, $\cos^2 t$ is a commonly used shorthand notation for $(\cos(t))^2$. Be aware that many calculators and computers do not recognize the shorthand notation. When in doubt, use the extra parentheses when entering calculations into a calculator or computer.

sine and cosine functions

If t is a real number and a point (x, y) on the unit circle corresponds to an angle of t, then

$$\cos t = x$$
$$\sin t = y$$

How To...

Given a point $P(x, y)$ on the unit circle corresponding to an angle of t, find the sine and cosine.

1. The sine of t is equal to the y-coordinate of point P: $\sin t = y$.
2. The cosine of t is equal to the x-coordinate of point P: $\cos t = x$.

Example 1 Finding Function Values for Sine and Cosine

Point P is a point on the unit circle corresponding to an angle of t, as shown in **Figure 4.** Find cos(t) and sin(t).

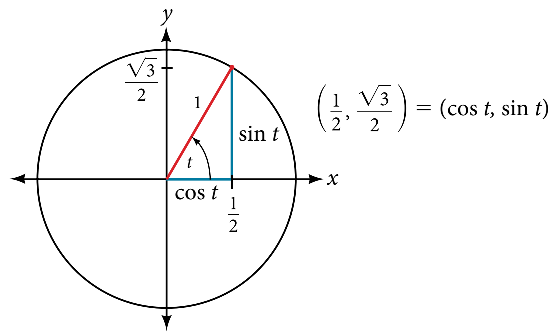


Figure 4

Solution We know that cos t is the x-coordinate of the corresponding point on the unit circle and sin t is the y-coordinate of the corresponding point on the unit circle. So:

$$x = \cos t = \frac{1}{2} \qquad y = \sin t = \frac{\sqrt{3}}{2}$$

Try It #1

A certain angle t corresponds to a point on the unit circle at $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ as shown in **Figure 5**. Find cos t and sin t.

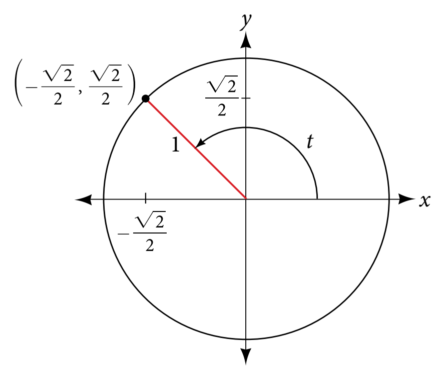


Figure 5

Finding Sines and Cosines of Angles on an Axis

For quadrantral angles, the corresponding point on the unit circle falls on the x- or y-axis. In that case, we can easily calculate cosine and sine from the values of x and y.

Example 2 Calculating Sines and Cosines along an Axis

Find cos(90°) and sin(90°).

Solution Moving 90° counterclockwise around the unit circle from the positive x-axis brings us to the top of the circle, where the (x, y) coordinates are (0, 1), as shown in **Figure 6**.

When we use our definitions of cosine and sine,

$$x = \cos t = \cos(90°) = 0$$
$$y = \sin t = \sin(90°) = 1$$

The cosine of 90° is 0; the sine of 90° is 1.

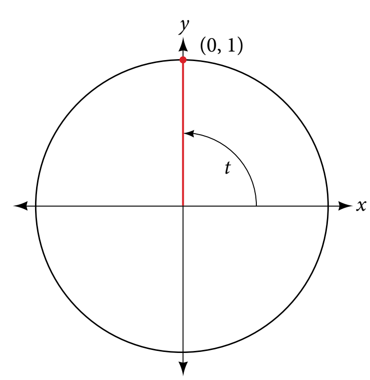


Figure 6

Try It #2

Find cosine and sine of the angle π.

The Pythagorean Identity

Now that we can define sine and cosine, we will learn how they relate to each other and the unit circle. Recall that the equation for the unit circle is $x^2 + y^2 = 1$. Because $x = \cos t$ and $y = \sin t$, we can substitute for x and y to get $\cos^2 t + \sin^2 t = 1$. This equation, $\cos^2 t + \sin^2 t = 1$, is known as the Pythagorean Identity. See **Figure 7**.

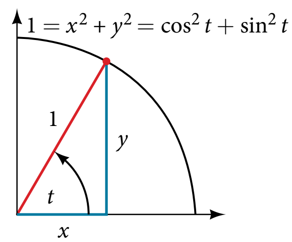


Figure 7

We can use the Pythagorean Identity to find the cosine of an angle if we know the sine, or vice versa. However, because the equation yields two solutions, we need additional knowledge of the angle to choose the solution with the correct sign. If we know the quadrant where the angle is, we can easily choose the correct solution.

Pythagorean Identity

The **Pythagorean Identity** states that, for any real number t,

$$\cos^2 t + \sin^2 t = 1$$

How To...

Given the sine of some angle t and its quadrant location, find the cosine of t.

1. Substitute the known value of $\sin(t)$ into the Pythagorean Identity.
2. Solve for $\cos(t)$.
3. Choose the solution with the appropriate sign for the x-values in the quadrant where t is located.

Example 3 Finding a Cosine from a Sine or a Sine from a Cosine

If $\sin(t) = \frac{3}{7}$ and t is in the second quadrant, find $\cos(t)$.

Solution If we drop a vertical line from the point on the unit circle corresponding to t, we create a right triangle, from which we can see that the Pythagorean Identity is simply one case of the Pythagorean Theorem. See **Figure 8**.

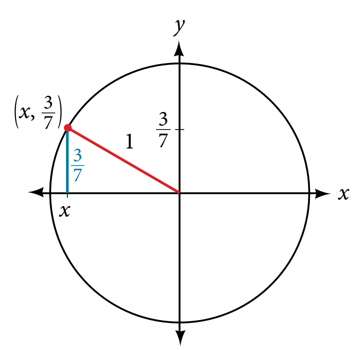


Figure 8

Substituting the known value for sine into the Pythagorean Identity,

$$\cos^2(t) + \sin^2(t) = 1$$

$$\cos^2(t) + \frac{9}{49} = 1$$

$$\cos^2(t) = \frac{40}{49}$$

$$\cos(t) = \pm\sqrt{\frac{40}{49}} = \pm\frac{\sqrt{40}}{7} = \pm\frac{2\sqrt{10}}{7}$$

Because the angle is in the second quadrant, we know the x-value is a negative real number, so the cosine is also negative. So $\cos(t) = -\frac{2\sqrt{10}}{7}$

Try It #3

If $\cos(t) = \frac{24}{25}$ and t is in the fourth quadrant, find $\sin(t)$.

Finding Sines and Cosines of Special Angles

We have already learned some properties of the special angles, such as the conversion from radians to degrees, and we found their sines and cosines using right triangles. We can also calculate sines and cosines of the special angles using the Pythagorean Identity.

Finding Sines and Cosines of 45° Angles

First, we will look at angles of 45° or $\frac{\pi}{4}$, as shown in **Figure 9**. A 45° – 45° – 90° triangle is an isosceles triangle, so the x- and y-coordinates of the corresponding point on the circle are the same. Because the x- and y-values are the same, the sine and cosine values will also be equal.

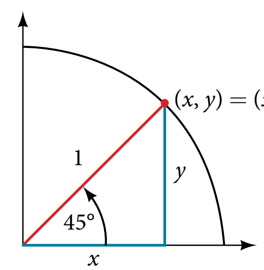


Figure 9

At $t = \frac{\pi}{4}$, which is 45 degrees, the radius of the unit circle bisects the first quadrantal angle. This means the radius lies along the line $y = x$. A unit circle has a radius equal to 1. So, the right triangle formed below the line $y = x$ has sides x and y $(y = x)$, and a radius $= 1$. See **Figure 10**.

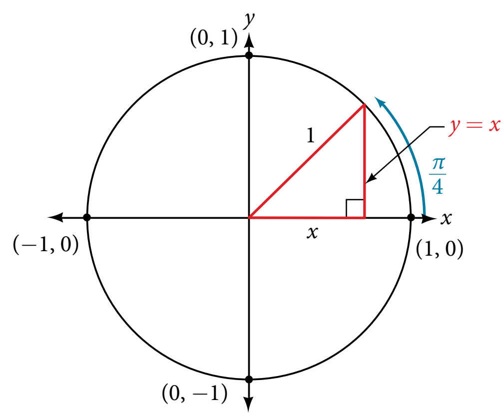


Figure 10

From the Pythagorean Theorem we get

$$x^2 + y^2 = 1$$

We can then substitute $y = x$.

$$x^2 + x^2 = 1$$

Next we combine like terms.

$$2x^2 = 1$$

And solving for x, we get

$$x^2 = \frac{1}{2}$$

$$x = \pm\frac{1}{\sqrt{2}}$$

In quadrant I, $x = \frac{1}{\sqrt{2}}$.

At $t = \frac{\pi}{4}$ or 45 degrees,

$$(x, y) = (x, x) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$x = \frac{1}{\sqrt{2}},\ y = \frac{1}{\sqrt{2}}$$

$$\cos t = \frac{1}{\sqrt{2}},\ \sin t = \frac{1}{\sqrt{2}}$$

If we then rationalize the denominators, we get

$$\begin{aligned}\cos t &= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} \\ &= \frac{\sqrt{2}}{2} \\ \sin t &= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} \\ &= \frac{\sqrt{2}}{2}\end{aligned}$$

Therefore, the (x, y) coordinates of a point on a circle of radius 1 at an angle of 45° are $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

Finding Sines and Cosines of 30° and 60° Angles

Next, we will find the cosine and sine at an angle of 30°, or $\frac{\pi}{6}$. First, we will draw a triangle inside a circle with one side at an angle of 30°, and another at an angle of −30°, as shown in **Figure 11**. If the resulting two right triangles are combined into one large triangle, notice that all three angles of this larger triangle will be 60°, as shown in **Figure 12**.

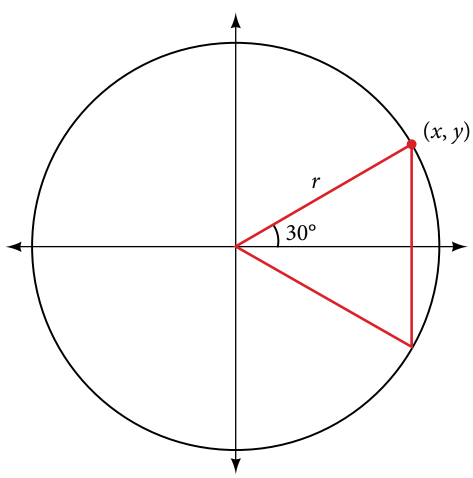


Figure 11

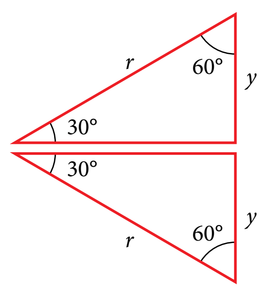


Figure 12

Because all the angles are equal, the sides are also equal. The vertical line has length $2y$, and since the sides are all equal, we can also conclude that $r = 2y$ or $y = \frac{1}{2}r$. Since $\sin t = y$,

$$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}r$$

And since $r = 1$ in our unit circle,

$$\begin{aligned}\sin\left(\frac{\pi}{6}\right) &= \frac{1}{2}(1) \\ &= \frac{1}{2}\end{aligned}$$

Using the Pythagorean Identity, we can find the cosine value.

$$\cos^2\frac{\pi}{6}+\sin^2\left(\frac{\pi}{6}\right)=1$$

$$\cos^2\left(\frac{\pi}{6}\right)+\left(\frac{1}{2}\right)^2=1$$

$$\cos^2\left(\frac{\pi}{6}\right)=\frac{3}{4} \qquad \text{Use the square root property.}$$

$$\cos\left(\frac{\pi}{6}\right)=\frac{\pm\sqrt{3}}{\pm\sqrt{4}}=\frac{\sqrt{3}}{2} \qquad \text{Since } y \text{ is positive, choose the positive root.}$$

The (x, y) coordinates for the point on a circle of radius 1 at an angle of 30° are $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. At $t=\frac{\pi}{3}$ (60°), the radius of the unit circle, 1, serves as the hypotenuse of a 30-60-90 degree right triangle, BAD, as shown in **Figure 13**. Angle A has measure 60°. At point B, we draw an angle ABC with measure of 60°. We know the angles in a triangle sum to 180°, so the measure of angle C is also 60°. Now we have an equilateral triangle. Because each side of the equilateral triangle ABC is the same length, and we know one side is the radius of the unit circle, all sides must be of length 1.

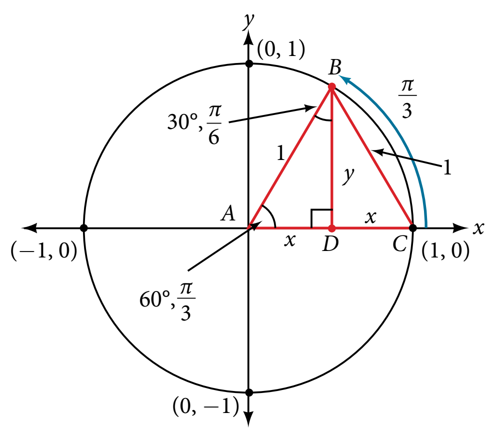


Figure 13

The measure of angle ABD is 30°. So, if double, angle ABC is 60°. BD is the perpendicular bisector of AC, so it cuts AC in half. This means that AD is $\frac{1}{2}$ the radius, or $\frac{1}{2}$. Notice that AD is the x-coordinate of point B, which is at the intersection of the 60° angle and the unit circle. This gives us a triangle BAD with hypotenuse of 1 and side x of length $\frac{1}{2}$.

From the Pythagorean Theorem, we get

$$x^2+y^2=1$$

Substituting $x=\frac{1}{2}$, we get

$$\left(\frac{1}{2}\right)^2+y^2=1$$

Solving for y, we get

$$\frac{1}{4}+y^2=1$$

$$y^2=1-\frac{1}{4}$$

$$y^2=\frac{3}{4}$$

$$y=\pm\frac{\sqrt{3}}{2}$$

Since $t=\frac{\pi}{3}$ has the terminal side in quadrant I where the y-coordinate is positive, we choose $y=\frac{\sqrt{3}}{2}$, the positive value.

At $t = \frac{\pi}{3}$ (60°), the (x, y) coordinates for the point on a circle of radius 1 at an angle of 60° are $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, so we can find the sine and cosine.

$$(x, y) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$

$$x = \frac{1}{2}, y = \frac{\sqrt{3}}{2}$$

$$\cos t = \frac{1}{2}, \sin t = \frac{\sqrt{3}}{2}$$

We have now found the cosine and sine values for all of the most commonly encountered angles in the first quadrant of the unit circle. **Table 1** summarizes these values.

Angle	0	$\frac{\pi}{6}$, or 30°	$\frac{\pi}{4}$, or 45°	$\frac{\pi}{3}$, or 60°	$\frac{\pi}{2}$, or 90°
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
Sine	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

Table 1

$\cos \pi/4$ and $\sin \pi/4$ are often written as $1/\sqrt{2}$ (see the first 10 lines of page 609).

Figure 14 shows the common angles in the first quadrant of the unit circle.

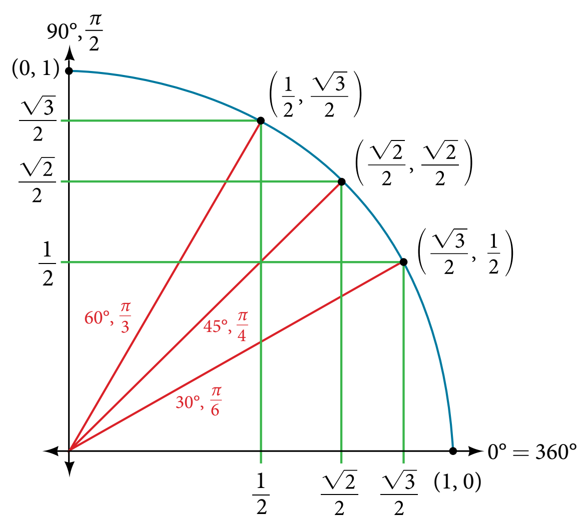


Figure 14

Using a Calculator to Find Sine and Cosine

To find the cosine and sine of angles other than the special angles, we turn to a computer or calculator. **Be aware**: Most calculators can be set into "degree" or "radian" mode, which tells the calculator the units for the input value. When we evaluate cos(30) on our calculator, it will evaluate it as the cosine of 30 degrees if the calculator is in degree mode, or the cosine of 30 radians if the calculator is in radian mode.

How To...

Given an angle in radians, use a graphing calculator to find the cosine.

1. If the calculator has degree mode and radian mode, set it to radian mode.
2. Press the **COS** key.
3. Enter the radian value of the angle and press the close-parentheses key ")".
4. Press **ENTER**.

Example 4 Using a Graphing Calculator to Find Sine and Cosine

Evaluate $\cos\left(\frac{5\pi}{3}\right)$ using a graphing calculator or computer.

Solution Enter the following keystrokes:

$$\textbf{COS}(5 \times \pi \div 3)\ \textbf{ENTER}$$

$$\cos\left(\frac{5\pi}{3}\right) = 0.5$$

Analysis *We can find the cosine or sine of an angle in degrees directly on a calculator with degree mode. For calculators or software that use only radian mode, we can find the sign of 20°, for example, by including the conversion factor to radians as part of the input:*

$$\textbf{SIN}(20 \times \pi \div 180)\ \textbf{ENTER}$$

Try It #4

Evaluate $\sin\left(\frac{\pi}{3}\right)$.

Identifying the Domain and Range of Sine and Cosine Functions

Now that we can find the sine and cosine of an angle, we need to discuss their domains and ranges. What are the domains of the sine and cosine functions? That is, what are the smallest and largest numbers that can be inputs of the functions? Because angles smaller than 0 and angles larger than 2π can still be graphed on the unit circle and have real values of x, y, and r, there is no lower or upper limit to the angles that can be inputs to the sine and cosine functions. The input to the sine and cosine functions is the rotation from the positive x-axis, and that may be any real number.

What are the ranges of the sine and cosine functions? What are the least and greatest possible values for their output? We can see the answers by examining the unit circle, as shown in **Figure 15**. The bounds of the x-coordinate are $[-1, 1]$.

The bounds of the y-coordinate are also $[-1, 1]$. Therefore, the range of both the sine and cosine functions is $[-1, 1]$.

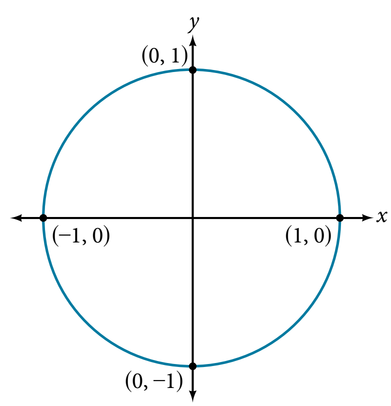


Figure 15

Finding Reference Angles

We have discussed finding the sine and cosine for angles in the first quadrant, but what if our angle is in another quadrant? For any given angle in the first quadrant, there is an angle in the second quadrant with the same sine value. Because the sine value is the y-coordinate on the unit circle, the other angle with the same sine will share the same y-value, but have the opposite x-value. Therefore, its cosine value will be the opposite of the first angle's cosine value.

Likewise, there will be an angle in the fourth quadrant with the same cosine as the original angle. The angle with the same cosine will share the same x-value but will have the opposite y-value. Therefore, its sine value will be the opposite of the original angle's sine value.

As shown in **Figure 16**, angle α has the same sine value as angle t; the cosine values are opposites. Angle β has the same cosine value as angle t; the sine values are opposites.

$$\sin(t) = \sin(\alpha) \quad \text{and } \cos(t) = -\cos(\alpha)$$

$$\sin(t) = -\sin(\beta) \text{ and } \cos(t) = \cos(\beta)$$

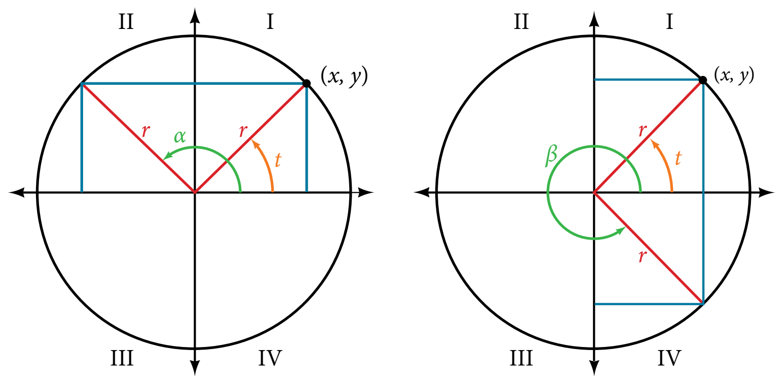


Figure 16

Recall that an angle's reference angle is the acute angle, t, formed by the terminal side of the angle t and the horizontal axis. A reference angle is always an angle between 0 and 90°, or 0 and $\frac{\pi}{2}$ radians. As we can see from **Figure 17**, for any angle in quadrants II, III, or IV, there is a reference angle in quadrant I.

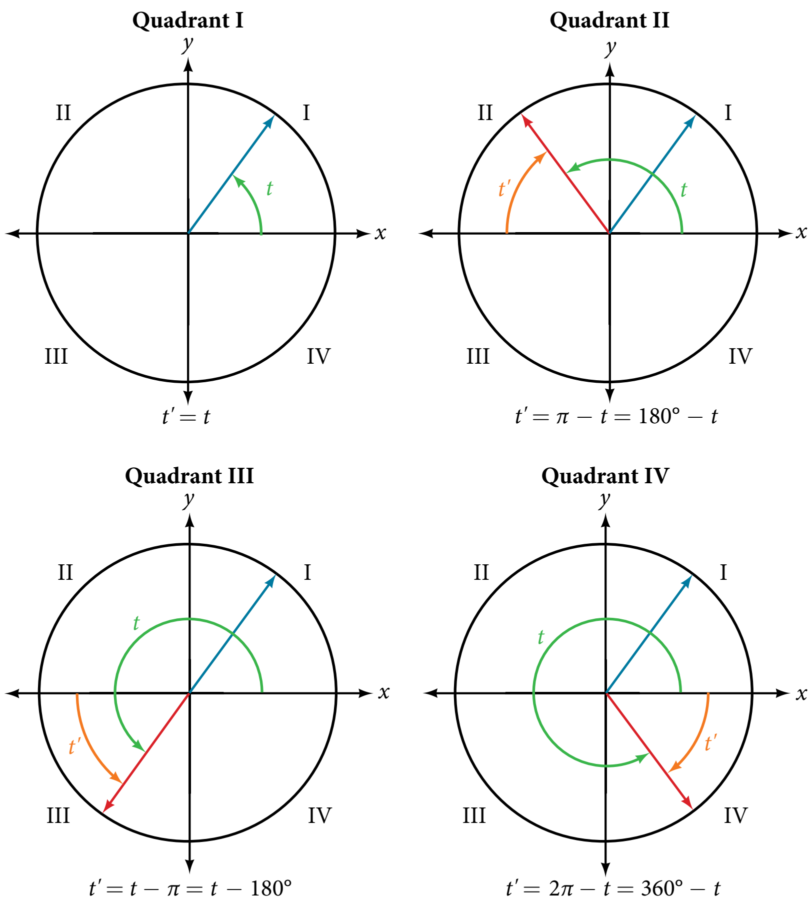


Figure 17

How To…

Given an angle between 0 and 2π, find its reference angle.

1. An angle in the first quadrant is its own reference angle.
2. For an angle in the second or third quadrant, the reference angle is $|\pi - t|$ or $|180° - t|$.
3. For an angle in the fourth quadrant, the reference angle is $2\pi - t$ or $360° - t$.
4. If an angle is less than 0 or greater than 2π, add or subtract 2π as many times as needed to find an equivalent angle between 0 and 2π.

Example 5 Finding a Reference Angle

Find the reference angle of 225° as shown in **Figure 18**.

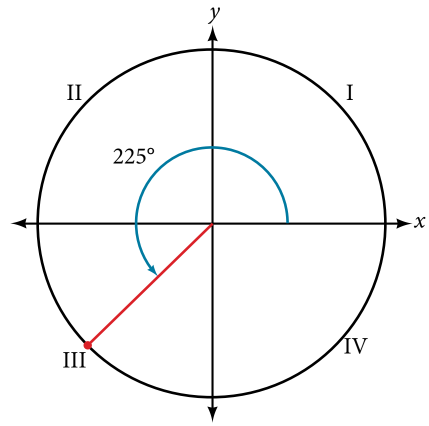


Figure 18

Solution Because 225° is in the third quadrant, the reference angle is

$$|(180° - 225°)| = |-45°| = 45°$$

Try It #5

Find the reference angle of $\frac{5\pi}{3}$.

Using Reference Angles

Now let's take a moment to reconsider the Ferris wheel introduced at the beginning of this section. Suppose a rider snaps a photograph while stopped twenty feet above ground level. The rider then rotates three-quarters of the way around the circle. What is the rider's new elevation? To answer questions such as this one, we need to evaluate the sine or cosine functions at angles that are greater than 90 degrees or at a negative angle. Reference angles make it possible to evaluate trigonometric functions for angles outside the first quadrant. They can also be used to find (x, y) coordinates for those angles. We will use the reference angle of the angle of rotation combined with the quadrant in which the terminal side of the angle lies.

Using Reference Angles to Evaluate Trigonometric Functions

We can find the cosine and sine of any angle in any quadrant if we know the cosine or sine of its reference angle. The absolute values of the cosine and sine of an angle are the same as those of the reference angle. The sign depends on the quadrant of the original angle. The cosine will be positive or negative depending on the sign of the x-values in that quadrant. The sine will be positive or negative depending on the sign of the y-values in that quadrant.

using reference angles to find cosine and sine

Angles have cosines and sines with the same absolute value as their reference angles. The sign (positive or negative) can be determined from the quadrant of the angle.

How To...

Given an angle in standard position, find the reference angle, and the cosine and sine of the original angle.

1. Measure the angle between the terminal side of the given angle and the horizontal axis. That is the reference angle.
2. Determine the values of the cosine and sine of the reference angle.
3. Give the cosine the same sign as the x-values in the quadrant of the original angle.
4. Give the sine the same sign as the y-values in the quadrant of the original angle.

Example 6 Using Reference Angles to Find Sine and Cosine

a. Using a reference angle, find the exact value of $\cos(150°)$ and $\sin(150°)$.

b. Using the reference angle, find $\cos \frac{5\pi}{4}$ and $\sin \frac{5\pi}{4}$.

Solution

a. 150° is located in the second quadrant. The angle it makes with the x-axis is $180° - 150° = 30°$, so the reference angle is 30°.

This tells us that 150° has the same sine and cosine values as 30°, except for the sign. We know that

$$\cos(30°) = \frac{\sqrt{3}}{2} \quad \text{and} \quad \sin(30°) = \frac{1}{2}.$$

Since 150° is in the second quadrant, the x-coordinate of the point on the circle is negative, so the cosine value is negative. The y-coordinate is positive, so the sine value is positive.

$$\cos(150°) = -\frac{\sqrt{3}}{2} \quad \text{and} \quad \sin(150°) = \frac{1}{2}$$

b. $\frac{5\pi}{4}$ is in the third quadrant. Its reference angle is $\frac{5\pi}{4} - \pi = \frac{\pi}{4}$. The cosine and sine of $\frac{\pi}{4}$ are both $\frac{\sqrt{2}}{2}$. In the third quadrant, both x and y are negative, so:

$$\cos \frac{5\pi}{4} = -\frac{\sqrt{2}}{2} \text{ and } \sin \frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$$

Try It #6

a. Use the reference angle of 315° to find $\cos(315°)$ and $\sin(315°)$.

b. Use the reference angle of $-\frac{\pi}{6}$ to find $\cos\left(-\frac{\pi}{6}\right)$ and $\sin\left(-\frac{\pi}{6}\right)$.

Using Reference Angles to Find Coordinates

Now that we have learned how to find the cosine and sine values for special angles in the first quadrant, we can use symmetry and reference angles to fill in cosine and sine values for the rest of the special angles on the unit circle. They are shown in **Figure 19**. Take time to learn the (x, y) coordinates of all of the major angles in the first quadrant.

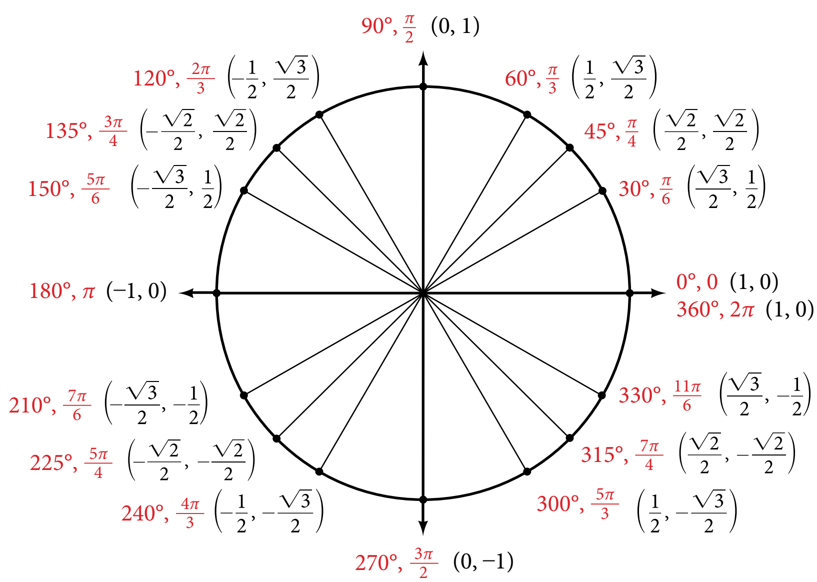


Figure 19 Special angles and coordinates of corresponding points on the unit circle

In addition to learning the values for special angles, we can use reference angles to find (x, y) coordinates of any point on the unit circle, using what we know of reference angles along with the identities

$$x = \cos t \qquad\qquad y = \sin t$$

First we find the reference angle corresponding to the given angle. Then we take the sine and cosine values of the reference angle, and give them the signs corresponding to the y- and x-values of the quadrant.

How To...

Given the angle of a point on a circle and the radius of the circle, find the (x, y) coordinates of the point.

1. Find the reference angle by measuring the smallest angle to the x-axis.
2. Find the cosine and sine of the reference angle.
3. Determine the appropriate signs for x and y in the given quadrant.

Example 7 Using the Unit Circle to Find Coordinates

Find the coordinates of the point on the unit circle at an angle of $\frac{7\pi}{6}$.

Solution We know that the angle $\frac{7\pi}{6}$ is in the third quadrant.

First, let's find the reference angle by measuring the angle to the x-axis. To find the reference angle of an angle whose terminal side is in quadrant III, we find the difference of the angle and π.

$$\frac{7\pi}{6} - \pi = \frac{\pi}{6}$$

Next, we will find the cosine and sine of the reference angle:

$$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \text{ and } \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

We must determine the appropriate signs for x and y in the given quadrant. Because our original angle is in the third quadrant, where both x and y are negative, both cosine and sine are negative.

$$\cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$$

$$\sin\left(\frac{7\pi}{6}\right) = -\frac{1}{2}$$

Now we can calculate the (x, y) coordinates using the identities $x = \cos\theta$ and $y = \sin\theta$.

The coordinates of the point are $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ on the unit circle.

Try It #7

Find the coordinates of the point on the unit circle at an angle of $\frac{5\pi}{3}$.

Access these online resources for additional instruction and practice with sine and cosine functions.

- Trigonometric Functions Using the Unit Circle (http://openstaxcollege.org/l/trigunitcir)
- Sine and Cosine from the Unit Circle (http://openstaxcollege.org/l/sincosuc)
- Sine and Cosine from the Unit Circle and Multiples of Pi Divided by Six (http://openstaxcollege.org/l/sincosmult)
- Sine and Cosine from the Unit Circle and Multiples of Pi Divided by Four (http://openstaxcollege.org/l/sincosmult4)
- Trigonometric Functions Using Reference Angles (http://openstaxcollege.org/l/trigrefang)

7.3 SECTION EXERCISES

VERBAL

1. Describe the unit circle.

2. What do the x- and y-coordinates of the points on the unit circle represent?

3. Discuss the difference between a coterminal angle and a reference angle.

4. Explain how the cosine of an angle in the second quadrant differs from the cosine of its reference angle in the unit circle.

5. Explain how the sine of an angle in the second quadrant differs from the sine of its reference angle in the unit circle.

ALGEBRAIC

For the following exercises, use the given sign of the sine and cosine functions to find the quadrant in which the terminal point determined by t lies.

6. $\sin(t) < 0$ and $\cos(t) < 0$

7. $\sin(t) > 0$ and $\cos(t) > 0$

8. $\sin(t) > 0$ and $\cos(t) < 0$

9. $\sin(t) < 0$ and $\cos(t) > 0$

For the following exercises, find the exact value of each trigonometric function.

10. $\sin \frac{\pi}{2}$

11. $\sin \frac{\pi}{3}$

12. $\cos \frac{\pi}{2}$

13. $\cos \frac{\pi}{3}$

14. $\sin \frac{\pi}{4}$

15. $\cos \frac{\pi}{4}$

16. $\sin \frac{\pi}{6}$

17. $\sin \pi$

18. $\sin \frac{3\pi}{2}$

19. $\cos \pi$

20. $\cos 0$

21. $\cos \frac{\pi}{6}$

22. $\sin 0$

NUMERIC

For the following exercises, state the reference angle for the given angle.

23. $240°$

24. $-170°$

25. $100°$

26. $-315°$

27. $135°$

28. $\frac{5\pi}{4}$

29. $\frac{2\pi}{3}$

30. $\frac{5\pi}{6}$

31. $\frac{-11\pi}{3}$

32. $\frac{-7\pi}{4}$

33. $\frac{-\pi}{8}$

For the following exercises, find the reference angle, the quadrant of the terminal side, and the sine and cosine of each angle. If the angle is not one of the angles on the unit circle, use a calculator and round to three decimal places.

34. $225°$

35. $300°$

36. $320°$

37. $135°$

38. $210°$

39. $120°$

40. $250°$

41. $150°$

42. $\frac{5\pi}{4}$

43. $\frac{7\pi}{6}$

44. $\frac{5\pi}{3}$

45. $\frac{3\pi}{4}$

46. $\frac{4\pi}{3}$

47. $\frac{2\pi}{3}$

48. $\frac{5\pi}{6}$

49. $\frac{7\pi}{4}$

For the following exercises, find the requested value.

50. If $\cos(t) = \frac{1}{7}$ and t is in the 4^{th} quadrant, find $\sin(t)$.

51. If $\cos(t) = \frac{2}{9}$ and t is in the 1^{st} quadrant, find $\sin(t)$.

52. If $\sin(t) = \frac{3}{8}$ and t is in the 2^{nd} quadrant, find $\cos(t)$.

53. If $\sin(t) = -\frac{1}{4}$ and t is in the 3^{rd} quadrant, find $\cos(t)$.

54. Find the coordinates of the point on a circle with radius 15 corresponding to an angle of $220°$.

55. Find the coordinates of the point on a circle with radius 20 corresponding to an angle of $120°$.

56. Find the coordinates of the point on a circle with radius 8 corresponding to an angle of $\frac{7\pi}{4}$.

57. Find the coordinates of the point on a circle with radius 16 corresponding to an angle of $\frac{5\pi}{9.}$

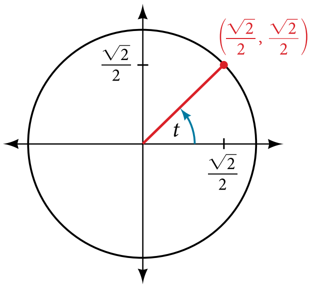
58. State the domain of the sine and cosine functions.

59. State the range of the sine and cosine functions.

GRAPHICAL

For the following exercises, use the given point on the unit circle to find the value of the sine and cosine of t.

60.



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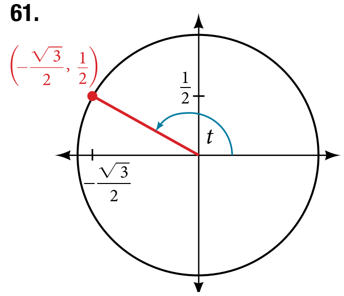

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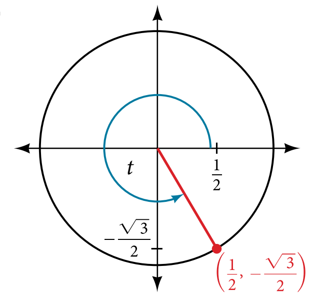

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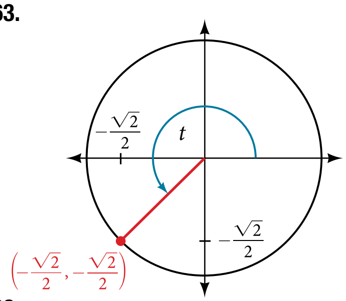

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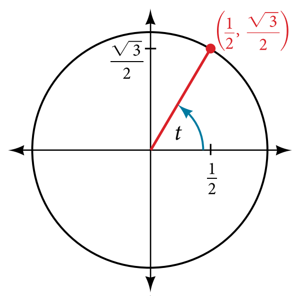

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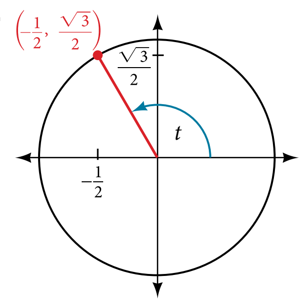

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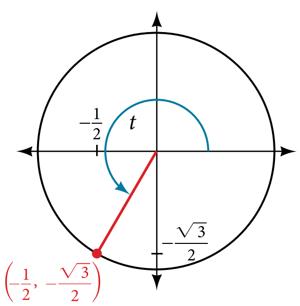

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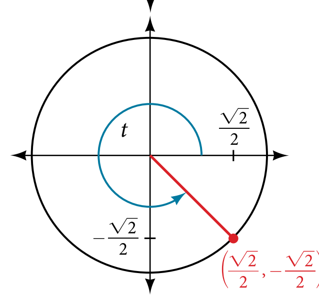

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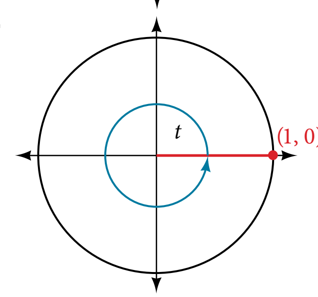

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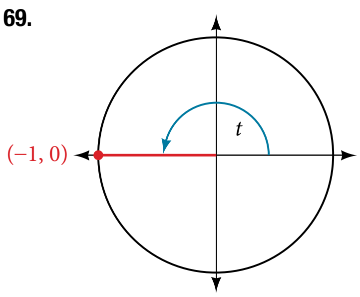

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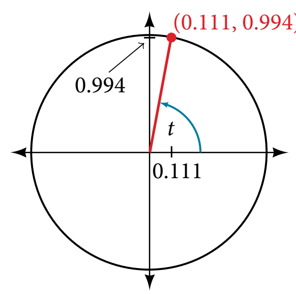

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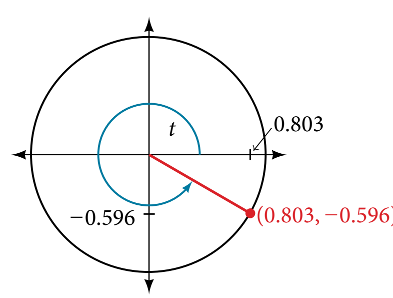

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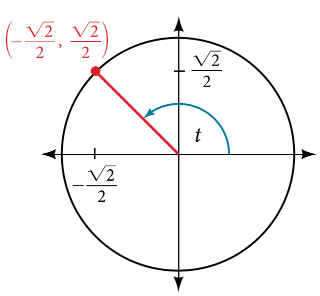

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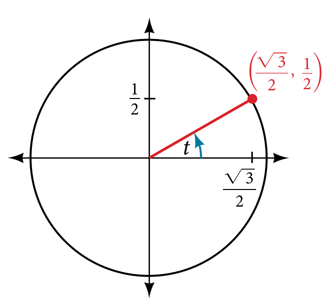

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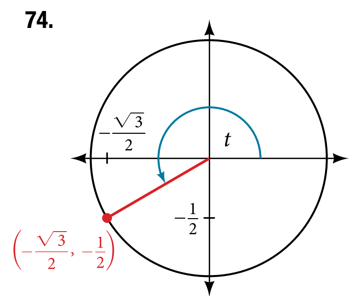

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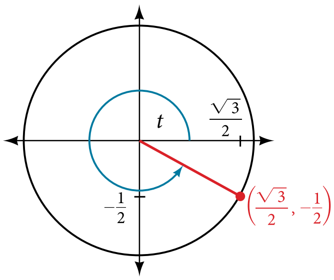

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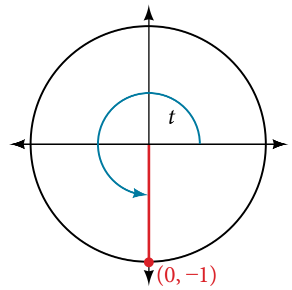

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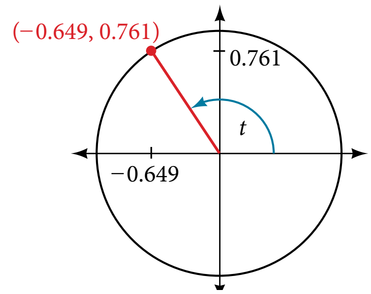

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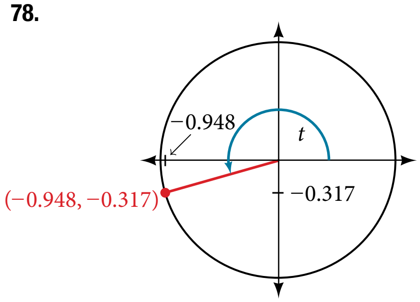

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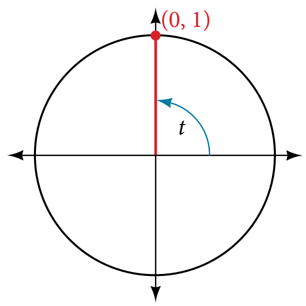


TECHNOLOGY

For the following exercises, use a graphing calculator to evaluate.

80. $\sin \frac{5\pi}{9}$

81. $\cos \frac{5\pi}{9}$

82. $\sin \frac{\pi}{10}$

83. $\cos \frac{\pi}{10}$

84. $\sin \frac{3\pi}{4}$

85. $\cos \frac{3\pi}{4}$

86. $\sin 98°$

87. $\cos 98°$

88. $\cos 310°$

89. $\sin 310°$

EXTENSIONS

For the following exercises, evaluate.

90. $\sin\left(\frac{11\pi}{3}\right)\cos\left(\frac{-5\pi}{6}\right)$

91. $\sin\left(\frac{3\pi}{4}\right)\cos\left(\frac{5\pi}{3}\right)$

92. $\sin\left(-\frac{4\pi}{3}\right)\cos\left(\frac{\pi}{2}\right)$

93. $\sin\left(\frac{-9\pi}{4}\right)\cos\left(\frac{-\pi}{6}\right)$

94. $\sin\left(\frac{\pi}{6}\right)\cos\left(\frac{-\pi}{3}\right)$

95. $\sin\left(\frac{7\pi}{4}\right)\cos\left(\frac{-2\pi}{3}\right)$

96. $\cos\left(\frac{5\pi}{6}\right)\cos\left(\frac{2\pi}{3}\right)$

97. $\cos\left(\frac{-\pi}{3}\right)\cos\left(\frac{\pi}{4}\right)$

98. $\sin\left(\frac{-5\pi}{4}\right)\sin\left(\frac{11\pi}{6}\right)$

99. $\sin(\pi)\sin\left(\frac{\pi}{6}\right)$

REAL-WORLD APPLICATIONS

For the following exercises, use this scenario: A child enters a carousel that takes one minute to revolve once around. The child enters at the point (0, 1), that is, on the due north position. Assume the carousel revolves counter clockwise.

100. What are the coordinates of the child after 45 seconds?

101. What are the coordinates of the child after 90 seconds?

102. What is the coordinates of the child after 125 seconds?

103. When will the child have coordinates (0.707, –0.707) if the ride lasts 6 minutes? (There are multiple answers.)

104. When will the child have coordinates $(-0.866, -0.5)$ if the ride last 6 minutes?

LEARNING OBJECTIVES

In this section, you will:

- Find exact values of the trigonometric functions secant, cosecant, tangent, and cotangent of $\frac{\pi}{3}$, $\frac{\pi}{4}$, and $\frac{\pi}{6}$.
- Use reference angles to evaluate the trigonometric functions secant, cosecant, tangent, and cotangent.
- Use properties of even and odd trigonometric functions.
- Recognize and use fundamental identities.
- Evaluate trigonometric functions with a calculator.

7.4 THE OTHER TRIGONOMETRIC FUNCTIONS

A wheelchair ramp that meets the standards of the Americans with Disabilities Act must make an angle with the ground whose tangent is $\frac{1}{12}$ or less, regardless of its length. A tangent represents a ratio, so this means that for every 1 inch of rise, the ramp must have 12 inches of run. Trigonometric functions allow us to specify the shapes and proportions of objects independent of exact dimensions. We have already defined the sine and cosine functions of an angle. Though sine and cosine are the trigonometric functions most often used, there are four others. Together they make up the set of six trigonometric functions. In this section, we will investigate the remaining functions.

Finding Exact Values of the Trigonometric Functions Secant, Cosecant, Tangent, and Cotangent

We can also define the remaining functions in terms of the unit circle with a point (x, y) corresponding to an angle of t, as shown in **Figure 1**. As with the sine and cosine, we can use the (x, y) coordinates to find the other functions.

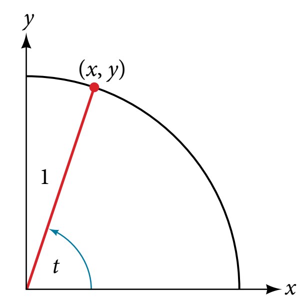


Figure 1

The first function we will define is the tangent. The **tangent** of an angle is the ratio of the y-value to the x-value of the corresponding point on the unit circle. In **Figure 1**, the tangent of angle t is equal to $\frac{y}{x}$, $x \neq 0$. Because the y-value is equal to the sine of t, and the x-value is equal to the cosine of t, the tangent of angle t can also be defined as $\frac{\sin t}{\cos t}$, $\cos t \neq 0$. The tangent function is abbreviated as tan. The remaining three functions can all be expressed as reciprocals of functions we have already defined.

- The **secant** function is the reciprocal of the cosine function. In **Figure 1**, the secant of angle t is equal to $\frac{1}{\cos t} = \frac{1}{x}$, $x \neq 0$. The secant function is abbreviated as sec.
- The **cotangent** function is the reciprocal of the tangent function. In **Figure 1**, the cotangent of angle t is equal to $\frac{\cos t}{\sin t} = \frac{x}{y}$, $y \neq 0$. The cotangent function is abbreviated as cot.
- The **cosecant** function is the reciprocal of the sine function. In **Figure 1**, the cosecant of angle t is equal to $\frac{1}{\sin t} = \frac{1}{y}$, $y \neq 0$. The cosecant function is abbreviated as csc.

tangent, secant, cosecant, and cotangent functions

If t is a real number and (x, y) is a point where the terminal side of an angle of t radians intercepts the unit circle, then

$$\tan t = \frac{y}{x},\ x \neq 0 \qquad \sec t = \frac{1}{x},\ x \neq 0$$

$$\csc t = \frac{1}{y},\ y \neq 0 \qquad \cot t = \frac{x}{y},\ y \neq 0$$

Example 1 Finding Trigonometric Functions from a Point on the Unit Circle

The point $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ is on the unit circle, as shown in **Figure 2**. Find sin t, cos t, tan t, sec t, csc t, and cot t.

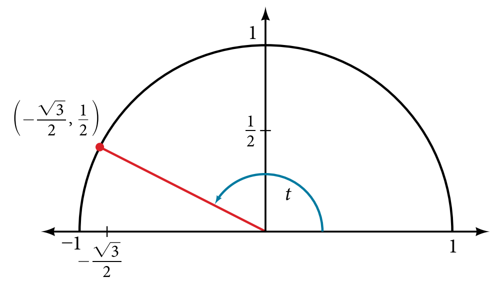


Figure 2

Solution Because we know the (x, y) coordinates of the point on the unit circle indicated by angle t, we can use those coordinates to find the six functions:

$$\sin t = y = \frac{1}{2}$$

$$\cos t = x = -\frac{\sqrt{3}}{2}$$

$$\tan t = \frac{y}{x} = \frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \frac{1}{2}\left(-\frac{2}{\sqrt{3}}\right) = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

$$\sec t = \frac{1}{x} = \frac{1}{-\frac{\sqrt{3}}{2}} = -\frac{2}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}$$

$$\csc t = \frac{1}{y} = \frac{1}{\frac{1}{2}} = 2$$

$$\cot t = \frac{x}{y} = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\frac{\sqrt{3}}{2}\left(\frac{2}{1}\right) = -\sqrt{3}$$

Try It #1

The point $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ is on the unit circle, as shown in **Figure 3**. Find sin t, cos t, tan t, sec t, csc t, and cot t.

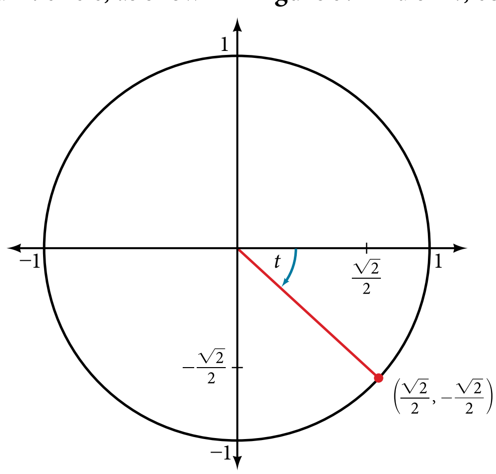


Figure 3

Example 2 Finding the Trigonometric Functions of an Angle

Find sin t, cos t, tan t, sec t, csc t, and cot t when $t = \frac{\pi}{6}$.

Solution We have previously used the properties of equilateral triangles to demonstrate that $\sin \frac{\pi}{6} = \frac{1}{2}$ and $\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$.

We can use these values and the definitions of tangent, secant, cosecant, and cotangent as functions of sine and cosine to find the remaining function values.

$$\tan \frac{\pi}{6} = \frac{\sin \frac{\pi}{6}}{\cos \frac{\pi}{6}}$$

$$= \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\sec \frac{\pi}{6} = \frac{1}{\cos \frac{\pi}{6}}$$

$$= \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

$$\csc \frac{\pi}{6} = \frac{1}{\sin \frac{\pi}{6}} = \frac{1}{\frac{1}{2}} = 2$$

$$\cot \frac{\pi}{6} = \frac{\cos \frac{\pi}{6}}{\sin \frac{\pi}{6}}$$

$$= \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$$

Try It #2

Find sin t, cos t, tan t, sec t, csc t, and cot t when $t = \frac{\pi}{3}$.

Because we know the sine and cosine values for the common first-quadrant angles, we can find the other function values for those angles as well by setting x equal to the cosine and y equal to the sine and then using the definitions of tangent, secant, cosecant, and cotangent. The results are shown in **Table 1**.

Angle	**0**	$\frac{\pi}{6}$**, or 30°**	$\frac{\pi}{4}$**, or 45°**	$\frac{\pi}{3}$**, or 60°**	$\frac{\pi}{2}$**, or 90°**
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
Sine	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
Tangent	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Undefined
Secant	1	$\frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	Undefined
Cosecant	Undefined	2	$\sqrt{2}$	$\frac{2\sqrt{3}}{3}$	1
Cotangent	Undefined	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0

Table 1

Using Reference Angles to Evaluate Tangent, Secant, Cosecant, and Cotangent

We can evaluate trigonometric functions of angles outside the first quadrant using reference angles as we have already done with the sine and cosine functions. The procedure is the same: Find the reference angle formed by the terminal side of the given angle with the horizontal axis. The trigonometric function values for the original angle will be the same as those for the reference angle, except for the positive or negative sign, which is determined by *x*- and *y*-values in the original quadrant. **Figure 4** shows which functions are positive in which quadrant.

To help us remember which of the six trigonometric functions are positive in each quadrant, we can use the mnemonic phrase "A Smart Trig Class." Each of the four words in the phrase corresponds to one of the four quadrants, starting with quadrant I and rotating counterclockwise. In quadrant I, which is "**A**," **a**ll of the six trigonometric functions are positive. In quadrant II, "**S**mart," only **s**ine and its reciprocal function, cosecant, are positive. In quadrant III, "**T**rig," only **t**angent and its reciprocal function, cotangent, are positive. Finally, in quadrant IV, "**C**lass," only **c**osine and its reciprocal function, secant, are positive.

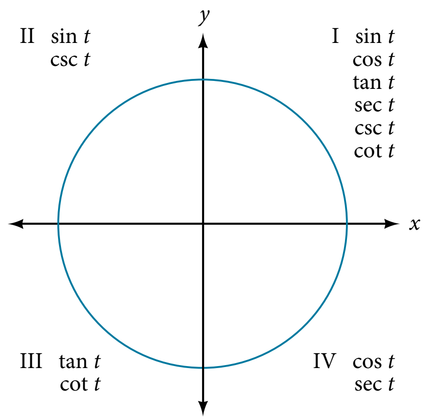


Figure 4 The trigonometric functions are each listed in the quadrants in which they are positive.

How To...

Given an angle not in the first quadrant, use reference angles to find all six trigonometric functions.

1. Measure the angle formed by the terminal side of the given angle and the horizontal axis. This is the reference angle.
2. Evaluate the function at the reference angle.
3. Observe the quadrant where the terminal side of the original angle is located. Based on the quadrant, determine whether the output is positive or negative.

Example 3 Using Reference Angles to Find Trigonometric Functions

Use reference angles to find all six trigonometric functions of $-\frac{5\pi}{6}$.

Solution The angle between this angle's terminal side and the *x*-axis is $\frac{\pi}{6}$, so that is the reference angle. Since $-\frac{5\pi}{6}$ is in the third quadrant, where both *x* and *y* are negative, cosine, sine, secant, and cosecant will be negative, while tangent and cotangent will be positive.

$$\cos\left(-\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}, \quad \sin\left(-\frac{5\pi}{6}\right) = -\frac{1}{2}, \quad \tan\left(-\frac{5\pi}{6}\right) = \frac{\sqrt{3}}{3}$$

$$\sec\left(-\frac{5\pi}{6}\right) = -\frac{2\sqrt{3}}{3}, \quad \csc\left(-\frac{5\pi}{6}\right) = -2, \quad \cot\left(-\frac{5\pi}{6}\right) = \sqrt{3}$$

Try It #3

Use reference angles to find all six trigonometric functions of $-\frac{7\pi}{4}$.

Let $t = 19\pi/4$. Since $t = 5\pi - \pi/4$, t is in quadrant II and the reference angle is $\pi/4$. Hence $\cos t = -1/\sqrt{2}$, $\sin t = 1/\sqrt{2}$, $\tan t = \cot t = -1$, $\sec t = -\sqrt{2}$, $\csc t = \sqrt{2}$.

Using Even and Odd Trigonometric Functions

To be able to use our six trigonometric functions freely with both positive and negative angle inputs, we should examine how each function treats a negative input. As it turns out, there is an important difference among the functions in this regard. Consider the function $f(x) = x^2$, shown in **Figure 5**. The graph of the function is symmetrical about the y-axis. All along the curve, any two points with opposite x-values have the same function value. This matches the result of calculation: $(4)^2 = (-4)^2$, $(-5)^2 = (5)^2$, and so on. So $f(x) = x^2$ is an even function, a function such that two inputs that are opposites have the same output. That means $f(-x) = f(x)$.

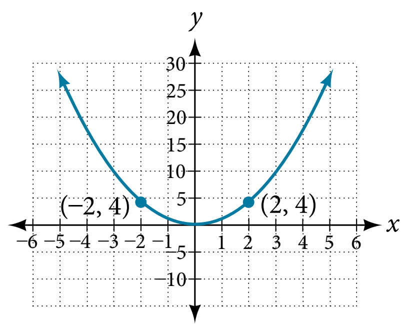


Figure 5 The function $f(x) = x^2$ is an even function.

Now consider the function $f(x) = x^3$, shown in **Figure 6**. The graph is not symmetrical about the y-axis. All along the graph, any two points with opposite x-values also have opposite y-values. So $f(x) = x^3$ is an odd function, one such that two inputs that are opposites have outputs that are also opposites. That means $f(-x) = -f(x)$.

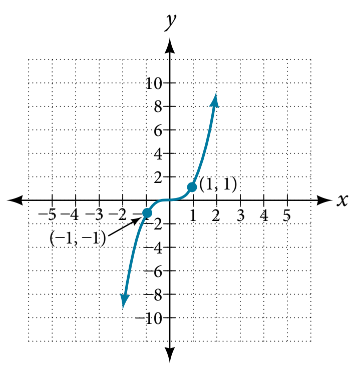


Figure 6 The function $f(x) = x^3$ is an odd function.

We can test whether a trigonometric function is even or odd by drawing a unit circle with a positive and a negative angle, as in **Figure 7**. The sine of the positive angle is y. The sine of the negative angle is $-y$. The sine function, then, is an odd function. We can test each of the six trigonometric functions in this fashion. The results are shown in **Table 2**.

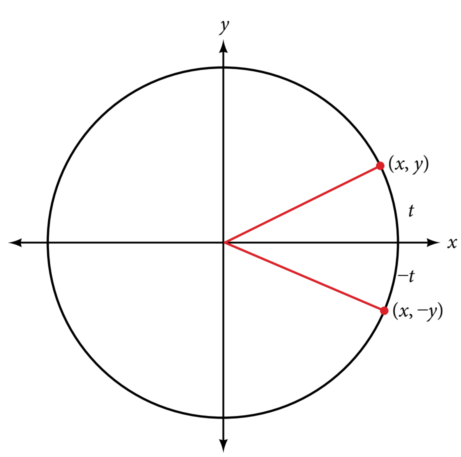


Figure 7

$\sin t = y$ $\sin(-t) = -y$ $\sin t \neq \sin(-t)$	$\cos t = x$ $\cos(-t) = x$ $\cos t = \cos(-t)$	$\tan(t) = \frac{y}{x}$ $\tan(-t) = -\frac{y}{x}$ $\tan t \neq \tan(-t)$
$\sec t = \frac{1}{x}$ $\sec(-t) = \frac{1}{x}$ $\sec t = \sec(-t)$	$\csc t = \frac{1}{y}$ $\csc(-t) = \frac{1}{-y}$ $\csc t \neq \csc(-t)$	$\cot t = \frac{x}{y}$ $\cot(-t) = \frac{x}{-y}$ $\cot t \neq \cot(-t)$

Table 2

even and odd trigonometric functions

An even function is one in which $f(-x) = f(x)$. An odd function is one in which $f(-x) = -f(x)$.
Cosine and secant are even:

$$\cos(-t) = \cos t$$
$$\sec(-t) = \sec t$$

Sine, tangent, cosecant, and cotangent are odd:

$$\sin(-t) = -\sin t$$
$$\tan(-t) = -\tan t$$
$$\csc(-t) = -\csc t$$
$$\cot(-t) = -\cot t$$

Example 4 Using Even and Odd Properties of Trigonometric Functions

If the secant of angle t is 2, what is the secant of $-t$?

Solution Secant is an even function. The secant of an angle is the same as the secant of its opposite. So if the secant of angle t is 2, the secant of $-t$ is also 2.

Try It #4

If the cotangent of angle t is $\sqrt{3}$, what is the cotangent of $-t$?

Recognizing and Using Fundamental Identities

We have explored a number of properties of trigonometric functions. Now, we can take the relationships a step further, and derive some fundamental identities. Identities are statements that are true for all values of the input on which they are defined. Usually, identities can be derived from definitions and relationships we already know. For example, the Pythagorean Identity we learned earlier was derived from the Pythagorean Theorem and the definitions of sine and cosine.

fundamental identities

We can derive some useful **identities** from the six trigonometric functions. The other four trigonometric functions can be related back to the sine and cosine functions using these basic relationships:

$$\tan t = \frac{\sin t}{\cos t} \qquad \sec t = \frac{1}{\cos t}$$

$$\csc t = \frac{1}{\sin t} \qquad \cot t = \frac{1}{\tan t} = \frac{\cos t}{\sin t}$$

Example 5 Using Identities to Evaluate Trigonometric Functions

a. Given $\sin(45°) = \frac{\sqrt{2}}{2,}$, $\cos(45°) = \frac{\sqrt{2}}{2}$, evaluate $\tan(45°)$.

b. Given $\sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}$, $\cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$, evaluate $\sec\left(\frac{5\pi}{6}\right)$.

Solution Because we know the sine and cosine values for these angles, we can use identities to evaluate the other functions.

a.

$$\begin{aligned} \tan(45°) &= \frac{\sin(45°)}{\cos(45°)} \\ &= \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} \\ &= 1 \end{aligned}$$

b.

$$\begin{aligned} \sec\left(\frac{5\pi}{6}\right) &= \frac{1}{\left(\cos\frac{5\pi}{6}\right)} \\ &= \frac{1}{-\frac{\sqrt{3}}{2}} \\ &= \frac{-2}{\sqrt{3}} \\ &= -\frac{2\sqrt{3}}{3} \end{aligned}$$

Try It #5

Evaluate $\csc\left(\frac{7\pi}{6}\right)$.

Example 6 Using Identities to Simplify Trigonometric Expressions

Simplify $\frac{\sec t}{\tan t}$.

Solution We can simplify this by rewriting both functions in terms of sine and cosine.

$$\begin{aligned} \frac{\sec t}{\tan t} &= \frac{\frac{1}{\cos t}}{\frac{\sin t}{\cos t}} \\ &= \frac{1}{\cos t} \cdot \frac{\cos t}{\sin t} && \text{Multiply by the reciprocal.} \\ &= \frac{1}{\sin t} = \csc t && \text{Simplify and use the identity.} \end{aligned}$$

By showing that $\frac{\sec t}{\tan t}$ can be simplified to $\csc t$, we have, in fact, established a new identity.

$$\frac{\sec t}{\tan t} = \csc t$$

Try It #6

Simplify $(\tan t)(\cos t)$.

Alternate Forms of the Pythagorean Identity

We can use these fundamental identities to derive alternative forms of the Pythagorean Identity, $\cos^2 t + \sin^2 t = 1$. One form is obtained by dividing both sides by $\cos^2 t$:

$$\frac{\cos^2 t}{\cos^2 t} + \frac{\sin^2 t}{\cos^2 t} = \frac{1}{\cos^2 t}$$

$$1 + \tan^2 t = \sec^2 t$$

The other form is obtained by dividing both sides by $\sin^2 t$:

$$\frac{\cos^2 t}{\sin^2 t} + \frac{\sin^2 t}{\sin^2 t} = \frac{1}{\sin^2 t}$$

$$\cot^2 t + 1 = \csc^2 t$$

alternate forms of the pythagorean identity

$$1 + \tan^2 t = \sec^2 t$$

$$\cot^2 t + 1 = \csc^2 t$$

Example 7 Using Identities to Relate Trigonometric Functions

If $\cos(t) = \frac{12}{13}$ and t is in quadrant IV, as shown in **Figure 8**, find the values of the other five trigonometric functions.

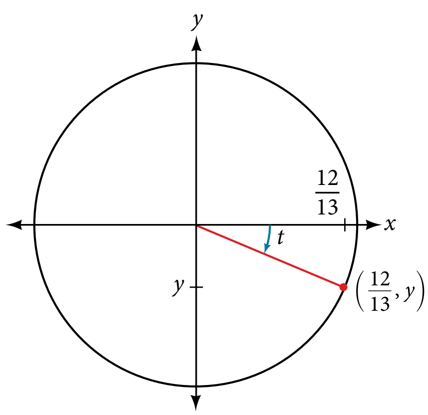


Figure 8

Solution We can find the sine using the Pythagorean Identity, $\cos^2 t + \sin^2 t = 1$, and the remaining functions by relating them to sine and cosine.

$$\left(\frac{12}{13}\right)^2 + \sin^2 t = 1$$

$$\sin^2 t = 1 - \left(\frac{12}{13}\right)^2$$

$$\sin^2 t = 1 - \frac{144}{169}$$

$$\sin^2 t = \frac{25}{169}$$

$$\sin t = \pm\sqrt{\frac{25}{169}}$$

$$\sin t = \pm\frac{\sqrt{25}}{\sqrt{169}}$$

$$\sin t = \pm\frac{5}{13}$$

The sign of the sine depends on the y-values in the quadrant where the angle is located. Since the angle is in quadrant IV, where the y-values are negative, its sine is negative, $-\frac{5}{13}$.

The remaining functions can be calculated using identities relating them to sine and cosine.

$$\tan t = \frac{\sin t}{\cos t} = \frac{-\frac{5}{13}}{\frac{12}{13}} = -\frac{5}{12}$$

$$\sec t = \frac{1}{\cos t} = \frac{1}{\frac{12}{13}} = \frac{13}{12}$$

$$\csc t = \frac{1}{\sin t} = \frac{1}{-\frac{5}{13}} = -\frac{13}{5}$$

$$\cot t = \frac{1}{\tan \text{t}} = \frac{1}{-\frac{5}{12}} = -\frac{12}{5}$$

Try It #7

If $\sec(t) = -\frac{17}{8}$ and $0 < t < \pi$, find the values of the other five functions.

As we discussed in the chapter opening, a function that repeats its values in regular intervals is known as a periodic function. The trigonometric functions are periodic. For the four trigonometric functions, sine, cosine, cosecant and secant, a revolution of one circle, or 2π, will result in the same outputs for these functions. And for tangent and cotangent, only a half a revolution will result in the same outputs.

Other functions can also be periodic. For example, the lengths of months repeat every four years. If x represents the length time, measured in years, and $f(x)$ represents the number of days in February, then $f(x + 4) = f(x)$. This pattern repeats over and over through time. In other words, every four years, February is guaranteed to have the same number of days as it did 4 years earlier. The positive number 4 is the smallest positive number that satisfies this condition and is called the period. A **period** is the shortest interval over which a function completes one full cycle—in this example, the period is 4 and represents the time it takes for us to be certain February has the same number of days.

period of a function

The **period** P of a repeating function f is the number representing the interval such that $f(x + P) = f(x)$ for any value of x.
The period of the cosine, sine, secant, and cosecant functions is 2π.
The period of the tangent and cotangent functions is π.

Example 8 Finding the Values of Trigonometric Functions

Find the values of the six trigonometric functions of angle t based on **Figure 9**.

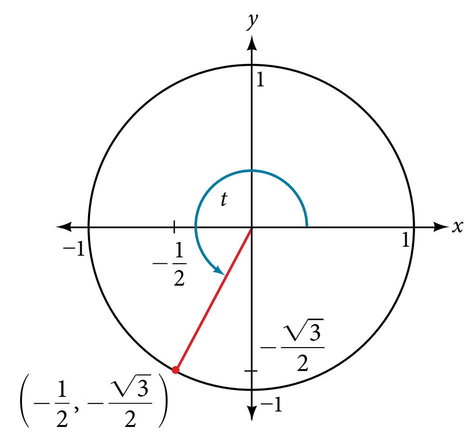


Figure 9

Solution

$$\sin t = y = -\frac{\sqrt{3}}{2}$$

$$\cos t = x = -\frac{1}{2}$$

$$\tan t = \frac{\sin t}{\cos t} = \frac{-\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = \sqrt{3}$$

$$\sec t = \frac{1}{\cos t} = \frac{1}{-\frac{1}{2}} = -2$$

$$\csc t = \frac{1}{\sin t} = \frac{1}{-\frac{\sqrt{3}}{2}} = -\frac{2\sqrt{3}}{3}$$

$$\cot t = \frac{1}{\tan t} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

Try It #8

Find the values of the six trigonometric functions of angle t based on **Figure 10**.

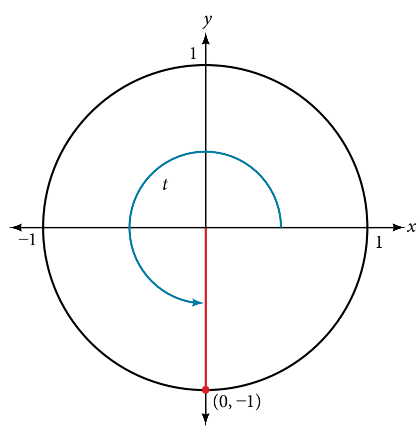


Figure 10

Example 9 Finding the Value of Trigonometric Functions

If $\sin(t) = -\frac{\sqrt{3}}{2}$ and $\cos(t) = \frac{1}{2}$, find $\sec(t)$, $\csc(t)$, $\tan(t)$, $\cot(t)$.

Solution

$$\sec t = \frac{1}{\cos t} = \frac{1}{\frac{1}{2}} = 2$$

$$\csc t = \frac{1}{\sin t} = \frac{1}{-\frac{\sqrt{3}}{2}} = -\frac{2\sqrt{3}}{3}$$

$$\tan t = \frac{\sin t}{\cos t} = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\sqrt{3}$$

$$\cot t = \frac{1}{\tan t} = \frac{1}{-\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

Try It #9

If $\sin(t) = \frac{\sqrt{2}}{2}$ and $\cos(t) = \frac{\sqrt{2}}{2}$, find $\sec(t)$, $\csc(t)$, $\tan(t)$, and $\cot(t)$.

Evaluating Trigonometric Functions with a Calculator

We have learned how to evaluate the six trigonometric functions for the common first-quadrant angles and to use them as reference angles for angles in other quadrants. To evaluate trigonometric functions of other angles, we use a scientific or graphing calculator or computer software. If the calculator has a degree mode and a radian mode, confirm the correct mode is chosen before making a calculation.

Evaluating a tangent function with a scientific calculator as opposed to a graphing calculator or computer algebra system is like evaluating a sine or cosine: Enter the value and press the **TAN** key. For the reciprocal functions, there may not be any dedicated keys that say **CSC**, **SEC**, or **COT**. In that case, the function must be evaluated as the reciprocal of a sine, cosine, or tangent.

If we need to work with degrees and our calculator or software does not have a degree mode, we can enter the degrees multiplied by the conversion factor $\frac{\pi}{180}$ to convert the degrees to radians. To find the secant of 30°, we could press

$$\text{(for a scientific calculator): } \frac{1}{30 \times \frac{\pi}{180}\ \text{COS}} \quad \text{or} \quad \text{(for a graphing calculator): } \frac{1}{\cos\left(\frac{30\pi}{180}\right)}$$

How To...

Given an angle measure in radians, use a scientific calculator to find the cosecant.

1. If the calculator has degree mode and radian mode, set it to radian mode.
2. Enter: **1 /**
3. Enter the value of the angle inside parentheses.
4. Press the **SIN** key.
5. Press the = key.

How To...

Given an angle measure in radians, use a graphing utility/calculator to find the cosecant.

1. If the graphing utility has degree mode and radian mode, set it to radian mode.
2. Enter: **1 /**
3. Press the **SIN** key.
4. Enter the value of the angle inside parentheses.
5. Press the **ENTER** key.

Example 10 Evaluating the Cosecant Using Technology

Evaluate the cosecant of $\frac{5\pi}{7}$.

Solution

For a scientific calculator, enter information as follows:

$$1\ /\ (\ 5 \times \pi\ /\ 7\ )\ \text{SIN} =$$

$$\csc\left(\frac{5\pi}{7}\right) \approx 1.279$$

Try It #10

Evaluate the cotangent of $-\frac{\pi}{8}$.

Access these online resources for additional instruction and practice with other trigonometric functions.

- Determining Trig Function Values (http://openstaxcollege.org/l/trigfuncval)
- More Examples of Determining Trig Functions (http://openstaxcollege.org/l/moretrigfun)
- Pythagorean Identities (http://openstaxcollege.org/l/pythagiden)
- Trig Functions on a Calculator (http://openstaxcollege.org/l/trigcalc)

7.4 SECTION EXERCISES

VERBAL

1. On an interval of $[0, 2\pi)$, can the sine and cosine values of a radian measure ever be equal? If so, where?

2. What would you estimate the cosine of π degrees to be? Explain your reasoning.

3. For any angle in quadrant II, if you knew the sine of the angle, how could you determine the cosine of the angle?

4. Describe the secant function.

5. Tangent and cotangent have a period of π. What does this tell us about the output of these functions?

ALGEBRAIC

For the following exercises, find the exact value of each expression.

6. $\tan \frac{\pi}{6}$

7. $\sec \frac{\pi}{6}$

8. $\csc \frac{\pi}{6}$

9. $\cot \frac{\pi}{6}$

10. $\tan \frac{\pi}{4}$

11. $\sec \frac{\pi}{4}$

12. $\csc \frac{\pi}{4}$

13. $\cot \frac{\pi}{4}$

14. $\tan \frac{\pi}{3}$

15. $\sec \frac{\pi}{3}$

16. $\csc \frac{\pi}{3}$

17. $\cot \frac{\pi}{3}$

For the following exercises, use reference angles to evaluate the expression.

18. $\tan \frac{5\pi}{6}$

19. $\sec \frac{7\pi}{6}$

20. $\csc \frac{11\pi}{6}$

21. $\cot \frac{13\pi}{6}$

22. $\tan \frac{7\pi}{4}$

23. $\sec \frac{3\pi}{4}$

24. $\csc \frac{5\pi}{4}$

25. $\cot \frac{11\pi}{4}$

26. $\tan \frac{8\pi}{3}$

27. $\sec \frac{4\pi}{3}$

28. $\csc \frac{2\pi}{3}$

29. $\cot \frac{5\pi}{3}$

30. $\tan 225°$

31. $\sec 300°$

32. $\csc 150°$

33. $\cot 240°$

34. $\tan 330°$

35. $\sec 120°$

36. $\csc 210°$

37. $\cot 315°$

38. If $\sin t = \frac{3}{4}$, and t is in quadrant II, find $\cos t$, $\sec t$, $\csc t$, $\tan t$, $\cot t$.

39. If $\cos t = -\frac{1}{3}$, and t is in quadrant III, find $\sin t$, $\sec t$, $\csc t$, $\tan t$, $\cot t$.

40. If $\tan t = \frac{12}{5}$, and $0 \le t < \frac{\pi}{2}$, find $\sin t$, $\cos t$, $\sec t$, $\csc t$, and $\cot t$.

41. If $\sin t = \frac{\sqrt{3}}{2}$ and $\cos t = \frac{1}{2}$, find $\sec t$, $\csc t$, $\tan t$, and $\cot t$.

42. If $\sin 40° \approx 0.643$ and $\cos 40° \approx 0.766$, find $\sec 40°$, $\csc 40°$, $\tan 40°$, and $\cot 40°$.

43. If $\sin t = \frac{\sqrt{2}}{2}$, what is the $\sin(-t)$?

44. If $\cos t = \frac{1}{2}$, what is the $\cos(-t)$?

45. If $\sec t = 3.1$, what is the $\sec(-t)$?

46. If $\csc t = 0.34$, what is the $\csc(-t)$?

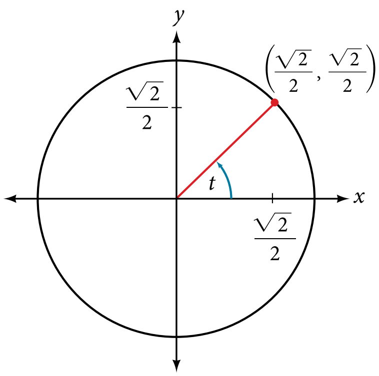
47. If $\tan t = -1.4$, what is the $\tan(-t)$?

48. If $\cot t = 9.23$, what is the $\cot(-t)$?

GRAPHICAL

For the following exercises, use the angle in the unit circle to find the value of the each of the six trigonometric functions.

49.



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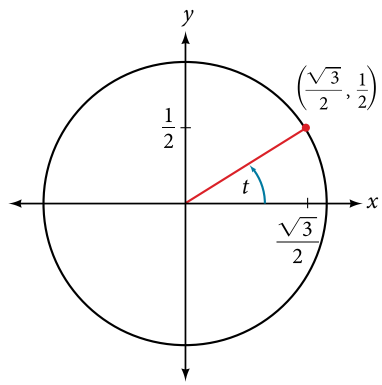

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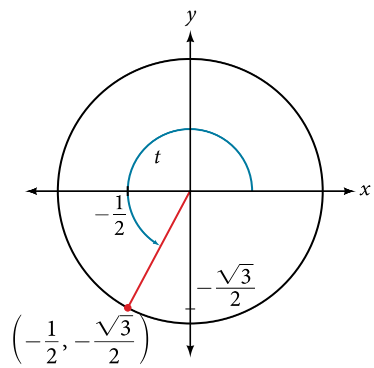


TECHNOLOGY

For the following exercises, use a graphing calculator to evaluate.

52. $\csc \frac{5\pi}{9}$

53. $\cot \frac{4\pi}{7}$

54. $\sec \frac{\pi}{10}$

55. $\tan \frac{5\pi}{8}$

56. $\sec \frac{3\pi}{4}$

57. $\csc \frac{\pi}{4}$

58. $\tan 98°$

59. $\cot 33°$

60. $\cot 140°$

61. $\sec 310°$

EXTENSIONS

For the following exercises, use identities to evaluate the expression.

62. If $\tan(t) \approx 2.7$, and $\sin(t) \approx 0.94$, find $\cos(t)$.

63. If $\tan(t) \approx 1.3$, and $\cos(t) \approx 0.61$, find $\sin(t)$.

64. If $\csc(t) \approx 3.2$, and $\cos(t) \approx 0.95$, find $\tan(t)$.

65. If $\cot(t) \approx 0.58$, and $\cos(t) \approx 0.5$, find $\csc(t)$.

66. Determine whether the function $f(x) = 2\sin x \cos x$ is even, odd, or neither.

67. Determine whether the function $f(x) = 3\sin^2 x \cos x + \sec x$ is even, odd, or neither.

68. Determine whether the function $f(x) = \sin x - 2\cos^2 x$ is even, odd, or neither.

69. Determine whether the function $f(x) = \csc^2 x + \sec x$ is even, odd, or neither.

For the following exercises, use identities to simplify the expression.

70. $\csc t \tan t$

71. $\frac{\sec t}{\csc t}$

REAL-WORLD APPLICATIONS

72. The amount of sunlight in a certain city can be modeled by the function $h = 15 \cos\left(\frac{1}{600}d\right)$, where h represents the hours of sunlight, and d is the day of the year. Use the equation to find how many hours of sunlight there are on February 10, the 42nd day of the year. State the period of the function.

73. The amount of sunlight in a certain city can be modeled by the function $h = 16\cos\left(\frac{1}{500}d\right)$, where h represents the hours of sunlight, and d is the day of the year. Use the equation to find how many hours of sunlight there are on September 24, the 267th day of the year. State the period of the function.

74. The equation $P = 20\sin(2\pi t) + 100$ models the blood pressure, P, where t represents time in seconds. **a.** Find the blood pressure after 15 seconds. **b.** What are the maximum and minimum blood pressures?

75. The height of a piston, h, in inches, can be modeled by the equation $y = 2\cos x + 6$, where x represents the crank angle. Find the height of the piston when the crank angle is 55°.

76. The height of a piston, h, in inches, can be modeled by the equation $y = 2\cos x + 5$, where x represents the crank angle. Find the height of the piston when the crank angle is 55°.

CHAPTER 7 REVIEW

Key Terms

adjacent side in a right triangle, the side between a given angle and the right angle

angle the union of two rays having a common endpoint

angle of depression the angle between the horizontal and the line from the object to the observer's eye, assuming the object is positioned lower than the observer

angle of elevation the angle between the horizontal and the line from the object to the observer's eye, assuming the object is positioned higher than the observer

angular speed the angle through which a rotating object travels in a unit of time

arc length the length of the curve formed by an arc

area of a sector area of a portion of a circle bordered by two radii and the intercepted arc; the fraction $\frac{\theta}{2\pi}$ multiplied by the area of the entire circle

cosecant the reciprocal of the sine function: on the unit circle, $\csc t = \frac{1}{y}, y \neq 0$

cosine function the x-value of the point on a unit circle corresponding to a given angle

cotangent the reciprocal of the tangent function: on the unit circle, $\cot t = \frac{x}{y}, y \neq 0$

coterminal angles description of positive and negative angles in standard position sharing the same terminal side

degree a unit of measure describing the size of an angle as one-360th of a full revolution of a circle

hypotenuse the side of a right triangle opposite the right angle

identities statements that are true for all values of the input on which they are defined

initial side the side of an angle from which rotation begins

linear speed the distance along a straight path a rotating object travels in a unit of time; determined by the arc length

measure of an angle the amount of rotation from the initial side to the terminal side

negative angle description of an angle measured clockwise from the positive x-axis

opposite side in a right triangle, the side most distant from a given angle

period the smallest interval P of a repeating function f such that $f(x + P) = f(x)$

positive angle description of an angle measured counterclockwise from the positive x-axis

Pythagorean Identity a corollary of the Pythagorean Theorem stating that the square of the cosine of a given angle plus the square of the sine of that angle equals 1

quadrantal angle an angle whose terminal side lies on an axis

radian the measure of a central angle of a circle that intercepts an arc equal in length to the radius of that circle

radian measure the ratio of the arc length formed by an angle divided by the radius of the circle

ray one point on a line and all points extending in one direction from that point; one side of an angle

reference angle the measure of the acute angle formed by the terminal side of the angle and the horizontal axis

secant the reciprocal of the cosine function: on the unit circle, $\sec t = \frac{1}{x}, x \neq 0$

sine function the y-value of the point on a unit circle corresponding to a given angle

standard position the position of an angle having the vertex at the origin and the initial side along the positive x-axis

tangent the quotient of the sine and cosine: on the unit circle, $\tan t = \frac{y}{x}, x \neq 0$

terminal side the side of an angle at which rotation ends

unit circle a circle with a center at (0, 0) and radius 1.

vertex the common endpoint of two rays that form an angle

Key Equations

arc length	$s = r\theta$
area of a sector	$A = \frac{1}{2}\theta r^2$
angular speed	$\omega = \frac{\theta}{t}$
linear speed	$v = \frac{s}{t}$
linear speed related to angular speed	$v = r\omega$
trigonometric functions	Sine $\sin t = \frac{\text{opposite}}{\text{hypotenuse}}$ Cosine $\cos t = \frac{\text{adjacent}}{\text{hypotenuse}}$ Tangent $\tan t = \frac{\text{opposite}}{\text{adjacent}}$ Secant $\sec t = \frac{\text{hypotenuse}}{\text{adjacent}}$ Cosecant $\csc t = \frac{\text{hypotenuse}}{\text{opposite}}$ Cotangent $\cot t = \frac{\text{adjacent}}{\text{opposite}}$
reciprocal trigonometric functions	$\sin t = \frac{1}{\csc t}$ $\csc t = \frac{1}{\sin t}$ $\cos t = \frac{1}{\sec t}$ $\sec t = \frac{1}{\cos t}$ $\tan t = \frac{1}{\cot t}$ $\cot t = \frac{1}{\tan t}$
cofunction identities	$\cos t = \sin\left(\frac{\pi}{2} - t\right)$ $\sin t = \cos\left(\frac{\pi}{2} - t\right)$ $\tan t = \cot\left(\frac{\pi}{2} - t\right)$ $\cot t = \tan\left(\frac{\pi}{2} - t\right)$ $\sec t = \csc\left(\frac{\pi}{2} - t\right)$ $\csc t = \sec\left(\frac{\pi}{2} - t\right)$
cosine	$\cos t = x$
sine	$\sin t = y$
Pythagorean Identity	$\cos^2 t + \sin^2 t = 1$
tangent function	$\tan t = \frac{\sin t}{\cos t}$
secant function	$\sec t = \frac{1}{\cos t}$
cosecant function	$\csc t = \frac{1}{\sin t}$
cotangent function	$\cot t = \frac{1}{\tan t} = \frac{\cos t}{\sin t}$

Key Concepts

7.1 Angles

- An angle is formed from the union of two rays, by keeping the initial side fixed and rotating the terminal side. The amount of rotation determines the measure of the angle.
- An angle is in standard position if its vertex is at the origin and its initial side lies along the positive x-axis. A positive angle is measured counterclockwise from the initial side and a negative angle is measured clockwise.
- To draw an angle in standard position, draw the initial side along the positive x-axis and then place the terminal side according to the fraction of a full rotation the angle represents. See **Example 1**.
- In addition to degrees, the measure of an angle can be described in radians. See **Example 2**.
- To convert between degrees and radians, use the proportion $\frac{\theta}{180} = \frac{\theta_R}{\pi}$. See **Example 3** and **Example 4**.
- Two angles that have the same terminal side are called coterminal angles.
- We can find coterminal angles by adding or subtracting $360°$ or 2π. See **Example 5** and **Example 6**.
- Coterminal angles can be found using radians just as they are for degrees. See **Example 7**.
- The length of a circular arc is a fraction of the circumference of the entire circle. See **Example 8**.
- The area of sector is a fraction of the area of the entire circle. See **Example 9**.
- An object moving in a circular path has both linear and angular speed.
- The angular speed of an object traveling in a circular path is the measure of the angle through which it turns in a unit of time. See **Example 10**.
- The linear speed of an object traveling along a circular path is the distance it travels in a unit of time. See **Example 11**.

7.2 Right Triangle Trigonometry

- We can define trigonometric functions as ratios of the side lengths of a right triangle. See **Example 1**.
- The same side lengths can be used to evaluate the trigonometric functions of either acute angle in a right triangle. See **Example 2**.
- We can evaluate the trigonometric functions of special angles, knowing the side lengths of the triangles in which they occur. See **Example 3**.
- Any two complementary angles could be the two acute angles of a right triangle.
- If two angles are complementary, the cofunction identities state that the sine of one equals the cosine of the other and vice versa. See **Example 4**.
- We can use trigonometric functions of an angle to find unknown side lengths.
- Select the trigonometric function representing the ratio of the unknown side to the known side. See **Example 5**.
- Right-triangle trigonometry permits the measurement of inaccessible heights and distances.
- The unknown height or distance can be found by creating a right triangle in which the unknown height or distance is one of the sides, and another side and angle are known. See **Example 6**.

7.3 Unit Circle

- Finding the function values for the sine and cosine begins with drawing a unit circle, which is centered at the origin and has a radius of 1 unit.
- Using the unit circle, the sine of an angle t equals the y-value of the endpoint on the unit circle of an arc of length t whereas the cosine of an angle t equals the x-value of the endpoint. See **Example 1**.
- The sine and cosine values are most directly determined when the corresponding point on the unit circle falls on an axis. See **Example 2**.
- When the sine or cosine is known, we can use the Pythagorean Identity to find the other. The Pythagorean Identity is also useful for determining the sines and cosines of special angles. See **Example 3**.
- Calculators and graphing software are helpful for finding sines and cosines if the proper procedure for entering information is known. See **Example 4**.
- The domain of the sine and cosine functions is all real numbers.

- The range of both the sine and cosine functions is $[-1, 1]$.
- The sine and cosine of an angle have the same absolute value as the sine and cosine of its reference angle.
- The signs of the sine and cosine are determined from the x- and y-values in the quadrant of the original angle.
- An angle's reference angle is the size angle, t, formed by the terminal side of the angle t and the horizontal axis. See **Example 5**.
- Reference angles can be used to find the sine and cosine of the original angle. See **Example 6**.
- Reference angles can also be used to find the coordinates of a point on a circle. See **Example 7**.

7.4 The Other Trigonometric Functions

- The tangent of an angle is the ratio of the y-value to the x-value of the corresponding point on the unit circle.
- The secant, cotangent, and cosecant are all reciprocals of other functions. The secant is the reciprocal of the cosine function, the cotangent is the reciprocal of the tangent function, and the cosecant is the reciprocal of the sine function.
- The six trigonometric functions can be found from a point on the unit circle. See **Example 1**.
- Trigonometric functions can also be found from an angle. See **Example 2**.
- Trigonometric functions of angles outside the first quadrant can be determined using reference angles. See **Example 3**.
- A function is said to be even if $f(-x) = f(x)$ and odd if $f(-x) = -f(x)$.
- Cosine and secant are even; sine, tangent, cosecant, and cotangent are odd.
- Even and odd properties can be used to evaluate trigonometric functions. See **Example 4**.
- The Pythagorean Identity makes it possible to find a cosine from a sine or a sine from a cosine.
- Identities can be used to evaluate trigonometric functions. See **Example 5** and **Example 6**.
- Fundamental identities such as the Pythagorean Identity can be manipulated algebraically to produce new identities. See **Example 7**.
- The trigonometric functions repeat at regular intervals.
- The period P of a repeating function f is the smallest interval such that $f(x + P) = f(x)$ for any value of x.
- The values of trigonometric functions of special angles can be found by mathematical analysis.
- To evaluate trigonometric functions of other angles, we can use a calculator or computer software. See **Example 8**.

CHAPTER 7 REVIEW EXERCISES

ANGLES

For the following exercises, convert the angle measures to degrees.

1. $\frac{\pi}{4}$

2. $-\frac{5\pi}{3}$

For the following exercises, convert the angle measures to radians.

3. $-210°$

4. $180°$

5. Find the length of an arc in a circle of radius 7 meters subtended by the central angle of $85°$.

6. Find the area of the sector of a circle with diameter 32 feet and an angle of $\frac{3\pi}{5}$ radians.

For the following exercises, find the angle between $0°$ and $360°$ that is coterminal with the given angle.

7. $420°$

8. $-80°$

For the following exercises, find the angle between 0 and 2π in radians that is coterminal with the given angle.

9. $-\frac{20\pi}{11}$

10. $\frac{14\pi}{5}$

For the following exercises, draw the angle provided in standard position on the Cartesian plane.

11. $-210°$

12. $75°$

13. $\frac{5\pi}{4}$

14. $-\frac{\pi}{3}$

15. Find the linear speed of a point on the equator of the earth if the earth has a radius of 3,960 miles and the earth rotates on its axis every 24 hours. Express answer in miles per hour.

16. A car wheel with a diameter of 18 inches spins at the rate of 10 revolutions per second. What is the car's speed in miles per hour?

RIGHT TRIANGLE TRIGONOMETRY

For the following exercises, use side lengths to evaluate.

17. $\cos \frac{\pi}{4}$

18. $\cot \frac{\pi}{3}$

19. $\tan \frac{\pi}{6}$

20. $\cos\left(\frac{\pi}{2}\right) = \sin(\_\_\_\_\_°)$

21. $\csc(18°) = \sec(\_\_\_\_\_°)$

For the following exercises, use the given information to find the lengths of the other two sides of the right triangle.

22. $\cos B = \frac{3}{5}, a = 6$

23. $\tan A = \frac{5}{9}, b = 6$

For the following exercises, use **Figure 1** to evaluate each trigonometric function.

24. $\sin A$

25. $\tan B$

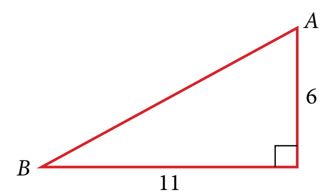
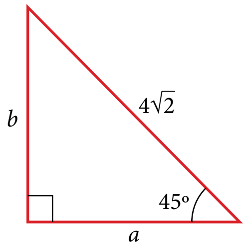


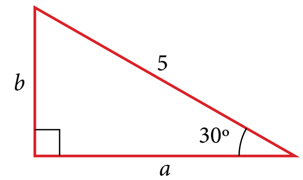
Figure 1

For the following exercises, solve for the unknown sides of the given triangle.

26.



27.



28. A 15-ft ladder leans against a building so that the angle between the ground and the ladder is 70°. How high does the ladder reach up the side of the building? Find the answer to four decimal places.

29. The angle of elevation to the top of a building in Baltimore is found to be 4 degrees from the ground at a distance of 1 mile from the base of the building. Using this information, find the height of the building. Find the answer to four decimal places.

UNIT CIRCLE

30. Find the exact value of $\sin \dfrac{\pi}{3}$.

31. Find the exact value of $\cos \dfrac{\pi}{4}$.

32. Find the exact value of $\cos \pi$.

33. State the reference angle for 300°.

34. State the reference angle for $\dfrac{3\pi}{4}$.

35. Compute cosine of 330°.

36. Compute sine of $\dfrac{5\pi}{4}$.

37. State the domain of the sine and cosine functions.

38. State the range of the sine and cosine functions.

THE OTHER TRIGONOMETRIC FUNCTIONS

For the following exercises, find the exact value of the given expression.

39. $\cos \dfrac{\pi}{6}$

40. $\tan \dfrac{\pi}{4}$

41. $\csc \dfrac{\pi}{3}$

42. $\sec \dfrac{\pi}{4}$

For the following exercises, use reference angles to evaluate the given expression.

43. $\sec \dfrac{11\pi}{3}$

44. $\sec 315°$

45. If $\sec(t) = -2.5$, what is the $\sec(-t)$?

46. If $\tan(t) = -0.6$, what is the $\tan(-t)$?

47. If $\tan(t) = \dfrac{1}{3}$, find $\tan(t - \pi)$.

48. If $\cos(t) = \dfrac{\sqrt{2}}{2}$, find $\sin(t + 2\pi)$.

49. Which trigonometric functions are even?

50. Which trigonometric functions are odd?

CHAPTER 7 PRACTICE TEST

1. Convert $\frac{5\pi}{6}$ radians to degrees.

2. Convert $-620°$ to radians.

3. Find the length of a circular arc with a radius 12 centimeters subtended by the central angle of $30°$.

4. Find the area of the sector with radius of 8 feet and an angle of $\frac{5\pi}{4}$ radians.

5. Find the angle between $0°$ and $360°$ that is coterminal with $375°$.

6. Find the angle between 0 and 2π in radians that is coterminal with $-\frac{4\pi}{7}$.

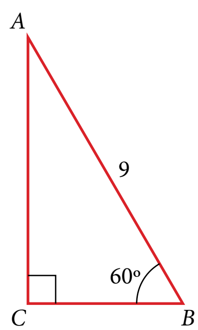
7. Draw the angle $315°$ in standard position on the Cartesian plane.

8. Draw the angle $-\frac{\pi}{6}$ in standard position on the Cartesian plane.

9. A carnival has a Ferris wheel with a diameter of 80 feet. The time for the Ferris wheel to make one revolution is 75 seconds. What is the linear speed in feet per second of a point on the Ferris wheel? What is the angular speed in radians per second?

10. Find the missing sides of the triangle ABC: $\sin B = \frac{3}{4}$, $c = 12$

11. Find the missing sides of the triangle.



12. The angle of elevation to the top of a building in Chicago is found to be 9 degrees from the ground at a distance of 2,000 feet from the base of the building. Using this information, find the height of the building.

13. Find the exact value of $\sin \frac{\pi}{6}$.

14. Compute sine of $240°$.

15. State the domain of the sine and cosine functions.

16. State the range of the sine and cosine functions.

17. Find the exact value of $\cot \frac{\pi}{4}$.

18. Find the exact value of $\tan \frac{\pi}{3}$.

19. Use reference angles to evaluate $\csc \frac{7\pi}{4}$.

20. Use reference angles to evaluate $\tan 210°$.

21. If $\csc t = 0.68$, what is the $\csc(-t)$?

22. If $\cos t = \frac{\sqrt{3}}{2}$, find $\cos(t - 2\pi)$.

23. Find the missing angle: $\cos\left(\frac{\pi}{6}\right) = \sin(\_\_\_\_\_\_)$

8

Periodic Functions

Figure 1 (credit: "Maxxer_", Flickr)

CHAPTER OUTLINE

Introduction

Each day, the sun rises in an easterly direction, approaches some maximum height relative to the celestial equator, and sets in a westerly direction. The celestial equator is an imaginary line that divides the visible universe into two halves in much the same way Earth's equator is an imaginary line that divides the planet into two halves. The exact path the sun appears to follow depends on the exact location on Earth, but each location observes a predictable pattern over time.

The pattern of the sun's motion throughout the course of a year is a periodic function. Creating a visual representation of a periodic function in the form of a graph can help us analyze the properties of the function. In this chapter, we will investigate graphs of sine, cosine, and other trigonometric functions.

LEARNING OBJECTIVES

In this section, you will:

- Graph variations of $y = \sin(x)$ and $y = \cos(x)$.
- Use phase shifts of sine and cosine curves.

8.1 GRAPHS OF THE SINE AND COSINE FUNCTIONS

Figure 1 Light can be separated into colors because of its wavelike properties. (credit: "wonderferret"/ Flickr)

White light, such as the light from the sun, is not actually white at all. Instead, it is a composition of all the colors of the rainbow in the form of waves. The individual colors can be seen only when white light passes through an optical prism that separates the waves according to their wavelengths to form a rainbow.

Light waves can be represented graphically by the sine function. In the chapter on **Trigonometric Functions**, we examined trigonometric functions such as the sine function. In this section, we will interpret and create graphs of sine and cosine functions.

Graphing Sine and Cosine Functions

Recall that the sine and cosine functions relate real number values to the x- and y-coordinates of a point on the unit circle. So what do they look like on a graph on a coordinate plane? Let's start with the sine function. We can create a table of values and use them to sketch a graph. **Table 1** lists some of the values for the sine function on a unit circle.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\sin(x)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

Table 1

Plotting the points from the table and continuing along the x-axis gives the shape of the sine function. See **Figure 2**.

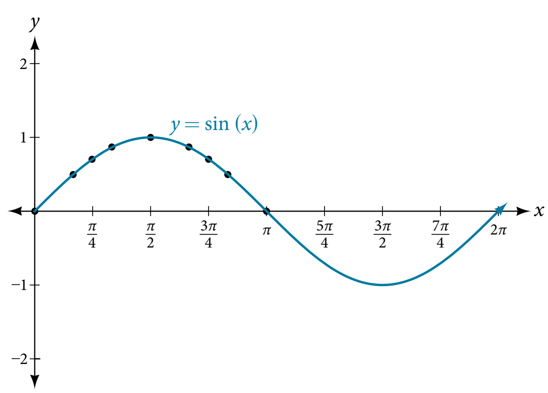


Figure 2 The sine function

Notice how the sine values are positive between 0 and π, which correspond to the values of the sine function in quadrants I and II on the unit circle, and the sine values are negative between π and 2π, which correspond to the values of the sine function in quadrants III and IV on the unit circle. See **Figure 3**.

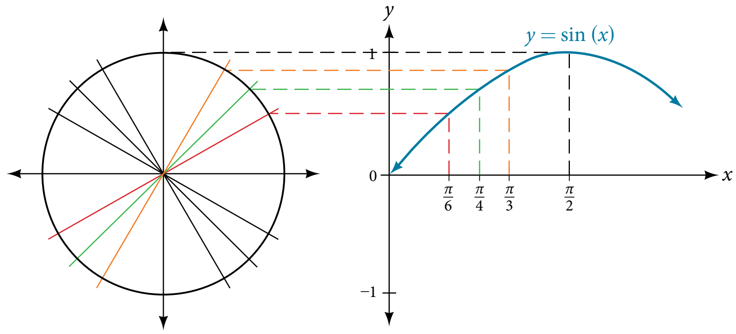


Figure 3 Plotting values of the sine function

Now let's take a similar look at the cosine function. Again, we can create a table of values and use them to sketch a graph. **Table 2** lists some of the values for the cosine function on a unit circle.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\cos(x)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1

Table 2

As with the sine function, we can plots points to create a graph of the cosine function as in *Figure 4*.

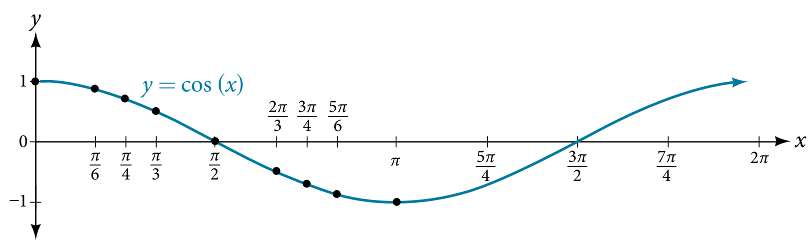


Figure 4 The cosine function

Because we can evaluate the sine and cosine of any real number, both of these functions are defined for all real numbers. By thinking of the sine and cosine values as coordinates of points on a unit circle, it becomes clear that the range of both functions must be the interval $[-1, 1]$.

In both graphs, the shape of the graph repeats after 2π, which means the functions are periodic with a period of 2π. A **periodic function** is a function for which a specific horizontal shift, P, results in a function equal to the original function: $f(x + P) = f(x)$ for all values of x in the domain of f. When this occurs, we call the smallest such horizontal shift with $P > 0$ the period of the function. **Figure 5** shows several periods of the sine and cosine functions.

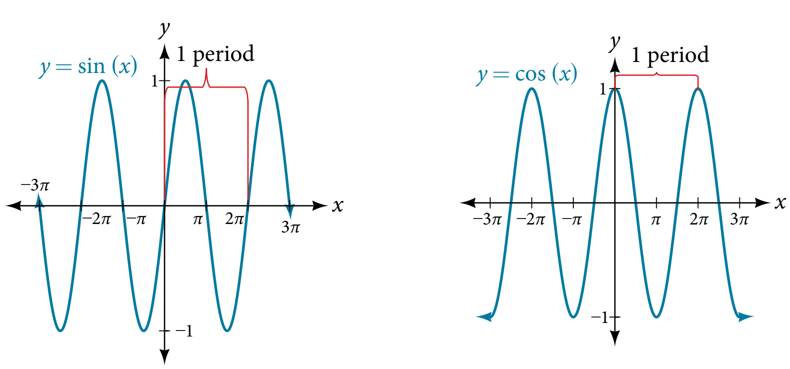


Figure 5

Looking again at the sine and cosine functions on a domain centered at the y-axis helps reveal symmetries. As we can see in **Figure 6**, the sine function is symmetric about the origin. Recall from **The Other Trigonometric Functions** that we determined from the unit circle that the sine function is an odd function because $\sin(-x) = -\sin x$. Now we can clearly see this property from the graph.

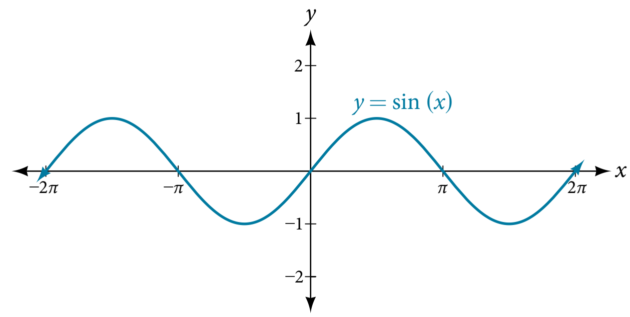


Figure 6 Odd symmetry of the sine function

Figure 7 shows that the cosine function is symmetric about the y-axis. Again, we determined that the cosine function is an even function. Now we can see from the graph that $\cos(-x) = \cos x$.

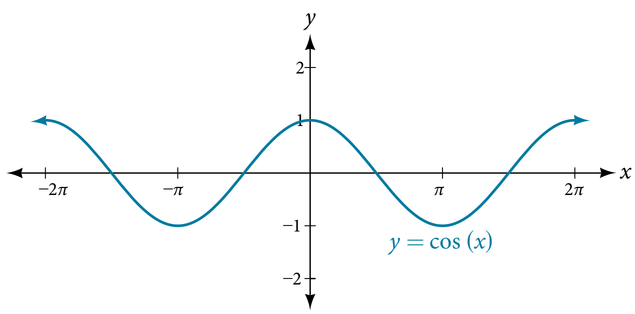


Figure 7 Even symmetry of the cosine function

characteristics of sine and cosine functions

The sine and cosine functions have several distinct characteristics:

- They are periodic functions with a period of 2π.
- The domain of each function is $(-\infty, \infty)$ and the range is $[-1, 1]$.
- The graph of $y = \sin x$ is symmetric about the origin, because it is an odd function.
- The graph of $y = \cos x$ is symmetric about the y- axis, because it is an even function.

Investigating Sinusoidal Functions

As we can see, sine and cosine functions have a regular period and range. If we watch ocean waves or ripples on a pond, we will see that they resemble the sine or cosine functions. However, they are not necessarily identical. Some are taller or longer than others. A function that has the same general shape as a sine or cosine function is known as a **sinusoidal function**. The general forms of sinusoidal functions are

$$y = A\sin(Bx - C) + D$$

and

$$y = A\cos(Bx - C) + D$$

Determining the Period of Sinusoidal Functions

Looking at the forms of sinusoidal functions, we can see that they are transformations of the sine and cosine functions. We can use what we know about transformations to determine the period.

In the general formula, B is related to the period by $P = \dfrac{2\pi}{|B|}$. If $|B| > 1$, then the period is less than 2π and the function undergoes a horizontal compression, whereas if $|B| < 1$, then the period is greater than 2π and the function undergoes a horizontal stretch. For example, $f(x) = \sin(x)$, $B = 1$, so the period is 2π, which we knew. If $f(x) = \sin(2x)$, then $B = 2$, so the period is π and the graph is compressed. If $f(x) = \sin\left(\dfrac{x}{2}\right)$, then $B = \dfrac{1}{2}$, so the period is 4π and the graph is stretched. Notice in **Figure 8** how the period is indirectly related to $|B|$.

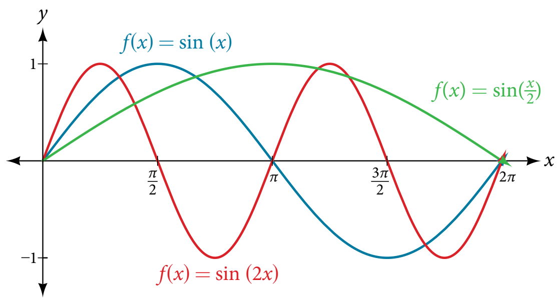


Figure 8

> ***period of sinusoidal functions***
>
> If we let $C = 0$ and $D = 0$ in the general form equations of the sine and cosine functions, we obtain the forms
>
> $$y = A\sin(Bx) \qquad y = A\cos(Bx)$$
>
> The period is $\dfrac{2\pi}{|B|}$.

Example 1 **Identifying the Period of a Sine or Cosine Function**

Determine the period of the function $f(x) = \sin\left(\dfrac{\pi}{6}x\right)$.

Solution Let's begin by comparing the equation to the general form $y = A\sin(Bx)$.

In the given equation, $B = \dfrac{\pi}{6}$, so the period will be

$$\begin{aligned} P &= \frac{2\pi}{|B|} \\ &= \frac{2\pi}{\frac{\pi}{6}} \\ &= 2\pi \cdot \frac{6}{\pi} \\ &= 12 \end{aligned}$$

Try It #1

Determine the period of the function $g(x) = \left(\cos\dfrac{x}{3}\right)$.

Determining Amplitude

Returning to the general formula for a sinusoidal function, we have analyzed how the variable B relates to the period. Now let's turn to the variable A so we can analyze how it is related to the **amplitude**, or greatest distance from rest. A represents the vertical stretch factor, and its absolute value $|A|$ is the amplitude. The local maxima will be a distance $|A|$ above the vertical **midline** of the graph, which is the line $x = D$; because $D = 0$ in this case, the midline is the x-axis. The local minima will be the same distance below the midline. If $|A| > 1$, the function is stretched. For example, the amplitude of $f(x) = 4\sin x$ is twice the amplitude of $f(x) = 2\sin x$. If $|A| < 1$, the function is compressed. **Figure 9** compares several sine functions with different amplitudes.

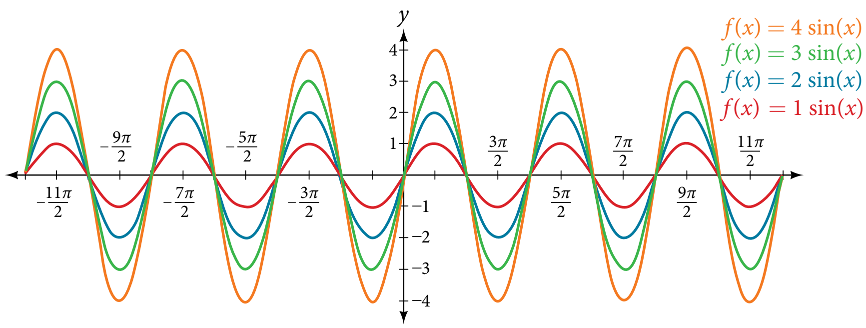


Figure 9

> ***amplitude of sinusoidal functions***
>
> If we let $C = 0$ and $D = 0$ in the general form equations of the sine and cosine functions, we obtain the forms
>
> $$y = A\sin(Bx) \text{ and } y = A\cos(Bx)$$
>
> The **amplitude** is A, and the vertical height from the **midline** is $|A|$. In addition, notice in the example that
>
> $$|A| = \text{amplitude} = \frac{1}{2}|\text{maximum} - \text{minimum}|$$

Example 2 Identifying the Amplitude of a Sine or Cosine Function

What is the amplitude of the sinusoidal function $f(x) = -4\sin(x)$? Is the function stretched or compressed vertically?

Solution Let's begin by comparing the function to the simplified form $y = A\sin(Bx)$.

In the given function, $A = -4$, so the amplitude is $|A| = |-4| = 4$. The function is stretched.

Analysis *The negative value of A results in a reflection across the x-axis of the sine function, as shown in* ***Figure 10****.*

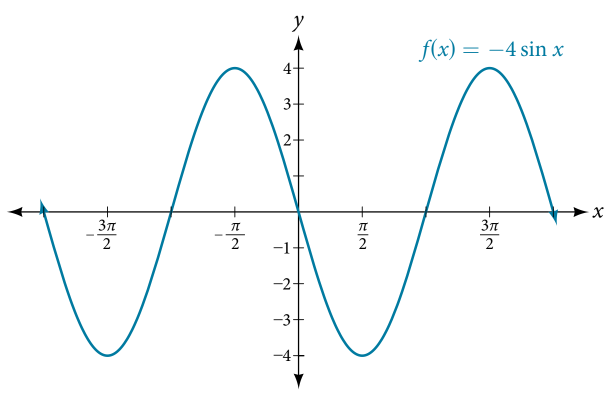


Figure 10

Try It #2

What is the amplitude of the sinusoidal function $f(x) = \frac{1}{2}\sin(x)$? Is the function stretched or compressed vertically?

Analyzing Graphs of Variations of $y = \sin x$ and $y = \cos x$

Now that we understand how A and B relate to the general form equation for the sine and cosine functions, we will explore the variables C and D. Recall the general form:

$$y = A\sin(Bx - C) + D \text{ and } y = A\cos(Bx - C) + D$$

or

$$y = A\sin\left(B\left(x - \frac{C}{B}\right)\right) + D \text{ and } y = A\cos\left(B\left(x - \frac{C}{B}\right)\right) + D$$

The value $\frac{C}{B}$ for a sinusoidal function is called the **phase shift**, or the horizontal displacement of the basic sine or cosine function. ~~If $C > 0$, the graph shifts to the right. If $C < 0$, the graph shifts to the left.~~ The greater the value of $|C|$, the more the graph is shifted. *Figure 11* shows that the graph of $f(x) = \sin(x - \pi)$ shifts to the right by π units, which is more than we see in the graph of $f(x) = \sin\left(x - \frac{\pi}{4}\right)$, which shifts to the right by $\frac{\pi}{4}$ units.

If $C/B > 0$ the graph shifts to the right. If $C/B < 0$ the graph shifts to the left.

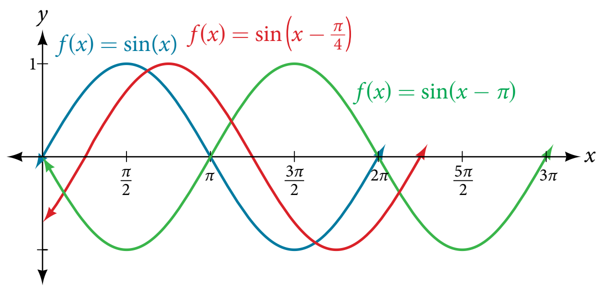


Figure 11

While C relates to the horizontal shift, D indicates the vertical shift from the midline in the general formula for a sinusoidal function. See **Figure 12**. The function $y = \cos(x) + D$ has its midline at $y = D$.

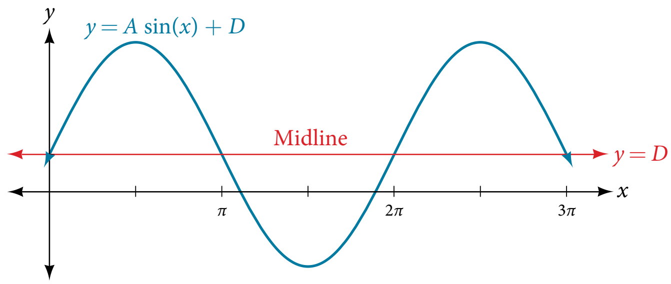


Figure 12

Any value of D other than zero shifts the graph up or down. *Figure 13* compares $f(x) = \sin x$ with $f(x) = \sin x + 2$, which is shifted 2 units up on a graph.

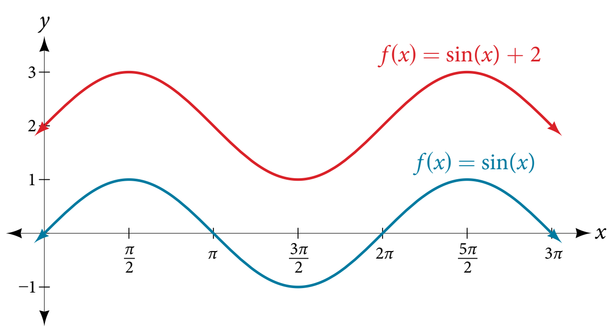


Figure 13

variations of sine and cosine functions

Given an equation in the form $f(x) = A\sin(Bx - C) + D$ or $f(x) = A\cos(Bx - C) + D$, $\frac{C}{B}$ is the **phase shift** and D is the vertical shift.

Example 3 Identifying the Phase Shift of a Function

Determine the direction and magnitude of the phase shift for $f(x) = \sin\left(x + \frac{\pi}{6}\right) - 2$.

Solution Let's begin by comparing the equation to the general form $y = A\sin(Bx - C) + D$.

In the given equation, notice that $B = 1$ and $C = -\frac{\pi}{6}$. So the phase shift is

$$\frac{C}{B} = -\frac{\frac{\pi}{6}}{1}$$
$$= -\frac{\pi}{6}$$

or $\frac{\pi}{6}$ units to the left.

Analysis *We must pay attention to the sign in the equation for the general form of a sinusoidal function. The equation shows a minus sign before C. Therefore* $f(x) = \sin\left(x + \frac{\pi}{6}\right) - 2$ *can be rewritten as* $f(x) = \sin\left(x - \left(-\frac{\pi}{6}\right)\right) - 2.$

~~*If the value of C is negative, the shift is to the left.*~~ ← If the value of C/B is negative, the shift is to the left.

Try It #3

Determine the direction and magnitude of the phase shift for $f(x) = 3\cos\left(x - \frac{\pi}{2}\right)$.

Example 4 Identifying the Vertical Shift of a Function

Determine the direction and magnitude of the vertical shift for $f(x) = \cos(x) - 3$.

Solution Let's begin by comparing the equation to the general form $y = A\cos(Bx - C) + D$.

In the given equation, $D = -3$ so the shift is 3 units downward.

Try It #4

Determine the direction and magnitude of the vertical shift for $f(x) = 3\sin(x) + 2$.

How To...

Given a sinusoidal function in the form $f(x) = A\sin(Bx - C) + D$, identify the midline, amplitude, period, and phase shift.

1. Determine the amplitude as $|A|$.
2. Determine the period as $P = \frac{2\pi}{|B|}$.
3. Determine the phase shift as $\frac{C}{B}$.
4. Determine the midline as $y = D$.

Example 5 Identifying the Variations of a Sinusoidal Function from an Equation

Determine the midline, amplitude, period, and phase shift of the function $y = 3\sin(2x) + 1$.

Solution Let's begin by comparing the equation to the general form $y = A\sin(Bx - C) + D$.

$A = 3$, so the amplitude is $|A| = 3$.

Next, $B = 2$, so the period is $P = \frac{2\pi}{|B|} = \frac{2\pi}{2} = \pi$.

There is no added constant inside the parentheses, so $C = 0$ and the phase shift is $\frac{C}{B} = \frac{0}{2} = 0$.

Finally, $D = 1$, so the midline is $y = 1$.

Analysis *Inspecting the graph, we can determine that the period is π, the midline is* $y = 1$, *and the amplitude is 3. See* ***Figure 14.***

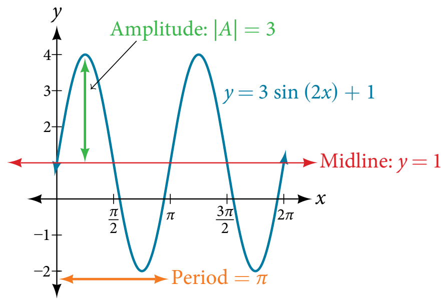


Figure 14

Try It #5

Determine the midline, amplitude, period, and phase shift of the function $y = \frac{1}{2}\cos\left(\frac{x}{3} - \frac{\pi}{3}\right)$.

Example 6 Identifying the Equation for a Sinusoidal Function from a Graph

Determine the formula for the cosine function in *Figure 15*.

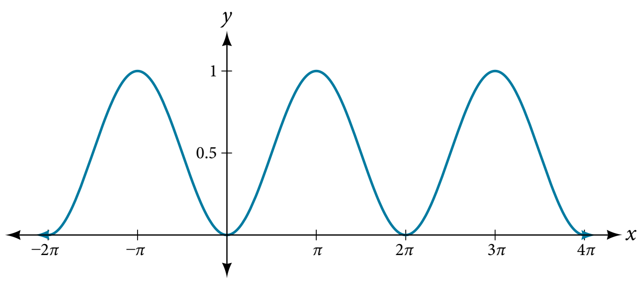


Figure 15

Solution To determine the equation, we need to identify each value in the general form of a sinusoidal function.

$$y = A\sin(Bx - C) + D \qquad y = A\cos(Bx - C) + D$$

The graph could represent either a sine or a cosine function that is shifted and/or reflected. When $x = 0$, the graph has an extreme point, $(0, 0)$. Since the cosine function has an extreme point for $x = 0$, let us write our equation in terms of a cosine function.

Let's start with the midline. We can see that the graph rises and falls an equal distance above and below $y = 0.5$. This value, which is the midline, is D in the equation, so $D = 0.5$.

The greatest distance above and below the midline is the amplitude. The maxima are 0.5 units above the midline and the minima are 0.5 units below the midline. So $|A| = 0.5$. Another way we could have determined the amplitude is by recognizing that the difference between the height of local maxima and minima is 1, so $|A| = \frac{1}{2} = 0.5$. Also, the graph is reflected about the x-axis so that $A = -0.5$.

The graph is not horizontally stretched or compressed, so $B = 1$; and the graph is not shifted horizontally, so $C = 0$.

Putting this all together,

$$g(x) = -0.5\cos(x) + 0.5$$

Try It #6

Determine the formula for the sine function in *Figure 16*.

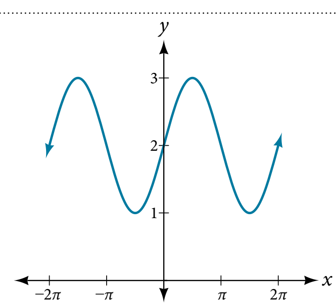


Figure 16

Example 7 Identifying the Equation for a Sinusoidal Function from a Graph

Determine the equation for the sinusoidal function in *Figure 17.*

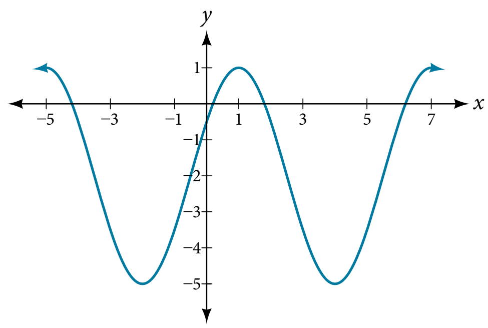


Figure 17

Solution With the highest value at 1 and the lowest value at −5, the midline will be halfway between at −2. So $D = -2$.

The distance from the midline to the highest or lowest value gives an amplitude of $|A| = 3$.

The period of the graph is 6, which can be measured from the peak at $x = 1$ to the next peak at $x = 7$, or from the distance between the lowest points. Therefore, $P = \frac{2\pi}{|B|} = 6$. Using the positive value for B, we find that

$$B = \frac{2\pi}{P} = \frac{2\pi}{6} = \frac{\pi}{3}$$

So far, our equation is either $y = 3\sin\left(\frac{\pi}{3}x - C\right) - 2$ or $y = 3\cos\left(\frac{\pi}{3}x - C\right) - 2$. For the shape and shift, we have more than one option. We could write this as any one of the following:

- a cosine shifted to the right
- a negative cosine shifted to the left
- a sine shifted to the left
- a negative sine shifted to the right

While any of these would be correct, the cosine shifts are easier to work with than the sine shifts in this case because they involve integer values. So our function becomes

$$y = 3\cos\left(\frac{\pi}{3}x - \frac{\pi}{3}\right) - 2 \text{ or } y = -3\cos\left(\frac{\pi}{3}x + \frac{2\pi}{3}\right) - 2$$

Again, these functions are equivalent, so both yield the same graph.

Try It #7

Write a formula for the function graphed in *Figure 18.*

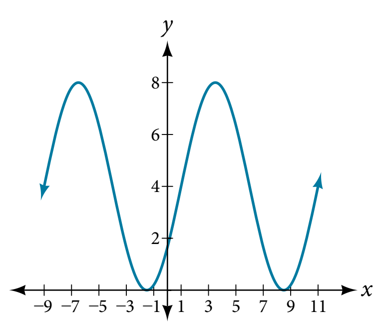


Figure 18

Graphing Variations of $y = \sin x$ and $y = \cos x$

Throughout this section, we have learned about types of variations of sine and cosine functions and used that information to write equations from graphs. Now we can use the same information to create graphs from equations.

Instead of focusing on the general form equations

$$y = A\sin(Bx - C) + D \text{ and } y = A\cos(Bx - C) + D,$$

we will let $C = 0$ and $D = 0$ and work with a simplified form of the equations in the following examples.

How To...

Given the function $y = A\sin(Bx)$, sketch its graph.

1. Identify the amplitude, $|A|$.
2. Identify the period, $P = \frac{2\pi}{|B|}$.
3. Start at the origin, with the function increasing to the right if A is positive or decreasing if A is negative.
4. At $x = \frac{\pi}{2|B|}$ there is a local maximum for $A > 0$ or a minimum for $A < 0$, with $y = A$.
5. The curve returns to the x-axis at $x = \frac{\pi}{|B|}$.
6. There is a local minimum for $A > 0$ (maximum for $A < 0$) at $x = \frac{3\pi}{2|B|}$ with $y = -A$.
7. The curve returns again to the x-axis at $x = \frac{\pi}{2|B|}$.

Example 8 Graphing a Function and Identifying the Amplitude and Period

Sketch a graph of $f(x) = -2\sin\left(\frac{\pi x}{2}\right)$.

Solution Let's begin by comparing the equation to the form $y = A\sin(Bx)$.

Step 1. We can see from the equation that $A = -2$, so the amplitude is 2.

$$|A| = 2$$

Step 2. The equation shows that $B = \frac{\pi}{2}$, so the period is

$$\begin{aligned} P &= \frac{2\pi}{\frac{\pi}{2}} \\ &= 2\pi \cdot \frac{2}{\pi} \\ &= 4 \end{aligned}$$

Step 3. Because A is negative, the graph descends as we move to the right of the origin.

Step 4–7. The x-intercepts are at the beginning of one period, $x = 0$, the horizontal midpoints are at $x = 2$ and at the end of one period at $x = 4$.

The quarter points include the minimum at $x = 1$ and the maximum at $x = 3$. A local minimum will occur 2 units below the midline, at $x = 1$, and a local maximum will occur at 2 units above the midline, at $x = 3$. **Figure 19** shows the graph of the function.

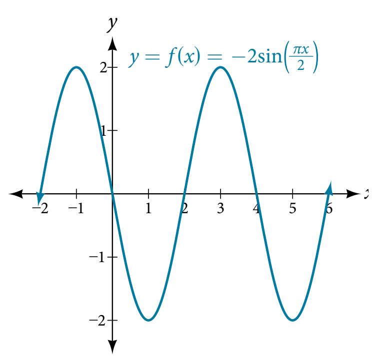


Figure 19

Try It #8

Sketch a graph of $g(x) = -0.8\cos(2x)$. Determine the midline, amplitude, period, and phase shift.

How To...

Given a sinusoidal function with a phase shift and a vertical shift, sketch its graph.

1. Express the function in the general form $y = A\sin(Bx - C) + D$ or $y = A\cos(Bx - C) + D$.
2. Identify the amplitude, $|A|$.
3. Identify the period, $P = \frac{2\pi}{|B|}$.
4. Identify the phase shift, $\frac{C}{B}$.
5. Draw the graph of $f(x) = A\sin(Bx)$ shifted to the right or left by $\frac{C}{B}$ and up or down by D.

Example 9 Graphing a Transformed Sinusoid

Sketch a graph of $f(x) = 3\sin\left(\frac{\pi}{4}x - \frac{\pi}{4}\right)$.

Solution

Step 1. The function is already written in general form: $f(x) = 3\sin\left(\frac{\pi}{4}x - \frac{\pi}{4}\right)$. This graph will have the shape of a sine function, starting at the midline and increasing to the right.

Step 2. $|A| = |3| = 3$. The amplitude is 3.

Step 3. Since $|B| = \left|\frac{\pi}{4}\right| = \frac{\pi}{4}$, we determine the period as follows.

$$P = \frac{2\pi}{|B|} = \frac{2\pi}{\frac{\pi}{4}} = 2\pi \cdot \frac{4}{\pi} = 8$$

The period is 8.

Step 4. Since $C = \frac{\pi}{4}$, the phase shift is

$$\frac{C}{B} = \frac{\frac{\pi}{4}}{\frac{\pi}{4}} = 1.$$

The phase shift is 1 unit.

Step 5. **Figure 20** shows the graph of the function.

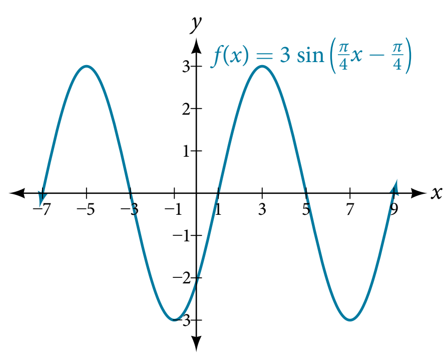


Figure 20 A horizontally compressed, vertically stretched, and horizontally shifted sinusoid

Try It #9

Draw a graph of $g(x) = -2\cos\left(\frac{\pi}{3}x + \frac{\pi}{6}\right)$. Determine the midline, amplitude, period, and phase shift.

Example 10 Identifying the Properties of a Sinusoidal Function

Given $y = -2\cos\left(\frac{\pi}{2}x + \pi\right) + 3$, determine the amplitude, period, phase shift, and horizontal shift. Then graph the function.

Solution Begin by comparing the equation to the general form and use the steps outlined in **Example 9**.

$$y = A\cos(Bx - C) + D$$

Step 1. The function is already written in general form.

Step 2. Since $A = -2$, the amplitude is $|A| = 2$.

Step 3. $|B| = \frac{\pi}{2}$, so the period is $P = \frac{2\pi}{|B|} = \frac{2\pi}{\frac{\pi}{2}} = 2\pi \cdot \frac{2}{\pi} = 4$. The period is 4.

Step 4. $C = -\pi$, so we calculate the phase shift as $\frac{C}{B} = \frac{-\pi}{\frac{\pi}{2}} = -\pi \cdot \frac{2}{\pi} = -2$. The phase shift is -2.

Step 5. $D = 3$, so the midline is $y = 3$, and the vertical shift is up 3.

Since A is negative, the graph of the cosine function has been reflected about the x-axis. **Figure 21** shows one cycle of the graph of the function.

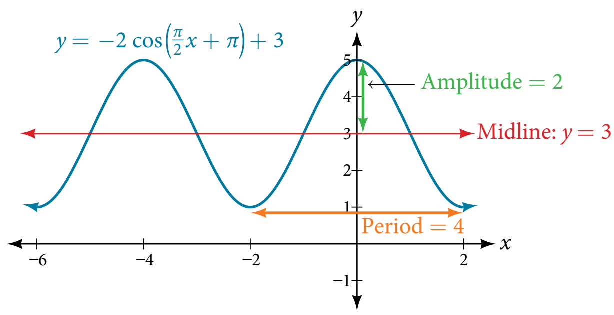


Figure 21

Using Transformations of Sine and Cosine Functions

We can use the transformations of sine and cosine functions in numerous applications. As mentioned at the beginning of the chapter, circular motion can be modeled using either the sine or cosine function.

Example 11 Finding the Vertical Component of Circular Motion

A point rotates around a circle of radius 3 centered at the origin. Sketch a graph of the y-coordinate of the point as a function of the angle of rotation.

Solution Recall that, for a point on a circle of radius r, the y-coordinate of the point is $y = r\sin(x)$, so in this case, we get the equation $y(x) = 3\sin(x)$. The constant 3 causes a vertical stretch of the y-values of the function by a factor of 3, which we can see in the graph in **Figure 22**.

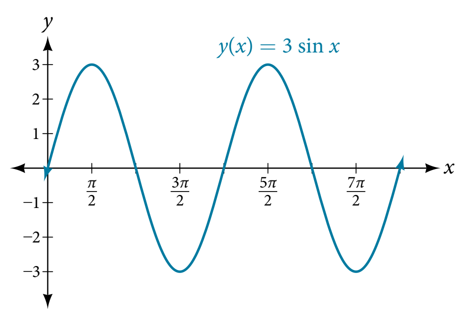


Figure 22

Analysis *Notice that the period of the function is still* 2π*; as we travel around the circle, we return to the point* $(3, 0)$ *for* $x = 2\pi, 4\pi, 6\pi, ...$ *Because the outputs of the graph will now oscillate between* -3 *and* 3*, the amplitude of the sine wave is 3.*

Try It #10

What is the amplitude of the function $f(x) = 7\cos(x)$? Sketch a graph of this function.

Example 12 Finding the Vertical Component of Circular Motion

A circle with radius 3 ft is mounted with its center 4 ft off the ground. The point closest to the ground is labeled P, as shown in **Figure 23**. Sketch a graph of the height above the ground of the point P as the circle is rotated; then find a function that gives the height in terms of the angle of rotation.

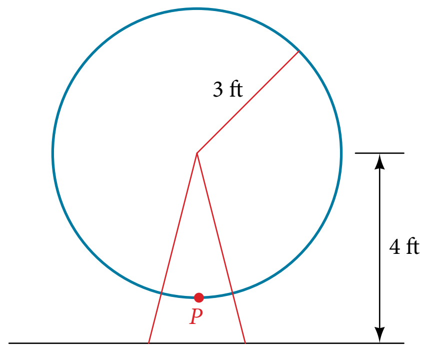


Figure 23

Solution Sketching the height, we note that it will start 1 ft above the ground, then increase up to 7 ft above the ground, and continue to oscillate 3 ft above and below the center value of 4 ft, as shown in **Figure 24**.

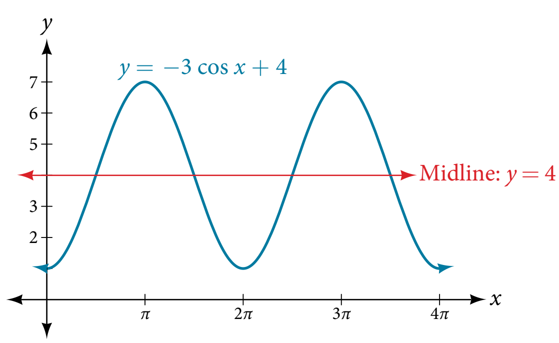


Figure 24

Although we could use a transformation of either the sine or cosine function, we start by looking for characteristics that would make one function easier to use than the other. Let's use a cosine function because it starts at the highest or lowest value, while a sine function starts at the middle value. A standard cosine starts at the highest value, and this graph starts at the lowest value, so we need to incorporate a vertical reflection.

Second, we see that the graph oscillates 3 above and below the center, while a basic cosine has an amplitude of 1, so this graph has been vertically stretched by 3, as in the last example.

Finally, to move the center of the circle up to a height of 4, the graph has been vertically shifted up by 4. Putting these transformations together, we find that

$$y = -3\cos(x) + 4$$

Try It #11

A weight is attached to a spring that is then hung from a board, as shown in **Figure 25**. As the spring oscillates up and down, the position y of the weight relative to the board ranges from -1 in. (at time $x = 0$) to -7 in. (at time $x = \pi$) below the board. Assume the position of y is given as a sinusoidal function of x. Sketch a graph of the function, and then find a cosine function that gives the position y in terms of x.

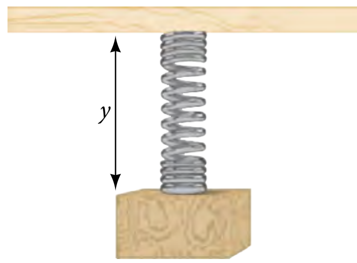


Figure 25

Example 13 Determining a Rider's Height on a Ferris Wheel

The London Eye is a huge Ferris wheel with a diameter of 135 meters (443 feet). It completes one rotation every 30 minutes. Riders board from a platform 2 meters above the ground. Express a rider's height above ground as a function of time in minutes.

Solution With a diameter of 135 m, the wheel has a radius of 67.5 m. The height will oscillate with amplitude 67.5 m above and below the center.

Passengers board 2 m above ground level, so the center of the wheel must be located $67.5 + 2 = 69.5$ m above ground level. The midline of the oscillation will be at 69.5 m.

The wheel takes 30 minutes to complete 1 revolution, so the height will oscillate with a period of 30 minutes.

Lastly, because the rider boards at the lowest point, the height will start at the smallest value and increase, following the shape of a vertically reflected cosine curve.

- Amplitude: 67.5, so $A = 67.5$
- Midline: 69.5, so $D = 69.5$
- Period: 30, so $B = \frac{2\pi}{30} = \frac{\pi}{15}$
- Shape: $-\cos(t)$

An equation for the rider's height would be

$$y = -67.5\cos\left(\frac{\pi}{15}t\right) + 69.5$$

where t is in minutes and y is measured in meters.

Access these online resources for additional instruction and practice with graphs of sine and cosine functions.

- Amplitude and Period of Sine and Cosine (http://openstaxcollege.org/l/ampperiod)
- Translations of Sine and Cosine (http://openstaxcollege.org/l/translasincos)
- Graphing Sine and Cosine Transformations (http://openstaxcollege.org/l/transformsincos)
- Graphing the Sine Function (http://openstaxcollege.org/l/graphsinefunc)

8.1 SECTION EXERCISES

VERBAL

1. Why are the sine and cosine functions called periodic functions?

2. How does the graph of $y = \sin x$ compare with the graph of $y = \cos x$? Explain how you could horizontally translate the graph of $y = \sin x$ to obtain $y = \cos x$.

3. For the equation $A\cos(Bx + C) + D$, what constants affect the range of the function and how do they affect the range?

4. How does the range of a translated sine function relate to the equation $y = A\sin(Bx + C) + D$?

5. How can the unit circle be used to construct the graph of $f(t) = \sin t$?

GRAPHICAL

For the following exercises, graph two full periods of each function and state the amplitude, period, and midline. State the maximum and minimum y-values and their corresponding x-values on one period for $x > 0$. Round answers to two decimal places if necessary.

6. $f(x) = 2\sin x$

7. $f(x) = \frac{2}{3}\cos x$

8. $f(x) = -3\sin x$

9. $f(x) = 4\sin x$

10. $f(x) = 2\cos x$

11. $f(x) = \cos(2x)$

12. $f(x) = 2\sin\left(\frac{1}{2}x\right)$

13. $f(x) = 4\cos(\pi x)$

14. $f(x) = 3\cos\left(\frac{6}{5}x\right)$

15. $y = 3\sin(8(x + 4)) + 5$

16. $y = 2\sin(3x - 21) + 4$

17. $y = 5\sin(5x + 20) - 2$

For the following exercises, graph one full period of each function, starting at $x = 0$. For each function, state the amplitude, period, and midline. State the maximum and minimum y-values and their corresponding x-values on one period for $x > 0$. State the phase shift and vertical translation, if applicable. Round answers to two decimal places if necessary.

18. $f(t) = 2\sin\left(t - \frac{5\pi}{6}\right)$

19. $f(t) = -\cos\left(t + \frac{\pi}{3}\right) + 1$

20. $f(t) = 4\cos\left(2\left(t + \frac{\pi}{4}\right)\right) - 3$

21. $f(t) = -\sin\left(\frac{1}{2}t + \frac{5\pi}{3}\right)$

22. $f(x) = 4\sin\left(\frac{\pi}{2}(x - 3)\right) + 7$

23. Determine the amplitude, midline, period, and an equation involving the sine function for the graph shown in **Figure 26.**

24. Determine the amplitude, period, midline, and an equation involving cosine for the graph shown in **Figure 27.**

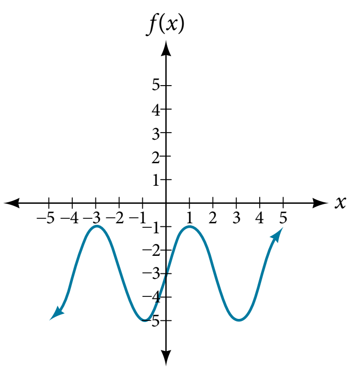


Figure 26

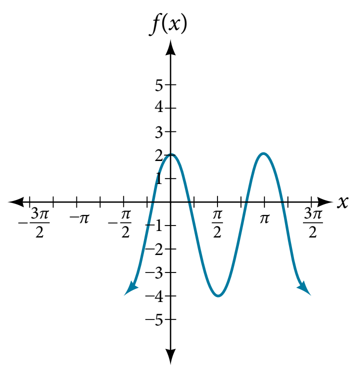


Figure 27

25. Determine the amplitude, period, midline, and an equation involving cosine for the graph shown in **Figure 28.**

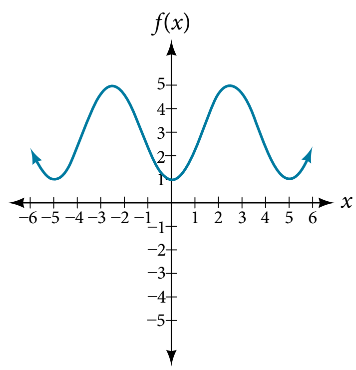


Figure 28

26. Determine the amplitude, period, midline, and an equation involving sine for the graph shown in **Figure 29.**

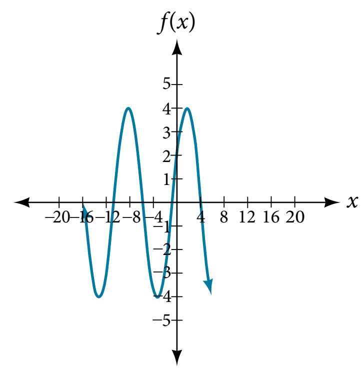


Figure 29

27. Determine the amplitude, period, midline, and an equation involving cosine for the graph shown in **Figure 30.**

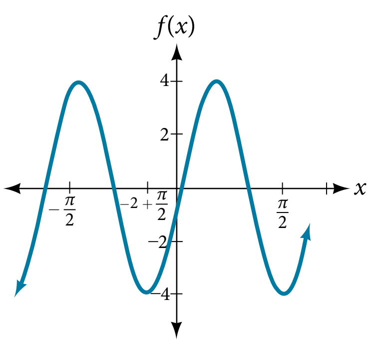


Figure 30

28. Determine the amplitude, period, midline, and an equation involving sine for the graph shown in **Figure 31.**

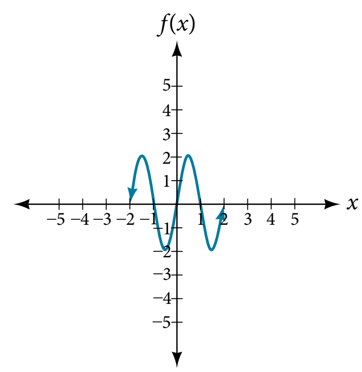


Figure 31

29. Determine the amplitude, period, midline, and an equation involving cosine for the graph shown in **Figure 32.**

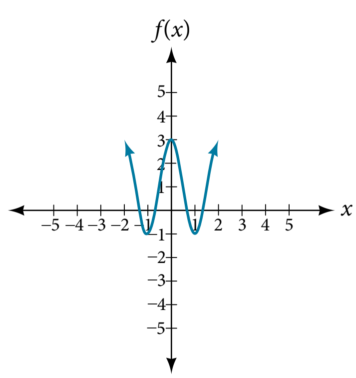


Figure 32

30. Determine the amplitude, period, midline, and an equation involving sine for the graph shown in **Figure 33.**

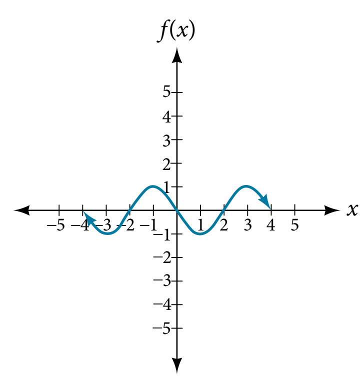


Figure 33

ALGEBRAIC

For the following exercises, let $f(x) = \sin x$.

31. On $[0, 2\pi)$, solve $f(x) = 0$.

32. On $[0, 2\pi)$, solve $f(x) = \frac{1}{2}$.

33. Evaluate $f\left(\frac{\pi}{2}\right)$.

34. On $[0, 2\pi)$, $f(x) = \frac{\sqrt{2}}{2}$. Find all values of x.

35. On $[0, 2\pi)$, the maximum value(s) of the function occur(s) at what x-value(s)?

36. On $[0, 2\pi)$, the minimum value(s) of the function occur(s) at what x-value(s)?

37. Show that $f(-x) = -f(x)$. This means that $f(x) = \sin x$ is an odd function and possesses symmetry with respect to ________________.

For the following exercises, let $f(x) = \cos x$.

38. On $[0, 2\pi)$, solve the equation $f(x) = \cos x = 0$.

39. On $[0, 2\pi)$, solve $f(x) = \frac{1}{2}$.

40. On $[0, 2\pi)$, find the x-intercepts of $f(x) = \cos x$.

41. On $[0, 2\pi)$, find the x-values at which the function has a maximum or minimum value.

42. On $[0, 2\pi)$, solve the equation $f(x) = \frac{\sqrt{3}}{2}$.

TECHNOLOGY

43. Graph $h(x) = x + \sin x$ on $[0, 2\pi]$. Explain why the graph appears as it does.

44. Graph $h(x) = x + \sin x$ on $[-100, 100]$. Did the graph appear as predicted in the previous exercise?

45. Graph $f(x) = x \sin x$ on $[0, 2\pi]$ and verbalize how the graph varies from the graph of $f(x) = \sin x$.

46. Graph $f(x) = x \sin x$ on the window $[-10, 10]$ and explain what the graph shows.

47. Graph $f(x) = \frac{\sin x}{x}$ on the window $[-5\pi, 5\pi]$ and explain what the graph shows.

REAL-WORLD APPLICATIONS

48. A Ferris wheel is 25 meters in diameter and boarded from a platform that is 1 meter above the ground. The six o'clock position on the Ferris wheel is level with the loading platform. The wheel completes 1 full revolution in 10 minutes. The function $h(t)$ gives a person's height in meters above the ground t minutes after the wheel begins to turn.

a. Find the amplitude, midline, and period of $h(t)$.

b. Find a formula for the height function $h(t)$.

c. How high off the ground is a person after 5 minutes?

LEARNING OBJECTIVES

In this section, you will:

- Analyze the graph of $y = \tan x$.
- Graph variations of $y = \tan x$.
- Analyze the graphs of $y = \sec x$ and $y = \csc x$.
- Graph variations of $y = \sec x$ and $y = \csc x$.
- Analyze the graph of $y = \cot x$.
- Graph variations of $y = \cot x$.

8.2 GRAPHS OF THE OTHER TRIGONOMETRIC FUNCTIONS

We know the tangent function can be used to find distances, such as the height of a building, mountain, or flagpole. But what if we want to measure repeated occurrences of distance? Imagine, for example, a police car parked next to a warehouse. The rotating light from the police car would travel across the wall of the warehouse in regular intervals. If the input is time, the output would be the distance the beam of light travels. The beam of light would repeat the distance at regular intervals. The tangent function can be used to approximate this distance. Asymptotes would be needed to illustrate the repeated cycles when the beam runs parallel to the wall because, seemingly, the beam of light could appear to extend forever. The graph of the tangent function would clearly illustrate the repeated intervals. In this section, we will explore the graphs of the tangent and other trigonometric functions.

Analyzing the Graph of $y = \tan x$

We will begin with the graph of the tangent function, plotting points as we did for the sine and cosine functions. Recall that

$$\tan x = \frac{\sin x}{\cos x}$$

The period of the tangent function is π because the graph repeats itself on intervals of $k\pi$ where k is a constant. If we graph the tangent function on $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, we can see the behavior of the graph on one complete cycle. If we look at any larger interval, we will see that the characteristics of the graph repeat.

We can determine whether tangent is an odd or even function by using the definition of tangent.

$$\begin{aligned} \tan(-x) &= \frac{\sin(-x)}{\cos(-x)} && \text{Definition of tangent.} \\ &= \frac{-\sin x}{\cos x} && \text{Sine is an odd function, cosine is even.} \\ &= -\frac{\sin x}{\cos x} && \text{The quotient of an odd and an even function is odd.} \\ &= -\tan x && \text{Definition of tangent.} \end{aligned}$$

Therefore, tangent is an odd function. We can further analyze the graphical behavior of the tangent function by looking at values for some of the special angles, as listed in **Table 1.**

x	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
tan(x)	undefined	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined

Table 1

These points will help us draw our graph, but we need to determine how the graph behaves where it is undefined. If we look more closely at values when $\frac{\pi}{3} < x < \frac{\pi}{2}$, we can use a table to look for a trend. Because $\frac{\pi}{3} \approx 1.05$ and $\frac{\pi}{2} \approx 1.57$, we will evaluate x at radian measures $1.05 < x < 1.57$ as shown in **Table 2.**

x	1.3	1.5	1.55	1.56
$\tan x$	3.6	14.1	48.1	92.6

Table 2

As x approaches $\frac{\pi}{2}$, the outputs of the function get larger and larger. Because $y = \tan x$ is an odd function, we see the corresponding table of negative values in **Table 3**.

x	-1.3	-1.5	-1.55	-1.56
$\tan x$	-3.6	-14.1	-48.1	-92.6

Table 3

We can see that, as x approaches $-\frac{\pi}{2}$, the outputs get smaller and smaller. Remember that there are some values of x for which $\cos x = 0$. For example, $\cos\left(\frac{\pi}{2}\right) = 0$ and $\cos\left(\frac{3\pi}{2}\right) = 0$. At these values, the tangent function is undefined, so the graph of $y = \tan x$ has discontinuities at $x = \frac{\pi}{2}$ and $\frac{3\pi}{2}$. At these values, the graph of the tangent has vertical asymptotes. **Figure 1** represents the graph of $y = \tan(x)$. The tangent is positive from 0 to $\frac{\pi}{2}$ and from π to $\frac{3\pi}{2}$, corresponding to quadrants I and III of the unit circle.

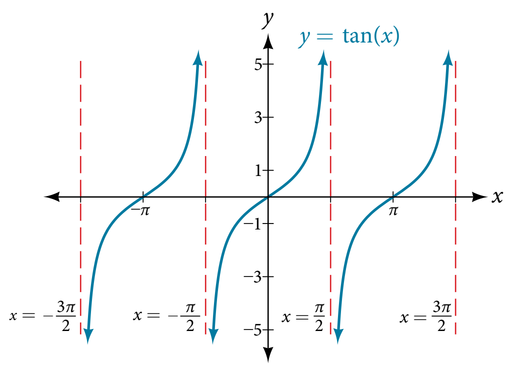


Figure 1 Graph of the tangent function

Graphing Variations of $y = \tan x$

As with the sine and cosine functions, the tangent function can be described by a general equation.

$$y = A\tan(Bx)$$

We can identify horizontal and vertical stretches and compressions using values of A and B. The horizontal stretch can typically be determined from the period of the graph. With tangent graphs, it is often necessary to determine a vertical stretch using a point on the graph.

Because there are no maximum or minimum values of a tangent function, the term *amplitude* cannot be interpreted as it is for the sine and cosine functions. Instead, we will use the phrase *stretching/compressing factor* when referring to the constant A.

features of the graph of $y = A\tan(Bx)$

- The stretching factor is $|A|$.
- The period is $P = \frac{\pi}{|B|}$. Thus the period of $y = \tan x$ is π .
- The domain is all real numbers x, where $x \neq \frac{\pi}{2|B|} + \frac{\pi}{|B|}k$ such that k is an integer.
- The range is $(-\infty, \infty)$.
- The asymptotes occur at $x = \frac{\pi}{2|B|} + \frac{\pi}{|B|}k$, where k is an integer.
- $y = A\tan(Bx)$ is an odd function.

Graphing One Period of a Stretched or Compressed Tangent Function

We can use what we know about the properties of the tangent function to quickly sketch a graph of any stretched and/or compressed tangent function of the form $f(x) = A\tan(Bx)$. We focus on a single period of the function including the origin, because the periodic property enables us to extend the graph to the rest of the function's domain if we wish. Our limited domain is then the interval $\left(-\frac{P}{2}, \frac{P}{2}\right)$ and the graph has vertical asymptotes at $\pm\frac{P}{2}$ where $P = \frac{\pi}{B}$. On $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, the graph will come up from the left asymptote at $x = -\frac{\pi}{2}$, cross through the origin, and continue to increase as it approaches the right asymptote at $x = \frac{\pi}{2}$. To make the function approach the asymptotes at the correct rate, we also need to set the vertical scale by actually evaluating the function for at least one point that the graph will pass through. For example, we can use

$$f\left(\frac{P}{4}\right) = A\tan\left(B\frac{P}{4}\right) = A\tan\left(B\frac{\pi}{4B}\right) = A$$

because $\tan\left(\frac{\pi}{4}\right) = 1$.

How To…

Given the function $f(x) = A\tan(Bx)$, graph one period.

1. Identify the stretching factor, $|A|$.
2. Identify B and determine the period, $P = \frac{\pi}{|B|}$.
3. Draw vertical asymptotes at $x = -\frac{P}{2}$ and $x = \frac{P}{2}$.
4. For $A > 0$, the graph approaches the left asymptote at negative output values and the right asymptote at positive output values (reverse for $A < 0$).
5. Plot reference points at $\left(\frac{P}{4}, A\right)$, $(0, 0)$, and $\left(-\frac{P}{4}, -A\right)$, and draw the graph through these points.

Example 1 Sketching a Compressed Tangent

Sketch a graph of one period of the function $y = 0.5\tan\left(\frac{\pi}{2}x\right)$.

Solution First, we identify A and B.

$$\begin{array}{c} y = 0.5\tan\left(\frac{\pi}{2}x\right) \\ \uparrow \quad \nearrow \\ y = A\tan(Bx) \end{array}$$

Because $A = 0.5$ and $B = \frac{\pi}{2}$, we can find the stretching/compressing factor and period. The period is $\frac{\pi}{\frac{\pi}{2}} = 2$, so the asymptotes are at $x = \pm 1$. At a quarter period from the origin, we have

$$\begin{aligned} f(0.5) &= 0.5\tan\left(\frac{0.5\pi}{2}\right) \\ &= 0.5\tan\left(\frac{\pi}{4}\right) \\ &= 0.5 \end{aligned}$$

This means the curve must pass through the points $(0.5, 0.5)$, $(0, 0)$, and $(-0.5, -0.5)$. The only inflection point is at the origin. **Figure 2** shows the graph of one period of the function.

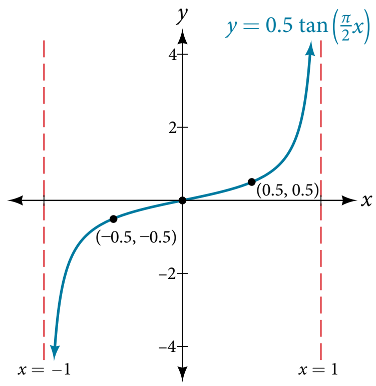


Figure 2

Try It #1

Sketch a graph of $f(x) = 3\tan\left(\frac{\pi}{6}x\right)$.

Graphing One Period of a Shifted Tangent Function

Now that we can graph a tangent function that is stretched or compressed, we will add a vertical and/or horizontal (or phase) shift. In this case, we add C and D to the general form of the tangent function.

$$f(x) = A\tan(Bx - C) + D$$

The graph of a transformed tangent function is different from the basic tangent function $\tan x$ in several ways:

features of the graph of $y = A\tan(Bx - C) + D$

- The stretching factor is $|A|$.
- The period is $\frac{\pi}{|B|}$.
- The domain is $x \neq \frac{C}{B} + \frac{\pi}{2|B|}k$, where k is an odd integer.
- The range is $(-\infty, \infty)$.
- The vertical asymptotes occur at $x = \frac{C}{B} + \frac{\pi}{2|B|}k$, where k is an odd integer.
- There is no amplitude.
- $y = A\tan(Bx)$ is an odd function because it is the quotient of odd and even functions (sine and cosine respectively).

Often facts can be either memorized or figured out. The 2nd, 3rd and 5th dots of the blue box can be figured out as follows: We are compressing horizontally by a factor of $|B|$ so new the period must be $\pi/|B|$. The domain of $\tan(x)$ does not include $k\pi/2$ where k is an odd integer. For $\tan(Bx - C)$ we set $Bx - C = k\pi/2$ and solve for x. This yields the x values that are not in the domain, which are also the locations of the vertical asymptotes. Since k can be either positive or negative, we can use either $(\pi/2|B|)k$ or $(\pi/2B)k$.

How To…

Given the function $y = A\tan(Bx - C) + D$, sketch the graph of one period.

1. Express the function given in the form $y = A\tan(Bx - C) + D$.
2. Identify the stretching/compressing factor, $|A|$.
3. Identify B and determine the period, $P = \frac{\pi}{|B|}$.
4. Identify C and determine the phase shift, $\frac{C}{B}$.
5. Draw the graph of $y = A\tan(Bx)$ shifted to the right by $\frac{C}{B}$ and up by D.
6. Sketch the vertical asymptotes, which occur at $x = \frac{C}{B} + \frac{\pi}{2|B|}k$, where k is an odd integer.
7. Plot any three reference points and draw the graph through these points.

Example 2 Graphing One Period of a Shifted Tangent Function

Graph one period of the function $y = -2\tan(\pi x + \pi) - 1$.

Solution

Step 1. The function is already written in the form $y = A\tan(Bx - C) + D$.

Step 2. $A = -2$, so the stretching factor is $|A| = 2$.

Step 3. $B = \pi$, so the period is $P = \frac{\pi}{|B|} = \frac{\pi}{\pi} = 1$.

Step 4. $C = -\pi$, so the phase shift is $\frac{C}{B} = \frac{-\pi}{\pi} = -1$.

Step 5-7. The asymptotes are at $x = -\frac{3}{2}$ and $x = -\frac{1}{2}$ and the three recommended reference points are $(-1.25, 1)$, $(-1, -1)$, and $(-0.75, -3)$. The graph is shown in **Figure 3**.

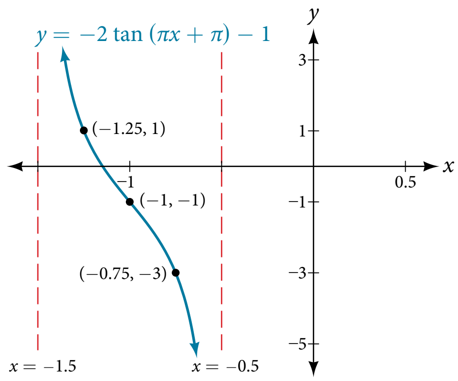


Figure 3

Analysis *Note that this is a decreasing function because $A < 0$.*

Try It #2

How would the graph in **Example 2** look different if we made $A = 2$ instead of -2?

How To…

Given the graph of a tangent function, identify horizontal and vertical stretches.

1. Find the period P from the spacing between successive vertical asymptotes or x-intercepts.
2. Write $f(x) = A\tan\left(\frac{\pi}{P}x\right)$.
3. Determine a convenient point $(x, f(x))$ on the given graph and use it to determine A.

Example 3 Identifying the Graph of a Stretched Tangent

Find a formula for the function graphed in **Figure 4.**

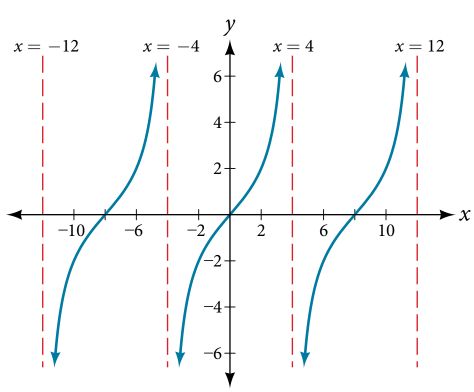


Figure 4 A stretched tangent function

Solution The graph has the shape of a tangent function.

Step 1. One cycle extends from -4 to 4, so the period is $P = 8$. Since $P = \frac{\pi}{|B|}$, we have $B = \frac{\pi}{P} = \frac{\pi}{8}$.

Step 2. The equation must have the form $f(x) = A\tan\left(\frac{\pi}{8}x\right)$.

Step 3. To find the vertical stretch A, we can use the point $(2, 2)$.

$$2 = A\tan\left(\frac{\pi}{8} \cdot 2\right) = A\tan\left(\frac{\pi}{4}\right)$$

Because $\tan\left(\frac{\pi}{4}\right) = 1$, $A = 2$.

This function would have a formula $f(x) = 2\tan\left(\frac{\pi}{8}x\right)$.

Try It #3

Find a formula for the function in **Figure 5**.

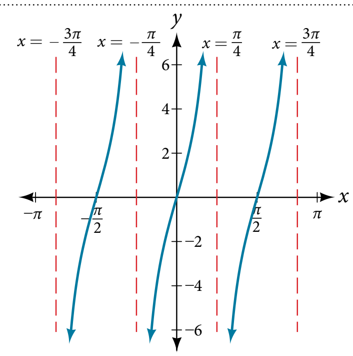


Figure 5

Analyzing the Graphs of $y = \sec x$ and $y = \csc x$

The secant was defined by the reciprocal identity $\sec x = \frac{1}{\cos x}$. Notice that the function is undefined when the cosine is 0, leading to vertical asymptotes at $\frac{\pi}{2}$, $\frac{3\pi}{2}$, etc. Because the cosine is never more than 1 in absolute value, the secant, being the reciprocal, will never be less than 1 in absolute value.

We can graph $y = \sec x$ by observing the graph of the cosine function because these two functions are reciprocals of one another. See **Figure 6**. The graph of the cosine is shown as a blue wave so we can see the relationship. Where the graph of the cosine function decreases, the graph of the secant function increases. Where the graph of the cosine function increases, the graph of the secant function decreases. When the cosine function is zero, the secant is undefined.

The secant graph has vertical asymptotes at each value of x where the cosine graph crosses the x-axis; we show these in the graph below with dashed vertical lines, but will not show all the asymptotes explicitly on all later graphs involving the secant and cosecant.

Note that, because cosine is an even function, secant is also an even function. That is, $\sec(-x) = \sec x$.

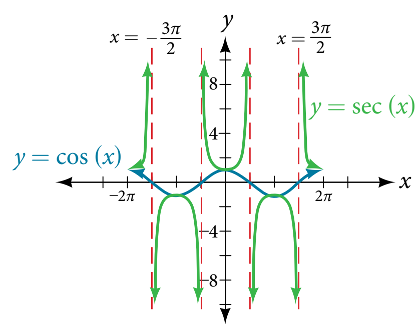


Figure 6 Graph of the secant function, $f(x) = \sec x = \frac{1}{\cos x}$

As we did for the tangent function, we will again refer to the constant $|A|$ as the stretching factor, not the amplitude.

features of the graph of $y = A\sec(Bx)$

- The stretching factor is $|A|$.
- The period is $\dfrac{2\pi}{|B|}$.
- The domain is $x \neq \dfrac{\pi}{2|B|}k$, where k is an odd integer.
- The range is $(-\infty, -|A|] \cup [|A|, \infty)$.
- The vertical asymptotes occur at $x = \dfrac{\pi}{2|B|}k$, where k is an odd integer.
- There is no amplitude.
- $y = A\sec(Bx)$ is an even function because cosine is an even function.

Similar to the secant, the cosecant is defined by the reciprocal identity $\csc x = \dfrac{1}{\sin x}$. Notice that the function is undefined when the sine is 0, leading to a vertical asymptote in the graph at 0, π, etc. Since the sine is never more than 1 in absolute value, the cosecant, being the reciprocal, will never be less than 1 in absolute value.

We can graph $y = \csc x$ by observing the graph of the sine function because these two functions are reciprocals of one another. See **Figure 7**. The graph of sine is shown as a blue wave so we can see the relationship. Where the graph of the sine function decreases, the graph of the cosecant function increases. Where the graph of the sine function increases, the graph of the cosecant function decreases.

The cosecant graph has vertical asymptotes at each value of x where the sine graph crosses the x-axis; we show these in the graph below with dashed vertical lines.

Note that, since sine is an odd function, the cosecant function is also an odd function. That is, $\csc(-x) = -\csc x$.

The graph of cosecant, which is shown in **Figure 7**, is similar to the graph of secant.

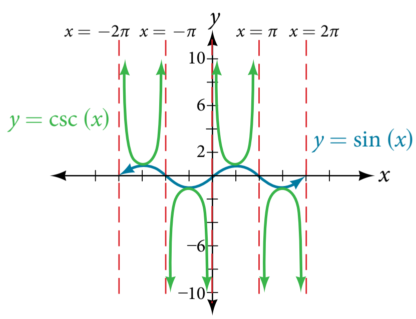


Figure 7 The graph of the cosecant function, $f(x) = \csc x = \dfrac{1}{\sin x}$

features of the graph of $y = A\csc(Bx)$

- The stretching factor is $|A|$.
- The period is $\dfrac{2\pi}{|B|}$.
- The domain is $x \neq \dfrac{\pi}{|B|}k$, where k is an integer.
- The range is $(-\infty, -|A|] \cup [|A|, \infty)$.
- The asymptotes occur at $x = \dfrac{\pi}{|B|}k$, where k is an integer.
- $y = A\csc(Bx)$ is an odd function because sine is an odd function.

Graphing Variations of $y = \sec x$ and $y = \csc x$

For shifted, compressed, and/or stretched versions of the secant and cosecant functions, we can follow similar methods to those we used for tangent and cotangent. That is, we locate the vertical asymptotes and also evaluate the functions for a few points (specifically the local extrema). If we want to graph only a single period, we can choose the interval for the period in more than one way. The procedure for secant is very similar, because the cofunction identity means that the secant graph is the same as the cosecant graph shifted half a period to the left. Vertical and phase shifts may be applied to the cosecant function in the same way as for the secant and other functions. The equations become the following.

$$y = A\sec(Bx - C) + D \qquad y = A\csc(Bx - C) + D$$

features of the graph of $y = A\sec(Bx - C) + D$

- The stretching factor is $|A|$.
- The period is $\frac{2\pi}{|B|}$.
- The domain is $x \neq \frac{C}{B} + \frac{\pi}{2|B|}k$, where k is an odd integer.
- The range is $(-\infty, -|A| + D] \cup [|A| + D, \infty)$.
- The vertical asymptotes occur at $x = \frac{C}{B} + \frac{\pi}{2|B|}k$, where k is an odd integer.
- There is no amplitude.
- $y = A\sec(Bx)$ is an even function because cosine is an even function.

As we saw with the tangent, this comes from $Bx - C = k\pi/2$.

The end points come from $\sec(Bx - C) = \pm 1$, which happens when $y = A(\pm 1) + D$.

features of the graph of $y = A\csc(Bx - C) + D$

- The stretching factor is $|A|$.
- The period is $\frac{2\pi}{|B|}$.
- The domain is $x \neq \frac{C}{B} + \frac{\pi}{|B|}k$, where k is an integer.
- The range is $(-\infty, -|A| + D] \cup [|A| + D, \infty)$.
- The vertical asymptotes occur at $x = \frac{C}{B} + \frac{\pi}{|B|}k$, where k is an integer.
- There is no amplitude.
- $y = A\csc(Bx)$ is an odd function because sine is an odd function.

How To...

Given a function of the form $y = A\sec(Bx)$, graph one period.

1. Express the function given in the form $y = A\sec(Bx)$.
2. Identify the stretching/compressing factor, $|A|$.
3. Identify B and determine the period, $P = \frac{2\pi}{|B|}$.
4. Sketch the graph of $y = A\cos(Bx)$.
5. Use the reciprocal relationship between $y = \cos x$ and $y = \sec x$ to draw the graph of $y = A\sec(Bx)$.
6. Sketch the asymptotes.
7. Plot any two reference points and draw the graph through these points.

Example 4 Graphing a Variation of the Secant Function

Graph one period of $f(x) = 2.5\sec(0.4x)$.

Solution

Step 1. The given function is already written in the general form, $y = A\sec(Bx)$.

Step 2. $A = 2.5$ so the stretching factor is 2.5.

Step 3. $B = 0.4$ so $P = \frac{2\pi}{0.4} = 5\pi$. The period is 5π units.

Step 4. Sketch the graph of the function $g(x) = 2.5\cos(0.4x)$.

Step 5. Use the reciprocal relationship of the cosine and secant functions to draw the cosecant function.

Steps 6–7. Sketch two asymptotes at $x = 1.25\pi$ and $x = 3.75\pi$. We can use two reference points, the local minimum at (0, 2.5) and the local maximum at $(2.5\pi, -2.5)$. **Figure 8** shows the graph.

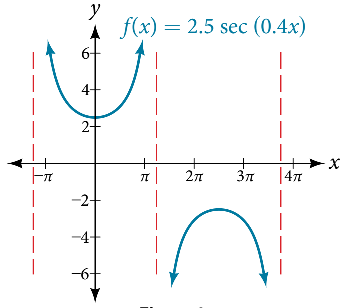


Figure 8

Try It #4

Graph one period of $f(x) = -2.5\sec(0.4x)$.

Q & A...

Do the vertical shift and stretch/compression affect the secant's range?

Yes. The range of $f(x) = A\sec(Bx - C) + D$ is $(-\infty, -|A| + D] \cup [|A| + D, \infty)$.

How To...

Given a function of the form $f(x) = A\sec(Bx - C) + D$, graph one period.

1. Express the function given in the form $y = A\sec(Bx - C) + D$.
2. Identify the stretching/compressing factor, $|A|$.
3. Identify B and determine the period, $\dfrac{2\pi}{|B|}$.
4. Identify C and determine the phase shift, $\dfrac{C}{B}$.
5. Draw the graph of $y = A\sec(Bx)$ but shift it to the right by $\dfrac{C}{B}$ and up by D.
6. Sketch the vertical asymptotes, which occur at $x = \dfrac{C}{B} + \dfrac{\pi}{2|B|}k$, where k is an odd integer.

Example 5 Graphing a Variation of the Secant Function

Graph one period of $y = 4\sec\left(\dfrac{\pi}{3}x - \dfrac{\pi}{2}\right) + 1$.

Solution

Step 1. Express the function given in the form $y = 4\sec\left(\dfrac{\pi}{3}x - \dfrac{\pi}{2}\right) + 1$.

Step 2. The stretching/compressing factor is $|A| = 4$.

Step 3. The period is

$$\begin{aligned} \frac{2\pi}{|B|} &= \frac{2\pi}{\frac{\pi}{3}} \\ &= \frac{2\pi}{1} \cdot \frac{3}{\pi} \\ &= 6 \end{aligned}$$

Step 4. The phase shift is

$$\begin{aligned} \frac{C}{B} &= \frac{\frac{\pi}{2}}{\frac{\pi}{3}} \\ &= \frac{\pi}{2} \cdot \frac{3}{\pi} \\ &= 1.5 \end{aligned}$$

Step 5. Draw the graph of $y = A\sec(Bx)$, but shift it to the right by $\frac{C}{B} = 1.5$ and up by $D = 6$.

Step 6. Sketch the vertical asymptotes, which occur at $x = 0$, $x = 3$, and $x = 6$. There is a local minimum at $(1.5, 5)$ and a local maximum at $(4.5, -3)$. **Figure 9** shows the graph.

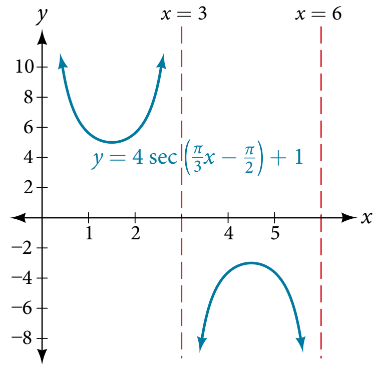


Figure 9

Try It #5

Graph one period of $f(x) = -6\sec(4x + 2) - 8$.

Q & A...

The domain of $\csc x$ was given to be all x such that $x \neq k\pi$ for any integer k. Would the domain of $y = A\csc(Bx - C) + D$ be $x \neq \frac{C + k\pi}{B}$?

Yes. The excluded points of the domain follow the vertical asymptotes. Their locations show the horizontal shift and compression or expansion implied by the transformation to the original function's input.

How To...

Given a function of the form $y = A\csc(Bx)$, graph one period.

1. Express the function given in the form $y = A\csc(Bx)$.
2. Identify the stretching/compressing factor, $|A|$.
3. Identify B and determine the period, $P = \frac{2\pi}{|B|}$.
4. Draw the graph of $y = A\sin(Bx)$.
5. Use the reciprocal relationship between $y = \sin x$ and $y = \csc x$ to draw the graph of $y = A\csc(Bx)$.
6. Sketch the asymptotes.
7. Plot any two reference points and draw the graph through these points.

Example 6 Graphing a Variation of the Cosecant Function

Graph one period of $f(x) = -3\csc(4x)$.

Solution

Step 1. The given function is already written in the general form, $y = A\csc(Bx)$.

Step 2. $|A| = |-3| = 3$, so the stretching factor is 3.

Step 3. $B = 4$, so $P = \frac{2\pi}{4} = \frac{\pi}{2}$. The period is $\frac{\pi}{2}$ units.

Step 4. Sketch the graph of the function $g(x) = -3\sin(4x)$.

Step 5. Use the reciprocal relationship of the sine and cosecant functions to draw the cosecant function.

Steps 6–7. Sketch three asymptotes at $x = 0$, $x = \frac{\pi}{4}$, and $x = \frac{\pi}{2}$. We can use two reference points, the local maximum at $\left(\frac{\pi}{8}, -3\right)$ and the local minimum at $\left(\frac{3\pi}{8}, 3\right)$. **Figure 10** shows the graph.

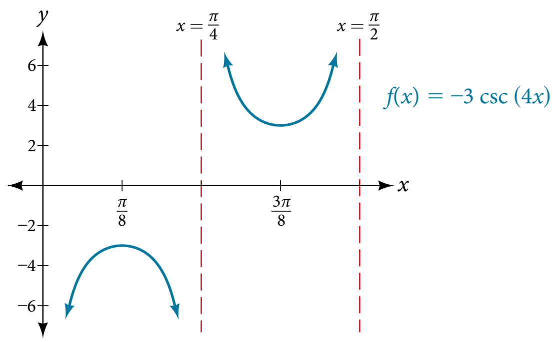


Figure 10

Try It #6

Graph one period of $f(x) = 0.5\csc(2x)$.

How To…

Given a function of the form $f(x) = A\csc(Bx - C) + D$, graph one period.

1. Express the function given in the form $y = A\csc(Bx - C) + D$.
2. Identify the stretching/compressing factor, $|A|$.
3. Identify B and determine the period, $\frac{2\pi}{|B|}$.
4. Identify C and determine the phase shift, $\frac{C}{B}$.
5. Draw the graph of $y = A\csc(Bx)$ but shift it to the right by $\frac{C}{B}$ and up by D.
6. Sketch the vertical asymptotes, which occur at $x = \frac{C}{B} + \frac{\pi}{|B|}k$, where k is an integer.

Example 7 Graphing a Vertically Stretched, Horizontally Compressed, and Vertically Shifted Cosecant

Sketch a graph of $y = 2\csc\left(\frac{\pi}{2}x\right) + 1$. What are the domain and range of this function?

Solution

Step 1. Express the function given in the form $y = 2\csc\left(\frac{\pi}{2}x\right) + 1$.

Step 2. Identify the stretching/compressing factor, $|A| = 2$.

Step 3. The period is $\frac{2\pi}{|B|} = \frac{2\pi}{\frac{\pi}{2}} = \frac{2\pi}{1} \cdot \frac{2}{\pi} = 4$.

Step 4. The phase shift is $\frac{0}{\frac{\pi}{2}} = 0$.

Step 5. Draw the graph of $y = A\csc(Bx)$ but shift it up $D = 1$.

Step 6. Sketch the vertical asymptotes, which occur at $x = 0$, $x = 2$, $x = 4$.

The graph for this function is shown in **Figure 11**.

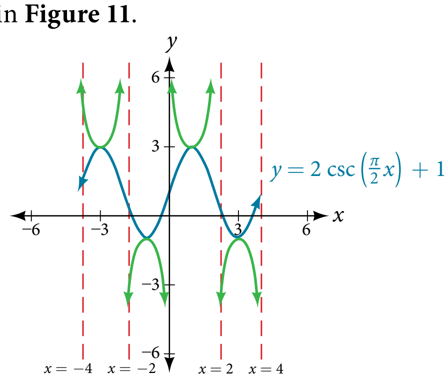


Figure 11 A transformed cosecant function

Analysis *The vertical asymptotes shown on the graph mark off one period of the function, and the local extrema in this interval are shown by dots. Notice how the graph of the transformed cosecant relates to the graph of* $f(x) = 2\sin\left(\frac{\pi}{2}x\right) + 1$, *shown as the blue wave.*

Try It #7

Given the graph of $f(x) = 2\cos\left(\frac{\pi}{2}x\right) + 1$ shown in **Figure 12**, sketch the graph of $g(x) = 2\sec\left(\frac{\pi}{2}x\right) + 1$ on the same axes.

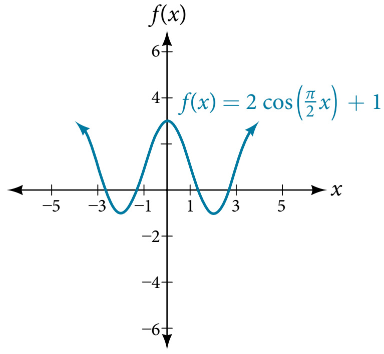


Figure 12

Analyzing the Graph of $y = \cot x$

The last trigonometric function we need to explore is cotangent. The cotangent is defined by the reciprocal identity $\cot x = \frac{1}{\tan x}$. Notice that the function is undefined when the tangent function is 0, leading to a vertical asymptote in the graph at 0, π, etc. Since the output of the tangent function is all real numbers, the output of the cotangent function is also all real numbers.

We can graph $y = \cot x$ by observing the graph of the tangent function because these two functions are reciprocals of one another. See **Figure 13.** Where the graph of the tangent function decreases, the graph of the cotangent function increases. Where the graph of the tangent function increases, the graph of the cotangent function decreases.

The cotangent graph has vertical asymptotes at each value of x where $\tan x = 0$; we show these in the graph below with dashed lines. Since the cotangent is the reciprocal of the tangent, $\cot x$ has vertical asymptotes at all values of x where $\tan x = 0$, and $\cot x = 0$ at all values of x where $\tan x$ has its vertical asymptotes.

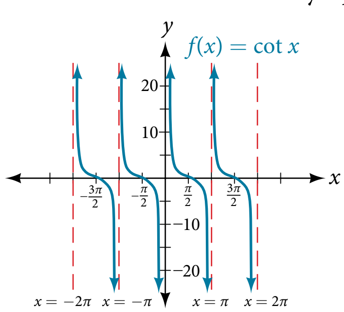


Figure 13 The cotangent function

features of the graph of $y = A\cot(Bx)$

- The stretching factor is $|A|$.
- The period is $P = \frac{\pi}{|B|}$.
- The domain is $x \neq \frac{\pi}{|B|}k$, where k is an integer.
- The range is $(-\infty, \infty)$.
- The asymptotes occur at $x = \frac{\pi}{|B|}k$, where k is an integer.
- $y = A\cot(Bx)$ is an odd function.

Graphing Variations of $y = \cot x$

We can transform the graph of the cotangent in much the same way as we did for the tangent. The equation becomes the following.

$$y = A\cot(Bx - C) + D$$

features of the graph of $y = A\cot(Bx - C) + D$

- The stretching factor is $|A|$.
- The period is $\dfrac{\pi}{|B|}$.
- The domain is $x \neq \dfrac{C}{B} + \dfrac{\pi}{|B|}k$, where k is an integer.
- The range is $(-\infty, \infty)$.
- The vertical asymptotes occur at $x = \dfrac{C}{B} + \dfrac{\pi}{|B|}k$, where k is an integer.
- There is no amplitude.
- $y = A\cot(Bx)$ is an odd function because it is the quotient of even and odd functions (cosine and sine, respectively)

How To…

Given a modified cotangent function of the form $f(x) = A\cot(Bx)$, graph one period.

1. Express the function in the form $f(x) = A\cot(Bx)$.
2. Identify the stretching factor, $|A|$.
3. Identify the period, $P = \dfrac{\pi}{|B|}$.
4. Draw the graph of $y = A\tan(Bx)$.
5. Plot any two reference points.
6. Use the reciprocal relationship between tangent and cotangent to draw the graph of $y = A\cot(Bx)$.
7. Sketch the asymptotes.

Example 8 Graphing Variations of the Cotangent Function

Determine the stretching factor, period, and phase shift of $y = 3\cot(4x)$, and then sketch a graph.

Solution

Step 1. Expressing the function in the form $f(x) = A\cot(Bx)$ gives $f(x) = 3\cot(4x)$.

Step 2. The stretching factor is $|A| = 3$.

Step 3. The period is $P = \dfrac{\pi}{4}$.

Step 4. Sketch the graph of $y = 3\tan(4x)$.

Step 5. Plot two reference points. Two such points are $\left(\dfrac{\pi}{16}, 3\right)$ and $\left(\dfrac{3\pi}{16}, -3\right)$.

Step 6. Use the reciprocal relationship to draw $y = 3\cot(4x)$.

Step 7. Sketch the asymptotes, $x = 0$, $x = \dfrac{\pi}{4}$.

The blue graph in **Figure 14** shows $y = 3\tan(4x)$ and the red graph shows $y = 3\cot(4x)$.

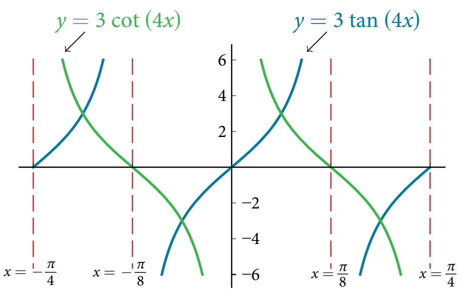


Figure 14

How To...

Given a modified cotangent function of the form $f(x) = A\cot(Bx - C) + D$, graph one period.

1. Express the function in the form $f(x) = A\cot(Bx - C) + D$.
2. Identify the stretching factor, $|A|$.
3. Identify the period, $P = \dfrac{\pi}{|B|}$.
4. Identify the phase shift, $\dfrac{C}{B}$.
5. Draw the graph of $y = A\tan(Bx)$ shifted to the right by $\dfrac{C}{B}$ and up by D.
6. Sketch the asymptotes $x = \dfrac{C}{B} + \dfrac{\pi}{|B|}k$, where k is an integer.
7. Plot any three reference points and draw the graph through these points.

Example 9 Graphing a Modified Cotangent

Sketch a graph of one period of the function $f(x) = 4\cot\left(\dfrac{\pi}{8}x - \dfrac{\pi}{2}\right) - 2$.

Solution

Step 1. The function is already written in the general form $f(x) = A\cot(Bx - C) + D$.

Step 2. $A = 4$, so the stretching factor is 4.

Step 3. $B = \dfrac{\pi}{8}$, so the period is $P = \dfrac{\pi}{|B|} = \dfrac{\pi}{\frac{\pi}{8}} = 8$.

Step 4. $C = \dfrac{\pi}{2}$, so the phase shift is $\dfrac{C}{B} = \dfrac{\frac{\pi}{2}}{\frac{\pi}{8}} = 4$.

Step 5. We draw $f(x) = 4\tan\left(\dfrac{\pi}{8}x - \dfrac{\pi}{2}\right) - 2$.

Step 6-7. Three points we can use to guide the graph are (6, 2), (8, −2), and (10, −6). We use the reciprocal relationship of tangent and cotangent to draw $f(x) = 4\cot\left(\dfrac{\pi}{8}x - \dfrac{\pi}{2}\right) - 2$.

Step 8. The vertical asymptotes are $x = 4$ and $x = 12$.

The graph is shown in **Figure 15.**

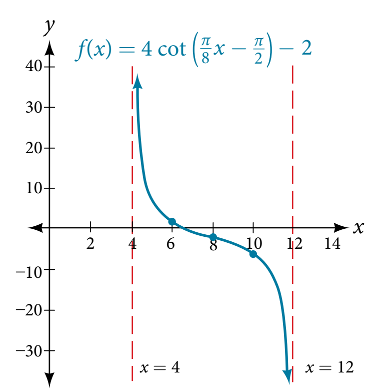


Figure 15 One period of a modified cotangent function

Using the Graphs of Trigonometric Functions to Solve Real-World Problems

Many real-world scenarios represent periodic functions and may be modeled by trigonometric functions. As an example, let's return to the scenario from the section opener. Have you ever observed the beam formed by the rotating light on a police car and wondered about the movement of the light beam itself across the wall? The periodic behavior of the distance the light shines as a function of time is obvious, but how do we determine the distance? We can use the tangent function.

Example 10 Using Trigonometric Functions to Solve Real-World Scenarios

Suppose the function $y = 5\tan\left(\frac{\pi}{4}t\right)$ marks the distance in the movement of a light beam from the top of a police car across a wall where t is the time in seconds and y is the distance in feet from a point on the wall directly across from the police car.

a. Find and interpret the stretching factor and period.

b. Graph on the interval [0, 5].

c. Evaluate $f(1)$ and discuss the function's value at that input.

Solution

a. We know from the general form of $y = A\tan(Bt)$ that $|A|$ is the stretching factor and $\frac{\pi}{B}$ is the period.

$$y = \underset{\substack{\uparrow \\ A}}{5}\tan\left(\underset{\substack{\uparrow \\ B}}{\frac{\pi}{4}}t\right)$$

Figure 16

We see that the stretching factor is 5. This means that the beam of light will have moved 5 ft after half the period.

The period is $\frac{\pi}{\frac{\pi}{4}} = \frac{\pi}{1} \cdot \frac{4}{\pi} = 4$. This means that every 4 seconds, the beam of light sweeps the wall. The distance from the spot across from the police car grows larger as the police car approaches.

b. To graph the function, we draw an asymptote at $t = 2$ and use the stretching factor and period. See **Figure 17.**

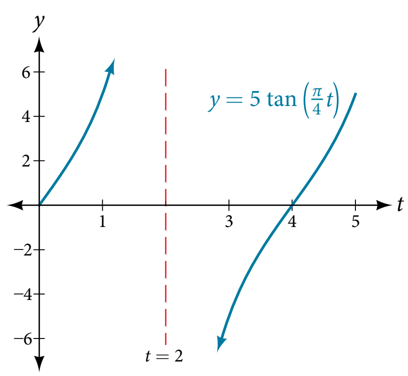


Figure 17

c. period: $f(1) = 5\tan\left(\frac{\pi}{4}(1)\right) = 5(1) = 5$; after 1 second, the beam of has moved 5 ft from the spot across from the police car.

Access these online resources for additional instruction and practice with graphs of other trigonometric functions.

- Graphing the Tangent (http://openstaxcollege.org/l/graphtangent)
- Graphing Cosecant and Secant (http://openstaxcollege.org/l/graphcscsec)
- Graphing the Cotangent (http://openstaxcollege.org/l/graphcot)

8.2 SECTION EXERCISES

VERBAL

1. Explain how the graph of the sine function can be used to graph $y = \csc x$.

2. How can the graph of $y = \cos x$ be used to construct the graph of $y = \sec x$?

3. Explain why the period of $\tan x$ is equal to π.

4. Why are there no intercepts on the graph of $y = \csc x$?

5. How does the period of $y = \csc x$ compare with the period of $y = \sin x$?

ALGEBRAIC

For the following exercises, match each trigonometric function with one of the graphs in **Figure 18**.

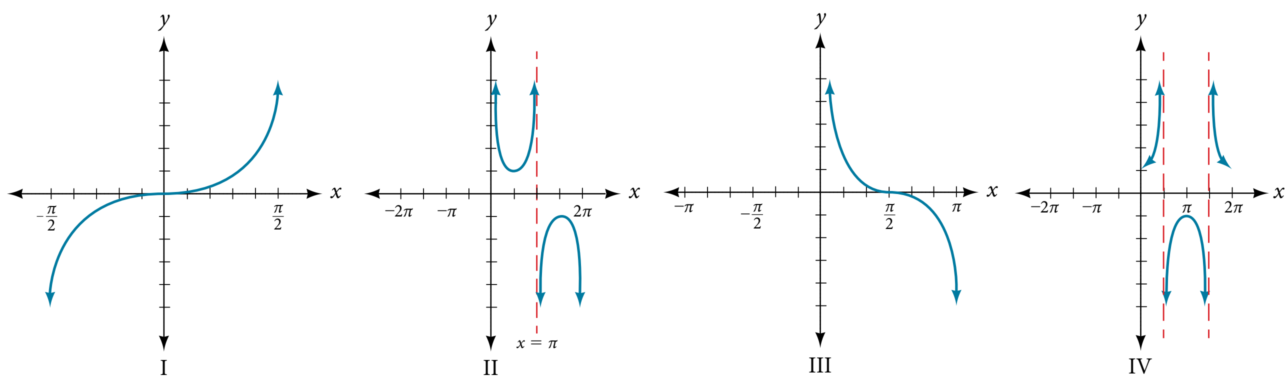


Figure 18

6. $f(x) = \tan x$

7. $f(x) = \sec x$

8. $f(x) = \csc x$

9. $f(x) = \cot x$

For the following exercises, find the period and horizontal shift of each of the functions.

10. $f(x) = 2\tan(4x - 32)$

11. $h(x) = 2\sec\left(\frac{\pi}{4}(x + 1)\right)$

12. $m(x) = 6\csc\left(\frac{\pi}{3}x + \pi\right)$

13. If $\tan x = -1.5$, find $\tan(-x)$.

14. If $\sec x = 2$, find $\sec(-x)$.

15. If $\csc x = -5$, find $\csc(-x)$.

16. If $x\sin x = 2$, find $(-x)\sin(-x)$.

For the following exercises, rewrite each expression such that the argument x is positive.

17. $\cot(-x)\cos(-x) + \sin(-x)$

18. $\cos(-x) + \tan(-x)\sin(-x)$

GRAPHICAL

For the following exercises, sketch two periods of the graph for each of the following functions. Identify the stretching factor, period, and asymptotes.

19. $f(x) = 2\tan(4x - 32)$

20. $h(x) = 2\sec\left(\frac{\pi}{4}(x + 1)\right)$

21. $m(x) = 6\csc\left(\frac{\pi}{3}x + \pi\right)$

22. $j(x) = \tan\left(\frac{\pi}{2}x\right)$

23. $p(x) = \tan\left(x - \frac{\pi}{2}\right)$

24. $f(x) = 4\tan(x)$

25. $f(x) = \tan\left(x + \frac{\pi}{4}\right)$

26. $f(x) = \pi\tan(\pi x - \pi) - \pi$

27. $f(x) = 2\csc(x)$

28. $f(x) = -\frac{1}{4}\csc(x)$

29. $f(x) = 4\sec(3x)$

30. $f(x) = -3\cot(2x)$

31. $f(x) = 7\sec(5x)$

32. $f(x) = \frac{9}{10}\csc(\pi x)$

33. $f(x) = 2\csc\left(x + \frac{\pi}{4}\right) - 1$

34. $f(x) = -\sec\left(x - \frac{\pi}{3}\right) - 2$

35. $f(x) = \frac{7}{5}\csc\left(x - \frac{\pi}{4}\right)$

36. $f(x) = 5\left(\cot\left(x + \frac{\pi}{2}\right) - 3\right)$

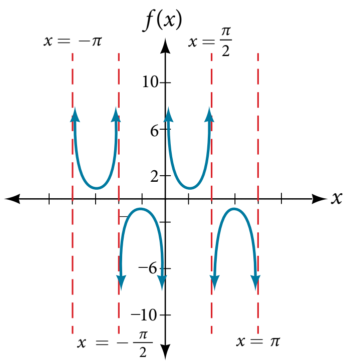
For the following exercises, find and graph two periods of the periodic function with the given stretching factor, $|A|$, period, and phase shift.

37. A tangent curve, $A = 1$, period of $\frac{\pi}{3}$; and phase shift $(h, k) = \left(\frac{\pi}{4}, 2\right)$

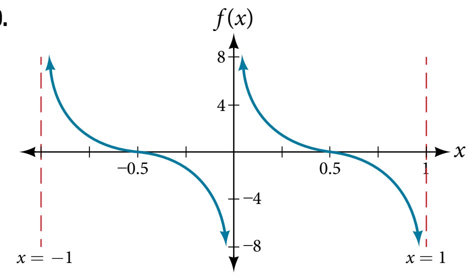
38. A tangent curve, $A = -2$, period of $\frac{\pi}{4}$, and phase shift $(h, k) = \left(-\frac{\pi}{4}, -2\right)$

For the following exercises, find an equation for the graph of each function.

39.



40.



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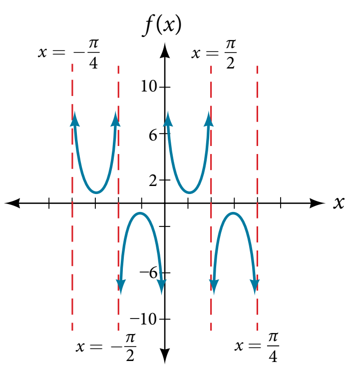

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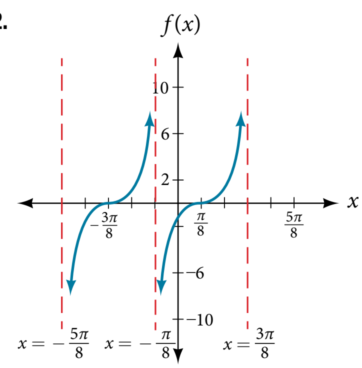

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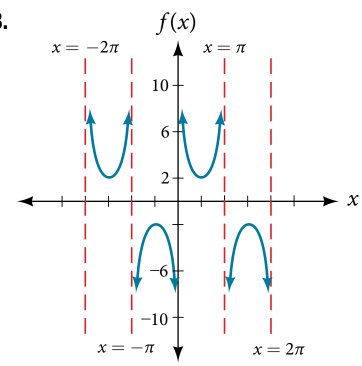

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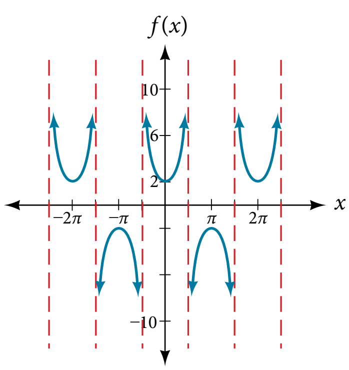

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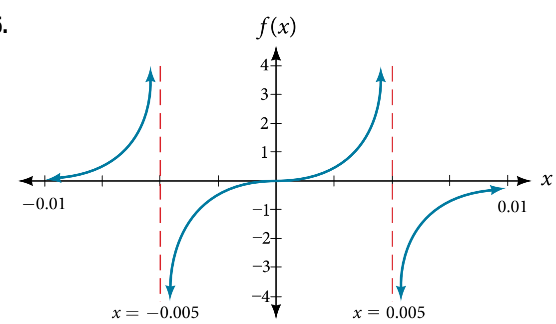


TECHNOLOGY

For the following exercises, use a graphing calculator to graph two periods of the given function. Note: most graphing calculators do not have a cosecant button; therefore, you will need to input $\csc x$ as $\frac{1}{\sin x}$.

46. $f(x) = |\csc(x)|$

47. $f(x) = |\cot(x)|$

48. $f(x) = 2^{\csc(x)}$

49. $f(x) = \dfrac{\csc(x)}{\sec(x)}$

50. Graph $f(x) = 1 + \sec^2(x) - \tan^2(x)$. What is the function shown in the graph?

51. $f(x) = \sec(0.001x)$

52. $f(x) = \cot(100\pi x)$

53. $f(x) = \sin^2 x + \cos^2 x$

REAL-WORLD APPLICATIONS

54. The function $f(x) = 20\tan\left(\frac{\pi}{10}x\right)$ marks the distance in the movement of a light beam from a police car across a wall for time x, in seconds, and distance $f(x)$, in feet.

a. Graph on the interval $[0, 5]$.

b. Find and interpret the stretching factor, period, and asymptote.

c. Evaluate $f(1)$ and $f(2.5)$ and discuss the function's values at those inputs.

55. Standing on the shore of a lake, a fisherman sights a boat far in the distance to his left. Let x, measured in radians, be the angle formed by the line of sight to the ship and a line due north from his position. Assume due north is 0 and x is measured negative to the left and positive to the right. (See **Figure 19**.) The boat travels from due west to due east and, ignoring the curvature of the Earth, the distance $d(x)$, in kilometers, from the fisherman to the boat is given by the function $d(x) = 1.5\sec(x)$.

a. What is a reasonable domain for $d(x)$?

b. Graph $d(x)$ on this domain.

c. Find and discuss the meaning of any vertical asymptotes on the graph of $d(x)$.

d. Calculate and interpret $d\left(-\frac{\pi}{3}\right)$. Round to the second decimal place.

e. Calculate and interpret $d\left(\frac{\pi}{6}\right)$. Round to the second decimal place.

f. What is the minimum distance between the fisherman and the boat? When does this occur?

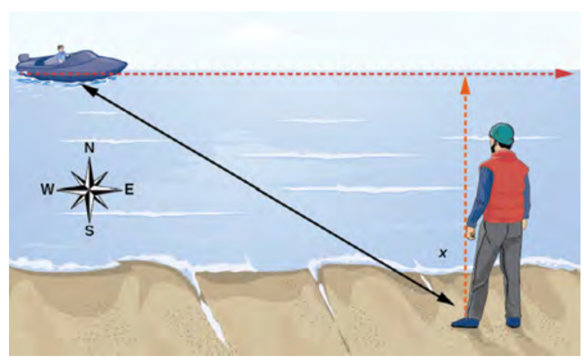


Figure 19

56. A laser rangefinder is locked on a comet approaching Earth. The distance $g(x)$, in kilometers, of the comet after x days, for x in the interval 0 to 30 days, is given by $g(x) = 250{,}000\csc\left(\frac{\pi}{30}x\right)$.

a. Graph $g(x)$ on the interval $[0, 35]$.

b. Evaluate $g(5)$ and interpret the information.

c. What is the minimum distance between the comet and Earth? When does this occur? To which constant in the equation does this correspond?

d. Find and discuss the meaning of any vertical asymptotes.

57. A video camera is focused on a rocket on a launching pad 2 miles from the camera. The angle of elevation from the ground to the rocket after x seconds is $\frac{\pi}{120}x$.

a. Write a function expressing the altitude $h(x)$, in miles, of the rocket above the ground after x seconds. Ignore the curvature of the Earth.

b. Graph $h(x)$ on the interval $(0, 60)$.

c. Evaluate and interpret the values $h(0)$ and $h(30)$.

d. What happens to the values of $h(x)$ as x approaches 60 seconds? Interpret the meaning of this in terms of the problem.

LEARNING OBJECTIVES

In this section, you will:

- Understand and use the inverse sine, cosine, and tangent functions.
- Find the exact value of expressions involving the inverse sine, cosine, and tangent functions.
- Use a calculator to evaluate inverse trigonometric functions.
- Find exact values of composite functions with inverse trigonometric functions.

8.3 INVERSE TRIGONOMETRIC FUNCTIONS

For any right triangle, given one other angle and the length of one side, we can figure out what the other angles and sides are. But what if we are given only two sides of a right triangle? We need a procedure that leads us from a ratio of sides to an angle. This is where the notion of an inverse to a trigonometric function comes into play. In this section, we will explore the inverse trigonometric functions.

Understanding and Using the Inverse Sine, Cosine, and Tangent Functions

In order to use inverse trigonometric functions, we need to understand that an inverse trigonometric function "undoes" what the original trigonometric function "does," as is the case with any other function and its inverse. In other words, the domain of the inverse function is the range of the original function, and vice versa, as summarized in **Figure 1.**

Trig Functions	Inverse Trig Functions
Domain: Measure of an angle	Domain: Ratio
Range: Ratio	Range: Measure of an angle

Figure 1

For example, if $f(x) = \sin x$, then we would write $f^{-1}(x) = \sin^{-1} x$. Be aware that $\sin^{-1} x$ does not mean $\frac{1}{\sin x}$. The following examples illustrate the inverse trigonometric functions:

- Since $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$, then $\frac{\pi}{6} = \sin^{-1}\left(\frac{1}{2}\right)$.
- Since $\cos(\pi) = -1$, then $\pi = \cos^{-1}(-1)$.
- Since $\tan\left(\frac{\pi}{4}\right) = 1$, then $\frac{\pi}{4} = \tan^{-1}(1)$.

In previous sections, we evaluated the trigonometric functions at various angles, but at times we need to know what angle would yield a specific sine, cosine, or tangent value. For this, we need inverse functions. Recall that, for a one-to-one function, if $f(a) = b$, then an inverse function would satisfy $f^{-1}(b) = a$.

Bear in mind that the sine, cosine, and tangent functions are not one-to-one functions. The graph of each function would fail the horizontal line test. In fact, no periodic function can be one-to-one because each output in its range corresponds to at least one input in every period, and there are an infinite number of periods. As with other functions that are not one-to-one, we will need to restrict the domain of each function to yield a new function that is one-to-one. We choose a domain for each function that includes the number 0. **Figure 2** shows the graph of the sine function limited to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and the graph of the cosine function limited to $[0, \pi]$.

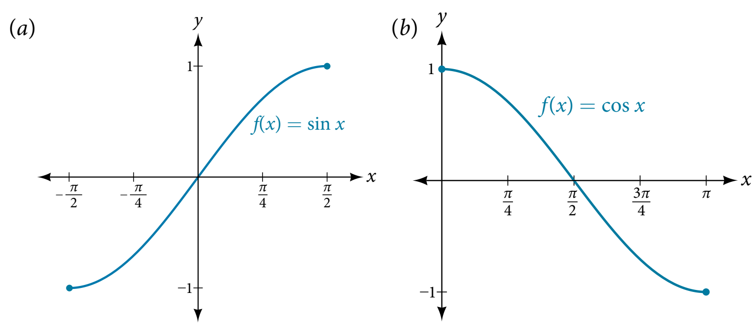


Figure 2 (a) Sine function on a restricted domain of $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$; (b) Cosine function on a restricted domain of $[0, \pi]$

Figure 3 shows the graph of the tangent function limited to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

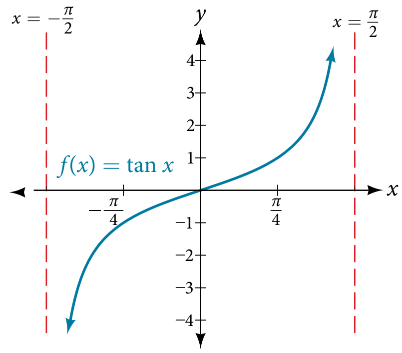


Figure 3 Tangent function on a restricted domain of $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

These conventional choices for the restricted domain are somewhat arbitrary, but they have important, helpful characteristics. Each domain includes the origin and some positive values, and most importantly, each results in a one-to-one function that is invertible. The conventional choice for the restricted domain of the tangent function also has the useful property that it extends from one vertical asymptote to the next instead of being divided into two parts by an asymptote.

On these restricted domains, we can define the inverse trigonometric functions.

- The **inverse sine function** $y = \sin^{-1} x$ means $x = \sin y$. The inverse sine function is sometimes called the **arcsine function**, and notated arcsin$x$.

$$y = \sin^{-1} x \text{ has domain } [-1, 1] \text{ and range } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

- The **inverse cosine function** $y = \cos^{-1} x$ means $x = \cos y$. The inverse cosine function is sometimes called the **arccosine** function, and notated arccos x.

$$y = \cos^{-1} x \text{ has domain } [-1, 1] \text{ and range } [0, \pi]$$

- The **inverse tangent function** $y = \tan^{-1} x$ means $x = \tan y$. The inverse tangent function is sometimes called the **arctangent** function, and notated arctan x.

$$y = \tan^{-1} x \text{ has domain } (-\infty, \infty) \text{ and range } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

The graphs of the inverse functions are shown in **Figure 4, Figure 5,** and **Figure 6**. Notice that the output of each of these inverse functions is a number, an angle in radian measure. We see that $\sin^{-1} x$ has domain $[-1, 1]$ and range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\cos^{-1} x$ has domain $[-1, 1]$ and range $[0, \pi]$, and $\tan^{-1} x$ has domain of all real numbers and range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. To find the domain and range of inverse trigonometric functions, switch the domain and range of the original functions. Each graph of the inverse trigonometric function is a reflection of the graph of the original function about the line $y = x$.

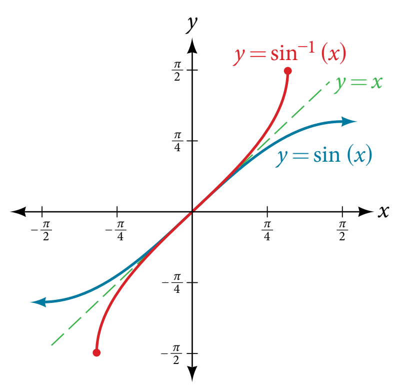


Figure 4 The sine function and inverse sine (or arcsine) function

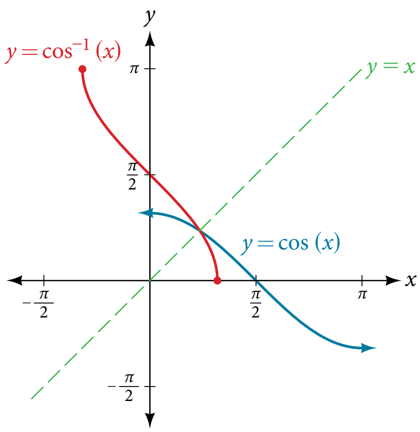


Figure 5 The cosine function and inverse cosine (or arccosine) function

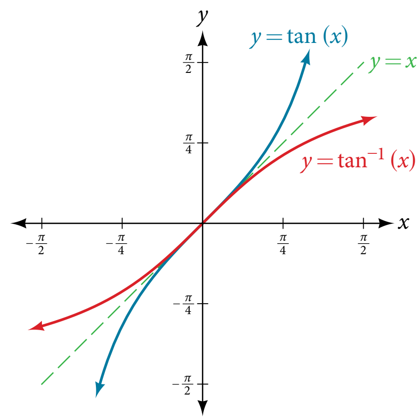


Figure 6 The tangent function and inverse tangent (or arctangent) function

relations for inverse sine, cosine, and tangent functions

For angles in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, if $\sin y = x$, then $\sin^{-1} x = y$.

For angles in the interval $[0, \pi]$, if $\cos y = x$, then $\cos^{-1} x = y$.

For angles in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, if $\tan y = x$, then $\tan^{-1} x = y$.

Example 1 Writing a Relation for an Inverse Function

Given $\sin\left(\frac{5\pi}{12}\right) \approx 0.96593$, write a relation involving the inverse sine.

Solution Use the relation for the inverse sine. If $\sin y = x$, then $\sin^{-1} x = y$.

In this problem, $x = 0.96593$, and $y = \frac{5\pi}{12}$.

$$\sin^{-1}(0.96593) \approx \frac{5\pi}{12}$$

Try It #1

Given $\cos(0.5) \approx 0.8776$, write a relation involving the inverse cosine.

Finding the Exact Value of Expressions Involving the Inverse Sine, Cosine, and Tangent Functions

Now that we can identify inverse functions, we will learn to evaluate them. For most values in their domains, we must evaluate the inverse trigonometric functions by using a calculator, interpolating from a table, or using some other numerical technique. Just as we did with the original trigonometric functions, we can give exact values for the inverse functions when we are using the special angles, specifically $\frac{\pi}{6}$ (30°), $\frac{\pi}{4}$ (45°), and $\frac{\pi}{3}$ (60°), and their reflections into other quadrants.

How To...

Given a "special" input value, evaluate an inverse trigonometric function.

1. Find angle x for which the original trigonometric function has an output equal to the given input for the inverse trigonometric function.
2. If x is not in the defined range of the inverse, find another angle y that is in the defined range and has the same sine, cosine, or tangent as x, depending on which corresponds to the given inverse function.

Example 2 Evaluating Inverse Trigonometric Functions for Special Input Values

Evaluate each of the following.

a. $\sin^{-1}\left(\frac{1}{2}\right)$ **b.** $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$ **c.** $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ **d.** $\tan^{-1}(1)$

Solution

a. Evaluating $\sin^{-1}\left(\frac{1}{2}\right)$ is the same as determining the angle that would have a sine value of $\frac{1}{2}$. In other words, what angle x would satisfy $\sin(x) = \frac{1}{2}$? There are multiple values that would satisfy this relationship, such as $\frac{\pi}{6}$ and $\frac{5\pi}{6}$, but we know we need the angle in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, so the answer will be $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$. Remember that the inverse is a function, so for each input, we will get exactly one output.

b. To evaluate $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$, we know that $\frac{5\pi}{4}$ and $\frac{7\pi}{4}$ both have a sine value of $-\frac{\sqrt{2}}{2}$, but neither is in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. For that, we need the negative angle coterminal with $\frac{7\pi}{4}$: $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$.

c. To evaluate $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$, we are looking for an angle in the interval $[0, \pi]$ with a cosine value of $-\frac{\sqrt{3}}{2}$. The angle that satisfies this is $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$.

d. Evaluating $\tan^{-1}(1)$, we are looking for an angle in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ with a tangent value of 1. The correct angle is $\tan^{-1}(1) = \frac{\pi}{4}$.

Try It #2

Evaluate each of the following.

a. $\sin^{-1}(-1)$ **b.** $\tan^{-1}(-1)$ **c.** $\cos^{-1}(-1)$ **d.** $\cos^{-1}\left(\frac{1}{2}\right)$

Using a Calculator to Evaluate Inverse Trigonometric Functions

To evaluate inverse trigonometric functions that do not involve the special angles discussed previously, we will need to use a calculator or other type of technology. Most scientific calculators and calculator-emulating applications have specific keys or buttons for the inverse sine, cosine, and tangent functions. These may be labeled, for example, **SIN-1**, **ARCSIN**, or **ASIN**.

In the previous chapter, we worked with trigonometry on a right triangle to solve for the sides of a triangle given one side and an additional angle. Using the inverse trigonometric functions, we can solve for the angles of a right triangle given two sides, and we can use a calculator to find the values to several decimal places.

In these examples and exercises, the answers will be interpreted as angles and we will use θ as the independent variable. The value displayed on the calculator may be in degrees or radians, so be sure to set the mode appropriate to the application.

Example 3 **Evaluating the Inverse Sine on a Calculator**

Evaluate $\sin^{-1}(0.97)$ using a calculator.

Solution Because the output of the inverse function is an angle, the calculator will give us a degree value if in degree mode and a radian value if in radian mode. Calculators also use the same domain restrictions on the angles as we are using.

In radian mode, $\sin^{-1}(0.97) \approx 1.3252$. In degree mode, $\sin^{-1}(0.97) \approx 75.93°$. Note that in calculus and beyond we will use radians in almost all cases.

Try It #3

Evaluate $\cos^{-1}(-0.4)$ using a calculator.

How To...

Given two sides of a right triangle like the one shown in **Figure 7**, find an angle.

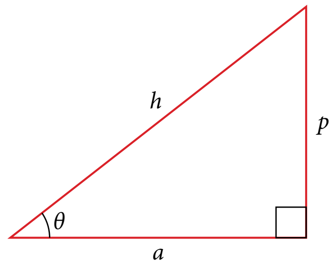


Figure 7

1. If one given side is the hypotenuse of length h and the side of length a adjacent to the desired angle is given, use the equation $\theta = \cos^{-1}\left(\frac{a}{h}\right)$.
2. If one given side is the hypotenuse of length h and the side of length p opposite to the desired angle is given, use the equation $\theta = \sin^{-1}\left(\frac{p}{h}\right)$
3. If the two legs (the sides adjacent to the right angle) are given, then use the equation $\theta = \tan^{-1}\left(\frac{p}{a}\right)$.

Example 4 Applying the Inverse Cosine to a Right Triangle

Solve the triangle in **Figure 8** for the angle θ.

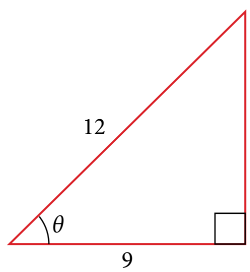


Figure 8

Solution Because we know the hypotenuse and the side adjacent to the angle, it makes sense for us to use the cosine function.

$$\cos\theta = \frac{9}{12}$$

$$\theta = \cos^{-1}\left(\frac{9}{12}\right) \qquad \text{Apply definition of the inverse.}$$

$$\theta \approx 0.7227 \text{ or about } 41.4096^\circ \qquad \text{Evaluate.}$$

Try It #4

Solve the triangle in **Figure 9** for the angle θ.

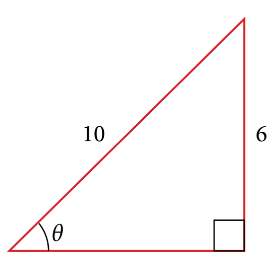


Figure 9

Finding Exact Values of Composite Functions with Inverse Trigonometric Functions

There are times when we need to compose a trigonometric function with an inverse trigonometric function. In these cases, we can usually find exact values for the resulting expressions without resorting to a calculator. Even when the input to the composite function is a variable or an expression, we can often find an expression for the output. To help sort out different cases, let $f(x)$ and $g(x)$ be two different trigonometric functions belonging to the set $\{\sin(x), \cos(x), \tan(x)\}$ and let $f^{-1}(y)$ and $g^{-1}(y)$ be their inverses.

Evaluating Compositions of the Form $f(f^{-1}(y))$ and $f^{-1}(f(x))$

For any trigonometric function, $f(f^{-1}(y)) = y$ for all y in the proper domain for the given function. This follows from the definition of the inverse and from the fact that the range of f was defined to be identical to the domain of f^{-1}. However, we have to be a little more careful with expressions of the form $f^{-1}(f(x))$.

compositions of a trigonometric function and its inverse

$$\sin(\sin^{-1} x) = x \text{ for } -1 \leq x \leq 1$$
$$\cos(\cos^{-1} x) = x \text{ for } -1 \leq x \leq 1$$
$$\tan(\tan^{-1} x) = x \text{ for } -\infty < x < \infty$$
$$\sin^{-1}(\sin x) = x \text{ only for } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$
$$\cos^{-1}(\cos x) = x \text{ only for } 0 \leq x \leq \pi$$
$$\tan^{-1}(\tan x) = x \text{ only for } -\frac{\pi}{2} < x < \frac{\pi}{2}$$

In the first 3 equations, whenever the left-hand side exists the equations are true. The last 3 equations are different. In the 4th and 5th equations the left-hand side exists for all x, but the equations are false unless x belongs to the given intervals. The intervals are the ranges of the arcsine and the arccosine. Similar statements hold for the 6th equation.

Q & A...

Is it correct that $\sin^{-1}(\sin x) = x$?

No. This equation is correct if x belongs to the restricted domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, but sine is defined for all real input values, and for x outside the restricted interval, the equation is not correct because its inverse always returns a value in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. The situation is similar for cosine and tangent and their inverses. For example, $\sin^{-1}\left(\sin\left(\frac{3\pi}{4}\right)\right) = \frac{\pi}{4}$.

How To...

Given an expression of the form $f^{-1}(f(\theta))$ where $f(\theta) = \sin\theta, \cos\theta$, or $\tan\theta$, evaluate.

1. If θ is in the restricted domain of f, then $f^{-1}(f(\theta)) = \theta$.
2. If not, then find an angle ϕ within the restricted domain of f such that $f(\phi) = f(\theta)$. Then $f^{-1}(f(\theta)) = \phi$.

Example 5 Using Inverse Trigonometric Functions

Evaluate the following:

a. $\sin^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right)$ **b.** $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$ **c.** $\cos^{-1}\left(\cos\left(\frac{2\pi}{3}\right)\right)$ **d.** $\cos^{-1}\left(\cos\left(-\frac{\pi}{3}\right)\right)$

Solution

a. $\frac{\pi}{3}$ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, so $\sin^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right) = \frac{\pi}{3}$.

b. $\frac{2\pi}{3}$ is not in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, but $\sin\left(\frac{2\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right)$, so $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right) = \frac{\pi}{3}$.

c. $\frac{2\pi}{3}$ is in $[0, \pi]$, so $\cos^{-1}\left(\cos\left(\frac{2\pi}{3}\right)\right) = \frac{2\pi}{3}$.

d. $-\frac{\pi}{3}$ is not in $[0, \pi]$, but $\cos\left(-\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right)$ because cosine is an even function. $\frac{\pi}{3}$ is in $[0, \pi]$, so $\cos^{-1}\left(\cos\left(-\frac{\pi}{3}\right)\right) = \frac{\pi}{3}$.

Try It #5

Evaluate $\tan^{-1}\left(\tan\left(\frac{\pi}{8}\right)\right)$ and $\tan^{-1}\left(\tan\left(\frac{11\pi}{9}\right)\right)$.

Evaluating Compositions of the Form $f^{-1}(g(x))$

Now that we can compose a trigonometric function with its inverse, we can explore how to evaluate a composition of a trigonometric function and the inverse of another trigonometric function. We will begin with compositions of the form $f^{-1}(g(x))$. For special values of x, we can exactly evaluate the inner function and then the outer, inverse function. However, we can find a more general approach by considering the relation between the two acute angles of a right triangle where one is θ, making the other $\frac{\pi}{2} - \theta$. Consider the sine and cosine of each angle of the right triangle in **Figure 10.**

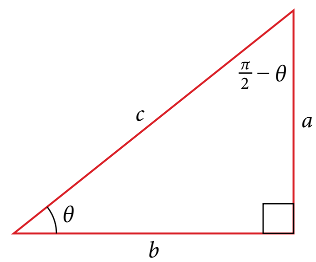


Figure 10 Right triangle illustrating the cofunction relationships

Because $\cos\theta = \frac{b}{c} = \sin\left(\frac{\pi}{2} - \theta\right)$, we have $\sin^{-1}(\cos\theta) = \frac{\pi}{2} - \theta$ if $0 \leq \theta \leq \pi$. If θ is not in this domain, then we need to find another angle that has the same cosine as θ and does belong to the restricted domain; we then subtract this angle from $\frac{\pi}{2}$. Similarly, $\sin\theta = \frac{a}{c} = \cos\left(\frac{\pi}{2} - \theta\right)$, so $\cos^{-1}(\sin\theta) = \frac{\pi}{2} - \theta$ if $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. These are just the function-cofunction relationships presented in another way.

How To...

Given functions of the form $\sin^{-1}(\cos x)$ and $\cos^{-1}(\sin x)$, evaluate them.

1. If x is in $[0, \pi]$, then $\sin^{-1}(\cos x) = \frac{\pi}{2} - x$.
2. If x is not in $[0, \pi]$, then find another angle y in $[0, \pi]$ such that $\cos y = \cos x$.
$$\sin^{-1}(\cos x) = \frac{\pi}{2} - y$$
3. If x is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then $\cos^{-1}(\sin x) = \frac{\pi}{2} - x$.
4. If x is not in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then find another angle y in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that $\sin y = \sin x$.
$$\cos^{-1}(\sin x) = \frac{\pi}{2} - y$$

Example 6 Evaluating the Composition of an Inverse Sine with a Cosine

Evaluate $\sin^{-1}\left(\cos\left(\frac{13\pi}{6}\right)\right)$

a. by direct evaluation. **b.** by the method described previously.

Solution

a. Here, we can directly evaluate the inside of the composition.

$$\begin{aligned}\cos\left(\frac{13\pi}{6}\right) &= \cos\left(\frac{\pi}{6} + 2\pi\right)\\ &= \cos\left(\frac{\pi}{6}\right)\\ &= \frac{\sqrt{3}}{2}\end{aligned}$$

Now, we can evaluate the inverse function as we did earlier.

$$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$$

b. We have $x = \frac{13\pi}{6}$, $y = \frac{\pi}{6}$, and

$$\sin^{-1}\left(\cos\left(\frac{13\pi}{6}\right)\right) = \frac{\pi}{2} - \frac{\pi}{6}$$
$$= \frac{\pi}{3}$$

Try It #6

Evaluate $\cos^{-1}\left(\sin\left(-\frac{11\pi}{4}\right)\right)$.

Evaluating Compositions of the Form $f(g^{-1}(x))$

To evaluate compositions of the form $f(g^{-1}(x))$, where f and g are any two of the functions sine, cosine, or tangent and x is any input in the domain of g^{-1}, we have exact formulas, such as $\sin(\cos^{-1} x) = \sqrt{1 - x^2}$. When we need to use them, we can derive these formulas by using the trigonometric relations between the angles and sides of a right triangle, together with the use of Pythagoras's relation between the lengths of the sides. We can use the Pythagorean identity, $\sin^2 x + \cos^2 x = 1$, to solve for one when given the other. We can also use the inverse trigonometric functions to find compositions involving algebraic expressions.

Example 7 Evaluating the Composition of a Sine with an Inverse Cosine

Find an exact value for $\sin\left(\cos^{-1}\left(\frac{4}{5}\right)\right)$.

Solution Beginning with the inside, we can say there is some angle such that $\theta = \cos^{-1}\left(\frac{4}{5}\right)$, which means $\cos\theta = \frac{4}{5}$, and we are looking for $\sin\theta$. We can use the Pythagorean identity to do this.

$$\sin^2\theta + \cos^2\theta = 1 \qquad \text{Use our known value for cosine.}$$
$$\sin^2\theta + \left(\frac{4}{5}\right)^2 = 1 \qquad \text{Solve for sine.}$$
$$\sin^2\theta = 1 - \frac{16}{25}$$
$$\sin\theta = \pm\sqrt{\frac{9}{25}} = \pm\frac{3}{5}$$

Since $\theta = \cos^{-1}\left(\frac{4}{5}\right)$ is in quadrant I, $\sin\theta$ must be positive, so the solution is $\frac{3}{5}$. See **Figure 11.**

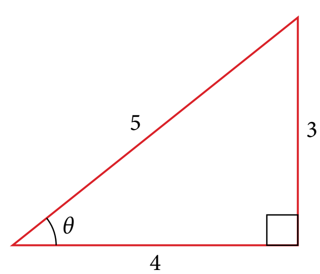


Figure 11 Right triangle illustrating that if $\cos\theta = \frac{4}{5}$, then $\sin\theta = \frac{3}{5}$

We know that the inverse cosine always gives an angle on the interval $[0, \pi]$, so we know that the sine of that angle must be positive; therefore $\sin\left(\cos^{-1}\left(\frac{4}{5}\right)\right) = \sin\theta = \frac{3}{5}$.

Try It #7

Evaluate $\cos\left(\tan^{-1}\left(\frac{5}{12}\right)\right)$.

Example 8 Evaluating the Composition of a Sine with an Inverse Tangent

Find an exact value for $\sin\left(\tan^{-1}\left(\frac{7}{4}\right)\right)$.

Solution While we could use a similar technique as in **Example 6**, we will demonstrate a different technique here. From the inside, we know there is an angle such that $\tan\theta = \frac{7}{4}$. We can envision this as the opposite and adjacent sides on a right triangle, as shown in **Figure 12.**

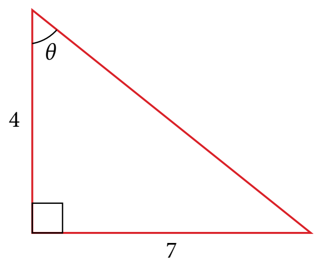


Figure 12 A right triangle with two sides known

Using the Pythagorean Theorem, we can find the hypotenuse of this triangle.

$$4^2 + 7^2 = \text{hypotenuse}^2$$
$$\text{hypotenuse} = \sqrt{65}$$

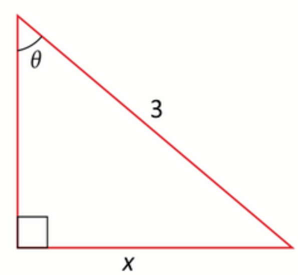
Now, we can evaluate the sine of the angle as the opposite side divided by the hypotenuse.

$$\sin\theta = \frac{7}{\sqrt{65}}$$

This gives us our desired composition.

$$\begin{aligned}\sin\left(\tan^{-1}\left(\frac{7}{4}\right)\right) &= \sin\theta \\ &= \frac{7}{\sqrt{65}} \\ &= \frac{7\sqrt{65}}{65}\end{aligned}$$

In Example 9, if we temporarily assume $0 \le x \le 3$ and $\theta = \sin^{-1}(x/3)$, we can use the triangle below as a guide. By the Pythagorean Theorem the adjacent side is $\sqrt{9-x^2}$, so $\cos(\theta) = \sqrt{9-x^2}/3$. In the book's algebraic approach we do not need extra assumptions.



Try It #8

Evaluate $\cos\left(\sin^{-1}\left(\frac{7}{9}\right)\right)$.

Example 9 Finding the Cosine of the Inverse Sine of an Algebraic Expression

Find a simplified expression for $\cos\left(\sin^{-1}\left(\frac{x}{3}\right)\right)$ for $-3 \le x \le 3$.

Solution We know there is an angle θ such that $\sin\theta = \frac{x}{3}$.

$$\begin{aligned}\sin^2\theta + \cos^2\theta &= 1 && \text{Use the Pythagorean Theorem.} \\ \left(\frac{x}{3}\right)^2 + \cos^2\theta &= 1 && \text{Solve for cosine.} \\ \cos^2\theta &= 1 - \frac{x^2}{9} \\ \cos\theta &= \pm\sqrt{\frac{9-x^2}{9}} = \pm\frac{\sqrt{9-x^2}}{3}\end{aligned}$$

Because we know that the inverse sine must give an angle on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we can deduce that the cosine of that angle must be positive.

$$\cos\left(\sin^{-1}\left(\frac{x}{3}\right)\right) = \frac{\sqrt{9-x^2}}{3}$$

Try It #9

Find a simplified expression for $\sin(\tan^{-1}(4x))$ for $-\frac{1}{4} \le x \le \frac{1}{4}$.

Access this online resource for additional instruction and practice with inverse trigonometric functions.

- Evaluate Expressions Involving Inverse Trigonometric Functions (http://openstaxcollege.org/l/evalinverstrig)

8.3 SECTION EXERCISES

VERBAL

1. Why do the functions $f(x) = \sin^{-1} x$ and $g(x) = \cos^{-1} x$ have different ranges?

2. Since the functions $y = \cos x$ and $y = \cos^{-1} x$ are inverse functions, why is $\cos^{-1}\left(\cos\left(-\frac{\pi}{6}\right)\right)$ not equal to $-\frac{\pi}{6}$?

3. Explain the meaning of $\frac{\pi}{6} = \arcsin(0.5)$.

4. Most calculators do not have a key to evaluate $\sec^{-1}(2)$. Explain how this can be done using the cosine function or the inverse cosine function.

5. Why must the domain of the sine function, $\sin x$, be restricted to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ for the inverse sine function to exist?

6. Discuss why this statement is incorrect: $\arccos(\cos x) = x$ for all x.

7. Determine whether the following statement is true or false and explain your answer: $\arccos(-x) = \pi - \arccos x$.

ALGEBRAIC

For the following exercises, evaluate the expressions.

8. $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$

9. $\sin^{-1}\left(-\frac{1}{2}\right)$

10. $\cos^{-1}\left(\frac{1}{2}\right)$

11. $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

12. $\tan^{-1}(1)$

13. $\tan^{-1}(-\sqrt{3})$

14. $\tan^{-1}(-1)$

15. $\tan^{-1}(\sqrt{3})$

16. $\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$

For the following exercises, use a calculator to evaluate each expression. Express answers to the nearest hundredth.

17. $\cos^{-1}(-0.4)$

18. $\arcsin(0.23)$

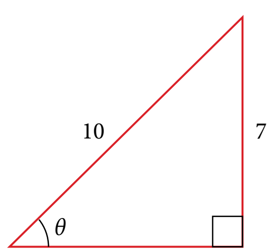
19. $\arccos\left(\frac{3}{5}\right)$

20. $\cos^{-1}(0.8)$

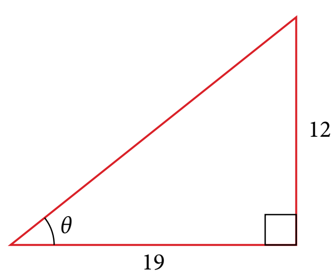
21. $\tan^{-1}(6)$

For the following exercises, find the angle θ in the given right triangle. Round answers to the nearest hundredth.

22.



23.



For the following exercises, find the exact value, if possible, without a calculator. If it is not possible, explain why.

24. $\sin^{-1}(\cos(\pi))$

25. $\tan^{-1}(\sin(\pi))$

26. $\cos^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right)$

27. $\tan^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right)$

28. $\sin^{-1}\left(\cos\left(\frac{-\pi}{2}\right)\right)$

29. $\tan^{-1}\left(\sin\left(\frac{4\pi}{3}\right)\right)$

30. $\sin^{-1}\left(\sin\left(\frac{5\pi}{6}\right)\right)$

31. $\tan^{-1}\left(\sin\left(\frac{-5\pi}{2}\right)\right)$

32. $\cos\left(\sin^{-1}\left(\frac{4}{5}\right)\right)$

33. $\sin\left(\cos^{-1}\left(\frac{3}{5}\right)\right)$

34. $\sin\left(\tan^{-1}\left(\frac{4}{3}\right)\right)$

35. $\cos\left(\tan^{-1}\left(\frac{12}{5}\right)\right)$

36. $\cos\left(\sin^{-1}\left(\frac{1}{2}\right)\right)$

For the following exercises, find the exact value of the expression in terms of x with the help of a reference triangle.

37. $\tan(\sin^{-1}(x-1))$

38. $\sin(\cos^{-1}(1-x))$

39. $\cos\left(\sin^{-1}\left(\frac{1}{x}\right)\right)$

40. $\cos(\tan^{-1}(3x-1))$

41. $\tan\left(\sin^{-1}\left(x+\frac{1}{2}\right)\right)$

EXTENSIONS

For the following exercise, evaluate the expression without using a calculator. Give the exact value.

42. $\dfrac{\sin^{-1}\left(\frac{1}{2}\right)-\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)+\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)-\cos^{-1}(1)}{\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)-\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)+\cos^{-1}\left(\frac{1}{2}\right)-\sin^{-1}(0)}$

For the following exercises, find the function if $\sin t=\dfrac{x}{x+1}$.

43. $\cos t$

44. $\sec t$

45. $\cot t$

46. $\cos\left(\sin^{-1}\left(\frac{x}{x+1}\right)\right)$

47. $\tan^{-1}\left(\frac{x}{\sqrt{2x+1}}\right)$

GRAPHICAL

48. Graph $y=\sin^{-1}x$ and state the domain and range of the function.

49. Graph $y=\arccos x$ and state the domain and range of the function.

50. Graph one cycle of $y=\tan^{-1}x$ and state the domain and range of the function.

51. For what value of x does $\sin x=\sin^{-1}x$? Use a graphing calculator to approximate the answer.

52. For what value of x does $\cos x=\cos^{-1}x$? Use a graphing calculator to approximate the answer.

REAL-WORLD APPLICATIONS

53. Suppose a 13-foot ladder is leaning against a building, reaching to the bottom of a second-floor window 12 feet above the ground. What angle, in radians, does the ladder make with the building?

54. Suppose you drive 0.6 miles on a road so that the vertical distance changes from 0 to 150 feet. What is the angle of elevation of the road?

55. An isosceles triangle has two congruent sides of length 9 inches. The remaining side has a length of 8 inches. Find the angle that a side of 9 inches makes with the 8-inch side.

56. Without using a calculator, approximate the value of arctan(10,000). Explain why your answer is reasonable.

57. A truss for the roof of a house is constructed from two identical right triangles. Each has a base of 12 feet and height of 4 feet. Find the measure of the acute angle adjacent to the 4-foot side.

58. The line $y=\frac{3}{5}x$ passes through the origin in the x,y-plane. What is the measure of the angle that the line makes with the positive x-axis?

59. The line $y=-\frac{3}{7}x$ passes through the origin in the x,y-plane. What is the measure of the angle that the line makes with the negative x-axis?

60. What percentage grade should a road have if the angle of elevation of the road is 4 degrees? (The percentage grade is defined as the change in the altitude of the road over a 100-foot horizontal distance. For example a 5% grade means that the road rises 5 feet for every 100 feet of horizontal distance.)

61. A 20-foot ladder leans up against the side of a building so that the foot of the ladder is 10 feet from the base of the building. If specifications call for the ladder's angle of elevation to be between 35 and 45 degrees, does the placement of this ladder satisfy safety specifications?

62. Suppose a 15-foot ladder leans against the side of a house so that the angle of elevation of the ladder is 42 degrees. How far is the foot of the ladder from the side of the house?

CHAPTER 8 REVIEW

Key Terms

amplitude the vertical height of a function; the constant A appearing in the definition of a sinusoidal function

arccosine another name for the inverse cosine; $\arccos x = \cos^{-1} x$

arcsine another name for the inverse sine; $\arcsin x = \sin^{-1} x$

arctangent another name for the inverse tangent; $\arctan x = \tan^{-1} x$

inverse cosine function the function $\cos^{-1} x$, which is the inverse of the cosine function and the angle that has a cosine equal to a given number

inverse sine function the function $\sin^{-1} x$, which is the inverse of the sine function and the angle that has a sine equal to a given number

inverse tangent function the function $\tan^{-1} x$, which is the inverse of the tangent function and the angle that has a tangent equal to a given number

midline the horizontal line $y = D$, where D appears in the general form of a sinusoidal function

periodic function a function $f(x)$ that satisfies $f(x + P) = f(x)$ for a specific constant P and any value of x

phase shift the horizontal displacement of the basic sine or cosine function; the constant $\frac{C}{B}$

sinusoidal function any function that can be expressed in the form $f(x) = A\sin(Bx - C) + D$ or $f(x) = A\cos(Bx - C) + D$

Key Equations

Sinusoidal functions	$f(x) = A\sin(Bx - C) + D$
	$f(x) = A\cos(Bx - C) + D$
Shifted, compressed, and/or stretched tangent function	$y = A\tan(Bx - C) + D$
Shifted, compressed, and/or stretched secant function	$y = A\sec(Bx - C) + D$
Shifted, compressed, and/or stretched cosecant function	$y = A\csc(Bx - C) + D$
Shifted, compressed, and/or stretched cotangent function	$y = A\cot(Bx - C) + D$

Key Concepts

8.1 Graphs of the Sine and Cosine Functions

- Periodic functions repeat after a given value. The smallest such value is the period. The basic sine and cosine functions have a period of 2π.
- The function $\sin x$ is odd, so its graph is symmetric about the origin. The function $\cos x$ is even, so its graph is symmetric about the y-axis.
- The graph of a sinusoidal function has the same general shape as a sine or cosine function.
- In the general formula for a sinusoidal function, the period is $P = \frac{2\pi}{|B|}$ See **Example 1**.
- In the general formula for a sinusoidal function, $|A|$ represents amplitude. If $|A| > 1$, the function is stretched, whereas if $|A| < 1$, the function is compressed. See **Example 2**.
- The value $\frac{C}{B}$ in the general formula for a sinusoidal function indicates the phase shift. See **Example 3**.
- The value D in the general formula for a sinusoidal function indicates the vertical shift from the midline. See **Example 4**.
- Combinations of variations of sinusoidal functions can be detected from an equation. See **Example 5**.
- The equation for a sinusoidal function can be determined from a graph. See **Example 6** and **Example 7**.
- A function can be graphed by identifying its amplitude and period. See **Example 8** and **Example 9**.
- A function can also be graphed by identifying its amplitude, period, phase shift, and horizontal shift. See **Example 10**.
- Sinusoidal functions can be used to solve real-world problems. See **Example 11**, **Example 12**, and **Example 13**.

8.2 Graphs of the Other Trigonometric Functions

- The tangent function has period π.
- $f(x) = A\tan(Bx - C) + D$ is a tangent with vertical and/or horizontal stretch/compression and shift. See **Example 1**, **Example 2**, and **Example 3**.
- The secant and cosecant are both periodic functions with a period of 2π. $f(x) = A\sec(Bx - C) + D$ gives a shifted, compressed, and/or stretched secant function graph. See **Example 4** and **Example 5**.
- $f(x) = A\csc(Bx - C) + D$ gives a shifted, compressed, and/or stretched cosecant function graph. See **Example 6** and **Example 7**.
- The cotangent function has period π and vertical asymptotes at $0, \pm\pi, \pm 2\pi, ...$
- The range of cotangent is $(-\infty, \infty)$, and the function is decreasing at each point in its range.
- The cotangent is zero at $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, ...$
- $f(x) = A\cot(Bx - C) + D$ is a cotangent with vertical and/or horizontal stretch/compression and shift. See **Example 8** and **Example 9**.
- Real-world scenarios can be solved using graphs of trigonometric functions. See **Example 10**.

8.3 Inverse Trigonometric Functions

- An inverse function is one that "undoes" another function. The domain of an inverse function is the range of the original function and the range of an inverse function is the domain of the original function.
- Because the trigonometric functions are not one-to-one on their natural domains, inverse trigonometric functions are defined for restricted domains.
- For any trigonometric function $f(x)$, if $x = f^{-1}(y)$, then $f(x) = y$. However, $f(x) = y$ only implies $x = f^{-1}(y)$ if x is in the restricted domain of f. See **Example 1.**
- Special angles are the outputs of inverse trigonometric functions for special input values; for example, $\frac{\pi}{4} = \tan^{-1}(1)$ and $\frac{\pi}{6} = \sin^{-1}\left(\frac{1}{2}\right)$. See **Example 2**.
- A calculator will return an angle within the restricted domain of the original trigonometric function. See **Example 3**.
- Inverse functions allow us to find an angle when given two sides of a right triangle. See **Example 4**.
- In function composition, if the inside function is an inverse trigonometric function, then there are exact expressions; for example, $\sin(\cos^{-1}(x)) = \sqrt{1 - x^2}$. See **Example 5**.
- If the inside function is a trigonometric function, then the only possible combinations are $\sin^{-1}(\cos x) = \frac{\pi}{2} - x$ if $0 \le x \le \pi$ and $\cos^{-1}(\sin x) = \frac{\pi}{2} - x$ if $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. See **Example 6** and **Example 7**.
- When evaluating the composition of a trigonometric function with an inverse trigonometric function, draw a reference triangle to assist in determining the ratio of sides that represents the output of the trigonometric function. See **Example 8**.
- When evaluating the composition of a trigonometric function with an inverse trigonometric function, you may use trig identities to assist in determining the ratio of sides. See **Example 9**.

CHAPTER 8 REVIEW EXERCISES

GRAPHS OF THE SINE AND COSINE FUNCTIONS

For the following exercises, graph the functions for two periods and determine the amplitude or stretching factor, period, midline equation, and asymptotes.

1. $f(x) = -3\cos x + 3$

2. $f(x) = \frac{1}{4}\sin x$

3. $f(x) = 3\cos\left(x + \frac{\pi}{6}\right)$

4. $f(x) = -2\sin\left(x - \frac{2\pi}{3}\right)$

5. $f(x) = 3\sin\left(x - \frac{\pi}{4}\right) - 4$

6. $f(x) = 2\left(\cos\left(x - \frac{4\pi}{3}\right) + 1\right)$

7. $f(x) = 6\sin\left(3x - \frac{\pi}{6}\right) - 1$

8. $f(x) = -100\sin(50x - 20)$

GRAPHS OF THE OTHER TRIGONOMETRIC FUNCTIONS

For the following exercises, graph the functions for two periods and determine the amplitude or stretching factor, period, midline equation, and asymptotes.

9. $f(x) = \tan x - 4$

10. $f(x) = 2\tan\left(x - \frac{\pi}{6}\right)$

11. $f(x) = -3\tan(4x) - 2$

12. $f(x) = 0.2\cos(0.1x) + 0.3$

For the following exercises, graph two full periods. Identify the period, the phase shift, the amplitude, and asymptotes.

13. $f(x) = \frac{1}{3}\sec x$

14. $f(x) = 3\cot x$

15. $f(x) = 4\csc(5x)$

16. $f(x) = 8\sec\left(\frac{1}{4}x\right)$

17. $f(x) = \frac{2}{3}\csc\left(\frac{1}{2}x\right)$

18. $f(x) = -\csc(2x + \pi)$

For the following exercises, use this scenario: The population of a city has risen and fallen over a 20-year interval. Its population may be modeled by the following function: $y = 12{,}000 + 8{,}000\sin(0.628x)$, where the domain is the years since 1980 and the range is the population of the city.

19. What is the largest and smallest population the city may have?

20. Graph the function on the domain of [0, 40].

21. What are the amplitude, period, and phase shift for the function?

22. Over this domain, when does the population reach 18,000? 13,000?

23. What is the predicted population in 2007? 2010?

For the following exercises, suppose a weight is attached to a spring and bobs up and down, exhibiting symmetry.

24. Suppose the graph of the displacement function is shown in **Figure 1**, where the values on the x-axis represent the time in seconds and the y-axis represents the displacement in inches. Give the equation that models the vertical displacement of the weight on the spring.

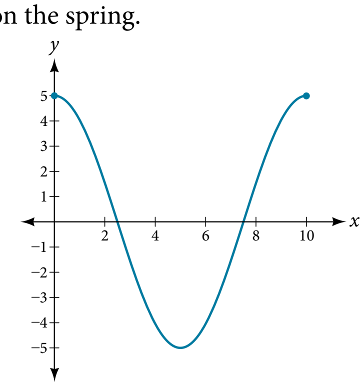


Figure 1

25. At time $= 0$, what is the displacement of the weight?

26. At what time does the displacement from the equilibrium point equal zero?

27. What is the time required for the weight to return to its initial height of 5 inches? In other words, what is the period for the displacement function?

INVERSE TRIGONOMETRIC FUNCTIONS

For the following exercises, find the exact value without the aid of a calculator.

28. $\sin^{-1}(1)$

29. $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

30. $\tan^{-1}(-1)$

31. $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$

32. $\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right)$

33. $\sin^{-1}\left(\cos\left(\frac{\pi}{6}\right)\right)$

34. $\cos^{-1}\left(\tan\left(\frac{3\pi}{4}\right)\right)$

35. $\sin\left(\sec^{-1}\left(\frac{3}{5}\right)\right)$

36. $\cot\left(\sin^{-1}\left(\frac{3}{5}\right)\right)$

37. $\tan\left(\cos^{-1}\left(\frac{5}{13}\right)\right)$

38. $\sin\left(\cos^{-1}\left(\frac{x}{x+1}\right)\right)$

39. Graph $f(x) = \cos x$ and $f(x) = \sec x$ on the interval $[0, 2\pi)$ and explain any observations.

40. Graph $f(x) = \sin x$ and $f(x) = \csc x$ and explain any observations.

41. Graph the function $f(x) = \frac{x}{1} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$ on the interval $[-1, 1]$ and compare the graph to the graph of $f(x) = \sin x$ on the same interval. Describe any observations.

CHAPTER 8 PRACTICE TEST

For the following exercises, sketch the graph of each function for two full periods. Determine the amplitude, the period, and the equation for the midline.

1. $f(x) = 0.5\sin x$

2. $f(x) = 5\cos x$

3. $f(x) = 5\sin x$

4. $f(x) = \sin(3x)$

5. $f(x) = -\cos\left(x + \frac{\pi}{3}\right) + 1$

6. $f(x) = 5\sin\left(3\left(x - \frac{\pi}{6}\right)\right) + 4$

7. $f(x) = 3\cos\left(\frac{1}{3}x - \frac{5\pi}{6}\right)$

8. $f(x) = \tan(4x)$

9. $f(x) = -2\tan\left(x - \frac{7\pi}{6}\right) + 2$

10. $f(x) = \pi\cos(3x + \pi)$

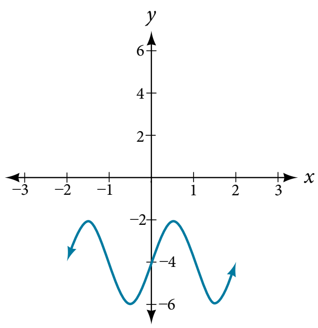
11. $f(x) = 5\csc(3x)$

12. $f(x) = \pi\sec\left(\frac{\pi}{2}x\right)$

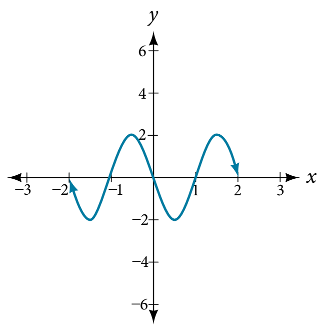
13. $f(x) = 2\csc\left(x + \frac{\pi}{4}\right) - 3$

For the following exercises, determine the amplitude, period, and midline of the graph, and then find a formula for the function.

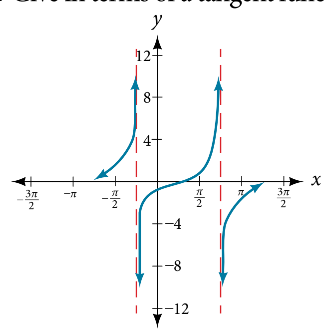
14. Give in terms of a sine function.



15. Give in terms of a sine function.



16. Give in terms of a tangent function.



For the following exercises, find the amplitude, period, phase shift, and midline.

17. $y = \sin\left(\frac{\pi}{6}x + \pi\right) - 3$

18. $y = 8\sin\left(\frac{7\pi}{6}x + \frac{7\pi}{2}\right) + 6$

19. The outside temperature over the course of a day can be modeled as a sinusoidal function. Suppose you know the temperature is 68°F at midnight and the high and low temperatures during the day are 80°F and 56°F, respectively. Assuming t is the number of hours since midnight, find a function for the temperature, D, in terms of t.

20. Water is pumped into a storage bin and empties according to a periodic rate. The depth of the water is 3 feet at its lowest at 2:00 a.m. and 71 feet at its highest, which occurs every 5 hours. Write a cosine function that models the depth of the water as a function of time, and then graph the function for one period.

For the following exercises, find the period and horizontal shift of each function.

21. $g(x) = 3\tan(6x + 42)$

22. $n(x) = 4\csc\left(\frac{5\pi}{3}x - \frac{20\pi}{3}\right)$

23. Write the equation for the graph in **Figure 1** in terms of the secant function and give the period and phase shift.

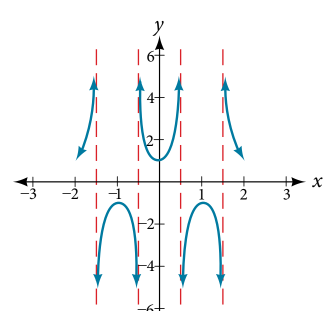


Figure 1

24. If $\tan x = 3$, find $\tan(-x)$.

25. If $\sec x = 4$, find $\sec(-x)$.

For the following exercises, graph the functions on the specified window and answer the questions.

26. Graph $m(x) = \sin(2x) + \cos(3x)$ on the viewing window $[-10, 10]$ by $[-3, 3]$. Approximate the graph's period.

27. Graph $n(x) = 0.02\sin(50\pi x)$ on the following domains in x: $[0, 1]$ and $[0, 3]$. Suppose this function models sound waves. Why would these views look so different?

28. Graph $f(x) = \dfrac{\sin x}{x}$ on $[-0.5, 0.5]$ and explain any observations.

For the following exercises, let $f(x) = \dfrac{3}{5}\cos(6x)$.

29. What is the largest possible value for $f(x)$?

30. What is the smallest possible value for $f(x)$?

31. Where is the function increasing on the interval $[0, 2\pi]$?

For the following exercises, find and graph one period of the periodic function with the given amplitude, period, and phase shift.

32. Sine curve with amplitude 3, period $\dfrac{\pi}{3}$, and phase shift $(h, k) = \left(\dfrac{\pi}{4}, 2\right)$

33. Cosine curve with amplitude 2, period $\dfrac{\pi}{6}$, and phase shift $(h, k) = \left(-\dfrac{\pi}{4}, 3\right)$

For the following exercises, graph the function. Describe the graph and, wherever applicable, any periodic behavior, amplitude, asymptotes, or undefined points.

34. $f(x) = 5\cos(3x) + 4\sin(2x)$

35. $f(x) = e^{\sin t}$

For the following exercises, find the exact value.

36. $\sin^{-1}\left(\dfrac{\sqrt{3}}{2}\right)$

37. $\tan^{-1}(\sqrt{3})$

38. $\cos^{-1}\left(-\dfrac{\sqrt{3}}{2}\right)$

39. $\cos^{-1}(\sin(\pi))$

40. $\cos^{-1}\left(\tan\left(\dfrac{7\pi}{4}\right)\right)$

41. $\cos(\sin^{-1}(1 - 2x))$

42. $\cos^{-1}(-0.4)$

43. $\cos(\tan^{-1}(x^2))$

For the following exercises, suppose $\sin t = \dfrac{x}{x+1}$.

44. $\tan t$

45. $\csc t$

46. Given **Figure 2**, find the measure of angle θ to three decimal places. Answer in radians.

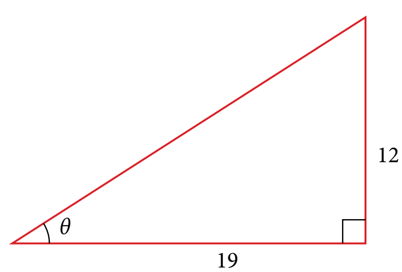


Figure 2

For the following exercises, determine whether the equation is true or false.

47. $\arcsin\left(\sin\left(\dfrac{5\pi}{6}\right)\right) = \dfrac{5\pi}{6}$

48. $\arccos\left(\cos\left(\dfrac{5\pi}{6}\right)\right) = \dfrac{5\pi}{6}$

49. The grade of a road is 7%. This means that for every horizontal distance of 100 feet on the road, the vertical rise is 7 feet. Find the angle the road makes with the horizontal in radians.

9

Trigonometric Identities and Equations

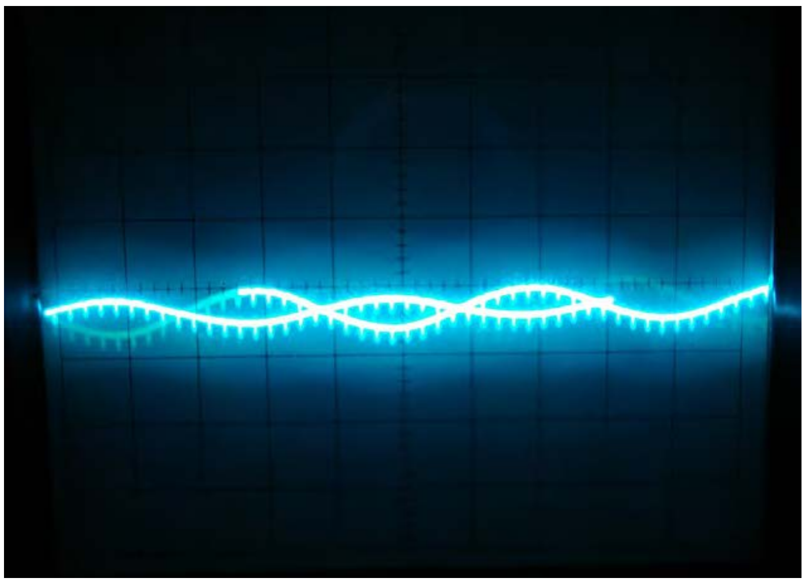


Figure 1 A sine wave models disturbance. (credit: modification of work by Mikael Altemark, Flickr).

CHAPTER OUTLINE

Introduction

Math is everywhere, even in places we might not immediately recognize. For example, mathematical relationships describe the transmission of images, light, and sound. The sinusoidal graph in **Figure 1** models music playing on a phone, radio, or computer. Such graphs are described using trigonometric equations and functions. In this chapter, we discuss how to manipulate trigonometric equations algebraically by applying various formulas and trigonometric identities. We will also investigate some of the ways that trigonometric equations are used to model real-life phenomena.

LEARNING OBJECTIVES

In this section, you will:

- Verify the fundamental trigonometric identities.
- Simplify trigonometric expressions using algebra and the identities.

9.1 SOLVING TRIGONOMETRIC EQUATIONS WITH IDENTITIES

Figure 1 International passports and travel documents

In espionage movies, we see international spies with multiple passports, each claiming a different identity. However, we know that each of those passports represents the same person. The trigonometric identities act in a similar manner to multiple passports—there are many ways to represent the same trigonometric expression. Just as a spy will choose an Italian passport when traveling to Italy, we choose the identity that applies to the given scenario when solving a trigonometric equation.

In this section, we will begin an examination of the fundamental trigonometric identities, including how we can verify them and how we can use them to simplify trigonometric expressions.

Verifying the Fundamental Trigonometric Identities

Identities enable us to simplify complicated expressions. They are the basic tools of trigonometry used in solving trigonometric equations, just as factoring, finding common denominators, and using special formulas are the basic tools of solving algebraic equations. In fact, we use algebraic techniques constantly to simplify trigonometric expressions. Basic properties and formulas of algebra, such as the difference of squares formula and the perfect squares formula, will simplify the work involved with trigonometric expressions and equations. We already know that all of the trigonometric functions are related because they all are defined in terms of the unit circle. Consequently, any trigonometric identity can be written in many ways.

To verify the trigonometric identities, we usually start with the more complicated side of the equation and essentially rewrite the expression until it has been transformed into the same expression as the other side of the equation. Sometimes we have to factor expressions, expand expressions, find common denominators, or use other algebraic strategies to obtain the desired result. In this first section, we will work with the fundamental identities: the Pythagorean identities, the even-odd identities, the reciprocal identities, and the quotient identities.

We will begin with the **Pythagorean identities** (see **Table 1**), which are equations involving trigonometric functions based on the properties of a right triangle. We have already seen and used the first of these identifies, but now we will also use additional identities.

Pythagorean Identities		
$\sin^2\theta + \cos^2\theta = 1$	$1 + \cot^2\theta = \csc^2\theta$	$1 + \tan^2\theta = \sec^2\theta$

Table 1

The second and third identities can be obtained by manipulating the first. The identity $1 + \cot^2 \theta = \csc^2 \theta$ is found by rewriting the left side of the equation in terms of sine and cosine.

Prove: $1 + \cot^2 \theta = \csc^2 \theta$

$$\begin{aligned} 1 + \cot^2 \theta &= \left(1 + \frac{\cos^2 \theta}{\sin^2 \theta}\right) && \text{Rewrite the left side.} \\ &= \left(\frac{\sin^2 \theta}{\sin^2 \theta}\right) + \left(\frac{\cos^2 \theta}{\sin^2 \theta}\right) && \text{Write both terms with the common denominator.} \\ &= \frac{\sin^2 \theta + \cos^2 \theta}{\sin^2 \theta} \\ &= \frac{1}{\sin^2 \theta} \\ &= \csc^2 \theta \end{aligned}$$

Similarly, $1 + \tan^2 \theta = \sec^2 \theta$ can be obtained by rewriting the left side of this identity in terms of sine and cosine. This gives

$$\begin{aligned} 1 + \tan^2 \theta &= 1 + \left(\frac{\sin \theta}{\cos \theta}\right)^2 && \text{Rewrite left side.} \\ &= \left(\frac{\cos \theta}{\cos \theta}\right)^2 + \left(\frac{\sin \theta}{\cos \theta}\right)^2 && \text{Write both terms with the common denominator.} \\ &= \frac{\cos^2 \theta + \sin^2 \theta}{\cos^2 \theta} \\ &= \frac{1}{\cos^2 \theta} \\ &= \sec^2 \theta \end{aligned}$$

Recall that we determined which trigonometric functions are odd and which are even. The next set of fundamental identities is the set of **even-odd identities**. The even-odd identities relate the value of a trigonometric function at a given angle to the value of the function at the opposite angle. (See **Table 2**).

Even-Odd Identities		
$\tan(-\theta) = -\tan \theta$	$\sin(-\theta) = -\sin \theta$	$\cos(-\theta) = \cos \theta$
$\cot(-\theta) = -\cot \theta$	$\csc(-\theta) = -\csc \theta$	$\sec(-\theta) = \sec \theta$

Table 2

Recall that an odd function is one in which $f(-x) = -f(x)$ for all x in the domain of f. The sine function is an odd function because $\sin(-\theta) = -\sin \theta$. The graph of an odd function is symmetric about the origin. For example, consider corresponding inputs of $\frac{\pi}{2}$ and $-\frac{\pi}{2}$. The output of $\sin\left(\frac{\pi}{2}\right)$ is opposite the output of $\sin\left(-\frac{\pi}{2}\right)$. Thus,

$$\sin\left(\frac{\pi}{2}\right) = 1$$

and

$$\begin{aligned} \sin\left(-\frac{\pi}{2}\right) &= -\sin\left(\frac{\pi}{2}\right) \\ &= -1 \end{aligned}$$

This is shown in **Figure 2**.

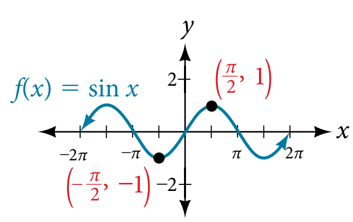


Figure 2 Graph of $y = \sin \theta$

Recall that an even function is one in which

$$f(-x) = f(x) \text{ for all } x \text{ in the domain of } f$$

The graph of an even function is symmetric about the y-axis. The cosine function is an even function because $\cos(-\theta) = \cos\theta$. For example, consider corresponding inputs $\frac{\pi}{4}$ and $-\frac{\pi}{4}$. The output of $\cos\left(\frac{\pi}{4}\right)$ is the same as the output of $\cos\left(-\frac{\pi}{4}\right)$. Thus,

$$\cos\left(-\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) \approx 0.707$$

See **Figure 3**.

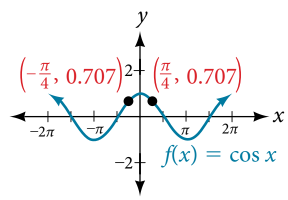


Figure 3 Graph of $y = \cos\theta$

For all θ in the domain of the sine and cosine functions, respectively, we can state the following:

- Since $\sin(-\theta) = -\sin\theta$, sine is an odd function.
- Since, $\cos(-\theta) = \cos\theta$, cosine is an even function.

The other even-odd identities follow from the even and odd nature of the sine and cosine functions. For example, consider the tangent identity, $\tan(-\theta) = -\tan\theta$. We can interpret the tangent of a negative angle as $\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin\theta}{\cos\theta} = -\tan\theta$. Tangent is therefore an odd function, which means that $\tan(-\theta) = -\tan(\theta)$ for all θ in the domain of the tangent function.

The cotangent identity, $\cot(-\theta) = -\cot\theta$, also follows from the sine and cosine identities. We can interpret the cotangent of a negative angle as $\cot(-\theta) = \frac{\cos(-\theta)}{\sin(-\theta)} = \frac{\cos\theta}{-\sin\theta} = -\cot\theta$. Cotangent is therefore an odd function, which means that $\cot(-\theta) = -\cot(\theta)$ for all θ in the domain of the cotangent function.

The cosecant function is the reciprocal of the sine function, which means that the cosecant of a negative angle will be interpreted as $\csc(-\theta) = \frac{1}{\sin(-\theta)} = \frac{1}{-\sin\theta} = -\csc\theta$. The cosecant function is therefore odd.

Finally, the secant function is the reciprocal of the cosine function, and the secant of a negative angle is interpreted as $\sec(-\theta) = \frac{1}{\cos(-\theta)} = \frac{1}{\cos\theta} = \sec\theta$. The secant function is therefore even.

To sum up, only two of the trigonometric functions, cosine and secant, are even. The other four functions are odd, verifying the even-odd identities.

The next set of fundamental identities is the set of **reciprocal identities**, which, as their name implies, relate trigonometric functions that are reciprocals of each other. See **Table 3**. Recall that we first encountered these identities when defining trigonometric functions from right angles in **Right Angle Trigonometry**.

Reciprocal Identities	
$\sin\theta = \frac{1}{\csc\theta}$	$\csc\theta = \frac{1}{\sin\theta}$
$\cos\theta = \frac{1}{\sec\theta}$	$\sec\theta = \frac{1}{\cos\theta}$
$\tan\theta = \frac{1}{\cot\theta}$	$\cot\theta = \frac{1}{\tan\theta}$

Table 3

The final set of identities is the set of quotient identities, which define relationships among certain trigonometric functions and can be very helpful in verifying other identities. See **Table 4**.

Quotient Identities	
$\tan\theta = \frac{\sin\theta}{\cos\theta}$	$\cot\theta = \frac{\cos\theta}{\sin\theta}$

Table 4

The reciprocal and quotient identities are derived from the definitions of the basic trigonometric functions.

summarizing trigonometric identities

The **Pythagorean identities** are based on the properties of a right triangle.

$$\cos^2\theta + \sin^2\theta = 1$$
$$1 + \cot^2\theta = \csc^2\theta$$
$$1 + \tan^2\theta = \sec^2\theta$$

The **even-odd identities** relate the value of a trigonometric function at a given angle to the value of the function at the opposite angle.

$$\tan(-\theta) = -\tan\theta$$
$$\cot(-\theta) = -\cot\theta$$
$$\sin(-\theta) = -\sin\theta$$
$$\csc(-\theta) = -\csc\theta$$
$$\cos(-\theta) = \cos\theta$$
$$\sec(-\theta) = \sec\theta$$

The **reciprocal identities** define reciprocals of the trigonometric functions.

$$\sin\theta = \frac{1}{\csc\theta}$$
$$\cos\theta = \frac{1}{\sec\theta}$$
$$\tan\theta = \frac{1}{\cot\theta}$$
$$\csc\theta = \frac{1}{\sin\theta}$$
$$\sec\theta = \frac{1}{\cos\theta}$$
$$\cot\theta = \frac{1}{\tan\theta}$$

The **quotient identities** define the relationship among the trigonometric functions.

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$
$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

Example 1 Graphing the Equations of an Identity

Graph both sides of the identity $\cot\theta = \frac{1}{\tan\theta}$. In other words, on the graphing calculator, graph $y = \cot\theta$ and $y = \frac{1}{\tan\theta}$.

Solution See **Figure 4**.

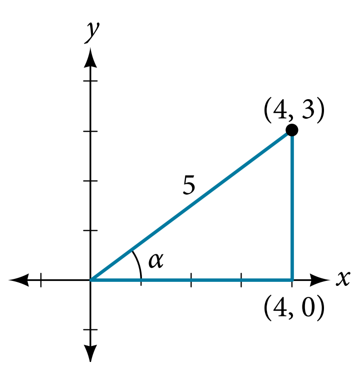


Figure 4

Analysis *We see only one graph because both expressions generate the same image. One is on top of the other. This is a good way to prove any identity. If both expressions give the same graph, then they must be identities.*

How To…

Given a trigonometric identity, verify that it is true.

1. Work on one side of the equation. It is usually better to start with the more complex side, as it is easier to simplify than to build.
2. Look for opportunities to factor expressions, square a binomial, or add fractions.
3. Noting which functions are in the final expression, look for opportunities to use the identities and make the proper substitutions.
4. If these steps do not yield the desired result, try converting all terms to sines and cosines.

Example 2 Verifying a Trigonometric Identity

Verify $\tan\theta\cos\theta = \sin\theta$.

Solution We will start on the left side, as it is the more complicated side:

$$\begin{aligned}\tan\theta\cos\theta &= \left(\frac{\sin\theta}{\cos\theta}\right)\cos\theta\\ &= \left(\frac{\sin\theta}{\cancel{\cos\theta}}\right)\cancel{\cos\theta}\\ &= \sin\theta\end{aligned}$$

Analysis *This identity was fairly simple to verify, as it only required writing $\tan\theta$ in terms of $\sin\theta$ and $\cos\theta$.*

Try It #1

Verify the identity $\csc\theta\cos\theta\tan\theta = 1$.

Example 3 Verifying the Equivalency Using the Even-Odd Identities

Verify the following equivalency using the even-odd identities:

$$(1 + \sin x)[1 + \sin(-x)] = \cos^2 x$$

Solution Working on the left side of the equation, we have

$$\begin{aligned}(1 + \sin x)[1 + \sin(-x)] &= (1 + \sin x)(1 - \sin x) && \text{Since } \sin(-x) = -\sin x\\ &= 1 - \sin^2 x && \text{Difference of squares}\\ &= \cos^2 x && \cos^2 x = 1 - \sin^2 x\end{aligned}$$

Example 4 Verifying a Trigonometric Identity Involving $\sec^2\theta$

Verify the identity $\dfrac{\sec^2\theta - 1}{\sec^2\theta} = \sin^2\theta$

Solution As the left side is more complicated, let's begin there.

$$\begin{aligned}\frac{\sec^2\theta - 1}{\sec^2\theta} &= \frac{(\tan^2\theta + 1) - 1}{\sec^2\theta} && \sec^2\theta = \tan^2\theta + 1\\ &= \frac{\tan^2\theta}{\sec^2\theta}\\ &= \tan^2\theta\left(\frac{1}{\sec^2\theta}\right)\\ &= \tan^2\theta(\cos^2\theta) && \cos^2\theta = \frac{1}{\sec^2\theta}\\ &= \left(\frac{\sin^2\theta}{\cos^2\theta}\right)(\cos^2\theta) && \tan^2\theta = \frac{\sin^2\theta}{\cos^2\theta}\\ &= \left(\frac{\sin^2\theta}{\cancel{\cos^2\theta}}\right)(\cancel{\cos^2\theta})\\ &= \sin^2\theta\end{aligned}$$

There is more than one way to verify an identity. Here is another possibility. Again, we can start with the left side.

$$\begin{aligned}\frac{\sec^2\theta - 1}{\sec^2\theta} &= \frac{\sec^2\theta}{\sec^2\theta} - \frac{1}{\sec^2\theta}\\ &= 1 - \cos^2\theta\\ &= \sin^2\theta\end{aligned}$$

Analysis *In the first method, we used the identity* $\sec^2\theta = \tan^2\theta + 1$ *and continued to simplify. In the second method, we split the fraction, putting both terms in the numerator over the common denominator. This problem illustrates that there are multiple ways we can verify an identity. Employing some creativity can sometimes simplify a procedure. As long as the substitutions are correct, the answer will be the same.*

Try It #2

Show that $\dfrac{\cot\theta}{\csc\theta} = \cos\theta$.

Example 5 Creating and Verifying an Identity

Create an identity for the expression $2\tan\theta\sec\theta$ by rewriting strictly in terms of sine.

Solution There are a number of ways to begin, but here we will use the quotient and reciprocal identities to rewrite the expression:

$$\begin{aligned}2\tan\theta\sec\theta &= 2\left(\frac{\sin\theta}{\cos\theta}\right)\left(\frac{1}{\cos\theta}\right)\\ &= \frac{2\sin\theta}{\cos^2\theta}\\ &= \frac{2\sin\theta}{1-\sin^2\theta} && \text{Substitute } 1-\sin^2\theta \text{ for } \cos^2\theta\end{aligned}$$

Thus,

$$2\tan\theta\sec\theta = \frac{2\sin\theta}{1-\sin^2\theta}$$

Example 6 Verifying an Identity Using Algebra and Even/Odd Identities

Verify the identity:

$$\frac{\sin^2(-\theta) - \cos^2(-\theta)}{\sin(-\theta) - \cos(-\theta)} = \cos\theta - \sin\theta$$

Solution Let's start with the left side and simplify:

$$\begin{aligned}\frac{\sin^2(-\theta) - \cos^2(-\theta)}{\sin(-\theta) - \cos(-\theta)} &= \frac{[\sin(-\theta)]^2 - [\cos(-\theta)]^2}{\sin(-\theta) - \cos(-\theta)}\\ &= \frac{(-\sin\theta)^2 - (\cos\theta)^2}{-\sin\theta - \cos\theta} && \sin(-x) = -\sin x \text{ and } \cos(-x) = \cos x\\ &= \frac{(\sin\theta)^2 - (\cos\theta)^2}{-\sin\theta - \cos\theta} && \text{Difference of squares}\\ &= \frac{(\sin\theta - \cos\theta)(\sin\theta + \cos\theta)}{-(\sin\theta + \cos\theta)}\\ &= \frac{(\sin\theta - \cos\theta)\cancel{(\sin\theta + \cos\theta)}}{-\cancel{(\sin\theta + \cos\theta)}}\\ &= \cos\theta - \sin\theta\end{aligned}$$

Try It #3

Verify the identity $\dfrac{\sin^2\theta - 1}{\tan\theta\sin\theta - \tan\theta} = \dfrac{\sin\theta + 1}{\tan\theta}$.

Example 7 **Verifying an Identity Involving Cosines and Cotangents**

Verify the identity: $(1 - \cos^2 x)(1 + \cot^2 x) = 1$.

Solution We will work on the left side of the equation

$$\begin{aligned}(1 - \cos^2 x)(1 + \cot^2 x) &= (1 - \cos^2 x)\left(1 + \frac{\cos^2 x}{\sin^2 x}\right) \\ &= (1 - \cos^2 x)\left(\frac{\sin^2 x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x}\right) && \text{Find the common denominator.} \\ &= (1 - \cos^2 x)\left(\frac{\sin^2 x + \cos^2 x}{\sin^2 x}\right) \\ &= (\sin^2 x)\left(\frac{1}{\sin^2 x}\right) \\ &= 1\end{aligned}$$

Using Algebra to Simplify Trigonometric Expressions

We have seen that algebra is very important in verifying trigonometric identities, but it is just as critical in simplifying trigonometric expressions before solving. Being familiar with the basic properties and formulas of algebra, such as the difference of squares formula, the perfect square formula, or substitution, will simplify the work involved with trigonometric expressions and equations.

For example, the equation $(\sin x + 1)(\sin x - 1) = 0$ resembles the equation $(x + 1)(x - 1) = 0$, which uses the factored form of the difference of squares. Using algebra makes finding a solution straightforward and familiar. We can set each factor equal to zero and solve. This is one example of recognizing algebraic patterns in trigonometric expressions or equations.

Another example is the difference of squares formula, $a^2 - b^2 = (a - b)(a + b)$, which is widely used in many areas other than mathematics, such as engineering, architecture, and physics. We can also create our own identities by continually expanding an expression and making the appropriate substitutions. Using algebraic properties and formulas makes many trigonometric equations easier to understand and solve.

Example 8 **Writing the Trigonometric Expression as an Algebraic Expression**

Write the following trigonometric expression as an algebraic expression: $2\cos^2 \theta + \cos \theta - 1$.

Solution Notice that the pattern displayed has the same form as a standard quadratic expression, $ax^2 + bx + c$. Letting $\cos \theta = x$, we can rewrite the expression as follows:

$$2x^2 + x - 1$$

This expression can be factored as $(2x + 1)(x - 1)$. If it were set equal to zero and we wanted to solve the equation, we would use the zero factor property and solve each factor for x. At this point, we would replace x with $\cos \theta$ and solve for θ.

Example 9 **Rewriting a Trigonometric Expression Using the Difference of Squares**

Rewrite the trigonometric expression: $4 \cos^2 \theta - 1$.

Solution Notice that both the coefficient and the trigonometric expression in the first term are squared, and the square of the number 1 is 1. This is the difference of squares.

$$\begin{aligned}4 \cos^2 \theta - 1 &= (2 \cos \theta)^2 - 1 \\ &= (2 \cos \theta - 1)(2 \cos \theta + 1)\end{aligned}$$

Analysis *If this expression were written in the form of an equation set equal to zero, we could solve each factor using the zero factor property. We could also use substitution like we did in the previous problem and let* $\cos \theta = x$*, rewrite the expression as* $4x^2 - 1$*, and factor* $(2x - 1)(2x + 1)$*. Then replace x with* $\cos \theta$ *and solve for the angle.*

Try It #4

Rewrite the trigonometric expression using the difference of squares: $25 - 9 \sin^2 \theta$.

Example 10 Simplify by Rewriting and Using Substitution

Simplify the expression by rewriting and using identities:

$$\csc^2\theta - \cot^2\theta$$

Solution We can start with the Pythagorean Identity.

$$1 + \cot^2\theta = \csc^2\theta$$

Now we can simplify by substituting $1 + \cot^2\theta$ for $\csc^2\theta$. We have

$$\begin{aligned}\csc^2\theta - \cot^2\theta &= 1 + \cot^2\theta - \cot^2\theta \\ &= 1\end{aligned}$$

Try It #5

Use algebraic techniques to verify the identity: $\dfrac{\cos\theta}{1+\sin\theta} = \dfrac{1-\sin\theta}{\cos\theta}$.

(Hint: Multiply the numerator and denominator on the left side by $1 - \sin\theta$.)

Access these online resources for additional instruction and practice with the fundamental trigonometric identities.

- Fundamental Trigonometric Identities (http://openstaxcollege.org/l/funtrigiden)
- Verifying Trigonometric Identities (http://openstaxcollege.org/l/verifytrigiden)

Example 11 Either establish that the following identity is true, or that it is false:

$$\frac{2+\sec^2 x}{\tan x} - \cot x = \sec x \csc x.$$

Solution We will algebraically manipulate the left-hand side and try to transform it into the right-hand side. There is no clear first step and it looks complicated, so our first step will be to convert everything into $\sin x$ and $\cos x$. (The second step will be to clear out the double fractions by multiplying the first term by $\frac{\cos^2 x}{\cos^2 x}$. The third step will be to obtain a common denominator.)

$$\frac{2+\sec^2 x}{\tan x} - \cot x = \frac{2+\frac{1}{\cos^2 x}}{\frac{\sin x}{\cos x}} - \frac{\cos x}{\sin x} = \frac{2\cos^2 x + 1}{\sin x\,\cos x} - \frac{\cos x}{\sin x} =$$

$$\frac{2\cos^2 x + 1}{\sin x\,\cos x} - \frac{\cos^2 x}{\sin x \cos x} = \frac{\cos^2 x + 1}{\sin x\,\cos x} = \cot x + \sec x\,\csc x.$$

This cannot possibly be equal to the right-hand side of the original equation, which is $\sec x \csc x$. We conclude that the original equation is false.

9.1 SECTION EXERCISES

VERBAL

1. We know $g(x) = \cos x$ is an even function, and $f(x) = \sin x$ and $h(x) = \tan x$ are odd functions. What about $G(x) = \cos^2 x$, $F(x) = \sin^2 x$, and $H(x) = \tan^2 x$? Are they even, odd, or neither? Why?

2. Examine the graph of $f(x) = \sec x$ on the interval $[-\pi, \pi]$. How can we tell whether the function is even or odd by only observing the graph of $f(x) = \sec x$?

3. After examining the reciprocal identity for $\sec t$, explain why the function is undefined at certain points.

4. All of the Pythagorean identities are related. Describe how to manipulate the equations to get from $\sin^2 t + \cos^2 t = 1$ to the other forms.

ALGEBRAIC

For the following exercises, use the fundamental identities to fully simplify the expression.

5. $\sin x \cos x \sec x$

6. $\sin(-x)\cos(-x)\csc(-x)$

7. $\tan x \sin x + \sec x \cos^2 x$

8. $\csc x + \cos x \cot(-x)$

9. $\dfrac{\cot t + \tan t}{\sec(-t)}$

10. $3\sin^3 t \csc t + \cos^2 t + 2\cos(-t)\cos t$

11. $-\tan(-x)\cot(-x)$

12. $\dfrac{-\sin(-x)\cos x \sec x \csc x \tan x}{\cot x}$

13. $\dfrac{1 + \tan^2 \theta}{\csc^2 \theta} + \sin^2 \theta + \dfrac{1}{\sec^2 \theta}$

14. $\left(\dfrac{\tan x}{\csc^2 x} + \dfrac{\tan x}{\sec^2 x}\right)\left(\dfrac{1 + \tan x}{1 + \cot x}\right) - \dfrac{1}{\cos^2 x}$

15. $\dfrac{1 - \cos^2 x}{\tan^2 x} + 2\sin^2 x$

For the following exercises, simplify the first trigonometric expression by writing the simplified form in terms of the second expression.

16. $\dfrac{\tan x + \cot x}{\csc x}$; $\cos x$

17. $\dfrac{\sec x + \csc x}{1 + \tan x}$; $\sin x$

18. $\dfrac{\cos x}{1 + \sin x} + \tan x$; $\cos x$

19. $\dfrac{1}{\sin x \cos x} - \cot x$; $\cot x$

20. $\dfrac{1}{1 - \cos x} - \dfrac{\cos x}{1 + \cos x}$; $\csc x$

21. $(\sec x + \csc x)(\sin x + \cos x) - 2 - \cot x$; $\tan x$

22. $\dfrac{1}{\csc x - \sin x}$; $\sec x$ and $\tan x$

23. $\dfrac{1 - \sin x}{1 + \sin x} - \dfrac{1 + \sin x}{1 - \sin x}$; $\sec x$ and $\tan x$

24. $\tan x$; $\sec x$

25. $\sec x$; $\cot x$

26. $\sec x$; $\sin x$

27. $\cot x$; $\sin x$

28. $\cot x$; $\csc x$

For the following exercises, verify the identity.

29. $\cos x - \cos^3 x = \cos x \sin^2 x$

30. $\cos x(\tan x - \sec(-x)) = \sin x - 1$

31. $\dfrac{1 + \sin^2 x}{\cos^2 x} = \dfrac{1}{\cos^2 x} + \dfrac{\sin^2 x}{\cos^2 x} = 1 + 2\tan^2 x$

32. $(\sin x + \cos x)^2 = 1 + 2\sin x \cos x$

33. $\cos^2 x - \tan^2 x = 2 - \sin^2 x - \sec^2 x$

EXTENSIONS

For the following exercises, prove or disprove the identity.

34. $\dfrac{1}{1+\cos x} - \dfrac{1}{1-\cos(-x)} = -2\cot x \csc x$

35. $\csc^2 x(1+\sin^2 x) = \cot^2 x$

36. $\left(\dfrac{\sec^2(-x)-\tan^2 x}{\tan x}\right)\left(\dfrac{2+2\tan x}{2+2\cot x}\right) - 2\sin^2 x = \cos 2x$

37. $\dfrac{\tan x}{\sec x}\sin(-x) = \cos^2 x$

38. $\dfrac{\sec(-x)}{\tan x + \cot x} = -\sin(-x)$

39. $\dfrac{1+\sin x}{\cos x} = \dfrac{\cos x}{1+\sin(-x)}$

For the following exercises, determine whether the identity is true or false. If false, find an appropriate equivalent expression.

40. $\dfrac{\cos^2\theta - \sin^2\theta}{1-\tan^2\theta} = \sin^2\theta$

41. $3\sin^2\theta + 4\cos^2\theta = 3 + \cos^2\theta$

42. $\dfrac{\sec\theta + \tan\theta}{\cot\theta + \cos\theta} = \sec^2\theta$

LEARNING OBJECTIVES

In this section, you will:

- Use sum and difference formulas for cosine.
- Use sum and difference formulas for sine.
- Use sum and difference formulas for tangent.
- Use sum and difference formulas for cofunctions.
- Use sum and difference formulas to verify identities.

9.2 SUM AND DIFFERENCE IDENTITIES

Figure 1 Mount McKinley, in Denali National Park, Alaska, rises 20,237 feet (6,168 m) above sea level. It is the highest peak in North America. (credit: Daniel A. Leifheit, Flickr)

How can the height of a mountain be measured? What about the distance from Earth to the sun? Like many seemingly impossible problems, we rely on mathematical formulas to find the answers. The trigonometric identities, commonly used in mathematical proofs, have had real-world applications for centuries, including their use in calculating long distances.

The trigonometric identities we will examine in this section can be traced to a Persian astronomer who lived around 950 AD, but the ancient Greeks discovered these same formulas much earlier and stated them in terms of chords. These are special equations or postulates, true for all values input to the equations, and with innumerable applications.

In this section, we will learn techniques that will enable us to solve problems such as the ones presented above. The formulas that follow will simplify many trigonometric expressions and equations. Keep in mind that, throughout this section, the term *formula* is used synonymously with the word *identity*.

Using the Sum and Difference Formulas for Cosine

Finding the exact value of the sine, cosine, or tangent of an angle is often easier if we can rewrite the given angle in terms of two angles that have known trigonometric values. We can use the special angles, which we can review in the unit circle shown in **Figure 2**.

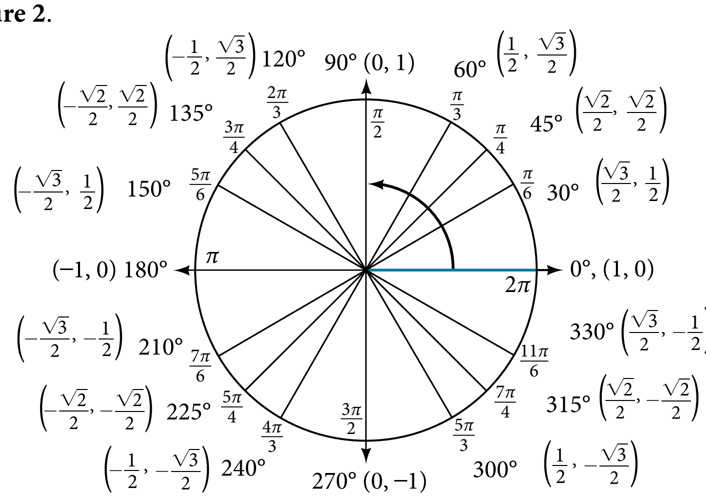


Figure 2 The Unit Circle

We will begin with the sum and difference formulas for cosine, so that we can find the cosine of a given angle if we can break it up into the sum or difference of two of the special angles. See **Table 1**.

Sum formula for cosine	$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$
Difference formula for cosine	$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

Table 1

First, we will prove the difference formula for cosines. Let's consider two points on the unit circle. See **Figure 3**. Point P is at an angle α from the positive x-axis with coordinates $(\cos\alpha, \sin\alpha)$ and point Q is at an angle of β from the positive x-axis with coordinates $(\cos\beta, \sin\beta)$. Note the measure of angle POQ is $\alpha-\beta$.

Label two more points: A at an angle of $(\alpha-\beta)$ from the positive x-axis with coordinates $(\cos(\alpha-\beta), \sin(\alpha-\beta))$; and point B with coordinates $(1, 0)$. Triangle POQ is a rotation of triangle AOB and thus the distance from P to Q is the same as the distance from A to B.

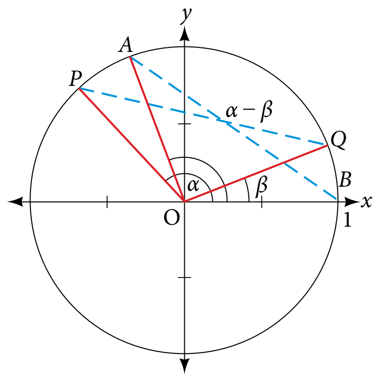


Figure 3

We can find the distance from P to Q using the distance formula.

$$\begin{aligned} d_{PQ} &= \sqrt{(\cos\alpha-\cos\beta)^2+(\sin\alpha-\sin\beta)^2} \\ &= \sqrt{\cos^2\alpha-2\cos\alpha\cos\beta+\cos^2\beta+\sin^2\alpha-2\sin\alpha\sin\beta+\sin^2\beta} \end{aligned}$$

Then we apply the Pythagorean Identity and simplify.

$$\begin{aligned} &= \sqrt{(\cos^2\alpha+\sin^2\alpha)+(\cos^2\beta+\sin^2\beta)-2\cos\alpha\cos\beta-2\sin\alpha\sin\beta} \\ &= \sqrt{1+1-2\cos\alpha\cos\beta-2\sin\alpha\sin\beta} \\ &= \sqrt{2-2\cos\alpha\cos\beta-2\sin\alpha\sin\beta} \end{aligned}$$

Similarly, using the distance formula we can find the distance from A to B.

$$\begin{aligned} d_{AB} &= \sqrt{(\cos(\alpha-\beta)-1)^2+(\sin(\alpha-\beta)-0)^2} \\ &= \sqrt{\cos^2(\alpha-\beta)-2\cos(\alpha-\beta)+1+\sin^2(\alpha-\beta)} \end{aligned}$$

Applying the Pythagorean Identity and simplifying we get:

$$\begin{aligned} &= \sqrt{(\cos^2(\alpha-\beta)+\sin^2(\alpha-\beta))-2\cos(\alpha-\beta)+1} \\ &= \sqrt{1-2\cos(\alpha-\beta)+1} \\ &= \sqrt{2-2\cos(\alpha-\beta)} \end{aligned}$$

Because the two distances are the same, we set them equal to each other and simplify.

$$\sqrt{2-2\cos\alpha\cos\beta-2\sin\alpha\sin\beta}=\sqrt{2-2\cos(\alpha-\beta)}$$
$$2-2\cos\alpha\cos\beta-2\sin\alpha\sin\beta=2-2\cos(\alpha-\beta)$$

Finally we subtract 2 from both sides and divide both sides by -2.

$$\cos\alpha\cos\beta+\sin\alpha\sin\beta=\cos(\alpha-\beta)$$

Thus, we have the difference formula for cosine. We can use similar methods to derive the cosine of the sum of two angles.

sum and difference formulas for cosine

These formulas can be used to calculate the cosine of sums and differences of angles.

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \qquad \cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

How To…

Given two angles, find the cosine of the difference between the angles.

1. Write the difference formula for cosine.
2. Substitute the values of the given angles into the formula.
3. Simplify.

Example 1 Finding the Exact Value Using the Formula for the Cosine of the Difference of Two Angles

Using the formula for the cosine of the difference of two angles, find the exact value of $\cos\left(\frac{5\pi}{4} - \frac{\pi}{6}\right)$.

Solution Begin by writing the formula for the cosine of the difference of two angles. Then substitute the given values.

$$\begin{aligned}
\cos(\alpha - \beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta \\
\cos\left(\frac{5\pi}{4} - \frac{\pi}{6}\right) &= \cos\left(\frac{5\pi}{4}\right)\cos\left(\frac{\pi}{6}\right) + \sin\left(\frac{5\pi}{4}\right)\sin\left(\frac{\pi}{6}\right) \\
&= \left(-\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) \\
&= -\frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} \\
&= \frac{-\sqrt{6} - \sqrt{2}}{4}
\end{aligned}$$

Keep in mind that we can always check the answer using a graphing calculator in radian mode.

Try It #1

Find the exact value of $\cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$.

Example 2 Finding the Exact Value Using the Formula for the Sum of Two Angles for Cosine

Find the exact value of $\cos(75°)$.

Solution As $75° = 45° + 30°$, we can evaluate $\cos(75°)$ as $\cos(45° + 30°)$.

$$\begin{aligned}
\cos(\alpha + \beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\
\cos(45° + 30°) &= \cos(45°)\cos(30°) - \sin(45°)\sin(30°) \\
&= \frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2}\right) - \frac{\sqrt{2}}{2}\left(\frac{1}{2}\right) \\
&= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} \\
&= \frac{\sqrt{6} - \sqrt{2}}{4}
\end{aligned}$$

Keep in mind that we can always check the answer using a graphing calculator in degree mode.

Analysis *Note that we could have also solved this problem using the fact that* $75° = 135° - 60°$.

$$\begin{aligned}
\cos(\alpha - \beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta \\
\cos(135° - 60°) &= \cos(135°)\cos(60°) - \sin(135°)\sin(60°) \\
&= \left(-\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) \\
&= \left(-\frac{\sqrt{2}}{4}\right) + \left(\frac{\sqrt{6}}{4}\right) \\
&= \frac{\sqrt{6} - \sqrt{2}}{4}
\end{aligned}$$

Try It #2

Find the exact value of $\cos(105°)$.

Using the Sum and Difference Formulas for Sine

The sum and difference formulas for sine can be derived in the same manner as those for cosine, and they resemble the cosine formulas.

sum and difference formulas for sine

These formulas can be used to calculate the sines of sums and differences of angles.

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \qquad \sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

How To...

Given two angles, find the sine of the difference between the angles.

1. Write the difference formula for sine.
2. Substitute the given angles into the formula.
3. Simplify.

Example 3 Using Sum and Difference Identities to Evaluate the Difference of Angles

Use the sum and difference identities to evaluate the difference of the angles and show that part *a* equals part *b*.

a. $\sin(45° - 30°)$ **b.** $\sin(135° - 120°)$

Solution

a. Let's begin by writing the formula and substitute the given angles.

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$\sin(45° - 30°) = \sin(45°)\cos(30°) - \cos(45°)\sin(30°)$$

Next, we need to find the values of the trigonometric expressions.

$$\sin(45°) = \frac{\sqrt{2}}{2},\ \cos(30°) = \frac{\sqrt{3}}{2},\ \cos(45°) = \frac{\sqrt{2}}{2},\ \sin(30°) = \frac{1}{2}$$

Now we can substitute these values into the equation and simplify.

$$\begin{aligned}\sin(45° - 30°) &= \frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2}\right) - \frac{\sqrt{2}}{2}\left(\frac{1}{2}\right)\\ &= \frac{\sqrt{6} - \sqrt{2}}{4}\end{aligned}$$

b. Again, we write the formula and substitute the given angles.

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$\sin(135° - 120°) = \sin(135°)\cos(120°) - \cos(135°)\sin(120°)$$

Next, we find the values of the trigonometric expressions.

$$\sin(135°) = \frac{\sqrt{2}}{2},\ \cos(120°) = -\frac{1}{2},\ \cos(135°) = -\frac{\sqrt{2}}{2},\ \sin(120°) = \frac{\sqrt{3}}{2}$$

Now we can substitute these values into the equation and simplify.

$$\begin{aligned}\sin(135° - 120°) &= \frac{\sqrt{2}}{2}\left(-\frac{1}{2}\right) - \left(-\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)\\ &= \frac{-\sqrt{2} + \sqrt{6}}{4}\\ &= \frac{\sqrt{6} - \sqrt{2}}{4}\end{aligned}$$

$$\begin{aligned}\sin(135° - 120°) &= \frac{\sqrt{2}}{2}\left(-\frac{1}{2}\right) - \left(-\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) \\ &= \frac{-\sqrt{2}+\sqrt{6}}{4} \\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

Example 4 **Finding the Exact Value of an Expression Involving an Inverse Trigonometric Function**

Find the exact value of $\sin\left(\cos^{-1}\frac{1}{2} + \sin^{-1}\frac{3}{5}\right)$. Then check the answer with a graphing calculator.

Solution The pattern displayed in this problem is $\sin(\alpha + \beta)$. Let $\alpha = \cos^{-1}\frac{1}{2}$ and $\beta = \sin^{-1}\frac{3}{5}$. Then we can write

$$\cos\alpha = \frac{1}{2},\ 0 \le \alpha \le \pi$$

$$\sin\beta = \frac{3}{5},\ -\frac{\pi}{2} \le \beta \le \frac{\pi}{2}$$

We will use the Pythagorean identities to find $\sin\alpha$ and $\cos\beta$.

$$\begin{aligned}\sin\alpha &= \sqrt{1-\cos^2\alpha} \\ &= \sqrt{1-\frac{1}{4}} \\ &= \sqrt{\frac{3}{4}} \\ &= \frac{\sqrt{3}}{2}\end{aligned} \qquad \begin{aligned}\cos\beta &= \sqrt{1-\sin^2\beta} \\ &= \sqrt{1-\frac{9}{25}} \\ &= \sqrt{\frac{16}{25}} \\ &= \frac{4}{5}\end{aligned}$$

Using the sum formula for sine,

$$\begin{aligned}\sin\left(\cos^{-1}\frac{1}{2} + \sin^{-1}\frac{3}{5}\right) &= \sin(\alpha+\beta) \\ &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ &= \frac{\sqrt{3}}{2}\cdot\frac{4}{5} + \frac{1}{2}\cdot\frac{3}{5} \\ &= \frac{4\sqrt{3}+3}{10}\end{aligned}$$

Using the Sum and Difference Formulas for Tangent

Finding exact values for the tangent of the sum or difference of two angles is a little more complicated, but again, it is a matter of recognizing the pattern.

Finding the sum of two angles formula for tangent involves taking quotient of the sum formulas for sine and cosine and simplifying. Recall, $\tan x = \frac{\sin x}{\cos x}$, $\cos x \neq 0$.

Let's derive the sum formula for tangent.

$$\begin{aligned}\tan(\alpha+\beta) &= \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} \\ &= \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta} \\ &= \frac{\frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\frac{\cos\alpha\cos\beta - \sin\alpha\sin\beta}{\cos\alpha\cos\beta}} \qquad \text{Divide the numerator and denominator by } \cos\alpha\cos\beta \\ &= \frac{\frac{\sin\alpha\cancel{\cos\beta}}{\cos\alpha\cancel{\cos\beta}} + \frac{\cancel{\cos\alpha}\sin\beta}{\cancel{\cos\alpha}\cos\beta}}{\frac{\cancel{\cos\alpha\cos\beta}}{\cancel{\cos\alpha\cos\beta}} - \frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}\end{aligned}$$

$$= \frac{\frac{\sin\alpha}{\cos\alpha} + \frac{\sin\beta}{\cos\beta}}{1 - \frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}$$

$$= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

We can derive the difference formula for tangent in a similar way.

sum and difference formulas for tangent

The sum and difference formulas for tangent are:

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \qquad \tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

How To…

Given two angles, find the tangent of the sum of the angles.

1. Write the sum formula for tangent.
2. Substitute the given angles into the formula.
3. Simplify.

Example 5 Finding the Exact Value of an Expression Involving Tangent

Find the exact value of $\tan\left(\frac{\pi}{6} + \frac{\pi}{4}\right)$.

Solution Let's first write the sum formula for tangent and substitute the given angles into the formula.

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$\tan\left(\frac{\pi}{6} + \frac{\pi}{4}\right) = \frac{\tan\left(\frac{\pi}{6}\right) + \tan\left(\frac{\pi}{4}\right)}{1 - \left(\tan\left(\frac{\pi}{6}\right)\tan\left(\frac{\pi}{4}\right)\right)}$$

Next, we determine the individual tangents within the formulas:

$$\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}} \qquad \tan\left(\frac{\pi}{4}\right) = 1$$

So we have

$$\tan\left(\frac{\pi}{6} + \frac{\pi}{4}\right) = \frac{\frac{1}{\sqrt{3}} + 1}{1 - \left(\frac{1}{\sqrt{3}}\right)(1)}$$

$$= \frac{\frac{1 + \sqrt{3}}{\sqrt{3}}}{\frac{\sqrt{3} - 1}{\sqrt{3}}}$$

$$= \frac{1 + \sqrt{3}}{\sqrt{3}}\left(\frac{\sqrt{3}}{\sqrt{3} - 1}\right)$$

$$= \frac{\sqrt{3} + 1}{\sqrt{3} - 1}$$

Try It #3

Find the exact value of $\tan\left(\frac{2\pi}{3} + \frac{\pi}{4}\right)$.

Example 6 Finding Multiple Sums and Differences of Angles

Given $\sin\alpha = \frac{3}{5}, 0 < \alpha < \frac{\pi}{2}, \cos\beta = -\frac{5}{13}, \pi < \beta < \frac{3\pi}{2}$, find

a. $\sin(\alpha + \beta)$ **b.** $\cos(\alpha + \beta)$ **c.** $\tan(\alpha + \beta)$ **d.** $\tan(\alpha - \beta)$

Solution We can use the sum and difference formulas to identify the sum or difference of angles when the ratio of sine, cosine, or tangent is provided for each of the individual angles. To do so, we construct what is called a reference triangle to help find each component of the sum and difference formulas.

a. To find $\sin(\alpha + \beta)$, we begin with $\sin\alpha = \frac{3}{5}$ and $0 < \alpha < \frac{\pi}{2}$. The side opposite α has length 3, the hypotenuse has length 5, and α is in the first quadrant. See **Figure 4**. Using the Pythagorean Theorem,we can find the length of side a:

$$\begin{aligned} a^2 + 3^2 &= 5^2 \\ a^2 &= 16 \\ a &= 4 \end{aligned}$$

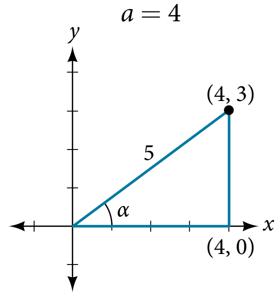


Figure 4

Since $\cos\beta = -\frac{5}{13}$ and $\pi < \beta < \frac{3\pi}{2}$, the side adjacent to β is -5, the hypotenuse is 13, and β is in the third quadrant. See **Figure 5**. Again, using the Pythagorean Theorem, we have

$$\begin{aligned} (-5)^2 + a^2 &= 13^2 \\ 25 + a^2 &= 169 \\ a^2 &= 144 \\ a &= \pm 12 \end{aligned}$$

Since β is in the third quadrant, $a = -12$.

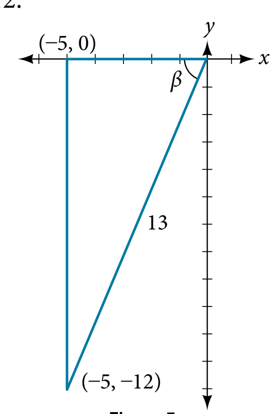


Figure 5

The next step is finding the cosine of α and the sine of β. The cosine of α is the adjacent side over the hypotenuse. We can find it from the triangle in **Figure 5**: $\cos\alpha = \frac{4}{5}$. We can also find the sine of β from the triangle in **Figure 5**, as opposite side over the hypotenuse: $\sin\beta = -\frac{12}{13}$. Now we are ready to evaluate $\sin(\alpha + \beta)$.

$$\begin{aligned}\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ &= \left(\frac{3}{5}\right)\left(-\frac{5}{13}\right) + \left(\frac{4}{5}\right)\left(-\frac{12}{13}\right) \\ &= -\frac{15}{65} - \frac{48}{65} \\ &= -\frac{63}{65}\end{aligned}$$

b. We can find $\cos(\alpha+\beta)$ in a similar manner. We substitute the values according to the formula.

$$\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ &= \left(\frac{4}{5}\right)\left(-\frac{5}{13}\right) - \left(\frac{3}{5}\right)\left(-\frac{12}{13}\right) \\ &= -\frac{20}{65} + \frac{36}{65} \\ &= \frac{16}{65}\end{aligned}$$

c. For $\tan(\alpha+\beta)$, if $\sin\alpha = \frac{3}{5}$ and $\cos\alpha = \frac{4}{5}$, then

$$\tan\alpha = \frac{\frac{3}{5}}{\frac{4}{5}} = \frac{3}{4}$$

If $\sin\beta = -\frac{12}{13}$ and $\cos\beta = -\frac{5}{13}$, then

$$\tan\beta = \frac{\frac{-12}{13}}{\frac{-5}{13}} = \frac{12}{5}$$

Then,

$$\begin{aligned}\tan(\alpha+\beta) &= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \\ &= \frac{\frac{3}{4} + \frac{12}{5}}{1 - \frac{3}{4}\left(\frac{12}{5}\right)} \\ &= \frac{\frac{63}{20}}{-\frac{16}{20}} \\ &= -\frac{63}{16}\end{aligned}$$

d. To find $\tan(\alpha-\beta)$, we have the values we need. We can substitute them in and evaluate.

$$\begin{aligned}\tan(\alpha-\beta) &= \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta} \\ &= \frac{\frac{3}{4} - \frac{12}{5}}{1 + \frac{3}{4}\left(\frac{12}{5}\right)} \\ &= \frac{-\frac{33}{20}}{\frac{56}{20}} \\ &= -\frac{33}{56}\end{aligned}$$

Analysis *A common mistake when addressing problems such as this one is that we may be tempted to think that α and β are angles in the same triangle, which of course, they are not. Also note that*

$$\tan(\alpha + \beta) = \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)}$$

Using Sum and Difference Formulas for Cofunctions

Now that we can find the sine, cosine, and tangent functions for the sums and differences of angles, we can use them to do the same for their cofunctions. You may recall from **Right Triangle Trigonometry** that, if the sum of two positive angles is $\frac{\pi}{2}$, those two angles are complements, and the sum of the two acute angles in a right triangle is $\frac{\pi}{2}$, so they are also complements. In **Figure 6**, notice that if one of the acute angles is labeled as θ, then the other acute angle must be labeled $\left(\frac{\pi}{2} - \theta\right)$.

Notice also that $\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$: opposite over hypotenuse. Thus, when two angles are complimentary, we can say that the sine of θ equals the cofunction of the complement of θ. Similarly, tangent and cotangent are cofunctions, and secant and cosecant are cofunctions.

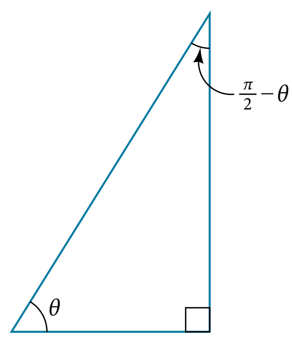


Figure 6

From these relationships, the cofunction identities are formed. Recall that you first encountered these identities in **The Unit Circle: Sine and Cosine Functions**.

cofunction identities

The cofunction identities are summarized in **Table 2**.

$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$	$\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$	$\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right)$
$\sec\theta = \csc\left(\frac{\pi}{2} - \theta\right)$	$\csc\theta = \sec\left(\frac{\pi}{2} - \theta\right)$	$\cot\theta = \tan\left(\frac{\pi}{2} - \theta\right)$

Table 2

Notice that the formulas in the table may also be justified algebraically using the sum and difference formulas. For example, using

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta,$$

we can write

$$\begin{aligned}\cos\left(\frac{\pi}{2} - \theta\right) &= \cos\frac{\pi}{2}\cos\theta + \sin\frac{\pi}{2}\sin\theta \\ &= (0)\cos\theta + (1)\sin\theta \\ &= \sin\theta\end{aligned}$$

Example 7 **Finding a Cofunction with the Same Value as the Given Expression**

Write $\tan \frac{\pi}{9}$ in terms of its cofunction.

Solution The cofunction of $\tan \theta = \cot\left(\frac{\pi}{2} - \theta\right)$. Thus,

$$\begin{aligned} \tan\left(\frac{\pi}{9}\right) &= \cot\left(\frac{\pi}{2} - \frac{\pi}{9}\right) \\ &= \cot\left(\frac{9\pi}{18} - \frac{2\pi}{18}\right) \\ &= \cot\left(\frac{7\pi}{18}\right) \end{aligned}$$

Try It #4

Write $\sin \frac{\pi}{7}$ in terms of its cofunction.

Using the Sum and Difference Formulas to Verify Identities

Verifying an identity means demonstrating that the equation holds for all values of the variable. It helps to be very familiar with the identities or to have a list of them accessible while working the problems. Reviewing the general rules presented earlier may help simplify the process of verifying an identity.

How To…

Given an identity, verify using sum and difference formulas.

1. Begin with the expression on the side of the equal sign that appears most complex. Rewrite that expression until it matches the other side of the equal sign. Occasionally, we might have to alter both sides, but working on only one side is the most efficient.
2. Look for opportunities to use the sum and difference formulas.
3. Rewrite sums or differences of quotients as single quotients.
4. If the process becomes cumbersome, rewrite the expression in terms of sines and cosines.

Example 8 **Verifying an Identity Involving Sine**

Verify the identity $\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2 \sin \alpha \cos \beta$.

Solution We see that the left side of the equation includes the sines of the sum and the difference of angles.

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

We can rewrite each using the sum and difference formulas.

$$\begin{aligned} \sin(\alpha + \beta) + \sin(\alpha - \beta) &= \sin \alpha \cos \beta + \cos \alpha \sin \beta + \sin \alpha \cos \beta - \cos \alpha \sin \beta \\ &= 2 \sin \alpha \cos \beta \end{aligned}$$

We see that the identity is verified.

Example 9 Verifying an Identity Involving Tangent

Verify the following identity.

$$\frac{\sin(\alpha - \beta)}{\cos\alpha\cos\beta} = \tan\alpha - \tan\beta$$

Solution We can begin by rewriting the numerator on the left side of the equation.

$$\begin{aligned}
\frac{\sin(\alpha - \beta)}{\cos\alpha\cos\beta} &= \frac{\sin\alpha\cos\beta - \cos\alpha\sin\beta}{\cos\alpha\cos\beta} \\
&= \frac{\sin\alpha\cancel{\cos\beta}}{\cos\alpha\cancel{\cos\beta}} - \frac{\cancel{\cos\alpha}\sin\beta}{\cancel{\cos\alpha}\cos\beta} && \text{Rewrite using a common denominator.} \\
&= \frac{\sin\alpha}{\cos\alpha} - \frac{\sin\beta}{\cos\beta} && \text{Cancel.} \\
&= \tan\alpha - \tan\beta && \text{Rewrite in terms of tangent.}
\end{aligned}$$

We see that the identity is verified. In many cases, verifying tangent identities can successfully be accomplished by writing the tangent in terms of sine and cosine.

Try It #5

Verify the identity: $\tan(\pi - \theta) = -\tan\theta$.

Example 10 Using Sum and Difference Formulas to Solve an Application Problem

Let L_1 and L_2 denote two non-vertical intersecting lines, and let θ denote the acute angle between L_1 and L_2. See **Figure 7.** Show that

$$\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2}$$

where m_1 and m_2 are the slopes of L_1 and L_2 respectively. (**Hint:** Use the fact that $\tan\theta_1 = m_1$ and $\tan\theta_2 = m_2$.)

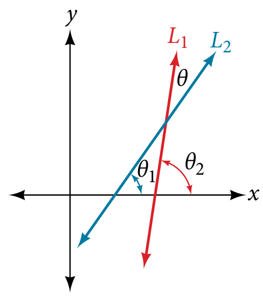


Figure 7

Solution Using the difference formula for tangent, this problem does not seem as daunting as it might.

$$\begin{aligned}
\tan\theta &= \tan(\theta_2 - \theta_1) \\
&= \frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_1\tan\theta_2} \\
&= \frac{m_2 - m_1}{1 + m_1 m_2}
\end{aligned}$$

Example 11 Investigating a Guy-wire Problem

For a climbing wall, a guy-wire R is attached 47 feet high on a vertical pole. Added support is provided by another guy-wire S attached 40 feet above ground on the same pole. If the wires are attached to the ground 50 feet from the pole, find the angle α between the wires. See **Figure 8**.

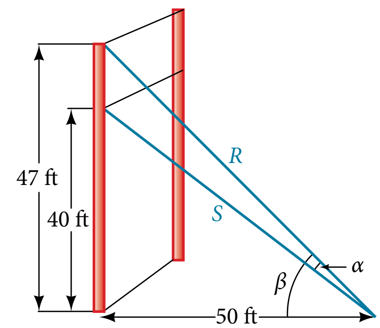


Figure 8

Solution Let's first summarize the information we can gather from the diagram. As only the sides adjacent to the right angle are known, we can use the tangent function. Notice that $\tan \beta = \frac{47}{50}$, and $\tan(\beta - \alpha) = \frac{40}{50} = \frac{4}{5}$. We can then use difference formula for tangent.

$$\tan(\beta - \alpha) = \frac{\tan \beta - \tan \alpha}{1 + \tan \beta \tan \alpha}$$

Now, substituting the values we know into the formula, we have

$$\frac{4}{5} = \frac{\frac{47}{50} - \tan \alpha}{1 + \frac{47}{50}\tan \alpha}$$

$$4\left(1 + \frac{47}{50}\tan \alpha\right) = 5\left(\frac{47}{50} - \tan \alpha\right)$$

Use the distributive property, and then simplify the functions.

$$\begin{aligned} 4(1) + 4\left(\frac{47}{50}\right)\tan \alpha &= 5\left(\frac{47}{50}\right) - 5\tan \alpha \\ 4 + 3.76\tan \alpha &= 4.7 - 5\tan \alpha \\ 5\tan \alpha + 3.76\tan \alpha &= 0.7 \\ 8.76\tan \alpha &= 0.7 \\ \tan \alpha &\approx 0.07991 \\ \tan^{-1}(0.07991) &\approx .079741 \end{aligned}$$

Now we can calculate the angle in degrees.

$$\alpha \approx 0.079741\left(\frac{180}{\pi}\right) \approx 4.57°$$

Analysis *Occasionally, when an application appears that includes a right triangle, we may think that solving is a matter of applying the Pythagorean Theorem. That may be partially true, but it depends on what the problem is asking and what information is given.*

Access these online resources for additional instruction and practice with sum and difference identities.

- Sum and Difference Identities for Cosine (http://openstaxcollege.org/l/sumdifcos)
- Sum and Difference Identities for Sine (http://openstaxcollege.org/l/sumdifsin)
- Sum and Difference Identities for Tangent (http://openstaxcollege.org/l/sumdiftan)

9.2 SECTION EXERCISES

VERBAL

1. Explain the basis for the cofunction identities and when they apply.

2. Is there only one way to evaluate $\cos\left(\frac{5\pi}{4}\right)$? Explain how to set up the solution in two different ways, and then compute to make sure they give the same answer.

3. Explain to someone who has forgotten the even-odd properties of sinusoidal functions how the addition and subtraction formulas can determine this characteristic for $f(x) = \sin(x)$ and $g(x) = \cos(x)$. (Hint: $0 - x = -x$)

ALGEBRAIC

For the following exercises, find the exact value.

4. $\cos\left(\frac{7\pi}{12}\right)$

5. $\cos\left(\frac{\pi}{12}\right)$

6. $\sin\left(\frac{5\pi}{12}\right)$

7. $\sin\left(\frac{11\pi}{12}\right)$

8. $\tan\left(-\frac{\pi}{12}\right)$

9. $\tan\left(\frac{19\pi}{12}\right)$

For the following exercises, rewrite in terms of $\sin x$ and $\cos x$.

10. $\sin\left(x + \frac{11\pi}{6}\right)$

11. $\sin\left(x - \frac{3\pi}{4}\right)$

12. $\cos\left(x - \frac{5\pi}{6}\right)$

13. $\cos\left(x + \frac{2\pi}{3}\right)$

For the following exercises, simplify the given expression.

14. $\csc\left(\frac{\pi}{2} - t\right)$

15. $\sec\left(\frac{\pi}{2} - \theta\right)$

16. $\cot\left(\frac{\pi}{2} - x\right)$

17. $\tan\left(\frac{\pi}{2} - x\right)$

18. $\sin(2x)\cos(5x) - \sin(5x)\cos(2x)$

19. $\dfrac{\tan\left(\frac{3}{2}x\right) - \tan\left(\frac{7}{5}x\right)}{1 + \tan\left(\frac{3}{2}x\right)\tan\left(\frac{7}{5}x\right)}$

For the following exercises, find the requested information.

20. Given that $\sin a = \frac{2}{3}$ and $\cos b = -\frac{1}{4}$, with a and b both in the interval $\left[\frac{\pi}{2}, \pi\right)$, find $\sin(a + b)$ and $\cos(a - b)$.

21. Given that $\sin a = \frac{4}{5}$, and $\cos b = \frac{1}{3}$, with a and b both in the interval $\left[0, \frac{\pi}{2}\right)$, find $\sin(a - b)$ and $\cos(a + b)$.

For the following exercises, find the exact value of each expression.

22. $\sin\left(\cos^{-1}(0) - \cos^{-1}\left(\frac{1}{2}\right)\right)$

23. $\cos\left(\cos^{-1}\left(\frac{\sqrt{2}}{2}\right) + \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)$

24. $\tan\left(\sin^{-1}\left(\frac{1}{2}\right) - \cos^{-1}\left(\frac{1}{2}\right)\right)$

GRAPHICAL

For the following exercises, simplify the expression, and then graph both expressions as functions to verify the graphs are identical. Confirm your answer using a graphing calculator.

25. $\cos\left(\frac{\pi}{2}-x\right)$

26. $\sin(\pi-x)$

27. $\tan\left(\frac{\pi}{3}+x\right)$

28. $\sin\left(\frac{\pi}{3}+x\right)$

29. $\tan\left(\frac{\pi}{4}-x\right)$

30. $\cos\left(\frac{7\pi}{6}+x\right)$

31. $\sin\left(\frac{\pi}{4}+x\right)$

32. $\cos\left(\frac{5\pi}{4}+x\right)$

For the following exercises, use a graph to determine whether the functions are the same or different. If they are the same, show why. If they are different, replace the second function with one that is identical to the first. (Hint: think $2x = x + x$.)

33. $f(x)=\sin(4x)-\sin(3x)\cos x$, $g(x)=\sin x\cos(3x)$

34. $f(x)=\cos(4x)+\sin x\sin(3x)$, $g(x)=-\cos x\cos(3x)$

35. $f(x)=\sin(3x)\cos(6x)$, $g(x)=-\sin(3x)\cos(6x)$

36. $f(x)=\sin(4x)$, $g(x)=\sin(5x)\cos x-\cos(5x)\sin x$

37. $f(x)=\sin(2x)$, $\mathrm{g}(x)=2\sin x\cos x$

38. $f(\theta)=\cos(2\theta)$, $g(\theta)=\cos^2\theta-\sin^2\theta$

39. $f(\theta)=\tan(2\theta)$, $g(\theta)=\dfrac{\tan\theta}{1+\tan^2\theta}$

40. $f(x)=\sin(3x)\sin x$,
$g(x)=\sin^2(2x)\cos^2 x-\cos^2(2x)\sin^2 x$

41. $f(x)=\tan(-x)$, $g(x)=\dfrac{\tan x-\tan(2x)}{1-\tan x\tan(2x)}$

TECHNOLOGY

For the following exercises, find the exact value algebraically, and then confirm the answer with a calculator to the fourth decimal point.

42. $\sin(75°)$

43. $\sin(195°)$

44. $\cos(165°)$

45. $\cos(345°)$

46. $\tan(-15°)$

EXTENSIONS

For the following exercises, prove the identities provided.

47. $\tan\left(x+\frac{\pi}{4}\right)=\dfrac{\tan x+1}{1-\tan x}$

48. $\dfrac{\tan(a+b)}{\tan(a-b)}=\dfrac{\sin a\cos a+\sin b\cos b}{\sin a\cos a-\sin b\cos b}$

49. $\dfrac{\cos(a+b)}{\cos a\cos b}=1-\tan a\tan b$

50. $\cos(x+y)\cos(x-y)=\cos^2 x-\sin^2 y$

51. $\dfrac{\cos(x+h)-\cos x}{h}=\cos x\dfrac{\cos h-1}{h}-\sin x\dfrac{\sin h}{h}$

For the following exercises, prove or disprove the statements.

52. $\tan(u+v)=\dfrac{\tan u+\tan v}{1-\tan u\tan v}$

53. $\tan(u-v)=\dfrac{\tan u-\tan v}{1+\tan u\tan v}$

54. $\dfrac{\tan(x+y)}{1+\tan x\tan x}=\dfrac{\tan x+\tan y}{1-\tan^2 x\tan^2 y}$

55. If α, β, and γ are angles in the same triangle, then prove or disprove $\sin(\alpha+\beta)=\sin\gamma$.

56. If α, β, and γ are angles in the same triangle, then prove or disprove:
$\tan\alpha+\tan\beta+\tan\gamma=\tan\alpha\tan\beta\tan\gamma$.

LEARNING OBJECTIVES

In this section, you will:

- Use double-angle formulas to find exact values.
- Use double-angle formulas to verify identities.
- Use reduction formulas to simplify an expression.
- Use half-angle formulas to find exact values.

9.3 DOUBLE-ANGLE, HALF-ANGLE, AND REDUCTION FORMULAS

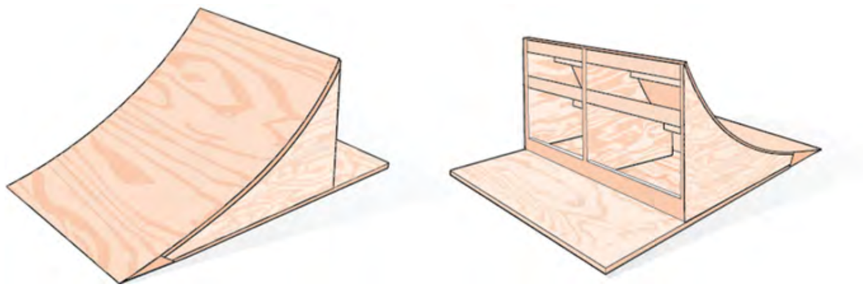


Figure 1 Bicycle ramps for advanced riders have a steeper incline than those designed for novices.

Bicycle ramps made for competition (see **Figure 1**) must vary in height depending on the skill level of the competitors. For advanced competitors, the angle formed by the ramp and the ground should be θ such that $\tan\theta = \frac{5}{3}$. The angle is divided in half for novices. What is the steepness of the ramp for novices? In this section, we will investigate three additional categories of identities that we can use to answer questions such as this one.

Using Double-Angle Formulas to Find Exact Values

In the previous section, we used addition and subtraction formulas for trigonometric functions. Now, we take another look at those same formulas. The double-angle formulas are a special case of the sum formulas, where $\alpha = \beta$. Deriving the double-angle formula for sine begins with the sum formula,

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

If we let $\alpha = \beta = \theta$, then we have

$$\begin{aligned}\sin(\theta + \theta) &= \sin\theta\cos\theta + \cos\theta\sin\theta \\ \sin(2\theta) &= 2\sin\theta\cos\theta\end{aligned}$$

Deriving the double-angle for cosine gives us three options. First, starting from the sum formula, $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$, and letting $\alpha = \beta = \theta$, we have

$$\begin{aligned}\cos(\theta + \theta) &= \cos\theta\cos\theta - \sin\theta\sin\theta \\ \cos(2\theta) &= \cos^2\theta - \sin^2\theta\end{aligned}$$

Using the Pythagorean properties, we can expand this double-angle formula for cosine and get two more interpretations. The first one is:

$$\begin{aligned}\cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ &= (1 - \sin^2\theta) - \sin^2\theta \\ &= 1 - 2\sin^2\theta\end{aligned}$$

The second variation is:

$$\begin{aligned}\cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ &= \cos^2\theta - (1 - \cos^2\theta) \\ &= 2\cos^2\theta - 1\end{aligned}$$

Similarly, to derive the double-angle formula for tangent, replacing $\alpha = \beta = \theta$ in the sum formula gives

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$\tan(\theta + \theta) = \frac{\tan\theta + \tan\theta}{1 - \tan\theta\tan\theta}$$

$$\tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta}$$

double-angle formulas

The **double-angle formulas** are summarized as follows:

$$\begin{aligned}\sin(2\theta) &= 2\sin\theta\cos\theta \\ \cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ &= 1 - 2\sin^2\theta \\ &= 2\cos^2\theta - 1 \\ \tan(2\theta) &= \frac{2\tan\theta}{1 - \tan^2\theta}\end{aligned}$$

How To...

Given the tangent of an angle and the quadrant in which it is located, use the double-angle formulas to find the exact value.

1. Draw a triangle to reflect the given information.
2. Determine the correct double-angle formula.
3. Substitute values into the formula based on the triangle.
4. Simplify.

Example 1 Using a Double-Angle Formula to Find the Exact Value Involving Tangent

Given that $\tan\theta = -\frac{3}{4}$ and θ is in quadrant II, find the following:

a. $\sin(2\theta)$ **b.** $\cos(2\theta)$ **c.** $\tan(2\theta)$

Solution If we draw a triangle to reflect the information given, we can find the values needed to solve the problems on the image. We are given $\tan\theta = -\frac{3}{4}$, such that θ is in quadrant II. The tangent of an angle is equal to the opposite side over the adjacent side, and because θ is in the second quadrant, the adjacent side is on the x-axis and is negative. Use the Pythagorean Theorem to find the length of the hypotenuse:

$$\begin{aligned}(-4)^2 + (3)^2 &= c^2 \\ 16 + 9 &= c^2 \\ 25 &= c^2 \\ c &= 5\end{aligned}$$

Now we can draw a triangle similar to the one shown in **Figure 2**.

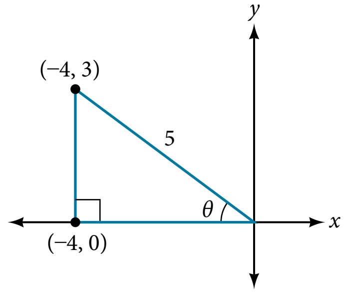


Figure 2

a. Let's begin by writing the double-angle formula for sine.

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

We see that we to need to find $\sin\theta$ and $\cos\theta$. Based on **Figure 2**, we see that the hypotenuse equals 5, so $\sin\theta = \frac{3}{5}$, and $\cos\theta = -\frac{4}{5}$. Substitute these values into the equation, and simplify.

Thus,

$$\begin{aligned}\sin(2\theta) &= 2\left(\frac{3}{5}\right)\left(-\frac{4}{5}\right)\\ &= -\frac{24}{25}\end{aligned}$$

b. Write the double-angle formula for cosine.

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

Again, substitute the values of the sine and cosine into the equation, and simplify.

$$\begin{aligned}\cos(2\theta) &= \left(-\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2\\ &= \frac{16}{25} - \frac{9}{25}\\ &= \frac{7}{25}\end{aligned}$$

c. Write the double-angle formula for tangent.

$$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$$

In this formula, we need the tangent, which we were given as $\tan\theta = -\frac{3}{4}$. Substitute this value into the equation, and simplify.

$$\begin{aligned}\tan(2\theta) &= \frac{2\left(-\frac{3}{4}\right)}{1-\left(-\frac{3}{4}\right)^2}\\ &= \frac{-\frac{3}{2}}{1-\frac{9}{16}}\\ &= -\frac{3}{2}\left(\frac{16}{7}\right)\\ &= -\frac{24}{7}\end{aligned}$$

Try It #1

Given $\sin\alpha = \frac{5}{8}$, with θ in quadrant I, find $\cos(2\alpha)$.

Example 2 Using the Double-Angle Formula for Cosine without Exact Values

Use the double-angle formula for cosine to write $\cos(6x)$ in terms of $\cos(3x)$.

Solution

$$\begin{aligned}\cos(6x) &= \cos(2(3x))\\ &= \cos^2 3x - \sin^2 3x\\ &= 2\cos^2 3x - 1\end{aligned}$$

Analysis *This example illustrates that we can use the double-angle formula without having exact values. It emphasizes that the pattern is what we need to remember and that identities are true for all values in the domain of the trigonometric function.*

Using Double-Angle Formulas to Verify Identities

Establishing identities using the double-angle formulas is performed using the same steps we used to derive the sum and difference formulas. Choose the more complicated side of the equation and rewrite it until it matches the other side.

Example 3 Using the Double-Angle Formulas to Verify an Identity

Verify the following identity using double-angle formulas: $1 + \sin(2\theta) = (\sin\theta + \cos\theta)^2$

Solution We will work on the right side of the equal sign and rewrite the expression until it matches the left side.

$$\begin{aligned}(\sin\theta+\cos\theta)^2 &= \sin^2\theta+2\sin\theta\cos\theta+\cos^2\theta\\ &= (\sin^2\theta+\cos^2\theta)+2\sin\theta\cos\theta\\ &= 1+2\sin\theta\cos\theta\\ &= 1+\sin(2\theta)\end{aligned}$$

Analysis *This process is not complicated, as long as we recall the perfect square formula from algebra:*

$$(a\pm b)^2=a^2\pm 2ab+b^2$$

where $a=\sin\theta$ and $b=\cos\theta$. Part of being successful in mathematics is the ability to recognize patterns. While the terms or symbols may change, the algebra remains consistent.

Try It #2

Verify the identity: $\cos^4\theta-\sin^4\theta=\cos(2\theta)$.

Example 4 Verifying a Double-Angle Identity for Tangent

Verify the identity:

$$\tan(2\theta)=\frac{2}{\cot\theta-\tan\theta}$$

Solution In this case, we will work with the left side of the equation and simplify or rewrite until it equals the right side of the equation.

$$\begin{aligned}\tan(2\theta) &= \frac{2\tan\theta}{1-\tan^2\theta} && \text{Double-angle formula}\\ &= \frac{2\tan\theta\left(\frac{1}{\tan\theta}\right)}{(1-\tan^2\theta)\left(\frac{1}{\tan\theta}\right)} && \text{Multiply by a term that results in desired numerator.}\\ &= \frac{2}{\frac{1}{\tan\theta}-\frac{\tan^2\theta}{\tan\theta}}\\ &= \frac{2}{\cot\theta-\tan\theta} && \text{Use reciprocal identity for } \frac{1}{\tan\theta}.\end{aligned}$$

Analysis *Here is a case where the more complicated side of the initial equation appeared on the right, but we chose to work the left side. However, if we had chosen the left side to rewrite, we would have been working backwards to arrive at the equivalency. For example, suppose that we wanted to show*

$$\frac{2\tan\theta}{1-\tan^2\theta}=\frac{2}{\cot\theta-\tan\theta}$$

Let's work on the right side.

$$\begin{aligned}\frac{2}{\cot\theta-\tan\theta} &= \frac{2}{\frac{1}{\tan\theta}-\tan\theta}\left(\frac{\tan\theta}{\tan\theta}\right)\\ &= \frac{2\tan\theta}{\frac{1}{\cancel{\tan\theta}}(\cancel{\tan\theta})-\tan\theta(\tan\theta)}\\ &= \frac{2\tan\theta}{1-\tan^2\theta}\end{aligned}$$

When using the identities to simplify a trigonometric expression or solve a trigonometric equation, there are usually several paths to a desired result. There is no set rule as to what side should be manipulated. However, we should begin with the guidelines set forth earlier.

Try It #3

Verify the identity: $\cos(2\theta)\cos\theta=\cos^3\theta-\cos\theta\sin^2\theta$.

Use Reduction Formulas to Simplify an Expression

The double-angle formulas can be used to derive the reduction formulas, which are formulas we can use to reduce the power of a given expression involving even powers of sine or cosine. They allow us to rewrite the even powers of sine or cosine in terms of the first power of cosine. These formulas are especially important in higher-level math courses, calculus in particular. Also called the power-reducing formulas, three identities are included and are easily derived from the double-angle formulas.

We can use two of the three double-angle formulas for cosine to derive the reduction formulas for sine and cosine. Let's begin with $\cos(2\theta) = 1 - 2\sin^2\theta$. Solve for $\sin^2\theta$:

$$\begin{aligned}\cos(2\theta) &= 1 - 2\sin^2\theta\\ 2\sin^2\theta &= 1 - \cos(2\theta)\\ \sin^2\theta &= \frac{1 - \cos(2\theta)}{2}\end{aligned}$$

Next, we use the formula $\cos(2\theta) = 2\cos^2\theta - 1$. Solve for $\cos^2\theta$:

$$\begin{aligned}\cos(2\theta) &= 2\cos^2\theta - 1\\ 1 + \cos(2\theta) &= 2\cos^2\theta\\ \frac{1 + \cos(2\theta)}{2} &= \cos^2\theta\end{aligned}$$

The last reduction formula is derived by writing tangent in terms of sine and cosine:

$$\begin{aligned}\tan^2\theta &= \frac{\sin^2\theta}{\cos^2\theta}\\ &= \frac{\frac{1 - \cos(2\theta)}{2}}{\frac{1 + \cos(2\theta)}{2}} && \text{Substitute the reduction formulas.}\\ &= \left(\frac{1 - \cos(2\theta)}{2}\right)\left(\frac{2}{1 + \cos(2\theta)}\right)\\ &= \frac{1 - \cos(2\theta)}{1 + \cos(2\theta)}\end{aligned}$$

reduction formulas

The **reduction formulas** are summarized as follows:

$$\sin^2\theta = \frac{1 - \cos(2\theta)}{2} \qquad \cos^2\theta = \frac{1 + \cos(2\theta)}{2} \qquad \tan^2\theta = \frac{1 - \cos(2\theta)}{1 + \cos(2\theta)}$$

Example 5 Writing an Equivalent Expression Not Containing Powers Greater Than 1

Write an equivalent expression for $\cos^4 x$ that does not involve any powers of sine or cosine greater than 1.

Solution We will apply the reduction formula for cosine twice.

$$\begin{aligned}\cos^4 x &= (\cos^2 x)^2\\ &= \left(\frac{1 + \cos(2x)}{2}\right)^2 && \text{Substitute reduction formula for } \cos^2 x.\\ &= \frac{1}{4}(1 + 2\cos(2x) + \cos^2(2x))\\ &= \frac{1}{4} + \frac{1}{2}\cos(2x) + \frac{1}{4}\left(\frac{1 + \cos2(2x)}{2}\right) && \text{Substitute reduction formula for } \cos^2 x.\\ &= \frac{1}{4} + \frac{1}{2}\cos(2x) + \frac{1}{8} + \frac{1}{8}\cos(4x)\\ &= \frac{3}{8} + \frac{1}{2}\cos(2x) + \frac{1}{8}\cos(4x)\end{aligned}$$

Analysis *The solution is found by using the reduction formula twice, as noted, and the perfect square formula from algebra.*

Example 6 Using the Power-Reducing Formulas to Prove an Identity

Use the power-reducing formulas to prove

$$\sin^3(2x) = \left[\frac{1}{2}\sin(2x)\right][1 - \cos(4x)]$$

Solution We will work on simplifying the left side of the equation:

$$\begin{aligned} \sin^3(2x) &= [\sin(2x)][\sin^2(2x)] \\ &= \sin(2x)\left[\frac{1 - \cos(4x)}{2}\right] && \text{Substitute the power-reduction formula.} \\ &= \sin(2x)\left(\frac{1}{2}\right)[1 - \cos(4x)] \\ &= \frac{1}{2}[\sin(2x)][1 - \cos(4x)] \end{aligned}$$

Analysis *Note that in this example, we substituted*

$$\frac{1 - \cos(4x)}{2}$$

for $\sin^2(2x)$. *The formula states*

$$\sin^2\theta = \frac{1 - \cos(2\theta)}{2}$$

We let $\theta = 2x$, *so* $2\theta = 4x$.

Try It #4

Use the power-reducing formulas to prove that $10\cos^4 x = \frac{15}{4} + 5\cos(2x) + \frac{5}{4}\cos(4x)$.

Using Half-Angle Formulas to Find Exact Values

The next set of identities is the set of **half-angle formulas**, which can be derived from the reduction formulas and we can use when we have an angle that is half the size of a special angle. If we replace θ with $\frac{\alpha}{2}$, the half-angle formula for sine is found by simplifying the equation and solving for $\sin\left(\frac{\alpha}{2}\right)$. Note that the half-angle formulas are preceded by a $\pm$ sign.

This does not mean that both the positive and negative expressions are valid. Rather, it depends on the quadrant in which $\frac{\alpha}{2}$ terminates.

The half-angle formula for sine is derived as follows:

$$\begin{aligned} \sin^2\theta &= \frac{1 - \cos(2\theta)}{2} \\ \sin^2\left(\frac{\alpha}{2}\right) &= \frac{1 - \cos\left(2\cdot\frac{\alpha}{2}\right)}{2} \\ &= \frac{1 - \cos\alpha}{2} \\ \sin\left(\frac{\alpha}{2}\right) &= \pm\sqrt{\frac{1 - \cos\alpha}{2}} \end{aligned}$$

To derive the half-angle formula for cosine, we have

$$\begin{aligned} \cos^2\theta &= \frac{1 + \cos(2\theta)}{2} \\ \cos^2\left(\frac{\alpha}{2}\right) &= \frac{1 + \cos\left(2\cdot\frac{\alpha}{2}\right)}{2} \\ &= \frac{1 + \cos\alpha}{2} \\ \cos\left(\frac{\alpha}{2}\right) &= \pm\sqrt{\frac{1 + \cos\alpha}{2}} \end{aligned}$$

For the tangent identity, we have

$$\tan^2\theta = \frac{1-\cos(2\theta)}{1+\cos(2\theta)}$$

$$\tan^2\left(\frac{\alpha}{2}\right) = \frac{1-\cos\left(2\cdot\frac{\alpha}{2}\right)}{1+\cos\left(2\cdot\frac{\alpha}{2}\right)}$$

$$= \frac{1-\cos\alpha}{1+\cos\alpha}$$

$$\tan\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$$

half-angle formulas

The **half-angle formulas** are as follows:

$$\sin\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos\alpha}{2}}$$

$$\cos\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1+\cos\alpha}{2}}$$

$$\tan\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$$

$$= \frac{\sin\alpha}{1+\cos\alpha}$$

$$= \frac{1-\cos\alpha}{\sin\alpha}$$

Example 7 Using a Half-Angle Formula to Find the Exact Value of a Sine Function

Find $\sin(15°)$ using a half-angle formula.

Solution Since $15° = \frac{30°}{2}$, we use the half-angle formula for sine:

$$\sin\frac{30°}{2} = \sqrt{\frac{1-\cos 30°}{2}}$$

$$= \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}}$$

$$= \sqrt{\frac{\frac{2-\sqrt{3}}{2}}{2}}$$

$$= \sqrt{\frac{2-\sqrt{3}}{4}}$$

$$= \frac{\sqrt{2-\sqrt{3}}}{2}$$

Remember that we can check the answer with a graphing calculator.

Analysis *Notice that we used only the positive root because* $\sin(15°)$ *is positive.*

How To…

Given the tangent of an angle and the quadrant in which the angle lies, find the exact values of trigonometric functions of half of the angle.

1. Draw a triangle to represent the given information.
2. Determine the correct half-angle formula.
3. Substitute values into the formula based on the triangle.
4. Simplify.

Example 8 Finding Exact Values Using Half-Angle Identities

Given that $\tan \alpha = \frac{8}{15}$ and α lies in quadrant III, find the exact value of the following:

a. $\sin\left(\frac{\alpha}{2}\right)$ **b.** $\cos\left(\frac{\alpha}{2}\right)$ **c.** $\tan\left(\frac{\alpha}{2}\right)$

Solution Using the given information, we can draw the triangle shown in **Figure 3**. Using the Pythagorean Theorem, we find the hypotenuse to be 17. Therefore, we can calculate $\sin \alpha = -\frac{8}{17}$ and $\cos \alpha = -\frac{15}{17}$.

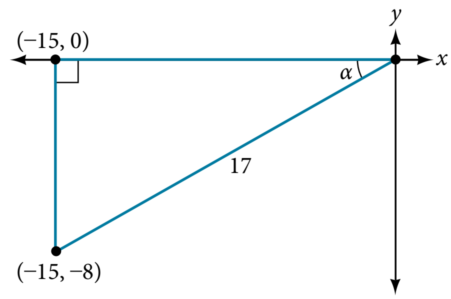


Figure 3

a. Before we start, we must remember that, if α is in quadrant III, then $180° < \alpha < 270°$, so $\frac{180°}{2} < \frac{\alpha}{2} < \frac{270°}{2}$. This means that the terminal side of $\frac{\alpha}{2}$ is in quadrant II, since $90° < \frac{\alpha}{2} < 135°$. To find $\sin \frac{\alpha}{2}$, we begin by writing the half-angle formula for sine. Then we substitute the value of the cosine we found from the triangle in **Figure 3** and simplify.

$$\begin{aligned}
\sin \frac{\alpha}{2} &= \pm \sqrt{\frac{1 - \cos \alpha}{2}} \\
&= \pm \sqrt{\frac{1 - \left(-\frac{15}{17}\right)}{2}} \\
&= \pm \sqrt{\frac{\frac{32}{17}}{2}} \\
&= \pm \sqrt{\frac{32}{17} \cdot \frac{1}{2}} \\
&= \pm \sqrt{\frac{16}{17}} \\
&= \pm \frac{4}{\sqrt{17}} \\
&= \frac{4\sqrt{17}}{17}
\end{aligned}$$

We choose the positive value of $\sin \frac{\alpha}{2}$ because the angle terminates in quadrant II and sine is positive in quadrant II.

b. To find $\cos \frac{\alpha}{2}$, we will write the half-angle formula for cosine, substitute the value of the cosine we found from the triangle in **Figure 3**, and simplify.

$$\begin{aligned}
\cos\left(\frac{\alpha}{2}\right) &= \pm\sqrt{\frac{1+\cos\alpha}{2}} \\
&= \pm\sqrt{\frac{1+\left(-\frac{15}{17}\right)}{2}} \\
&= \pm\sqrt{\frac{\frac{2}{17}}{2}} \\
&= \pm\sqrt{\frac{2}{17}\cdot\frac{1}{2}} \\
&= \pm\sqrt{\frac{1}{17}} \\
&= -\frac{\sqrt{17}}{17}
\end{aligned}$$

We choose the negative value of $\cos\frac{\alpha}{2}$ because the angle is in quadrant II because cosine is negative in quadrant II.

c. To find $\tan\frac{\alpha}{2}$, we write the half-angle formula for tangent. Again, we substitute the value of the cosine we found from the triangle in **Figure 3** and simplify.

$$\begin{aligned}
\tan\frac{\alpha}{2} &= \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} \\
&= \pm\sqrt{\frac{1-\left(-\frac{15}{17}\right)}{1+\left(-\frac{15}{17}\right)}} \\
&= \pm\sqrt{\frac{\frac{32}{17}}{\frac{2}{17}}} \\
&= \pm\sqrt{\frac{32}{2}} \\
&= -\sqrt{16} \\
&= -4
\end{aligned}$$

We choose the negative value of $\tan\frac{\alpha}{2}$ because $\frac{\alpha}{2}$ lies in quadrant II, and tangent is negative in quadrant II.

Try It #5

Given that $\sin\alpha = -\frac{4}{5}$ and α lies in quadrant IV, find the exact value of $\cos\left(\frac{\alpha}{2}\right)$.

Example 9 Finding the Measurement of a Half Angle

Now, we will return to the problem posed at the beginning of the section. A bicycle ramp is constructed for high-level competition with an angle of θ formed by the ramp and the ground. Another ramp is to be constructed half as steep for novice competition. If $\tan\theta = \frac{5}{3}$ for higher-level competition, what is the measurement of the angle for novice competition?

Solution Since the angle for novice competition measures half the steepness of the angle for the high-level competition, and $\tan \theta = \frac{5}{3}$ for high-level competition, we can find $\cos \theta$ from the right triangle and the Pythagorean theorem so that we can use the half-angle identities. See **Figure 4**.

$$3^2 + 5^2 = 34$$
$$c = \sqrt{34}$$

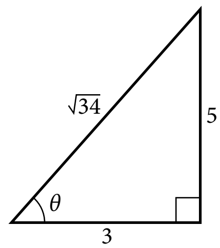


Figure 4

We see that $\cos \theta = \frac{3}{\sqrt{34}} = \frac{3\sqrt{34}}{34}$. We can use the half-angle formula for tangent: $\tan \frac{\theta}{2} = \sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}}$. Since $\tan \theta$ is in the first quadrant, so is $\tan \frac{\theta}{2}$. Thus,

$$\begin{aligned} \tan \frac{\theta}{2} &= \sqrt{\frac{1 - \frac{3\sqrt{34}}{34}}{1 + \frac{3\sqrt{34}}{34}}} \\ &= \sqrt{\frac{\frac{34 - 3\sqrt{34}}{34}}{\frac{34 + 3\sqrt{34}}{34}}} \\ &= \sqrt{\frac{34 - 3\sqrt{34}}{34 + 3\sqrt{34}}} \\ &\approx 0.57 \end{aligned}$$

We can take the inverse tangent to find the angle: $\tan^{-1}(0.57) \approx 29.7°$. So the angle of the ramp for novice competition is $\approx 29.7°$.

Access these online resources for additional instruction and practice with double-angle, half-angle, and reduction formulas.

- Double-Angle Identities (http://openstaxcollege.org/l/doubleangiden)
- Half-Angle Identities (http://openstaxcollege.org/l/halfangleident)

9.3 SECTION EXERCISES

VERBAL

1. Explain how to determine the reduction identities from the double-angle identity $\cos(2x) = \cos^2 x - \sin^2 x$.

2. Explain how to determine the double-angle formula for $\tan(2x)$ using the double-angle formulas for $\cos(2x)$ and $\sin(2x)$.

3. We can determine the half-angle formula for $\tan\left(\frac{x}{2}\right) = \pm \frac{\sqrt{1 - \cos x}}{\sqrt{1 + \cos x}}$ by dividing the formula for $\sin\left(\frac{x}{2}\right)$ by $\cos\left(\frac{x}{2}\right)$. Explain how to determine two formulas for $\tan\left(\frac{x}{2}\right)$ that do not involve any square roots.

4. For the half-angle formula given in the previous exercise for $\tan\left(\frac{x}{2}\right)$, explain why dividing by 0 is not a concern. (Hint: examine the values of $\cos x$ necessary for the denominator to be 0.)

ALGEBRAIC

For the following exercises, find the exact values of a) $\sin(2x)$, b) $\cos(2x)$, and c) $\tan(2x)$ without solving for x.

5. If $\sin x = \frac{1}{8}$, and x is in quadrant I.

6. If $\cos x = \frac{2}{3}$, and x is in quadrant I.

7. If $\cos x = -\frac{1}{2}$, and x is in quadrant III.

8. If $\tan x = -8$, and x is in quadrant IV.

For the following exercises, find the values of the six trigonometric functions if the conditions provided hold.

9. $\cos(2\theta) = \frac{3}{5}$ and $90° \leq \theta \leq 180°$

10. $\cos(2\theta) = \frac{1}{\sqrt{2}}$ and $180° \leq \theta \leq 270°$

For the following exercises, simplify to one trigonometric expression.

11. $2\sin\left(\frac{\pi}{4}\right) 2\cos\left(\frac{\pi}{4}\right)$

12. $4\sin\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{8}\right)$

For the following exercises, find the exact value using half-angle formulas.

13. $\sin\left(\frac{\pi}{8}\right)$

14. $\cos\left(-\frac{11\pi}{12}\right)$

15. $\sin\left(\frac{11\pi}{12}\right)$

16. $\cos\left(\frac{7\pi}{8}\right)$

17. $\tan\left(\frac{5\pi}{12}\right)$

18. $\tan\left(-\frac{3\pi}{12}\right)$

19. $\tan\left(-\frac{3\pi}{8}\right)$

For the following exercises, find the exact values of a) $\sin\left(\frac{x}{2}\right)$, b) $\cos\left(\frac{x}{2}\right)$, and c) $\tan\left(\frac{x}{2}\right)$ without solving for x, when $0 \leq x \leq 360°$.

20. If $\tan x = -\frac{4}{3}$, and x is in quadrant IV.

21. If $\sin x = -\frac{12}{13}$, and x is in quadrant III.

22. If $\csc x = 7$, and x is in quadrant II.

23. If $\sec x = -4$, and x is in quadrant II.

For the following exercises, use **Figure 5** to find the requested half and double angles.

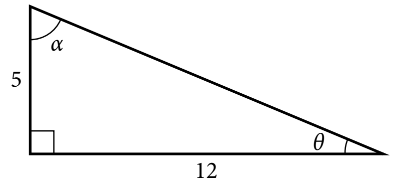


Figure 5

24. Find $\sin(2\theta)$, $\cos(2\theta)$, and $\tan(2\theta)$.

25. Find $\sin(2\alpha)$, $\cos(2\alpha)$, and $\tan(2\alpha)$.

26. Find $\sin\left(\frac{\theta}{2}\right)$, $\cos\left(\frac{\theta}{2}\right)$, and $\tan\left(\frac{\theta}{2}\right)$.

27. Find $\sin\left(\frac{\alpha}{2}\right)$, $\cos\left(\frac{\alpha}{2}\right)$, and $\tan\left(\frac{\alpha}{2}\right)$.

For the following exercises, simplify each expression. Do not evaluate.

28. $\cos^2(28°) - \sin^2(28°)$

29. $2\cos^2(37°) - 1$

30. $1 - 2\sin^2(17°)$

31. $\cos^2(9x) - \sin^2(9x)$

32. $4\sin(8x)\cos(8x)$

33. $6\sin(5x)\cos(5x)$

For the following exercises, prove the identity given.

34. $(\sin t - \cos t)^2 = 1 - \sin(2t)$

35. $\sin(2x) = -2\sin(-x)\cos(-x)$

36. $\cot x - \tan x = 2\cot(2x)$

37. $\dfrac{\sin(2\theta)}{1+\cos(2\theta)}\tan^2\theta = \tan^3\theta$

For the following exercises, rewrite the expression with an exponent no higher than 1.

38. $\cos^2(5x)$

39. $\cos^2(6x)$

40. $\sin^4(8x)$

41. $\sin^4(3x)$

42. $\cos^2 x \sin^4 x$

43. $\cos^4 x \sin^2 x$

44. $\tan^2 x \sin^2 x$

TECHNOLOGY

For the following exercises, reduce the equations to powers of one, and then check the answer graphically.

45. $\tan^4 x$

46. $\sin^2(2x)$

47. $\sin^2 x \cos^2 x$

48. $\tan^2 x \sin x$

49. $\tan^4 x \cos^2 x$

50. $\cos^2 x \sin(2x)$

51. $\cos^2(2x)\sin x$

52. $\tan^2\left(\frac{x}{2}\right)\sin x$

For the following exercises, algebraically find an equivalent function, only in terms of $\sin x$ and/or $\cos x$, and then check the answer by graphing both equations.

53. $\sin(4x)$

54. $\cos(4x)$

EXTENSIONS

For the following exercises, prove the identities.

55. $\sin(2x) = \dfrac{2\tan x}{1+\tan^2 x}$

56. $\cos(2\alpha) = \dfrac{1-\tan^2\alpha}{1+\tan^2\alpha}$

57. $\tan(2x) = \dfrac{2\sin x\cos x}{2\cos^2 x - 1}$

58. $(\sin^2 x - 1)^2 = \cos(2x) + \sin^4 x$

59. $\sin(3x) = 3\sin x\cos^2 x - \sin^3 x$

60. $\cos(3x) = \cos^3 x - 3\sin^2 x\cos x$

61. $\dfrac{1+\cos(2t)}{\sin(2t)-\cos t} = \dfrac{2\cos t}{2\sin t - 1}$

62. $\sin(16x) = 16\sin x\cos x\cos(2x)\cos(4x)\cos(8x)$

63. $\cos(16x) = (\cos^2(4x) - \sin^2(4x) - \sin(8x))(\cos^2(4x) - \sin^2(4x) + \sin(8x))$

LEARNING OBJECTIVES

In this section, you will:

- Express products as sums.
- Express sums as products.

9.4 SUM-TO-PRODUCT AND PRODUCT-TO-SUM FORMULAS

Figure 1 The UCLA marching band (credit: Eric Chan, Flickr).

A band marches down the field creating an amazing sound that bolsters the crowd. That sound travels as a wave that can be interpreted using trigonometric functions. For example, **Figure 2** represents a sound wave for the musical note A. In this section, we will investigate trigonometric identities that are the foundation of everyday phenomena such as sound waves.

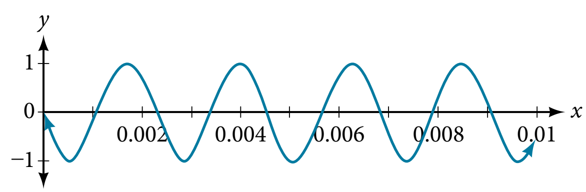


Figure 2

Expressing Products as Sums

We have already learned a number of formulas useful for expanding or simplifying trigonometric expressions, but sometimes we may need to express the product of cosine and sine as a sum. We can use the product-to-sum formulas, which express products of trigonometric functions as sums. Let's investigate the cosine identity first and then the sine identity.

Expressing Products as Sums for Cosine

We can derive the product-to-sum formula from the sum and difference identities for cosine. If we add the two equations, we get:

$$\begin{array}{rl} \cos\alpha\cos\beta + \sin\alpha\sin\beta &= \cos(\alpha-\beta) \\ +\ \cos\alpha\cos\beta - \sin\alpha\sin\beta &= \cos(\alpha+\beta) \\ \hline 2\cos\alpha\cos\beta &= \cos(\alpha-\beta)+\cos(\alpha+\beta) \end{array}$$

Then, we divide by 2 to isolate the product of cosines:

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$$

How To…

Given a product of cosines, express as a sum.

1. Write the formula for the product of cosines.
2. Substitute the given angles into the formula.
3. Simplify.

Example 1 Writing the Product as a Sum Using the Product-to-Sum Formula for Cosine

Write the following product of cosines as a sum: $2\cos\left(\frac{7x}{2}\right)\cos\frac{3x}{2}$.

Solution We begin by writing the formula for the product of cosines:

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)]$$

We can then substitute the given angles into the formula and simplify.

$$\begin{aligned} 2\cos\left(\frac{7x}{2}\right)\cos\left(\frac{3x}{2}\right) &= (2)\left(\frac{1}{2}\right)\left[\cos\left(\frac{7x}{2} - \frac{3x}{2}\right) + \cos\left(\frac{7x}{2} + \frac{3x}{2}\right)\right] \\ &= \left[\cos\left(\frac{4x}{2}\right) + \cos\left(\frac{10x}{2}\right)\right] \\ &= \cos 2x + \cos 5x \end{aligned}$$

Try It #1

Use the product-to-sum formula to write the product as a sum or difference: $\cos(2\theta)\cos(4\theta)$.

Expressing the Product of Sine and Cosine as a Sum

Next, we will derive the product-to-sum formula for sine and cosine from the sum and difference formulas for sine. If we add the sum and difference identities, we get:

$$\begin{array}{rl} \sin(\alpha + \beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ + \quad \sin(\alpha - \beta) &= \sin\alpha\cos\beta - \cos\alpha\sin\beta \\ \hline \sin(\alpha + \beta) + \sin(\alpha - \beta) &= 2\sin\alpha\cos\beta \end{array}$$

Then, we divide by 2 to isolate the product of cosine and sine:

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha + \beta) + \sin(\alpha - \beta)]$$

Example 2 Writing the Product as a Sum Containing only Sine or Cosine

Express the following product as a sum containing only sine or cosine and no products: $\sin(4\theta)\cos(2\theta)$.

Solution Write the formula for the product of sine and cosine. Then substitute the given values into the formula and simplify.

$$\begin{aligned} \sin\alpha\cos\beta &= \frac{1}{2}[\sin(\alpha + \beta) + \sin(\alpha - \beta)] \\ \sin(4\theta)\cos(2\theta) &= \frac{1}{2}[\sin(4\theta + 2\theta) + \sin(4\theta - 2\theta)] \\ &= \frac{1}{2}[\sin(6\theta) + \sin(2\theta)] \end{aligned}$$

Try It #2

Use the product-to-sum formula to write the product as a sum: $\sin(x + y)\cos(x - y)$.

Expressing Products of Sines in Terms of Cosine

Expressing the product of sines in terms of cosine is also derived from the sum and difference identities for cosine. In this case, we will first subtract the two cosine formulas:

$$\begin{array}{rl} \cos(\alpha - \beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta \\ - \quad \cos(\alpha + \beta) &= -\,(\cos\alpha\cos\beta - \sin\alpha\sin\beta) \\ \hline \cos(\alpha - \beta) - \cos(\alpha + \beta) &= 2\sin\alpha\sin\beta \end{array}$$

Then, we divide by 2 to isolate the product of sines:

$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$$

Similarly we could express the product of cosines in terms of sine or derive other product-to-sum formulas.

the product-to-sum formulas

The **product-to-sum formulas** are as follows:

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)] \qquad \sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha + \beta) + \sin(\alpha - \beta)]$$

$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)] \qquad \cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha + \beta) - \sin(\alpha - \beta)]$$

Example 3 Express the Product as a Sum or Difference

Write $\cos(3\theta)\cos(5\theta)$ as a sum or difference.

Solution We have the product of cosines, so we begin by writing the related formula. Then we substitute the given angles and simplify.

$$\begin{aligned} \cos\alpha\cos\beta &= \frac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)] \\ \cos(3\theta)\cos(5\theta) &= \frac{1}{2}[\cos(3\theta - 5\theta) + \cos(3\theta + 5\theta)] \\ &= \frac{1}{2}[\cos(2\theta) + \cos(8\theta)] \qquad \text{Use even-odd identity.} \end{aligned}$$

Try It #3

Use the product-to-sum formula to evaluate $\cos\frac{11\pi}{12}\cos\frac{\pi}{12}$.

Expressing Sums as Products

Some problems require the reverse of the process we just used. The sum-to-product formulas allow us to express sums of sine or cosine as products. These formulas can be derived from the product-to-sum identities. For example, with a few substitutions, we can derive the sum-to-product identity for sine. Let $\frac{u+v}{2} = \alpha$ and $\frac{u-v}{2} = \beta$.

Then,

$$\begin{aligned} \alpha + \beta &= \frac{u+v}{2} + \frac{u-v}{2} \\ &= \frac{2u}{2} \\ &= u \\ \alpha - \beta &= \frac{u+v}{2} - \frac{u-v}{2} \\ &= \frac{2v}{2} \\ &= v \end{aligned}$$

Thus, replacing α and β in the product-to-sum formula with the substitute expressions, we have

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$$

$$\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right) = \frac{1}{2}[\sin u + \sin v] \qquad \text{Substitute for } (\alpha+\beta) \text{ and } (\alpha-\beta)$$

$$2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right) = \sin u + \sin v$$

The other sum-to-product identities are derived similarly.

sum-to-product formulas

The **sum-to-product formulas** are as follows:

$$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \qquad \sin\alpha - \sin\beta = 2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$$

$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) \qquad \cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

Example 4 Writing the Difference of Sines as a Product

Write the following difference of sines expression as a product: $\sin(4\theta) - \sin(2\theta)$.

Solution We begin by writing the formula for the difference of sines.

$$\sin\alpha - \sin\beta = 2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$$

Substitute the values into the formula, and simplify.

$$\begin{aligned}\sin(4\theta) - \sin(2\theta) &= 2\sin\left(\frac{4\theta-2\theta}{2}\right)\cos\left(\frac{4\theta+2\theta}{2}\right)\\ &= 2\sin\left(\frac{2\theta}{2}\right)\cos\left(\frac{6\theta}{2}\right)\\ &= 2\sin\theta\cos(3\theta)\end{aligned}$$

Try It #4

Use the sum-to-product formula to write the sum as a product: $\sin(3\theta) + \sin(\theta)$.

Example 5 Evaluating Using the Sum-to-Product Formula

Evaluate $\cos(15°) - \cos(75°)$. Check the answer with a graphing calculator.

Solution We begin by writing the formula for the difference of cosines.

$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

Then we substitute the given angles and simplify.

$$\begin{aligned}\cos(15°) - \cos(75°) &= -2\sin\left(\frac{15°+75°}{2}\right)\sin\left(\frac{15°-75°}{2}\right)\\ &= -2\sin(45°)\sin(-30°)\\ &= -2\left(\frac{\sqrt{2}}{2}\right)\left(-\frac{1}{2}\right)\\ &= \frac{\sqrt{2}}{2}\end{aligned}$$

Example 6 Proving an Identity

Prove the identity:

$$\frac{\cos(4t) - \cos(2t)}{\sin(4t) + \sin(2t)} = -\tan t$$

Solution We will start with the left side, the more complicated side of the equation, and rewrite the expression until it matches the right side.

$$\begin{aligned}\frac{\cos(4t) - \cos(2t)}{\sin(4t) + \sin(2t)} &= \frac{-2\sin\left(\frac{4t+2t}{2}\right)\sin\left(\frac{4t-2t}{2}\right)}{2\sin\left(\frac{4t+2t}{2}\right)\cos\left(\frac{4t-2t}{2}\right)} \\ &= \frac{-2\sin(3t)\sin t}{2\sin(3t)\cos t} \\ &= \frac{-\cancel{2}\,\cancel{\sin(3t)}\sin t}{\cancel{2}\,\cancel{\sin(3t)}\cos t} \\ &= -\frac{\sin t}{\cos t} \\ &= -\tan t\end{aligned}$$

Analysis *Recall that verifying trigonometric identities has its own set of rules. The procedures for solving an equation are not the same as the procedures for verifying an identity. When we prove an identity, we pick one side to work on and make substitutions until that side is transformed into the other side.*

Example 7 Verifying the Identity Using Double-Angle Formulas and Reciprocal Identities

Verify the identity $\csc^2\theta - 2 = \frac{\cos(2\theta)}{\sin^2\theta}$.

Solution For verifying this equation, we are bringing together several of the identities. We will use the double-angle formula and the reciprocal identities. We will work with the right side of the equation and rewrite it until it matches the left side.

$$\begin{aligned}\frac{\cos(2\theta)}{\sin^2\theta} &= \frac{1 - 2\sin^2\theta}{\sin^2\theta} \\ &= \frac{1}{\sin^2\theta} - \frac{2\sin^2\theta}{\sin^2\theta} \\ &= \csc^2\theta - 2\end{aligned}$$

Try It #5

Verify the identity $\tan\theta\cot\theta - \cos^2\theta = \sin^2\theta$.

Access these online resources for additional instruction and practice with the product-to-sum and sum-to-product identities.

- Sum to Product Identities (http://openstaxcollege.org/l/sumtoprod)
- Sum to Product and Product to Sum Identities (http://openstaxcollege.org/l/sumtpptsum)

9.4 SECTION EXERCISES

VERBAL

1. Starting with the product to sum formula $\sin \alpha \cos \beta = \frac{1}{2}[\sin(\alpha + \beta) + \sin(\alpha - \beta)]$, explain how to determine the formula for $\cos \alpha \sin \beta$.

2. Provide two different methods of calculating $\cos(195°)\cos(105°)$, one of which uses the product to sum. Which method is easier?

3. Describe a situation where we would convert an equation from a sum to a product and give an example.

4. Describe a situation where we would convert an equation from a product to a sum, and give an example.

ALGEBRAIC

For the following exercises, rewrite the product as a sum or difference.

5. $16\sin(16x)\sin(11x)$

6. $20\cos(36t)\cos(6t)$

7. $2\sin(5x)\cos(3x)$

8. $10\cos(5x)\sin(10x)$

9. $\sin(-x)\sin(5x)$

10. $\sin(3x)\cos(5x)$

For the following exercises, rewrite the sum or difference as a product.

11. $\cos(6t) + \cos(4t)$

12. $\sin(3x) + \sin(7x)$

13. $\cos(7x) + \cos(-7x)$

14. $\sin(3x) - \sin(-3x)$

15. $\cos(3x) + \cos(9x)$

16. $\sin h - \sin(3h)$

For the following exercises, evaluate the product using a sum or difference of two functions. Evaluate exactly.

17. $\cos(45°)\cos(15°)$

18. $\cos(45°)\sin(15°)$

19. $\sin(-345°)\sin(-15°)$

20. $\sin(195°)\cos(15°)$

21. $\sin(-45°)\sin(-15°)$

For the following exercises, evaluate the product using a sum or difference of two functions. Leave in terms of sine and cosine.

22. $\cos(23°)\sin(17°)$

23. $2\sin(100°)\sin(20°)$

24. $2\sin(-100°)\sin(-20°)$

25. $\sin(213°)\cos(8°)$

26. $2\cos(56°)\cos(47°)$

For the following exercises, rewrite the sum as a product of two functions. Leave in terms of sine and cosine.

27. $\sin(76°) + \sin(14°)$

28. $\cos(58°) - \cos(12°)$

29. $\sin(101°) - \sin(32°)$

30. $\cos(100°) + \cos(200°)$

31. $\sin(-1°) + \sin(-2°)$

For the following exercises, prove the identity.

32. $\dfrac{\cos(a + b)}{\cos(a - b)} = \dfrac{1 - \tan a \tan b}{1 + \tan a \tan b}$

33. $4\sin(3x)\cos(4x) = 2\sin(7x) - 2\sin x$

34. $\dfrac{6\cos(8x)\sin(2x)}{\sin(-6x)} = -3\sin(10x)\csc(6x) + 3$

35. $\sin x + \sin(3x) = 4\sin x \cos^2 x$

36. $2(\cos^3 x - \cos x \sin^2 x) = \cos(3x) + \cos x$

37. $2\tan x \cos(3x) = \sec x(\sin(4x) - \sin(2x))$

38. $\cos(a + b) + \cos(a - b) = 2\cos a \cos b$

NUMERIC

For the following exercises, rewrite the sum as a product of two functions or the product as a sum of two functions. Give your answer in terms of sines and cosines. Then evaluate the final answer numerically, rounded to four decimal places.

39. $\cos(58°) + \cos(12°)$

40. $\sin(2°) - \sin(3°)$

41. $\cos(44°) - \cos(22°)$

42. $\cos(176°)\sin(9°)$

43. $\sin(-14°)\sin(85°)$

TECHNOLOGY

For the following exercises, algebraically determine whether each of the given expressions is a true identity. If it is not an identity, replace the right-hand side with an expression equivalent to the left side. Verify the results by graphing both expressions on a calculator.

44. $2\sin(2x)\sin(3x) = \cos x - \cos(5x)$

45. $\dfrac{\cos(10\theta) + \cos(6\theta)}{\cos(6\theta) - \cos(10\theta)} = \cot(2\theta)\cot(8\theta)$

46. $\dfrac{\sin(3x) - \sin(5x)}{\cos(3x) + \cos(5x)} = \tan x$

47. $2\cos(2x)\cos x + \sin(2x)\sin x = 2\sin x$

48. $\dfrac{\sin(2x) + \sin(4x)}{\sin(2x) - \sin(4x)} = -\tan(3x)\cot x$

For the following exercises, simplify the expression to one term, then graph the original function and your simplified version to verify they are identical.

49. $\dfrac{\sin(9t) - \sin(3t)}{\cos(9t) + \cos(3t)}$

50. $2\sin(8x)\cos(6x) - \sin(2x)$

51. $\dfrac{\sin(3x) - \sin x}{\sin x}$

52. $\dfrac{\cos(5x) + \cos(3x)}{\sin(5x) + \sin(3x)}$

53. $\sin x\cos(15x) - \cos x\sin(15x)$

EXTENSIONS

For the following exercises, prove the following sum-to-product formulas.

54. $\sin x - \sin y = 2\sin\left(\dfrac{x-y}{2}\right)\cos\left(\dfrac{x+y}{2}\right)$

55. $\cos x + \cos y = 2\cos\left(\dfrac{x+y}{2}\right)\cos\left(\dfrac{x-y}{2}\right)$

For the following exercises, prove the identity.

56. $\dfrac{\sin(6x) + \sin(4x)}{\sin(6x) - \sin(4x)} = \tan(5x)\cot x$

57. $\dfrac{\cos(3x) + \cos x}{\cos(3x) - \cos x} = -\cot(2x)\cot x$

58. $\dfrac{\cos(6y) + \cos(8y)}{\sin(6y) - \sin(4y)} = \cot y\cos(7y)\sec(5y)$

59. $\dfrac{\cos(2y) - \cos(4y)}{\sin(2y) + \sin(4y)} = \tan y$

60. $\dfrac{\sin(10x) - \sin(2x)}{\cos(10x) + \cos(2x)} = \tan(4x)$

61. $\cos x - \cos(3x) = 4\sin^2 x\cos x$

62. $(\cos(2x) - \cos(4x))^2 + (\sin(4x) + \sin(2x))^2 = 4\sin^2(3x)$

63. $\tan\left(\dfrac{\pi}{4} - t\right) = \dfrac{1 - \tan t}{1 + \tan t}$

LEARNING OBJECTIVES

In this section, you will:

- Solve linear trigonometric equations in sine and cosine.
- Solve equations involving a single trigonometric function.
- Solve trigonometric equations using a calculator.
- Solve trigonometric equations that are quadratic in form.
- Solve trigonometric equations using fundamental identities.
- Solve trigonometric equations with multiple angles.
- Solve right triangle problems.

9.5 SOLVING TRIGONOMETRIC EQUATIONS

Figure 1 Egyptian pyramids standing near a modern city. (credit: Oisin Mulvihill)

Thales of Miletus (circa 625–547 BC) is known as the founder of geometry. The legend is that he calculated the height of the Great Pyramid of Giza in Egypt using the theory of *similar triangles*, which he developed by measuring the shadow of his staff. Based on proportions, this theory has applications in a number of areas, including fractal geometry, engineering, and architecture. Often, the angle of elevation and the angle of depression are found using similar triangles.

In earlier sections of this chapter, we looked at trigonometric identities. Identities are true for all values in the domain of the variable. In this section, we begin our study of trigonometric equations to study real-world scenarios such as the finding the dimensions of the pyramids.

Solving Linear Trigonometric Equations in Sine and Cosine

Trigonometric equations are, as the name implies, equations that involve trigonometric functions. Similar in many ways to solving polynomial equations or rational equations, only specific values of the variable will be solutions, if there are solutions at all. Often we will solve a trigonometric equation over a specified interval. However, just as often, we will be asked to find all possible solutions, and as trigonometric functions are periodic, solutions are repeated within each period. In other words, trigonometric equations may have an infinite number of solutions. Additionally, like rational equations, the domain of the function must be considered before we assume that any solution is valid. The period of both the sine function and the cosine function is 2π. In other words, every 2π units, the y-values repeat. If we need to find all possible solutions, then we must add $2\pi k$, where k is an integer, to the initial solution. Recall the rule that gives the format for stating all possible solutions for a function where the period is 2π:

$$\sin \theta = \sin(\theta \pm 2k\pi)$$

There are similar rules for indicating all possible solutions for the other trigonometric functions. Solving trigonometric equations requires the same techniques as solving algebraic equations. We read the equation from left to right, horizontally, like a sentence. We look for known patterns, factor, find common denominators, and substitute certain expressions with a variable to make solving a more straightforward process. However, with trigonometric equations, we also have the advantage of using the identities we developed in the previous sections.

Example 1 Solving a Linear Trigonometric Equation Involving the Cosine Function

Find all possible exact solutions for the equation $\cos \theta = \frac{1}{2}$.

Solution From the unit circle, we know that

$$\cos \theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{3}, \frac{5\pi}{3}$$

These are the solutions in the interval $[0, 2\pi]$. All possible solutions are given by

$$\theta = \frac{\pi}{3} \pm 2k\pi \text{ and } \theta = \frac{5\pi}{3} \pm 2k\pi$$

where k is an integer.

Example 2 Solving a Linear Equation Involving the Sine Function

Find all possible exact solutions for the equation $\sin t = \frac{1}{2}$.

Solution Solving for all possible values of t means that solutions include angles beyond the period of 2π. From **Section 9.2 Figure 2**, we can see that the solutions are $t = \frac{\pi}{6}$ and $t = \frac{5\pi}{6}$. But the problem is asking for all possible values that solve the equation. Therefore, the answer is

$$t = \frac{\pi}{6} \pm 2\pi k \quad \text{and} \quad t = \frac{5\pi}{6} \pm 2\pi k$$

where k is an integer.

How To...

Given a trigonometric equation, solve using algebra.

1. Look for a pattern that suggests an algebraic property, such as the difference of squares or a factoring opportunity.
2. Substitute the trigonometric expression with a single variable, such as x or u.
3. Solve the equation the same way an algebraic equation would be solved.
4. Substitute the trigonometric expression back in for the variable in the resulting expressions.
5. Solve for the angle.

Example 3 Solve the Trigonometric Equation in Linear Form

Solve the equation exactly: $2\cos \theta - 3 = -5, 0 \leq \theta < 2\pi$.

Solution Use algebraic techniques to solve the equation.

$$2\cos \theta - 3 = -5$$

$$2\cos \theta = -2$$

$$\cos \theta = -1$$

$$\theta = \pi$$

Try It #1

Solve exactly the following linear equation on the interval $[0, 2\pi)$: $2\sin x + 1 = 0$.

Solving Equations Involving a Single Trigonometric Function

When we are given equations that involve only one of the six trigonometric functions, their solutions involve using algebraic techniques and the unit circle (see **Section 9.2 Figure 2**). We need to make several considerations when the equation involves trigonometric functions other than sine and cosine. Problems involving the reciprocals of the primary trigonometric functions need to be viewed from an algebraic perspective. In other words, we will write the reciprocal function, and solve for the angles using the function. Also, an equation involving the tangent function is slightly different from one containing a sine or cosine function. First, as we know, the period of tangent is π, not 2π. Further, the domain of tangent is all real numbers with the exception of odd integer multiples of $\frac{\pi}{2}$, unless, of course, a problem places its own restrictions on the domain.

Example 4 Solving a Problem Involving a Single Trigonometric Function

Solve the problem exactly: $2\sin^2\theta - 1 = 0$, $0 \leq \theta < 2\pi$.

Solution As this problem is not easily factored, we will solve using the square root property. First, we use algebra to isolate $\sin\theta$. Then we will find the angles.

$$2\sin^2\theta - 1 = 0$$
$$2\sin^2\theta = 1$$
$$\sin^2\theta = \frac{1}{2}$$
$$\sqrt{\sin^2\theta} = \pm\sqrt{\frac{1}{2}}$$
$$\sin\theta = \pm\frac{1}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2}$$
$$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

Example 5 Solving a Trigonometric Equation Involving Cosecant

Solve the following equation exactly: $\csc\theta = -2$, $0 \leq \theta < 4\pi$.

Solution We want all values of θ for which $\csc\theta = -2$ over the interval $0 \leq \theta < 4\pi$.

$$\csc\theta = -2$$
$$\frac{1}{\sin\theta} = -2$$
$$\sin\theta = -\frac{1}{2}$$
$$\theta = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}$$

Analysis *As* $\sin\theta = -\frac{1}{2}$, *notice that all four solutions are in the third and fourth quadrants.*

Example 6 Solving an Equation Involving Tangent

Solve the equation exactly: $\tan\left(\theta - \frac{\pi}{2}\right) = 1$, $0 \leq \theta < 2\pi$.

Solution Recall that the tangent function has a period of π. On the interval $[0, \pi)$, and at the angle of $\frac{\pi}{4}$, the tangent has a value of 1. However, the angle we want is $\left(\theta - \frac{\pi}{2}\right)$. Thus, if $\tan\left(\frac{\pi}{4}\right) = 1$, then

$$\theta - \frac{\pi}{2} = \frac{\pi}{4}$$
$$\theta = \frac{3\pi}{4} \pm k\pi$$

Over the interval $[0, 2\pi)$, we have two solutions:

$$\theta = \frac{3\pi}{4} \text{ and } \theta = \frac{3\pi}{4} + \pi = \frac{7\pi}{4}$$

Try It #2

Find all solutions for $\tan x = \sqrt{3}$.

Example 7 Identify all Solutions to the Equation Involving Tangent

Identify all exact solutions to the equation $2(\tan x + 3) = 5 + \tan x$, $0 \leq x < 2\pi$.

Solution We can solve this equation using only algebra. Isolate the expression tan x on the left side of the equals sign.

$$\begin{aligned} 2(\tan x) + 2(3) &= 5 + \tan x \\ 2\tan x + 6 &= 5 + \tan x \\ 2\tan x - \tan x &= 5 - 6 \\ \tan x &= -1 \end{aligned}$$

There are two angles on the unit circle that have a tangent value of -1: $\theta = \frac{3\pi}{4}$ and $\theta = \frac{7\pi}{4}$.

Solve Trigonometric Equations Using a Calculator

Not all functions can be solved exactly using only the unit circle. When we must solve an equation involving an angle other than one of the special angles, we will need to use a calculator. Make sure it is set to the proper mode, either degrees or radians, depending on the criteria of the given problem.

Example 8 Using a Calculator to Solve a Trigonometric Equation Involving Sine

Use a calculator to solve the equation $\sin\theta = 0.8$, where θ is in radians.

Solution Make sure mode is set to radians. To find θ, use the inverse sine function. On most calculators, you will need to push the **2ND** button and then the **SIN** button to bring up the $\mathbf{sin^{-1}}$ function. What is shown on the screen is $\mathbf{sin^{-1}}$**(** . The calculator is ready for the input within the parentheses. For this problem, we enter $\mathbf{sin^{-1}}$ **(0.8)**, and press **ENTER**. Thus, to four decimals places,

$$\sin^{-1}(0.8) \approx 0.9273$$

The solution is

$$\theta \approx 0.9273 \pm 2\pi k$$

The angle measurement in degrees is

$$\begin{aligned} \theta &\approx 53.1° \\ \theta &\approx 180° - 53.1° \\ &\approx 126.9° \end{aligned}$$

Analysis *Note that a calculator will only return an angle in quadrants I or IV for the sine function, since that is the range of the inverse sine. The other angle is obtained by using $\pi - \theta$.*

Example 9 Using a Calculator to Solve a Trigonometric Equation Involving Secant

Use a calculator to solve the equation $\sec\theta = -4$, giving your answer in radians.

Solution We can begin with some algebra.

$$\begin{aligned} \sec\theta &= -4 \\ \frac{1}{\cos\theta} &= -4 \\ \cos\theta &= -\frac{1}{4} \end{aligned}$$

Check that the MODE is in radians. Now use the inverse cosine function.

$$\cos^{-1}\left(-\frac{1}{4}\right) \approx 1.8235$$

$$\theta \approx 1.8235 + 2\pi k$$

Since $\frac{\pi}{2} \approx 1.57$ and $\pi \approx 3.14$, 1.8235 is between these two numbers, thus $\theta \approx 1.8235$ is in quadrant II.

Cosine is also negative in quadrant III. Note that a calculator will only return an angle in quadrants I or II for the cosine function, since that is the range of the inverse cosine. See **Figure 2**.

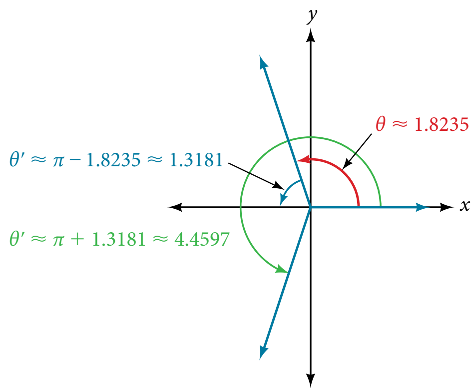


Figure 2

So, we also need to find the measure of the angle in quadrant III. In quadrant III, the reference angle is $\theta' \approx \pi - 1.8235 \approx 1.3181$. The other solution in quadrant III is $\theta' \approx \pi + 1.3181 \approx 4.4597$.

The solutions are $\theta \approx 1.8235 \pm 2\pi k$ and $\theta \approx 4.4597 \pm 2\pi k$.

Try It #3

Solve $\cos\theta = -0.2$.

Solving Trigonometric Equations in Quadratic Form

Solving a quadratic equation may be more complicated, but once again, we can use algebra as we would for any quadratic equation. Look at the pattern of the equation. Is there more than one trigonometric function in the equation, or is there only one? Which trigonometric function is squared? If there is only one function represented and one of the terms is squared, think about the standard form of a quadratic. Replace the trigonometric function with a variable such as x or u. If substitution makes the equation look like a quadratic equation, then we can use the same methods for solving quadratics to solve the trigonometric equations.

Example 10 Solving a Trigonometric Equation in Quadratic Form

Solve the equation exactly: $\cos^2\theta + 3\cos\theta - 1 = 0$, $0 \le \theta < 2\pi$.

Solution We begin by using substitution and replacing $\cos\theta$ with x. It is not necessary to use substitution, but it may make the problem easier to solve visually. Let $\cos\theta = x$. We have

$$x^2 + 3x - 1 = 0$$

The equation cannot be factored, so we will use the quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

$$\begin{aligned} x &= \frac{-3 \pm \sqrt{(-3)^2 - 4(1)(-1)}}{2} \\ &= \frac{-3 \pm \sqrt{13}}{2} \end{aligned}$$

Replace x with $\cos\theta$, and solve. Thus,

$$\begin{aligned} \cos\theta &= \frac{-3 \pm \sqrt{13}}{2} \\ \theta &= \cos^{-1}\left(\frac{-3 + \sqrt{13}}{2}\right) \end{aligned}$$

Note that only the + sign is used. This is because we get an error when we solve $\theta = \cos^{-1}\left(\dfrac{-3 - \sqrt{13}}{2}\right)$ on a calculator, since the domain of the inverse cosine function is $[-1, 1]$. However, there is a second solution:

$$\begin{aligned} \theta &= \cos^{-1}\left(\frac{-3 + \sqrt{13}}{2}\right) \\ &\approx 1.26 \end{aligned}$$

This terminal side of the angle lies in quadrant I. Since cosine is also positive in quadrant IV, the second solution is

$$\begin{aligned}\theta &= 2\pi - \cos^{-1}\left(\frac{-3+\sqrt{13}}{2}\right)\\ &\approx 5.02\end{aligned}$$

Example 11 Solving a Trigonometric Equation in Quadratic Form by Factoring

Solve the equation exactly: $2\sin^2\theta - 5\sin\theta + 3 = 0,\ 0 \le \theta \le 2\pi$.

Solution Using grouping, this quadratic can be factored. Either make the real substitution, $\sin\theta = u$, or imagine it, as we factor:

$$\begin{aligned}2\sin^2\theta - 5\sin\theta + 3 &= 0\\ (2\sin\theta - 3)(\sin\theta - 1) &= 0\end{aligned}$$

Now set each factor equal to zero.

$$\begin{aligned}2\sin\theta - 3 &= 0\\ 2\sin\theta &= 3\\ \sin\theta &= \frac{3}{2}\\ \sin\theta - 1 &= 0\\ \sin\theta &= 1\end{aligned}$$

Next solve for θ: $\sin\theta \ne \frac{3}{2}$, as the range of the sine function is $[-1, 1]$. However, $\sin\theta = 1$, giving the solution $\theta = \frac{\pi}{2}$.

Analysis *Make sure to check all solutions on the given domain as some factors have no solution.*

Try It #4

Solve $\sin^2\theta = 2\cos\theta + 2,\ 0 \le \theta \le 2\pi$. [Hint: Make a substitution to express the equation only in terms of cosine.]

Example 12 Solving a Trigonometric Equation Using Algebra

Solve exactly:

$$2\sin^2\theta + \sin\theta = 0;\ 0 \le \theta < 2\pi$$

Solution This problem should appear familiar as it is similar to a quadratic. Let $\sin\theta = x$. The equation becomes $2x^2 + x = 0$. We begin by factoring:

$$\begin{aligned}2x^2 + x &= 0\\ x(2x+1) &= 0\end{aligned}$$

Set each factor equal to zero.

$$\begin{aligned}x &= 0\\ (2x+1) &= 0\\ x &= -\frac{1}{2}\end{aligned}$$

Then, substitute back into the equation the original expression $\sin\theta$ for x. Thus,

$$\begin{aligned}\sin\theta &= 0\\ \theta &= 0, \pi\\ \sin\theta &= -\frac{1}{2}\\ \theta &= \frac{7\pi}{6}, \frac{11\pi}{6}\end{aligned}$$

The solutions within the domain $0 \le \theta < 2\pi$ are $\theta = 0, \pi, \frac{7\pi}{6}, \frac{11\pi}{6}$.

If we prefer not to substitute, we can solve the equation by following the same pattern of factoring and setting each factor equal to zero.

$$\begin{aligned}2\sin^2\theta + \sin\theta &= 0\\ \sin\theta(2\sin\theta + 1) &= 0\end{aligned}$$

$$\begin{aligned} \sin\theta &= 0 \\ \theta &= 0, \pi \\ 2\sin\theta + 1 &= 0 \\ 2\sin\theta &= -1 \\ \sin\theta &= -\frac{1}{2} \\ \theta &= \frac{7\pi}{6}, \frac{11\pi}{6} \end{aligned}$$

Analysis *We can see the solutions on the graph in **Figure 3**. On the interval $0 \le \theta < 2\pi$, the graph crosses the x-axis four times, at the solutions noted. Notice that trigonometric equations that are in quadratic form can yield up to four solutions instead of the expected two that are found with quadratic equations. In this example, each solution (angle) corresponding to a positive sine value will yield two angles that would result in that value.*

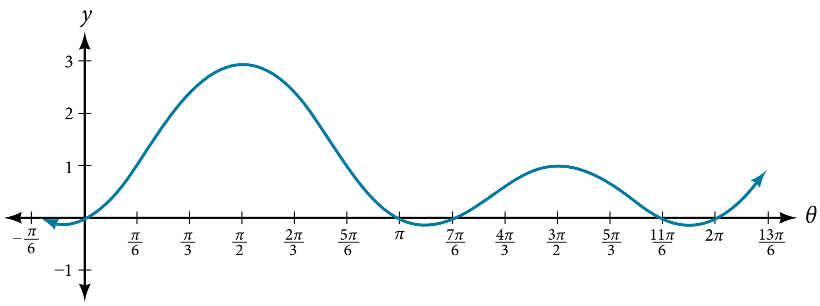


Figure 3

*We can verify the solutions on the unit circle in **Section 9.2 Figure 2** as well.*

Example 13 Solving a Trigonometric Equation Quadratic in Form

Solve the equation quadratic in form exactly: $2\sin^2\theta - 3\sin\theta + 1 = 0, 0 \le \theta < 2\pi$.

Solution We can factor using grouping. Solution values of θ can be found on the unit circle:

$$\begin{aligned} (2\sin\theta - 1)(\sin\theta - 1) &= 0 \\ 2\sin\theta - 1 &= 0 \\ \sin\theta &= \frac{1}{2} \\ \theta &= \frac{\pi}{6}, \frac{5\pi}{6} \\ \sin\theta &= 1 \\ \theta &= \frac{\pi}{2} \end{aligned}$$

Try It #5

Solve the quadratic equation $2\cos^2\theta + \cos\theta = 0$.

Solving Trigonometric Equations Using Fundamental Identities

While algebra can be used to solve a number of trigonometric equations, we can also use the fundamental identities because they make solving equations simpler. Remember that the techniques we use for solving are not the same as those for verifying identities. The basic rules of algebra apply here, as opposed to rewriting one side of the identity to match the other side. In the next example, we use two identities to simplify the equation.

Example 14 Use Identities to Solve an Equation

Use identities to solve exactly the trigonometric equation over the interval $0 \le x < 2\pi$.

$$\cos x \cos(2x) + \sin x \sin(2x) = \frac{\sqrt{3}}{2}$$

Solution Notice that the left side of the equation is the difference formula for cosine.

$$\begin{aligned} \cos x \cos(2x) + \sin x \sin(2x) &= \frac{\sqrt{3}}{2} \\ \cos(x - 2x) &= \frac{\sqrt{3}}{2} && \text{Difference formula for cosine} \\ \cos(-x) &= \frac{\sqrt{3}}{2} && \text{Use the negative angle identity.} \\ \cos x &= \frac{\sqrt{3}}{2} \end{aligned}$$

From the unit circle in **Section 9.2 Figure 2,** we see that $\cos x = \frac{\sqrt{3}}{2}$ when $x = \frac{\pi}{6}, \frac{11\pi}{6}$.

Example 15 Solving the Equation Using a Double-Angle Formula

Solve the equation exactly using a double-angle formula: $\cos(2\theta) = \cos\theta$.

Solution We have three choices of expressions to substitute for the double-angle of cosine. As it is simpler to solve for one trigonometric function at a time, we will choose the double-angle identity involving only cosine:

$$\begin{aligned} \cos(2\theta) &= \cos\theta \\ 2\cos^2\theta - 1 &= \cos\theta \\ 2\cos^2\theta - \cos\theta - 1 &= 0 \\ (2\cos\theta + 1)(\cos\theta - 1) &= 0 \\ 2\cos\theta + 1 &= 0 \\ \cos\theta &= -\frac{1}{2} \\ \cos\theta - 1 &= 0 \\ \cos\theta &= 1 \end{aligned}$$

So, if $\cos\theta = -\frac{1}{2}$, then $\theta = \frac{2\pi}{3} \pm 2\pi k$ and $\theta = \frac{4\pi}{3} \pm 2\pi k$; if $\cos\theta = 1$, then $\theta = 0 \pm 2\pi k$.

Example 16 Solving an Equation Using an Identity

Solve the equation exactly using an identity: $3\cos\theta + 3 = 2\sin^2\theta$, $0 \le \theta < 2\pi$.

Solution If we rewrite the right side, we can write the equation in terms of cosine:

$$\begin{aligned} 3\cos\theta + 3 &= 2\sin^2\theta \\ 3\cos\theta + 3 &= 2(1 - \cos^2\theta) \\ 3\cos\theta + 3 &= 2 - 2\cos^2\theta \\ 2\cos^2\theta + 3\cos\theta + 1 &= 0 \\ (2\cos\theta + 1)(\cos\theta + 1) &= 0 \\ 2\cos\theta + 1 &= 0 \\ \cos\theta &= -\frac{1}{2} \\ \theta &= \frac{2\pi}{3}, \frac{4\pi}{3} \\ \cos\theta + 1 &= 0 \\ \cos\theta &= -1 \\ \theta &= \pi \end{aligned}$$

Our solutions are $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}, \pi$

Solving Trigonometric Equations with Multiple Angles

Sometimes it is not possible to solve a trigonometric equation with identities that have a multiple angle, such as $\sin(2x)$ or $\cos(3x)$. When confronted with these equations, recall that $y = \sin(2x)$ is a horizontal compression by a factor of 2 of the function $y = \sin x$. On an interval of 2π, we can graph two periods of $y = \sin(2x)$, as opposed to one cycle of $y = \sin x$. This compression of the graph leads us to believe there may be twice as many x-intercepts or solutions to $\sin(2x) = 0$ compared to $\sin x = 0$. This information will help us solve the equation.

Example 17 **Solving a Multiple Angle Trigonometric Equation**

Solve exactly: $\cos(2x) = \frac{1}{2}$ on $[0, 2\pi)$.

Solution We can see that this equation is the standard equation with a multiple of an angle. If $\cos(\alpha) = \frac{1}{2}$, we know α is in quadrants I and IV. While $\theta = \cos^{-1}\frac{1}{2}$ will only yield solutions in quadrants I and II, we recognize that the solutions to the equation $\cos\theta = \frac{1}{2}$ will be in quadrants I and IV.

Therefore, the possible angles are $\theta = \frac{\pi}{3}$ and $\theta = \frac{5\pi}{3}$. So, $2x = \frac{\pi}{3}$ or $2x = \frac{5\pi}{3}$, which means that $x = \frac{\pi}{6}$ or $x = \frac{5\pi}{6}$. Does this make sense? Yes, because $\cos\left(2\left(\frac{\pi}{6}\right)\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$.

Are there any other possible answers? Let us return to our first step.

In quadrant I, $2x = \frac{\pi}{3}$, so $x = \frac{\pi}{6}$ as noted. Let us revolve around the circle again:

$$\begin{aligned} 2x &= \frac{\pi}{3} + 2\pi \\ &= \frac{\pi}{3} + \frac{6\pi}{3} \\ &= \frac{7\pi}{3} \end{aligned}$$

so $x = \frac{7\pi}{6}$.

One more rotation yields

$$\begin{aligned} 2x &= \frac{\pi}{3} + 4\pi \\ &= \frac{\pi}{3} + \frac{12\pi}{3} \\ &= \frac{13\pi}{3} \end{aligned}$$

$x = \frac{13\pi}{6} > 2\pi$, so this value for x is larger than 2π, so it is not a solution on $[0, 2\pi)$.

In quadrant IV, $2x = \frac{5\pi}{3}$, so $x = \frac{5\pi}{6}$ as noted. Let us revolve around the circle again:

$$\begin{aligned} 2x &= \frac{5\pi}{3} + 2\pi \\ &= \frac{5\pi}{3} + \frac{6\pi}{3} \\ &= \frac{11\pi}{3} \end{aligned}$$

so $x = \frac{11\pi}{6}$.

One more rotation yields

$$\begin{aligned} 2x &= \frac{5\pi}{3} + 4\pi \\ &= \frac{5\pi}{3} + \frac{12\pi}{3} \\ &= \frac{17\pi}{3} \end{aligned}$$

$x = \frac{17\pi}{6} > 2\pi$, so this value for x is larger than 2π, so it is not a solution on $[0, 2\pi)$.

Our solutions are $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}$, and $\frac{11\pi}{6}$. Note that whenever we solve a problem in the form of $\sin(nx) = c$, we must go around the unit circle n times.

Solving Right Triangle Problems

We can now use all of the methods we have learned to solve problems that involve applying the properties of right triangles and the Pythagorean Theorem. We begin with the familiar Pythagorean Theorem, $a^2 + b^2 = c^2$, and model an equation to fit a situation.

Example 18 Using the Pythagorean Theorem to Model an Equation

Use the Pythagorean Theorem, and the properties of right triangles to model an equation that fits the problem. One of the cables that anchors the center of the London Eye Ferris wheel to the ground must be replaced. The center of the Ferris wheel is 69.5 meters above the ground, and the second anchor on the ground is 23 meters from the base of the Ferris wheel. Approximately how long is the cable, and what is the angle of elevation (from ground up to the center of the Ferris wheel)? See **Figure 4.**

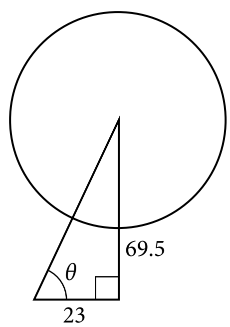


Figure 4

Solution Using the information given, we can draw a right triangle. We can find the length of the cable with the Pythagorean Theorem.

$$\begin{aligned} a^2 + b^2 &= c^2 \\ (23)^2 + (69.5)^2 &\approx 5359 \\ \sqrt{5359} &\approx 73.2 \text{ m} \end{aligned}$$

The angle of elevation is θ, formed by the second anchor on the ground and the cable reaching to the center of the wheel. We can use the tangent function to find its measure. Round to two decimal places.

$$\begin{aligned} \tan\theta &= \frac{69.5}{23} \\ \tan^{-1}\left(\frac{69.5}{23}\right) &\approx 1.2522 \\ &\approx 71.69° \end{aligned}$$

The angle of elevation is approximately 71.7°, and the length of the cable is 73.2 meters.

Example 19 Using the Pythagorean Theorem to Model an Abstract Problem

OSHA safety regulations require that the base of a ladder be placed 1 foot from the wall for every 4 feet of ladder length. Find the angle that a ladder of any length forms with the ground and the height at which the ladder touches the wall.

Solution For any length of ladder, the base needs to be a distance from the wall equal to one fourth of the ladder's length. Equivalently, if the base of the ladder is "a" feet from the wall, the length of the ladder will be $4a$ feet. See **Figure 5**.

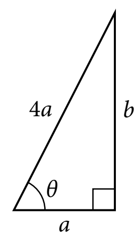


Figure 5

The side adjacent to θ is a and the hypotenuse is $4a$. Thus,

$$\cos \theta = \frac{a}{4a} = \frac{1}{4}$$

$$\cos^{-1}\left(\frac{1}{4}\right) \approx 75.5°$$

The elevation of the ladder forms an angle of 75.5° with the ground. The height at which the ladder touches the wall can be found using the Pythagorean Theorem:

$$\begin{aligned} a^2 + b^2 &= (4a)^2 \\ b^2 &= (4a)^2 - a^2 \\ b^2 &= 16a^2 - a^2 \\ b^2 &= 15a^2 \\ b &= a\sqrt{15} \end{aligned}$$

Thus, the ladder touches the wall at $a\sqrt{15}$ feet from the ground.

Access these online resources for additional instruction and practice with solving trigonometric equations.

- Solving Trigonometric Equations I (http://openstaxcollege.org/l/solvetrigeqI)
- Solving Trigonometric Equations II (http://openstaxcollege.org/l/solvetrigeqII)
- Solving Trigonometric Equations III (http://openstaxcollege.org/l/solvetrigeqIII)
- Solving Trigonometric Equations IV (http://openstaxcollege.org/l/solvetrigeqIV)
- Solving Trigonometric Equations V (http://openstaxcollege.org/l/solvetrigeqV)
- Solving Trigonometric Equations VI (http://openstaxcollege.org/l/solvetrigeqVI)

9.5 SECTION EXERCISES

VERBAL

1. Will there always be solutions to trigonometric function equations? If not, describe an equation that would not have a solution. Explain why or why not.

2. When solving a trigonometric equation involving more than one trig function, do we always want to try to rewrite the equation so it is expressed in terms of one trigonometric function? Why or why not?

3. When solving linear trig equations in terms of only sine or cosine, how do we know whether there will be solutions?

ALGEBRAIC

For the following exercises, find all solutions exactly on the interval $0 \leq \theta < 2\pi$.

4. $2\sin\theta = -\sqrt{2}$

5. $2\sin\theta = \sqrt{3}$

6. $2\cos\theta = 1$

7. $2\cos\theta = -\sqrt{2}$

8. $\tan\theta = -1$

9. $\tan x = 1$

10. $\cot x + 1 = 0$

11. $4\sin^2 x - 2 = 0$

12. $\csc^2 x - 4 = 0$

For the following exercises, solve exactly on $[0, 2\pi)$.

13. $2\cos\theta = \sqrt{2}$

14. $2\cos\theta = -1$

15. $2\sin\theta = -1$

16. $2\sin\theta = -\sqrt{3}$

17. $2\sin(3\theta) = 1$

18. $2\sin(2\theta) = \sqrt{3}$

19. $2\cos(3\theta) = -\sqrt{2}$

20. $\cos(2\theta) = -\dfrac{\sqrt{3}}{2}$

21. $2\sin(\pi\theta) = 1$

22. $2\cos\left(\dfrac{\pi}{5}\theta\right) = \sqrt{3}$

For the following exercises, find all exact solutions on $[0, 2\pi)$.

23. $\sec(x)\sin(x) - 2\sin(x) = 0$

24. $\tan(x) - 2\sin(x)\tan(x) = 0$

25. $2\cos^2 t + \cos(t) = 1$

26. $2\tan^2(t) = 3\sec(t)$

27. $2\sin(x)\cos(x) - \sin(x) + 2\cos(x) - 1 = 0$

28. $\cos^2\theta = \dfrac{1}{2}$

29. $\sec^2 x = 1$

30. $\tan^2(x) = -1 + 2\tan(-x)$

31. $8\sin^2(x) + 6\sin(x) + 1 = 0$

32. $\tan^5(x) = \tan(x)$

For the following exercises, solve with the methods shown in this section exactly on the interval $[0, 2\pi)$.

33. $\sin(3x)\cos(6x) - \cos(3x)\sin(6x) = -0.9$

34. $\sin(6x)\cos(11x) - \cos(6x)\sin(11x) = -0.1$

35. $\cos(2x)\cos x + \sin(2x)\sin x = 1$

36. $6\sin(2t) + 9\sin t = 0$

37. $9\cos(2\theta) = 9\cos^2\theta - 4$

38. $\sin(2t) = \cos t$

39. $\cos(2t) = \sin t$

40. $\cos(6x) - \cos(3x) = 0$

For the following exercises, solve exactly on the interval $[0, 2\pi)$. Use the quadratic formula if the equations do not factor.

41. $\tan^2 x - \sqrt{3}\tan x = 0$

42. $\sin^2 x + \sin x - 2 = 0$

43. $\sin^2 x - 2\sin x - 4 = 0$

44. $5\cos^2 x + 3\cos x - 1 = 0$

45. $3\cos^2 x - 2\cos x - 2 = 0$

46. $5\sin^2 x + 2\sin x - 1 = 0$

47. $\tan^2 x + 5\tan x - 1 = 0$

48. $\cot^2 x = -\cot x$

49. $-\tan^2 x - \tan x - 2 = 0$

For the following exercises, find exact solutions on the interval $[0, 2\pi)$. Look for opportunities to use trigonometric identities.

50. $\sin^2 x - \cos^2 x - \sin x = 0$

51. $\sin^2 x + \cos^2 x = 0$

52. $\sin(2x) - \sin x = 0$

53. $\cos(2x) - \cos x = 0$

54. $\dfrac{2\tan x}{2 - \sec^2 x} - \sin^2 x = \cos^2 x$

55. $1 - \cos(2x) = 1 + \cos(2x)$

56. $\sec^2 x = 7$

57. $10\sin x\cos x = 6\cos x$

58. $-3\sin t = 15\cos t\sin t$

59. $4\cos^2 x - 4 = 15\cos x$

60. $8\sin^2 x + 6\sin x + 1 = 0$

61. $8\cos^2\theta = 3 - 2\cos\theta$

62. $6\cos^2 x + 7\sin x - 8 = 0$

63. $12\sin^2 t + \cos t - 6 = 0$

64. $\tan x = 3\sin x$

65. $\cos^3 t = \cos \mathrm{t}$

GRAPHICAL

For the following exercises, algebraically determine all solutions of the trigonometric equation exactly, then verify the results by graphing the equation and finding the zeros.

66. $6\sin^2 x - 5\sin x + 1 = 0$

67. $8\cos^2 x - 2\cos x - 1 = 0$

68. $100\tan^2 x + 20\tan x - 3 = 0$

69. $2\cos^2 x - \cos x + 15 = 0$

70. $20\sin^2 x - 27\sin x + 7 = 0$

71. $2\tan^2 x + 7\tan x + 6 = 0$

72. $130\tan^2 x + 69\tan x - 130 = 0$

TECHNOLOGY

For the following exercises, use a calculator to find all solutions to four decimal places.

73. $\sin x = 0.27$

74. $\sin x = -0.55$

75. $\tan x = -0.34$

76. $\cos x = 0.71$

For the following exercises, solve the equations algebraically, and then use a calculator to find the values on the interval $[0, 2\pi)$. Round to four decimal places.

77. $\tan^2 x + 3\tan x - 3 = 0$

78. $6\tan^2 x + 13\tan x = -6$

79. $\tan^2 x - \sec x = 1$

80. $\sin^2 x - 2\cos^2 x = 0$

81. $2\tan^2 x + 9\tan x - 6 = 0$

82. $4\sin^2 x + \sin(2x)\sec x - 3 = 0$

EXTENSIONS

For the following exercises, find all solutions exactly to the equations on the interval $[0, 2\pi)$.

83. $\csc^2 x - 3\csc x - 4 = 0$

84. $\sin^2 x - \cos^2 x - 1 = 0$

85. $\sin^2 x(1 - \sin^2 x) + \cos^2 x(1 - \sin^2 x) = 0$

86. $3\sec^2 x + 2 + \sin^2 x - \tan^2 x + \cos^2 x = 0$

87. $\sin^2 x - 1 + 2\cos(2x) - \cos^2 x = 1$

88. $\tan^2 x - 1 - \sec^3 x\cos x = 0$

89. $\dfrac{\sin(2x)}{\sec^2 x} = 0$

90. $\dfrac{\sin(2x)}{2\csc^2 x} = 0$

91. $2\cos^2 x - \sin^2 x - \cos x - 5 = 0$

92. $\dfrac{1}{\sec^2 x} + 2 + \sin^2 x + 4\cos^2 x = 4$

REAL-WORLD APPLICATIONS

93. An airplane has only enough gas to fly to a city 200 miles northeast of its current location. If the pilot knows that the city is 25 miles north, how many degrees north of east should the airplane fly?

94. If a loading ramp is placed next to a truck, at a height of 4 feet, and the ramp is 15 feet long, what angle does the ramp make with the ground?

95. If a loading ramp is placed next to a truck, at a height of 2 feet, and the ramp is 20 feet long, what angle does the ramp make with the ground?

96. A woman is watching a launched rocket currently 11 miles in altitude. If she is standing 4 miles from the launch pad, at what angle is she looking up from horizontal?

97. An astronaut is in a launched rocket currently 15 miles in altitude. If a man is standing 2 miles from the launch pad, at what angle is she looking down at him from horizontal? (Hint: this is called the angle of depression.)

98. A woman is standing 8 meters away from a 10-meter tall building. At what angle is she looking to the top of the building?

99. A man is standing 10 meters away from a 6-meter tall building. Someone at the top of the building is looking down at him. At what angle is the person looking at him?

100. A 20-foot tall building has a shadow that is 55 feet long. What is the angle of elevation of the sun?

101. A 90-foot tall building has a shadow that is 2 feet long. What is the angle of elevation of the sun?

102. A spotlight on the ground 3 meters from a 2-meter tall man casts a 6 meter shadow on a wall 6 meters from the man. At what angle is the light?

103. A spotlight on the ground 3 feet from a 5-foot tall woman casts a 15-foot tall shadow on a wall 6 feet from the woman. At what angle is the light?

For the following exercises, find a solution to the word problem algebraically. Then use a calculator to verify the result. Round the answer to the nearest tenth of a degree.

104. A person does a handstand with his feet touching a wall and his hands 1.5 feet away from the wall. If the person is 6 feet tall, what angle do his feet make with the wall?

105. A person does a handstand with her feet touching a wall and her hands 3 feet away from the wall. If the person is 5 feet tall, what angle do her feet make with the wall?

106. A 23-foot ladder is positioned next to a house. If the ladder slips at 7 feet from the house when there is not enough traction, what angle should the ladder make with the ground to avoid slipping?

CHAPTER 9 REVIEW

Key Terms

double-angle formulas identities derived from the sum formulas for sine, cosine, and tangent in which the angles are equal

even-odd identities set of equations involving trigonometric functions such that if $f(-x) = -f(x)$, the identity is odd, and if $f(-x) = f(x)$, the identity is even

half-angle formulas identities derived from the reduction formulas and used to determine half-angle values of trigonometric functions

product-to-sum formula a trigonometric identity that allows the writing of a product of trigonometric functions as a sum or difference of trigonometric functions

Pythagorean identities set of equations involving trigonometric functions based on the right triangle properties

quotient identities pair of identities based on the fact that tangent is the ratio of sine and cosine, and cotangent is the ratio of cosine and sine

reciprocal identities set of equations involving the reciprocals of basic trigonometric definitions

reduction formulas identities derived from the double-angle formulas and used to reduce the power of a trigonometric function

sum-to-product formula a trigonometric identity that allows, by using substitution, the writing of a sum of trigonometric functions as a product of trigonometric functions

Key Equations

Pythagorean identities	$\sin^2\theta + \cos^2\theta = 1$ $1 + \cot^2\theta = \csc^2\theta$ $1 + \tan^2\theta = \sec^2\theta$
Even-odd identities	$\tan(-\theta) = -\tan\theta$ $\cot(-\theta) = -\cot\theta$ $\sin(-\theta) = -\sin\theta$ $\csc(-\theta) = -\csc\theta$ $\cos(-\theta) = \cos\theta$ $\sec(-\theta) = \sec\theta$
Reciprocal identities	$\sin\theta = \dfrac{1}{\csc\theta}$ $\cos\theta = \dfrac{1}{\sec\theta}$ $\tan\theta = \dfrac{1}{\cot\theta}$ $\csc\theta = \dfrac{1}{\sin\theta}$ $\sec\theta = \dfrac{1}{\cos\theta}$ $\cot\theta = \dfrac{1}{\tan\theta}$
Quotient identities	$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

Sum Formula for Cosine	$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$
Difference Formula for Cosine	$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$
Sum Formula for Sine	$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$
Difference Formula for Sine	$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$
Sum Formula for Tangent	$\tan(\alpha+\beta)=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$
Difference Formula for Tangent	$\tan(\alpha-\beta)=\dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$
Cofunction identities	$\sin\theta=\cos\left(\frac{\pi}{2}-\theta\right)$ $\cos\theta=\sin\left(\frac{\pi}{2}-\theta\right)$ $\tan\theta=\cot\left(\frac{\pi}{2}-\theta\right)$ $\cot\theta=\tan\left(\frac{\pi}{2}-\theta\right)$ $\sec\theta=\csc\left(\frac{\pi}{2}-\theta\right)$ $\csc\theta=\sec\left(\frac{\pi}{2}-\theta\right)$
Double-angle formulas	$\sin(2\theta)=2\sin\theta\cos\theta$ $\cos(2\theta)=\cos^2\theta-\sin^2\theta$ $=1-2\sin^2\theta$ $=2\cos^2\theta-1$ $\tan(2\theta)=\dfrac{2\tan\theta}{1-\tan^2\theta}$
Reduction formulas	$\sin^2\theta=\dfrac{1-\cos(2\theta)}{2}$ $\cos^2\theta=\dfrac{1+\cos(2\theta)}{2}$ $\tan^2\theta=\dfrac{1-\cos(2\theta)}{1+\cos(2\theta)}$
Half-angle formulas	$\sin\dfrac{\alpha}{2}=\pm\sqrt{\dfrac{1-\cos\alpha}{2}}$ $\cos\dfrac{\alpha}{2}=\pm\sqrt{\dfrac{1+\cos\alpha}{2}}$ $\tan\dfrac{\alpha}{2}=\pm\sqrt{\dfrac{1-\cos\alpha}{1+\cos\alpha}}$ $=\dfrac{\sin\alpha}{1+\cos\alpha}$ $=\dfrac{1-\cos\alpha}{\sin\alpha}$

Product-to-sum Formulas

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$$

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$$

$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]$$

$$\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta)-\sin(\alpha-\beta)]$$

Sum-to-product Formulas

$$\sin\alpha+\sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\sin\alpha-\sin\beta = 2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$$

$$\cos\alpha-\cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

$$\cos\alpha+\cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

Key Concepts

9.1 Solving Trigonometric Equations with Identities

- There are multiple ways to represent a trigonometric expression. Verifying the identities illustrates how expressions can be rewritten to simplify a problem.
- Graphing both sides of an identity will verify it. See **Example 1**.
- Simplifying one side of the equation to equal the other side is another method for verifying an identity. See **Example 2** and **Example 3**.
- The approach to verifying an identity depends on the nature of the identity. It is often useful to begin on the more complex side of the equation. See **Example 4**.
- We can create an identity by simplifying an expression and then verifying it. See **Example 5**.
- Verifying an identity may involve algebra with the fundamental identities. See **Example 6** and **Example 7**.
- Algebraic techniques can be used to simplify trigonometric expressions. We use algebraic techniques throughout this text, as they consist of the fundamental rules of mathematics. See **Example 8**, **Example 9**, and **Example 10**.

9.2 Sum and Difference Identities

- The sum formula for cosines states that the cosine of the sum of two angles equals the product of the cosines of the angles minus the product of the sines of the angles. The difference formula for cosines states that the cosine of the difference of two angles equals the product of the cosines of the angles plus the product of the sines of the angles.
- The sum and difference formulas can be used to find the exact values of the sine, cosine, or tangent of an angle. See **Example 1** and **Example 2**.
- The sum formula for sines states that the sine of the sum of two angles equals the product of the sine of the first angle and cosine of the second angle plus the product of the cosine of the first angle and the sine of the second angle. The difference formula for sines states that the sine of the difference of two angles equals the product of the sine of the first angle and cosine of the second angle minus the product of the cosine of the first angle and the sine of the second angle. See **Example 3**.
- The sum and difference formulas for sine and cosine can also be used for inverse trigonometric functions. See **Example 4**.

- The sum formula for tangent states that the tangent of the sum of two angles equals the sum of the tangents of the angles divided by 1 minus the product of the tangents of the angles. The difference formula for tangent states that the tangent of the difference of two angles equals the difference of the tangents of the angles divided by 1 plus the product of the tangents of the angles. See **Example 5**.
- The Pythagorean Theorem along with the sum and difference formulas can be used to find multiple sums and differences of angles. See **Example 6**.
- The cofunction identities apply to complementary angles and pairs of reciprocal functions. See **Example 7**.
- Sum and difference formulas are useful in verifying identities. See **Example 8** and **Example 9**.
- Application problems are often easier to solve by using sum and difference formulas. See **Example 10** and **Example 11**.

9.3 Double-Angle, Half-Angle, and Reduction Formulas

- Double-angle identities are derived from the sum formulas of the fundamental trigonometric functions: sine, cosine, and tangent. See **Example 1**, **Example 2**, **Example 3**, and **Example 4**.
- Reduction formulas are especially useful in calculus, as they allow us to reduce the power of the trigonometric term. See **Example 5** and **Example 6**.
- Half-angle formulas allow us to find the value of trigonometric functions involving half-angles, whether the original angle is known or not. See **Example 7**, **Example 8**, and **Example 9**.

9.4 Sum-to-Product and Product-to-Sum Formulas

- From the sum and difference identities, we can derive the product-to-sum formulas and the sum-to-product formulas for sine and cosine.
- We can use the product-to-sum formulas to rewrite products of sines, products of cosines, and products of sine and cosine as sums or differences of sines and cosines. See **Example 1**, **Example 2**, and **Example 3**.
- We can also derive the sum-to-product identities from the product-to-sum identities using substitution.
- We can use the sum-to-product formulas to rewrite sum or difference of sines, cosines, or products sine and cosine as products of sines and cosines. See **Example 4**.
- Trigonometric expressions are often simpler to evaluate using the formulas. See **Example 5**.
- The identities can be verified using other formulas or by converting the expressions to sines and cosines. To verify an identity, we choose the more complicated side of the equals sign and rewrite it until it is transformed into the other side. See **Example 6** and **Example 7**.

9.5 Solving Trigonometric Equations

- When solving linear trigonometric equations, we can use algebraic techniques just as we do solving algebraic equations. Look for patterns, like the difference of squares, quadratic form, or an expression that lends itself well to substitution. See **Example 1**, **Example 2**, and **Example 3**.
- Equations involving a single trigonometric function can be solved or verified using the unit circle. See **Example 4**, **Example 5**, and **Example 6**, and **Example 7**.
- We can also solve trigonometric equations using a graphing calculator. See **Example 8** and **Example 9**.
- Many equations appear quadratic in form. We can use substitution to make the equation appear simpler, and then use the same techniques we use solving an algebraic quadratic: factoring, the quadratic formula, etc. See **Example 10**, **Example 11**, **Example 12**, and **Example 13**.
- We can also use the identities to solve trigonometric equation. See **Example 14**, **Example 15**, and **Example 16**.
- We can use substitution to solve a multiple-angle trigonometric equation, which is a compression of a standard trigonometric function. We will need to take the compression into account and verify that we have found all solutions on the given interval. See **Example 17**.
- Real-world scenarios can be modeled and solved using the Pythagorean Theorem and trigonometric functions. See **Example 18**.

CHAPTER 9 REVIEW EXERCISES

SOLVING TRIGONOMETRIC EQUATIONS WITH IDENTITIES

For the following exercises, find all solutions exactly that exist on the interval $[0, 2\pi)$.

1. $\csc^2 t = 3$

2. $\cos^2 x = \frac{1}{4}$

3. $2 \sin \theta = -1$

4. $\tan x \sin x + \sin(-x) = 0$

5. $9\sin \omega - 2 = 4\sin^2 \omega$

6. $1 - 2\tan(\omega) = \tan^2(\omega)$

For the following exercises, use basic identities to simplify the expression.

7. $\sec x \cos x + \cos x - \frac{1}{\sec x}$

8. $\sin^3 x + \cos^2 x \sin x$

For the following exercises, determine if the given identities are equivalent.

9. $\sin^2 x + \sec^2 x - 1 = \frac{(1 - \cos^2 x)(1 + \cos^2 x)}{\cos^2 x}$

10. $\tan^3 x \csc^2 x \cot^2 x \cos x \sin x = 1$

SUM AND DIFFERENCE IDENTITIES

For the following exercises, find the exact value.

11. $\tan\left(\frac{7\pi}{12}\right)$

12. $\cos\left(\frac{25\pi}{12}\right)$

13. $\sin(70°)\cos(25°) - \cos(70°)\sin(25°)$

14. $\cos(83°)\cos(23°) + \sin(83°)\sin(23°)$

For the following exercises, prove the identity.

15. $\cos(4x) - \cos(3x)\cos x = \sin^2 x - 4\cos^2 x \sin^2 x$

16. $\cos(3x) - \cos^3 x = -\cos x \sin^2 x - \sin x \sin(2x)$

For the following exercise, simplify the expression.

17. $\dfrac{\tan\left(\frac{1}{2}x\right) + \tan\left(\frac{1}{8}x\right)}{1 - \tan\left(\frac{1}{8}x\right)\tan\left(\frac{1}{2}x\right)}$

For the following exercises, find the exact value.

18. $\cos\left(\sin^{-1}(0) - \cos^{-1}\left(\frac{1}{2}\right)\right)$

19. $\tan\left(\sin^{-1}(0) + \sin^{-1}\left(\frac{1}{2}\right)\right)$

DOUBLE-ANGLE, HALF-ANGLE, AND REDUCTION FORMULAS

For the following exercises, find the exact value.

20. Find $\sin(2\theta)$, $\cos(2\theta)$, and $\tan(2\theta)$ given $\cos \theta = -\frac{1}{3}$ and θ is in the interval $\left[\frac{\pi}{2}, \pi\right]$

21. Find $\sin(2\theta)$, $\cos(2\theta)$, and $\tan(2\theta)$ given $\sec \theta = -\frac{5}{3}$ and θ is in the interval $\left[\frac{\pi}{2}, \pi\right]$

22. $\sin\left(\frac{7\pi}{8}\right)$

23. $\sec\left(\frac{3\pi}{8}\right)$

For the following exercises, use **Figure 1** to find the desired quantities.

24. $\sin(2\beta)$, $\cos(2\beta)$, $\tan(2\beta)$, $\sin(2\alpha)$, $\cos(2\alpha)$, and $\tan(2\alpha)$

25. $\sin\left(\frac{\beta}{2}\right)$, $\cos\left(\frac{\beta}{2}\right)$, $\tan\left(\frac{\beta}{2}\right)$, $\sin\left(\frac{\alpha}{2}\right)$, $\cos\left(\frac{\alpha}{2}\right)$, and $\tan\left(\frac{\alpha}{2}\right)$

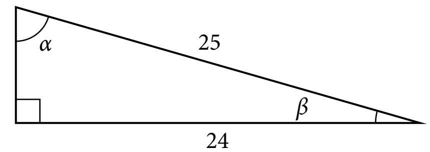


Figure 1

For the following exercises, prove the identity.

26. $\dfrac{2\cos(2x)}{\sin(2x)} = \cot x - \tan x$

27. $\cot x\cos(2x) = -\sin(2x) + \cot x$

For the following exercises, rewrite the expression with no powers.

28. $\cos^2 x\sin^4(2x)$

29. $\tan^2 x\sin^3 x$

SUM-TO-PRODUCT AND PRODUCT-TO-SUM FORMULAS

For the following exercises, evaluate the product for the given expression using a sum or difference of two functions. Write the exact answer.

30. $\cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right)$

31. $2\sin\left(\frac{2\pi}{3}\right)\sin\left(\frac{5\pi}{6}\right)$

32. $2\cos\left(\frac{\pi}{5}\right)\cos\left(\frac{\pi}{3}\right)$

For the following exercises, evaluate the sum by using a product formula. Write the exact answer.

33. $\sin\left(\frac{\pi}{12}\right) - \sin\left(\frac{7\pi}{12}\right)$

34. $\cos\left(\frac{5\pi}{12}\right) + \cos\left(\frac{7\pi}{12}\right)$

For the following exercises, change the functions from a product to a sum or a sum to a product.

35. $\sin(9x)\cos(3x)$

36. $\cos(7x)\cos(12x)$

37. $\sin(11x) + \sin(2x)$

38. $\cos(6x) + \cos(5x)$

SOLVING TRIGONOMETRIC EQUATIONS

For the following exercises, find all exact solutions on the interval $[0, 2\pi)$.

39. $\tan x + 1 = 0$

40. $2\sin(2x) + \sqrt{2} = 0$

For the following exercises, find all exact solutions on the interval $[0, 2\pi)$.

41. $2\sin^2 x - \sin x = 0$

42. $\cos^2 x - \cos x - 1 = 0$

43. $2\sin^2 x + 5\sin x + 3 = 0$

44. $\cos x - 5\sin(2x) = 0$

45. $\dfrac{1}{\sec^2 x} + 2 + \sin^2 x + 4\cos^2 x = 0$

For the following exercises, simplify the equation algebraically as much as possible. Then use a calculator to find the solutions on the interval $[0, 2\pi)$. Round to four decimal places.

46. $\sqrt{3}\cot^2 x + \cot x = 1$

47. $\csc^2 x - 3\csc x - 4 = 0$

For the following exercises, graph each side of the equation to find the zeroes on the interval $[0, 2\pi)$.

48. $20\cos^2 x + 21\cos x + 1 = 0$

49. $\sec^2 x - 2\sec x = 15$

CHAPTER 9 PRACTICE TEST

For the following exercises, simplify the given expression.

1. $\cos(-x)\sin x \cot x + \sin^2 x$
2. $\sin(-x)\cos(-2x) - \sin(-x)\cos(-2x)$
3. $\csc(\theta)\cot(\theta)(\sec^2 \theta - 1)$
4. $\cos^2(\theta)\sin^2(\theta)\left(1 + \cot^2(\theta)\right)\left(1 + \tan^2(\theta)\right)$

For the following exercises, find the exact value.

5. $\cos\left(\frac{7\pi}{12}\right)$
6. $\tan\left(\frac{3\pi}{8}\right)$
7. $\tan\left(\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) + \tan^{-1}\sqrt{3}\right)$
8. $2\sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right)$
9. $\cos\left(\frac{4\pi}{3} + \theta\right)$
10. $\tan\left(-\frac{\pi}{4} + \theta\right)$

For the following exercises, simplify each expression. Do not evaluate.

11. $\cos^2(32°)\tan^2(32°)$
12. $\cot\left(\frac{\theta}{2}\right)$

For the following exercises, find all exact solutions to the equation on $[0, 2\pi)$.

13. $\cos^2 x - \sin^2 x - 1 = 0$
14. $\cos^2 x = \cos x\, 4\sin^2 x + 2\sin x - 3 = 0$
15. $\cos(2x) + \sin^2 x = 0$
16. $2\sin^2 x - \sin x = 0$
17. Rewrite the sum as a product: $\cos(2x) + \cos(-8x)$.
18. Rewrite the product as a sum or difference: $8\cos(15x)\sin(3x)$
19. Rewrite the difference as a product: $2\sin(8\theta) - \sin(4\theta)$
20. Find all solutions of $\tan(x) - \sqrt{3} = 0$.
21. Find the solutions of $\sec^2 x - 2\sec x = 15$ on the interval $[0, 2\pi)$ algebraically; then graph both sides of the equation to determine the answer.

For the following exercises, find all exact solutions to the equation on $[0, 2\pi)$.

22. $2\cos\left(\frac{\theta}{2}\right) = 1$
23. $\sqrt{3}\cot(y) = 1$
24. Find $\sin(2\theta)$, $\cos(2\theta)$, and $\tan(2\theta)$ given $\cot\theta = -\frac{3}{4}$ and θ is on the interval $\left[\frac{\pi}{2}, \pi\right]$.
25. Find $\sin\left(\frac{\theta}{2}\right)$, $\cos\left(\frac{\theta}{2}\right)$, and $\tan\left(\frac{\theta}{2}\right)$ given $\cos\theta = \frac{7}{25}$ and θ is in quadrant IV.
26. Rewrite the expression $\sin^4 x$ with no powers greater than 1.

For the following exercises, prove the identity.

27. $\tan^3 x - \tan x \sec^2 x = \tan(-x)$
28. $\sin(3x) - \cos x \sin(2x) = \cos^2 x \sin x - \sin^3 x$
29. $\frac{\sin(2x)}{\sin x} - \frac{\cos(2x)}{\cos x} = \sec x$
30. Plot the points and find a function of the form $y = A\cos(Bx + C) + D$ that fits the given data.

x	0	1	2	3	4	5
y	−2	2	−2	2	−2	2

31. The displacement $h(t)$ in centimeters of a mass suspended by a spring is modeled by the function $h(t) = \frac{1}{4}\sin(120\pi t)$, where t is measured in seconds. Find the amplitude, period, and frequency of this displacement.

32. A woman is standing 300 feet away from a 2,000-foot building. If she looks to the top of the building, at what angle above horizontal is she looking? A bored worker looks down at her from the 15^{th} floor (1,500 feet above her). At what angle is he looking down at her? Round to the nearest tenth of a degree.

33. Two frequencies of sound are played on an instrument governed by the equation $n(t) = 8\cos(20\pi t)\cos(1{,}000\pi t)$. What are the period and frequency of the "fast" and "slow" oscillations? What is the amplitude?

34. The average monthly snowfall in a small village in the Himalayas is 6 inches, with the low of 1 inch occurring in July. Construct a function that models this behavior. During what period is there more than 10 inches of snowfall?

35. A spring attached to a ceiling is pulled down 20 cm. After 3 seconds, wherein it completes 6 full periods, the amplitude is only 15 cm. Find the function modeling the position of the spring t seconds after being released. At what time will the spring come to rest? In this case, use 1 cm amplitude as rest.

36. Water levels near a glacier currently average 9 feet, varying seasonally by 2 inches above and below the average and reaching their highest point in January. Due to global warming, the glacier has begun melting faster than normal. Every year, the water levels rise by a steady 3 inches. Find a function modeling the depth of the water t months from now. If the docks are 2 feet above current water levels, at what point will the water first rise above the docks?

10
Further Applications of Trigonometry

Figure 1 General Sherman, the world's largest living tree. (credit: Mike Baird, Flickr)

CHAPTER OUTLINE

Introduction

The world's largest tree by volume, named General Sherman, stands 274.9 feet tall and resides in Northern California.[27] Just how do scientists know its true height? A common way to measure the height involves determining the angle of elevation, which is formed by the tree and the ground at a point some distance away from the base of the tree. This method is much more practical than climbing the tree and dropping a very long tape measure.

In this chapter, we will explore applications of trigonometry that will enable us to solve many different kinds of problems, including finding the height of a tree. We extend topics we introduced in **Trigonometric Functions** and investigate applications more deeply and meaningfully.

27 Source: National Park Service. "The General Sherman Tree." http://www.nps.gov/seki/naturescience/sherman.htm. Accessed April 25, 2014.

LEARNING OBJECTIVES

In this section, you will:

- Use the Law of Sines to solve oblique triangles.
- Find the area of an oblique triangle using the sine function.
- Solve applied problems using the Law of Sines.

10.1 NON-RIGHT TRIANGLES: LAW OF SINES

Suppose two radar stations located 20 miles apart each detect an aircraft between them. The angle of elevation measured by the first station is 35 degrees, whereas the angle of elevation measured by the second station is 15 degrees. How can we determine the altitude of the aircraft? We see in **Figure 1** that the triangle formed by the aircraft and the two stations is not a right triangle, so we cannot use what we know about right triangles. In this section, we will find out how to solve problems involving non-right triangles.

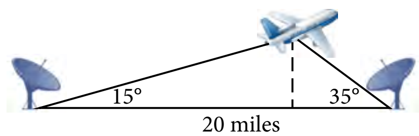


Figure 1

Using the Law of Sines to Solve Oblique Triangles

In any triangle, we can draw an **altitude**, a perpendicular line from one vertex to the opposite side, forming two right triangles. It would be preferable, however, to have methods that we can apply directly to non-right triangles without first having to create right triangles.

Any triangle that is not a right triangle is an **oblique triangle**. Solving an oblique triangle means finding the measurements of all three angles and all three sides. To do so, we need to start with at least three of these values, including at least one of the sides. We will investigate three possible oblique triangle problem situations:

1. **ASA (angle-side-angle)** We know the measurements of two angles and the included side. See **Figure 2**.

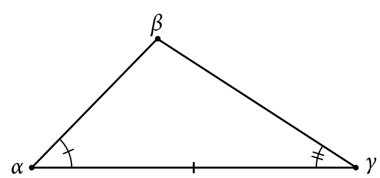


Figure 2

2. **AAS (angle-angle-side)** We know the measurements of two angles and a side that is not between the known angles. See **Figure 3**.

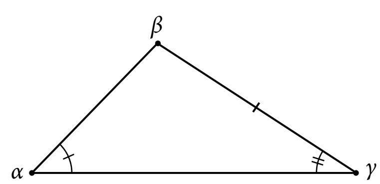


Figure 3

3. **SSA (side-side-angle)** We know the measurements of two sides and an angle that is not between the known sides. See **Figure 4**.

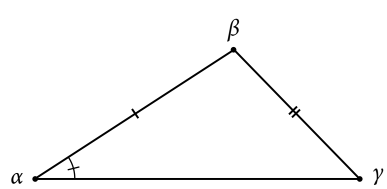


Figure 4

Knowing how to approach each of these situations enables us to solve oblique triangles without having to drop a perpendicular to form two right triangles. Instead, we can use the fact that the ratio of the measurement of one of the angles to the length of its opposite side will be equal to the other two ratios of angle measure to opposite side. Let's see how this statement is derived by considering the triangle shown in **Figure 5**.

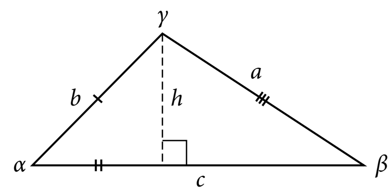


Figure 5

Using the right triangle relationships, we know that $\sin \alpha = \frac{h}{b}$ and $\sin \beta = \frac{h}{a}$. Solving both equations for h gives two different expressions for h.

$$h = b\sin \alpha \text{ and } h = a\sin \beta$$

We then set the expressions equal to each other.

$$b\sin \alpha = a\sin \beta$$

$$\left(\frac{1}{ab}\right)(b\sin \alpha) = (a\sin \beta)\left(\frac{1}{ab}\right) \qquad \text{Multiply both sides by } \frac{1}{ab}.$$

$$\frac{\sin \alpha}{a} = \frac{\sin \beta}{b}$$

Similarly, we can compare the other ratios.

$$\frac{\sin \alpha}{a} = \frac{\sin \gamma}{c} \text{ and } \frac{\sin \beta}{b} = \frac{\sin \gamma}{c}$$

Collectively, these relationships are called the **Law of Sines.**

$$\frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin \gamma}{c}$$

Note the standard way of labeling triangles: angle α (alpha) is opposite side a; angle β (beta) is opposite side b; and angle γ (gamma) is opposite side c. See **Figure 6**.

While calculating angles and sides, be sure to carry the exact values through to the final answer. Generally, final answers are rounded to the nearest tenth, unless otherwise specified.

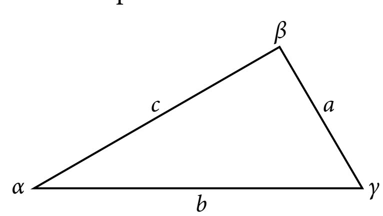


Figure 6

Law of Sines

Given a triangle with angles and opposite sides labeled as in **Figure 6**, the ratio of the measurement of an angle to the length of its opposite side will be equal to the other two ratios of angle measure to opposite side. All proportions will be equal. The **Law of Sines** is based on proportions and is presented symbolically two ways.

$$\frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin \gamma}{c}$$

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma}$$

To solve an oblique triangle, use any pair of applicable ratios.

Example 1 **Solving for Two Unknown Sides and Angle of an AAS Triangle**

Solve the triangle shown in **Figure 7** to the nearest tenth.

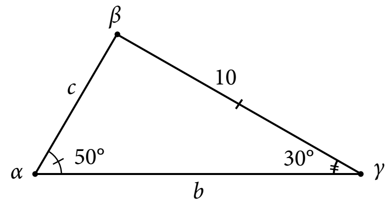


Figure 7

Solution The three angles must add up to 180 degrees. From this, we can determine that

$$\begin{aligned}\beta &= 180° - 50° - 30° \\ &= 100°\end{aligned}$$

To find an unknown side, we need to know the corresponding angle and a known ratio. We know that angle $\alpha = 50°$ and its corresponding side $a = 10$. We can use the following proportion from the Law of Sines to find the length of c.

$$\begin{aligned}\frac{\sin(50°)}{10} &= \frac{\sin(30°)}{c} \\ c\frac{\sin(50°)}{10} &= \sin(30°) && \text{Multiply both sides by } c. \\ c &= \sin(30°)\frac{10}{\sin(50°)} && \text{Multiply by the reciprocal to isolate } c. \\ c &\approx 6.5\end{aligned}$$

Similarly, to solve for b, we set up another proportion.

$$\begin{aligned}\frac{\sin(50°)}{10} &= \frac{\sin(100°)}{b} \\ b\sin(50°) &= 10\sin(100°) && \text{Multiply both sides by } b. \\ b &= \frac{10\sin(100°)}{\sin(50°)} && \text{Multiply by the reciprocal to isolate } b. \\ b &\approx 12.9\end{aligned}$$

Therefore, the complete set of angles and sides is

$$\begin{aligned}\alpha &= 50° & a &= 10 \\ \beta &= 100° & b &\approx 12.9 \\ \gamma &= 30° & c &\approx 6.5\end{aligned}$$

See page 764.5

Try It #1

Solve the triangle shown in **Figure 8** to the nearest tenth.

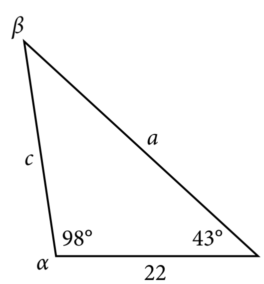


Figure 8

Using The Law of Sines to Solve SSA Triangles

We can use the Law of Sines to solve any oblique triangle, but some solutions may not be straightforward. In some cases, more than one triangle may satisfy the given criteria, which we describe as an **ambiguous case**. Triangles classified as SSA, those in which we know the lengths of two sides and the measurement of the angle opposite one of the given sides, may result in one or two solutions, or even no solution.

Example 1.5 In a triangle assume that the angle α satisfies $\sin\alpha = 3/8$, β satisfies $\sin\beta = 9/16$, and $a = 24$. Find b.

Solution We are ready to use the Law of Sines. $a/\sin\alpha = b/\sin\beta \Rightarrow b = (a\sin\beta)/\sin\alpha = (24)\,(9/16)/(3/8) = 36$.

Example 1.6 In a triangle assume that $\alpha = \pi/4$, that $\gamma = \pi/12$, and that $a = 6$. Find b.

Solution Before we can use the Law of Sines to find b we must obtain β. $\beta = \pi - \alpha - \gamma = \pi - \pi/4 - \pi/12 = 2\pi/3$. Now we are ready. $a/\sin\alpha = b/\sin\beta \Leftrightarrow b = (a\sin\beta)/\sin\alpha = (6)\,(\sqrt{3}/2)\,/\,(1/\sqrt{2}) = 3\sqrt{6}$.

possible outcomes for SSA triangles

Oblique triangles in the category SSA may have four different outcomes. **Figure 9** illustrates the solutions with the known sides a and b and known angle α.

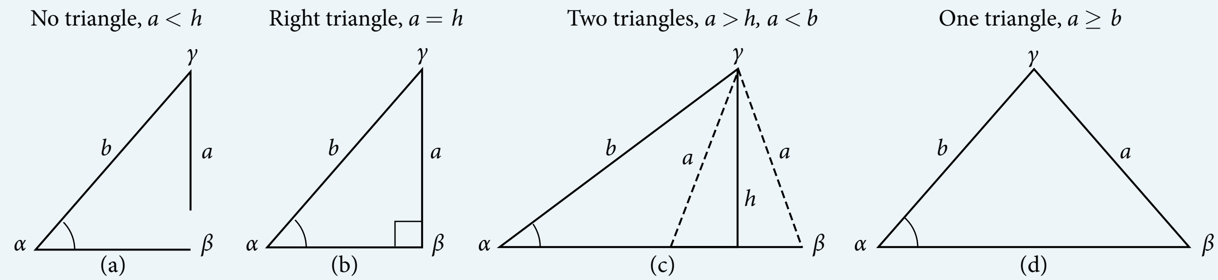


Figure 9

Figure 9 is for acute angles α. For obtuse angles α see page 767.5.

Example 2 Solving an Oblique SSA Triangle

Solve the triangle in **Figure 10** for the missing side and find the missing angle measures to the nearest tenth.

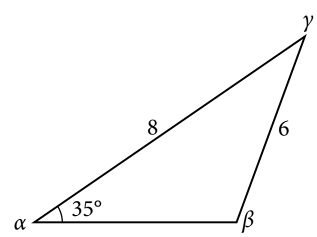


Figure 10

Solution Use the Law of Sines to find angle β and angle γ, and then side c. Solving for β, we have the proportion

$$\begin{aligned}
\frac{\sin \alpha}{a} &= \frac{\sin \beta}{b} \\
\frac{\sin(35°)}{6} &= \frac{\sin \beta}{8} \\
\frac{8\sin(35°)}{6} &= \sin \beta \\
0.7648 &\approx \sin \beta \\
\sin^{-1}(0.7648) &\approx 49.9° \\
\beta &\approx 49.9°
\end{aligned}$$

An obtuse angle is an angle whose measure θ satisfies $90° < \theta < 180°$ or, in radians, $\pi/2 < \theta < \pi$.

However, in the diagram, angle β appears to be an obtuse angle and may be greater than 90°. How did we get an acute angle, and how do we find the measurement of β? Let's investigate further. Dropping a perpendicular from γ and viewing the triangle from a right angle perspective, we have **Figure 11**. It appears that there may be a second triangle that will fit the given criteria.

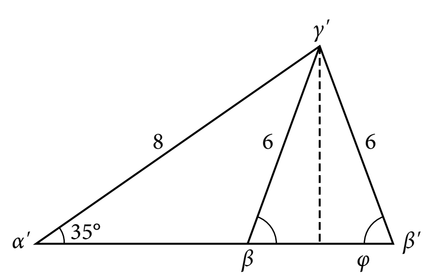


Figure 11

The angle supplementary to β is approximately equal to 49.9°, which means that $\beta = 180° - 49.9° = 130.1°$. (Remember that the sine function is positive in both the first and second quadrants.) Solving for γ, we have

$$\gamma = 180° - 35° - 130.1° \approx 14.9°$$

We can then use these measurements to solve the other triangle. Since γ' is supplementary to α' and β', we have

$$\gamma' = 180° - 35° - 49.9° \approx 95.1°$$

Now we need to find c and c'.

We have

$$\frac{c}{\sin(14.9°)} = \frac{6}{\sin(35°)}$$

$$c = \frac{6\sin(14.9°)}{\sin(35°)} \approx 2.7$$

Finally,

$$\frac{c'}{\sin(95.1°)} = \frac{6}{\sin(35°)}$$

$$c' = \frac{6\sin(95.1°)}{\sin(35°)} \approx 10.4$$

To summarize, there are two triangles with an angle of 35°, an adjacent side of 8, and an opposite side of 6, as shown in **Figure 12**.

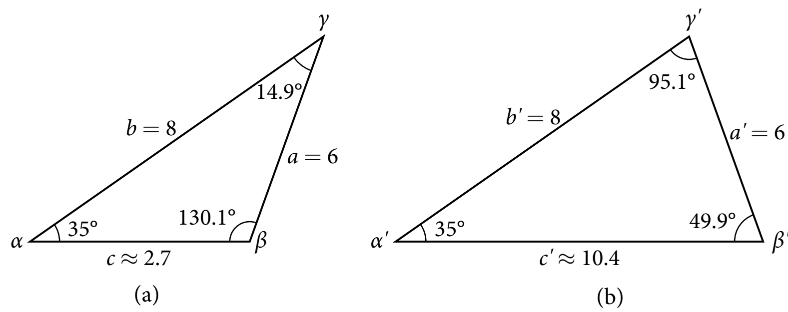


Figure 12

However, we were looking for the values for the triangle with an obtuse angle β. We can see them in the first triangle (a) in **Figure 12**.

Try It #2

Given $\alpha = 80°$, $a = 120$, and $b = 121$, find the missing side and angles. If there is more than one possible solution, show both.

Example 3 Solving for the Unknown Sides and Angles of a SSA Triangle

In the triangle shown in **Figure 13**, solve for the unknown side and angles. Round your answers to the nearest tenth.

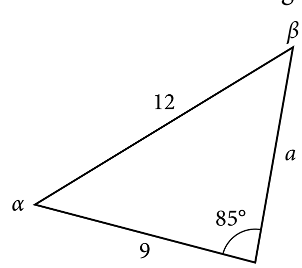


Figure 13

Solution In choosing the pair of ratios from the Law of Sines to use, look at the information given. In this case, we know the angle $\gamma = 85°$, and its corresponding side $c = 12$, and we know side $b = 9$. We will use this proportion to solve for β.

$$\frac{\sin(85°)}{12} = \frac{\sin\beta}{9} \quad \text{Isolate the unknown.}$$

$$\frac{9\sin(85°)}{12} = \sin\beta$$

To find β, apply the inverse sine function. The inverse sine will produce a single result, but keep in mind that there may be two values for β. It is important to verify the result, as there may be two viable solutions, only one solution (the usual case), or no solutions.

$$\beta = \sin^{-1}\left(\frac{9\sin(85°)}{12}\right)$$
$$\beta \approx \sin^{-1}(0.7471)$$
$$\beta \approx 48.3°$$

In this case, if we subtract β from 180°, we find that there may be a second possible solution. Thus, $\beta = 180° - 48.3° \approx 131.7°$. To check the solution, subtract both angles, 131.7° and 85°, from 180°. This gives

$$\alpha = 180° - 85° - 131.7° \approx -36.7°,$$

which is impossible, and so $\beta \approx 48.3°$.

To find the remaining missing values, we calculate $\alpha = 180° - 85° - 48.3° \approx 46.7°$. Now, only side a is needed. Use the Law of Sines to solve for a by one of the proportions.

$$\frac{\sin(85°)}{12} = \frac{\sin(46.7°)}{a}$$
$$a\frac{\sin(85°)}{12} = \sin(46.7°)$$
$$a = \frac{12\sin(46.7°)}{\sin(85°)} \approx 8.8$$

The complete set of solutions for the given triangle is

$$\begin{array}{ll} \alpha \approx 46.7° & a \approx 8.8 \\ \beta \approx 48.3° & b = 9 \\ \gamma = 85° & c = 12 \end{array}$$

Try It #3

Given $\alpha = 80°$, $a = 100$, $b = 10$, find the missing side and angles. If there is more than one possible solution, show both. Round your answers to the nearest tenth.

Example 4 **Finding the Triangles That Meet the Given Criteria**

Find all possible triangles if one side has length 4 opposite an angle of 50°, and a second side has length 10.

Solution Using the given information, we can solve for the angle opposite the side of length 10. See **Figure 14**.

$$\frac{\sin\alpha}{10} = \frac{\sin(50°)}{4}$$
$$\sin\alpha = \frac{10\sin(50°)}{4}$$
$$\sin\alpha \approx 1.915$$

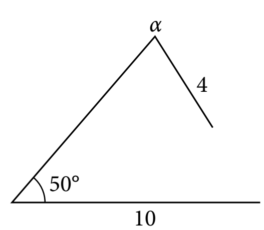


Figure 14

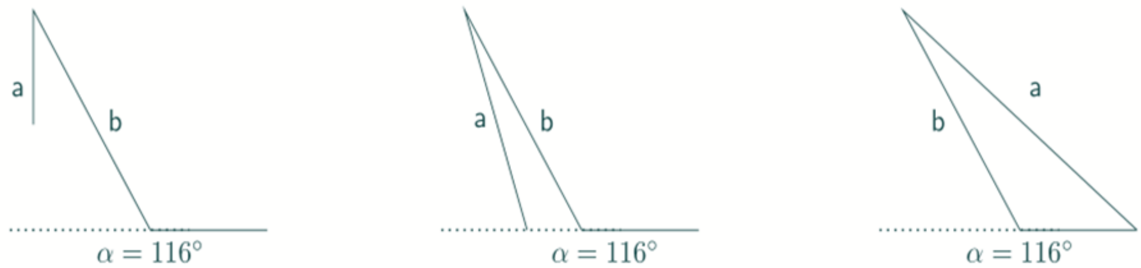
We can stop here without finding the value of α. Because the range of the sine function is $[-1, 1]$, it is impossible for the sine value to be 1.915. In fact, inputting $\sin^{-1}(1.915)$ in a graphing calculator generates an **ERROR DOMAIN**. Therefore, no triangles can be drawn with the provided dimensions.

Try It #4

Determine the number of triangles possible given $a = 31$, $b = 26$, $\beta = 48°$.

The content of the blue box on page 765 on possible outcomes for SSA triangles is perfect if the angle α is acute. If $90^\circ \le \alpha < 180^\circ$ the situation is different, but simpler. Figure 9.5 below illustrates what can happen in an SSA triangle when $90^\circ \le \alpha$. The quick summary is: If α is obtuse then there is one triangle if $a > b$, and otherwise there is no triangle.

No Triangle, $a < h$ | No Triangle, $a \ge h, a \le b$ | One Triangle, $a < b$



In the middle case, $a \ge h, a \le b$, the drawing contains a triangle. But 116° is not one of the interior angles of the triangle, so there is no triangle with the given values of α, a and b.

Example 4.5 Assume that $\alpha = 2\pi/3$ and $b = 7\sqrt{2}$. Solve the triangle if: (a) $a = 7$ (b) $a = 7\sqrt{3}$

(a) Since $a < b$ there is no solution. (It does not matter, but we note that $h = b \sin 2\pi/3 = (7\sqrt{2})\,(\sqrt{3}/2) = 7\sqrt{3}/\sqrt{2} > a = 7$. So in the figure above, the left-hand picture applies.)

(b) Since $a > b$ there is one triangle. $\sin\beta = (b/a)\sin\alpha = \left((7\sqrt{2})\,/\,(7\sqrt{3})\right)(\sqrt{3}/2) = 1/\sqrt{2}$. Since α is obtuse, β is necessarily acute, so $\beta = \pi/4$. We lack γ and c. $\gamma = \pi - \alpha - \beta = \pi - 2\pi/3 - \pi/4 = \pi/12$. $c = b\sin\gamma\,/\sin\beta = (7\sqrt{2})\,(\sin(\pi/12))/\,(1\,/\sqrt{2}) = 14\sin(\pi/12)$. (We note that we can use a half-angle formula and get an exact value for $\sin(\pi/12)$, but in this section we will leave our answers in this form.)

Example 4.6 Assume that α is obtuse, that $\sin\alpha = 3/4$ and that $b = 5$. Obtain β if: (a) $a = 4$ (b) $a = 11$

(a) Since $a < b$ there is no solution. (It does not matter, but we note that $h = b\sin\alpha = 5\,(3/4) = 15/4 < a = 4$. So in the figure above, the middle picture applies.)

(b) Since $a > b$ there is one triangle. $\sin\beta = (b/a)\sin\alpha = (5/11)\,(3/4) = 15/44$. Since α is obtuse, β is necessarily acute, so $\beta = \arcsin(15/44)$.

Finding the Area of an Oblique Triangle Using the Sine Function

Now that we can solve a triangle for missing values, we can use some of those values and the sine function to find the area of an oblique triangle. Recall that the area formula for a triangle is given as Area $= \frac{1}{2}bh$, where b is base and h is height. For oblique triangles, we must find h before we can use the area formula. Observing the two triangles in **Figure 15**, one acute and one obtuse, we can drop a perpendicular to represent the height and then apply the trigonometric property $\sin\alpha = \frac{\text{opposite}}{\text{hypotenuse}}$ to write an equation for area in oblique triangles. In the acute triangle, we have $\sin\alpha = \frac{h}{c}$ or $c\sin\alpha = h$. However, in the obtuse triangle, we drop the perpendicular outside the triangle and extend the base b to form a right triangle. The angle used in calculation is α', or $180 - \alpha$.

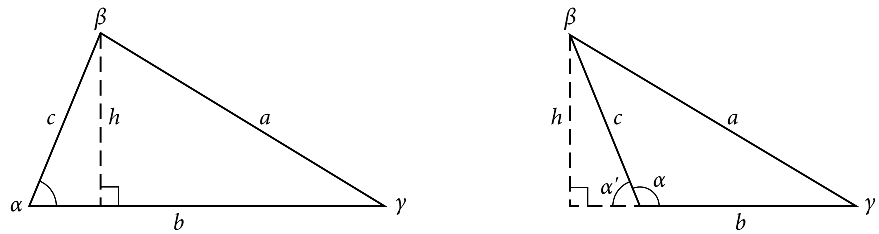


Figure 15

Thus,

$$\text{Area} = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}b(c\sin\alpha)$$

Similarly,

$$\text{Area} = \frac{1}{2}a(b\sin\gamma) = \frac{1}{2}a(c\sin\beta)$$

area of an oblique triangle

The formula for the area of an oblique triangle is given by

$$\begin{aligned}\text{Area} &= \frac{1}{2}bc\sin\alpha\\ &= \frac{1}{2}ac\sin\beta\\ &= \frac{1}{2}ab\sin\gamma\end{aligned}$$

This is equivalent to one-half of the product of two sides and the sine of their included angle.

Example 5 Finding the Area of an Oblique Triangle

Find the area of a triangle with sides $a = 90$, $b = 52$, and angle $\gamma = 102°$. Round the area to the nearest integer.

Solution Using the formula, we have

$$\text{Area} = \frac{1}{2}ab\sin\gamma$$

$$\text{Area} = \frac{1}{2}(90)(52)\sin(102°)$$

$$\text{Area} \approx 2289 \text{ square units}$$

Try It #5

Find the area of the triangle given $\beta = 42°$, $a = 7.2$ ft, $c = 3.4$ ft. Round the area to the nearest tenth.

Look at the next page, page 768.5

Solving Applied Problems Using the Law of Sines

The more we study trigonometric applications, the more we discover that the applications are countless. Some are flat, diagram-type situations, but many applications in calculus, engineering, and physics involve three dimensions and motion.

In the first page of this section we mentioned ASA, AAS and SSA triangles. The formula we just saw for computing areas works only for SAS triangles, i.e., triangles in which we know two sides and the angle between them. However, if we want the area of either an ASA or AAS triangle we can use the Law of Sines to get a second side, and then find the area using this formula. If we want the area of a SSA triangle we can use the Law of Sines to get a second angle, and then get the area with this formula.

For example, if we seek the area of the AAS triangle of Example 1 we use the method shown there to get $\beta = 100°$ and $c \approx 6.5$. Then we obtain the area using Area $= ac\sin(\beta) = (10)(6.5)\cos(100°)$.

If we seek the area of the SSA triangle of Example 3 we use the book's method to obtain β and then α. Then we have Area $= bc\sin(\alpha) = (9)(12)\cos(46.7°)$.

Example 6 **Finding an Altitude**

Find the altitude of the aircraft in the problem introduced at the beginning of this section, shown in **Figure 16**. Round the altitude to the nearest tenth of a mile.

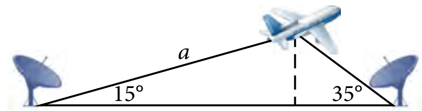


Figure 16

Solution To find the elevation of the aircraft, we first find the distance from one station to the aircraft, such as the side *a*, and then use right triangle relationships to find the height of the aircraft, *h*.

Because the angles in the triangle add up to 180 degrees, the unknown angle must be $180° - 15° - 35° = 130°$. This angle is opposite the side of length 20, allowing us to set up a Law of Sines relationship.

$$\frac{\sin(130°)}{20} = \frac{\sin(35°)}{a}$$
$$a\sin(130°) = 20\sin(35°)$$
$$a = \frac{20\sin(35°)}{\sin(130°)}$$
$$a \approx 14.98$$

The distance from one station to the aircraft is about 14.98 miles.

Now that we know *a*, we can use right triangle relationships to solve for *h*.

$$\sin(15°) = \frac{\text{opposite}}{\text{hypotenuse}}$$
$$\sin(15°) = \frac{h}{a}$$
$$\sin(15°) = \frac{h}{14.98}$$
$$h = 14.98\sin(15°)$$
$$h \approx 3.88$$

The aircraft is at an altitude of approximately 3.9 miles.

Try It #6

The diagram shown in **Figure 17** represents the height of a blimp flying over a football stadium. Find the height of the blimp if the angle of elevation at the southern end zone, point *A*, is 70°, the angle of elevation from the northern end zone, point *B*, is 62°, and the distance between the viewing points of the two end zones is 145 yards.

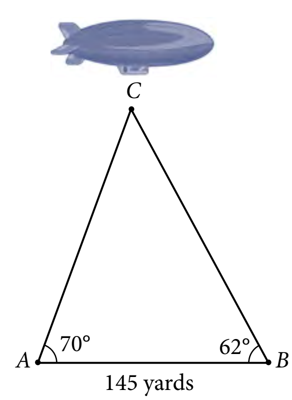


Figure 17

Access the following online resources for additional instruction and practice with trigonometric applications.

- Law of Sines: The Basics (http://openstaxcollege.org/l/sinesbasic)
- Law of Sines: The Ambiguous Case (http://openstaxcollege.org/l/sinesambiguous)

10.1 SECTION EXERCISES

VERBAL

1. Describe the altitude of a triangle.

2. Compare right triangles and oblique triangles.

3. When can you use the Law of Sines to find a missing angle?

4. In the Law of Sines, what is the relationship between the angle in the numerator and the side in the denominator?

5. What type of triangle results in an ambiguous case?

ALGEBRAIC

For the following exercises, assume α is opposite side a, β is opposite side b, and γ is opposite side c. Solve each triangle, if possible. Round each answer to the nearest tenth.

6. $\alpha = 43°, \gamma = 69°, a = 20$

7. $\alpha = 35°, \gamma = 73°, \mathrm{c} = 20$

8. $\alpha = 60°, \beta = 60°, \gamma = 60°$

9. $a = 4, \alpha = 60°, \beta = 100°$

10. $b = 10, \beta = 95°, \gamma = 30°$

For the following exercises, use the Law of Sines to solve for the missing side for each oblique triangle. Round each answer to the nearest hundredth. Assume that angle A is opposite side a, angle B is opposite side b, and angle C is opposite side c.

11. Find side b when $A = 37°, B = 49°, c = 5$.

12. Find side a when $A = 132°, C = 23°, b = 10$.

13. Find side c when $B = 37°, C = 21, b = 23$.

For the following exercises, assume α is opposite side a, β is opposite side b, and γ is opposite side c. Determine whether there is no triangle, one triangle, or two triangles. Then solve each triangle, if possible. Round each answer to the nearest tenth.

14. $\alpha = 119°, a = 14, b = 26$

15. $\gamma = 113°, b = 10, c = 32$

16. $b = 3.5, c = 5.3, \gamma = 80°$

17. $a = 12, c = 17, \alpha = 35°$

18. $a = 20.5, b = 35.0, \beta = 25°$

19. $a = 7, c = 9, \alpha = 43°$

20. $a = 7, b = 3, \beta = 24°$

21. $b = 13, c = 5, \gamma = 10°$

22. $a = 2.3, c = 1.8, \gamma = 28°$

23. $\beta = 119°, b = 8.2, a = 11.3$

For the following exercises, use the Law of Sines to solve, if possible, the missing side or angle for each triangle or triangles in the ambiguous case. Round each answer to the nearest tenth.

24. Find angle A when $a = 24, b = 5, B = 22°$.

25. Find angle A when $a = 13, b = 6, B = 20°$.

26. Find angle B when $A = 12°, a = 2, b = 9$.

For the following exercises, find the area of the triangle with the given measurements. Round each answer to the nearest tenth.

27. $a = 5, c = 6, \beta = 35°$

28. $b = 11, c = 8, \alpha = 28°$

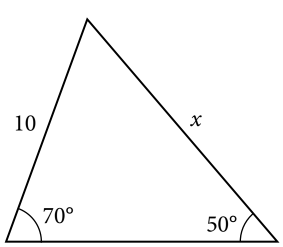
29. $a = 32, b = 24, \gamma = 75°$

30. $a = 7.2, b = 4.5, \gamma = 43°$

GRAPHICAL

For the following exercises, find the length of side x. Round to the nearest tenth.

31.



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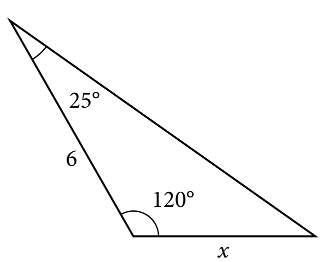

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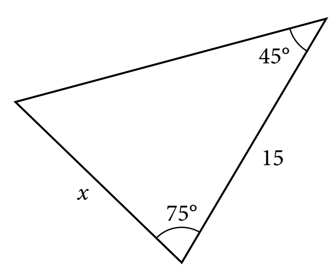

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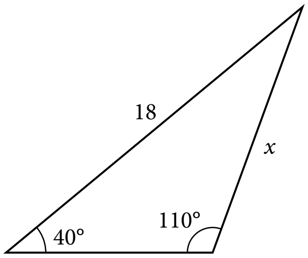

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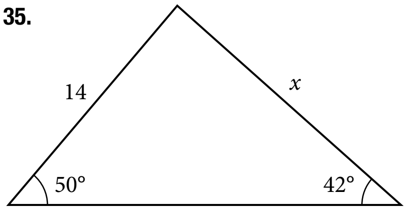

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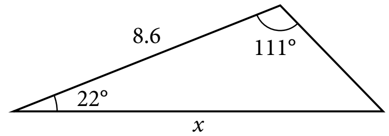
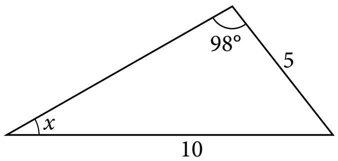


For the following exercises, find the measure of angle x, if possible. Round to the nearest tenth.

37.



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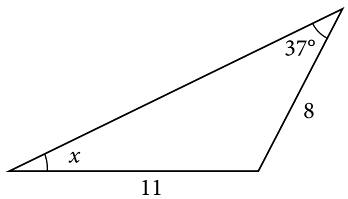

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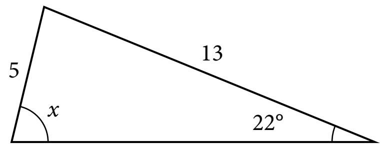
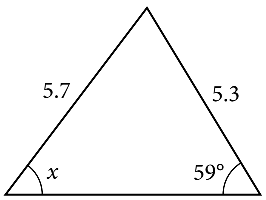
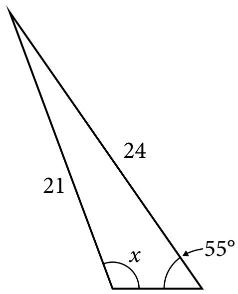


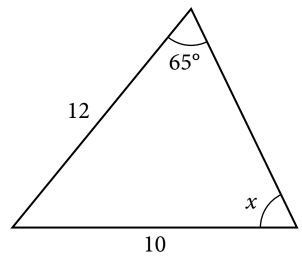
40.



41. Notice that x is an obtuse angle.

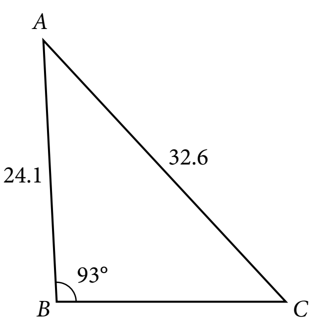


42.



For the following exercises, find the area of each triangle. Round each answer to the nearest tenth.

43.



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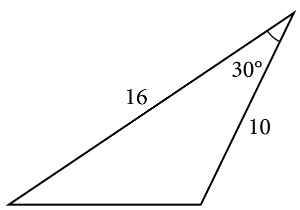

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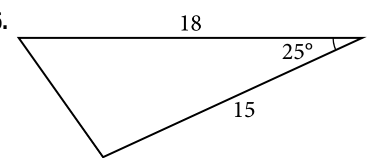

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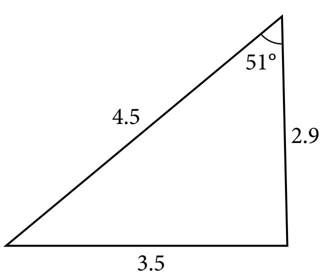

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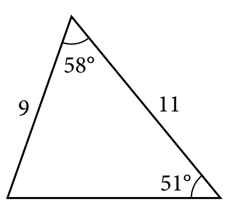

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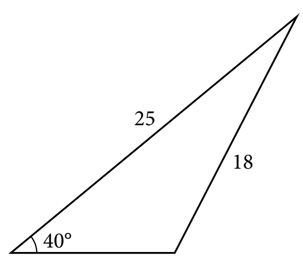

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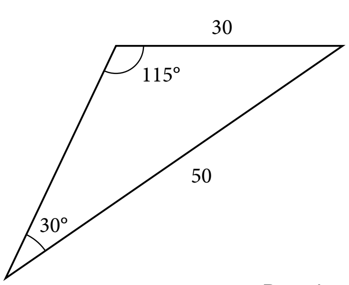


EXTENSIONS

50. Find the radius of the circle in **Figure 18**. Round to the nearest tenth.

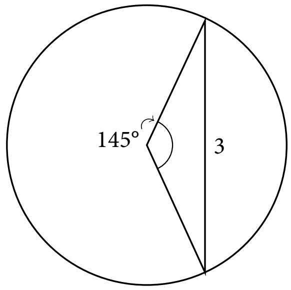


Figure 18

51. Find the diameter of the circle in **Figure 19**. Round to the nearest tenth.

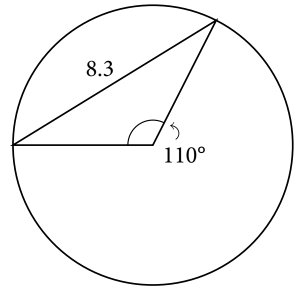


Figure 19

52. Find $m\ \angle ADC$ in **Figure 20**. Round to the nearest tenth.

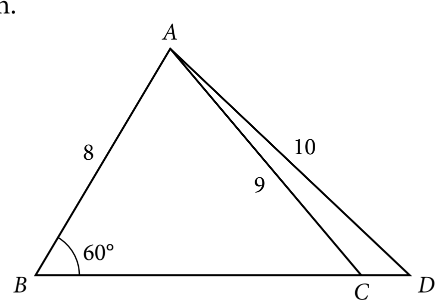


Figure 20

53. Find AD in **Figure 21**. Round to the nearest tenth.

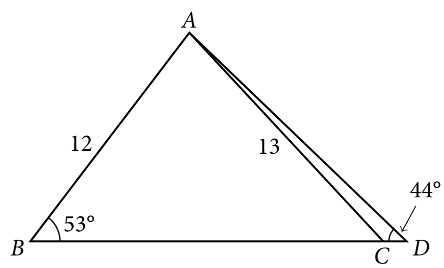


Figure 21

54. Solve both triangles in **Figure 22**. Round each answer to the nearest tenth.

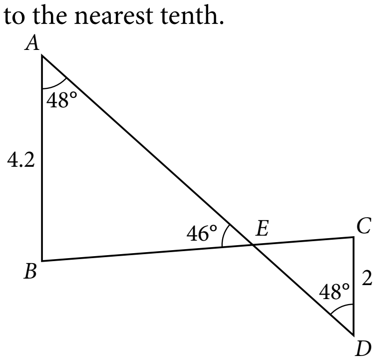


Figure 22

55. Find AB in the parallelogram shown in **Figure 23**.

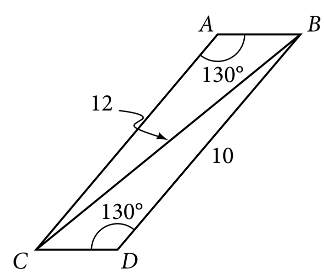


Figure 23

56. Solve the triangle in **Figure 24**. (Hint: Draw a perpendicular from H to JK). Round each answer to the nearest tenth.

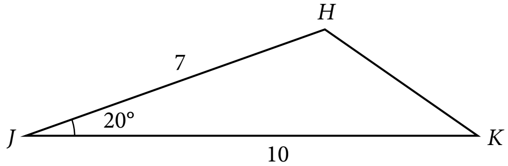


Figure 24

57. Solve the triangle in **Figure 25**. (Hint: Draw a perpendicular from N to LM). Round each answer to the nearest tenth.

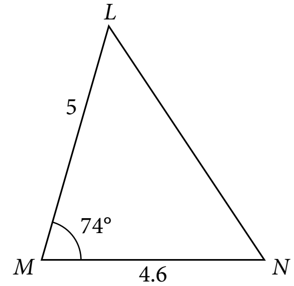


Figure 25

58. In **Figure 26**, $ABCD$ is not a parallelogram. $\angle m$ is obtuse. Solve both triangles. Round each answer to the nearest tenth.

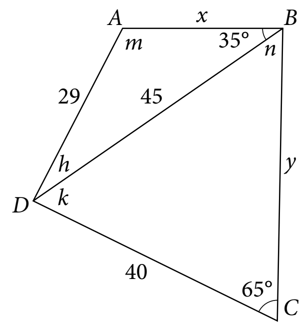


Figure 26

REAL-WORLD APPLICATIONS

59. A pole leans away from the sun at an angle of 7° to the vertical, as shown in **Figure 27**. When the elevation of the sun is 55°, the pole casts a shadow 42 feet long on the level ground. How long is the pole? Round the answer to the nearest tenth.

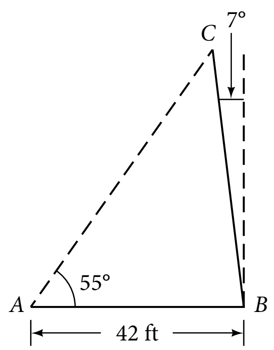


Figure 27

60. To determine how far a boat is from shore, two radar stations 500 feet apart find the angles out to the boat, as shown in **Figure 28**. Determine the distance of the boat from station A and the distance of the boat from shore. Round your answers to the nearest whole foot.

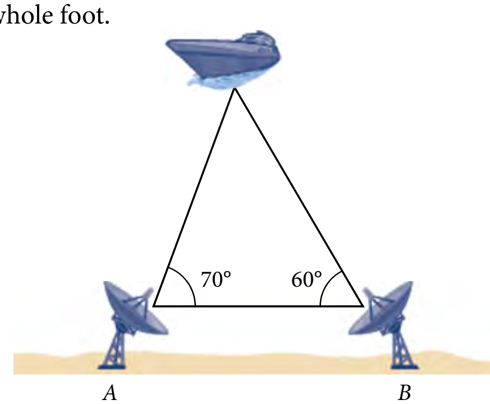


Figure 28

61. **Figure 29** shows *a* satellite orbiting Earth. The satellite passes directly over two tracking stations A and B, which are 69 miles apart. When the satellite is on one side of the two stations, the angles of elevation at A and B are measured to be 86.2° and 83.9°, respectively. How far is the satellite from station A and how high is the satellite above the ground? Round answers to the nearest whole mile.

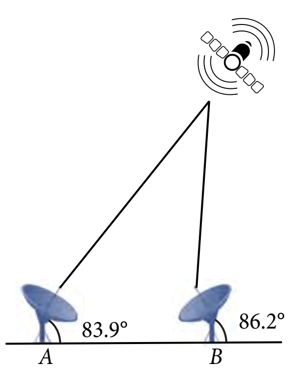


Figure 29

62. A communications tower is located at the top of *a* steep hill, as shown in **Figure 30**. The angle of inclination of the hill is 67°. *A* guy wire is to be attached to the top of the tower and to the ground, 165 meters downhill from the base of the tower. The angle formed by the guy wire and the hill is 16°. Find the length of the cable required for the guy wire to the nearest whole meter.

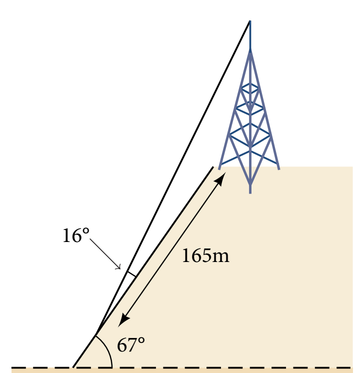


Figure 30

63. The roof of a house is at a 20° angle. An 8-foot solar panel is to be mounted on the roof and should be angled 38° relative to the horizontal for optimal results. (See **Figure 31**). How long does the vertical support holding up the back of the panel need to be? Round to the nearest tenth.

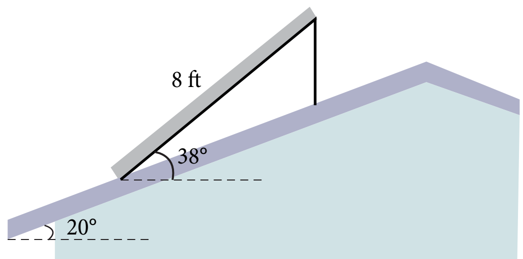


Figure 31

64. Similar to an angle of elevation, an *angle of depression* is the acute angle formed by a horizontal line and an observer's line of sight to an object below the horizontal. A pilot is flying over a straight highway. He determines the angles of depression to two mileposts, 6.6 km apart, to be 37° and 44°, as shown in **Figure 32**. Find the distance of the plane from point A to the nearest tenth of a kilometer.

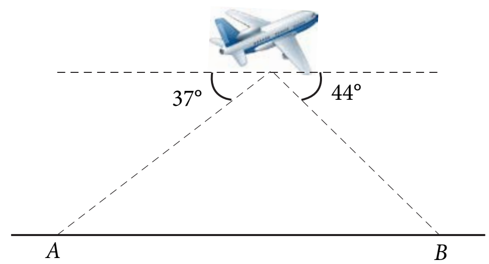


Figure 32

65. A pilot is flying over a straight highway. He determines the angles of depression to two mileposts, 4.3 km apart, to be 32° and 56°, as shown in **Figure 33**. Find the distance of the plane from point A to the nearest tenth of a kilometer.

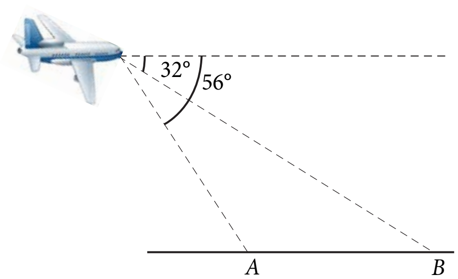


Figure 33

66. In order to estimate the height of a building, two students stand at a certain distance from the building at street level. From this point, they find the angle of elevation from the street to the top of the building to be 39°. They then move 300 feet closer to the building and find the angle of elevation to be 50°. Assuming that the street is level, estimate the height of the building to the nearest foot.

67. In order to estimate the height of a building, two students stand at a certain distance from the building at street level. From this point, they find the angle of elevation from the street to the top of the building to be 35°. They then move 250 feet closer to the building and find the angle of elevation to be 53°. Assuming that the street is level, estimate the height of the building to the nearest foot.

68. Points A and B are on opposite sides of a lake. Point C is 97 meters from A. The measure of angle BAC is determined to be 101°, and the measure of angle ACB is determined to be 53°. What is the distance from A to B, rounded to the nearest whole meter?

69. A man and a woman standing $3\frac{1}{2}$ miles apart spot a hot air balloon at the same time. If the angle of elevation from the man to the balloon is 27°, and the angle of elevation from the woman to the balloon is 41°, find the altitude of the balloon to the nearest foot.

70. Two search teams spot a stranded climber on a mountain. The first search team is 0.5 miles from the second search team, and both teams are at an altitude of 1 mile. The angle of elevation from the first search team to the stranded climber is 15°. The angle of elevation from the second search team to the climber is 22°. What is the altitude of the climber? Round to the nearest tenth of a mile.

71. A street light is mounted on a pole. *A* 6-foot-tall man is standing on the street a short distance from the pole, casting a shadow. The angle of elevation from the tip of the man's shadow to the top of his head of 28°. *A* 6-foot-tall woman is standing on the same street on the opposite side of the pole from the man. The angle of elevation from the tip of her shadow to the top of her head is 28°. If the man and woman are 20 feet apart, how far is the street light from the tip of the shadow of each person? Round the distance to the nearest tenth of a foot.

72. Three cities, *A*, *B*, and *C*, are located so that city *A* is due east of city *B*. If city *C* is located 35° west of north from city *B* and is 100 miles from city *A* and 70 miles from city *B*, how far is city *A* from city *B*? Round the distance to the nearest tenth of a mile.

73. Two streets meet at an 80° angle. At the corner, a park is being built in the shape of a triangle. Find the area of the park if, along one road, the park measures 180 feet, and along the other road, the park measures 215 feet.

74. Brian's house is on a corner lot. Find the area of the front yard if the edges measure 40 and 56 feet, as shown in **Figure 34**.

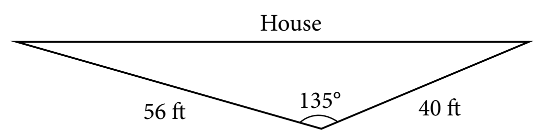


Figure 34

75. The Bermuda triangle is a region of the Atlantic Ocean that connects Bermuda, Florida, and Puerto Rico. Find the area of the Bermuda triangle if the distance from Florida to Bermuda is 1030 miles, the distance from Puerto Rico to Bermuda is 980 miles, and the angle created by the two distances is 62°.

76. A yield sign measures 30 inches on all three sides. What is the area of the sign?

77. Naomi bought a modern dining table whose top is in the shape of a triangle. Find the area of the table top if two of the sides measure 4 feet and 4.5 feet, and the smaller angles measure 32° and 42°, as shown in **Figure 35**.

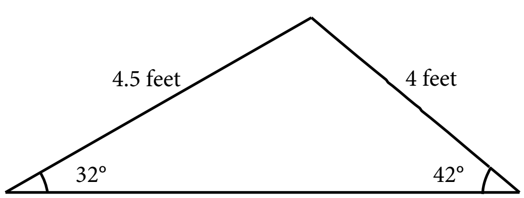


Figure 35

LEARNING OBJECTIVES

In this section, you will:

- Use the Law of Cosines to solve oblique triangles.
- Solve applied problems using the Law of Cosines.
- Use Heron's formula to find the area of a triangle.

10.2 NON-RIGHT TRIANGLES: LAW OF COSINES

Suppose a boat leaves port, travels 10 miles, turns 20 degrees, and travels another 8 miles as shown in **Figure 1**. How far from port is the boat?

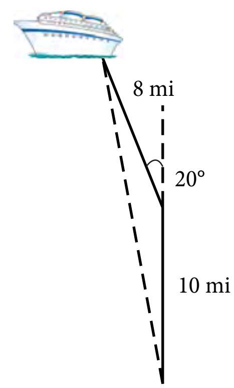


Figure 1

Unfortunately, while the Law of Sines enables us to address many non-right triangle cases, it does not help us with triangles where the known angle is between two known sides, a SAS (side-angle-side) triangle, or when all three sides are known, but no angles are known, a SSS (side-side-side) triangle. In this section, we will investigate another tool for solving oblique triangles described by these last two cases.

Using the Law of Cosines to Solve Oblique Triangles

The tool we need to solve the problem of the boat's distance from the port is the **Law of Cosines**, which defines the relationship among angle measurements and side lengths in oblique triangles. Three formulas make up the Law of Cosines. At first glance, the formulas may appear complicated because they include many variables. However, once the pattern is understood, the Law of Cosines is easier to work with than most formulas at this mathematical level.

Understanding how the Law of Cosines is derived will be helpful in using the formulas. The derivation begins with the **Generalized Pythagorean Theorem**, which is an extension of the Pythagorean Theorem to non-right triangles. Here is how it works: An arbitrary non-right triangle ABC is placed in the coordinate plane with vertex A at the origin, side c drawn along the x-axis, and vertex C located at some point (x, y) in the plane, as illustrated in **Figure 2**. Generally, triangles exist anywhere in the plane, but for this explanation we will place the triangle as noted.

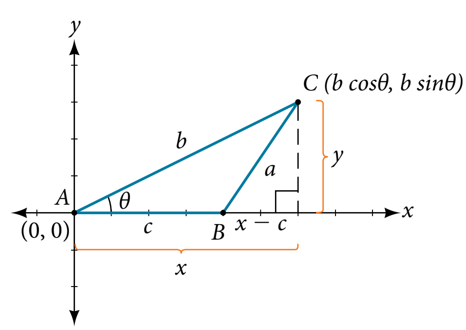


Figure 2

We can drop a perpendicular from C to the x-axis (this is the altitude or height). Recalling the basic trigonometric identities, we know that

$$\cos\theta = \frac{x(\text{adjacent})}{b(\text{hypotenuse})} \text{ and } \sin\theta = \frac{y(\text{opposite})}{b(\text{hypotenuse})}$$

In terms of θ, $x = b\cos\theta$ and $y = b\sin\theta$. The (x, y) point located at C has coordinates $(b\cos\theta, b\sin\theta)$. Using the side $(x - c)$ as one leg of a right triangle and y as the second leg, we can find the length of hypotenuse a using the Pythagorean Theorem. Thus,

$$\begin{aligned}
a^2 &= (x-c)^2 + y^2 \\
&= (b\cos\theta - c)^2 + (b\sin\theta)^2 && \text{Substitute } (b\cos\theta) \text{ for } x \text{ and } (b\sin\theta) \text{ for } y. \\
&= (b^2\cos^2\theta - 2bc\cos\theta + c^2) + b^2\sin^2\theta && \text{Expand the perfect square.} \\
&= b^2\cos^2\theta + b^2\sin^2\theta + c^2 - 2bc\cos\theta && \text{Group terms noting that } \cos^2\theta + \sin^2\theta = 1. \\
&= b^2(\cos^2\theta + \sin^2\theta) + c^2 - 2bc\cos\theta && \text{Factor out } b^2. \\
a^2 &= b^2 + c^2 - 2bc\cos\theta
\end{aligned}$$

The formula derived is one of the three equations of the Law of Cosines. The other equations are found in a similar fashion.

Keep in mind that it is always helpful to sketch the triangle when solving for angles or sides. In a real-world scenario, try to draw a diagram of the situation. As more information emerges, the diagram may have to be altered. Make those alterations to the diagram and, in the end, the problem will be easier to solve.

Law of Cosines

The **Law of Cosines** states that the square of any side of a triangle is equal to the sum of the squares of the other two sides minus twice the product of the other two sides and the cosine of the included angle. For triangles labeled as in **Figure 3**, with angles α, β, and γ, and opposite corresponding sides a, b, and c, respectively, the Law of Cosines is given as three equations.

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$

$$b^2 = a^2 + c^2 - 2ac\cos\beta$$

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

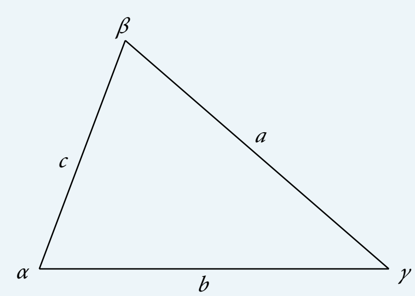


Figure 3

To solve for a missing side measurement, the corresponding opposite angle measure is needed.

When solving for an angle, the corresponding opposite side measure is needed. We can use another version of the Law of Cosines to solve for an angle.

$$\cos\alpha = \frac{b^2 + c^2 - a^2}{2bc} \qquad \cos\beta = \frac{a^2 + c^2 - b^2}{2ac} \qquad \cos\gamma = \frac{a^2 + b^2 - c^2}{2ab}$$

How To...

Given two sides and the angle between them (SAS), find the measures of the remaining side and angles of a triangle.

1. Sketch the triangle. Identify the measures of the known sides and angles. Use variables to represent the measures of the unknown sides and angles.
2. Apply the Law of Cosines to find the length of the unknown side or angle.
3. Apply the Law of Sines or Cosines to find the measure of a second angle.
4. Compute the measure of the remaining angle.

Example 1 Finding the Unknown Side and Angles of a SAS Triangle

Find the unknown side and angles of the triangle in **Figure 4**.

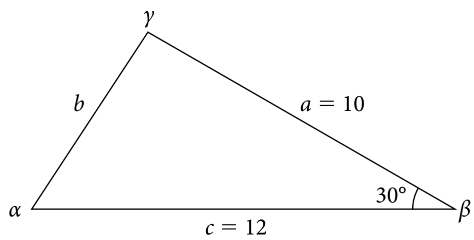


Figure 4

Solution First, make note of what is given: two sides and the angle between them. This arrangement is classified as SAS and supplies the data needed to apply the Law of Cosines.

Each one of the three laws of cosines begins with the square of an unknown side opposite a known angle. For this example, the first side to solve for is side b, as we know the measurement of the opposite angle β.

$$b^2 = a^2 + c^2 - 2ac\cos\beta$$

$$b^2 = 10^2 + 12^2 - 2(10)(12)\cos(30°)$$ Substitute the measurements for the known quantities.

$$b^2 = 100 + 144 - 240\left(\frac{\sqrt{3}}{2}\right)$$ Evaluate the cosine and begin to simplify.

$$b^2 = 244 - 120\sqrt{3}$$

$$b = \sqrt{244 - 120\sqrt{3}}$$ Use the square root property.

$$b \approx 6.013$$

Because we are solving for a length, we use only the positive square root. Now that we know the length b, we can use the Law of Sines to fill in the remaining angles of the triangle. Solving for angle α, we have

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b}$$

$$\frac{\sin\alpha}{10} = \frac{\sin(30°)}{6.013}$$

$$\sin\alpha = \frac{10\sin(30°)}{6.013}$$ Multiply both sides of the equation by 10.

$$\alpha = \sin^{-1}\left(\frac{10\sin(30°)}{6.013)}\right)$$ Find the inverse sine of $\frac{10\sin(30°)}{6.013}$.

$$\alpha \approx 56.3°$$

The other possibility for α would be $\alpha = 180° - 56.3° \approx 123.7°$. In the original diagram, α is adjacent to the longest side, so α is an acute angle and, therefore, 123.7° does not make sense. Notice that if we choose to apply the Law of Cosines, we arrive at a unique answer. We do not have to consider the other possibilities, as cosine is unique for angles between 0° and 180°. Proceeding with $\alpha \approx 56.3°$, we can then find the third angle of the triangle.

$$\gamma = 180° - 30° - 56.3° \approx 93.7°$$

The complete set of angles and sides is

$$\begin{aligned} \alpha &\approx 56.3° & a &= 10 \\ \beta &= 30° & b &\approx 6.013 \\ \gamma &\approx 93.7° & c &= 12 \end{aligned}$$

Look at the next page, page 778.5

Try It #1

Find the missing side and angles of the given triangle: $\alpha = 30°$, $b = 12$, $c = 24$.

A consequence of the law of sines is that in any triangle, the largest angle is always opposite the longest side, and the smallest angle always is opposite the shortest side.

When we desire to solve either a SAS or a SSS triangle, the first step is necessarily to use the Law of Cosines. Then we have a SSSA triangle and are looking for a second angle. At this point most people use the Law of Sines because it is easier than the Law of Cosines. Most people solve first for an angle that is not the largest of the three angles (i.e., is not opposite the longest side). This angle is necessarily actue, because a triangle cannot have 2 obtuse angles. This is a nice way to eliminate the potential ambiguity in the law of sines. (Knowing $\sin\theta$ does not tell us whether θ is acute or obtuse; recall the discussion of SSA triangles in Section 10.1.)

See Example 1 on the previous page. After getting the third side the authors observed that the longest side was opposite the angle γ. So they chose to go after α first. They refer to the logic we just explained, in the two sentences that start with "The other possibility for α would be ...".

See Example 2 on the next page. After obtaining α, if the authors wanted to solve the triangle they would note that β is opposite the longest side, so it is not known whether β is obtuse or acute. So they would use the Law of Sines to get γ, which is known to be acute. Then they would compute $\beta = 180° - \alpha - \gamma$.

Example 2 Solving for an Angle of a SSS Triangle

Find the angle α for the given triangle if side $a = 20$, side $b = 25$, and side $c = 18$.

Solution For this example, we have no angles. We can solve for any angle using the Law of Cosines. To solve for angle α, we have

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$

$$20^2 = 25^2 + 18^2 - 2(25)(18)\cos\alpha \quad \text{Substitute the appropriate measurements.}$$

$$400 = 625 + 324 - 900\cos\alpha \quad \text{Simplify in each step.}$$

$$400 = 949 - 900\cos\alpha$$

$$-549 = -900\cos\alpha \quad \text{Isolate } \cos\alpha.$$

$$\frac{-549}{-900} = \cos\alpha$$

$$0.61 \approx \cos\alpha$$

$$\cos^{-1}(0.61) \approx \alpha \quad \text{Find the inverse cosine.}$$

$$\alpha \approx 52.4°$$

See **Figure 5**.

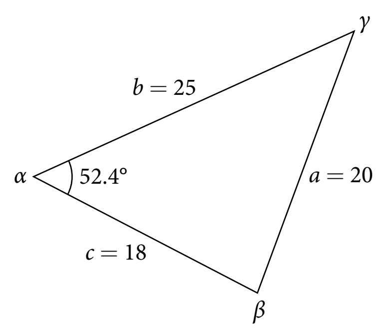


Figure 5

Analysis *Because the inverse cosine can return any angle between* 0 *and* 180 *degrees, there will not be any ambiguous cases using this method.*

Try It #2

Given $a = 5$, $b = 7$, and $c = 10$, find the missing angles.

> With SSS triangles, if the length of the longest side is longer than the sum of the lengths of the other two sides, there is no solution. For example, there is no triangle that has sides of lengths 3, 4, 8.

Solving Applied Problems Using the Law of Cosines

Just as the Law of Sines provided the appropriate equations to solve a number of applications, the Law of Cosines is applicable to situations in which the given data fits the cosine models. We may see these in the fields of navigation, surveying, astronomy, and geometry, just to name a few.

Example 3 Using the Law of Cosines to Solve a Communication Problem

On many cell phones with GPS, an approximate location can be given before the GPS signal is received. This is accomplished through a process called triangulation, which works by using the distances from two known points. Suppose there are two cell phone towers within range of a cell phone. The two towers are located 6,000 feet apart along a straight highway, running east to west, and the cell phone is north of the highway. Based on the signal delay, it can be determined that the signal is 5,050 feet from the first tower and 2,420 feet from the second tower. Determine the position of the cell phone north and east of the first tower, and determine how far it is from the highway.

Solution For simplicity, we start by drawing a diagram similar to **Figure 6** and labeling our given information.

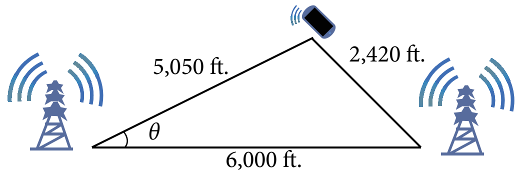


Figure 6

Using the Law of Cosines, we can solve for the angle θ. Remember that the Law of Cosines uses the square of one side to find the cosine of the opposite angle. For this example, let $a = 2420$, $b = 5050$, and $c = 6000$. Thus, θ corresponds to the opposite side $a = 2420$.

$$a^2 = b^2 + c^2 - 2bc\cos\theta$$
$$(2420)^2 = (5050)^2 + (6000)^2 - 2(5050)(6000)\cos\theta$$
$$(2420)^2 - (5050)^2 - (6000)^2 = -2(5050)(6000)\cos\theta$$
$$\frac{(2420)^2 - (5050)^2 - (6000)^2}{-2(5050)(6000)} = \cos\theta$$
$$\cos\theta \approx 0.9183$$
$$\theta \approx \cos^{-1}(0.9183)$$
$$\theta \approx 23.3°$$

To answer the questions about the phone's position north and east of the tower, and the distance to the highway, drop a perpendicular from the position of the cell phone, as in **Figure 7**. This forms two right triangles, although we only need the right triangle that includes the first tower for this problem.

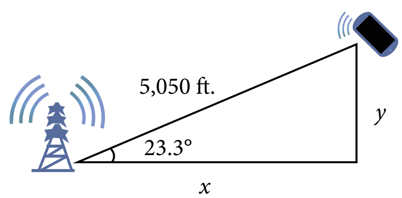


Figure 7

Using the angle $\theta = 23.3°$ and the basic trigonometric identities, we can find the solutions. Thus

$$\cos(23.3°) = \frac{x}{5050}$$
$$x = 5050\cos(23.3°)$$
$$x \approx 4638.15 \text{ feet}$$

$$\sin(23.3°) = \frac{y}{5050}$$
$$y = 5050\sin(23.3°)$$
$$y \approx 1997.5 \text{ feet}$$

The cell phone is approximately 4,638 feet east and 1,998 feet north of the first tower, and 1,998 feet from the highway.

Example 4 Calculating Distance Traveled Using a SAS Triangle

Returning to our problem at the beginning of this section, suppose a boat leaves port, travels 10 miles, turns 20 degrees, and travels another 8 miles. How far from port is the boat? The diagram is repeated here in **Figure 8**.

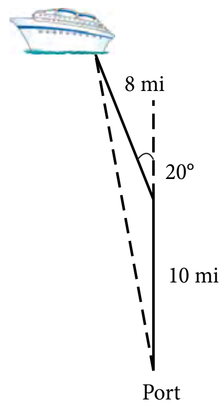


Figure 8

Solution The boat turned 20 degrees, so the obtuse angle of the non-right triangle is the supplemental angle, $180° - 20° = 160°$. With this, we can utilize the Law of Cosines to find the missing side of the obtuse triangle—the distance of the boat to the port.

$$x^2 = 8^2 + 10^2 - 2(8)(10)\cos(160°)$$
$$x^2 = 314.35$$
$$x = \sqrt{314.35}$$
$$x \approx 17.7 \text{ miles}$$

The boat is about 17.7 miles from port.

> The Law of Cosines can be used to get the third side of an SSA triangle, but it is unnecessarily difficult and is not recommended. The Law of Sines is much easier. In the homework for section 10.2 a few SSA triangles appear, as a reminder.

Using Heron's Formula to Find the Area of a Triangle

We already learned how to find the area of an oblique triangle when we know two sides and an angle. We also know the formula to find the area of a triangle using the base and the height. When we know the three sides, however, we can use Heron's formula instead of finding the height. Heron of Alexandria was a geometer who lived during the first century *A*.D. He discovered a formula for finding the area of oblique triangles when three sides are known.

Heron's formula

Heron's formula finds the area of oblique triangles in which sides a, b, and c are known.

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \dfrac{(a+b+c)}{2}$ is one half of the perimeter of the triangle, sometimes called the semi-perimeter.

Example 5 **Using Heron's Formula to Find the Area of a Given Triangle**

Find the area of the triangle in **Figure 9** using Heron's formula.

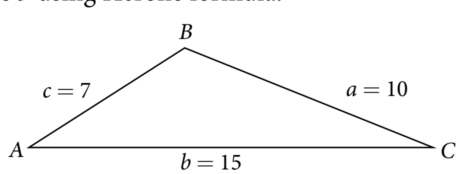


Figure 9

Solution First, we calculate s.

$$s = \frac{(a+b+c)}{2}$$
$$s = \frac{(10+15+7)}{2} = 16$$

> In Section 10.1 we got the area of a SAS triangle. We saw how to combine that formula with the Law of Sines to get the area of ASA, AAS and SSA triangles. To get the area of a SSS triangle, most people prefer Heron's Formula.

Then we apply the formula.

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$
$$\text{Area} = \sqrt{16(16-10)(16-15)(16-7)}$$
$$\text{Area} \approx 29.4$$

The area is approximately 29.4 square units.

Try It #3

Use Heron's formula to find the area of a triangle with sides of lengths $a = 29.7$ ft, $b = 42.3$ ft, and $c = 38.4$ ft.

Example 6 Applying Heron's Formula to a Real-World Problem

A Chicago city developer wants to construct a building consisting of artist's lofts on a triangular lot bordered by Rush Street, Wabash Avenue, and Pearson Street. The frontage along Rush Street is approximately 62.4 meters, along Wabash Avenue it is approximately 43.5 meters, and along Pearson Street it is approximately 34.1 meters. How many square meters are available to the developer? See **Figure 10** for a view of the city property.

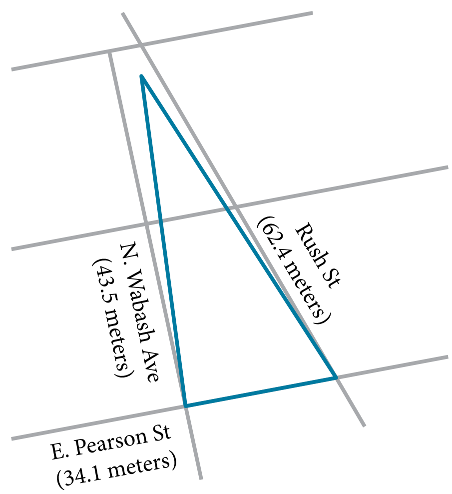


Figure 10

Solution Find the measurement for s, which is one-half of the perimeter.

$$s = \frac{62.4 + 43.5 + 34.1}{2}$$
$$s = 70 \text{ m}$$

Apply Heron's formula.

$$\text{Area} = \sqrt{70(70-62.4)(70-43.5)(70-34.1)}$$
$$\text{Area} = \sqrt{506{,}118.2}$$
$$\text{Area} \approx 711.4$$

The developer has about 711.4 square meters.

Try It #4

Find the area of a triangle given $a = 4.38$ ft , $b = 3.79$ ft, and $c = 5.22$ ft.

Access these online resources for additional instruction and practice with the Law of Cosines.

- Law of Cosines (http://openstaxcollege.org/l/lawcosines)
- Law of Cosines: Applications (http://openstaxcollege.org/l/cosineapp)
- Law of Cosines: Applications 2 (http://openstaxcollege.org/l/cosineapp2)

10.2 SECTION EXERCISES

VERBAL

1. If you are looking for a missing side of a triangle, what do you need to know when using the Law of Cosines?

2. If you are looking for a missing angle of a triangle, what do you need to know when using the Law of Cosines?

3. Explain what s represents in Heron's formula.

4. Explain the relationship between the Pythagorean Theorem and the Law of Cosines.

5. When must you use the Law of Cosines instead of the Pythagorean Theorem?

ALGEBRAIC

For the following exercises, assume α is opposite side a, β is opposite side b, and γ is opposite side c. If possible, solve each triangle for the unknown side. Round to the nearest tenth.

6. $\gamma = 41.2°, a = 2.49, b = 3.13$

7. $\alpha = 120°, b = 6, c = 7$

8. $\beta = 58.7°, a = 10.6, c = 15.7$

9. $\gamma = 115°, a = 18, b = 23$

10. $\alpha = 119°, a = 26, b = 14$

11. $\gamma = 113°, b = 10, c = 32$

12. $\beta = 67°, a = 49, b = 38$

13. $\alpha = 43.1°, a = 184.2, b = 242.8$

14. $\alpha = 36.6°, a = 186.2, b = 242.2$

15. $\beta = 50°, a = 105, b = 45$

For the following exercises, use the Law of Cosines to solve for the missing angle of the oblique triangle. Round to the nearest tenth.

16. $a = 42, b = 19, c = 30$; find angle A.

17. $a = 14, b = 13, c = 20$; find angle C.

18. $a = 16, b = 31, c = 20$; find angle B.

19. $a = 13, b = 22, c = 28$; find angle A.

20. $a = 108, b = 132, c = 160$; find angle C.

For the following exercises, solve the triangle. Round to the nearest tenth.

21. $A = 35°, b = 8, c = 11$

22. $B = 88°, a = 4.4, c = 5.2$

23. $C = 121°, a = 21, b = 37$

24. $a = 13, b = 11, c = 15$

25. $a = 3.1, b = 3.5, c = 5$

26. $a = 51, b = 25, c = 29$

For the following exercises, use Heron's formula to find the area of the triangle. Round to the nearest hundredth.

27. Find the area of a triangle with sides of length 18 in, 21 in, and 32 in. Round to the nearest tenth.

28. Find the area of a triangle with sides of length 20 cm, 26 cm, and 37 cm. Round to the nearest tenth.

29. $a = \frac{1}{2}\ m, b = \frac{1}{3}\ m, c = \frac{1}{4}\ m$

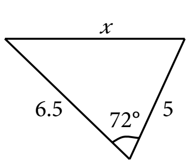
30. $a = 12.4$ ft, $b = 13.7$ ft, $c = 20.2$ ft

31. $a = 1.6$ yd, $b = 2.6$ yd, $c = 4.1$ yd

GRAPHICAL

For the following exercises, find the length of side x. Round to the nearest tenth.

32.



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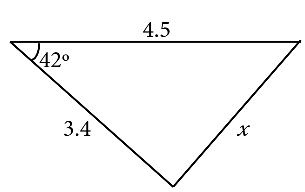

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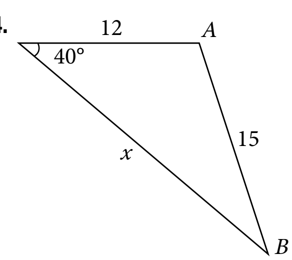

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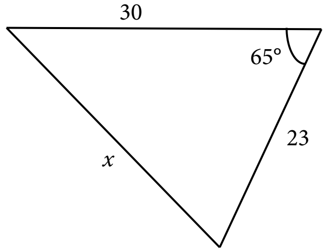

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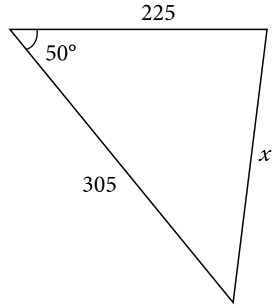

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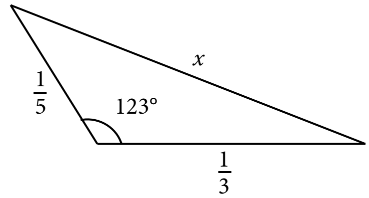
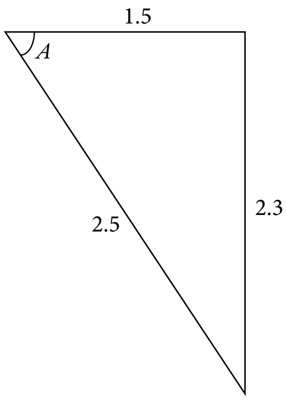


For the following exercises, find the measurement of angle *A*.

38.



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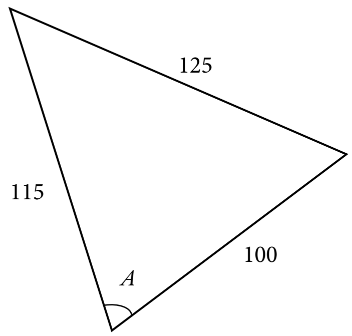

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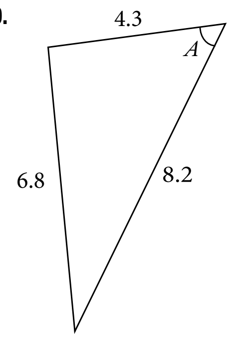
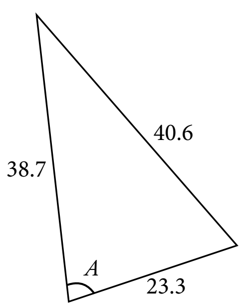


41.



42. Find the measure of each angle in the triangle shown in **Figure 11**. Round to the nearest tenth.

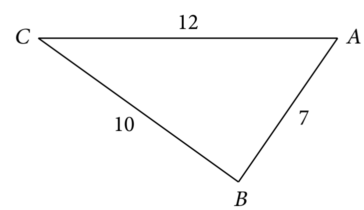
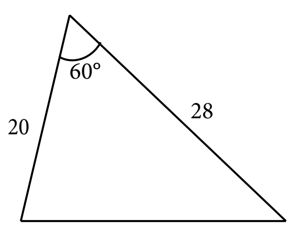


Figure 11

For the following exercises, solve for the unknown side. Round to the nearest tenth.

43.



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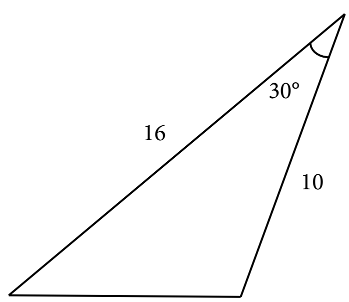

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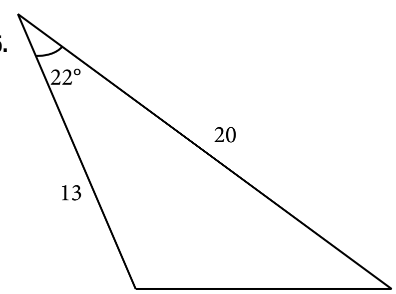

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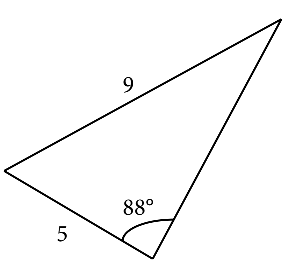
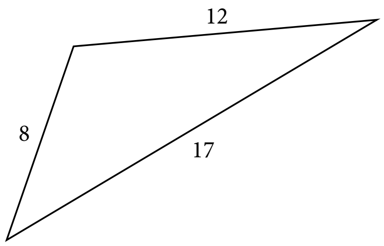


For the following exercises, find the area of the triangle. Round to the nearest hundredth.

47.



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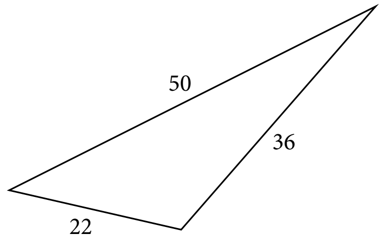

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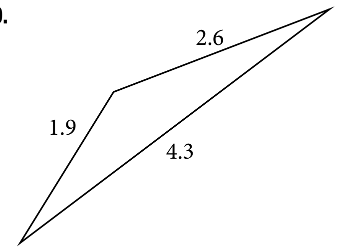

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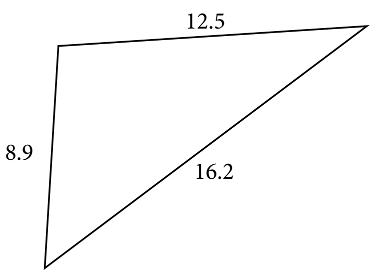
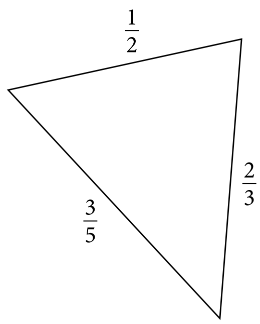


51.



EXTENSIONS

52. A parallelogram has sides of length 16 units and 10 units. The shorter diagonal is 12 units. Find the measure of the longer diagonal.

53. The sides of a parallelogram are 11 feet and 17 feet. The longer diagonal is 22 feet. Find the length of the shorter diagonal.

54. The sides of a parallelogram are 28 centimeters and 40 centimeters. The measure of the larger angle is 100°. Find the length of the shorter diagonal.

55. A regular octagon is inscribed in a circle with a radius of 8 inches. (See **Figure 12**.) Find the perimeter of the octagon.

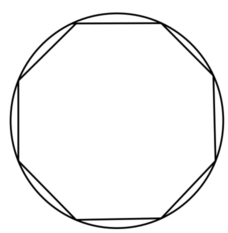


Figure 12

56. A regular pentagon is inscribed in a circle of radius 12 cm. (See **Figure 13**.) Find the perimeter of the pentagon. Round to the nearest tenth of a centimeter.

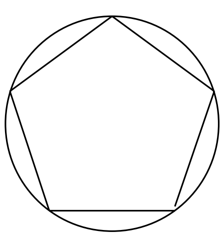


Figure 13

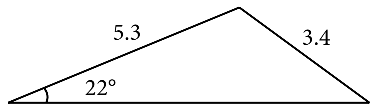
For the following exercises, suppose that $x^2 = 25 + 36 - 60 \cos(52)$ represents the relationship of three sides of a triangle and the cosine of an angle.

57. Draw the triangle.

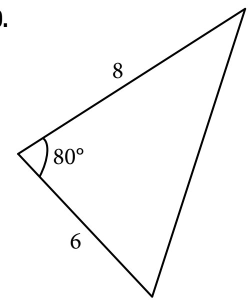
58. Find the length of the third side.

For the following exercises, find the area of the triangle.

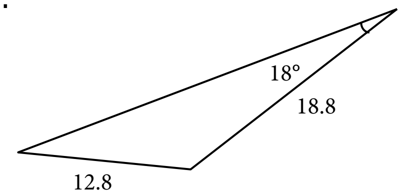
59.



60.



61.



REAL-WORLD APPLICATIONS

62. A surveyor has taken the measurements shown in **Figure 14**. Find the distance across the lake. Round answers to the nearest tenth.

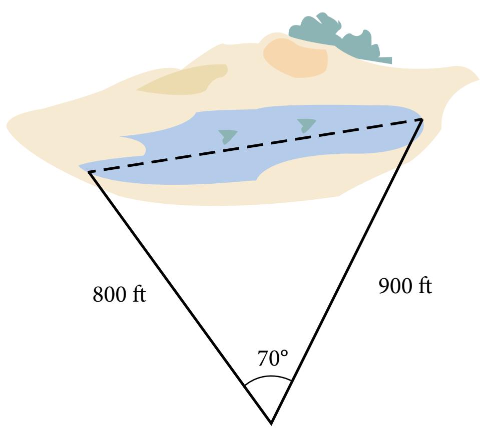


Figure 14

63. A satellite calculates the distances and angle shown in **Figure 15** (not to scale). Find the distance between the two cities. Round answers to the nearest tenth.

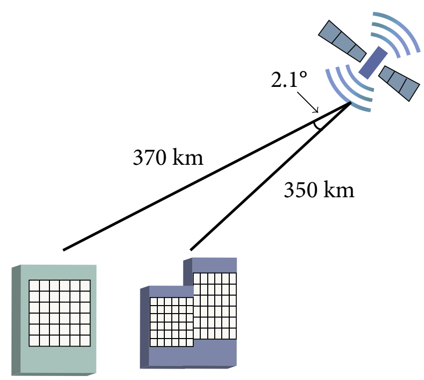


Figure 15

64. An airplane flies 220 miles with a heading of 40°, and then flies 180 miles with a heading of 170°. How far is the plane from its starting point, and at what heading? Round answers to the nearest tenth.

65. A 113-foot tower is located on a hill that is inclined 34° to the horizontal, as shown in **Figure 16**. *A* guy-wire is to be attached to the top of the tower and anchored at a point 98 feet uphill from the base of the tower. Find the length of wire needed.

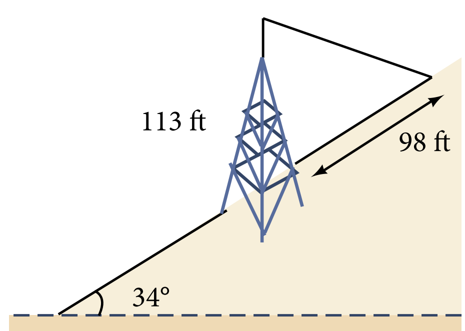


Figure 16

66. Two ships left a port at the same time. One ship traveled at a speed of 18 miles per hour at a heading of 320°. The other ship traveled at a speed of 22 miles per hour at a heading of 194°. Find the distance between the two ships after 10 hours of travel.

67. The graph in **Figure 17** represents two boats departing at the same time from the same dock. The first boat is traveling at 18 miles per hour at a heading of 327° and the second boat is traveling at 4 miles per hour at a heading of 60°. Find the distance between the two boats after 2 hours.

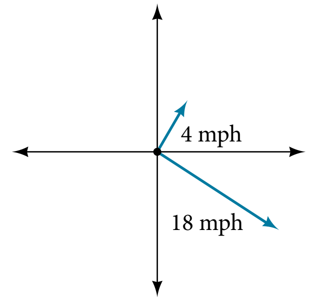


Figure 17

68. A triangular swimming pool measures 40 feet on one side and 65 feet on another side. These sides form an angle that measures 50°. How long is the third side (to the nearest tenth)?

69. A pilot flies in a straight path for 1 hour 30 min. She then makes a course correction, heading 10° to the right of her original course, and flies 2 hours in the new direction. If she maintains a constant speed of 680 miles per hour, how far is she from her starting position?

70. Los Angeles is 1,744 miles from Chicago, Chicago is 714 miles from New York, and New York is 2,451 miles from Los Angeles. Draw a triangle connecting these three cities, and find the angles in the triangle.

71. Philadelphia is 140 miles from Washington, *D*.C., Washington, *D*.C. is 442 miles from Boston, and Boston is 315 miles from Philadelphia. Draw a triangle connecting these three cities and find the angles in the triangle.

72. Two planes leave the same airport at the same time. One flies at 20° east of north at 500 miles per hour. The second flies at 30° east of south at 600 miles per hour. How far apart are the planes after 2 hours?

73. Two airplanes take off in different directions. One travels 300 mph due west and the other travels 25° north of west at 420 mph. After 90 minutes, how far apart are they, assuming they are flying at the same altitude?

74. A parallelogram has sides of length 15.4 units and 9.8 units. Its area is 72.9 square units. Find the measure of the longer diagonal.

75. The four sequential sides of a quadrilateral have lengths 4.5 cm, 7.9 cm, 9.4 cm, and 12.9 cm. The angle between the two smallest sides is 117°. What is the area of this quadrilateral?

76. The four sequential sides of a quadrilateral have lengths 5.7 cm, 7.2 cm, 9.4 cm, and 12.8 cm. The angle between the two smallest sides is 106°. What is the area of this quadrilateral?

77. Find the area of a triangular piece of land that measures 30 feet on one side and 42 feet on another; the included angle measures 132°. Round to the nearest whole square foot.

78. Find the area of a triangular piece of land that measures 110 feet on one side and 250 feet on another; the included angle measures 85°. Round to the nearest whole square foot.

LEARNING OBJECTIVES

In this section, you will:

- Plot points using polar coordinates.
- Convert from polar coordinates to rectangular coordinates.
- Convert from rectangular coordinates to polar coordinates.
- Transform equations between polar and rectangular forms.
- Identify and graph polar equations by converting to rectangular equations.

10.3 POLAR COORDINATES

Over 12 kilometers from port, a sailboat encounters rough weather and is blown off course by a 16-knot wind (see **Figure 1**). How can the sailor indicate his location to the Coast Guard? In this section, we will investigate a method of representing location that is different from a standard coordinate grid.

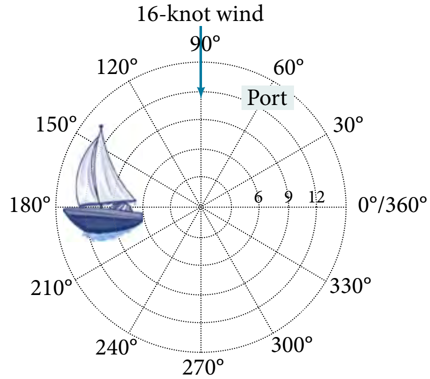


Figure 1

Plotting Points Using Polar Coordinates

When we think about plotting points in the plane, we usually think of rectangular coordinates (x, y) in the Cartesian coordinate plane. However, there are other ways of writing a coordinate pair and other types of grid systems. In this section, we introduce to **polar coordinates**, which are points labeled (r, θ) and plotted on a polar grid. The polar grid is represented as a series of concentric circles radiating out from the **pole**, or the origin of the coordinate plane.

The polar grid is scaled as the unit circle with the positive x-axis now viewed as the **polar axis** and the origin as the pole. The first coordinate r is the radius or length of the directed line segment from the pole. The angle θ, measured in radians, indicates the direction of r. We move counterclockwise from the polar axis by an angle of θ, and measure a directed line segment the length of r in the direction of θ. Even though we measure θ first and then r, the polar point is written with the r-coordinate first. For example, to plot the point $\left(2, \frac{\pi}{4}\right)$, we would move $\frac{\pi}{4}$ units in the counterclockwise direction and then a length of 2 from the pole. This point is plotted on the grid in **Figure 2**.

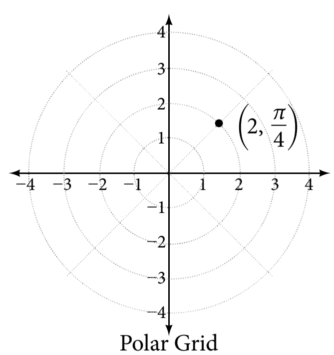


Figure 2

Example 1 Plotting a Point on the Polar Grid

Plot the point $\left(3, \frac{\pi}{2}\right)$ on the polar grid.

Solution The angle $\frac{\pi}{2}$ is found by sweeping in a counterclockwise direction 90° from the polar axis. The point is located at a length of 3 units from the pole in the $\frac{\pi}{2}$ direction, as shown in **Figure 3**.

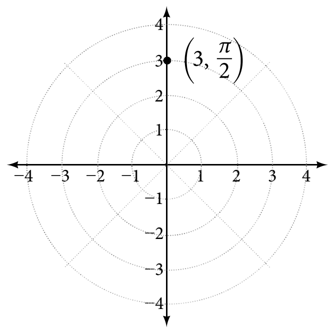


Figure 3

Try It #1

Plot the point $\left(2, \frac{\pi}{3}\right)$ in the polar grid.

Example 2 Plotting a Point in the Polar Coordinate System with a Negative Component

Plot the point $\left(-2, \frac{\pi}{6}\right)$ on the polar grid.

Solution We know that $\frac{\pi}{6}$ is located in the first quadrant. However, $r = -2$. We can approach plotting a point with a negative r in two ways:

1. Plot the point $\left(2, \frac{\pi}{6}\right)$ by moving $\frac{\pi}{6}$ in the counterclockwise direction and extending a directed line segment 2 units into the first quadrant. Then retrace the directed line segment back through the pole, and continue 2 units into the third quadrant;
2. Move $\frac{\pi}{6}$ in the counterclockwise direction, and draw the directed line segment from the pole 2 units in the negative direction, into the third quadrant.

See **Figure 4(a)**. Compare this to the graph of the polar coordinate $\left(2, \frac{\pi}{6}\right)$ shown in **Figure 4(b)**.

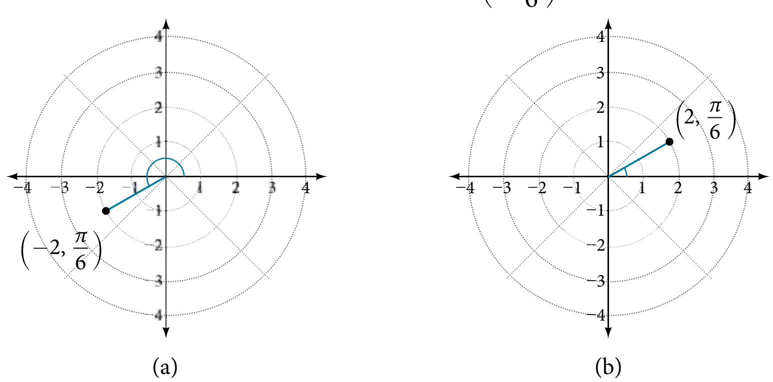


Figure 4

Try It #2

Plot the points $\left(3, -\frac{\pi}{6}\right)$ and $\left(2, \frac{9\pi}{4}\right)$ on the same polar grid.

Converting from Polar Coordinates to Rectangular Coordinates

When given a set of polar coordinates, we may need to convert them to rectangular coordinates. To do so, we can recall the relationships that exist among the variables x, y, r, and θ.

$$\cos\theta = \frac{x}{r} \rightarrow x = r\cos\theta$$
$$\sin\theta = \frac{y}{r} \rightarrow y = r\sin\theta$$

Dropping a perpendicular from the point in the plane to the x-axis forms a right triangle, as illustrated in **Figure 5**. An easy way to remember the equations above is to think of $\cos\theta$ as the adjacent side over the hypotenuse and $\sin\theta$ as the opposite side over the hypotenuse.

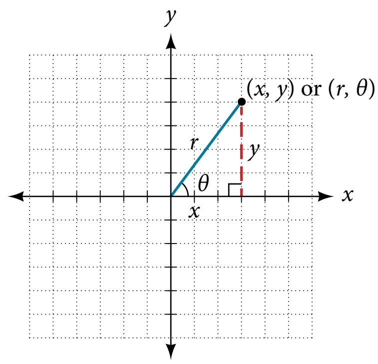


Figure 5

> ***converting from polar coordinates to rectangular coordinates***
>
> To convert polar coordinates (r, θ) to rectangular coordinates (x, y), let
>
> $$\cos\theta = \frac{x}{r} \rightarrow x = r\cos\theta$$
>
> $$\sin\theta = \frac{y}{r} \rightarrow y = r\sin\theta$$

How To...

Given polar coordinates, convert to rectangular coordinates.

1. Given the polar coordinate (r, θ), write $x = r\cos\theta$ and $y = r\sin\theta$.
2. Evaluate $\cos\theta$ and $\sin\theta$.
3. Multiply $\cos\theta$ by r to find the x-coordinate of the rectangular form.
4. Multiply $\sin\theta$ by r to find the y-coordinate of the rectangular form.

Example 3 Writing Polar Coordinates as Rectangular Coordinates

Write the polar coordinates $\left(3, \frac{\pi}{2}\right)$ as rectangular coordinates.

Solution Use the equivalent relationships.

$$x = r\cos\theta$$
$$x = 3\cos\frac{\pi}{2} = 0$$
$$y = r\sin\theta$$
$$y = 3\sin\frac{\pi}{2} = 3$$

The rectangular coordinates are $(0, 3)$. See **Figure 6**.

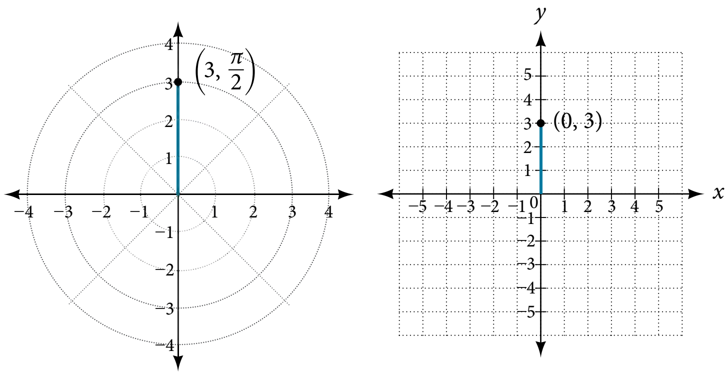


Figure 6

Example 4 Writing Polar Coordinates as Rectangular Coordinates

Write the polar coordinates $(-2, 0)$ as rectangular coordinates.

Solution See **Figure 7**. Writing the polar coordinates as rectangular, we have

$$x = r\cos\theta$$
$$x = -2\cos(0) = -2$$
$$y = r\sin\theta$$
$$y = -2\sin(0) = 0$$

The rectangular coordinates are also $(-2, 0)$.

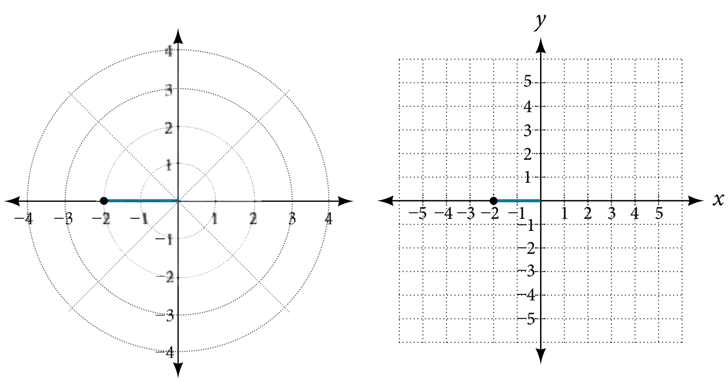


Figure 7

Try It #3

Write the polar coordinates $\left(-1, \frac{2\pi}{3}\right)$ as rectangular coordinates.

Converting from Rectangular Coordinates to Polar Coordinates

To convert rectangular coordinates to polar coordinates, we will use two other familiar relationships. With this conversion, however, we need to be aware that a set of rectangular coordinates will yield more than one polar point.

converting from rectangular coordinates to polar coordinates

Converting from rectangular coordinates to polar coordinates requires the use of one or more of the relationships illustrated in **Figure 8**.

$$\cos\theta = \frac{x}{r} \text{ or } x = r\cos\theta$$

$$\sin\theta = \frac{y}{r} \text{ or } y = r\sin\theta$$

$$r^2 = x^2 + y^2$$

$$\tan\theta = \frac{y}{x}$$

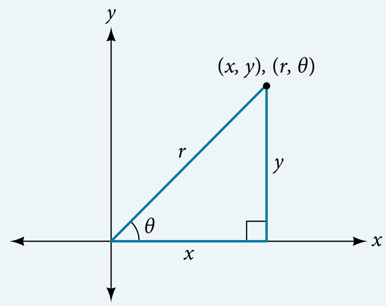


Figure 8

Example 5 Writing Rectangular Coordinates as Polar Coordinates

Convert the rectangular coordinates (3, 3) to polar coordinates.

Solution We see that the original point (3, 3) is in the first quadrant. To find θ, use the formula $\tan\theta = \frac{y}{x}$. This gives

$$\tan\theta = \frac{3}{3}$$

$$\tan\theta = 1$$

$$\theta = \tan^{-1}(1)$$

$$\theta = \frac{\pi}{4}$$

To find r, we substitute the values for x and y into the formula $r = \sqrt{x^2 + y^2}$. We know that r must be positive, as $\frac{\pi}{4}$ is in the first quadrant. Thus

$$r = \sqrt{3^2 + 3^2}$$

$$r = \sqrt{9 + 9}$$

$$r = \sqrt{18} = 3\sqrt{2}$$

So, $r = 3\sqrt{2}$ and $\theta = \frac{\pi}{4}$, giving us the polar point $\left(3\sqrt{2}, \frac{\pi}{4}\right)$. See **Figure 9**.

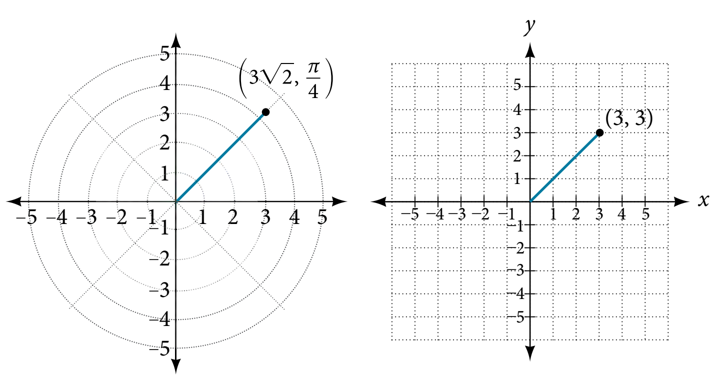


Figure 9

Analysis *There are other sets of polar coordinates that will be the same as our first solution. For example, the points* $\left(-3\sqrt{2}, \frac{5\pi}{4}\right)$ *and* $\left(3\sqrt{2}, -\frac{7\pi}{4}\right)$ *and will coincide with the original solution of* $\left(3\sqrt{2}, \frac{\pi}{4}\right)$. *The point* $\left(-3\sqrt{2}, \frac{5\pi}{4}\right)$ *indicates a move further counterclockwise by* π, *which is directly opposite* $\frac{\pi}{4}$. *The radius is expressed as* $-3\sqrt{2}$. *However, the angle* $\frac{5\pi}{4}$ *is located in the third quadrant and, as* r *is negative, we extend the directed line segment in the opposite direction, into the first quadrant. This is the same point as* $\left(3\sqrt{2}, \frac{\pi}{4}\right)$. *The point* $\left(3\sqrt{2}, -\frac{7\pi}{4}\right)$ *is a move further clockwise by* $-\frac{7\pi}{4}$, *from* $\frac{\pi}{4}$. *The radius,* $3\sqrt{2}$, *is the same.*

The blue box on the previous page accurately states that $\tan\theta = y/x$. In the Example 5 we applied this knowledge by stating that $\theta = \tan^{-1}(1)$ and consequently $\theta = \pi/4$. This method works beautifully when (x, y) is in either Quadrant I or Quadrant IV and is not on the y axis. A Quadrant IV example: if $(x, y) = (1, -\sqrt{3})$ then $r = \sqrt{1^2 + (\sqrt{3})^2} = 2$ and $\theta = \tan^{-1}(-\sqrt{3}/1) = -\pi/3$. (If we do not like negative angles we can use $\theta = -\pi/3 + 2\pi = 5\pi/3$, which is coterminal with $\theta = -\pi/3$.)

There are two cases in which the method of the previous page does not work. The first one is when $\tan\theta$ does not exist, i.e., when $\theta = \pi/2$ or $\theta = 3\pi/2$. In those cases (x, y) is on the y axis, i.e., $x = 0$. We can get r as before and we can easily get θ by inspection. For example, if $(x, y) = (0, -5)$ then $r = \sqrt{0^2 + (-5)^2} = 5$. Since the point $(0, -5)$ is on the negative part of the y axis we know that $\theta = 3\pi/2$.

The other case is when $x < 0$, meaning that (x, y) is in either Quadrant II or III, and $\pi/2 < \theta < 3\pi/2$. Thus, $\theta - \pi \in (-\pi/2, \pi/2)$, which is the range of the arctangent. Since the tangent is periodic with period π, $y/x = (r\sin\theta)/(r\cos\theta) = \tan(\theta) = \tan(\theta - \pi)$. This leads to $\tan^{-1}(y/x) = \theta - \pi$, i.e., $\theta = \tan^{-1}(y/x) + \pi$. We conclude that the bottom row in the blue box on the previous page should read

$$\theta = \tan^{-1}(y/x) \text{ if } x > 0, \text{ and } \theta = \tan^{-1}(y/x) + \pi \text{ if } x < 0.$$

Example 5.5 Convert $(-7\sqrt{3}, 7)$ into rectangular coordinates.

Solution $r = \sqrt{(-7\sqrt{3})^2 + 7^2} = \sqrt{49 \cdot 3 + 49} = 14$. Since $x = -7\sqrt{3} < 0$ we have $\theta = \tan^{-1}(y/x) + \pi = \tan^{-1}(-1/\sqrt{3}) + \pi = -\pi/6 + \pi = 5\pi/6$. Thus $(-7\sqrt{3}, 7)$ is, in polar coordinates, $(14, 5\pi/6)$.

Transforming Equations between Polar and Rectangular Forms

We can now convert coordinates between polar and rectangular form. Converting equations can be more difficult, but it can be beneficial to be able to convert between the two forms. Since there are a number of polar equations that cannot be expressed clearly in Cartesian form, and vice versa, we can use the same procedures we used to convert points between the coordinate systems. We can then use a graphing calculator to graph either the rectangular form or the polar form of the equation.

How To...

Given an equation in polar form, graph it using a graphing calculator.

1. Change the **MODE** to **POL**, representing polar form.
2. Press the **Y=** button to bring up a screen allowing the input of six equations: $r_1, r_2, \ldots, r_6$.
3. Enter the polar equation, set equal to r.
4. Press **GRAPH**.

Example 6 Writing a Cartesian Equation in Polar Form

Write the Cartesian equation $x^2 + y^2 = 9$ in polar form.

Solution The goal is to eliminate x and y from the equation and introduce r and θ. Ideally, we would write the equation r as a function of θ. To obtain the polar form, we will use the relationships between (x, y) and (r, θ). Since $x = r\cos\theta$ and $y = r\sin\theta$, we can substitute and solve for r.

$$(r\cos\theta)^2 + (r\sin\theta)^2 = 9$$
$$r^2\cos^2\theta + r^2\sin^2\theta = 9$$
$$r^2(\cos^2\theta + \sin^2\theta) = 9$$
$$r^2(1) = 9 \qquad \text{Substitute } \cos^2\theta + \sin^2\theta = 1.$$
$$r = \pm 3 \qquad \text{Use the square root property.}$$

Thus, $x^2 + y^2 = 9$, $r = 3$, and $r = -3$ should generate the same graph. See **Figure 10**.

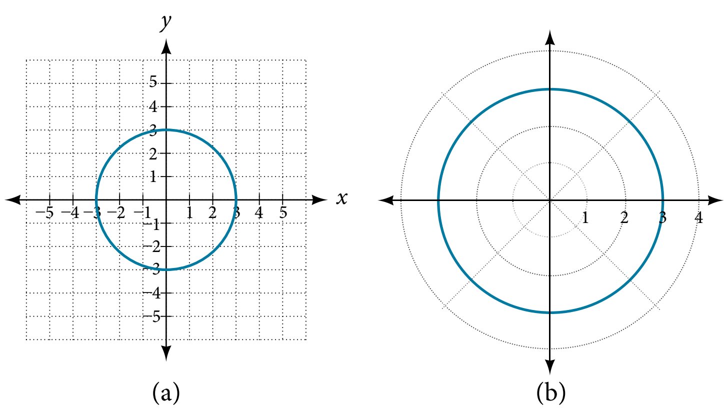


Figure 10 (a) Cartesian form $x^2 + y^2 = 9$ (b) Polar form $r = 3$

To graph a circle in rectangular form, we must first solve for y.

$$x^2 + y^2 = 9$$
$$y^2 = 9 - x^2$$
$$y = \pm\sqrt{9 - x^2}$$

Note that this is two separate functions, since a circle fails the vertical line test. Therefore, we need to enter the positive and negative square roots into the calculator separately, as two equations in the form $Y_1 = \sqrt{9 - x^2}$ and $Y_2 = -\sqrt{9 - x^2}$. Press **GRAPH**.

Example 7 Rewriting a Cartesian Equation as a Polar Equation

Rewrite the Cartesian equation $x^2 + y^2 = 6y$ as a polar equation.

Solution This equation appears similar to the previous example, but it requires different steps to convert the equation. We can still follow the same procedures we have already learned and make the following substitutions:

$$\begin{aligned} r^2 &= 6y && \text{Use } x^2 + y^2 = r^2. \\ r^2 &= 6r\sin\theta && \text{Substitute } y = r\sin\theta. \\ r^2 - 6r\sin\theta &= 0 && \text{Set equal to 0.} \\ r(r - 6\sin\theta) &= 0 && \text{Factor and solve.} \\ r &= 0 && \text{We reject } r = 0 \text{, as it only represents one point, (0, 0).} \\ \text{or } r &= 6\sin\theta \end{aligned}$$

Therefore, the equations $x^2 + y^2 = 6y$ and $r = 6\sin\theta$ should give us the same graph. See **Figure 11**.

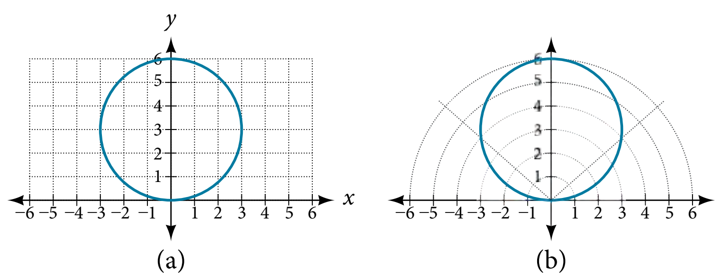


Figure 11 (a) Cartesian form $x^2 + y^2 = 6y$ (b) polar form $r = 6\sin\theta$

The Cartesian or rectangular equation is plotted on the rectangular grid, and the polar equation is plotted on the polar grid. Clearly, the graphs are identical.

Example 8 Rewriting a Cartesian Equation in Polar Form

Rewrite the Cartesian equation $y = 3x + 2$ as a polar equation.

Solution We will use the relationships $x = r\cos\theta$ and $y = r\sin\theta$.

$$\begin{aligned} y &= 3x + 2 \\ r\sin\theta &= 3r\cos\theta + 2 \\ r\sin\theta - 3r\cos\theta &= 2 \\ r(\sin\theta - 3\cos\theta) &= 2 && \text{Isolate } r. \\ r &= \frac{2}{\sin\theta - 3\cos\theta} && \text{Solve for } r. \end{aligned}$$

Try It #4

Rewrite the Cartesian equation $y^2 = 3 - x^2$ in polar form.

Identify and Graph Polar Equations by Converting to Rectangular Equations

We have learned how to convert rectangular coordinates to polar coordinates, and we have seen that the points are indeed the same. We have also transformed polar equations to rectangular equations and vice versa. Now we will demonstrate that their graphs, while drawn on different grids, are identical.

Example 9 Graphing a Polar Equation by Converting to a Rectangular Equation

Covert the polar equation $r = 2\sec\theta$ to a rectangular equation, and draw its corresponding graph.

Solution The conversion is

$$\begin{aligned} r &= 2\sec\theta \\ r &= \frac{2}{\cos\theta} \\ r\cos\theta &= 2 \\ x &= 2 \end{aligned}$$

Notice that the equation $r = 2\sec\theta$ drawn on the polar grid is clearly the same as the vertical line $x = 2$ drawn on the rectangular grid (see **Figure 12**). Just as $x = c$ is the standard form for a vertical line in rectangular form, $r = c\sec\theta$ is the standard form for a vertical line in polar form.

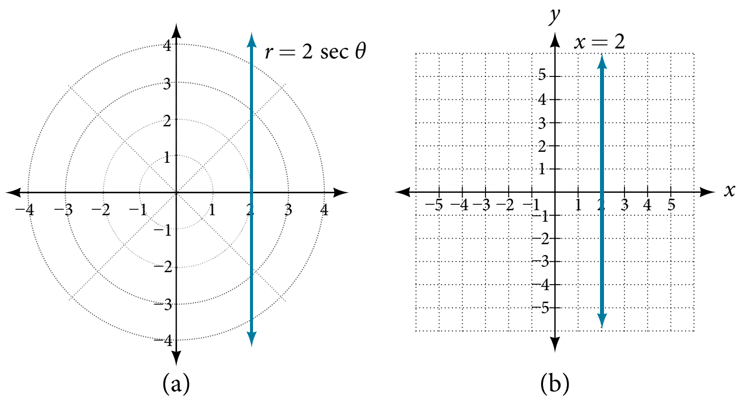


Figure 12 (a) Polar grid (b) Rectangular coordinate system

A similar discussion would demonstrate that the graph of the function $r = 2\csc\theta$ will be the horizontal line $y = 2$. In fact, $r = c\csc\theta$ is the standard form for a horizontal line in polar form, corresponding to the rectangular form $y = c$.

Example 10 Rewriting a Polar Equation in Cartesian Form

Rewrite the polar equation $r = \dfrac{3}{1 - 2\cos\theta}$ as a Cartesian equation.

Solution The goal is to eliminate θ and r, and introduce x and y. We clear the fraction, and then use substitution. In order to replace r with x and y, we must use the expression $x^2 + y^2 = r^2$.

$$\begin{aligned} r &= \frac{3}{1 - 2\cos\theta} \\ r(1 - 2\cos\theta) &= 3 \\ r\left(1 - 2\left(\frac{x}{r}\right)\right) &= 3 && \text{Use } \cos\theta = \frac{x}{r} \text{ to eliminate } \theta. \\ r - 2x &= 3 \\ r &= 3 + 2x && \text{Isolate } r. \\ r^2 &= (3 + 2x)^2 && \text{Square both sides.} \\ x^2 + y^2 &= (3 + 2x)^2 && \text{Use } x^2 + y^2 = r^2. \end{aligned}$$

The Cartesian equation is $x^2 + y^2 = (3 + 2x)^2$. However, to graph it, especially using a graphing calculator or computer program, we want to isolate y.

$$\begin{aligned} x^2 + y^2 &= (3 + 2x)^2 \\ y^2 &= (3 + 2x)^2 - x^2 \\ y &= \pm\sqrt{(3 + 2x)^2 - x^2} \end{aligned}$$

When our entire equation has been changed from r and θ to x and y, we can stop, unless asked to solve for y or simplify. See **Figure 13**.

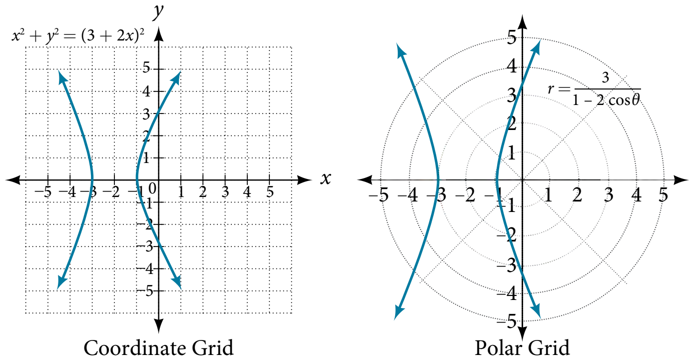


Figure 13

The "hour-glass" shape of the graph is called a *hyperbola*. Hyperbolas have many interesting geometric features and applications, which we will investigate further in **Analytic Geometry**.

Analysis *In this example, the right side of the equation can be expanded and the equation simplified further, as shown above. However, the equation cannot be written as a single function in Cartesian form. We may wish to write the rectangular equation in the hyperbola's standard form. To do this, we can start with the initial equation.*

$$\begin{aligned} x^2 + y^2 &= (3 + 2x)^2 \\ x^2 + y^2 - (3 + 2x)^2 &= 0 \\ x^2 + y^2 - (9 + 12x + 4x^2) &= 0 \\ x^2 + y^2 - 9 - 12x - 4x^2 &= 0 \\ -3x^2 - 12x + y^2 &= 9 && \text{Multiply through by } -1. \\ 3x^2 + 12x - y^2 &= -9 \\ 3(x^2 + 4x + \quad) - y^2 &= -9 && \text{Organize terms to complete the square for } x. \\ 3(x^2 + 4x + 4) - y^2 &= -9 + 12 \\ 3(x + 2)^2 - y^2 &= 3 \\ (x + 2)^2 - \frac{y^2}{3} &= 1 \end{aligned}$$

Try It #5

Rewrite the polar equation $r = 2\sin\theta$ in Cartesian form.

Example 11 Rewriting a Polar Equation in Cartesian Form

Rewrite the polar equation $r = \sin(2\theta)$ in Cartesian form.

Solution

$$\begin{aligned} r &= \sin(2\theta) && \text{Use the double angle identity for sine.} \\ r &= 2\sin\theta\cos\theta && \text{Use } \cos\theta = \frac{x}{r} \text{ and } \sin\theta = \frac{y}{r}. \\ r &= 2\left(\frac{x}{r}\right)\left(\frac{y}{r}\right) && \text{Simplify.} \\ r &= \frac{2xy}{r^2} && \text{Multiply both sides by } r^2. \\ r^3 &= 2xy \\ \left(\sqrt{x^2 + y^2}\right)^3 &= 2xy && \text{As } x^2 + y^2 = r^2, r = \sqrt{x^2 + y^2}. \end{aligned}$$

This equation can also be written as

$$(x^2 + y^2)^{\frac{3}{2}} = 2xy \text{ or } x^2 + y^2 = (2xy)^{\frac{2}{3}}.$$

Access these online resources for additional instruction and practice with polar coordinates.

- Introduction to Polar Coordinates (http://openstaxcollege.org/l/intropolar)
- Comparing Polar and Rectangular Coordinates (http://openstaxcollege.org/l/polarrect)

10.3 SECTION EXERCISES

VERBAL

1. How are polar coordinates different from rectangular coordinates?

2. How are the polar axes different from the x- and y-axes of the Cartesian plane?

3. Explain how polar coordinates are graphed.

4. How are the points $\left(3, \frac{\pi}{2}\right)$ and $\left(-3, \frac{\pi}{2}\right)$ related?

5. Explain why the points $\left(-3, \frac{\pi}{2}\right)$ and $\left(3, -\frac{\pi}{2}\right)$ are the same.

ALGEBRAIC

For the following exercises, convert the given polar coordinates to Cartesian coordinates with $r > 0$ and $0 \leq \theta \leq 2\pi$. Remember to consider the quadrant in which the given point is located when determining θ for the point.

6. $\left(7, \frac{7\pi}{6}\right)$

7. $(5, \pi)$

8. $\left(6, -\frac{\pi}{4}\right)$

9. $\left(-3, \frac{\pi}{6}\right)$

10. $\left(4, \frac{7\pi}{4}\right)$

For the following exercises, convert the given Cartesian coordinates to polar coordinates with $r > 0$, $0 \leq \theta < 2\pi$. Remember to consider the quadrant in which the given point is located.

11. $(4, 2)$

12. $(-4, 6)$

13. $(3, -5)$

14. $(-10, -13)$

15. $(8, 8)$

For the following exercises, convert the given Cartesian equation to a polar equation.

16. $x = 3$

17. $y = 4$

18. $y = 4x^2$

19. $y = 2x^4$

20. $x^2 + y^2 = 4y$

21. $x^2 + y^2 = 3x$

22. $x^2 - y^2 = x$

23. $x^2 - y^2 = 3y$

24. $x^2 + y^2 = 9$

25. $x^2 = 9y$

26. $y^2 = 9x$

27. $9xy = 1$

For the following exercises, convert the given polar equation to a Cartesian equation. Write in the standard form of a conic if possible, and identify the conic section represented.

28. $r = 3\sin\theta$

29. $r = 4\cos\theta$

30. $r = \dfrac{4}{\sin\theta + 7\cos\theta}$

31. $r = \dfrac{6}{\cos\theta + 3\sin\theta}$

32. $r = 2\sec\theta$

33. $r = 3\csc\theta$

34. $r = \sqrt{r\cos\theta + 2}$

35. $r^2 = 4\sec\theta\csc\theta$

36. $r = 4$

37. $r^2 = 4$

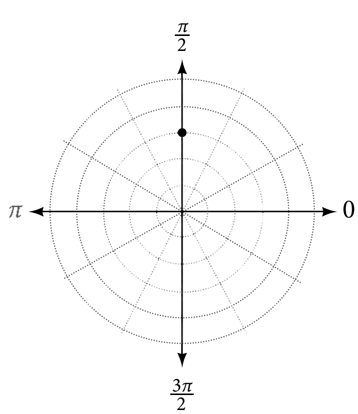
38. $r = \dfrac{1}{4\cos\theta - 3\sin\theta}$

39. $r = \dfrac{3}{\cos\theta - 5\sin\theta}$

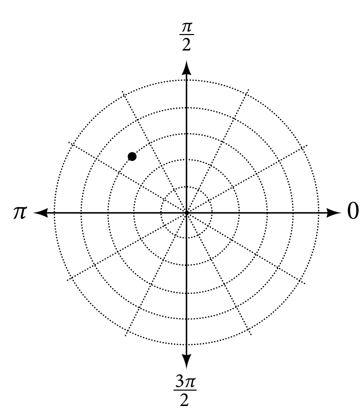
GRAPHICAL

For the following exercises, find the polar coordinates of the point.

40.



41.



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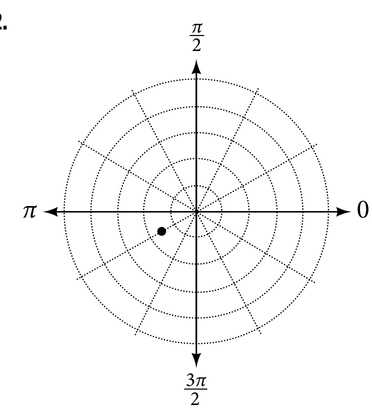

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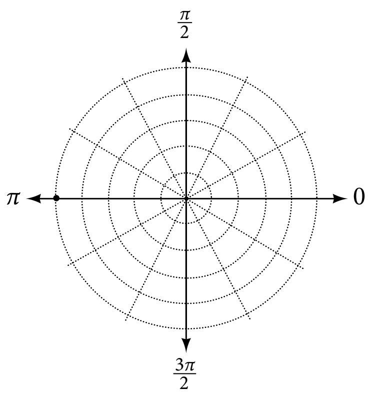

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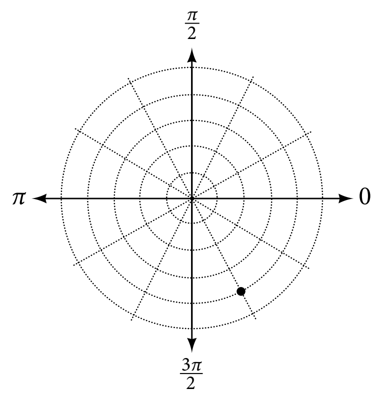


For the following exercises, plot the points.

45. $\left(-2, \frac{\pi}{3}\right)$ **46.** $\left(-1, -\frac{\pi}{2}\right)$ **47.** $\left(3.5, \frac{7\pi}{4}\right)$ **48.** $\left(-4, \frac{\pi}{3}\right)$ **49.** $\left(5, \frac{\pi}{2}\right)$

50. $\left(4, \frac{-5\pi}{4}\right)$ **51.** $\left(3, \frac{5\pi}{6}\right)$ **52.** $\left(-1.5, \frac{7\pi}{6}\right)$ **53.** $\left(-2, \frac{\pi}{4}\right)$ **54.** $\left(1, \frac{3\pi}{2}\right)$

For the following exercises, convert the equation from rectangular to polar form and graph on the polar axis.

55. $5x - y = 6$ **56.** $2x + 7y = -3$ **57.** $x^2 + (y - 1)^2 = 1$ **58.** $(x + 2)^2 + (y + 3)^2 = 13$

59. $x = 2$ **60.** $x^2 + y^2 = 5y$ **61.** $x^2 + y^2 = 3x$

For the following exercises, convert the equation from polar to rectangular form and graph on the rectangular plane.

62. $r = 6$ **63.** $r = -4$ **64.** $\theta = -\frac{2\pi}{3}$ **65.** $\theta = \frac{\pi}{4}$

66. $r = \sec\theta$ **67.** $r = -10\sin\theta$ **68.** $r = 3\cos\theta$

TECHNOLOGY

69. Use a graphing calculator to find the rectangular coordinates of $\left(2, -\frac{\pi}{5}\right)$. Round to the nearest thousandth.

70. Use a graphing calculator to find the rectangular coordinates of $\left(-3, \frac{3\pi}{7}\right)$. Round to the nearest thousandth.

71. Use a graphing calculator to find the polar coordinates of $(-7, 8)$ in degrees. Round to the nearest thousandth.

72. Use a graphing calculator to find the polar coordinates of $(3, -4)$ in degrees. Round to the nearest hundredth.

73. Use a graphing calculator to find the polar coordinates of $(-2, 0)$ in radians. Round to the nearest hundredth.

EXTENSIONS

74. Describe the graph of $r = a\sec\theta$; $a > 0$.

75. Describe the graph of $r = a\sec\theta$; $a < 0$.

76. Describe the graph of $r = a\csc\theta$; $a > 0$.

77. Describe the graph of $r = a\csc\theta$; $a < 0$.

78. What polar equations will give an oblique line?

For the following exercises, graph the polar inequality.

79. $r < 4$

80. $0 \le \theta \le \frac{\pi}{4}$

81. $\theta = \frac{\pi}{4}, r \ge 2$

82. $\theta = \frac{\pi}{4}, r \ge -3$

83. $0 \le \theta \le \frac{\pi}{3}, r < 2$

84. $-\frac{\pi}{6} < \theta \le \frac{\pi}{3}, -3 < r < 2$

LEARNING OBJECTIVES

In this section, you will:

- Test polar equations for symmetry.
- Graph polar equations by plotting points.

10.4 POLAR COORDINATES: GRAPHS

The planets move through space in elliptical, periodic orbits about the sun, as shown in **Figure 1**. They are in constant motion, so fixing an exact position of any planet is valid only for a moment. In other words, we can fix only a planet's *instantaneous* position. This is one application of polar coordinates, represented as (r, θ). We interpret r as the distance from the sun and θ as the planet's angular bearing, or its direction from a fixed point on the sun. In this section, we will focus on the polar system and the graphs that are generated directly from polar coordinates.

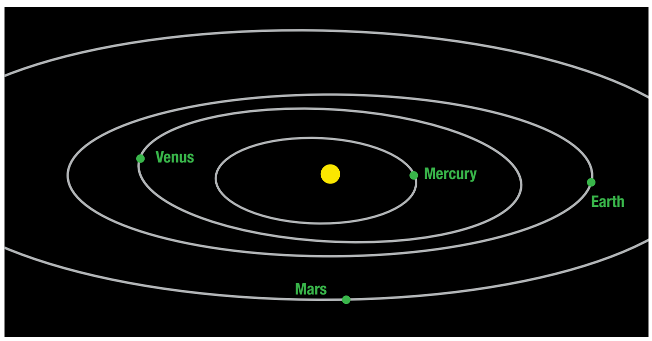


Figure 1 Planets follow elliptical paths as they orbit around the Sun. (credit: modification of work by NASA/JPL-Caltech)

Testing Polar Equations for Symmetry

Just as a rectangular equation such as $y = x^2$ describes the relationship between x and y on a Cartesian grid, a **polar equation** describes a relationship between r and θ on a polar grid. Recall that the coordinate pair (r, θ) indicates that we move counterclockwise from the polar axis (positive x-axis) by an angle of θ, and extend a ray from the pole (origin) r units in the direction of θ. All points that satisfy the polar equation are on the graph.

Symmetry is a property that helps us recognize and plot the graph of any equation. If an equation has a graph that is symmetric with respect to an axis, it means that if we folded the graph in half over that axis, the portion of the graph on one side would coincide with the portion on the other side. By performing three tests, we will see how to apply the properties of symmetry to polar equations. Further, we will use symmetry (in addition to plotting key points, zeros, and maximums of r) to determine the graph of a polar equation.

In the first test, we consider symmetry with respect to the line $\theta = \frac{\pi}{2}$ (y-axis). We replace (r, θ) with $(-r, -\theta)$ to determine if the new equation is equivalent to the original equation. For example, suppose we are given the equation $r = 2\sin\theta$;

$$\begin{aligned} r &= 2\sin\theta & & \\ -r &= 2\sin(-\theta) & & \text{Replace } (r, \theta) \text{ with } (-r, -\theta). \\ -r &= -2\sin\theta & & \text{Identity: } \sin(-\theta) = -\sin\theta. \\ r &= 2\sin\theta & & \text{Multiply both sides by} -1. \end{aligned}$$

This equation exhibits symmetry with respect to the line $\theta = \frac{\pi}{2}$.

In the second test, we consider symmetry with respect to the polar axis (x -axis). We replace (r, θ) with $(r, -\theta)$ or $(-r, \pi - \theta)$ to determine equivalency between the tested equation and the original. For example, suppose we are given the equation $r = 1 - 2\cos\theta$.

$$\begin{aligned} r &= 1 - 2\cos\theta & & \\ r &= 1 - 2\cos(-\theta) & & \text{Replace } (r, \theta) \text{ with } (r, -\theta). \\ r &= 1 - 2\cos\theta & & \text{Even/Odd identity} \end{aligned}$$

The graph of this equation exhibits symmetry with respect to the polar axis. In the third test, we consider symmetry with respect to the pole (origin). We replace (r, θ) with $(-r, \theta)$ to determine if the tested equation is equivalent to the original equation. For example, suppose we are given the equation $r = 2\sin(3\theta)$.

$$r = 2\sin(3\theta)$$
$$-r = 2\sin(3\theta)$$

The equation has failed the symmetry test, but that does not mean that it is not symmetric with respect to the pole. Passing one or more of the symmetry tests verifies that symmetry will be exhibited in a graph. However, failing the symmetry tests does not necessarily indicate that a graph will not be symmetric about the line $\theta = \frac{\pi}{2}$, the polar axis, or the pole. In these instances, we can confirm that symmetry exists by plotting reflecting points across the apparent axis of symmetry or the pole. Testing for symmetry is a technique that simplifies the graphing of polar equations, but its application is not perfect.

symmetry tests

A **polar equation** describes a curve on the polar grid. The graph of a polar equation can be evaluated for three types of symmetry, as shown in **Figure 2**.

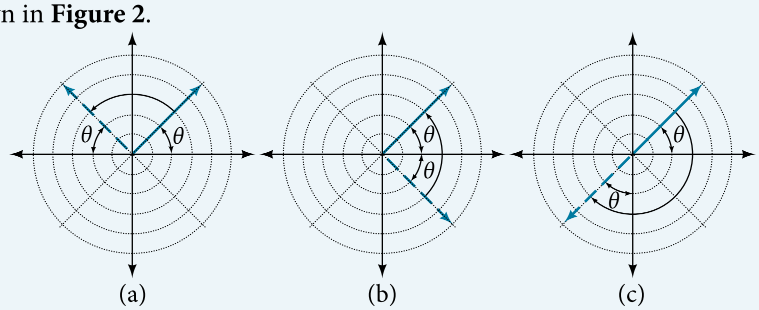


Figure 2 (a) A graph is symmetric with respect to the line $\theta = \frac{\pi}{2}$ (y-axis) if replacing (r, θ) with $(-r, -\theta)$ yields an equivalent equation. (b) A graph is symmetric with respect to the polar axis (x-axis) if replacing (r, θ) with $(r, -\theta)$ or $(-r, \pi-\theta)$ yields an equivalent equation. (c) A graph is symmetric with respect to the pole (origin) if replacing (r, θ) with $(-r, \theta)$ yields an equivalent equation.

How To…

Given a polar equation, test for symmetry.

1. Substitute the appropriate combination of components for (r, θ): $(-r, -\theta)$ for $\theta = \frac{\pi}{2}$ symmetry; $(r, -\theta)$ for polar axis symmetry; and $(-r, \theta)$ for symmetry with respect to the pole.
2. If the resulting equations are equivalent in one or more of the tests, the graph produces the expected symmetry.

Example 1 Testing a Polar Equation for Symmetry

Test the equation $r = 2\sin\theta$ for symmetry.

Solution Test for each of the three types of symmetry.

1) Replacing (r, θ) with $(-r, -\theta)$ yields the same result. Thus, the graph is symmetric with respect to the line $\theta = \frac{\pi}{2}$.	$-r = 2\sin(-\theta)$ $-r = -2\sin\theta$ Even-odd identity $r = 2\sin\theta$ Multiply by -1 Passed
2) Replacing θ with $-\theta$ does not yield the same equation. Therefore, the graph fails the test and may or may not be symmetric with respect to the polar axis.	$r = 2\sin(-\theta)$ $r = -2\sin\theta$ Even-odd identity $r = -2\sin\theta \neq 2\sin\theta$ Failed
3) Replacing r with $-r$ changes the equation and fails the test. The graph may or may not be symmetric with respect to the pole.	$-r = 2\sin\theta$ $r = -2\sin\theta \neq 2\sin\theta$ Failed

Table 1

Analysis *Using a graphing calculator, we can see that the equation $r = 2\sin\theta$ is a circle centered at (0, 1) with radius $r = 1$ and is indeed symmetric to the line $\theta = \frac{\pi}{2}$. We can also see that the graph is not symmetric with the polar axis or the pole. See* **Figure 3.**

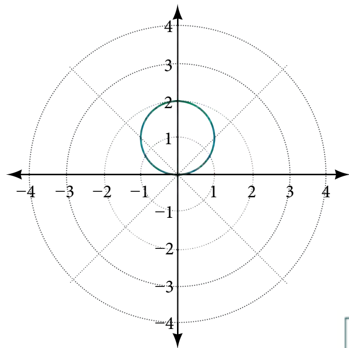


Figure 3

Warning: We can destroy symmetry without realizing it when we restrict the domain of a polar function. For example, consider the graph of $r = 2\sin\theta$ in Figure 3. The graph of the related function $r = 2\sin\theta, 0 \le \theta \le \pi/2$ is the right half of the circle, and it is not symmetric to the line $\theta = \pi/2$. The curve of Example 10 would be symmetric about the line $\theta = \pi/2$ if the domain were not restricted

Try It #1

Test the equation for symmetry: $r = -2\cos\theta$.

Graphing Polar Equations by Plotting Points

To graph in the rectangular coordinate system we construct a table of x and y values. To graph in the polar coordinate system we construct a table of θ and r values. We enter values of θ into a polar equation and calculate r. However, using the properties of symmetry and finding key values of θ and r means fewer calculations will be needed.

Finding Zeros and Maxima

Look at the next page, page 801.5

To find the zeros of a polar equation, we solve for the values of θ that result in $r = 0$. Recall that, to find the zeros of polynomial functions, we set the equation equal to zero and then solve for x. We use the same process for polar equations. Set $r = 0$, and solve for θ.

For many of the forms we will encounter, the maximum value of a polar equation is found by substituting those values of θ into the equation that result in the maximum value of the trigonometric functions. Consider $r = 5\cos\theta$; the maximum distance between the curve and the pole is 5 units. The maximum value of the cosine function is 1 when $\theta = 0$, so our polar equation is $5\cos\theta$, and the value $\theta = 0$ will yield the maximum $|r|$.

Similarly, the maximum value of the sine function is 1 when $\theta = \frac{\pi}{2}$, and if our polar equation is $r = 5\sin\theta$, the value $\theta = \frac{\pi}{2}$ will yield the maximum $|r|$. We may find additional information by calculating values of r when $\theta = 0$. These points would be polar axis intercepts, which may be helpful in drawing the graph and identifying the curve of a polar equation.

Example 2 **Finding Zeros and Maximum Values for a Polar Equation**

Using the equation in **Example 1**, find the zeros and maximum $|r|$ and, if necessary, the polar axis intercepts of $r = 2\sin\theta$.

Solution To find the zeros, set r equal to zero and solve for θ.

$$\begin{aligned} 2\sin\theta &= 0 \\ \sin\theta &= 0 \\ \theta &= \sin^{-1} 0 \\ \theta &= n\pi \qquad \text{where } n \text{ is an integer} \end{aligned}$$

Substitute any one of the θ values into the equation. We will use 0.

$$\begin{aligned} r &= 2\sin(0) \\ r &= 0 \end{aligned}$$

The points (0, 0) and $(0, \pm n\pi)$ are the zeros of the equation. They all coincide, so only one point is visible on the graph. This point is also the only polar axis intercept.

Example 1.5 Plot $r = \tan\theta - \cos\theta$, for $-\pi/2 < \theta < \pi/2$.

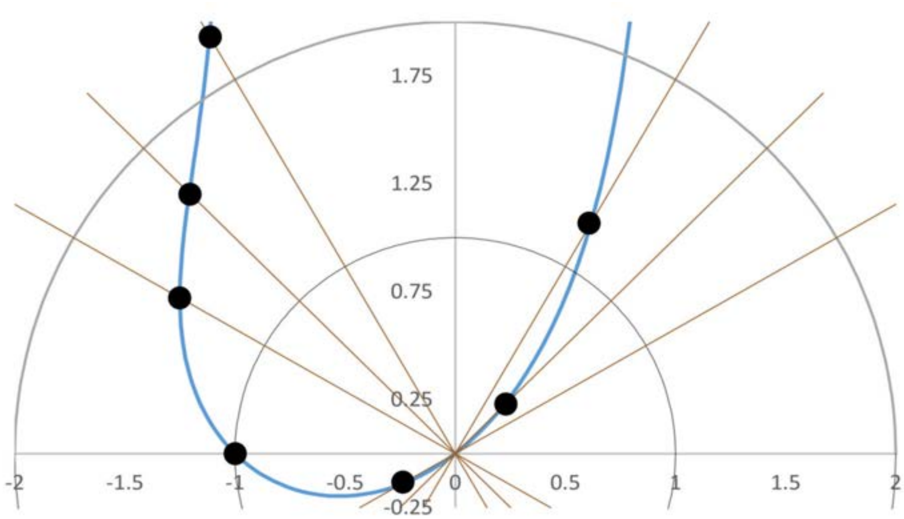
Solution We test for symmetry, but all three symmetry tests fail. We now plot the points that come from the standard angles. In this case one decimal point of accuracy is good enough, so we will round our numbers to the nearest tenth. Useful information: $1/\sqrt{3} \approx 0.6$, $1/\sqrt{2} \approx 0.7$, $\sqrt{2} \approx 1.4$ and $\sqrt{3} \approx 1.7$.

We start with $\theta = 0$ and proceed to $\theta = \pi/3$, the last 4 columns in Table 0 below. The values in the first three columns follow from the fact that the cosine is even and the tangent is odd.

θ	$-\pi/3$	$-\pi/4$	$-\pi/6$	0	$\pi/6$	$\pi/4$	$\pi/3$
$\cos\theta$	0.5	0.7	0.8	1	0.8	0.7	0.5
$\tan\theta$	-1.7	-1	-0.6	0	0.6	1	1.7
r	-2.2	-1.7	-1.4	-1	-0.2	0.3	1.2

Table 0

Using the first and fourth rows of Table 0 we plot the black points. Sketching a smooth curve through them in the sequence in which θ advances, we can approximately sketch the curve.



To find the maximum value of the equation, look at the maximum value of the trigonometric function $\sin\theta$, which occurs when $\theta = \frac{\pi}{2} \pm 2k\pi$ resulting in $\sin\left(\frac{\pi}{2}\right) = 1$. Substitute $\frac{\pi}{2}$ for θ.

$$r = 2\sin\left(\frac{\pi}{2}\right)$$
$$r = 2(1)$$
$$r = 2$$

Analysis *The point* $\left(2, \frac{\pi}{2}\right)$ *will be the maximum value on the graph. Let's plot a few more points to verify the graph of a circle. See* ***Table 2*** *and* ***Figure 4***.

θ	$r = 2\sin\theta$	r
0	$r = 2\sin(0) = 0$	0
$\frac{\pi}{6}$	$r = 2\sin\left(\frac{\pi}{6}\right) = 1$	1
$\frac{\pi}{3}$	$r = 2\sin\left(\frac{\pi}{3}\right) \approx 1.73$	1.73
$\frac{\pi}{2}$	$r = 2\sin\left(\frac{\pi}{2}\right) = 2$	2
$\frac{2\pi}{3}$	$r = 2\sin\left(\frac{2\pi}{3}\right) \approx 1.73$	1.73
$\frac{5\pi}{6}$	$r = 2\sin\left(\frac{5\pi}{6}\right) = 1$	1
π	$r = 2\sin(\pi) = 0$	0

Table 2

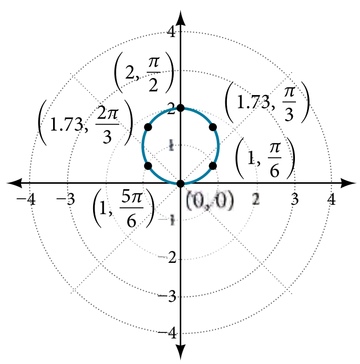


Figure 4

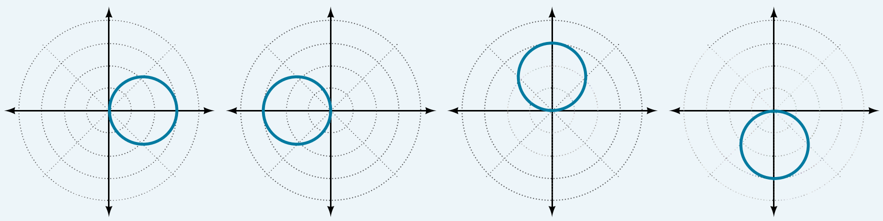
Try It #2

Without converting to Cartesian coordinates, test the given equation for symmetry and find the zeros and maximum values of $|r|$: $r = 3\cos\theta$.

Investigating Circles

Now we have seen the equation of a circle in the polar coordinate system. In the last two examples, the same equation was used to illustrate the properties of symmetry and demonstrate how to find the zeros, maximum values, and plotted points that produced the graphs. However, the circle is only one of many shapes in the set of polar curves.

There are five classic polar curves: **cardioids, limaçons, lemniscates, rose curves,** and **Archimedes' spirals**. We will briefly touch on the polar formulas for the circle before moving on to the classic curves and their variations.

> ***formulas for the equation of a circle***
> Some of the formulas that produce the graph of a circle in polar coordinates are given by $r = a\cos\theta$ and $r = a\sin\theta$, where a is the diameter of the circle or the distance from the pole to the farthest point on the circumference. The radius is $\frac{|a|}{2}$, or one-half the diameter. For $r = a\cos\theta$, the center is $\left(\frac{a}{2}, 0\right)$. For $r = a\sin\theta$, the center is $\left(\frac{a}{2}, \pi\right)$. **Figure 5** shows the graphs of these four circles.
>
> 
>
>
> **Figure 5**

Example 3 Sketching the Graph of a Polar Equation for a Circle

Sketch the graph of $r = 4\cos\theta$.

Solution First, testing the equation for symmetry, we find that the graph is symmetric about the polar axis. Next, we find the zeros and maximum $|r|$ for $r = 4\cos\theta$. First, set $r = 0$, and solve for θ. Thus, a zero occurs at $\theta = \frac{\pi}{2} \pm k\pi$. A key point to plot is $\left(0, \frac{\pi}{2}\right)$

To find the maximum value of r, note that the maximum value of the cosine function is 1 when $\theta = 0 \pm 2k\pi$. Substitute $\theta = 0$ into the equation:

$$r = 4\cos\theta$$
$$r = 4\cos(0)$$
$$r = 4(1) = 4$$

The maximum value of the equation is 4. A key point to plot is (4, 0).

As $r = 4\cos\theta$ is symmetric with respect to the polar axis, we only need to calculate r-values for θ over the interval $[0, \pi]$. Points in the upper quadrant can then be reflected to the lower quadrant. Make a table of values similar to **Table 3**. The graph is shown in **Figure 6**.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
r	4	3.46	2.83	2	0	−2	−2.83	−3.46	4

Table 3

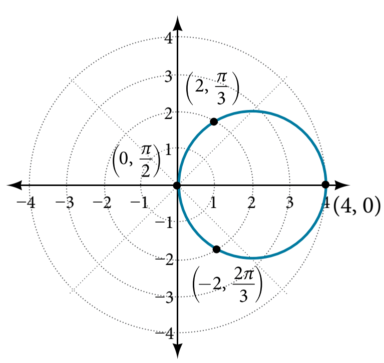


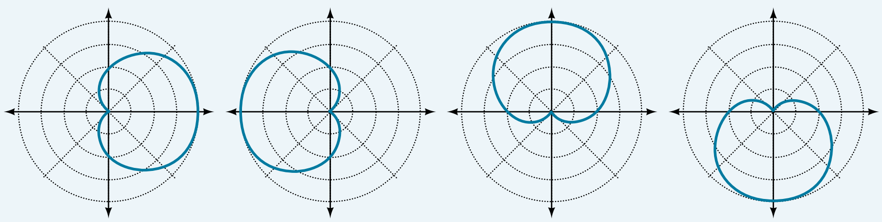
Figure 6

Investigating Cardioids

While translating from polar coordinates to Cartesian coordinates may seem simpler in some instances, graphing the classic curves is actually less complicated in the polar system. The next curve is called a cardioid, as it resembles a heart. This shape is often included with the family of curves called limaçons, but here we will discuss the cardioid on its own.

formulas for a cardioid

The formulas that produce the graphs of a **cardioid** are given by $r = a \pm b\cos\theta$ and $r = a \pm b\sin\theta$ where $a > 0$, $b > 0$, and $\frac{a}{b} = 1$. The cardioid graph passes through the pole, as we can see in **Figure 7**.



$r = a + b\cos\theta$ (a) $r = a - b\cos\theta$ (b) $r = a + b\sin\theta$ (c) $r = a - b\sin\theta$ (d)

Figure 7

How To...

Given the polar equation of a cardioid, sketch its graph.

1. Check equation for the three types of symmetry.
2. Find the zeros. Set $r = 0$.
3. Find the maximum value of the equation according to the maximum value of the trigonometric expression.
4. Make a table of values for r and θ.
5. Plot the points and sketch the graph.

Example 4 Sketching the Graph of a Cardioid

Sketch the graph of $r = 2 + 2\cos\theta$.

Solution First, testing the equation for symmetry, we find that the graph of this equation will be symmetric about the polar axis. Next, we find the zeros and maximums. Setting $r = 0$, we have $\theta = \pi + 2k\pi$. The zero of the equation is located at $(0, \pi)$. The graph passes through this point.

The maximum value of $r = 2 + 2\cos\theta$ occurs when $\cos\theta$ is a maximum, which is when $\cos\theta = 1$ or when $\theta = 0$. Substitute $\theta = 0$ into the equation, and solve for r.

$$r = 2 + 2\cos(0)$$
$$r = 2 + 2(1) = 4$$

The point $(4, 0)$ is the maximum value on the graph.

We found that the polar equation is symmetric with respect to the polar axis, but as it extends to all four quadrants, we need to plot values over the interval $[0, \pi]$. The upper portion of the graph is then reflected over the polar axis. Next, we make a table of values, as in **Table 4**, and then we plot the points and draw the graph. See **Figure 8**.

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	4	3.41	2	1	0

Table 4

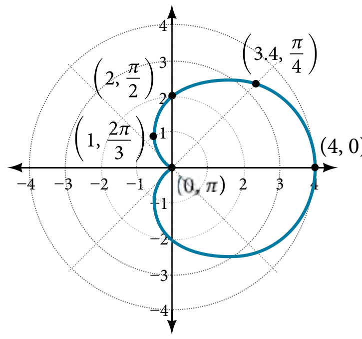
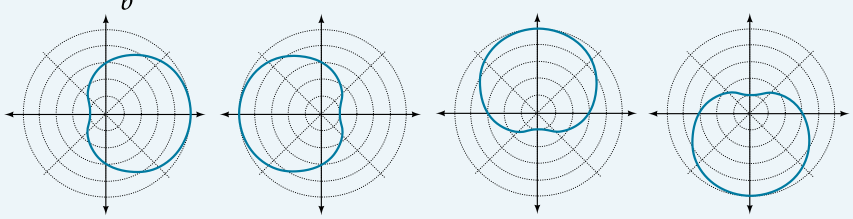


Figure 8

Investigating Limaçons

The word limaçon is Old French for "snail," a name that describes the shape of the graph. As mentioned earlier, the cardioid is a member of the limaçon family, and we can see the similarities in the graphs. The other images in this category include the one-loop limaçon and the two-loop (or inner-loop) limaçon. **One-loop limaçons** are sometimes referred to as **dimpled limaçons** when $1 < \frac{a}{b} < 2$ and **convex limaçons** when $\frac{a}{b} \geq 2$.

> ***formulas for one-loop limaçons***
>
> The formulas that produce the graph of a dimpled **one-loop limaçon** are given by $r = a \pm b\cos\theta$ and $r = a \pm b\sin\theta$ where $a > 0$, $b > 0$, and $1 < \frac{a}{b} < 2$. All four graphs are shown in **Figure 9**.
>
> 
>
> $r = a + b\cos\theta$ (a) | $r = a - b\cos\theta$ (b) | $r = a + b\sin\theta$ (c) | $r = a - b\sin\theta$ (d)
>
> **Figure 9** Dimpled limaçons

How To…

Given a polar equation for a one-loop limaçon, sketch the graph.

1. Test the equation for symmetry. Remember that failing a symmetry test does not mean that the shape will not exhibit symmetry. Often the symmetry may reveal itself when the points are plotted.
2. Find the zeros.
3. Find the maximum values according to the trigonometric expression.
4. Make a table.
5. Plot the points and sketch the graph.

Example 5 **Sketching the Graph of a One-Loop Limaçon**

Graph the equation $r = 4 - 3\sin\theta$.

Solution First, testing the equation for symmetry, we find that it fails all three symmetry tests, meaning that the graph may or may not exhibit symmetry, so we cannot use the symmetry to help us graph it. However, this equation has a graph that clearly displays symmetry with respect to the line $\theta = \frac{\pi}{2}$, yet it fails all the three symmetry tests. A graphing calculator will immediately illustrate the graph's reflective quality.

Next, we find the zeros and maximum, and plot the reflecting points to verify any symmetry. Setting $r = 0$ results in θ being undefined. What does this mean? How could θ be undefined? The angle θ is undefined for any value of $\sin\theta > 1$. Therefore, θ is undefined because there is no value of θ for which $\sin\theta > 1$. Consequently, the graph does not pass through the pole. Perhaps the graph does cross the polar axis, but not at the pole. We can investigate other intercepts by calculating r when $\theta = 0$.

$$r(0) = 4 - 3\sin(0)$$
$$r = 4 - 3 \cdot 0 = 4$$

So, there is at least one polar axis intercept at (4, 0).

Next, as the maximum value of the sine function is 1 when $\theta = \frac{\pi}{2}$, we will substitute $\theta = \frac{\pi}{2}$ into the equation and solve for r. Thus, $r = 1$.

Make a table of the coordinates similar to **Table 5**.

$\boldsymbol{\theta}$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$\boldsymbol{r}$	4	2.5	1.4	1	1.4	2.5	4	5.5	6.6	7	6.6	5.5	4

Table 5

The graph is shown in **Figure 10**.

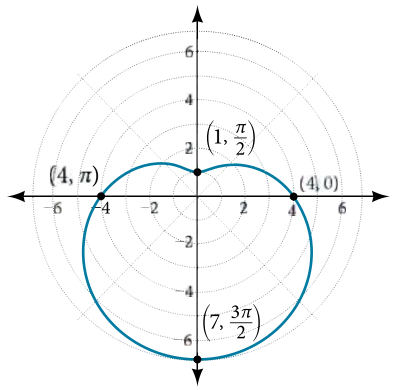


Figure 10

Analysis *This is an example of a curve for which making a table of values is critical to producing an accurate graph. The symmetry tests fail; the zero is undefined. While it may be apparent that an equation involving* $\sin\theta$ *is likely symmetric with respect to the line* $\theta = \frac{\pi}{2}$*, evaluating more points helps to verify that the graph is correct.*

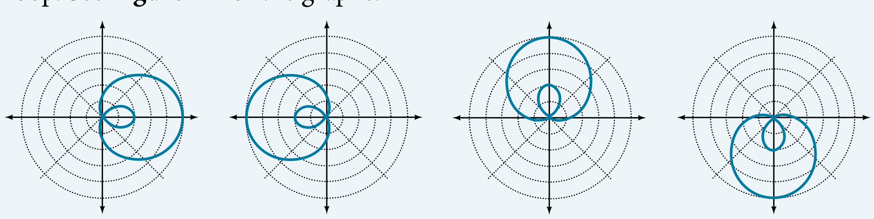
Try It #3

Sketch the graph of $r = 3 - 2\cos\theta$.

Another type of limaçon, the **inner-loop limaçon**, is named for the loop formed inside the general limaçon shape. It was discovered by the German artist Albrecht Dürer(1471-1528), who revealed a method for drawing the inner-loop limaçon in his 1525 book *Underweysung der Messing*. A century later, the father of mathematician Blaise Pascal, Étienne Pascal(1588-1651), rediscovered it.

formulas for inner-loop limaçons

The formulas that generate the **inner-loop limaçons** are given by $r = a \pm b\cos\theta$ and $r = a \pm b\sin\theta$ where $a > 0$, $b > 0$, and $a < b$. The graph of the inner-loop limaçon passes through the pole twice: once for the outer loop, and once for the inner loop. See **Figure 11** for the graphs.



$r = a + b\cos\theta,\ a < b$ (a) $\quad r = a - b\cos\theta,\ a < b$ (b) $\quad r = a + b\sin\theta,\ a < b$ (c) $\quad r = a - b\sin\theta,\ a < b$ (d)

Figure 11

Example 6 Sketching the Graph of an Inner-Loop Limaçon

Sketch the graph of $r = 2 + 5\cos\theta$.

Solution Testing for symmetry, we find that the graph of the equation is symmetric about the polar axis. Next, finding the zeros reveals that when $r = 0$, $\theta = 1.98$. The maximum $|r|$ is found when $\cos\theta = 1$ or when $\theta = 0$. Thus, the maximum is found at the point $(7, 0)$.

Even though we have found symmetry, the zero, and the maximum, plotting more points will help to define the shape, and then a pattern will emerge.

See **Table 6**.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
r	7	6.3	4.5	2	−0.5	−2.3	−3	−2.3	−0.5	2	4.5	6.3	7

Table 6

As expected, the values begin to repeat after $\theta = \pi$. The graph is shown in **Figure 12**.

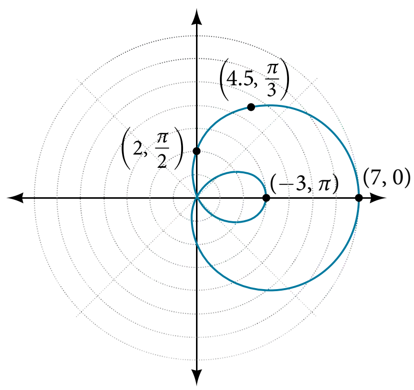
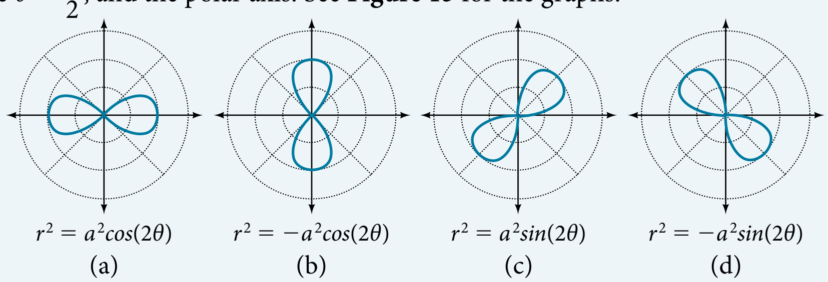


Figure 12 Inner-loop limaçon

Investigating Lemniscates

The lemniscate is a polar curve resembling the infinity symbol ∞ or a figure 8. Centered at the pole, a lemniscate is symmetrical by definition.

> ***formulas for lemniscates***
> The formulas that generate the graph of a **lemniscate** are given by $r^2 = a^2\cos 2\theta$ and $r^2 = a^2\sin 2\theta$ where $a \neq 0$. The formula $r^2 = a^2\sin 2\theta$ is symmetric with respect to the pole. The formula $r^2 = a^2\cos 2\theta$ is symmetric with respect to the pole, the line $\theta = \frac{\pi}{2}$, and the polar axis. See **Figure 13** for the graphs.
>
> 
>
>
> Figure 13

Example 7 Sketching the Graph of a Lemniscate

Sketch the graph of $r^2 = 4\cos 2\theta$.

Solution The equation exhibits symmetry with respect to the line $\theta = \frac{\pi}{2}$, the polar axis, and the pole.

Let's find the zeros. It should be routine by now, but we will approach this equation a little differently by making the substitution $u = 2\theta$.

$$0 = 4\cos 2\theta$$
$$0 = 4\cos u$$
$$0 = \cos u$$
$$\cos^{-1} 0 = \frac{\pi}{2}$$
$$u = \frac{\pi}{2} \qquad \text{Substitute } 2\theta \text{ back in for } u.$$
$$2\theta = \frac{\pi}{2}$$
$$\theta = \frac{\pi}{4}$$

So, the point $\left(0, \frac{\pi}{4}\right)$ is a zero of the equation.

Now let's find the maximum value. Since the maximum of $\cos u = 1$ when $u = 0$, the maximum $\cos 2\theta = 1$ when $2\theta = 0$. Thus,

$$r^2 = 4\cos(0)$$
$$r^2 = 4(1) = 4$$
$$r = \pm\sqrt{4} = 2$$

We have a maximum at (2, 0). Since this graph is symmetric with respect to the pole, the line $\theta = \frac{\pi}{2}$, and the polar axis, we only need to plot points in the first quadrant.

Make a table similar to **Table 7**.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
r	2	$\sqrt{2}$	0	$\sqrt{2}$	0

Table 7

Plot the points on the graph, such as the one shown in **Figure 14**.

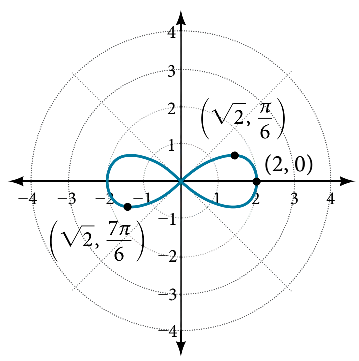


Figure 14 Lemniscate

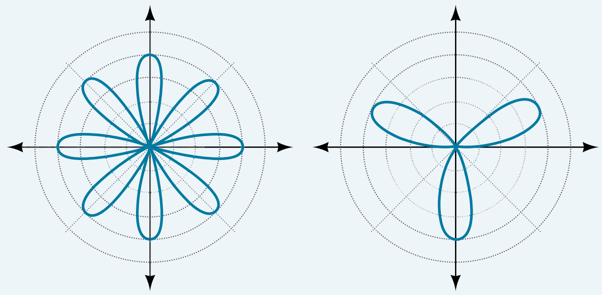
Analysis *Making a substitution such as $u = 2\theta$ is a common practice in mathematics because it can make calculations simpler. However, we must not forget to replace the substitution term with the original term at the end, and then solve for the unknown. Some of the points on this graph may not show up using the Trace function on the TI-84 graphing calculator, and the calculator table may show an error for these same points of r. This is because there are no real square roots for these values of θ. In other words, the corresponding r-values of $\sqrt{4\cos(2\theta)}$ are complex numbers because there is a negative number under the radical.*

Investigating Rose Curves

The next type of polar equation produces a petal-like shape called a rose curve. Although the graphs look complex, a simple polar equation generates the pattern.

rose curves

The formulas that generate the graph of a **rose curve** are given by $r = a\cos n\theta$ and $r = a\sin n\theta$ where $a \neq 0$. If n is even, the curve has $2n$ petals. If n is odd, the curve has n petals. See **Figure 15**.



$r = a\cos(n\theta)$, n even (a) $r = a\sin(n\theta)$, n odd (b)

Figure 15

Example 8 Sketching the Graph of a Rose Curve (*n* Even)

Sketch the graph of $r = 2\cos 4\theta$.

Solution Testing for symmetry, we find again that the symmetry tests do not tell the whole story. The graph is not only symmetric with respect to the polar axis, but also with respect to the line $\theta = \frac{\pi}{2}$ and the pole.

Now we will find the zeros. First make the substitution $u = 4\theta$.

$$\begin{aligned} 0 &= 2\cos 4\theta \\ 0 &= \cos 4\theta \\ 0 &= \cos u \\ \cos^{-1} 0 &= u \\ u &= \frac{\pi}{2} \\ 4\theta &= \frac{\pi}{2} \\ \theta &= \frac{\pi}{8} \end{aligned}$$

The zero is $\theta = \frac{\pi}{8}$. The point $\left(0, \frac{\pi}{8}\right)$ is on the curve.

Next, we find the maximum $|r|$. We know that the maximum value of $\cos u = 1$ when $\theta = 0$. Thus,

$$\begin{aligned} r &= 2\cos(4 \cdot 0) \\ r &= 2\cos(0) \\ r &= 2(1) = 2 \end{aligned}$$

The point $(2, 0)$ is on the curve.

The graph of the rose curve has unique properties, which are revealed in **Table 8**.

$\boldsymbol{\theta}$	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$	$\frac{5\pi}{8}$	$\frac{3\pi}{4}$
$\boldsymbol{r}$	2	0	-2	0	2	0	-2

Table 8

As $r = 0$ when $\theta = \frac{\pi}{8}$, it makes sense to divide values in the table by $\frac{\pi}{8}$ units. A definite pattern emerges. Look at the range of r-values: 2, 0, -2, 0, 2, 0, -2, and so on. This represents the development of the curve one petal at a time. Starting at $r = 0$, each petal extends out a distance of $r = 2$, and then turns back to zero $2n$ times for a total of eight petals. See the graph in **Figure 16**.

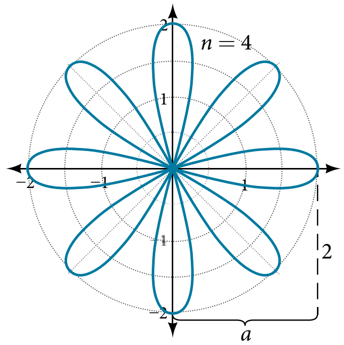


Figure 16 Rose curve, n even

Analysis *When these curves are drawn, it is best to plot the points in order, as in the* ***Table 8****. This allows us to see how the graph hits a maximum (the tip of a petal), loops back crossing the pole, hits the opposite maximum, and loops back to the pole. The action is continuous until all the petals are drawn.*

Try It #4

Sketch the graph of $r = 4\sin(2\theta)$.

Example 9 Sketching the Graph of a Rose Curve (n Odd)

Sketch the graph of $r = 2\sin(5\theta)$.

Solution The graph of the equation shows symmetry with respect to the line $\theta = \frac{\pi}{2}$. Next, find the zeros and maximum. We will want to make the substitution $u = 5\theta$.

$$\begin{aligned} 0 &= 2\sin(5\theta) \\ 0 &= \sin u \\ \sin^{-1} 0 &= 0 \\ u &= 0 \\ 5\theta &= 0 \\ \theta &= 0 \end{aligned}$$

The maximum value is calculated at the angle where $\sin\theta$ is a maximum. Therefore,

$$\begin{aligned} r &= 2\sin\left(5 \cdot \frac{\pi}{2}\right) \\ r &= 2(1) = 2 \end{aligned}$$

Thus, the maximum value of the polar equation is 2. This is the length of each petal. As the curve for n odd yields the same number of petals as n, there will be five petals on the graph. See **Figure 17**.

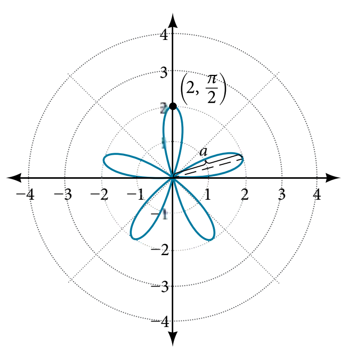


Figure 17 Rose curve, n odd

Create a table of values similar to **Table 9**.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
r	0	1	−1.73	2	−1.73	1	0

Table 9

Try It #5

Sketch the graph of $r = 3\cos(3\theta)$.

Investigating the Archimedes' Spiral

The final polar equation we will discuss is the Archimedes' spiral, named for its discoverer, the Greek mathematician Archimedes (c. 287 BCE–c. 212 BCE), who is credited with numerous discoveries in the fields of geometry and mechanics.

Archimedes' spiral

The formula that generates the graph of the **Archimedes' spiral** is given by $r = \theta$ for $\theta \geq 0$. As θ increases, r increases at a constant rate in an ever-widening, never-ending, spiraling path. See **Figure 18**.

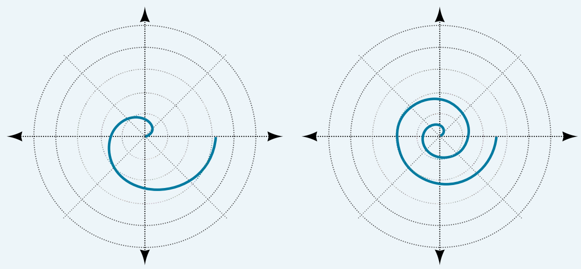


Figure 18

How To…

Given an Archimedes' spiral over $[0, 2\pi]$, sketch the graph.

1. Make a table of values for r and θ over the given domain.
2. Plot the points and sketch the graph.

Example 10 **Sketching the Graph of an Archimedes' Spiral**

Sketch the graph of $r = \theta$ over $[0, 2\pi]$.

Solution As r is equal to θ, the plot of the Archimedes' spiral begins at the pole at the point $(0, 0)$. While the graph hints of symmetry, there is no formal symmetry with regard to passing the symmetry tests. Further, there is no maximum value, unless the domain is restricted.

Create a table such as **Table 10**.

$\boldsymbol{\theta}$	$\frac{\pi}{4}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
$\boldsymbol{r}$	0.785	1.57	3.14	4.71	5.50	6.28

Table 10

Notice that the r-values are just the decimal form of the angle measured in radians. We can see them on a graph in **Figure 19.**

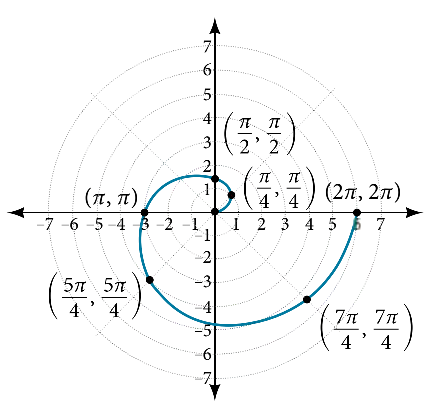


Figure 19 Archimedes' spiral

Analysis *The domain of this polar curve is* $[0, 2\pi]$. *In general, however, the domain of this function is* $(-\infty, \infty)$. *Graphing the equation of the Archimedes' spiral is rather simple, although the image makes it seem like it would be complex.*

Try It #6

Sketch the graph of $r = -\theta$ over the interval $[0, 4\pi]$.

Summary of Curves

We have explored a number of seemingly complex polar curves in this section. **Figure 20** and **Figure 21** summarize the graphs and equations for each of these curves.

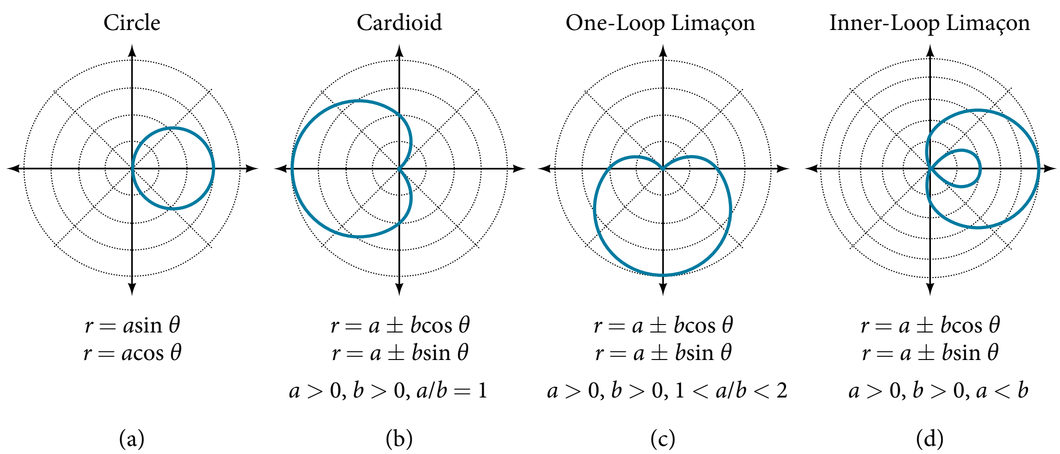


Figure 20

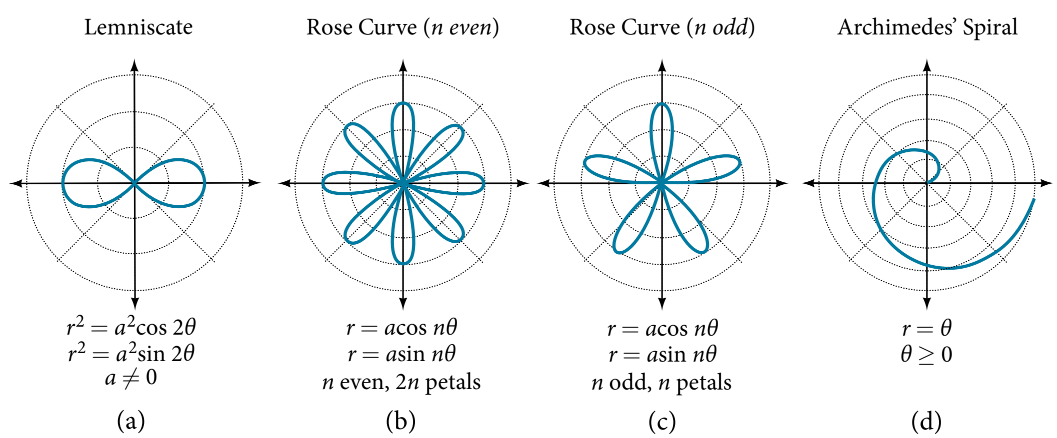


Figure 21

Access these online resources for additional instruction and practice with graphs of polar coordinates.

- Graphing Polar Equations Part 1 (http://openstaxcollege.org/l/polargraph1)
- Graphing Polar Equations Part 2 (http://openstaxcollege.org/l/polargraph2)
- Animation: The Graphs of Polar Equations (http://openstaxcollege.org/l/polaranim)
- Graphing Polar Equations on the TI-84 (http://openstaxcollege.org/l/polarTI84)

10.4 SECTION EXERCISES

VERBAL

1. Describe the three types of symmetry in polar graphs, and compare them to the symmetry of the Cartesian plane.

2. Which of the three types of symmetries for polar graphs correspond to the symmetries with respect to the x-axis, y-axis, and origin?

3. What are the steps to follow when graphing polar equations?

4. Describe the shapes of the graphs of cardioids, limaçons, and lemniscates.

5. What part of the equation determines the shape of the graph of a polar equation?

GRAPHICAL

For the following exercises, test the equation for symmetry.

6. $r = 5\cos 3\theta$

7. $r = 3 - 3\cos \theta$

8. $r = 3 + 2\sin \theta$

9. $r = 3\sin 2\theta$

10. $r = 4$

11. $r = 2\theta$

12. $r = 4\cos \dfrac{\theta}{2}$

13. $r = \dfrac{2}{\theta}$

14. $r = 3\sqrt{1 - \cos^2\theta}$

15. $r = \sqrt{5\sin 2\theta}$

For the following exercises, graph the polar equation. Identify the name of the shape.

16. $r = 3\cos \theta$

17. $r = 4\sin \theta$

18. $r = 2 + 2\cos \theta$

19. $r = 2 - 2\cos \theta$

20. $r = 5 - 5\sin \theta$

21. $r = 3 + 3\sin \theta$

22. $r = 3 + 2\sin \theta$

23. $r = 7 + 4\sin \theta$

24. $r = 4 + 3\cos \theta$

25. $r = 5 + 4\cos \theta$

26. $r = 10 + 9\cos \theta$

27. $r = 1 + 3\sin \theta$

28. $r = 2 + 5\sin \theta$

29. $r = 5 + 7\sin \theta$

30. $r = 2 + 4\cos \theta$

31. $r = 5 + 6\cos \theta$

32. $r^2 = 36\cos(2\theta)$

33. $r^2 = 10\cos(2\theta)$

34. $r^2 = 4\sin(2\theta)$

35. $r^2 = 10\sin(2\theta)$

36. $r = 3\sin(2\theta)$

37. $r = 3\cos(2\theta)$

38. $r = 5\sin(3\theta)$

39. $r = 4\sin(4\theta)$

40. $r = 4\sin(5\theta)$

41. $r = -\theta$

42. $r = 2\theta$

43. $r = -3\theta$

TECHNOLOGY

For the following exercises, use a graphing calculator to sketch the graph of the polar equation.

44. $r = \dfrac{1}{\theta}$

45. $r = \dfrac{1}{\sqrt{\theta}}$

46. $r = 2\sin \theta \tan \theta$, *a* cissoid

47. $r = 2\sqrt{1 - \sin^2 \theta}$, *a* hippopede

48. $r = 5 + \cos(4\theta)$

49. $r = 2 - \sin(2\theta)$

50. $r = \theta^2$

51. $r = \theta + 1$

52. $r = \theta\sin \theta$

53. $r = \theta\cos \theta$

For the following exercises, use a graphing utility to graph each pair of polar equations on a domain of $[0, 4\pi]$ and then explain the differences shown in the graphs.

54. $r = \theta, r = -\theta$

55. $r = \theta, r = \theta + \sin \theta$

56. $r = \sin \theta + \theta, r = \sin \theta - \theta$

57. $r = 2\sin\left(\dfrac{\theta}{2}\right), r = \theta\sin\left(\dfrac{\theta}{2}\right)$

58. $r = \sin(\cos(3\theta))\ r = \sin(3\theta)$

59. On a graphing utility, graph $r = \sin\left(\frac{16}{5}\theta\right)$ on $[0, 4\pi]$, $[0, 8\pi]$, $[0, 12\pi]$, and $[0, 16\pi]$. Describe the effect of increasing the width of the domain.

60. On a graphing utility, graph and sketch $r = \sin\theta + \left(\sin\left(\frac{5}{2}\theta\right)\right)^3$ on $[0, 4\pi]$.

61. On a graphing utility, graph each polar equation. Explain the similarities and differences you observe in the graphs.

$$r_1 = 3\sin(3\theta)$$
$$r_2 = 2\sin(3\theta)$$
$$r_3 = \sin(3\theta)$$

62. On a graphing utility, graph each polar equation. Explain the similarities and differences you observe in the graphs.

$$r_1 = 3 + 3\cos\theta$$
$$r_2 = 2 + 2\cos\theta$$
$$r_3 = 1 + \cos\theta$$

63. On a graphing utility, graph each polar equation. Explain the similarities and differences you observe in the graphs.

$$r_1 = 3\theta$$
$$r_2 = 2\theta$$
$$r_3 = \theta$$

EXTENSIONS

For the following exercises, draw each polar equation on the same set of polar axes, and find the points of intersection.

64. $r_1 = 3 + 2\sin\theta$, $r_2 = 2$

65. $r_1 = 6 - 4\cos\theta$, $r_2 = 4$

66. $r_1 = 1 + \sin\theta$, $r_2 = 3\sin\theta$

67. $r_1 = 1 + \cos\theta$, $r_2 = 3\cos\theta$

68. $r_1 = \cos(2\theta)$, $r_2 = \sin(2\theta)$

69. $r_1 = \sin^2(2\theta)$, $r_2 = 1 - \cos(4\theta)$

70. $r_1 = \sqrt{3}$, $r_2 = 2\sin(\theta)$

71. $r_1^{\,2} = \sin\theta$, $r_2^{\,2} = \cos\theta$

72. $r_1 = 1 + \cos\theta$, $r_2 = 1 - \sin\theta$

LEARNING OBJECTIVES

In this section, you will:

- Plot complex numbers in the complex plane.
- Find the absolute value of a complex number.
- Write complex numbers in polar form.
- Convert a complex number from polar to rectangular form.
- Find products of complex numbers in polar form.
- Find quotients of complex numbers in polar form.
- Find powers of complex numbers in polar form.
- Find roots of complex numbers in polar form.

10.5 POLAR FORM OF COMPLEX NUMBERS

"God made the integers; all else is the work of man." This rather famous quote by nineteenth-century German mathematician Leopold Kronecker sets the stage for this section on the polar form of a complex number. Complex numbers were invented by people and represent over a thousand years of continuous investigation and struggle by mathematicians such as Pythagoras, Descartes, De Moivre, Euler, Gauss, and others. Complex numbers answered questions that for centuries had puzzled the greatest minds in science.

We first encountered complex numbers in **Complex Numbers**. In this section, we will focus on the mechanics of working with complex numbers: translation of complex numbers from polar form to rectangular form and vice versa, interpretation of complex numbers in the scheme of applications, and application of De Moivre's Theorem.

Plotting Complex Numbers in the Complex Plane

Plotting a complex number $a + bi$ is similar to plotting a real number, except that the horizontal axis represents the real part of the number, a, and the vertical axis represents the imaginary part of the number, bi.

How To...

Given a complex number $a + bi$, plot it in the complex plane.

1. Label the horizontal axis as the *real* axis and the vertical axis as the *imaginary axis*.
2. Plot the point in the complex plane by moving a units in the horizontal direction and b units in the vertical direction.

Example 1 **Plotting a Complex Number in the Complex Plane**

Plot the complex number $2 - 3i$ in the complex plane.

Solution From the origin, move two units in the positive horizontal direction and three units in the negative vertical direction. See **Figure 1.**

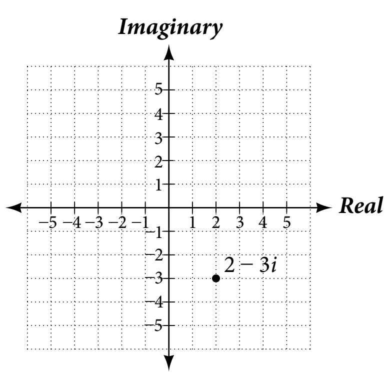


Figure 1

Try It #1

Plot the point $1 + 5i$ in the complex plane.

Finding the Absolute Value of a Complex Number

The first step toward working with a complex number in polar form is to find the absolute value. The absolute value of a complex number is the same as its magnitude, or $|z|$. It measures the distance from the origin to a point in the plane. For example, the graph of $z = 2 + 4i$, in **Figure 2**, shows $|z|$.

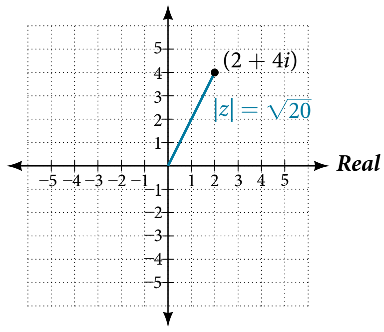


Figure 2

absolute value of a complex number

Given $z = x + yi$, a complex number, the absolute value of z is defined as

$$|z| = \sqrt{x^2 + y^2}$$

It is the distance from the origin to the point (x, y).

Notice that the absolute value of a real number gives the distance of the number from 0, while the absolute value of a complex number gives the distance of the number from the origin, $(0, 0)$.

Example 2 **Finding the Absolute Value of a Complex Number with a Radical**

Find the absolute value of $z = \sqrt{5} - i$.

Solution Using the formula, we have

$$\begin{aligned} |z| &= \sqrt{x^2 + y^2} \\ |z| &= \sqrt{\sqrt{5}^2 + (-1)^2} \\ |z| &= \sqrt{5 + 1} \\ |z| &= \sqrt{6} \end{aligned}$$

See **Figure 3**.

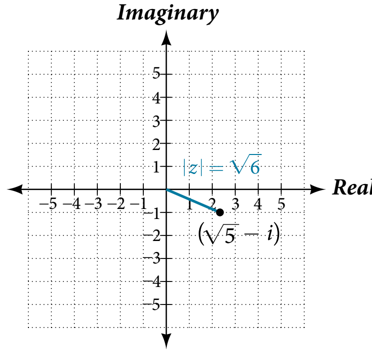


Figure 3

Try It #2

Find the absolute value of the complex number $z = 12 - 5i$.

Example 3 **Finding the Absolute Value of a Complex Number**

Given $z = 3 - 4i$, find $|z|$.

Solution Using the formula, we have

$$|z| = \sqrt{x^2 + y^2}$$
$$|z| = \sqrt{(3)^2 + (-4)^2}$$
$$|z| = \sqrt{9 + 16}$$
$$|z| = \sqrt{25}$$
$$|z| = 5$$

The absolute value of z is 5. See **Figure 4**.

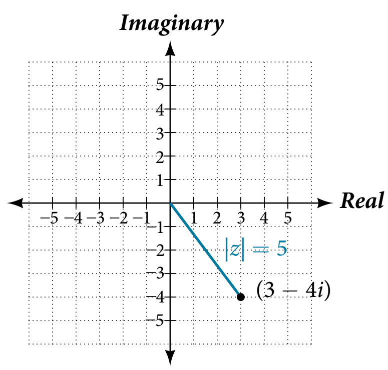


Figure 4

Try It #3

Given $z = 1 - 7i$, find $|z|$.

Writing Complex Numbers in Polar Form

The **polar form of a complex number** expresses a number in terms of an angle θ and its distance from the origin r. Given a complex number in rectangular form expressed as $z = x + yi$, we use the same conversion formulas as we do to write the number in trigonometric form:

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$r = \sqrt{x^2 + y^2}$$

We review these relationships in **Figure 5**.

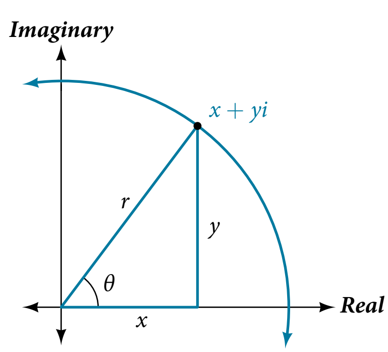


Figure 5

We use the term **modulus** to represent the absolute value of a complex number, or the distance from the origin to the point (x, y). The modulus, then, is the same as r, the radius in polar form. We use θ to indicate the angle of direction (just as with polar coordinates). Substituting, we have

$$z = x + yi$$
$$z = r\cos\theta + (r\sin\theta)i$$
$$z = r(\cos\theta + i\sin\theta)$$

polar form of a complex number

Writing a complex number in polar form involves the following conversion formulas:

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$r = \sqrt{x^2 + y^2}$$

Making a direct substitution, we have

$$z = x + yi$$
$$z = (r\cos\theta) + i(r\sin\theta)$$
$$z = r(\cos\theta + i\sin\theta)$$

where r is the **modulus** and θ is the **argument**. We often use the abbreviation $r\text{cis}\,\theta$ to represent $r(\cos\theta + isin\,\theta)$.

Example 4 Expressing a Complex Number Using Polar Coordinates

Express the complex number $4i$ using polar coordinates.

Solution On the complex plane, the number $z = 4i$ is the same as $z = 0 + 4i$. Writing it in polar form, we have to calculate r first.

$$r = \sqrt{x^2 + y^2}$$
$$r = \sqrt{0^2 + 4^2}$$
$$r = \sqrt{16}$$
$$r = 4$$

Next, we look at x. If $x = r\cos\theta$, and $x = 0$, then $\theta = \frac{\pi}{2}$. In polar coordinates, the complex number $z = 0 + 4i$ can be written as $z = 4\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right)$ or $4\text{cis}\left(\frac{\pi}{2}\right)$. See **Figure 6**.

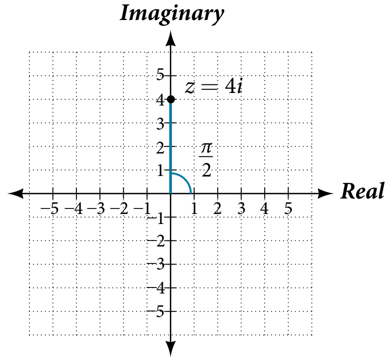


Figure 6

Try It #4

Express $z = 3i$ as $r\text{cis}\,\theta$ in polar form.

Example 5 Finding the Polar Form of a Complex Number

Find the polar form of $-4 + 4i$.

Solution First, find the value of r.

$$r = \sqrt{x^2 + y^2}$$
$$r = \sqrt{(-4)^2 + (4^2)}$$
$$r = \sqrt{32}$$
$$r = 4\sqrt{2}$$

Find the angle θ using the formula:

$$\cos\theta = \frac{x}{r}$$
$$\cos\theta = \frac{-4}{4\sqrt{2}}$$
$$\cos\theta = -\frac{1}{\sqrt{2}}$$
$$\theta = \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = \frac{3\pi}{4}$$

Thus, the solution is $4\sqrt{2}\text{cis}\left(\frac{3\pi}{4}\right)$.

Since the range of $\cos^{-1}(\cdot)$ is $[0, \pi]$, the approach we just used in Example 5 will be accurate if z is in Quadrant I or II. We just saw that it works for $z = -4 + 4i$, which is in Quadrant II. But it cannot possibly work if z is in either of Quadrants III, IV. The approach we used in Section 10.3 page 792.5 always works. If we apply it to this example we get $\tan\theta = 4/(-4) = -1$. Thus $\theta = \tan^{-1}(-1) = -\pi/4$ if $x > 0$, or if $x < 0$ then $\theta = \tan^{-1}(-1) + \pi = 3\pi/4$. In Example 5 $x = -4 < 0$, so $\theta = 3\pi/4$.

Try It #5

Write $z = \sqrt{3} + i$ in polar form.

Converting a Complex Number from Polar to Rectangular Form

Converting a complex number from polar form to rectangular form is a matter of evaluating what is given and using the distributive property. In other words, given $z = r(\cos\theta + i\sin\theta)$, first evaluate the trigonometric functions $\cos\theta$ and $\sin\theta$. Then, multiply through by r.

Example 6 Converting from Polar to Rectangular Form

Convert the polar form of the given complex number to rectangular form:

$$z = 12\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)$$

Solution We begin by evaluating the trigonometric expressions.

$$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \text{ and } \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

After substitution, the complex number is

$$z = 12\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$$

We apply the distributive property:

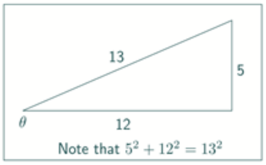
$$z = 12\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$$
$$= (12)\frac{\sqrt{3}}{2} + (12)\frac{1}{2}i$$
$$= 6\sqrt{3} + 6i$$

The rectangular form of the given point in complex form is $6\sqrt{3} + 6i$.

Example 7 Finding the Rectangular Form of a Complex Number

Find the rectangular form of the complex number given $r = 13$ and $\tan\theta = \frac{5}{12}$, and z is in the first quadrant.

Solution If $\tan\theta = \frac{5}{12}$, and $\tan\theta = \frac{y}{x}$, we first determine $r = \sqrt{x^2 + y^2} = \sqrt{12^2 + 5^2} = 13$. We then find $\cos\theta = \frac{x}{r}$ and $\sin\theta = \frac{y}{r}$.



$$z = 13(\cos\theta + i\sin\theta)$$
$$= 13\left(\frac{12}{13} + \frac{5}{13}i\right)$$
$$= 12 + 5i$$

The rectangular form of the given number in complex form is $12 + 5i$.

Example 7.5 We alter Example 7 by removing our assumption that z is in the first quadrant and by assuming that y is negative.

Solution Since $\tan\theta = y/x > 0$ and $y < 0$ we must have $x < 0$. As before, $r = 13$. $z = r\,\mathrm{cis}\,\theta = 13(\cos\theta + i\sin\theta) = 13\,(-12/13 + i\,(-5/13)) = -12 - 5\,i$.

Try It #6

Convert the complex number to rectangular form:

$$z = 4\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right)$$

Finding Products of Complex Numbers in Polar Form

Now that we can convert complex numbers to polar form we will learn how to perform operations on complex numbers in polar form. For the rest of this section, we will work with formulas developed by French mathematician Abraham De Moivre (1667–1754). These formulas have made working with products, quotients, powers, and roots of complex numbers much simpler than they appear. The rules are based on multiplying the moduli and adding the arguments.

products of complex numbers in polar form

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then the product of these numbers is given as:

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

$$z_1 z_2 = r_1 r_2 \mathrm{cis}(\theta_1 + \theta_2)$$

Notice that the product calls for multiplying the moduli and adding the angles.

Example 8 Finding the Product of Two Complex Numbers in Polar Form

Find the product of $z_1 z_2$, given $z_1 = 4(\cos(80°) + i\sin(80°))$ and $z_2 = 2(\cos(145°) + i\sin(145°))$.

Solution Follow the formula

$$z_1 z_2 = 4 \cdot 2[\cos(80° + 145°) + i\sin(80° + 145°)]$$

$$z_1 z_2 = 8[\cos(225°) + i\sin(225°)]$$

$$z_1 z_2 = 8\left[\cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right)\right]$$

$$z_1 z_2 = 8\left[-\frac{\sqrt{2}}{2} + i\left(-\frac{\sqrt{2}}{2}\right)\right]$$

$$z_1 z_2 = -4\sqrt{2} - 4i\sqrt{2}$$

Finding Quotients of Complex Numbers in Polar Form

The quotient of two complex numbers in polar form is the quotient of the two moduli and the difference of the two arguments.

quotients of complex numbers in polar form

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then the quotient of these numbers is

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)],\ z_2 \neq 0$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\operatorname{cis}(\theta_1 - \theta_2),\ z_2 \neq 0$$

Notice that the moduli are divided, and the angles are subtracted.

How To...

Given two complex numbers in polar form, find the quotient.

1. Divide $\frac{r_1}{r_2}$.
2. Find $\theta_1 - \theta_2$.
3. Substitute the results into the formula: $z = r(\cos\theta + i\sin\theta)$. Replace r with $\frac{r_1}{r_2}$, and replace θ with $\theta_1 - \theta_2$.
4. Calculate the new trigonometric expressions and multiply through by r.

Example 9 Finding the Quotient of Two Complex Numbers

Find the quotient of $z_1 = 2(\cos(213°) + i\sin(213°))$ and $z_2 = 4(\cos(33°) + i\sin(33°))$.

Solution Using the formula, we have

$$\frac{z_1}{z_2} = \frac{2}{4}[\cos(213° - 33°) + i\sin(213° - 33°)]$$

$$\frac{z_1}{z_2} = \frac{1}{2}[\cos(180°) + i\sin(180°)]$$

$$\frac{z_1}{z_2} = \frac{1}{2}[-1 + 0i]$$

$$\frac{z_1}{z_2} = -\frac{1}{2} + 0i$$

$$\frac{z_1}{z_2} = -\frac{1}{2}$$

Try It #7

Find the product and the quotient of $z_1 = 2\sqrt{3}(\cos(150°) + i\sin(150°))$ and $z_2 = 2(\cos(30°) + i\sin(30°))$.

Finding Powers of Complex Numbers in Polar Form

Finding powers of complex numbers is greatly simplified using **De Moivre's Theorem**. It states that, for a positive integer n, z^n is found by raising the modulus to the nth power and multiplying the argument by n. It is the standard method used in modern mathematics.

De Moivre's Theorem

If $z = r(\cos\theta + i\sin\theta)$ is a complex number, then

$$z^n = r^n[\cos(n\theta) + i\sin(n\theta)]$$

$$z^n = r^n\operatorname{cis}(n\theta)$$

where n is a positive integer.

Example 10 Evaluating an Expression Using De Moivre's Theorem

Evaluate the expression $(1 + i)^5$ using De Moivre's Theorem.

Solution Since De Moivre's Theorem applies to complex numbers written in polar form, we must first write $(1 + i)$ in polar form. Let us find r.

$$r = \sqrt{x^2 + y^2}$$
$$r = \sqrt{(1)^2 + (1)^2}$$
$$r = \sqrt{2}$$

Then we find θ. Using the formula $\tan \theta = \frac{y}{x}$ gives

$$\tan \theta = \frac{1}{1}$$
$$\tan \theta = 1 \qquad \boxed{x > 0. \quad \text{(Recall the box adjacent to Example 5.)}}$$
$$\theta = \frac{\pi}{4}$$

Use De Moivre's Theorem to evaluate the expression.

$$(a + bi)^n = r^n[\cos(n\theta) + i\sin(n\theta)]$$
$$(1 + i)^5 = (\sqrt{2})^5\left[\cos\left(5 \cdot \frac{\pi}{4}\right) + i\sin\left(5 \cdot \frac{\pi}{4}\right)\right]$$
$$(1 + i)^5 = 4\sqrt{2}\left[\cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right)\right]$$
$$(1 + i)^5 = 4\sqrt{2}\left[-\frac{\sqrt{2}}{2} + i\left(-\frac{\sqrt{2}}{2}\right)\right]$$
$$(1 + i)^5 = -4 - 4i$$

Finding Roots of Complex Numbers in Polar Form

To find the nth root of a complex number in polar form, we use the nth Root Theorem or De Moivre's Theorem and raise the complex number to a power with a rational exponent. There are several ways to represent a formula for finding nth roots of complex numbers in polar form.

the nth root theorem

To find the nth root of a complex number in polar form, use the formula given as

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}\left[\cos\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)\right]$$

where $k = 0, 1, 2, 3, \ldots, n - 1$. We add $\frac{2k\pi}{n}$ to $\frac{\theta}{n}$ in order to obtain the periodic roots.

Example 11 **Finding the *n*th Root of a Complex Number**

Evaluate the cube roots of $z = 8\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)$.

Solution We have

$$z^{\frac{1}{3}} = 8^{\frac{1}{3}}\left[\cos\left(\frac{\frac{2\pi}{3}}{3} + \frac{2k\pi}{3}\right) + i\sin\left(\frac{\frac{2\pi}{3}}{3} + \frac{2k\pi}{3}\right)\right]$$

$$z^{\frac{1}{3}} = 2\left[\cos\left(\frac{2\pi}{9} + \frac{2k\pi}{3}\right) + i\sin\left(\frac{2\pi}{9} + \frac{2k\pi}{3}\right)\right]$$

There will be three roots: $k = 0, 1, 2$. When $k = 0$, we have

$$z^{\frac{1}{3}} = 2\left(\cos\left(\frac{2\pi}{9}\right) + i\sin\left(\frac{2\pi}{9}\right)\right)$$

When $k = 1$, we have

$$z^{\frac{1}{3}} = 2\left[\cos\left(\frac{2\pi}{9} + \frac{6\pi}{9}\right) + i\sin\left(\frac{2\pi}{9} + \frac{6\pi}{9}\right)\right] \quad \text{Add } \frac{2(1)\pi}{3} \text{ to each angle.}$$

$$z^{\frac{1}{3}} = 2\left(\cos\left(\frac{8\pi}{9}\right) + i\sin\left(\frac{8\pi}{9}\right)\right)$$

When $k = 2$, we have

$$z^{\frac{1}{3}} = 2\left[\cos\left(\frac{2\pi}{9} + \frac{12\pi}{9}\right) + i\sin\left(\frac{2\pi}{9} + \frac{12\pi}{9}\right)\right] \quad \text{Add } \frac{2(2)\pi}{3} \text{ to each angle.}$$

$$z^{\frac{1}{3}} = 2\left(\cos\left(\frac{14\pi}{9}\right) + i\sin\left(\frac{14\pi}{9}\right)\right)$$

Remember to find the common denominator to simplify fractions in situations like this one. For $k = 1$, the angle simplification is

$$\begin{aligned}\frac{\frac{2\pi}{3}}{3} + \frac{2(1)\pi}{3} &= \frac{2\pi}{3}\left(\frac{1}{3}\right) + \frac{2(1)\pi}{3}\left(\frac{3}{3}\right) \\ &= \frac{2\pi}{9} + \frac{6\pi}{9} \\ &= \frac{8\pi}{9}\end{aligned}$$

Try It #8

Find the four fourth roots of $16(\cos(120°) + i\sin(120°))$.

We note that if z_0 is any given complex number with $|z_0| > 0$, the equation $z^n = z_0$ is satisfied by exactly n different complex numbers z. In other words, if $|z_0| > 0$ then z_0 has exactly n distinct complex n-th roots. The real numbers are not like that. For example, -1 has no real square roots, but it has two complex square roots, i and $-i$.

Access these online resources for additional instruction and practice with polar forms of complex numbers.

- The Product and Quotient of Complex Numbers in Trigonometric Form (http://openstaxcollege.org/l/prodquocomplex)
- De Moivre's Theorem (http://openstaxcollege.org/l/demoivre)

10.5 SECTION EXERCISES

VERBAL

1. A complex number is $a + bi$. Explain each part.

2. What does the absolute value of a complex number represent?

3. How is a complex number converted to polar form?

4. How do we find the product of two complex numbers?

5. What is De Moivre's Theorem and what is it used for?

ALGEBRAIC

For the following exercises, find the absolute value of the given complex number.

6. $5 + 3i$

7. $-7 + i$

8. $-3 - 3i$

9. $\sqrt{2} - 6i$

10. $2i$

11. $2.2 - 3.1i$

For the following exercises, write the complex number in polar form.

12. $2 + 2i$

13. $8 - 4i$

14. $-\frac{1}{2} - \frac{1}{2}i$

15. $\sqrt{3} + i$

16. $3i$

For the following exercises, convert the complex number from polar to rectangular form.

17. $z = 7\text{cis}\left(\frac{\pi}{6}\right)$

18. $z = 2\text{cis}\left(\frac{\pi}{3}\right)$

19. $z = 4\text{cis}\left(\frac{7\pi}{6}\right)$

20. $z = 7\text{cis}(25°)$

21. $z = 3\text{cis}(240°)$

22. $z = \sqrt{2}\text{cis}(100°)$

For the following exercises, find $z_1 z_2$ in polar form.

23. $z_1 = 2\sqrt{3}\text{cis}(116°)$; $z_2 = 2\text{cis}(82°)$

24. $z_1 = \sqrt{2}\text{cis}(205°)$; $z_2 = 2\sqrt{2}\text{cis}(118°)$

25. $z_1 = 3\text{cis}(120°)$; $z_2 = \frac{1}{4}\text{cis}(60°)$

26. $z_1 = 3\text{cis}\left(\frac{\pi}{4}\right)$; $z_2 = 5\text{cis}\left(\frac{\pi}{6}\right)$

27. $z_1 = \sqrt{5}\text{cis}\left(\frac{5\pi}{8}\right)$; $z_2 = \sqrt{15}\,\text{cis}\left(\frac{\pi}{12}\right)$

28. $z_1 = 4\text{cis}\left(\frac{\pi}{2}\right)$; $z_2 = 2\text{cis}\left(\frac{\pi}{4}\right)$

For the following exercises, find $\frac{z_1}{z_2}$ in polar form.

29. $z_1 = 21\text{cis}(135°)$; $z_2 = 3\text{cis}(65°)$

30. $z_1 = \sqrt{2}\text{cis}(90°)$; $z_2 = 2\text{cis}(60°)$

31. $z_1 = 15\text{cis}(120°)$; $z_2 = 3\text{cis}(40°)$

32. $z_1 = 6\text{cis}\left(\frac{\pi}{3}\right)$; $z_2 = 2\text{cis}\left(\frac{\pi}{4}\right)$

33. $z_1 = 5\sqrt{2}\text{cis}(\pi)$; $z_2 = \sqrt{2}\text{cis}\left(\frac{2\pi}{3}\right)$

34. $z_1 = 2\text{cis}\left(\frac{3\pi}{5}\right)$; $z_2 = 3\text{cis}\left(\frac{\pi}{4}\right)$

For the following exercises, find the powers of each complex number in polar form.

35. Find z^3 when $z = 5\text{cis}(45°)$.

36. Find z^4 when $z = 2\text{cis}(70°)$.

37. Find z^2 when $z = 3\text{cis}(120°)$.

38. Find z^2 when $z = 4\text{cis}\left(\frac{\pi}{4}\right)$.

39. Find z_4 when $z = \text{cis}\left(\frac{3\pi}{16}\right)$.

40. Find z^3 when $z = 3\text{cis}\left(\frac{5\pi}{3}\right)$.

For the following exercises, evaluate each root.

41. Evaluate the cube root of z when $z = 27\text{cis}(240°)$.

42. Evaluate the square root of z when $z = 16\text{cis}(100°)$.

43. Evaluate the cube root of z when $z = 32\text{cis}\left(\frac{2\pi}{3}\right)$.

44. Evaluate the square root of z when $z = 32\text{cis}(\pi)$.

45. Evaluate the cube root of z when $z = 8\text{cis}\left(\frac{7\pi}{4}\right)$.

GRAPHICAL

For the following exercises, plot the complex number in the complex plane.

46. $2 + 4i$

47. $-3 - 3i$

48. $5 - 4i$

49. $-1 - 5i$

50. $3 + 2i$

51. $2i$

52. -4

53. $6 - 2i$

54. $-2 + i$

55. $1 - 4i$

TECHNOLOGY

For the following exercises, find all answers rounded to the nearest hundredth.

56. Use the rectangular to polar feature on the graphing calculator to change $5 + 5i$ to polar form.

57. Use the rectangular to polar feature on the graphing calculator to change $3 - 2i$ to polar form.

58. Use the rectangular to polar feature on the graphing calculator to change $-3 - 8i$ to polar form.

59. Use the polar to rectangular feature on the graphing calculator to change $4\text{cis}(120°)$ to rectangular form.

60. Use the polar to rectangular feature on the graphing calculator to change $2\text{cis}(45°)$ to rectangular form.

61. Use the polar to rectangular feature on the graphing calculator to change $5\text{cis}(210°)$ to rectangular form.

LEARNING OBJECTIVES

In this section, you will:

- Parameterize a curve.
- Eliminate the parameter.
- Find a rectangular equation for a curve defined parametrically.
- Find parametric equations for curves defined by rectangular equations.

10.6 PARAMETRIC EQUATIONS

Consider the path a moon follows as it orbits a planet, which simultaneously rotates around the sun, as seen in **Figure 1**. At any moment, the moon is located at a particular spot relative to the planet. But how do we write and solve the equation for the position of the moon when the distance from the planet, the speed of the moon's orbit around the planet, and the speed of rotation around the sun are all unknowns? We can solve only for one variable at a time.

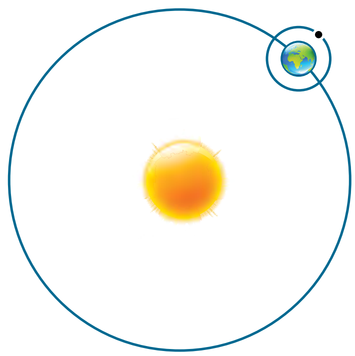


Figure 1

In this section, we will consider sets of equations given by $x(t)$ and $y(t)$ where t is the independent variable of time.

We can use these parametric equations in a number of applications when we are looking for not only a particular position but also the direction of the movement. As we trace out successive values of t, the orientation of the curve becomes clear.

This is one of the primary advantages of using parametric equations: we are able to trace the movement of an object along a path according to time. We begin this section with a look at the basic components of parametric equations and what it means to parameterize a curve. Then we will learn how to eliminate the parameter, translate the equations of a curve defined parametrically into rectangular equations, and find the parametric equations for curves defined by rectangular equations.

Parameterizing a Curve

When an object moves along a curve—or curvilinear path—in a given direction and in a given amount of time, the position of the object in the plane is given by the x-coordinate and the y-coordinate. However, both x and y vary over time and so are functions of time. For this reason, we add another variable, the **parameter**, upon which both x and y are dependent functions. In the example in the section opener, the parameter is time, t. The x position of the moon at time, t, is represented as the function $x(t)$, and the y position of the moon at time, t, is represented as the function $y(t)$. Together, $x(t)$ and $y(t)$ are called parametric equations, and generate an ordered pair $(x(t), y(t))$. Parametric equations primarily describe motion and direction.

When we parameterize a curve, we are translating a single equation in two variables, such as x and y, into an equivalent pair of equations in three variables, x, y, and t. One of the reasons we parameterize a curve is because the parametric equations yield more information: specifically, the direction of the object's motion over time.

When we graph parametric equations, we can observe the individual behaviors of x and of y. There are a number of shapes that cannot be represented in the form $y = f(x)$, meaning that they are not functions. For example, consider the graph of a circle, given as $r^2 = x^2 + y^2$. Solving for y gives $y = \pm\sqrt{r^2 - x^2}$, or two equations: $y_1 = \sqrt{r^2 - x^2}$ and $y_2 = -\sqrt{r^2 - x^2}$. If we graph y_1 and y_2 together, the graph will not pass the vertical line test, as shown in **Figure 2**. Thus, the equation for the graph of a circle is not a function.

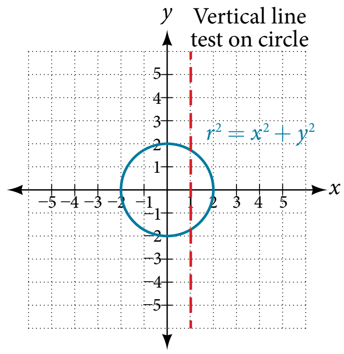


Figure 2

However, if we were to graph each equation on its own, each one would pass the vertical line test and therefore would represent a function. In some instances, the concept of breaking up the equation for a circle into two functions is similar to the concept of creating parametric equations, as we use two functions to produce a non-function. This will become clearer as we move forward.

parametric equations

Suppose t is a number on an interval, I. The set of ordered pairs, $(x(t), y(t))$, where $x = f(t)$ and $y = g(t)$, forms a plane curve based on the parameter t. The equations $x = f(t)$ and $y = g(t)$ are the parametric equations.

Example 1 **Parameterizing a Curve**

Parameterize the curve $y = x^2 - 1$ letting $x(t) = t$. Graph both equations.

Solution If $x(t) = t$, then to find $y(t)$ we replace the variable x with the expression given in $x(t)$. In other words, $y(t) = t^2 - 1$. Make a table of values similar to **Table 1**, and sketch the graph.

t	$x(t)$	$y(t)$
-4	-4	$y(-4) = (-4)^2 - 1 = 15$
-3	-3	$y(-3) = (-3)^2 - 1 = 8$
-2	-2	$y(-2) = (-2)^2 - 1 = 3$
-1	-1	$y(-1) = (-1)^2 - 1 = 0$
0	0	$y(0) = (0)^2 - 1 = -1$
1	1	$y(1) = (1)^2 - 1 = 0$
2	2	$y(2) = (2)^2 - 1 = 3$
3	3	$y(3) = (3)^2 - 1 = 8$
4	4	$y(4) = (4)^2 - 1 = 15$

Table 1

See the graphs in **Figure 3**. It may be helpful to use the **TRACE** feature of a graphing calculator to see how the points are generated as t increases.

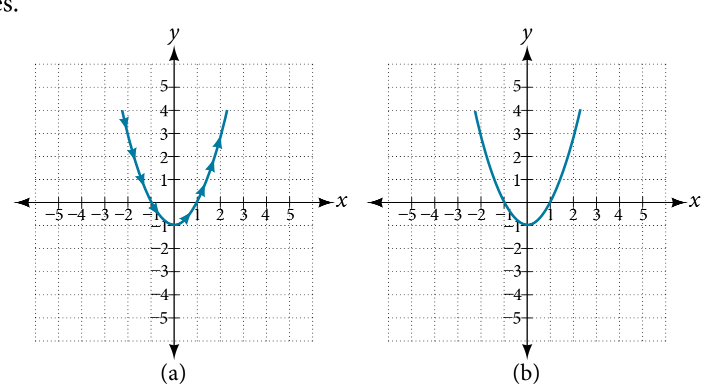


Figure 3 (a) Parametric $y(t) = t^2 - 1$ (b) Rectangular $y = x^2 - 1$

Analysis *The arrows indicate the direction in which the curve is generated. Notice the curve is identical to the curve of* $y = x^2 - 1$.

Try It #1

Construct a table of values and plot the parametric equations: $x(t) = t - 3$, $y(t) = 2t + 4$; $-1 \le t \le 2$.

Example 2 Finding a Pair of Parametric Equations

Find a pair of parametric equations that models the graph of $y = 1 - x^2$, using the parameter $x(t) = t$. Plot some points and sketch the graph.

Solution If $x(t) = t$ and we substitute t for x into the y equation, then $y(t) = 1 - t^2$. Our pair of parametric equations is

$$x(t) = t$$
$$y(t) = 1 - t^2$$

To graph the equations, first we construct a table of values like that in **Table 2**. We can choose values around $t = 0$, from $t = -3$ to $t = 3$. The values in the $x(t)$ column will be the same as those in the t column because $x(t) = t$. Calculate values for the column $y(t)$.

t	$x(t) = t$	$y(t) = 1 - t^2$
-3	-3	$y(-3) = 1 - (-3)^2 = -8$
-2	-2	$y(-2) = 1 - (-2)^2 = -3$
-1	-1	$y(-1) = 1 - (-1)^2 = 0$
0	0	$y(0) = 1 - 0 = 1$
1	1	$y(1) = 1 - (1)^2 = 0$
2	2	$y(2) = 1 - (2)^2 = -3$
3	3	$y(3) = 1 - (3)^2 = -8$

Table 2

The graph of $y = 1 - t^2$ is a parabola facing downward, as shown in **Figure 4**. We have mapped the curve over the interval $[-3, 3]$, shown as a solid line with arrows indicating the orientation of the curve according to t. Orientation refers to the path traced along the curve in terms of increasing values of t. As this parabola is symmetric with respect to the line $x = 0$, the values of x are reflected across the y-axis.

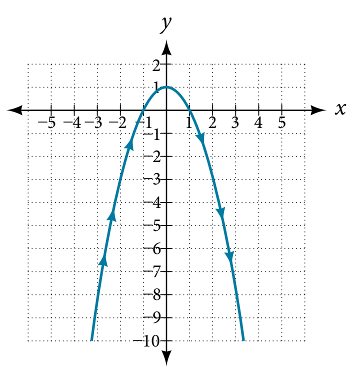


Figure 4

Try It #2

Parameterize the curve given by $x = y^3 - 2y$.

Example 3 Finding Parametric Equations That Model Given Criteria

An object travels at a steady rate along a straight path $(-5, 3)$ to $(3, -1)$ in the same plane in four seconds. The coordinates are measured in meters. Find parametric equations for the position of the object.

Solution The parametric equations are simple linear expressions, but we need to view this problem in a step-by-step fashion. The x-value of the object starts at -5 meters and goes to 3 meters. This means the distance x has changed by 8 meters in 4 seconds, which is a rate of $\frac{8\text{m}}{4\text{s}}$, or 2 m/s. We can write the x-coordinate as a linear function with respect to time as $x(t) = 2t - 5$. In the linear function template $y = mx + b$, $2t = mx$ and $-5 = b$. Similarly, the y-value of the object starts at 3 and goes to -1, which is a change in the distance y of -4 meters in 4 seconds, which is a rate of $\frac{-4\text{m}}{4\text{s}}$, or -1 m/s. We can also write the y-coordinate as the linear function $y(t) = -t + 3$. Together, these are the parametric equations for the position of the object, where x and y are expressed in meters and t represents time:

$$x(t) = 2t - 5$$

$$y(t) = -t + 3$$

Using these equations, we can build a table of values for t, x, and y (see **Table 3**). In this example, we limited values of t to non-negative numbers. In general, any value of t can be used.

t	$x(t) = 2t - 5$	$y(t) = -t + 3$
0	$x = 2(0) - 5 = -5$	$y = -(0) + 3 = 3$
1	$x = 2(1) - 5 = -3$	$y = -(1) + 3 = 2$
2	$x = 2(2) - 5 = -1$	$y = -(2) + 3 = 1$
3	$x = 2(3) - 5 = 1$	$y = -(3) + 3 = 0$
4	$x = 2(4) - 5 = 3$	$y = -(4) + 3 = -1$

Table 3

From this table, we can create three graphs, as shown in **Figure 5**.

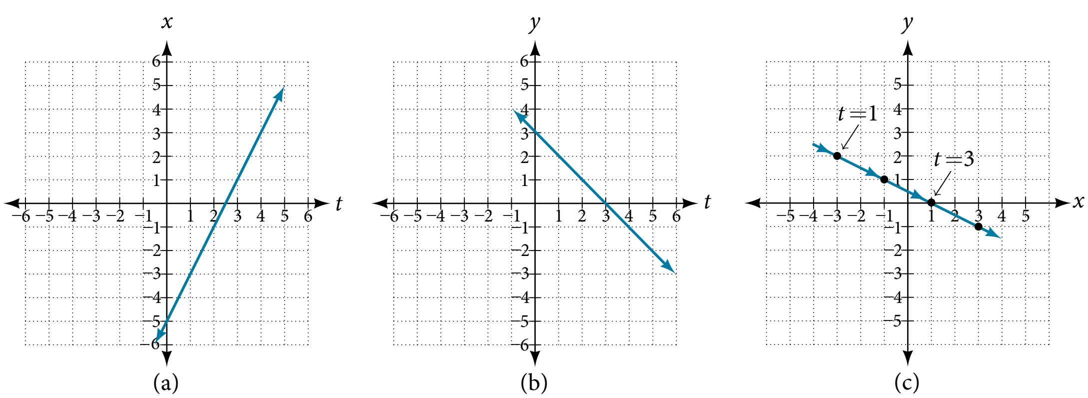


Figure 5 (a) A graph of x vs. t, representing the horizontal position over time. (b) A graph of y vs. t, representing the vertical position over time. (c) A graph of y vs. x, representing the position of the object in the plane at time t.

Analysis *Again, we see that, in* ***Figure 5(c)****, when the parameter represents time, we can indicate the movement of the object along the path with arrows.*

Eliminating the Parameter

In many cases, we may have a pair of parametric equations but find that it is simpler to draw a curve if the equation involves only two variables, such as x and y. Eliminating the parameter is a method that may make graphing some curves easier.

However, if we are concerned with the mapping of the equation according to time, then it will be necessary to indicate the orientation of the curve as well. There are various methods for eliminating the parameter t from a set of parametric equations; not every method works for every type of equation. Here we will review the methods for the most common types of equations.

Eliminating the Parameter from Polynomial, Exponential, and Logarithmic Equations

For polynomial, exponential, or logarithmic equations expressed as two parametric equations, we choose the equation that is most easily manipulated and solve for t. We substitute the resulting expression for t into the second equation. This gives one equation in x and y.

Example 4 Eliminating the Parameter in Polynomials

Given $x(t) = t^2 + 1$ and $y(t) = 2 + t$, eliminate the parameter, and write the parametric equations as a Cartesian equation.

Solution We will begin with the equation for y because the linear equation is easier to solve for t.

$$y = 2 + t$$
$$y - 2 = t$$

Next, substitute $y - 2$ for t in $x(t)$.

$$x = t^2 + 1$$
$$x = (y - 2)^2 + 1 \qquad \text{Substitute the expression for } t \text{ into } x.$$
$$x = y^2 - 4y + 4 + 1$$
$$x = y^2 - 4y + 5$$
$$x = y^2 - 4y + 5$$

The Cartesian form is $x = y^2 - 4y + 5$.

Analysis *This is an equation for a parabola in which, in rectangular terms, x is dependent on y. From the curve's vertex at (1, 2), the graph sweeps out to the right. See* ***Figure 6****. In this section, we consider sets of equations given by the functions x(t) and y(t), where t is the independent variable of time. Notice, both x and y are functions of time; so in general y is not a function of x.*

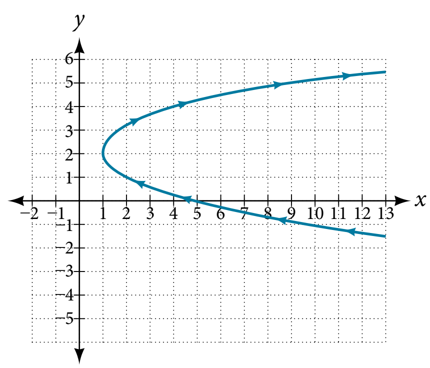


Figure 6

Try It #3

Given the equations below, eliminate the parameter and write as a rectangular equation for y as a function of x.

$$x(t) = 2t^2 + 6$$
$$y(t) = 5 - t$$

Example 5 Eliminating the Parameter in Exponential Equations

Eliminate the parameter and write as a Cartesian equation: $x(t) = e^{-t}$ and $y(t) = 3e^t$, $t > 0$.

Solution Isolate e^t.

$$x = e^{-t}$$
$$e^t = \frac{1}{x}$$

Substitute the expression into $y(t)$.

$$y = 3e^t$$
$$y = 3\left(\frac{1}{x}\right)$$
$$y = \frac{3}{x}$$

The Cartesian form is $y = \frac{3}{x}$

Analysis *The graph of the parametric equation is shown in **Figure 7(a)**. The domain is restricted to $t > 0$. The Cartesian equation, $y = 3x$ is shown in **Figure 7(b)** and has only one restriction on the domain, $x \neq 0$.*

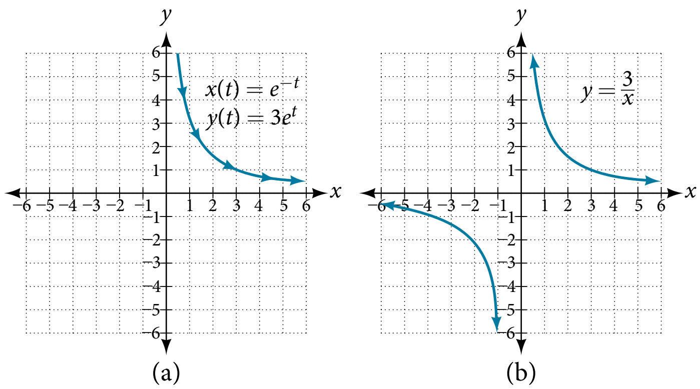


Figure 7

Example 6 Eliminating the Parameter in Logarithmic Equations

Eliminate the parameter and write as a Cartesian equation: $x(t) = \sqrt{t} + 2$ and $y(t) = \log(t)$.

Solution Solve the first equation for t.

$$x = \sqrt{t} + 2$$
$$x - 2 = \sqrt{t}$$
$$(x - 2)^2 = t \qquad \text{Square both sides.}$$

Then, substitute the expression for t into the y equation.

$$y = \log(t)$$
$$y = \log(x - 2)^2$$

The Cartesian form is $y = \log(x - 2)^2$.

Analysis *To be sure that the parametric equations are equivalent to the Cartesian equation, check the domains. The parametric equations restrict the domain on $x = \sqrt{t} + 2$ to $t > 0$; we restrict the domain on x to $x > 2$. The domain for the parametric equation $y = \log(t)$ is restricted to $t > 0$; we limit the domain on $y = \log(x - 2)^2$ to $x > 2$.*

Try It #4

Eliminate the parameter and write as a rectangular equation.

$$x(t) = t^2$$
$$y(t) = \ln(t) \qquad t > 0$$

Eliminating the Parameter from Trigonometric Equations

Eliminating the parameter from trigonometric equations is a straightforward substitution. We can use a few of the familiar trigonometric identities and the Pythagorean Theorem.

First, we use the identities:

$$x(t) = a\cos t$$
$$y(t) = b\sin t$$

Solving for cos t and sin t, we have

$$\frac{x}{a} = \cos t$$
$$\frac{y}{a} = \sin t$$

Then, use the Pythagorean Theorem:

$$\cos^2 t + \sin^2 t = 1$$

Substituting gives

$$\cos^2 t + \sin^2 t = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

Example 7 Eliminating the Parameter from a Pair of Trigonometric Parametric Equations

Eliminate the parameter from the given pair of trigonometric equations where $0 \le t \le 2\pi$ and sketch the graph.

$$x(t) = 4\cos t$$
$$y(t) = 3\sin t$$

Solution Solving for cos t and sin t, we have

$$x = 4\cos t$$
$$\frac{x}{4} = \cos t$$
$$y = 3\sin t$$
$$\frac{y}{3} = \sin t$$

Next, use the Pythagorean identity and make the substitutions.

$$\cos^2 t + \sin^2 t = 1$$
$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$
$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

The graph for the equation is shown in **Figure 8**.

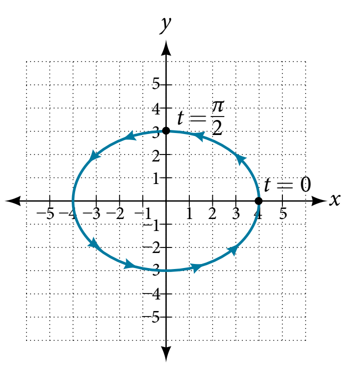


Figure 8

Analysis *Applying the general equations for conic sections (introduced in **Analytic Geometry**, we can identify $\frac{x^2}{16} + \frac{y^2}{9} = 1$ as an ellipse centered at (0, 0). Notice that when $t = 0$ the coordinates are (4, 0), and when $t = \frac{\pi}{2}$ the coordinates are (0, 3). This shows the orientation of the curve with increasing values of t.*

Try It #5

Eliminate the parameter from the given pair of parametric equations and write as a Cartesian equation: $x(t) = 2\cos t$ and $y(t) = 3\sin t$.

Finding Cartesian Equations from Curves Defined Parametrically

When we are given a set of parametric equations and need to find an equivalent Cartesian equation, we are essentially "eliminating the parameter." However, there are various methods we can use to rewrite a set of parametric equations as a Cartesian equation. The simplest method is to set one equation equal to the parameter, such as $x(t) = t$. In this case, $y(t)$ can be any expression. For example, consider the following pair of equations.

$$x(t) = t$$
$$y(t) = t^2 - 3$$

Rewriting this set of parametric equations is a matter of substituting x for t. Thus, the Cartesian equation is $y = x^2 - 3$.

Example 8 **Finding a Cartesian Equation Using Alternate Methods**

Use two different methods to find the Cartesian equation equivalent to the given set of parametric equations.

$$x(t) = 3t - 2$$
$$y(t) = t + 1$$

Solution

Method 1. First, let's solve the x equation for t. Then we can substitute the result into the y equation.

$$\begin{aligned} x &= 3t - 2 \\ x + 2 &= 3t \\ \frac{x+2}{3} &= t \end{aligned}$$

Now substitute the expression for t into the y equation.

$$\begin{aligned} y &= t + 1 \\ y &= \left(\frac{x+2}{3}\right) + 1 \\ y &= \frac{x}{3} + \frac{2}{3} + 1 \\ y &= \frac{1}{3}x + \frac{5}{3} \end{aligned}$$

Method 2. Solve the y equation for t and substitute this expression in the x equation.

$$\begin{aligned} y &= t + 1 \\ y - 1 &= t \end{aligned}$$

Make the substitution and then solve for y.

$$\begin{aligned} x &= 3(y - 1) - 2 \\ x &= 3y - 3 - 2 \\ x &= 3y - 5 \\ x + 5 &= 3y \\ \frac{x+5}{3} &= y \\ y &= \frac{1}{3}x + \frac{5}{3} \end{aligned}$$

Try It #6

Write the given parametric equations as a Cartesian equation: $x(t) = t^3$ and $y(t) = t^6$.

Finding Parametric Equations for Curves Defined by Rectangular Equations

Although we have just shown that there is only one way to interpret a set of parametric equations as a rectangular equation, there are multiple ways to interpret a rectangular equation as a set of parametric equations. Any strategy we may use to find the parametric equations is valid if it produces equivalency. In other words, if we choose an expression to represent x, and then substitute it into the y equation, and it produces the same graph over the same domain as the rectangular equation, then the set of parametric equations is valid. If the domain becomes restricted in the set of parametric equations, and the function does not allow the same values for x as the domain of the rectangular equation, then the graphs will be different.

Example 9 Finding a Set of Parametric Equations for Curves Defined by Rectangular Equations

Find a set of equivalent parametric equations for $y = (x + 3)^2 + 1$.

Solution An obvious choice would be to let $x(t) = t$. Then $y(t) = (t + 3)^2 + 1$. But let's try something more interesting. What if we let $x = t + 3$? Then we have

$$y = (x + 3)^2 + 1$$
$$y = ((t + 3) + 3)^2 + 1$$
$$y = (t + 6)^2 + 1$$

The set of parametric equations is

$$x(t) = t + 3$$
$$y(t) = (t + 6)^2 + 1$$

See **Figure 9**.

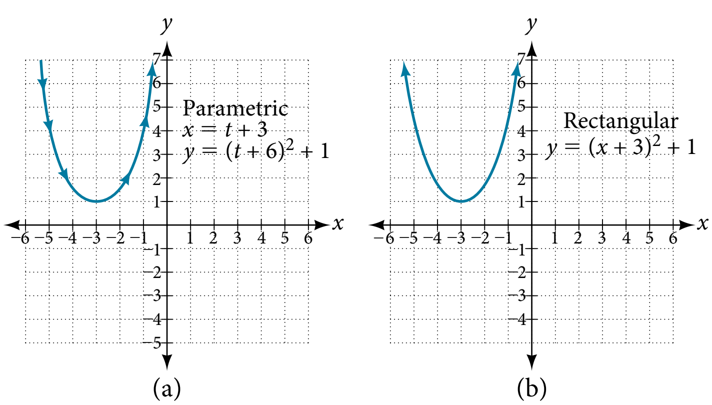


Figure 9

Access these online resources for additional instruction and practice with parametric equations.

- Introduction to Parametric Equations (http://openstaxcollege.org/l/introparametric)
- Converting Parametric Equations to Rectangular Form (http://openstaxcollege.org/l/convertpara)

10.6 SECTION EXERCISES

VERBAL

1. What is a system of parametric equations?

2. Some examples of a third parameter are time, length, speed, and scale. Explain when time is used as a parameter.

3. Explain how to eliminate a parameter given a set of parametric equations.

4. What is a benefit of writing a system of parametric equations as a Cartesian equation?

5. What is a benefit of using parametric equations?

6. Why are there many sets of parametric equations to represent on Cartesian function?

ALGEBRAIC

For the following exercises, eliminate the parameter t to rewrite the parametric equation as a Cartesian equation.

7. $\begin{cases} x(t) = 5 - t \\ y(t) = 8 - 2t \end{cases}$

8. $\begin{cases} x(t) = 6 - 3t \\ y(t) = 10 - t \end{cases}$

9. $\begin{cases} x(t) = 2t + 1 \\ y(t) = 3\sqrt{t} \end{cases}$

10. $\begin{cases} x(t) = 3t - 1 \\ y(t) = 2t^2 \end{cases}$

11. $\begin{cases} x(t) = 2e^t \\ y(t) = 1 - 5t \end{cases}$

12. $\begin{cases} x(t) = e^{-2t} \\ y(t) = 2e^{-t} \end{cases}$

13. $\begin{cases} x(t) = 4\log(t) \\ y(t) = 3 + 2t \end{cases}$

14. $\begin{cases} x(t) = \log(2t) \\ y(t) = \sqrt{t - 1} \end{cases}$

15. $\begin{cases} x(t) = t^3 - t \\ y(t) = 2t \end{cases}$

16. $\begin{cases} x(t) = t - t^4 \\ y(t) = t + 2 \end{cases}$

17. $\begin{cases} x(t) = e^{2t} \\ y(t) = e^{6t} \end{cases}$

18. $\begin{cases} x(t) = t^5 \\ y(t) = t^{10} \end{cases}$

19. $\begin{cases} x(t) = 4\cos t \\ y(t) = 5\sin t \end{cases}$

20. $\begin{cases} x(t) = 3\sin t \\ y(t) = 6\cos t \end{cases}$

21. $\begin{cases} x(t) = 2\cos^2 t \\ y(t) = -\sin t \end{cases}$

22. $\begin{cases} x(t) = \cos t + 4 \\ y(t) = 2\sin^2 t \end{cases}$

23. $\begin{cases} x(t) = t - 1 \\ y(t) = t^2 \end{cases}$

24. $\begin{cases} x(t) = -t \\ y(t) = t^3 + 1 \end{cases}$

25. $\begin{cases} x(t) = 2t - 1 \\ y(t) = t^3 - 2 \end{cases}$

For the following exercises, rewrite the parametric equation as a Cartesian equation by building an x-y table.

26. $\begin{cases} x(t) = 2t - 1 \\ y(t) = t + 4 \end{cases}$

27. $\begin{cases} x(t) = 4 - t \\ y(t) = 3t + 2 \end{cases}$

28. $\begin{cases} x(t) = 2t - 1 \\ y(t) = 5t \end{cases}$

29. $\begin{cases} x(t) = 4t - 1 \\ y(t) = 4t + 2 \end{cases}$

For the following exercises, parameterize (write parametric equations for) each Cartesian equation by setting $x(t) = t$ or by setting $y(t) = t$.

30. $y(x) = 3x^2 + 3$

31. $y(x) = 2\sin x + 1$

32. $x(y) = 3\log(y) + y$

33. $x(y) = \sqrt{y} + 2y$

For the following exercises, parameterize (write parametric equations for) each Cartesian equation by using $x(t) = a\cos t$ and $y(t) = b\sin t$. Identify the curve.

34. $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$

35. $\dfrac{x^2}{16} + \dfrac{y^2}{36} = 1$

36. $x^2 + y^2 = 16$

37. $x^2 + y^2 = 10$

38. Parameterize the line from $(3, 0)$ to $(-2, -5)$ so that the line is at $(3, 0)$ at $t = 0$, and at $(-2, -5)$ at $t = 1$.

39. Parameterize the line from $(-1, 0)$ to $(3, -2)$ so that the line is at $(-1, 0)$ at $t = 0$, and at $(3, -2)$ at $t = 1$.

40. Parameterize the line from $(-1, 5)$ to $(2, 3)$ so that the line is at $(-1, 5)$ at $t = 0$, and at $(2, 3)$ at $t = 1$.

41. Parameterize the line from $(4, 1)$ to $(6, -2)$ so that the line is at $(4, 1)$ at $t = 0$, and at $(6, -2)$ at $t = 1$.

TECHNOLOGY

For the following exercises, use the table feature in the graphing calculator to determine whether the graphs intersect.

42. $\begin{cases} x_1(t) = 3t \\ y_1(t) = 2t - 1 \end{cases}$ and $\begin{cases} x_2(t) = t + 3 \\ y_2(t) = 4t - 4 \end{cases}$

43. $\begin{cases} x_1(t) = t^2 \\ y_1(t) = 2t - 1 \end{cases}$ and $\begin{cases} x_2(t) = -t + 6 \\ y_2(t) = t + 1 \end{cases}$

For the following exercises, use a graphing calculator to complete the table of values for each set of parametric equations.

44. $\begin{cases} x_1(t) = 3t^2 - 3t + 7 \\ y_1(t) = 2t + 3 \end{cases}$

t	x	y
-1		
0		
1		

45. $\begin{cases} x_1(t) = t^2 - 4 \\ y_1(t) = 2t^2 - 1 \end{cases}$

t	x	y
1		
2		
3		

46. $\begin{cases} x_1(t) = t^4 \\ y_1(t) = t^3 + 4 \end{cases}$

t	x	y
-1		
0		
1		
2		

EXTENSIONS

47. Find two different sets of parametric equations for $y = (x + 1)^2$.

48. Find two different sets of parametric equations for $y = 3x - 2$.

49. Find two different sets of parametric equations for $y = x^2 - 4x + 4$.

LEARNING OBJECTIVES

In this section, you will:

- Graph plane curves described by parametric equations by plotting points.
- Graph parametric equations.

10.7 PARAMETRIC EQUATIONS: GRAPHS

It is the bottom of the ninth inning, with two outs and two men on base. The home team is losing by two runs. The batter swings and hits the baseball at 140 feet per second and at an angle of approximately 45° to the horizontal. How far will the ball travel? Will it clear the fence for a game-winning home run? The outcome may depend partly on other factors (for example, the wind), but mathematicians can model the path of a projectile and predict approximately how far it will travel using parametric equations. In this section, we'll discuss parametric equations and some common applications, such as projectile motion problems.

Figure 1 Parametric equations can model the path of a projectile. (credit: Paul Kreher, Flickr)

Graphing Parametric Equations by Plotting Points

In lieu of a graphing calculator or a computer graphing program, plotting points to represent the graph of an equation is the standard method. As long as we are careful in calculating the values, point-plotting is highly dependable.

How To...

Given a pair of parametric equations, sketch a graph by plotting points.

1. Construct a table with three columns: t, $x(t)$, and $y(t)$.
2. Evaluate x and y for values of t over the interval for which the functions are defined.
3. Plot the resulting pairs (x, y).

Example 1 **Sketching the Graph of a Pair of Parametric Equations by Plotting Points**

Sketch the graph of the parametric equations $x(t) = t^2 + 1$, $y(t) = 2 + t$.

Solution Construct a table of values for t, $x(t)$, and $y(t)$, as in **Table 1**, and plot the points in a plane.

t	$x(t) = t^2 + 1$	$y(t) = 2 + t$
-5	26	-3
-4	17	-2
-3	10	-1
-2	5	0
-1	2	1
0	1	2
1	2	3
2	5	4
3	10	5
4	17	6
5	26	7

Table 1

The graph is a parabola with vertex at the point (1, 2), opening to the right. See **Figure 2**.

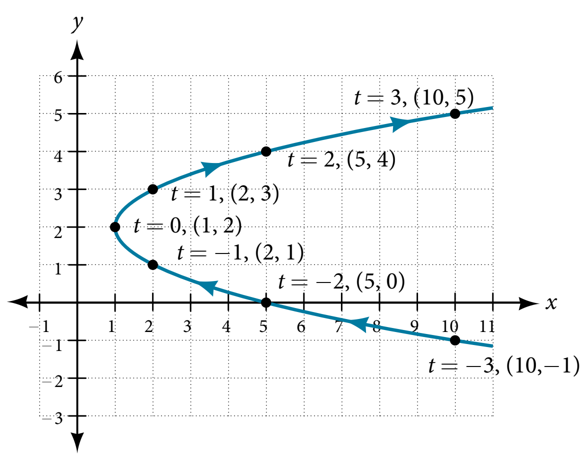


Figure 2

Analysis *As values for t progress in a positive direction from* 0 *to* 5, *the plotted points trace out the top half of the parabola. As values of t become negative, they trace out the lower half of the parabola. There are no restrictions on the domain. The arrows indicate direction according to increasing values of t. The graph does not represent a function, as it will fail the vertical line test. The graph is drawn in two parts: the positive values for t, and the negative values for t.*

Try It #1

Sketch the graph of the parametric equations $x = \sqrt{t}$, $y = 2t + 3$, $0 \leq t \leq 3$.

Example 2 Sketching the Graph of Trigonometric Parametric Equations

Construct a table of values for the given parametric equations and sketch the graph:

$$x = 2\cos t$$
$$y = 4\sin t$$

Solution Construct a table like that in **Table 2** using angle measure in radians as inputs for t, and evaluating x and y. Using angles with known sine and cosine values for t makes calculations easier.

t	$x = 2\cos t$	$y = 4\sin t$
0	$x = 2\cos(0) = 2$	$y = 4\sin(0) = 0$
$\frac{\pi}{6}$	$x = 2\cos\left(\frac{\pi}{6}\right) = \sqrt{3}$	$y = 4\sin\left(\frac{\pi}{6}\right) = 2$
$\frac{\pi}{3}$	$x = 2\cos\left(\frac{\pi}{3}\right) = 1$	$y = 4\sin\left(\frac{\pi}{3}\right) = 2\sqrt{3}$
$\frac{\pi}{2}$	$x = 2\cos\left(\frac{\pi}{2}\right) = 0$	$y = 4\sin\left(\frac{\pi}{2}\right) = 4$
$\frac{2\pi}{3}$	$x = 2\cos\left(\frac{2\pi}{3}\right) = -1$	$y = 4\sin\left(\frac{2\pi}{3}\right) = -2\sqrt{3}$
$\frac{5\pi}{6}$	$x = 2\cos\left(\frac{5\pi}{6}\right) = -\sqrt{3}$	$y = 4\sin\left(\frac{5\pi}{6}\right) = 2$
π	$x = 2\cos(\pi) = -2$	$y = 4\sin(\pi) = 0$
$\frac{7\pi}{6}$	$x = 2\cos\left(\frac{7\pi}{6}\right) = -\sqrt{3}$	$y = 4\sin\left(\frac{7\pi}{6}\right) = -2$
$\frac{4\pi}{3}$	$x = 2\cos\left(\frac{4\pi}{3}\right) = -1$	$y = 4\sin\left(\frac{4\pi}{3}\right) = -2\sqrt{3}$
$\frac{3\pi}{2}$	$x = 2\cos\left(\frac{3\pi}{2}\right) = 0$	$y = 4\sin\left(\frac{3\pi}{2}\right) = -4$
$\frac{5\pi}{3}$	$x = 2\cos\left(\frac{5\pi}{3}\right) = 1$	$y = 4\sin\left(\frac{5\pi}{3}\right) = -2\sqrt{3}$
$\frac{11\pi}{6}$	$x = 2\cos\left(\frac{11\pi}{6}\right) = \sqrt{3}$	$y = 4\sin\left(\frac{11\pi}{6}\right) = -2$
2π	$x = 2\cos(2\pi) = 2$	$y = 4\sin(2\pi) = 0$

Table 2

Figure 3 shows the graph.

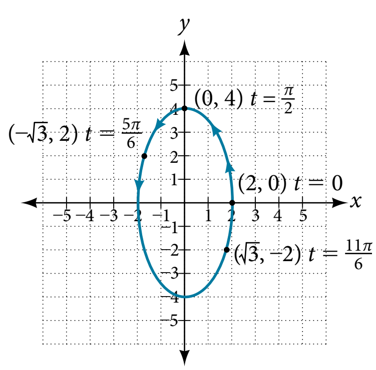


Figure 3

By the symmetry shown in the values of x and y, we see that the parametric equations represent an ellipse.

The ellipse is mapped in a counterclockwise direction as shown by the arrows indicating increasing t values.

Analysis *We have seen that parametric equations can be graphed by plotting points. However, a graphing calculator will save some time and reveal nuances in a graph that may be too tedious to discover using only hand calculations.*

Make sure to change the mode on the calculator to parametric (**PAR**). *To confirm, the* **Y=** *window should show*

$$X_{1T} =$$

$$Y_{1T} =$$

instead of $Y_1 =$.

Try It #2

Graph the parametric equations: $x = 5\cos t$, $y = 3\sin t$.

Example 3 Graphing Parametric Equations and Rectangular Form Together

Graph the parametric equations $x = 5\cos t$ and $y = 2\sin t$. First, construct the graph using data points generated from the parametric form. Then graph the rectangular form of the equation. Compare the two graphs.

Solution Construct a table of values like that in **Table 3**.

t	$x = 5\cos t$	$y = 2\sin t$
0	$x = 5\cos(0) = 5$	$y = 2\sin(0) = 0$
1	$x = 5\cos(1) \approx 2.7$	$y = 2\sin(1) \approx 1.7$
2	$x = 5\cos(2) \approx -2.1$	$y = 2\sin(2) \approx 1.8$
3	$x = 5\cos(3) \approx -4.95$	$y = 2\sin(3) \approx 0.28$
4	$x = 5\cos(4) \approx -3.3$	$y = 2\sin(4) \approx -1.5$
5	$x = 5\cos(5) \approx 1.4$	$y = 2\sin(5) \approx -1.9$
-1	$x = 5\cos(-1) \approx 2.7$	$y = 2\sin(-1) \approx -1.7$
-2	$x = 5\cos(-2) \approx -2.1$	$y = 2\sin(-2) \approx -1.8$
-3	$x = 5\cos(-3) \approx -4.95$	$y = 2\sin(-3) \approx -0.28$
-4	$x = 5\cos(-4) \approx -3.3$	$y = 2\sin(-4) \approx 1.5$
-5	$x = 5\cos(-5) \approx 1.4$	$y = 2\sin(-5) \approx 1.9$

Table 3

Plot the (x, y) values from the table. See **Figure 4**.

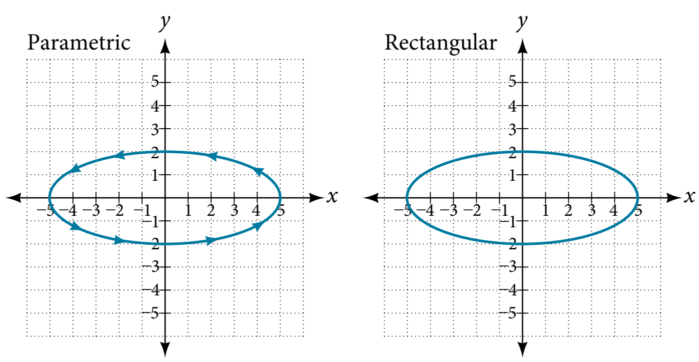


Figure 4

Next, translate the parametric equations to rectangular form. To do this, we solve for t in either $x(t)$ or $y(t)$, and then substitute the expression for t in the other equation. The result will be a function $y(x)$ if solving for t as a function of x, or $x(y)$ if solving for t as a function of y.

$$x = 5\cos t$$

$$\frac{x}{5} = \cos t \qquad \text{Solve for } \cos t.$$

$$y = 2\sin t \qquad \text{Solve for } \sin t.$$

$$\frac{y}{2} = \sin t$$

Then, use the Pythagorean Theorem.

$$\cos^2 t + \sin^2 t = 1$$

$$\left(\frac{x}{5}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$$

$$\frac{x^2}{25} + \frac{y^2}{4} = 1$$

Analysis *In **Figure 5**, the data from the parametric equations and the rectangular equation are plotted together. The parametric equations are plotted in blue; the graph for the rectangular equation is drawn on top of the parametric in a dashed style colored red. Clearly, both forms produce the same graph.*

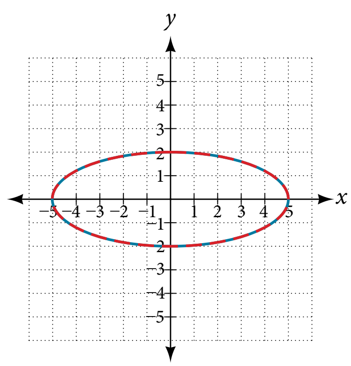


Figure 5

Example 4 Graphing Parametric Equations and Rectangular Equations on the Coordinate System

Graph the parametric equations $x = t + 1$ and $y = \sqrt{t}$, $t \geq 0$, and the rectangular equivalent $y = \sqrt{x - 1}$ on the same coordinate system.

Solution Construct a table of values for the parametric equations, as we did in the previous example, and graph $y = \sqrt{t}$, $t \geq 0$ on the same grid, as in **Figure 6**.

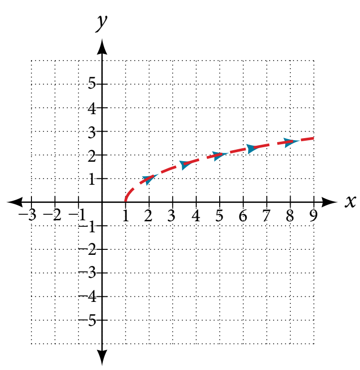


Figure 6

Analysis *With the domain on t restricted, we only plot positive values of t. The parametric data is graphed in blue and the graph of the rectangular equation is dashed in red. Once again, we see that the two forms overlap.*

Try It #3

Sketch the graph of the parametric equations $x = 2\cos\theta$ and $y = 4\sin\theta$, along with the rectangular equation on the same grid.

Applications of Parametric Equations

Many of the advantages of parametric equations become obvious when applied to solving real-world problems. Although rectangular equations in x and y give an overall picture of an object's path, they do not reveal the position of an object at a specific time. Parametric equations, however, illustrate how the values of x and y change depending on t, as the location of a moving object at a particular time.

A common application of parametric equations is solving problems involving projectile motion. In this type of motion, an object is propelled forward in an upward direction forming an angle of θ to the horizontal, with an initial speed of v_0, and at a height h above the horizontal.

The path of an object propelled at an inclination of θ to the horizontal, with initial speed v_0, and at a height h above the horizontal, is given by

$$x = (v_0 \cos \theta)t$$
$$y = -\frac{1}{2}gt^2 + (v_0 \sin \theta)t + h$$

where g accounts for the effects of gravity and h is the initial height of the object. Depending on the units involved in the problem, use $g = 32$ ft/s^2 or $g = 9.8$ m/s^2. The equation for x gives horizontal distance, and the equation for y gives the vertical distance.

How To...

Given a projectile motion problem, use parametric equations to solve.

1. The horizontal distance is given by $x = (v_0 \cos \theta)t$. Substitute the initial speed of the object for v_0.
2. The expression $\cos \theta$ indicates the angle at which the object is propelled. Substitute that angle in degrees for $\cos \theta$.
3. The vertical distance is given by the formula $y = -\frac{1}{2}gt^2 + (v_0 \sin \theta)t + h$. The term $-\frac{1}{2}gt^2$ represents the effect of gravity. Depending on units involved, use $g = 32$ ft/s^2 or $g = 9.8$ m/s^2. Again, substitute the initial speed for v_0, and the height at which the object was propelled for h.
4. Proceed by calculating each term to solve for t.

Example 5 Finding the Parametric Equations to Describe the Motion of a Baseball

Solve the problem presented at the beginning of this section. Does the batter hit the game-winning home run? Assume that the ball is hit with an initial velocity of 140 feet per second at an angle of 45° to the horizontal, making contact 3 feet above the ground.

- **a.** Find the parametric equations to model the path of the baseball.
- **b.** Where is the ball after 2 seconds?
- **c.** How long is the ball in the air?
- **d.** Is it a home run?

Solution

a. Use the formulas to set up the equations. The horizontal position is found using the parametric equation for x. Thus,

$$x = (v_0 \cos \theta)t$$
$$x = (140\cos(45°))t$$

The vertical position is found using the parametric equation for y. Thus,

$$y = -16t^2 + (v_0 \sin \theta)t + h$$
$$y = -16t^2 + (140\sin(45°))t + 3$$

b. Substitute 2 into the equations to find the horizontal and vertical positions of the ball.

$$x = (140\cos(45°))(2)$$
$$x = 198 \text{ feet}$$
$$y = -16(2)^2 + (140\sin(45°))(2) + 3$$
$$y = 137 \text{ feet}$$

After 2 seconds, the ball is 198 feet away from the batter's box and 137 feet above the ground.

c. To calculate how long the ball is in the air, we have to find out when it will hit ground, or when $y = 0$. Thus,

$$
\begin{aligned}
y &= -16t^2 + (140\sin(45°))t + 3 \\
y &= 0 && \text{Set } y(t) = 0 \text{ and solve the quadratic.} \\
t &= 6.2173
\end{aligned}
$$

When $t = 6.2173$ seconds, the ball has hit the ground. (The quadratic equation can be solved in various ways, but this problem was solved using a computer math program.)

d. We cannot confirm that the hit was a home run without considering the size of the outfield, which varies from field to field. However, for simplicity's sake, let's assume that the outfield wall is 400 feet from home plate in the deepest part of the park. Let's also assume that the wall is 10 feet high. In order to determine whether the ball clears the wall, we need to calculate how high the ball is when $x = 400$ feet. So we will set $x = 400$, solve for t, and input t into y.

$$
\begin{aligned}
x &= (140\cos(45°))t \\
400 &= (140\cos(45°))t \\
t &= 4.04 \\
y &= -16(4.04)^2 + (140\sin(45°))(4.04) + 3 \\
y &= 141.8
\end{aligned}
$$

The ball is 141.8 feet in the air when it soars out of the ballpark. It was indeed a home run. See **Figure 7**.

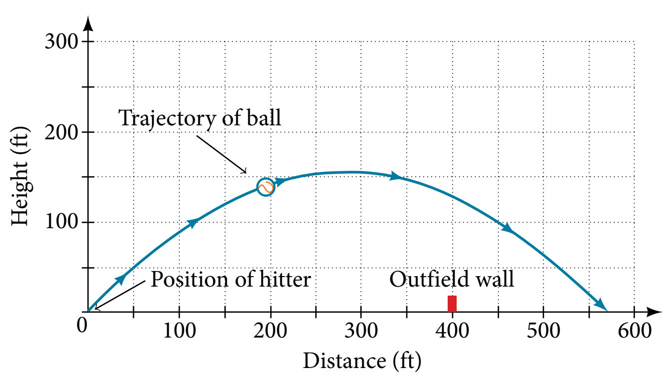


Figure 7

Access the following online resource for additional instruction and practice with graphs of parametric equations.

- Graphing Parametric Equations on the TI-84 (http://openstaxcollege.org/l/graphpara84)

10.7 SECTION EXERCISES

VERBAL

1. What are two methods used to graph parametric equations?
2. What is one difference in point-plotting parametric equations compared to Cartesian equations?
3. Why are some graphs drawn with arrows?
4. Name a few common types of graphs of parametric equations.
5. Why are parametric graphs important in understanding projectile motion?

GRAPHICAL

For the following exercises, graph each set of parametric equations by making a table of values. Include the orientation on the graph.

6. $\begin{cases} x(t) = t \\ y(t) = t^2 - 1 \end{cases}$

t	x	y
-3		
-2		
-1		
0		
1		
2		
3		

7. $\begin{cases} x(t) = t - 1 \\ y(t) = t^2 \end{cases}$

t	-3	-2	-1	0	1	2
x						
y						

8. $\begin{cases} x(t) = 2 + t \\ y(t) = 3 - 2t \end{cases}$

t	-2	-1	0	1	2	3
x						
y						

9. $\begin{cases} x(t) = -2 - 2t \\ y(t) = 3 + t \end{cases}$

t	-3	-2	-1	0	1
x					
y					

10. $\begin{cases} x(t) = t^3 \\ y(t) = t + 2 \end{cases}$

t	-2	-1	0	1	2
x					
y					

11. $\begin{cases} x(t) = t^2 \\ y(t) = t + 3 \end{cases}$

t	-2	-1	0	1	2
x					
y					

For the following exercises, sketch the curve and include the orientation.

12. $\begin{cases} x(t) = t \\ y(t) = \sqrt{t} \end{cases}$
13. $\begin{cases} x(t) = -\sqrt{t} \\ y(t) = t \end{cases}$
14. $\begin{cases} x(t) = 5 - |t| \\ y(t) = t + 2 \end{cases}$
15. $\begin{cases} x(t) = -t + 2 \\ y(t) = 5 - |t| \end{cases}$
16. $\begin{cases} x(t) = 4\sin t \\ y(t) = 2\cos t \end{cases}$
17. $\begin{cases} x(t) = 2\sin t \\ y(t) = 4\cos t \end{cases}$
18. $\begin{cases} x(t) = 3\cos^2 t \\ y(t) = -3\sin t \end{cases}$
19. $\begin{cases} x(t) = 3\cos^2 t \\ y(t) = -3\sin^2 t \end{cases}$
20. $\begin{cases} x(t) = \sec t \\ y(t) = \tan t \end{cases}$
21. $\begin{cases} x(t) = \sec t \\ y(t) = \tan^2 t \end{cases}$
22. $\begin{cases} x(t) = \dfrac{1}{e^{2t}} \\ y(t) = e^{-t} \end{cases}$

For the following exercises, graph the equation and include the orientation. Then, write the Cartesian equation.

23. $\begin{cases} x(t) = t - 1 \\ y(t) = -t^2 \end{cases}$
24. $\begin{cases} x(t) = t^3 \\ y(t) = t + 3 \end{cases}$
25. $\begin{cases} x(t) = 2\cos t \\ y(t) = -\sin t \end{cases}$

26. $\begin{cases} x(t) = 7\cos t \\ y(t) = 7\sin t \end{cases}$

27. $\begin{cases} x(t) = e^{2t} \\ y(t) = -e^{t} \end{cases}$

For the following exercises, graph the equation and include the orientation.

28. $x = t^2, y = 3t, 0 \leq t \leq 5$

29. $x = 2t, y = t^2, -5 \leq t \leq 5$

30. $x = t, y = \sqrt{25 - t^2}, 0 < t \leq 5$

31. $x(t) = -t, y(t) = \sqrt{t}, t \geq 5$

32. $x(t) = -2\cos t, y = 6\sin t \; 0 \leq t \leq \pi$

33. $x(t) = -\sec t, y = \tan t, -\frac{\pi}{2} < t < \frac{\pi}{2}$

For the following exercises, use the parametric equations for integers a and b:

$$x(t) = a\cos((a + b)t) \qquad y(t) = a\cos((a - b)t)$$

34. Graph on the domain $[-\pi, 0]$, where $a = 2$ and $b = 1$, and include the orientation.

35. Graph on the domain $[-\pi, 0]$, where $a = 3$ and $b = 2$, and include the orientation.

36. Graph on the domain $[-\pi, 0]$, where $a = 4$ and $b = 3$, and include the orientation.

37. Graph on the domain $[-\pi, 0]$, where $a = 5$ and $b = 4$, and include the orientation.

38. If a is 1 more than b, describe the effect the values of a and b have on the graph of the parametric equations.

39. Describe the graph if $a = 100$ and $b = 99$.

40. What happens if b is 1 more than a? Describe the graph.

41. If the parametric equations $x(t) = t^2$ and $y(t) = 6 - 3t$ have the graph of a horizontal parabola opening to the right, what would change the direction of the curve?

For the following exercises, describe the graph of the set of parametric equations.

42. $x(t) = -\, t^2$ and $y(t)$ is linear

43. $y(t) = t^2$ and $x(t)$ is linear

44. $y(\text{t}) = -\, t^2$ and $x(t)$ is linear

45. Write the parametric equations of a circle with center (0, 0), radius 5, and a counterclockwise orientation.

46. Write the parametric equations of an ellipse with center (0, 0), major axis of length 10, minor axis of length 6, and a counterclockwise orientation.

For the following exercises, use a graphing utility to graph on the window $[-3, 3]$ by $[-3, 3]$ on the domain $[0, 2\pi)$ for the following values of a and b , and include the orientation.

$$\begin{cases} x(t) = \sin(at) \\ y(t) = \sin(bt) \end{cases}$$

47. $a = 1, b = 2$

48. $a = 2, b = 1$

49. $a = 3, b = 3$

50. $a = 5, b = 5$

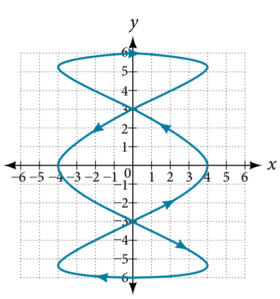
51. $a = 2, b = 5$

52. $a = 5, b = 2$

TECHNOLOGY

For the following exercises, look at the graphs that were created by parametric equations of the form $\begin{cases} x(t) = a\cos(bt) \\ y(t) = c\sin(dt) \end{cases}$ Use the parametric mode on the graphing calculator to find the values of a, b, c, and d to achieve each graph.

53.



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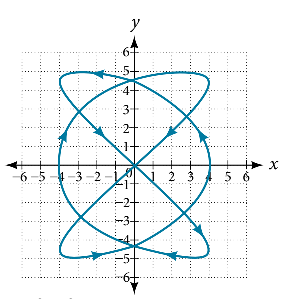

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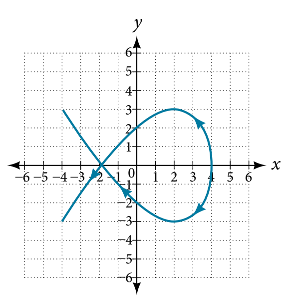
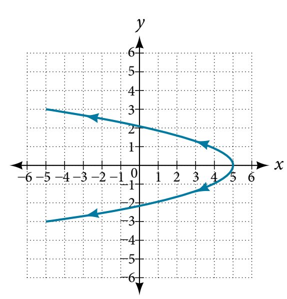


56.



For the following exercises, use a graphing utility to graph the given parametric equations.

a. $\begin{cases} x(t) = \cos t - 1 \\ y(t) = \sin t + t \end{cases}$ **b.** $\begin{cases} x(t) = \cos t + t \\ y(t) = \sin t - 1 \end{cases}$ **c.** $\begin{cases} x(t) = t - \sin t \\ y(t) = \cos t - 1 \end{cases}$

57. Graph all three sets of parametric equations on the domain $[0, 2\pi]$.

58. Graph all three sets of parametric equations on the domain $[0, 4\pi]$.

59. Graph all three sets of parametric equations on the domain $[-4\pi, 6\pi]$.

60. The graph of each set of parametric equations appears to "creep" along one of the axes. What controls which axis the graph creeps along?

61. Explain the effect on the graph of the parametric equation when we switched $\sin t$ and $\cos t$.

62. Explain the effect on the graph of the parametric equation when we changed the domain.

EXTENSIONS

63. An object is thrown in the air with vertical velocity of 20 ft/s and horizontal velocity of 15 ft/s. The object's height can be described by the equation $y(t) = -16t^2 + 20t$, while the object moves horizontally with constant velocity 15 ft/s. Write parametric equations for the object's position, and then eliminate time to write height as a function of horizontal position.

64. A skateboarder riding on a level surface at a constant speed of 9 ft/s throws a ball in the air, the height of which can be described by the equation $y(t) = -16t^2 + 10t + 5$. Write parametric equations for the ball's position, and then eliminate time to write height as a function of horizontal position.

For the following exercises, use this scenario: A dart is thrown upward with an initial velocity of 65 ft/s at an angle of elevation of 52°. Consider the position of the dart at any time t. Neglect air resistance.

65. Find parametric equations that model the problem situation.

66. Find all possible values of x that represent the situation.

67. When will the dart hit the ground?

68. Find the maximum height of the dart.

69. At what time will the dart reach maximum height?

For the following exercises, look at the graphs of each of the four parametric equations. Although they look unusual and beautiful, they are so common that they have names, as indicated in each exercise. Use a graphing utility to graph each on the indicated domain.

70. An epicycloid: $\begin{cases} x(t) = 14\cos t - \cos(14t) \\ y(t) = 14\sin t + \sin(14t) \end{cases}$ on the domain $[0, 2\pi]$.

71. An hypocycloid: $\begin{cases} x(t) = 6\sin t + 2\sin(6t) \\ y(t) = 6\cos t - 2\cos(6t) \end{cases}$ on the domain $[0, 2\pi]$.

72. An hypotrochoid: $\begin{cases} x(t) = 2\sin t + 5\cos(6t) \\ y(t) = 5\cos t - 2\sin(6t) \end{cases}$ on the domain $[0, 2\pi]$.

73. A rose: $\begin{cases} x(t) = 5\sin(2t) \sin t \\ y(t) = 5\sin(2t) \cos t \end{cases}$ on the domain $[0, 2\pi]$.

LEARNING OBJECTIVES

In this section, you will:

- View vectors geometrically.
- Find magnitude and direction.
- Perform vector addition and scalar multiplication.
- Find the component form of a vector.
- Find the unit vector in the direction of *v*.
- Perform operations with vectors in terms of *i* and *j*.
- Find the dot product of two vectors.

10.8 VECTORS

An airplane is flying at an airspeed of 200 miles per hour headed on a SE bearing of 140°. A north wind (from north to south) is blowing at 16.2 miles per hour, as shown in **Figure 1**. What are the ground speed and actual bearing of the plane?

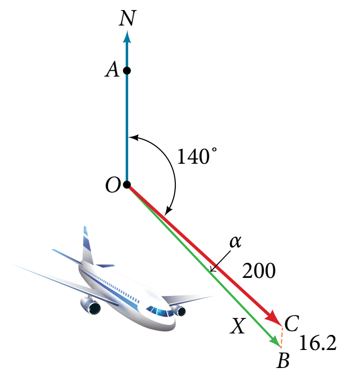


Figure 1

Ground speed refers to the speed of a plane relative to the ground. Airspeed refers to the speed a plane can travel relative to its surrounding air mass. These two quantities are not the same because of the effect of wind. In an earlier section, we used triangles to solve a similar problem involving the movement of boats. Later in this section, we will find the airplane's groundspeed and bearing, while investigating another approach to problems of this type. First, however, let's examine the basics of vectors.

A Geometric View of Vectors

A **vector** is a specific quantity drawn as a line segment with an arrowhead at one end. It has an **initial point**, where it begins, and a **terminal point**, where it ends. A vector is defined by its **magnitude**, or the length of the line, and its direction, indicated by an arrowhead at the terminal point. Thus, a vector is a directed line segment. There are various symbols that distinguish vectors from other quantities:

- Lower case, boldfaced type, with or without an arrow on top such as $\boldsymbol{v}$, $\boldsymbol{u}$, $\boldsymbol{w}$, $\overrightarrow{\boldsymbol{v}}$, $\overrightarrow{\boldsymbol{u}}$, $\overrightarrow{\boldsymbol{w}}$.
- Given initial point P and terminal point Q, a vector can be represented as $\overrightarrow{PQ}$. The arrowhead on top is what indicates that it is not just a line, but a directed line segment.
- Given an initial point of $(0, 0)$ and terminal point (a, b), a vector may be represented as $\langle a, b\rangle$.

This last symbol $\langle a, b\rangle$ has special significance. It is called the **standard position**. The position vector has an initial point $(0, 0)$ and a terminal point $\langle a, b\rangle$. To change any vector into the position vector, we think about the change in the x-coordinates and the change in the y-coordinates. Thus, if the initial point of a vector $\overrightarrow{CD}$ is $C(x_1, y_1)$ and the terminal point is $D(x_2, y_2)$, then the position vector is found by calculating

$$\begin{aligned}\overrightarrow{AB} &= \langle x_2 - x_1, y_2 - y_1\rangle \\ &= \langle a, b\rangle\end{aligned}$$

In **Figure 2**, we see the original vector $\overrightarrow{CD}$ and the position vector $\overrightarrow{AB}$.

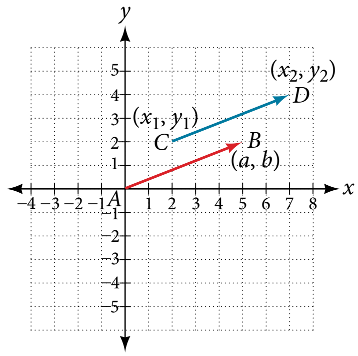


Figure 2

> ***properties of vectors***
> A vector is a directed line segment with an initial point and a terminal point. Vectors are identified by magnitude, or the length of the line, and direction, represented by the arrowhead pointing toward the terminal point. The position vector has an initial point at (0, 0) and is identified by its terminal point $\langle a, b\rangle$.

Example 1 Find the Position Vector

Consider the vector whose initial point is $P(2, 3)$ and terminal point is $Q(6, 4)$. Find the position vector.

Solution The position vector is found by subtracting one x-coordinate from the other x-coordinate, and one y-coordinate from the other y-coordinate. Thus

$$\begin{aligned} v &= \langle 6-2, 4-3\rangle \\ &= \langle 4, 1\rangle \end{aligned}$$

The position vector begins at (0, 0) and terminates at (4, 1). The graphs of both vectors are shown in **Figure 3**.

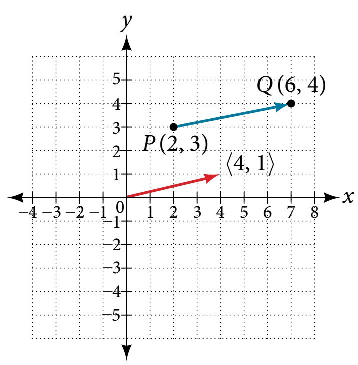


Figure 3

We see that the position vector is $\langle 4, 1\rangle$.

Example 2 Drawing a Vector with the Given Criteria and Its Equivalent Position Vector

Find the position vector given that vector v has an initial point at $(-3, 2)$ and a terminal point at (4, 5), then graph both vectors in the same plane.

Solution The position vector is found using the following calculation:

$$\begin{aligned} v &= \langle 4-(-3), 5-2\rangle \\ &= \langle 7, 3\rangle \end{aligned}$$

Thus, the position vector begins at (0, 0) and terminates at (7, 3). See **Figure 4**.

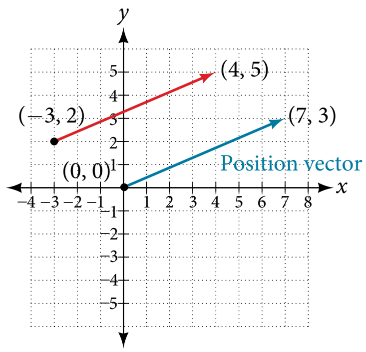


Figure 4

Try It #1

Draw a vector v that connects from the origin to the point (3, 5).

Finding Magnitude and Direction

To work with a vector, we need to be able to find its magnitude and its direction. We find its magnitude using the Pythagorean Theorem or the distance formula, and we find its direction using the inverse tangent function.

magnitude and direction of a vector

Given a position vector $v = \langle a, b \rangle$, the magnitude is found by $|v| = \sqrt{a^2 + b^2}$. The direction is equal to the angle formed with the x-axis, or with the y-axis, depending on the application. For a position vector, the direction is found by $\tan\theta = \left(\frac{b}{a}\right) \Rightarrow \theta = \tan^{-1}\left(\frac{b}{a}\right)$, as illustrated in **Figure 5.**

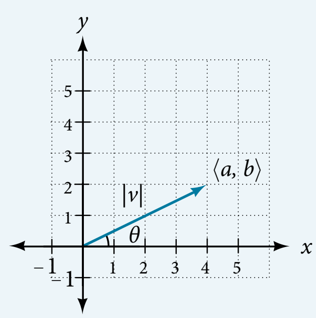


Figure 5

Two vectors v and u are considered equal if they have the same magnitude and the same direction. Additionally, if both vectors have the same position vector, they are equal.

Example 3 Finding the Magnitude and Direction of a Vector

Find the magnitude and direction of the vector with initial point $P(-8, 1)$ and terminal point $Q(-2, -5)$. Draw the vector.

Solution First, find the position vector.

$$\begin{aligned} u &= \langle -2, -(-8), -5-1 \rangle \\ &= \langle 6, -6 \rangle \end{aligned}$$

We use the Pythagorean Theorem to find the magnitude.

$$\begin{aligned} |u| &= \sqrt{(6)^2 + (-6)^2} \\ &= \sqrt{72} \\ &= 6\sqrt{2} \end{aligned}$$

The direction is given as

$$\tan\theta = \frac{-6}{6} = -1 \Rightarrow \theta = \tan^{-1}(-1)$$
$$= -45°$$

However, the angle terminates in the fourth quadrant, so we add 360° to obtain a positive angle. Thus, $-45° + 360° = 315°$. See **Figure 6**.

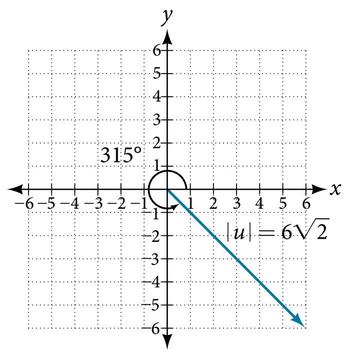


Figure 6

Example 4 Showing That Two Vectors Are Equal

Show that vector v with initial point at $(5, -3)$ and terminal point at $(-1, 2)$ is equal to vector u with initial point at $(-1, -3)$ and terminal point at $(-7, 2)$. Draw the position vector on the same grid as v and u. Next, find the magnitude and direction of each vector.

Solution As shown in **Figure 7**, draw the vector v starting at initial $(5, -3)$ and terminal point $(-1, 2)$. Draw the vector u with initial point $(-1, -3)$ and terminal point $(-7, 2)$. Find the standard position for each.

Next, find and sketch the position vector for v and u. We have

$$\begin{aligned} v &= \langle -1 - 5, 2 - (-3) \rangle \\ &= \langle -6, 5 \rangle \\ u &= \langle -7 - (-1), 2 - (-3) \rangle \\ &= \langle -6, 5 \rangle \end{aligned}$$

Since the position vectors are the same, v and u are the same.

An alternative way to check for vector equality is to show that the magnitude and direction are the same for both vectors. To show that the magnitudes are equal, use the Pythagorean Theorem.

$$\begin{aligned} |v| &= \sqrt{(-1-5)^2 + (2-(-3))^2} \\ &= \sqrt{(-6)^2 + (5)^2} \\ &= \sqrt{36 + 25} \\ &= \sqrt{61} \\ |u| &= \sqrt{(-7-(-1))^2 + (2-(-3))^2} \\ &= \sqrt{(-6)^2 + (5)^2} \\ &= \sqrt{36 + 25} \\ &= \sqrt{61} \end{aligned}$$

As the magnitudes are equal, we now need to verify the direction. Using the tangent function with the position vector gives

$$\tan\theta = -\frac{5}{6} \Rightarrow \theta = \tan^{-1}\left(-\frac{5}{6}\right)$$
$$= -39.8°$$

However, we can see that the position vector terminates in the second quadrant, so we add 180°. Thus, the direction is $-39.8° + 180° = 140.2°$.

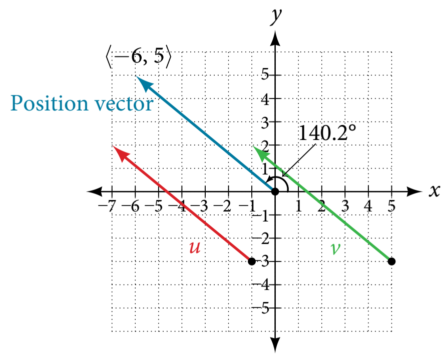


Figure 7

Performing Vector Addition and Scalar Multiplication

Now that we understand the properties of vectors, we can perform operations involving them. While it is convenient to think of the vector $u = \langle x, y \rangle$ as an arrow or directed line segment from the origin to the point (x, y), vectors can be situated anywhere in the plane. The sum of two vectors u and v, or **vector addition**, produces a third vector $u + v$, the **resultant** vector.

To find $u + v$, we first draw the vector u, and from the terminal end of u, we drawn the vector v. In other words, we have the initial point of v meet the terminal end of u. This position corresponds to the notion that we move along the first vector and then, from its terminal point, we move along the second vector. The sum $u + v$ is the resultant vector because it results from addition or subtraction of two vectors. The resultant vector travels directly from the beginning of u to the end of v in a straight path, as shown in **Figure 8**.

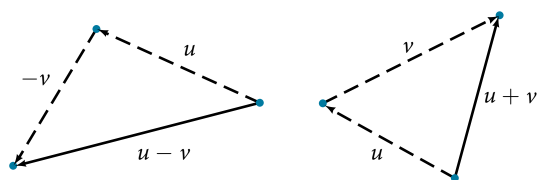


Figure 8

Vector subtraction is similar to vector addition. To find $u - v$, view it as $u + (-v)$. Adding $-v$ is reversing direction of v and adding it to the end of u. The new vector begins at the start of u and stops at the end point of $-v$. See **Figure 9** for a visual that compares vector addition and vector subtraction using parallelograms.

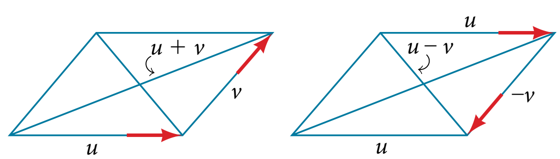


Figure 9

Example 5 Adding and Subtracting Vectors

Given $u = \langle 3, -2 \rangle$ and $v = \langle -1, 4 \rangle$, find two new vectors $u + v$, and $u - v$.

Solution To find the sum of two vectors, we add the components. Thus,

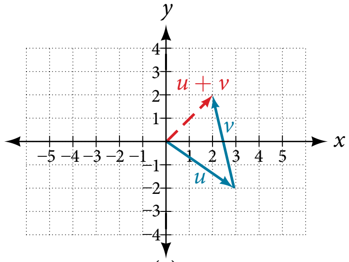
$$\begin{aligned} u + v &= \langle 3, -2 \rangle + \langle -1, 4 \rangle \\ &= \langle 3 + (-1), -2 + 4 \rangle \\ &= \langle 2, 2 \rangle \end{aligned}$$

See **Figure 10(a)**.

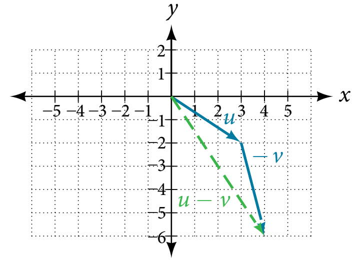
To find the difference of two vectors, add the negative components of v to u. Thus,

$$\begin{aligned} u + (-v) &= \langle 3, -2 \rangle + \langle 1, -4 \rangle \\ &= \langle 3 + 1, -2 + (-4) \rangle \\ &= \langle 4, -6 \rangle \end{aligned}$$

See **Figure 10(b)**.



(a)



(b)

Figure 10 (a) Sum of two vectors (b) Difference of two vectors

Multiplying By a Scalar

While adding and subtracting vectors gives us a new vector with a different magnitude and direction, the process of multiplying a vector by a **scalar**, a constant, changes only the magnitude of the vector or the length of the line. Scalar multiplication has no effect on the direction unless the scalar is negative, in which case the direction of the resulting vector is opposite the direction of the original vector.

> ***scalar multiplication***
> **Scalar multiplication** involves the product of a vector and a scalar. Each component of the vector is multiplied by the scalar. Thus, to multiply $v = \langle a, b\rangle$ by k, we have
>
> $$kv = \langle ka, kb\rangle$$
>
> Only the magnitude changes, unless k is negative, and then the vector reverses direction.

Example 6 Performing Scalar Multiplication

Given vector $v = \langle 3, 1\rangle$, find $3v$, $\frac{1}{2}v$, and $-v$.

Solution See **Figure 11** for a geometric interpretation. If $v = \langle 3, 1\rangle$, then

$$\begin{aligned} 3v &= \langle 3 \cdot 3, 3 \cdot 1\rangle \\ &= \langle 9, 3\rangle \\ \frac{1}{2}v &= \left\langle \frac{1}{2} \cdot 3, \frac{1}{2} \cdot 1\right\rangle \\ &= \left\langle \frac{3}{2}, \frac{1}{2}\right\rangle \\ -v &= \langle -3, -1\rangle \end{aligned}$$

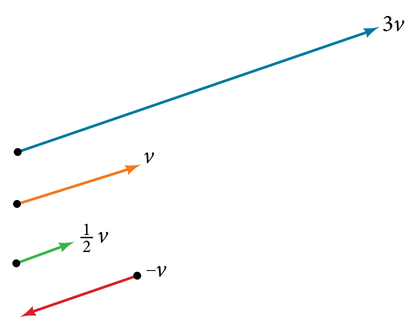


Figure 11

Analysis *Notice that the vector $3v$ is three times the length of v, $\frac{1}{2}v$ is half the length of v, and $-v$ is the same length of v, but in the opposite direction.*

Try It #2

Find the scalar multiple $3u$ given $u = \langle 5, 4 \rangle$.

Example 7 Using Vector Addition and Scalar Multiplication to Find a New Vector

Given $u = \langle 3, -2 \rangle$ and $v = \langle -1, 4 \rangle$, find a new vector $w = 3u + 2v$.

Solution First, we must multiply each vector by the scalar.

$$\begin{aligned} 3u &= 3\,\langle 3, -2 \rangle \\ &= \langle 9, -6 \rangle \\ 2v &= 2\,\langle -1, 4 \rangle \\ &= \langle -2, 8 \rangle \end{aligned}$$

Then, add the two together.

$$\begin{aligned} w &= 3u + 2v \\ &= \langle 9, -6 \rangle + \langle -2, 8 \rangle \\ &= \langle 9 - 2, -6 + 8 \rangle \\ &= \langle 7, 2 \rangle \end{aligned}$$

So, $w = \langle 7, 2 \rangle$.

Finding Component Form

In some applications involving vectors, it is helpful for us to be able to break a vector down into its components. Vectors are comprised of two components: the horizontal component is the x direction, and the vertical component is the y direction. For example, we can see in the graph in **Figure 12** that the position vector $\langle 2, 3 \rangle$ comes from adding the vectors v_1 and v_2. We have v_1 with initial point $(0, 0)$ and terminal point $(2, 0)$.

$$\begin{aligned} v_1 &= \langle 2 - 0, 0 - 0 \rangle \\ &= \langle 2, 0 \rangle \end{aligned}$$

We also have v_2 with initial point $(0, 0)$ and terminal point $(0, 3)$.

$$\begin{aligned} v_2 &= \langle 0 - 0, 3 - 0 \rangle \\ &= \langle 0, 3 \rangle \end{aligned}$$

Therefore, the position vector is

$$\begin{aligned} v &= \langle 2 + 0, 3 + 0 \rangle \\ &= \langle 2, 3 \rangle \end{aligned}$$

Using the Pythagorean Theorem, the magnitude of v_1 is 2, and the magnitude of v_2 is 3. To find the magnitude of v, use the formula with the position vector.

$$\begin{aligned} |v| &= \sqrt{|v_1|^2 + |v_2|^2} \\ &= \sqrt{2^2 + 3^2} \\ &= \sqrt{13} \end{aligned}$$

The magnitude of v is $\sqrt{13}$. To find the direction, we use the tangent function $\tan\theta = \frac{y}{x}$.

$$\tan\theta = \frac{v_2}{v_1}$$

$$\tan\theta = \frac{3}{2}$$

$$\theta = \tan^{-1}\left(\frac{3}{2}\right) = 56.3°$$

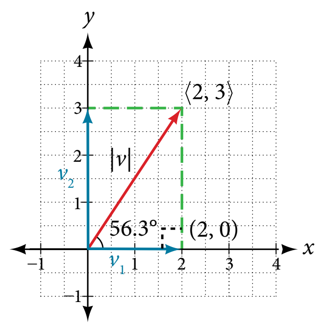


Figure 12

Thus, the magnitude of v is $\sqrt{13}$ and the direction is 56.3° off the horizontal.

Example 8 Finding the Components of the Vector

Find the components of the vector v with initial point (3, 2) and terminal point (7, 4).

Solution First find the standard position.

$$\begin{aligned} v &= \langle 7-3, 4-2 \rangle \\ &= \langle 4, 2 \rangle \end{aligned}$$

See the illustration in **Figure 13**.

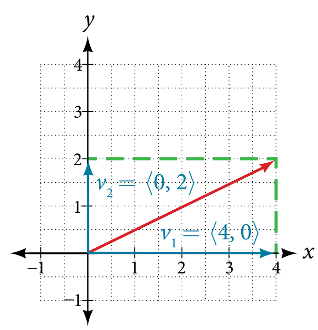


Figure 13

The horizontal component is $v_1 = \langle 4, 0 \rangle$ and the vertical component is $v_2 = \langle 0, 2 \rangle$.

Finding the Unit Vector in the Direction of *v*

In addition to finding a vector's components, it is also useful in solving problems to find a vector in the same direction as the given vector, but of magnitude 1. We call a vector with a magnitude of 1 a **unit vector**. We can then preserve the direction of the original vector while simplifying calculations.

Unit vectors are defined in terms of components. The horizontal unit vector is written as $i = \langle 1, 0 \rangle$ and is directed along the positive horizontal axis. The vertical unit vector is written as $j = \langle 0, 1 \rangle$ and is directed along the positive vertical axis. See **Figure 14**.

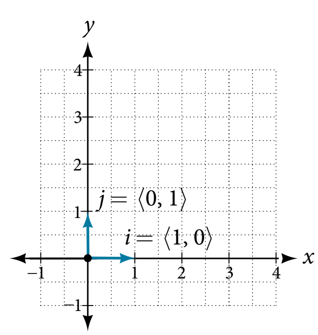


Figure 14

> ***the unit vectors***
>
> If v is a nonzero vector, then $\frac{v}{|v|}$ is a unit vector in the direction of v. Any vector divided by its magnitude is a unit vector. Notice that magnitude is always a scalar, and dividing by a scalar is the same as multiplying by the reciprocal of the scalar.

Example 9 **Finding the Unit Vector in the Direction of *v***

Find a unit vector in the same direction as $v = \langle -5, 12\rangle$.

Solution First, we will find the magnitude.

$$\begin{aligned} |v| &= \sqrt{(-5)^2 + (12)^2} \\ &= \sqrt{25 + 144} \\ &= \sqrt{169} \\ &= 13 \end{aligned}$$

Then we divide each component by $|v|$, which gives a unit vector in the same direction as v:

$$\frac{v}{|v|} = -\frac{5}{13}i + \frac{12}{13}j$$

or, in component form

$$\frac{v}{|v|} = \left\langle -\frac{5}{13}, \frac{12}{13} \right\rangle$$

See **Figure 15**.

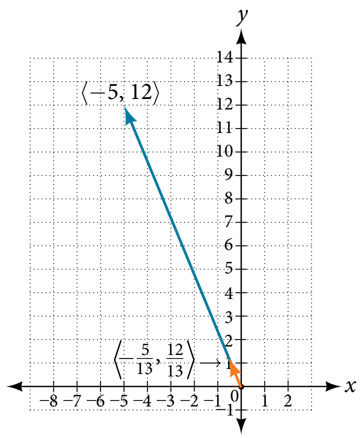


Figure 15

Verify that the magnitude of the unit vector equals 1. The magnitude of $-\frac{5}{13}i + \frac{12}{13}j$ is given as

$$\begin{aligned} \sqrt{\left(-\frac{5}{13}\right)^2 + \left(\frac{12}{13}\right)^2} &= \sqrt{\frac{25}{169} + \frac{144}{169}} \\ &= \sqrt{\frac{169}{169}} \\ &= 1 \end{aligned}$$

The vector $u = \frac{5}{13}i + \frac{12}{13}j$ is the unit vector in the same direction as $v = \langle -5, 12\rangle$.

Performing Operations with Vectors in Terms of *i* and *j*

So far, we have investigated the basics of vectors: magnitude and direction, vector addition and subtraction, scalar multiplication, the components of vectors, and the representation of vectors geometrically. Now that we are familiar with the general strategies used in working with vectors, we will represent vectors in rectangular coordinates in terms of *i* and *j*.

vectors in the rectangular plane

Given a vector v with initial point $P = (x_1, y_1)$ and terminal point $Q = (x_2, y_2)$, v is written as

$$v = (x_2 - x_1)i + (y_1 - y_2)j$$

The position vector from (0, 0) to (a, b), where $(x_2 - x_1) = a$ and $(y_2 - y_1) = b$, is written as $v = ai + bj$. This vector sum is called a linear combination of the vectors i and j.

The magnitude of $v = ai + bj$ is given as $|v| = \sqrt{a^2 + b^2}$. See **Figure 16**.

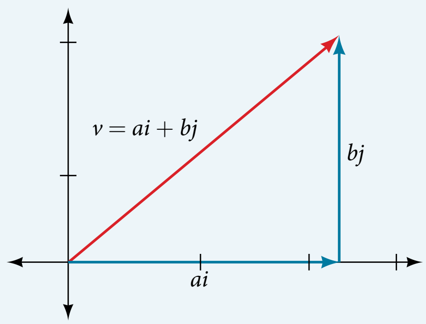


Figure 16

Example 10 Writing a Vector in Terms of *i* and *j*

Given a vector v with initial point $P = (2, -6)$ and terminal point $Q = (-6, 6)$, write the vector in terms of i and j.

Solution Begin by writing the general form of the vector. Then replace the coordinates with the given values.

$$\begin{aligned} v &= (x_2 - x_1)i + (y_2 - y_1)j \\ &= (-6 - 2)i + (6 - (-6))j \\ &= -8i + 12j \end{aligned}$$

Example 11 Writing a Vector in Terms of *i* and *j* Using Initial and Terminal Points

Given initial point $P_1 = (-1, 3)$ and terminal point $P_2 = (2, 7)$, write the vector v in terms of i and j.

Solution Begin by writing the general form of the vector. Then replace the coordinates with the given values.

$$\begin{aligned} v &= (x_2 - x_1)i + (y_2 - y_1)j \\ v &= (2 - (-1))i + (7 - 3)j \\ &= 3i + 4j \end{aligned}$$

Try It #3

Write the vector u with initial point $P = (-1, 6)$ and terminal point $Q = (7, -5)$ in terms of i and j.

Performing Operations on Vectors in Terms of *i* and *j*

When vectors are written in terms of i and j, we can carry out addition, subtraction, and scalar multiplication by performing operations on corresponding components.

adding and subtracting vectors in rectangular coordinates

Given $v = ai + bj$ and $u = ci + dj$, then

$$\begin{aligned} v + u &= (a + c)i + (b + d)j \\ v - u &= (a - c)i + (b - d)j \end{aligned}$$

Example 12 Finding the Sum of the Vectors

Find the sum of $v_1 = 2i - 3j$ and $v_2 = 4i + 5j$.

Solution According to the formula, we have

$$\begin{aligned} v_1 + v_2 &= (2 + 4)i + (-3 + 5)j \\ &= 6i + 2j \end{aligned}$$

Calculating the Component Form of a Vector: Direction

We have seen how to draw vectors according to their initial and terminal points and how to find the position vector. We have also examined notation for vectors drawn specifically in the Cartesian coordinate plane using i and j. For any of these vectors, we can calculate the magnitude. Now, we want to combine the key points, and look further at the ideas of magnitude and direction.

Calculating direction follows the same straightforward process we used for polar coordinates. We find the direction of the vector by finding the angle to the horizontal. We do this by using the basic trigonometric identities, but with $|v|$ replacing r.

vector components in terms of magnitude and direction

Given a position vector $v = \langle x, y\rangle$ and a direction angle θ,

$$\cos\theta = \frac{x}{|v|} \quad \text{and} \quad \sin\theta = \frac{y}{|v|}$$

$$x = |v|\cos\theta \qquad y = |v|\sin\theta$$

Thus, $v = xi + yj = |v|\cos\theta i + |v|\sin\theta j$, and magnitude is expressed as $|v| = \sqrt{x^2 + y^2}$.

Example 13 Writing a Vector in Terms of Magnitude and Direction

Write a vector with length 7 at an angle of 135° to the positive x-axis in terms of magnitude and direction.

Solution Using the conversion formulas $x = |v|\cos\theta i$ and $y = |v|\sin\theta j$, we find that

$$\begin{aligned} x &= 7\cos(135°)i \\ &= -\frac{7\sqrt{2}}{2} \\ y &= 7\sin(135°)j \\ &= \frac{7\sqrt{2}}{2} \end{aligned}$$

This vector can be written as $v = 7\cos(135°)i + 7\sin(135°)j$ or simplified as

$$v = -\frac{7\sqrt{2}}{2}i + \frac{7\sqrt{2}}{2}j$$

Try It #4

A vector travels from the origin to the point (3, 5). Write the vector in terms of magnitude and direction.

Finding the Dot Product of Two Vectors

As we discussed earlier in the section, scalar multiplication involves multiplying a vector by a scalar, and the result is a vector. As we have seen, multiplying a vector by a number is called scalar multiplication. If we multiply a vector by a vector, there are two possibilities: the *dot product* and the *cross product*. We will only examine the dot product here; you may encounter the cross product in more advanced mathematics courses.

The dot product of two vectors involves multiplying two vectors together, and the result is a scalar.

dot product

The **dot product** of two vectors $v = \langle a, b\rangle$ and $u = \langle c, d\rangle$ is the sum of the product of the horizontal components and the product of the vertical components.

$$v \cdot u = ac + bd$$

To find the angle between the two vectors, use the formula below.

$$\cos\theta = \frac{v}{|v|} \cdot \frac{u}{|u|}$$

Example 14 Finding the Dot Product of Two Vectors

Find the dot product of $v = \langle 5, 12\rangle$ and $u = \langle -3, 4\rangle$.

Solution Using the formula, we have

$$\begin{aligned} v \cdot u &= \langle 5, 12\rangle \cdot \langle -3, 4\rangle \\ &= 5 \cdot (-3) + 12 \cdot 4 \\ &= -15 + 48 \\ &= 33 \end{aligned}$$

Example 15 Finding the Dot Product of Two Vectors and the Angle between Them

Find the dot product of $v_1 = 5i + 2j$ and $v_2 = 3i + 7j$. Then, find the angle between the two vectors.

Solution Finding the dot product, we multiply corresponding components.

$$\begin{aligned} v_1 \cdot v_2 &= \langle 5, 2\rangle \cdot \langle 3, 7\rangle \\ &= 5 \cdot 3 + 2 \cdot 7 \\ &= 15 + 14 \\ &= 29 \end{aligned}$$

To find the angle between them, we use the formula $\cos\theta = \frac{v}{|v|} \cdot \frac{u}{|u|}$

$$\begin{aligned} \frac{v}{|v|} \cdot \frac{u}{|u|} &= \left\langle \frac{5}{\sqrt{29}} + \frac{2}{\sqrt{29}} \right\rangle \cdot \left\langle \frac{3}{\sqrt{58}} + \frac{7}{\sqrt{58}} \right\rangle \\ &= \frac{5}{\sqrt{29}} \cdot \frac{3}{\sqrt{58}} + \frac{2}{\sqrt{29}} \cdot \frac{7}{\sqrt{58}} \\ &= \frac{15}{\sqrt{1682}} + \frac{14}{\sqrt{1682}} = \frac{29}{\sqrt{1682}} \\ &= 0.707107 \end{aligned}$$

$$\cos^{-1}(0.707107) = 45°$$

See **Figure 17**.

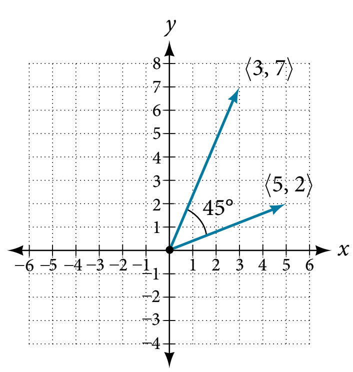


Figure 17

Example 16 Finding the Angle between Two Vectors

Find the angle between $u = \langle -3, 4 \rangle$ and $v = \langle 5, 12 \rangle$.

Solution Using the formula, we have

$$\theta = \cos^{-1}\left(\frac{u}{|u|} \cdot \frac{v}{|v|}\right)$$

$$\begin{aligned}\left(\frac{u}{|u|} \cdot \frac{v}{|v|}\right) &= \frac{-3i + 4j}{5} \cdot \frac{5i + 12j}{13} \\ &= \left(-\frac{3}{5} \cdot \frac{5}{13}\right) + \left(\frac{4}{5} \cdot \frac{12}{13}\right) \\ &= -\frac{15}{65} + \frac{48}{65} \\ &= \frac{33}{65} \\ \theta &= \cos^{-1}\left(\frac{33}{65}\right) \\ &= 59.5°\end{aligned}$$

See **Figure 18**.

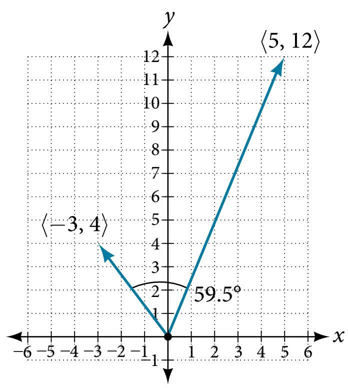


Figure 18

Example 17 Finding Ground Speed and Bearing Using Vectors

We now have the tools to solve the problem we introduced in the opening of the section.

An airplane is flying at an airspeed of 200 miles per hour headed on a SE bearing of 140°. A north wind (from north to south) is blowing at 16.2 miles per hour. What are the ground speed and actual bearing of the plane? See **Figure 19**.

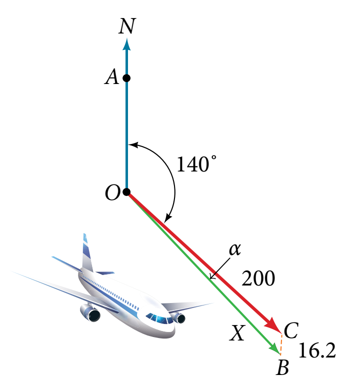


Figure 19

Solution The ground speed is represented by x in the diagram, and we need to find the angle α in order to calculate the adjusted bearing, which will be $140° + \alpha$.

Notice in **Figure 19**, that angle *BCO* must be equal to angle *AOC* by the rule of alternating interior angles, so angle *BCO* is 140°. We can find x by the Law of Cosines:

$$\begin{aligned} x^2 &= (16.2)^2 + (200)^2 - 2(16.2)(200)\cos(140°) \\ x^2 &= 45{,}226.41 \\ x &= \sqrt{45,226.41} \\ x &= 212.7 \end{aligned}$$

The ground speed is approximately 213 miles per hour. Now we can calculate the bearing using the Law of Sines.

$$\begin{aligned} \frac{\sin\alpha}{16.2} &= \frac{\sin(140°)}{212.7} \\ \sin\alpha &= \frac{16.2\sin(140°)}{212.7} \\ &= 0.04896 \\ \sin^{-1}(0.04896) &= 2.8° \end{aligned}$$

Therefore, the plane has a SE bearing of $140° + 2.8° = 142.8°$. The ground speed is 212.7 miles per hour.

Access these online resources for additional instruction and practice with vectors.

- Introduction to Vectors (http://openstaxcollege.org/l/introvectors)
- Vector Operations (http://openstaxcollege.org/l/vectoroperation)
- The Unit Vector (http://openstaxcollege.org/l/unitvector)

10.8 SECTION EXERCISES

VERBAL

1. What are the characteristics of the letters that are commonly used to represent vectors?

2. How is a vector more specific than a line segment?

3. What are i and j, and what do they represent?

4. What is component form?

5. When a unit vector is expressed as $\langle a, b\rangle$, which letter is the coefficient of the i and which the j?

ALGEBRAIC

6. Given a vector with initial point $(5, 2)$ and terminal point $(-1, -3)$, find an equivalent vector whose initial point is $(0, 0)$. Write the vector in component form $\langle a, b\rangle$.

7. Given a vector with initial point $(-4, 2)$ and terminal point $(3, -3)$, find an equivalent vector whose initial point is $(0, 0)$. Write the vector in component form $\langle a, b\rangle$.

8. Given a vector with initial point $(7, -1)$ and terminal point $(-1, -7)$, find an equivalent vector whose initial point is $(0, 0)$. Write the vector in component form $\langle a, b\rangle$.

For the following exercises, determine whether the two vectors u and v are equal, where u has an initial point P_1 and a terminal point P_2 and v has an initial point P_3 and a terminal point P_4.

9. $P_1 = (5, 1)$, $P_2 = (3, -2)$, $P_3 = (-1, 3)$, and $P_4 = (9, -4)$

10. $P_1 = (2, -3)$, $P_2 = (5, 1)$, $P_3 = (6, -1)$, and $P_4 = (9, 3)$

11. $P_1 = (-1, -1)$, $P_2 = (-4, 5)$, $P_3 = (-10, 6)$, and $P_4 = (-13, 12)$

12. $P_1 = (3, 7)$, $P_2 = (2, 1)$, $P_3 = (1, 2)$, and $P_4 = (-1, -4)$

13. $P_1 = (8, 3)$, $P_2 = (6, 5)$, $P_3 = (11, 8)$, and $P_4 = (9, 10)$

14. Given initial point $P_1 = (-3, 1)$ and terminal point $P_2 = (5, 2)$, write the vector v in terms of i and j.

15. Given initial point $P_1 = (6, 0)$ and terminal point $P_2 = (-1, -3)$, write the vector v in terms of i and j.

For the following exercises, use the vectors $u = i + 5j$, $v = -2i - 3j$, and $w = 4i - j$.

16. Find $u + (v - w)$

17. Find $4v + 2u$

For the following exercises, use the given vectors to compute $u + v$, $u - v$, and $2u - 3v$.

18. $u = \langle 2, -3\rangle$, $v = \langle 1, 5\rangle$

19. $u = \langle -3, 4\rangle$, $v = \langle -2, 1\rangle$

20. Let $v = -4i + 3j$. Find a vector that is half the length and points in the same direction as v.

21. Let $v = 5i + 2j$. Find a vector that is twice the length and points in the opposite direction as v.

For the following exercises, find a unit vector in the same direction as the given vector.

22. $a = 3i + 4j$

23. $b = -2i + 5j$

24. $c = 10i - j$

25. $d = -\frac{1}{3}i + \frac{5}{2}j$

26. $u = 100i + 200j$

27. $u = -14i + 2j$

For the following exercises, find the magnitude and direction of the vector, $0 \le \theta < 2\pi$.

28. $\langle 0, 4\rangle$

29. $\langle 6, 5\rangle$

30. $\langle 2, -5\rangle$

31. $\langle -4, -6\rangle$

32. Given $u = 3i - 4j$ and $v = -2i + 3j$, calculate $u \cdot v$.

33. Given $u = -i - j$ and $v = i + 5j$, calculate $u \cdot v$.

34. Given $u = \langle -2, 4\rangle$ and $v = \langle -3, 1\rangle$, calculate $u \cdot v$.

35. Given $u = \langle -1, 6\rangle$ and $v = \langle 6, -1\rangle$, calculate $u \cdot v$.

GRAPHICAL

For the following exercises, given v, draw v, $3v$ and $\frac{1}{2}v$.

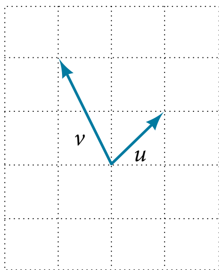
36. $\langle 2, -1 \rangle$

37. $\langle -1, 4 \rangle$

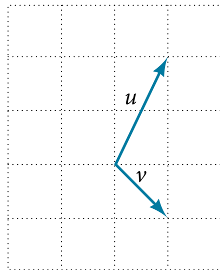
38. $\langle -3, -2 \rangle$

For the following exercises, use the vectors shown to sketch $u + v$, $u - v$, and $2u$.

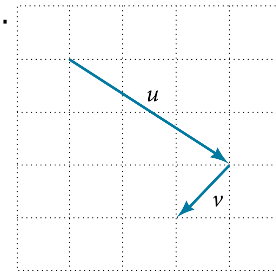
39.



40.

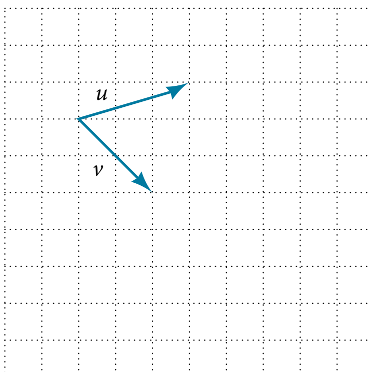


41.

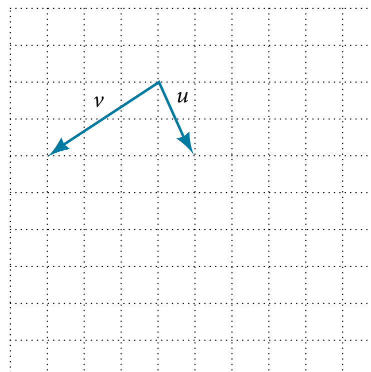


For the following exercises, use the vectors shown to sketch $2u + v$.

42.

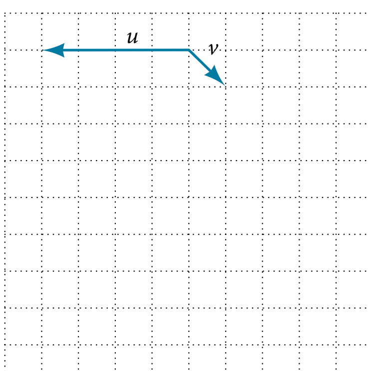


43.



For the following exercises, use the vectors shown to sketch $u - 3v$.

44.



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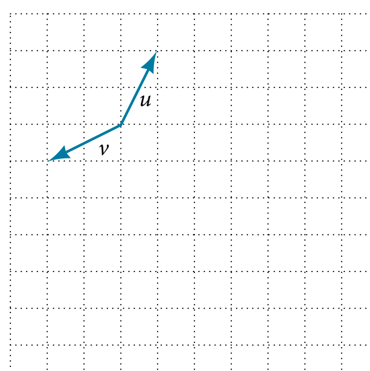
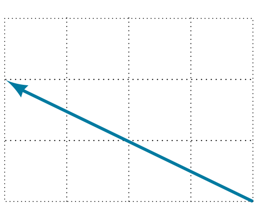
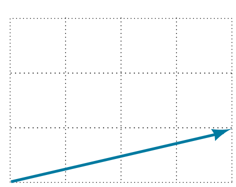


For the following exercises, write the vector shown in component form.

46.



47.



48. Given initial point $P_1 = (2, 1)$ and terminal point $P_2 = (-1, 2)$, write the vector v in terms of i and j, then draw the vector on the graph.

49. Given initial point $P_1 = (4, -1)$ and terminal point $P_2 = (-3, 2)$, write the vector v in terms of i and j. Draw the points and the vector on the graph.

50. Given initial point $P_1 = (3, 3)$ and terminal point $P_2 = (-3, 3)$, write the vector v in terms of i and j. Draw the points and the vector on the graph.

EXTENSIONS

For the following exercises, use the given magnitude and direction in standard position, write the vector in component form.

51. $|v| = 6, \theta = 45\,°$

52. $|v| = 8, \theta = 220°$

53. $|v| = 2, \theta = 300°$

54. $|v| = 5, \theta = 135°$

55. A 60-pound box is resting on a ramp that is inclined 12°. Rounding to the nearest tenth,

a. Find the magnitude of the normal (perpendicular) component of the force.

b. Find the magnitude of the component of the force that is parallel to the ramp.

56. A 25-pound box is resting on a ramp that is inclined 8°. Rounding to the nearest tenth,

a. Find the magnitude of the normal (perpendicular) component of the force.

b. Find the magnitude of the component of the force that is parallel to the ramp.

57. Find the magnitude of the horizontal and vertical components of a vector with magnitude 8 pounds pointed in a direction of 27° above the horizontal. Round to the nearest hundredth.

58. Find the magnitude of the horizontal and vertical components of the vector with magnitude 4 pounds pointed in a direction of 127° above the horizontal. Round to the nearest hundredth.

59. Find the magnitude of the horizontal and vertical components of a vector with magnitude 5 pounds pointed in a direction of 55° above the horizontal. Round to the nearest hundredth.

60. Find the magnitude of the horizontal and vertical components of the vector with magnitude 1 pound pointed in a direction of 8° above the horizontal. Round to the nearest hundredth.

REAL-WORLD APPLICATIONS

61. A woman leaves home and walks 3 miles west, then 2 miles southwest. How far from home is she, and in what direction must she walk to head directly home?

62. A boat leaves the marina and sails 6 miles north, then 2 miles northeast. How far from the marina is the boat, and in what direction must it sail to head directly back to the marina?

63. A man starts walking from home and walks 4 miles east, 2 miles southeast, 5 miles south, 4 miles southwest, and 2 miles east. How far has he walked? If he walked straight home, how far would he have to walk?

64. A woman starts walking from home and walks 4 miles east, 7 miles southeast, 6 miles south, 5 miles southwest, and 3 miles east. How far has she walked? If she walked straight home, how far would she have to walk?

65. A man starts walking from home and walks 3 miles at 20° north of west, then 5 miles at 10° west of south, then 4 miles at 15° north of east. If he walked straight home, how far would he have to the walk, and in what direction?

66. A woman starts walking from home and walks 6 miles at 40° north of east, then 2 miles at 15° east of south, then 5 miles at 30° south of west. If she walked straight home, how far would she have to walk, and in what direction?

67. An airplane is heading north at an airspeed of 600 km/hr, but there is a wind blowing from the southwest at 80 km/hr. How many degrees off course will the plane end up flying, and what is the plane's speed relative to the ground?

68. An airplane is heading north at an airspeed of 500 km/hr, but there is a wind blowing from the northwest at 50 km/hr. How many degrees off course will the plane end up flying, and what is the plane's speed relative to the ground?

69. An airplane needs to head due north, but there is a wind blowing from the southwest at 60 km/hr. The plane flies with an airspeed of 550 km/hr. To end up flying due north, how many degrees west of north will the pilot need to fly the plane?

70. An airplane needs to head due north, but there is a wind blowing from the northwest at 80 km/hr. The plane flies with an airspeed of 500 km/hr. To end up flying due north, how many degrees west of north will the pilot need to fly the plane?

71. As part of a video game, the point (5, 7) is rotated counterclockwise about the origin through an angle of 35°. Find the new coordinates of this point.

72. As part of a video game, the point (7, 3) is rotated counterclockwise about the origin through an angle of 40°. Find the new coordinates of this point.

73. Two children are throwing a ball back and forth straight across the back seat of a car. The ball is being thrown 10 mph relative to the car, and the car is traveling 25 mph down the road. If one child doesn't catch the ball, and it flies out the window, in what direction does the ball fly (ignoring wind resistance)?

74. Two children are throwing a ball back and forth straight across the back seat of a car. The ball is being thrown 8 mph relative to the car, and the car is traveling 45 mph down the road. If one child doesn't catch the ball, and it flies out the window, in what direction does the ball fly (ignoring wind resistance)?

75. A 50-pound object rests on a ramp that is inclined 19°. Find the magnitude of the components of the force parallel to and perpendicular to (normal) the ramp to the nearest tenth of a pound.

76. Suppose a body has a force of 10 pounds acting on it to the right, 25 pounds acting on it upward, and 5 pounds acting on it 45° from the horizontal. What single force is the resultant force acting on the body?

77. Suppose a body has a force of 10 pounds acting on it to the right, 25 pounds acting on it $-135°$ from the horizontal, and 5 pounds acting on it directed 150° from the horizontal. What single force is the resultant force acting on the body?

78. The condition of equilibrium is when the sum of the forces acting on a body is the zero vector. Suppose a body has a force of 2 pounds acting on it to the right, 5 pounds acting on it upward, and 3 pounds acting on it 45° from the horizontal. What single force is needed to produce a state of equilibrium on the body?

79. Suppose a body has a force of 3 pounds acting on it to the left, 4 pounds acting on it upward, and 2 pounds acting on it 30° from the horizontal. What single force is needed to produce a state of equilibrium on the body? Draw the vector.

CHAPTER 10 REVIEW

Key Terms

altitude a perpendicular line from one vertex of a triangle to the opposite side, or in the case of an obtuse triangle, to the line containing the opposite side, forming two right triangles

ambiguous case a scenario in which more than one triangle is a valid solution for a given oblique SSA triangle

Archimedes' spiral a polar curve given by $r = \theta$. When multiplied by a constant, the equation appears as $r = a\theta$. As $r = \theta$, the curve continues to widen in a spiral path over the domain.

argument the angle associated with a complex number; the angle between the line from the origin to the point and the positive real axis

cardioid a member of the limaçon family of curves, named for its resemblance to a heart; its equation is given as $r = a \pm b\cos\theta$ and $r = a \pm b\sin\theta$, where $\frac{a}{b} = 1$

convex limaçon a type of one-loop limaçon represented by $r = a \pm b\cos\theta$ and $r = a \pm b\sin\theta$ such that $\frac{a}{b} \geq 2$

De Moivre's Theorem formula used to find the nth power or nth roots of a complex number; states that, for a positive integer n, z^n is found by raising the modulus to the nth power and multiplying the angles by n

dimpled limaçon a type of one-loop limaçon represented by $r = a \pm b\cos\theta$ and $r = a \pm b\sin\theta$ such that $1 < \frac{a}{b} < 2$

dot product given two vectors, the sum of the product of the horizontal components and the product of the vertical components

Generalized Pythagorean Theorem an extension of the Law of Cosines; relates the sides of an oblique triangle and is used for SAS and SSS triangles

initial point the origin of a vector

inner-loop limaçon a polar curve similar to the cardioid, but with an inner loop; passes through the pole twice; represented by $r = a \pm b\cos\theta$ and $r = a \pm b\sin\theta$ where $a < b$

Law of Cosines states that the square of any side of a triangle is equal to the sum of the squares of the other two sides minus twice the product of the other two sides and the cosine of the included angle

Law of Sines states that the ratio of the measurement of one angle of a triangle to the length of its opposite side is equal to the remaining two ratios of angle measure to opposite side; any pair of proportions may be used to solve for a missing angle or side

lemniscate a polar curve resembling a **Figure 8** and given by the equation $r^2 = a^2\cos 2\theta$ and $r^2 = a^2\sin 2\theta$, $a \neq 0$

magnitude the length of a vector; may represent a quantity such as speed, and is calculated using the Pythagorean Theorem

modulus the absolute value of a complex number, or the distance from the origin to the point (x, y); also called the amplitude

oblique triangle any triangle that is not a right triangle

one-loop limaçon a polar curve represented by $r = a \pm b\cos\theta$ and $r = a \pm b\sin\theta$ such that $a > 0$, $b > 0$, and $\frac{a}{b} > 1$; may be dimpled or convex; does not pass through the pole

parameter a variable, often representing time, upon which x and y are both dependent

polar axis on the polar grid, the equivalent of the positive x-axis on the rectangular grid

polar coordinates on the polar grid, the coordinates of a point labeled (r, θ), where θ indicates the angle of rotation from the polar axis and r represents the radius, or the distance of the point from the pole in the direction of θ

polar equation an equation describing a curve on the polar grid

polar form of a complex number a complex number expressed in terms of an angle θ and its distance from the origin r; can be found by using conversion formulas $x = r\cos\theta$, $y = r\sin\theta$, and $r = \sqrt{x^2 + y^2}$

pole the origin of the polar grid

resultant a vector that results from addition or subtraction of two vectors, or from scalar multiplication

rose curve a polar equation resembling a flower, given by the equations $r = a\cos n\theta$ and $r = a\sin n\theta$; when n is even there are $2n$ petals, and the curve is highly symmetrical; when n is odd there are n petals.

scalar a quantity associated with magnitude but not direction; a constant

scalar multiplication the product of a constant and each component of a vector

standard position the placement of a vector with the initial point at $(0, 0)$ and the terminal point (a, b), represented by the change in the x-coordinates and the change in the y-coordinates of the original vector

terminal point the end point of a vector, usually represented by an arrow indicating its direction

unit vector a vector that begins at the origin and has magnitude of 1; the horizontal unit vector runs along the x-axis and is defined as $v_1 = \langle 1, 0 \rangle$ the vertical unit vector runs along the y-axis and is defined as $v_2 = \langle 0, 1 \rangle$.

vector a quantity associated with both magnitude and direction, represented as a directed line segment with a starting point (initial point) and an end point (terminal point)

vector addition the sum of two vectors, found by adding corresponding components

Key Equations

Law of Sines

$$\frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin \gamma}{c}$$
$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma}$$

Area for oblique triangles

$$\begin{aligned} \text{Area} &= \frac{1}{2}bc\sin \alpha \\ &= \frac{1}{2}ac\sin \beta \\ &= \frac{1}{2}ab\sin \gamma \end{aligned}$$

Law of Cosines

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos \alpha \\ b^2 &= a^2 + c^2 - 2ac\cos \beta \\ c^2 &= a^2 + b^2 - 2ab\cos \gamma \end{aligned}$$

Heron's formula

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)} \text{ where } s = \frac{(a+b+c)}{2}$$

Conversion formulas

$$\begin{aligned} \cos \theta &= \frac{x}{r} \rightarrow x = r\cos \theta \\ \sin \theta &= \frac{y}{r} \rightarrow y = r\sin \theta \\ r^2 &= x^2 + y^2 \\ \tan \theta &= \frac{y}{x} \end{aligned}$$

Key Concepts

10.1 Non-right Triangles: Law of Sines

- The Law of Sines can be used to solve oblique triangles, which are non-right triangles.
- According to the Law of Sines, the ratio of the measurement of one of the angles to the length of its opposite side equals the other two ratios of angle measure to opposite side.

- There are three possible cases: ASA, AAS, SSA. Depending on the information given, we can choose the appropriate equation to find the requested solution. See **Example 1**.
- The ambiguous case arises when an oblique triangle can have different outcomes.
- There are three possible cases that arise from SSA arrangement—a single solution, two possible solutions, and no solution. See **Example 2** and **Example 3**.
- The Law of Sines can be used to solve triangles with given criteria. See **Example 4**.
- The general area formula for triangles translates to oblique triangles by first finding the appropriate height value. See **Example 5**.
- There are many trigonometric applications. They can often be solved by first drawing a diagram of the given information and then using the appropriate equation. See **Example 6**.

10.2 Non-right Triangles: Law of Cosines

- The Law of Cosines defines the relationship among angle measurements and lengths of sides in oblique triangles.
- The Generalized Pythagorean Theorem is the Law of Cosines for two cases of oblique triangles: SAS and SSS. Dropping an imaginary perpendicular splits the oblique triangle into two right triangles or forms one right triangle, which allows sides to be related and measurements to be calculated. See **Example 1** and **Example 2**.
- The Law of Cosines is useful for many types of applied problems. The first step in solving such problems is generally to draw a sketch of the problem presented. If the information given fits one of the three models (the three equations), then apply the Law of Cosines to find a solution. See **Example 3** and **Example 4**.
- Heron's formula allows the calculation of area in oblique triangles. All three sides must be known to apply Heron's formula. See **Example 5** and See **Example 6**.

10.3 Polar Coordinates

- The polar grid is represented as a series of concentric circles radiating out from the pole, or origin.
- To plot a point in the form (r, θ), $\theta > 0$, move in a counterclockwise direction from the polar axis by an angle of θ, and then extend a directed line segment from the pole the length of r in the direction of θ. If θ is negative, move in a clockwise direction, and extend a directed line segment the length of r in the direction of θ. See **Example 1**.
- If r is negative, extend the directed line segment in the opposite direction of θ. See **Example 2**.
- To convert from polar coordinates to rectangular coordinates, use the formulas $x = r\cos\theta$ and $y = r\sin\theta$. See **Example 3** and **Example 4**.
- To convert from rectangular coordinates to polar coordinates, use one or more of the formulas: $\cos\theta = \frac{x}{r}$, $\sin\theta = \frac{y}{r}$, $\tan\theta = \frac{y}{x}$, and $r = \sqrt{x^2 + y^2}$. See **Example 5**.
- Transforming equations between polar and rectangular forms means making the appropriate substitutions based on the available formulas, together with algebraic manipulations. See **Example 6**, **Example 7**, and **Example 8**.
- Using the appropriate substitutions makes it possible to rewrite a polar equation as a rectangular equation, and then graph it in the rectangular plane. See **Example 9**, **Example 10**, and **Example 11**.

10.4 Polar Coordinates: Graphs

- It is easier to graph polar equations if we can test the equations for symmetry with respect to the line $\theta = \frac{\pi}{2}$, the polar axis, or the pole.
- There are three symmetry tests that indicate whether the graph of a polar equation will exhibit symmetry. If an equation fails a symmetry test, the graph may or may not exhibit symmetry. See **Example 1**.
- Polar equations may be graphed by making a table of values for θ and r.
- The maximum value of a polar equation is found by substituting the value θ that leads to the maximum value of the trigonometric expression.
- The zeros of a polar equation are found by setting $r = 0$ and solving for θ. See **Example 2**.
- Some formulas that produce the graph of a circle in polar coordinates are given by $r = a\cos\theta$ and $r = a\sin\theta$. See **Example 3**.
- The formulas that produce the graphs of a cardioid are given by $r = a \pm b\cos\theta$ and $r = a \pm b\sin\theta$, for $a > 0$, $b > 0$, and $\frac{a}{b} = 1$. See **Example 4**.

- The formulas that produce the graphs of a one-loop limaçon are given by $r = a \pm b\cos\theta$ and $r = a \pm b\sin\theta$ for $1 < \frac{a}{b} < 2$. See **Example 5**.
- The formulas that produce the graphs of an inner-loop limaçon are given by $r = a \pm b\cos\theta$ and $r = a \pm b\sin\theta$ for $a > 0$, $b > 0$, and $a < b$. See **Example 6**.
- The formulas that produce the graphs of a lemniscates are given by $r^2 = a^2\cos 2\theta$ and $r^2 = a^2\sin 2\theta$, where $a \neq 0$. See **Example 7**.
- The formulas that produce the graphs of rose curves are given by $r = a\cos n\theta$ and $r = a\sin n\theta$, where $a \neq 0$; if n is even, there are $2n$ petals, and if n is odd, there are n petals. See **Example 8** and **Example 9**.
- The formula that produces the graph of an Archimedes' spiral is given by $r = \theta$, $\theta \geq 0$. See **Example 10**.

10.5 Polar Form of Complex Numbers

- Complex numbers in the form $a + bi$ are plotted in the complex plane similar to the way rectangular coordinates are plotted in the rectangular plane. Label the x-axis as the real axis and the y-axis as the imaginary axis. See **Example 1**.
- The absolute value of a complex number is the same as its magnitude. It is the distance from the origin to the point: $|z| = \sqrt{a^2 + b^2}$. See **Example 2** and **Example 3**.
- To write complex numbers in polar form, we use the formulas $x = r\cos\theta$, $y = r\sin\theta$, and $r = \sqrt{x^2 + y^2}$. Then, $z = r(\cos\theta + i\sin\theta)$. See **Example 4** and **Example 5**.
- To convert from polar form to rectangular form, first evaluate the trigonometric functions. Then, multiply through by r. See **Example 6** and **Example 7**.
- To find the product of two complex numbers, multiply the two moduli and add the two angles. Evaluate the trigonometric functions, and multiply using the distributive property. See **Example 8**.
- To find the quotient of two complex numbers in polar form, find the quotient of the two moduli and the difference of the two angles. See **Example 9**.
- To find the power of a complex number z^n, raise r to the power n, and multiply θ by n. See **Example 10**.
- Finding the roots of a complex number is the same as raising a complex number to a power, but using a rational exponent. See **Example 11**.

10.6 Parametric Equations

- Parameterizing a curve involves translating a rectangular equation in two variables, x and y, into two equations in three variables, x, y, and t. Often, more information is obtained from a set of parametric equations. See **Example 1**, **Example 2**, and **Example 3**.
- Sometimes equations are simpler to graph when written in rectangular form. By eliminating t, an equation in x and y is the result.
- To eliminate t, solve one of the equations for t, and substitute the expression into the second equation. See **Example 4**, **Example 5**, **Example 6**, and **Example 7**.
- Finding the rectangular equation for a curve defined parametrically is basically the same as eliminating the parameter. Solve for t in one of the equations, and substitute the expression into the second equation. See **Example 8**.
- There are an infinite number of ways to choose a set of parametric equations for a curve defined as a rectangular equation.
- Find an expression for x such that the domain of the set of parametric equations remains the same as the original rectangular equation. See **Example 9**.

10.7 Parametric Equations: Graphs

- When there is a third variable, a third parameter on which x and y depend, parametric equations can be used.
- To graph parametric equations by plotting points, make a table with three columns labeled t, $x(t)$, and $y(t)$. Choose values for t in increasing order. Plot the last two columns for x and y. See **Example 1** and **Example 2**.
- When graphing a parametric curve by plotting points, note the associated t-values and show arrows on the graph indicating the orientation of the curve. See **Example 3** and **Example 4**.
- Parametric equations allow the direction or the orientation of the curve to be shown on the graph. Equations that are not functions can be graphed and used in many applications involving motion. See **Example 5**.
- Projectile motion depends on two parametric equations: $x = (v_0 \cos \theta)t$ and $y = -16t^2 + (v_0 \sin \theta)t + h$. Initial velocity is symbolized as v_0. θ represents the initial angle of the object when thrown, and h represents the height at which the object is propelled.

10.8 Vectors

- The position vector has its initial point at the origin. See **Example 1**.
- If the position vector is the same for two vectors, they are equal. See **Example 2**. Vectors are defined by their magnitude and direction. See **Example 3**.
- If two vectors have the same magnitude and direction, they are equal. See **Example 4**.
- Vector addition and subtraction result in a new vector found by adding or subtracting corresponding elements. See **Example 5**.
- Scalar multiplication is multiplying a vector by a constant. Only the magnitude changes; the direction stays the same. See **Example 6** and **Example 7**.
- Vectors are comprised of two components: the horizontal component along the positive x-axis, and the vertical component along the positive y-axis. See **Example 8**.
- The unit vector in the same direction of any nonzero vector is found by dividing the vector by its magnitude.
- The magnitude of a vector in the rectangular coordinate system is $|v| = \sqrt{a^2 + b^2}$. See **Example 9**.
- In the rectangular coordinate system, unit vectors may be represented in terms of i and j where i represents the horizontal component and j represents the vertical component. Then, $v = ai + bj$ is a scalar multiple of v by real numbers a and b. See **Example 10** and **Example 11**.
- Adding and subtracting vectors in terms of i and j consists of adding or subtracting corresponding coefficients of i and corresponding coefficients of j. See **Example 12**.
- A vector $v = ai + bj$ is written in terms of magnitude and direction as $v = |v|\cos \theta \mathrm{i} + |v|\sin \theta j$. See **Example 13**.
- The dot product of two vectors is the product of the i terms plus the product of the j terms. See **Example 14**.
- We can use the dot product to find the angle between two vectors. **Example 15** and **Example 16**.
- Dot products are useful for many types of physics applications. See **Example 17**.

CHAPTER 10 REVIEW EXERCISES

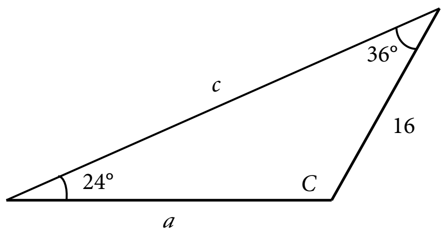
NON-RIGHT TRIANGLES: LAW OF SINES

For the following exercises, assume α is opposite side a, β is opposite side b, and γ is opposite side c. Solve each triangle, if possible. Round each answer to the nearest tenth.

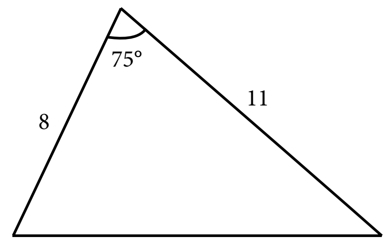
1. $\beta = 50°$, $a = 105$, $b = 45$

2. $\alpha = 43.1°$, $a = 184.2$, $b = 242.8$

3. Solve the triangle.



4. Find the area of the triangle.



5. A pilot is flying over a straight highway. He determines the angles of depression to two mileposts, 2.1 km apart, to be 25° and 49°, as shown in **Figure 1**. Find the distance of the plane from point A and the elevation of the plane.

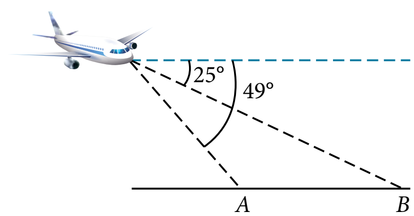


Figure 1

NON-RIGHT TRIANGLES: LAW OF COSINES

6. Solve the triangle, rounding to the nearest tenth, assuming α is opposite side a, β is opposite side b, and γ is opposite side c: $a = 4$, $b = 6$, $c = 8$.

7. Solve the triangle in **Figure 2**, rounding to the nearest tenth.

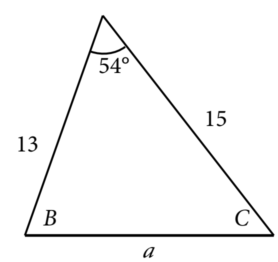


Figure 2

8. Find the area of a triangle with sides of length 8.3, 6.6, and 9.1.

9. To find the distance between two cities, a satellite calculates the distances and angle shown in **Figure 3** (not to scale). Find the distance between the cities. Round answers to the nearest tenth.

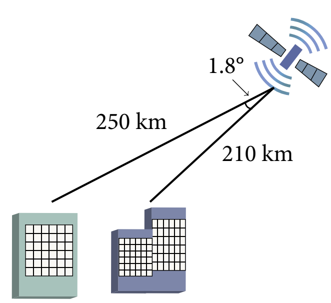


Figure 3

POLAR COORDINATES

10. Plot the point with polar coordinates $\left(3, \frac{\pi}{6}\right)$.

11. Plot the point with polar coordinates $\left(5, -\frac{2\pi}{3}\right)$

12. Convert $\left(6, -\frac{3\pi}{4}\right)$ to rectangular coordinates.

13. Convert $\left(-2, \frac{3\pi}{2}\right)$ to rectangular coordinates.

14. Convert $(7, -2)$ to polar coordinates.

15. Convert $(-9, -4)$ to polar coordinates.

For the following exercises, convert the given Cartesian equation to a polar equation.

16. $x = -2$

17. $x^2 + y^2 = 64$

18. $x^2 + y^2 = -2y$

For the following exercises, convert the given polar equation to a Cartesian equation.

19. $r = 7\cos\theta$

20. $r = \dfrac{-2}{4\cos\theta + \sin\theta}$

For the following exercises, convert to rectangular form and graph.

21. $\theta = \dfrac{3\pi}{4}$

22. $r = 5\sec\theta$

POLAR COORDINATES: GRAPHS

For the following exercises, test each equation for symmetry.

23. $r = 4 + 4\sin\theta$

24. $r = 7$

25. Sketch a graph of the polar equation $r = 1 - 5\sin\theta$. Label the axis intercepts.

26. Sketch a graph of the polar equation $r = 5\sin(7\theta)$.

27. Sketch a graph of the polar equation $r = 3 - 3\cos\theta$

POLAR FORM OF COMPLEX NUMBERS

For the following exercises, find the absolute value of each complex number.

28. $-2 + 6i$

29. $4 - 3i$

Write the complex number in polar form.

30. $5 + 9i$

31. $\frac{1}{2} - \frac{\sqrt{3}}{2}i$

For the following exercises, convert the complex number from polar to rectangular form.

32. $z = 5\text{cis}\left(\frac{5\pi}{6}\right)$

33. $z = 3\text{cis}(40°)$

For the following exercises, find the product $z_1 z_2$ in polar form.

34. $z_1 = 2\text{cis}(89°)$, $z_2 = 5\text{cis}(23°)$

35. $z_1 = 10\text{cis}\left(\frac{\pi}{6}\right)$, $z_2 = 6\text{cis}\left(\frac{\pi}{3}\right)$

For the following exercises, find the quotient $\frac{z_1}{z_2}$ in polar form.

36. $z_1 = 12\text{cis}(55°)$, $z_2 = 3\text{cis}(18°)$

37. $z_1 = 27\text{cis}\left(\frac{5\pi}{3}\right)$, $z_2 = 9\text{cis}\left(\frac{\pi}{3}\right)$

For the following exercises, find the powers of each complex number in polar form.

38. Find z^4 when $z = 2\text{cis}(70°)$

39. Find z^2 when $z = 5\text{cis}\left(\frac{3\pi}{4}\right)$

For the following exercises, evaluate each root.

40. Evaluate the cube root of z when $z = 64\text{cis}(210°)$.

41. Evaluate the square root of z when $z = 25\text{cis}\left(\frac{3\pi}{2}\right)$.

For the following exercises, plot the complex number in the complex plane.

42. $6 - 2i$

43. $-1 + 3i$

PARAMETRIC EQUATIONS

For the following exercises, eliminate the parameter t to rewrite the parametric equation as a Cartesian equation.

44. $\begin{cases} x(t) = 3t - 1 \\ y(t) = \sqrt{t} \end{cases}$

45. $\begin{cases} x(t) = -\cos t \\ y(t) = 2\sin^2 t \end{cases}$

46. Parameterize (write a parametric equation for) each Cartesian equation by using $x(t) = a\cos t$ and $y(t) = b\sin t$ for $\frac{x^2}{25} + \frac{y^2}{16} = 1$.

47. Parameterize the line from $(-2, 3)$ to $(4, 7)$ so that the line is at $(-2, 3)$ at $t = 0$ and $(4, 7)$ at $t = 1$.

PARAMETRIC EQUATIONS: GRAPHS

For the following exercises, make a table of values for each set of parametric equations, graph the equations, and include an orientation; then write the Cartesian equation.

48. $\begin{cases} x(t) = 3t^2 \\ y(t) = 2t - 1 \end{cases}$

49. $\begin{cases} x(t) = e^t \\ y(t) = -2e^{5t} \end{cases}$

50. $\begin{cases} x(t) = 3\cos t \\ y(t) = 2\sin t \end{cases}$

51. A ball is launched with an initial velocity of 80 feet per second at an angle of 40° to the horizontal. The ball is released at a height of 4 feet above the ground.

a. Find the parametric equations to model the path of the ball.

b. Where is the ball after 3 seconds?

c. How long is the ball in the air?

VECTORS

For the following exercises, determine whether the two vectors, u and v, are equal, where u has an initial point P_1 and a terminal point P_2, and v has an initial point P_3 and a terminal point P_4.

52. $P_1 = (-1, 4), P_2 = (3, 1), P_3 = (5, 5)$ and $P_4 = (9, 2)$

53. $P_1 = (6, 11), P_2 = (-2, 8), P_3 = (0, -1)$ and $P_4 = (-8, 2)$

For the following exercises, use the vectors $u = 2i - j$, $v = 4i - 3j$, and $w = -2i + 5j$ to evaluate the expression.

54. $u - v$

55. $2v - u + w$

For the following exercises, find a unit vector in the same direction as the given vector.

56. $a = 8i - 6j$

57. $b = -3i - j$

For the following exercises, find the magnitude and direction of the vector.

58. $\langle 6, -2 \rangle$

59. $\langle -3, -3 \rangle$

For the following exercises, calculate $u \cdot v$.

60. $u = -2i + j$ and $v = 3i + 7j$

61. $u = i + 4j$ and $v = 4i + 3j$

62. Given $v = \langle -3, 4 \rangle$ draw v, $2v$, and $\frac{1}{2}v$.

63. Given the vectors shown in **Figure 4**, sketch $u + v$, $u - v$ and $3v$.

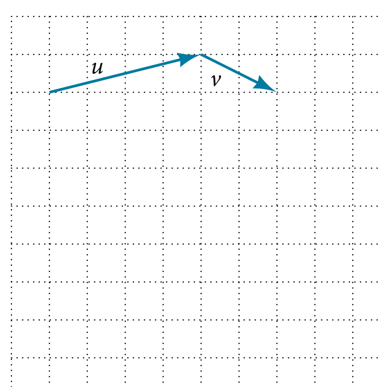


Figure 4

64. Given initial point $P_1 = (3, 2)$ and terminal point $P_2 = (-5, -1)$, write the vector v in terms of i and j. Draw the points and the vector on the graph.

CHAPTER 10 PRACTICE TEST

1. Assume α is opposite side a, β is opposite side b, and γ is opposite side c. Solve the triangle, if possible, and round each answer to the nearest tenth, given $\beta = 68°$, $b = 21$, $c = 16$.

2. Find the area of the triangle in **Figure 1**. Round each answer to the nearest tenth.

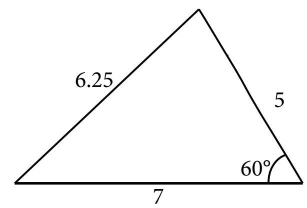


Figure 1

3. A pilot flies in a straight path for 2 hours. He then makes a course correction, heading 15° to the right of his original course, and flies 1 hour in the new direction. If he maintains a constant speed of 575 miles per hour, how far is he from his starting position?

4. Convert (2, 2) to polar coordinates, and then plot the point.

5. Convert $\left(2, \frac{\pi}{3}\right)$ to rectangular coordinates.

6. Convert the polar equation to a Cartesian equation: $x^2 + y^2 = 5y$.

7. Convert to rectangular form and graph: $r = -3\csc\theta$.

8. Test the equation for symmetry: $r = -4\sin(2\theta)$.

9. Graph $r = 3 + 3\cos\theta$.

10. Graph $r = 3 - 5\sin\theta$.

11. Find the absolute value of the complex number $5 - 9i$.

12. Write the complex number in polar form: $4 + i$.

13. Convert the complex number from polar to rectangular form: $z = 5\text{cis}\left(\frac{2\pi}{3}\right)$.

Given $z_1 = 8\text{cis}(36°)$ and $z_2 = 2\text{cis}(15°)$, evaluate each expression.

14. $z_1 z_2$

15. $\frac{z_1}{z_2}$

16. $(z_2)^3$

17. $\sqrt{z_1}$

18. Plot the complex number $-5 - i$ in the complex plane.

19. Eliminate the parameter t to rewrite the following parametric equations as a Cartesian equation:
$$\begin{cases} x(t) = t + 1 \\ y(t) = 2t^2 \end{cases}$$

20. Parameterize (write a parametric equation for) the following Cartesian equation by using $x(t) = a\cos t$ and $y(t) = b\sin t$: $\frac{x^2}{36} + \frac{y^2}{100} = 1$.

21. Graph the set of parametric equations and find the Cartesian equation:
$$\begin{cases} x(t) = -2\sin t \\ y(t) = 5\cos t \end{cases}$$

22. A ball is launched with an initial velocity of 95 feet per second at an angle of 52° to the horizontal. The ball is released at a height of 3.5 feet above the ground.

a. Find the parametric equations to model the path of the ball.
b. Where is the ball after 2 seconds?
c. How long is the ball in the air?

For the following exercises, use the vectors $u = i - 3j$ and $v = 2i + 3j$.

23. Find $2u - 3v$.

24. Calculate $u \cdot v$.

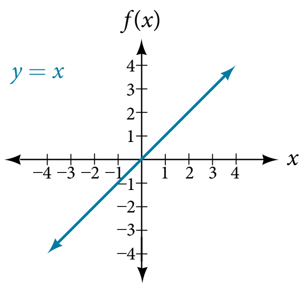
25. Find a unit vector in the same direction as v.

26. Given vector v has an initial point $P_1 = (2, 2)$ and terminal point $P_2 = (-1, 0)$, write the vector v in terms of i and j. On the graph, draw v, and $-v$.

Proofs, Identities, and Toolkit Functions

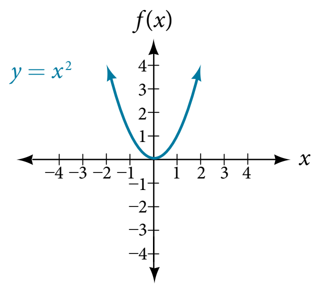
A1 Graphs of the Parent Functions

Identity



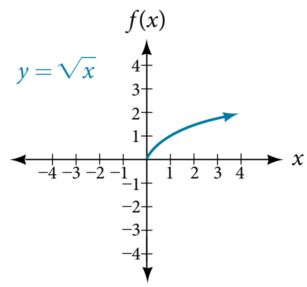
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

Square



Domain: $(-\infty, \infty)$
Range: $[0, \infty)$

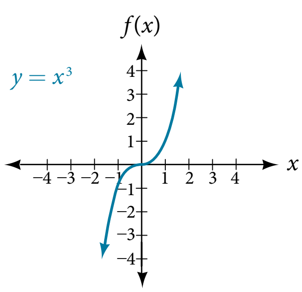
Square Root



Domain: $[0, \infty)$
Range: $[0, \infty)$

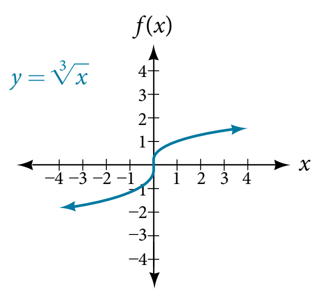
Figure A1

Cubic



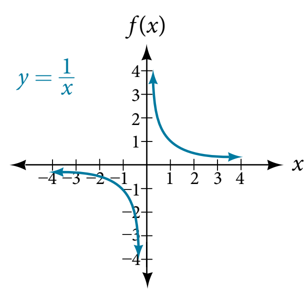
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

Cube Root



Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

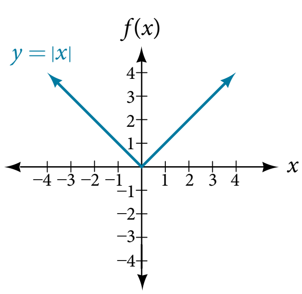
Reciprocal



Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(-\infty, 0) \cup (0, \infty)$

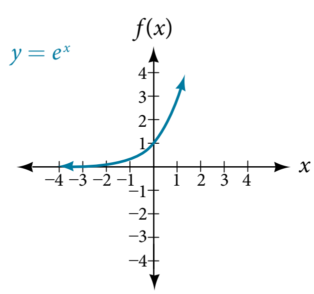
Figure A2

Absolute Value



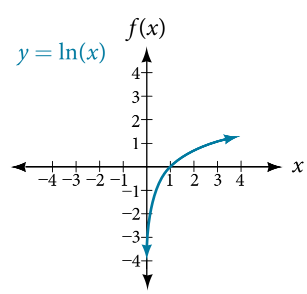
Domain: $(-\infty, \infty)$
Range: $[0, \infty)$

Exponential



Domain: $(-\infty, \infty)$
Range: $[0, \infty)$

Natural Logarithm

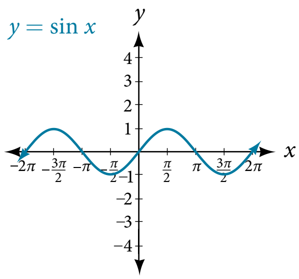


Domain: $(0, \infty)$
Range: $(-\infty, \infty)$

Figure A3

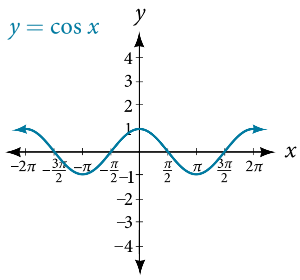
A2 Graphs of the Trigonometric Functions

Sine



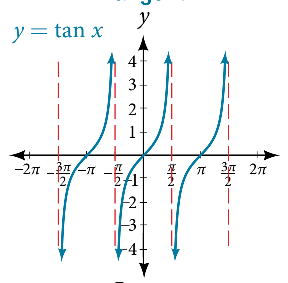
Domain: $(-\infty, \infty)$
Range: $(-1, 1)$

Cosine



Domain: $(-\infty, \infty)$
Range: $(-1, 1)$

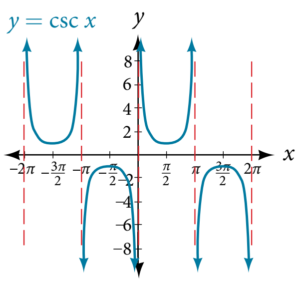
Tangent



Domain: $x \neq \frac{\pi}{2}k$ where k is an odd integer
Range: $(-\infty, -1] \cup [1, \infty)$

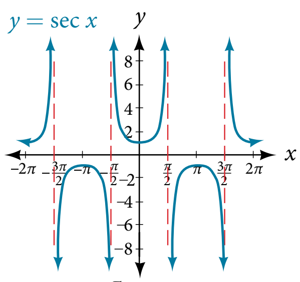
Figure A4

Cosecant



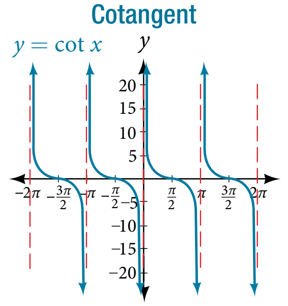
Domain: $x \neq \pi k$ where k is an integer
Range: $(-\infty, -1] \cup [1, \infty)$

Secant



Domain: $x \neq \frac{\pi}{2}k$ where k is an odd integer
Range: $(-\infty, -1] \cup [1, \infty)$

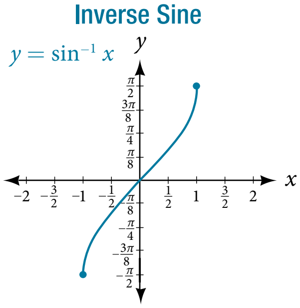
Cotangent



Domain: $x \neq \pi k$ where k is an integer
Range: $(-\infty, \infty)$

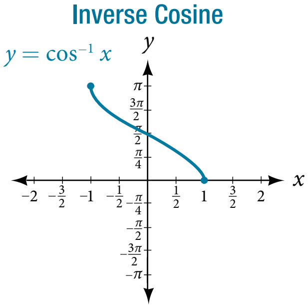
Figure A5

Inverse Sine



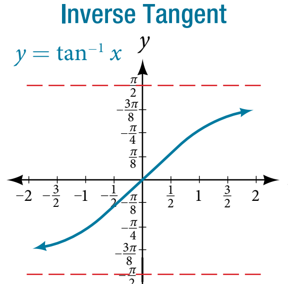
Domain: $[-1, 1]$
Range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Inverse Cosine



Domain: $[-1, 1]$
Range: $[0, \pi)$

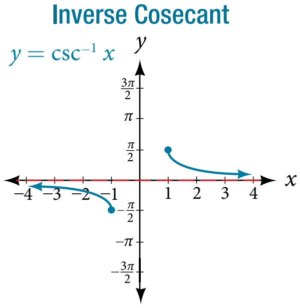
Inverse Tangent



Domain: $(-\infty, \infty)$
Range: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

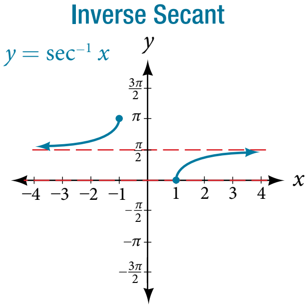
Figure A6

Inverse Cosecant



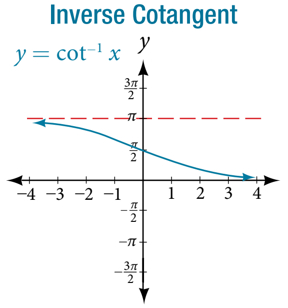
Domain: $(-\infty, -1] \cup [1, \infty)$
Range: $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

Inverse Secant



Domain: $(-\infty, -1] \cup [1, \infty)$
Range: $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$

Inverse Cotangent



Domain: $(-\infty, \infty)$
Range: $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

Figure A7

A3 Trigonometric Identities

Identities	Equations
Pythagorean Identities	$\sin^2\theta + \cos^2\theta = 1$ $1 + \tan^2\theta = \sec^2\theta$ $1 + \cot^2\theta = \csc^2\theta$
Even-odd Identities	$\cos(-\theta) = \cos\theta$ $\sec(-\theta) = \sec\theta$ $\sin(-\theta) = -\sin\theta$ $\tan(-\theta) = -\tan\theta$ $\csc(-\theta) = -\csc\theta$ $\cot(-\theta) = -\cot\theta$
Cofunction identities	$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$ $\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$ $\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right)$ $\cot\theta = \tan\left(\frac{\pi}{2} - \theta\right)$ $\sec\theta = \csc\left(\frac{\pi}{2} - \theta\right)$ $\csc\theta = \sec\left(\frac{\pi}{2} - \theta\right)$
Fundamental Identities	$\tan\theta = \frac{\sin\theta}{\cos\theta}$ $\sec\theta = \frac{1}{\cos\theta}$ $\csc\theta = \frac{1}{\sin\theta}$ $\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$
Sum and Difference Identities	$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$ $\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$ $\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$
Double-Angle Formulas	$\sin(2\theta) = 2\sin\theta\cos\theta$ $\cos(2\theta) = \cos^2\theta - \sin^2\theta$ $\cos(2\theta) = 1 - 2\sin^2\theta$ $\cos(2\theta) = 2\cos^2\theta - 1$ $\tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta}$

Table A1

Identities	Equations
Half-Angle formulas	$\sin \frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}$ $\cos \frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$ $\tan \frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$ $= \frac{\sin\alpha}{1-\cos\alpha}$ $= \frac{1-\cos\alpha}{\sin\alpha}$
Reduction Formulas	$\sin^2\theta = \frac{1-\cos(2\theta)}{2}$ $\cos^2\theta = \frac{1+\cos(2\theta)}{2}$ $\tan^2\theta = \frac{1-\cos(2\theta)}{1+\cos(2\theta)}$
Product-to-Sum Formulas	$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$ $\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$ $\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]$ $\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta)-\sin(\alpha-\beta)]$
Sum-to-Product Formulas	$\sin\alpha+\sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$ $\sin\alpha-\sin\beta = 2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$ $\cos\alpha-\cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$ $\cos\alpha+\cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$
Law of Sines	$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$ $\frac{\sin a}{\alpha} = \frac{\sin b}{\beta} = \frac{\sin c}{\gamma}$
Law of Cosines	$a^2 = b^2+c^2-2bc\cos\alpha$ $b^2 = a^2+c^2-2ac\cos\beta$ $c^2 = a^2+b^2-2aa\cos\gamma$

Table A1

Try It Answers

Chapter 1

Section 1.1

1. a. $\frac{11}{1}$ **b.** $\frac{3}{1}$ **c.** $-\frac{4}{1}$ **2. a.** 4 (or 4.0), terminating **b.** $0.\overline{615384}$, repeating **c.** -0.85, terminating **3. a.** rational and repeating **b.** rational and terminating **c.** irrational **d.** rational and repeating **e.** irrational **4. a.** positive, irrational; right **b.** negative, rational; left **c.** positive, rational; right **d.** negative, irrational; left **e.** positive, rational; right

5.

	N	W	I	Q	Q'
a. $-\frac{35}{7}$			×	×	
b. 0		×	×	×	
c. $\sqrt{169}$	×	×	×	×	
d. $\sqrt{24}$					×
e. 4.763763763…				×	

6. a. 10 **b.** 2 **c.** 4.5 **d.** 25 **e.** 26

7. a. 11, commutative property of multiplication, associative property of multiplication, inverse property of multiplication, identity property of multiplication; **b.** 33, distributive property; **c.** 26, distributive property; **d.** $\frac{4}{9}$, commutative property of addition, associative property of addition, inverse property of addition, identity property of addition; **e.** 0, distributive property, inverse property of addition, identity property of addition

8.

	Constants	*Variables*
a. $2\pi r(r+h)$	$2, \pi$	r, h
b. $2(L+W)$	2	L, W
c. $4y^3+y$	4	y

9. a. 5 **b.** 11 **c.** 9 **d.** 26 **10. a.** 4 **b.** 11 **c.** $\frac{121}{3}\pi$ **d.** 1,728 **e.** 3 **11.** 1,152 cm^2

12. a. $-2y-2z$ or $-2(y+z)$ **b.** $\frac{2}{t}-1$ **c.** $3pq-4p+q$ **d.** $7r-2s+6$ **13.** $A = P(1+rt)$

Section 1.2

1. a. k^{15} **b.** $\left(\frac{2}{y}\right)^5$ **c.** t^{14} **2. a.** s^7 **b.** $(-3)^5$ **c.** $(ef^2)^2$

3. a. $(3y)^{24}$ **b.** t^{35} **c.** $(-g)^{16}$ **4. a.** 1 **b.** $\frac{1}{2}$ **c.** 1 **d.** 1

5. a. $\frac{1}{(-3t)^6}$ **b.** $\frac{1}{f^3}$ **c.** $\frac{2}{5k^3}$ **6. a.** $t^{-5} = \frac{1}{t^5}$ **b.** $\frac{1}{25}$

7. a. $g^{10}h^{15}$ **b.** $125t^3$ **c.** $-27y^{15}$ **d.** $\frac{1}{a^{18}b^{21}}$ **e.** $\frac{r^{12}}{s^8}$

8. a. $\frac{b^{15}}{c^3}$ **b.** $\frac{625}{u^{32}}$ **c.** $-\frac{1}{w^{105}}$ **d.** $\frac{q^{24}}{p^{32}}$ **e.** $\frac{1}{c^{20}d^{12}}$

9. a. $\frac{v^6}{8u^3}$ **b.** $\frac{1}{x^3}$ **c.** $\frac{e^4}{f^4}$ **d.** $\frac{27r}{s}$ **e.** 1 **f.** $\frac{16h^{10}}{49}$ **10. a.** $\$1.52 \times 10^5$ **b.** 7.158×10^9 **c.** $\$8.55 \times 10^{13}$ **d.** 3.34×10^{-9} **e.** 7.15×10^{-8} **11. a.** 703,000 **b.** $-816{,}000{,}000{,}000$ **c.** -0.00000000000039 **d.** 0.000008 **12. a.** -8.475×10^6 **b.** 8×10^{-8} **c.** 2.976×10^{13} **d.** -4.3×10^6 **e.** $\approx 1.24 \times 10^{15}$ **13.** Number of cells: 3×10^{13}; length of a cell: 8×10^{-6} m; total length: 2.4×10^8 m or 240,000,000 m

Section 1.3

1. a. 15 **b.** 3 **c.** 4 **d.** 17 **2.** $5|x||y|\sqrt{2yz}$ Notice the absolute value signs around x and y? That's because their value must be positive. **3.** $10|x|$ **4.** $\frac{x\sqrt{2}}{3y^2}$ We do not need the absolute value signs for y^2 because that term will always be nonnegative.

5. $b^4\sqrt{3ab}$ **6.** $13\sqrt{5}$ **7.** 0 **8.** $6\sqrt{6}$ **9.** $14-7\sqrt{3}$ **10. a.** -6 **b.** 6 **c.** $88\sqrt[3]{9}$ **11.** $(\sqrt{9})^5 = 3^5 = 243$ **12.** $x(5y)^{\frac{9}{2}}$ **13.** $28x^{\frac{23}{15}}$

Section 1.4

1. The degree is 6, the leading term is $-x^6$,and the leading coefficient is -1. **2.** $2x^3+7x^2-4x-3$ **3.** $-11x^3-x^2+7x-9$ **4.** $3x^4-10x^3-8x^2+21x+14$ **5.** $3x^2+16x-35$ **6.** $16x^2-8x+1$ **7.** $4x^2-49$ **8.** $6x^2+21xy-29x-7y+9$

Section 1.5

1. $(b^2-a)(x+6)$ **2.** $(x-6)(x-1)$ **3. a.** $(2x+3)(x+3)$ **b.** $(3x-1)(2x+1)$ **4.** $(7x-1)^2$ **5.** $(9y+10)(9y-10)$ **6.** $(6a+b)(36a^2-6ab+b^2)$ **7.** $(10x-1)(100x^2+10x+1)$ **8.** $(5a-1)^{-\frac{1}{4}}(17a-2)$

Section 1.6

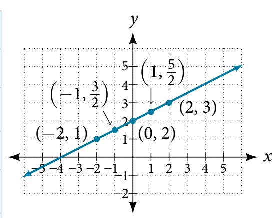
1. $\frac{1}{x+6}$ **2.** $\frac{(x+5)(x+6)}{(x+2)(x+4)}$ **3.** 1 **4.** $\frac{2(x-7)}{(x+5)(x-3)}$ **5.** $\frac{x^2-y^2}{xy^2}$

Chapter 2

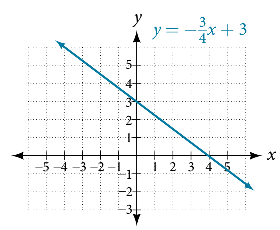
Section 2.1

1.

x	$y = \frac{1}{2}x + 2$	(x, y)
-2	$y = \frac{1}{2}(-2)+2 = 1$	$(-2, 1)$
-1	$y = \frac{1}{2}(-1)+2 = \frac{3}{2}$	$\left(-1, \frac{3}{2}\right)$
0	$y = \frac{1}{2}(0)+2 = 2$	$(0, 2)$
1	$y = \frac{1}{2}(1)+2 = \frac{5}{2}$	$\left(1, \frac{5}{2}\right)$
2	$y = \frac{1}{2}(2)+2 = 3$	$(2, 3)$



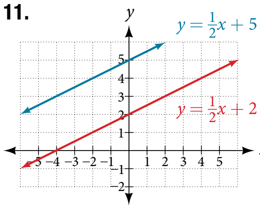
2. x-intercept is (4, 0); y-intercept is (0, 3).



3. $\sqrt{125} = 5\sqrt{5}$ **4.** $\left(-5, \frac{5}{2}\right)$

Section 2.2

1. $x = -5$ **2.** $x = -3$ **3.** $x = \frac{10}{3}$ **4.** $x = 1$
5. $x = -\frac{7}{17}$, excluded values are $-\frac{1}{2}$ and $-\frac{1}{3}$.
6. $x = \frac{1}{3}$ **7.** $m = -\frac{2}{3}$ **8.** $y = 4x - 3$ **9.** $x + 3y = 2$
10. Horizontal line: $y = 2$
11.



Parallel lines: equations are written in slope-intercept form.

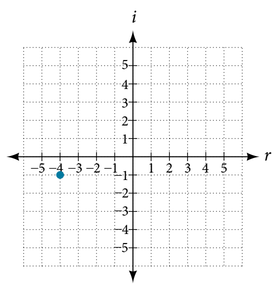
12. $y = 5x + 3$

Section 2.3

1. 11 and 25 **2.** $C = 2.5x + 3{,}650$ **3.** 45 mi/h
4. $L = 37$ cm, $W = 18$ cm **5.** 250 ft^2

Section 2.4

1. $\sqrt{-24} = 0 + 2i\sqrt{6}$ **2.**



3. $(3 - 4i) - (2 + 5i) = 1 - 9i$
4. $\frac{5}{2} - i$ **5.** $18 + i$
6. $-3 - 4i$ **7.** -1

Section 2.5

1. $(x - 6)(x + 1) = 0; x = 6, x = -1$ **2.** $(x - 7)(x + 3) = 0;$ $x = 7, x = -3$ **3.** $(x + 5)(x - 5) = 0; x = -5, x = 5$
4. $(3x + 2)(4x + 1) = 0; x = -\frac{2}{3}, x = -\frac{1}{4}$ **5.** $x = 0, x = -10, x = -1$ **6.** $x = 4 \pm \sqrt{5}$ **7.** $x = 3 \pm \sqrt{22}$ **8.** $x = -\frac{2}{3}$, $x = \frac{1}{3}$ **9.** 5 units

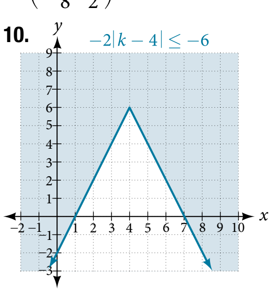
Section 2.6

1. $\frac{1}{4}$ **2.** 25 **3.** $\{-1\}$ **4.** $x = 0, x = \frac{1}{2}, x = -\frac{1}{2}$
5. $x = 1$; extraneous solution: $-\frac{2}{9}$ **6.** $x = -2$; extraneous solution: -1 **7.** $x = -1, x = \frac{3}{2}$ **8.** $x = -3, 3, -i, i$
9. $x = 2, x = 12$ **10.** $x = -1$, 0 is not a solution.

Section 2.7

1. $[-3, 5]$ **2.** $(-\infty, -2) \cup [3, \infty)$ **3.** $x < 1$ **4.** $x \geq -5$
5. $(2, \infty)$ **6.** $\left[-\frac{3}{14}, \infty\right)$ **7.** $6 < x \leq 9$ or $(6, 9]$
8. $\left(-\frac{1}{8}, \frac{1}{2}\right)$ **9.** $|x - 2| \leq 3$

10.



$k \leq 1$ or $k \geq 7$; in interval notation, this would be $(-\infty, 1] \cup [7, \infty)$.

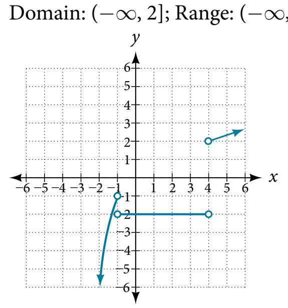
Chapter 3

Section 3.1

1. a. Yes **b.** Yes (Note: If two players had been tied for, say, 4th place, then the name would not have been a function of rank.)
2. $w = f(d)$ **3.** Yes **4.** $g(5) = 1$ **5.** $m = 8$
6. $y = f(x) = \frac{\sqrt[3]{x}}{2}$ **7.** $g(1) = 8$ **8.** $x = 0$ or $x = 2$
9. a. Yes, because each bank account has a single balance at any given time; **b.** No, because several bank account numbers may have the same balance; **c.** No, because the same output may correspond to more than one input. **10. a.** Yes, letter grade is a function of percent grade; **b.** No, it is not one-to-one. There are 100 different percent numbers we could get but only about five possible letter grades, so there cannot be only one percent number that corresponds to each letter grade. **11.** Yes
12. No, because it does not pass the horizontal line test.

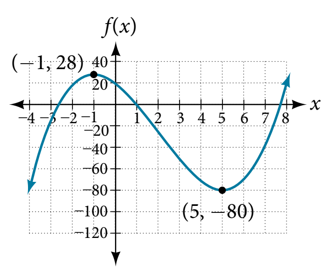
Section 3.2

1. $\{-5, 0, 5, 10, 15\}$ **2.** $(-\infty, \infty)$ **3.** $\left(-\infty, \frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$
4. $\left[-\frac{5}{2}, \infty\right)$ **5. a.** Values that are less than or equal to -2, or values that are greater than or equal to -1 and less than 3;
b. $\{x | x \leq -2 \text{ or } -1 \leq x < 3\}$ **c.** $(-\infty, -2] \cup [-1, 3)$
6. Domain $= [1950, 2002]$; Range $= [47{,}000{,}000, 89{,}000{,}000]$
7. Domain: $(-\infty, 2]$; Range: $(-\infty, 0]$
8.



Section 3.3

1. $\frac{\$2.84 - \$2.31}{5 \text{ years}} = \frac{\$0.53}{5 \text{ years}} = \$0.106$ per year. **2.** $\frac{1}{2}$
3. $a + 7$ **4.** The local maximum appears to occur at $(-1, 28)$, and the local minimum occurs at $(5, -80)$. The function is increasing on $(-\infty, -1) \cup (5, \infty)$ and decreasing on $(-1, 5)$.



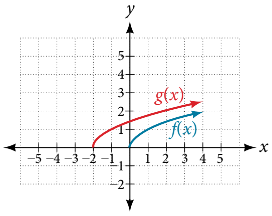
Section 3.4

1. a. $(fg)(x) = f(x)g(x) = (x - 1)(x^2 - 1) = x^3 - x^2 - x + 1$
$(f - g)(x) = f(x) - g(x) = (x - 1) - (x^2 - 1) = x - x^2$
b. No, the functions are not the same.

2. A gravitational force is still a force, so $a(G(r))$ makes sense as the acceleration of a planet at a distance r from the Sun (due to gravity), but $G(a(F))$ does not make sense.
3. $f(g(1)) = f(3) = 3$ and $g(f(4)) = g(1) = 3$ **4.** $g(f(2)) = g(5) = 3$
5. a. 8; **b.** 20 **6.** $[-4, 0) \cup (0, \infty)$
7. Possible answer: $g(x) = \sqrt{4 + x^2}$; $h(x) = \dfrac{4}{3 - x}$; $f = h \circ g$

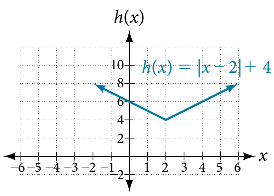
Section 3.5

1. $b(t) = h(t) + 10 = -4.9t^2 + 30t + 10$
2.



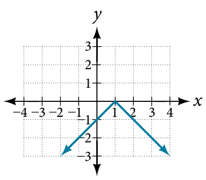
The graphs of $f(x)$ and $g(x)$ are shown here. The transformation is a horizontal shift. The function is shifted to the left by 2 units.

3.

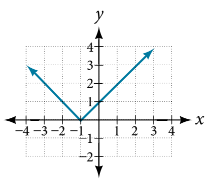


4. $g(x) = \dfrac{1}{x-1} + 1$

5. a.



b.



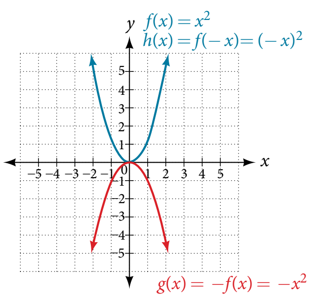
6. a. $g(x) = -f(x)$

x	−2	0	2	4
$g(x)$	−5	−10	−15	−20

b. $h(x) = f(-x)$

x	−2	0	2	4
$h(x)$	15	10	5	unknown

7.



Notice: $h(x) = f(-x)$ looks the same as $f(x)$.

8. Even

9.

x	2	4	6	8
$g(x)$	9	12	15	0

10. $g(x) = 3x - 2$

11. $g(x) = f\left(\frac{1}{3}x\right)$ so using the square root function we get $g(x) = \sqrt{\frac{1}{3}x}$

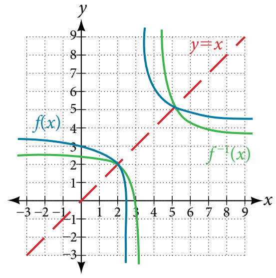
Section 3.6

1. Using the variable p for passing, $|p - 80| \le 20$
2. $f(x) = -|x + 2| + 3$ **3.** $x = -1$ or $x = 2$

Section 3.7

1. $h(2) = 6$ **2.** Yes **3.** Yes **4.** The domain of function f^{-1} is $(-\infty, -2)$ and the range of function f^{-1} is $(1, \infty)$.
5. a. $f(60) = 50$. In 60 minutes, 50 miles are traveled.
b. $f^{-1}(60) = 70$. To travel 60 miles, it will take 70 minutes.

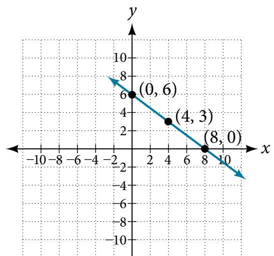
6. a. 3 **b.** 5.6 **7.** $x = 3y + 5$ **8.** $f^{-1}(x) = (2 - x)^2$; domain of f: $[0, \infty)$; domain of f^{-1}: $(-\infty, 2]$
9.



Chapter 4

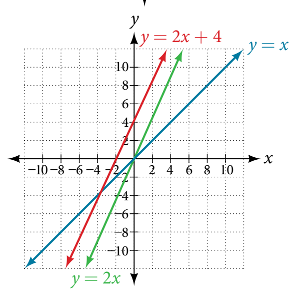
Section 4.1

1. $m = \dfrac{4-3}{0-2} = \dfrac{1}{-2} = -\dfrac{1}{2}$; decreasing because $m < 0$.
2. $m = \dfrac{1{,}868 - 1{,}442}{2{,}012 - 2{,}009} = \dfrac{426}{3} = 142$ people per year
3. $y = -7x + 3$ **4.** $H(x) = 0.5x + 12.5$
5.



6. Possible answers include $(-3, 7)$, $(-6, 9)$, or $(-9, 11)$

7.



8. $(16, 0)$
9. a. $f(x) = 2x$;
b. $g(x) = -\dfrac{1}{2}x$
10. $y = -\dfrac{1}{3}x + 6$

Section 4.2

1. $C(x) = 0.25x + 25{,}000$; The y-intercept is $(0, 25{,}000)$. If the company does not produce a single doughnut, they still incur a cost of \$25,000. **2. a.** 41,100 **b.** 2020 **3.** 21.15 miles

Section 4.3

1. 54° F **2.** 150.871 billion gallons; extrapolation

Chapter 5

Section 5.1

1. The path passes through the origin and has vertex at $(-4, 7)$, so $(h)x = -\frac{7}{16}(x + 4)^2 + 7$. To make the shot, $h(-7.5)$ would need to be about 4 but $h(-7.5) \approx 1.64$; he doesn't make it.
2. $g(x) = x^2 - 6x + 13$ in general form; $g(x) = (x - 3)^2 + 4$ in standard form

3. The domain is all real numbers. The range is $f(x) \geq \frac{8}{11}$, or $\left[\frac{8}{11}, \infty\right)$. **4.** y-intercept at (0, 13), No x-intercepts
5. 3 seconds; 256 feet; 7 seconds

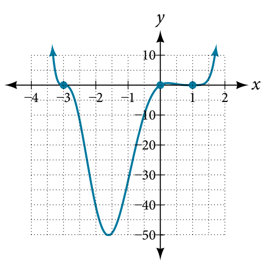
Section 5.2

1. $f(x)$ is a power function because it can be written as $f(x) = 8x^5$. The other functions are not power functions.
2. As x approaches positive or negative infinity, $f(x)$ decreases without bound: as $x \to \pm\infty, f(x) \to -\infty$ because of the negative coefficient. **3.** The degree is 6. The leading term is $-x^6$. The leading coefficient is -1. **4.** As $x \to \infty, f(x) \to -\infty$; as $x \to -\infty, f(x) \to -\infty$. It has the shape of an even degree power function with a negative coefficient. **5.** The leading term is $0.2x^3$, so it is a degree 3 polynomial. As x approaches positive infinity, $f(x)$ increases without bound; as x approaches negative infinity, $f(x)$ decreases without bound.
6. y-intercept (0, 0); x-intercepts (0, 0), $(-2, 0)$, and (5, 0)
7. There are at most 12 x-intercepts and at most 11 turning points.
8. The end behavior indicates an odd-degree polynomial function; there are 3 x-intercepts and 2 turning points, so the degree is odd and at least 3. Because of the end behavior, we know that the lead coefficient must be negative. **9.** The x-intercepts are (2, 0), $(-1, 0)$, and (5, 0), the y-intercept is (0, 2), and the graph has at most 2 turning points.

Section 5.3

1. y-intercept (0, 0); x-intercepts (0, 0), $(-5, 0)$, (2, 0), and (3, 0)
2. The graph has a zero of -5 with multiplicity 3, a zero of -1 with multiplicity 2, and a zero of 3 with multiplicity 4.

3.



4. Because f is a polynomial function and since $f(1)$ is negative and $f(2)$ is positive, there is at least one real zero between $x = 1$ and $x = 2$.

5. $f(x) = -\frac{1}{8}(x-2)^3(x+1)^2(x-4)$ **6.** The minimum occurs at approximately the point $(0, -6.5)$, and the maximum occurs at approximately the point (3.5, 7).

Section 5.4

1. $4x^2 - 8x + 15 - \frac{78}{4x+5}$ **2.** $3x^3 - 3x^2 + 21x - 150 + \frac{1{,}090}{x+7}$
3. $3x^2 - 4x + 1$

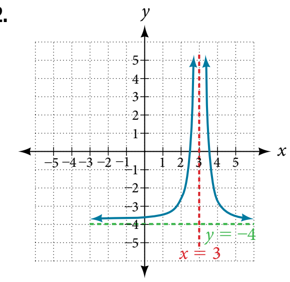
Section 5.5

1. $f(-3) = -412$ **2.** The zeros are 2, -2, and -4.
3. There are no rational zeros. **4.** The zeros are -4, $\frac{1}{2}$, and 1.
5. $f(x) = -\frac{1}{2}x^3 + \frac{5}{2}x^2 - 2x + 10$ **6.** There must be 4, 2, or 0 positive real roots and 0 negative real roots. The graph shows that there are 2 positive real zeros and 0 negative real zeros.
7. 3 meters by 4 meters by 7 meters

Section 5.6

1. End behavior: as $x \to \pm\infty$, $f(x) \to 0$; Local behavior: as $x \to 0$, $f(x) \to \infty$ (there are no x- or y-intercepts).

2.



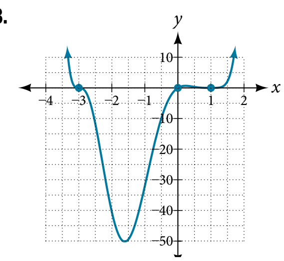
The function and the asymptotes are shifted 3 units right and 4 units down. As $x \to 3, f(x) \to \infty$, and as $x \to \pm\infty$, $f(x) \to -4$.
The function is $f(x) = \frac{1}{(x-3)^2} - 4$.

3. $\frac{12}{11}$ **4.** The domain is all real numbers except $x = 1$ and $x = 5$.
5. Removable discontinuity at $x = 5$. Vertical asymptotes: $x = 0$, $x = 1$. **6.** Vertical asymptotes at $x = 2$ and $x = -3$; horizontal asymptote at $y = 4$. **7.** For the transformed reciprocal squared function, we find the rational form.

$$\begin{aligned} f(x) &= \frac{1}{(x-3)^2} - 4 \\ &= \frac{1-4(x-3)^2}{(x-3)^2} \\ &= \frac{1-4(x^2-6x+9)}{(x-3)(x-3)} \\ &= \frac{-4x^2+24x-35}{x^2-6x+9} \end{aligned}$$

Because the numerator is the same degree as the denominator we know that as $x \to \pm\infty$, $f(x) \to -4$; so $y = -4$ is the horizontal asymptote. Next, we set the denominator equal to zero, and find that the vertical asymptote is $x = 3$, because as $x \to 3$, $f(x) \to \infty$. We then set the numerator equal to 0 and find the x-intercepts are at (2.5, 0) and (3.5, 0). Finally, we evaluate the function at 0 and find the y-intercept to be at $\left(0, -\frac{35}{9}\right)$.

8.



Horizontal asymptote at $y = \frac{1}{2}$. Vertical asymptotes at $x = 1$ and $x = 3$. y-intercept at $\left(0, \frac{4}{3}\right)$. x-intercepts at (2, 0) and $(-2, 0)$. $(-2, 0)$ is a zero with multiplicity 2, and the graph bounces off the x-axis at this point. (2, 0) is a single zero and the graph crosses the axis at this point.

Section 5.7

1. $f^{-1}(f(x)) = f^{-1}\left(\frac{x+5}{3}\right) = 3\left(\frac{x+5}{3}\right) - 5 = (x-5) + 5 = x$

and $f(f^{-1}(x)) = f(3x-5) = \frac{(3x-5)+5}{3} = \frac{3x}{3} = x$
2. $f^{-1}(x) = x^3 - 4$ **3.** $f^{-1}(x) = \sqrt{x-1}$
4. $f^{-1}(x) = \frac{x^2-3}{2}, x \geq 0$ **5.** $f^{-1}(x) = \frac{2x+3}{x-1}$

Section 5.8

1. $\frac{128}{3}$ **2.** $\frac{9}{2}$ **3.** $x = 20$

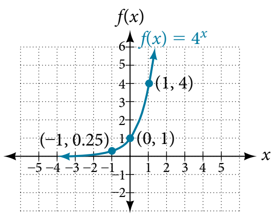
Chapter 6

Section 6.1

1. $g(x) = 0.875^x$ and $j(x) = 1095.6^{-2x}$ represent exponential functions. **2.** 5.5556 **3.** About 1.548 billion people; by the year 2031, India's population will exceed China's by about 0.001 billion, or 1 million people. **4.** (0, 129) and (2, 236); $N(t) = 129(1.3526)^t$ **5.** $f(x) = 2(1.5)^x$ **6.** $f(x) = \sqrt{2}(\sqrt{2})^x$; Answers may vary due to round-off error. the answer should be very close to $1.4142(1.4142)^x$. **7.** $y \approx 12 \cdot 1.85^x$ **8.** About \$3,644,675.88 **9.** \$13,693 **10.** $e^{-0.5} \approx 0.60653$ **11.** \$3,659,823.44 **12.** 3.77E-26(This is calculator notation for the number written as 3.77×10^{-26} in scientific notation. While the output of an exponential function is never zero, this number is so close to zero that for all practical purposes we can accept zero as the answer.)

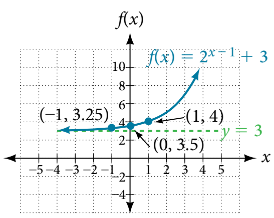
Section 6.2

1.



The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.

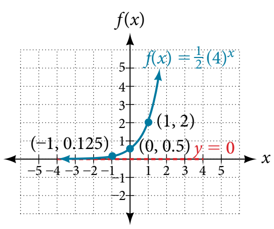
2.



The domain is $(-\infty, \infty)$; the range is $(3, \infty)$; the horizontal asymptote is $y = 3$.

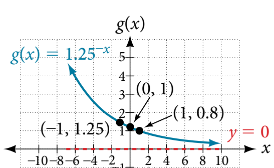
3. $x \approx -1.608$

4.



The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.

5.



The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.

6. $f(x) = -\frac{1}{3}e^x - 2$; the domain is $(-\infty, \infty)$; the range is $(-\infty, 2)$; the horizontal asymptote is $y = 2$.

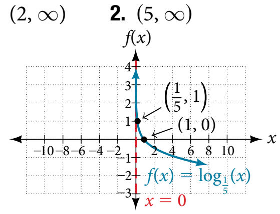
Section 6.3

1. a. $\log_{10}(1{,}000{,}000) = 6$ is equivalent to $10^6 = 1{,}000{,}000$ **b.** $\log_5(25) = 2$ is equivalent to $5^2 = 25$ **2. a.** $3^2 = 9$ is equivalent to $\log_3(9) = 2$ **b.** $5^3 = 125$ is equivalent to $\log_5(125) = 3$ **c.** $2^{-1} = \frac{1}{2}$ is equivalent to $\log_2\left(\frac{1}{2}\right) = -1$ **3.** $\log_{121}(11) = \frac{1}{2}$ (recalling that $\sqrt{121} = 121^{\frac{1}{2}} = 11$) **4.** $\log_2\left(\frac{1}{32}\right) = -5$ **5.** $\log(1{,}000{,}000) = 6$ **6.** $\log(123) \approx 2.0899$ **7.** The difference in magnitudes was about 3.929. **8.** It is not possible to take the logarithm of a negative number in the set of real numbers.

Section 6.4

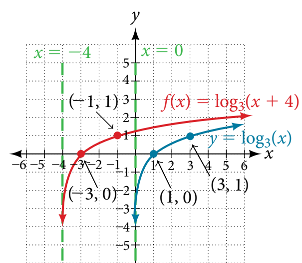
1. $(2, \infty)$ **2.** $(5, \infty)$

3.



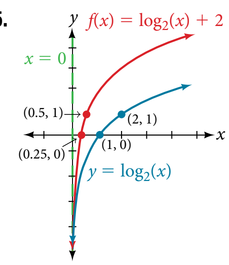
The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

4.



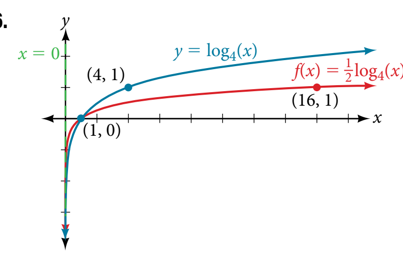
The domain is $(-4, \infty)$, the range $(-\infty, \infty)$, and the asymptote $x = -4$.

5.



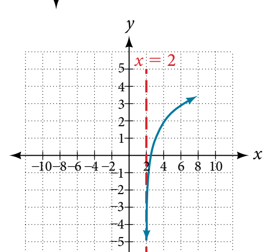
The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

6.



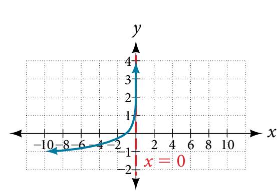
The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

7.



The domain is $(2, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 2$.

8.



The domain is $(-\infty, 0)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$. **9.** $x \approx 3.049$ **10.** $x = 1$ **11.** $f(x) = 2\ln(x + 3) - 1$

Section 6.5

1. $\log_b(2) + \log_b(2) + \log_b(2) + \log_b(k) = 3\log_b(2) + \log_b(k)$
2. $\log_3(x+3) - \log_3(x-1) - \log_3(x-2)$ **3.** $2\ln(x)$
4. $-2\ln(x)$ **5.** $\log_3(16)$ **6.** $2\log(x) + 3\log(y) - 4\log(z)$
7. $\frac{2}{3}\ln(x)$ **8.** $\frac{1}{2}\ln(x-1) + \ln(2x+1) - \ln(x+3) - \ln(x-3)$
9. $\log\left(\frac{3 \cdot 5}{4 \cdot 6}\right)$; can also be written $\log\left(\frac{5}{8}\right)$ by reducing the fraction to lowest terms. **10.** $\log\left(\frac{5(x-1)^3\sqrt{x}}{(7x-1)}\right)$
11. $\log\frac{x^{12}(x+5)^4}{(2x+3)^4}$; this answer could also be written $\log\left(\frac{x^3(x+5)}{(2x+3)}\right)^4$.
12. The pH increases by about 0.301. **13.** $\frac{\ln(8)}{\ln(0.5)}$
14. $\frac{\ln(100)}{\ln(5)} \approx \frac{4.6051}{1.6094} = 2.861$

Section 6.6

1. $x = -2$ **2.** $x = -1$ **3.** $x = \frac{1}{2}$
4. The equation has no solution. **5.** $x = \frac{\ln(3)}{\ln\left(\frac{2}{3}\right)}$
6. $t = 2\ln\left(\frac{11}{3}\right)$ or $\ln\left(\frac{11}{3}\right)^2$
7. $t = \ln\left(\frac{1}{\sqrt{2}}\right) = -\frac{1}{2}\ln(2)$ **8.** $x = \ln(2)$ **9.** $x = e^4$
10. $x = e^5 - 1$ **11.** $x \approx 9.97$ **12.** $x = 1$ or $x = -1$
13. $t = 703{,}800{,}000 \times \frac{\ln(0.8)}{\ln(0.5)}$ years $\approx 226{,}572{,}993$ years.

Section 6.7

1. $f(t) = A_0 e^{-0.0000000087t}$ **2.** Less than 230 years; 229.3157 to be exact
3. $f(t) = A_0 e^{\left(\frac{\ln(2)}{3}\right)t}$ **4.** 6.026 hours **5.** 895 cases on day 15
6. Exponential. $y = 2e^{0.5x}$ **7.** $y = 3e^{(\ln 0.5)x}$

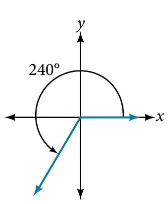
Section 6.8

1. a. The exponential regression model that fits these data is $y = 522.88585984(1.19645256)^x$. **b.** If spending continues at this rate, the graduate's credit card debt will be \$4,499.38 after one year.
2. a. The logarithmic regression model that fits these data is $y = 141.91242949 + 10.45366573\ln(x)$ **b.** If sales continue at this rate, about 171,000 games will be sold in the year 2015.
3. a. The logistic regression model that fits these data is $y = \frac{25.65665979}{1 + 6.113686306e^{-0.3852149008x}}$.
b. If the population continues to grow at this rate, there will be about 25,634 seals in 2020. **c.** To the nearest whole number, the carrying capacity is 25,657.

Chapter 7

Section 7.1

1.



2. $\frac{3\pi}{2}$ **3.** $-135°$ **4.** $\frac{7\pi}{10}$
5. $\alpha = 150°$ **6.** $\beta = 60°$ **7.** $\frac{7\pi}{6}$
8. $\frac{215\pi}{18} = 37.525$ units **9.** 1.88
10. $-\frac{3\pi}{2}$ rad/s
11. 1,655 kilometers per hour

Section 7.2

1. $\frac{7}{25}$
2. $\sin(t) = \frac{33}{65}$ $\cos(t) = \frac{56}{65}$ $\tan(t) = \frac{33}{56}$
$\sec(t) = \frac{65}{56}$ $\csc(t) = \frac{65}{33}$ $\cot(t) = \frac{56}{33}$
3. $\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$ $\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$ $\tan\left(\frac{\pi}{4}\right) = 1$
$\sec\left(\frac{\pi}{4}\right) = \sqrt{2}$ $\csc\left(\frac{\pi}{4}\right) = \sqrt{2}$ $\cot\left(\frac{\pi}{4}\right) = 1$
4. 2 **5.** Adjacent = 10; opposite = $10\sqrt{3}$; missing angle is $\frac{\pi}{6}$.
6. About 52 ft.

Section 7.3

1. $\cos(t) = -\frac{\sqrt{2}}{2}$, $\sin(t) = \frac{\sqrt{2}}{2}$ **2.** $\cos(\pi) = -1$, $\sin(\pi) = 0$
3. $\sin(t) = -\frac{7}{25}$ **4.** Approximately 0.866025403 **5.** $\frac{\pi}{3}$
6. a. $\cos(315°) = \frac{\sqrt{2}}{2}$, $\sin(315°) = -\frac{\sqrt{2}}{2}$
b. $\cos\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$, $\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$ **7.** $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

Section 7.4

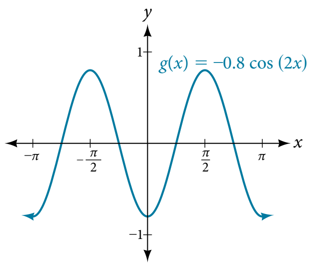
1. $\sin t = -\frac{\sqrt{2}}{2}$ $\cos t = \frac{\sqrt{2}}{2}$ $\tan t = -1$
$\sec t = \sqrt{2}$, $\csc t = -\sqrt{2}$ $\cot t = -1$
2. $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ $\cos\frac{\pi}{3} = \frac{1}{2}$ $\tan\frac{\pi}{3} = \sqrt{3}$
$\sec\frac{\pi}{3} = 2$ $\csc\frac{\pi}{3} = \frac{2\sqrt{3}}{3}$ $\cot\frac{\pi}{3} = \frac{\sqrt{3}}{3}$
3. $\sin\left(-\frac{7\pi}{4}\right) = \frac{\sqrt{2}}{2}$ $\cos\left(-\frac{7\pi}{4}\right) = \frac{\sqrt{2}}{2}$ $\tan\left(-\frac{7\pi}{4}\right) = 1$
$\sec\left(-\frac{7\pi}{4}\right) = \sqrt{2}$ $\csc\left(-\frac{7\pi}{4}\right) = \sqrt{2}$ $\cot\left(-\frac{7\pi}{4}\right) = 1$
4. $-\sqrt{3}$ **5.** -2 **6.** $\sin t$
7. $\cot t = -\frac{8}{17}$ $\sin t = \frac{15}{17}$ $\tan t = -\frac{15}{8}$
$\csc t = \frac{17}{15}$ $\cot t = -\frac{8}{15}$
8. $\sin t = -1$ $\cos t = 0$ $\tan t =$ Undefined
$\sec t =$ Undefined $\csc t = -1$ $\cot t = 0$
9. $\sec t = \sqrt{2}$ $\csc t = \sqrt{2}$ $\tan t = 1$
$\cot t = 1$
10. ≈ -2.414

Chapter 8

Section 8.1

1. 6π **2.** $\frac{1}{2}$ compressed **3.** $\frac{\pi}{2}$; right
4. 2 units up **5.** Midline: $y = 0$; Amplitude: $|A| = \frac{1}{2}$; Period: $P = \frac{2\pi}{|B|} = 6\pi$; Phase shift: $\frac{C}{B} = \pi$ **6.** $f(x) = \sin(x) + 2$
7. Two possibilities: $y = 4\sin\left(\frac{\pi}{5}x - \frac{\pi}{5}\right) + 4$ or $y = -4\sin\left(\frac{\pi}{5}x + \frac{4\pi}{5}\right) + 4$

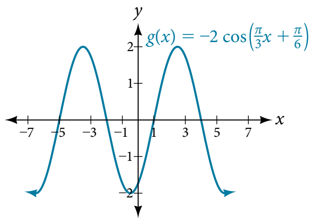
8.



Midline: $y = 0$;
Amplitude: $|A| = 0.8$;
Period: $P = \dfrac{2\pi}{|B|} = \pi$;
Phase shift: $\dfrac{C}{B} = 0$ or

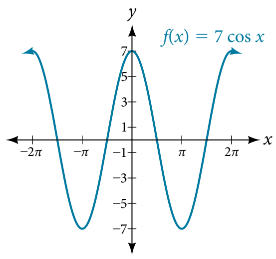
none

9.



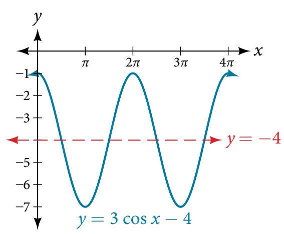
Midline: $y = 0$;
Amplitude: $|A| = 2$;
Period: $P = \dfrac{2\pi}{|B|} = 6$;
Phase shift: $\dfrac{C}{B} = -\dfrac{1}{2}$

10.



7

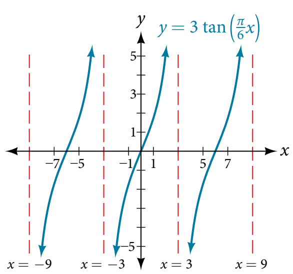
11.



$3\cos(x) - 4$

Section 8.2

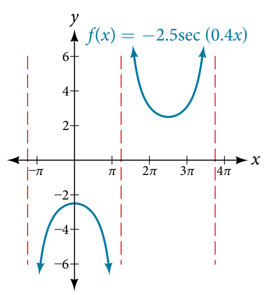
1.



2. It would be reflected across the line $y = -1$, becoming an increasing function.

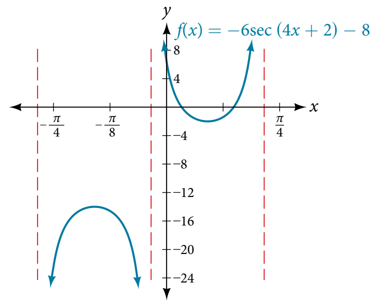
3. $g(x) = 4\tan(2x)$

4.

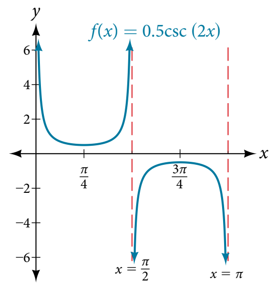


This is a vertical reflection of the preceding graph because A is negative.

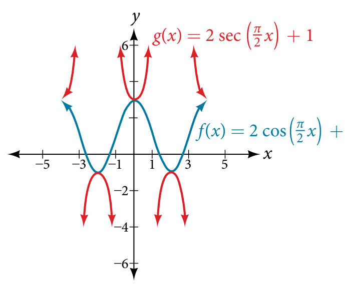
5.



6.



7.



Section 8.3

1. $\arccos(0.8776) \approx 0.5$ **2.** **a.** $-\dfrac{\pi}{2}$ **b.** $-\dfrac{\pi}{4}$ **c.** π **d.** $\dfrac{\pi}{3}$

3. 1.9823 or 113.578° **4.** $\sin^{-1}(0.6) = 36.87° = 0.6435$ radians

5. $\dfrac{\pi}{8}; \dfrac{2\pi}{9}$ **6.** $\dfrac{3\pi}{4}$ **7.** $\dfrac{12}{13}$ **8.** $\dfrac{4\sqrt{2}}{9}$ **9.** $\dfrac{4x}{\sqrt{16x^2 + 1}}$

Chapter 9

Section 9.1

1.
$$\begin{aligned}\csc\theta\cos\theta\tan\theta &= \left(\frac{1}{\sin\theta}\right)\cos\theta\left(\frac{\sin\theta}{\cos\theta}\right)\\ &= \frac{\cos\theta}{\sin\theta}\left(\frac{\sin\theta}{\cos\theta}\right)\\ &= \frac{\sin\theta\cos\theta}{\sin\theta\cos\theta}\\ &= 1\end{aligned}$$

2.
$$\frac{\cot\theta}{\csc\theta} = \frac{\frac{\cos\theta}{\sin\theta}}{\frac{1}{\sin\theta}} = \frac{\cos\theta}{\sin\theta}\cdot\frac{\sin\theta}{1} = \cos\theta$$

3.
$$\frac{\sin^2\theta - 1}{\tan\theta\sin\theta - \tan\theta} = \frac{(\sin\theta + 1)(\sin\theta - 1)}{\tan\theta(\sin\theta - 1)} = \frac{\sin\theta + 1}{\tan\theta}$$

4. This is a difference of squares formula:
$25 - 9\sin^2\theta = (5 - 3\sin\theta)(5 + 3\sin\theta)$.

5. $$\frac{\cos\theta}{1+\sin\theta}\left(\frac{1-\sin\theta}{1-\sin\theta}\right)=\frac{\cos\theta(1-\sin\theta)}{1-\sin^2\theta}$$
$$=\frac{\cos\theta(1-\sin\theta)}{\cos^2\theta}$$
$$=\frac{1-\sin\theta}{\cos\theta}$$

Section 9.2

1. $\frac{\sqrt{2}+\sqrt{6}}{4}$ **2.** $\frac{\sqrt{2}-\sqrt{6}}{4}$ **3.** $\frac{1-\sqrt{3}}{1+\sqrt{3}}$ **4.** $\cos\left(\frac{5\pi}{14}\right)$

5. $$\tan(\pi-\theta)=\frac{\tan(\pi)-\tan\theta}{1+\tan(\pi)\tan\theta}$$
$$=\frac{0-\tan\theta}{1+0\cdot\tan\theta}$$
$$=-\tan\theta$$

Section 9.3

1. $\cos(2\alpha)=\frac{7}{32}$

2. $\cos^4\theta-\sin^4\theta=(\cos^2\theta+\sin^2\theta)(\cos^2\theta-\sin^2\theta)=\cos(2\theta)$

3. $\cos(2\theta)\cos\theta=(\cos^2\theta-\sin^2\theta)\cos\theta=\cos^3\theta-\cos\theta\sin^2\theta$

4. $10\cos^4 x=10(\cos^2 x)^2$

$$=10\left[\frac{1+\cos(2x)}{2}\right]^2 \quad \text{Substitute reduction formula for } \cos^2 x.$$
$$=\frac{10}{4}\left[1+2\cos(2x)+\cos^2(2x)\right]$$
$$=\frac{10}{4}+\frac{10}{2}\cos(2x)+\frac{10}{4}\left(\frac{1+\cos^2(2x)}{2}\right) \quad \text{Substitute reduction formula for } \cos^2 x.$$
$$=\frac{10}{4}+\frac{10}{2}\cos(2x)+\frac{10}{8}+\frac{10}{8}\cos(4x)$$
$$=\frac{30}{8}+5\cos(2x)+\frac{10}{8}\cos(4x)$$
$$=\frac{15}{4}+5\cos(2x)+\frac{5}{4}\cos(4x)$$

5. $-\frac{2}{\sqrt{5}}$

Section 9.4

1. $\frac{1}{2}(\cos 6\theta+\cos 2\theta)$ **2.** $\frac{1}{2}(\sin 2x+\sin 2y)$ **3.** $\frac{-2-\sqrt{3}}{4}$

4. $2\sin(2\theta)\cos(\theta)$

5. $$\tan\theta\cot\theta-\cos^2\theta=\left(\frac{\sin\theta}{\cos\theta}\right)\left(\frac{\cos\theta}{\sin\theta}\right)-\cos^2\theta$$
$$=1-\cos^2\theta$$
$$=\sin^2\theta$$

Section 9.5

1. $x=\frac{7\pi}{6},\frac{11\pi}{6}$ **2.** $\frac{\pi}{3}\pm\pi k$ **3.** $\theta\approx 1.7722\pm 2\pi k$ and $\theta\approx 4.5110\pm 2\pi k$ **4.** $\cos\theta=-1, \theta=\pi$ **5.** $\frac{\pi}{2},\frac{2\pi}{3},\frac{4\pi}{3},\frac{3\pi}{2}$

Chapter 10

Section 10.1

1. $\alpha=98°, a=34.6; \beta=39°, b=22; \gamma=43°, c=23.8$

2. Solution 1 $\alpha=80°, a=120; \beta\approx 83.2°, b=121; \gamma\approx 16.8°, c\approx 35.2$
Solution 2 $\alpha'=80°, a'=120; \beta'\approx 96.8°, b'=121; \gamma'\approx 3.2°, c'\approx 6.8$

3. $\beta\approx 5.7°, \gamma\approx 94.3°, c\approx 101.3$ **4.** Two

5. About 8.2 square feet **6.** 161.9 yd.

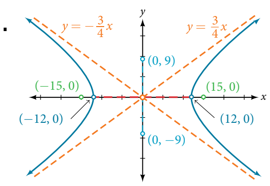
Section 10.2

1. $a\approx 14.9, \beta\approx 23.8°, \gamma\approx 126.2°$ **2.** $\alpha\approx 27.7°, \beta\approx 40.5°, \gamma\approx 111.8°$

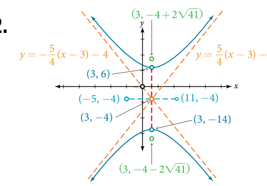
3. Area = 552 square feet **4.** About 8.15 square feet

Section 10.3

1.



2.

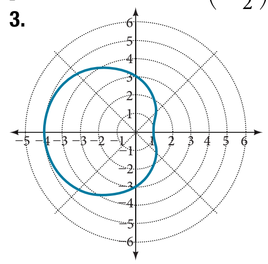


3. $(x,y)=\left(\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$ **4.** $r=\sqrt{3}$ **5.** $x^2+y^2=2y$ or, in the standard form for a circle, $x^2+(y-1)^2=1$

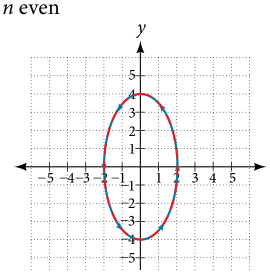
Section 10.4

1. The equation fails the symmetry test with respect to the line $\theta=\frac{\pi}{2}$ and with respect to the pole. It passes the polar axis symmetry test. **2.** Tests will reveal symmetry about the polar axis. The zero is $\left(\theta,\frac{\pi}{2}\right)$, and the maximum value is (3, 0).

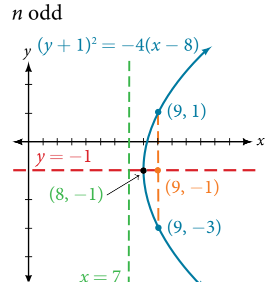
3.



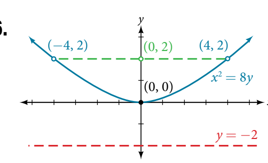
4. The graph is a rose curve, n even



5. The graph is a rose curve, n odd

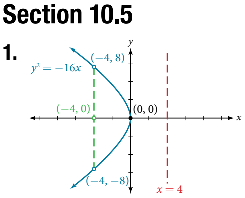


6.



Section 10.5

1.



2. 13 **3.** $|z|=\sqrt{50}=5\sqrt{2}$

4. $z=3\left(\cos\left(\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{2}\right)\right)$

5. $z=2\left(\cos\left(\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{6}\right)\right)$

6. $z=2\sqrt{3}-2i$

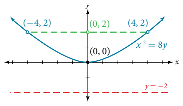
7. $z_1z_2=-4\sqrt{3}; \frac{z_1}{z_2}=-\frac{\sqrt{3}}{2}+\frac{3}{2}i$

8. $z_0=2(\cos(30°)+i\sin(30°)), z_1=2(\cos(120°)+i\sin(120°))$
$z_2=2(\cos(210°)+i\sin(210°)), z_3=2(\cos(300°)+i\sin(300°))$

Section 10.6

1.

t	$x(t)$	$y(t)$
-1	-4	2
0	-3	4
1	-2	6
2	-1	8

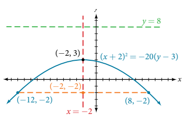


2. $x(t) = t^3 - 2t,\ y(t) = t$ **3.** $y = 5 - \sqrt{\frac{1}{2}x - 3}$

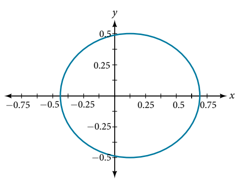
4. $y = \ln(\sqrt{x})$ **5.** $\frac{x^2}{4} + \frac{y^2}{9} = 1$ **6.** $y = x^2$

Section 10.7

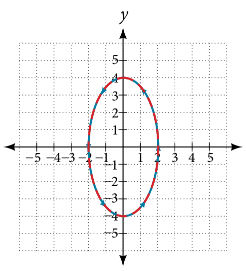
1.



2.



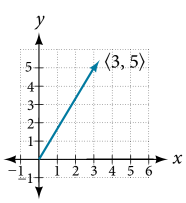
3.



The graph of the parametric equations is in red and the graph of the rectangular equation is drawn in blue dots on top of the parametric equations.

Section 10.8

1.



2. $3u = \langle 15, 12\rangle$

3. $u = 8i - 11j$

4. $v = \sqrt{34}\cos(59°)i + \sqrt{34}\sin(59°)j$

Magnitude $= \sqrt{34}$

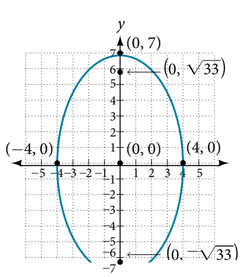
$\theta = \tan^{-1}\left(\frac{5}{3}\right) = 59.04°$

Chapter 11

Section 11.1

1. Not a solution

2. The solution to the system is the ordered pair$(-5, 3)$.



3. $(-2, -5)$ **4.** $(-6, -2)$ **5.** $(10, -4)$ **6.** No solution. It is an inconsistent system.

7. The system is dependent so there are infinite solutions of the form $(x, 2x + 5)$. **8.** 700 children, 950 adults

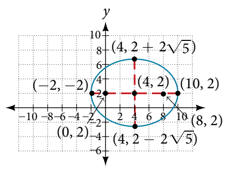
Section 11.2

1. $(1, -1, 1)$ **2.** No solution **3.** Infinite number of solutions of the form $(x, 4x - 11, -5x + 18)$

Section 11.3

1. $\left(-\frac{1}{2}, \frac{1}{2}\right)$ and $(2, 8)$ **2.** $(-1, 3)$ **3.** $\{(1, 3),(1, -3),(-1, 3),(-1, -3)\}$

4.



Section 11.4

1. $\frac{3}{x-3} - \frac{2}{x-2}$ **2.** $\frac{6}{x-1} - \frac{5}{(x-1)^2}$ **3.** $\frac{3}{x-1} + \frac{2x-4}{x^2+1}$

4. $\frac{x-2}{x^2-2x+3} + \frac{2x+1}{(x^2-2x+3)^2}$

Section 11.5

1. $A + B = \begin{bmatrix} 2 & 6 \\ 1 & 0 \\ 1 & -3 \end{bmatrix} + \begin{bmatrix} 3 & -2 \\ 1 & 5 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} 2+3 & 6+(-2) \\ 1+1 & 0+5 \\ 1+(-4) & -3+3 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 2 & 5 \\ -3 & 0 \end{bmatrix}$

2. $-2B = \begin{bmatrix} -8 & -2 \\ -6 & -4 \end{bmatrix}$

Section 11.6

1. $\left[\begin{array}{rr|r} 4 & -3 & 11 \\ 3 & 2 & 4 \end{array}\right]$ **2.** $\begin{aligned} x - y + z &= 5 \\ 2x - y + 3z &= 1 \\ y + z &= -9 \end{aligned}$ **3.** $(2, 1)$

4. $\left[\begin{array}{rrr|r} 1 & -\frac{5}{2} & \frac{5}{2} & \frac{17}{2} \\ 0 & 1 & 5 & 9 \\ 0 & 0 & 1 & 2 \end{array}\right]$ **5.** $(1, 1, 1)$

6. \$150,000 at 7%, \$750,000 at 8%, \$600,000 at 10%

Section 11.7

1. $AB = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1(-3)+4(1) & 1(-4)+4(1) \\ -1(-3)+-3(1) & -1(-4)+-3(1) \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$BA = \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix} = \begin{bmatrix} -3(1)+-4(-1) & -3(4)+-4(-3) \\ 1(1)+1(-1) & 1(4)+1(-3) \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

2. $A^{-1} = \begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix}$ **3.** $A^{-1} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix}$ **4.** $X = \begin{bmatrix} 4 \\ 38 \\ 58 \end{bmatrix}$

Section 11.8

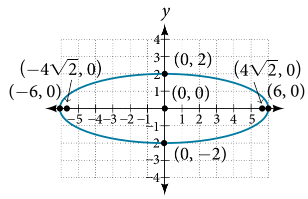
1. $(3, -7)$ **2.** -10 **3.** $\left(-2, \frac{3}{5}, \frac{12}{5}\right)$

Chapter 12

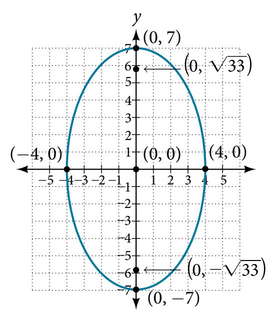
Section 12.1

1. $x^2 + \frac{y^2}{16} = 1$ **2.** $\frac{(x-1)^2}{16} + \frac{(y-3)^2}{4} = 1$

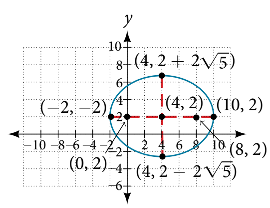
3. Center: (0, 0); vertices: $(\pm 6, 0)$; co-vertices: $(0, \pm 2)$; foci: $(\pm 4\sqrt{2}, 0)$



4. Standard form: $\frac{x^2}{16} + \frac{y^2}{49} = 1$; center: (0, 0); vertices: $(0, \pm 7)$; co-vertices: $(\pm 4, 0)$; foci: $(0, \pm\sqrt{33})$



5. Center: (4, 2); vertices: (−2, 2) and (10, 2); co-vertices: $(4, 2 - 2\sqrt{5})$ and $(4, 2 + 2\sqrt{5})$; foci: (0, 2) and (8, 2)



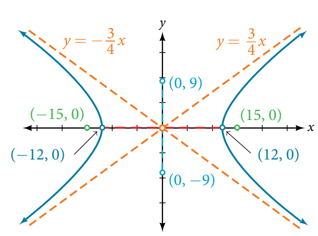
6. $\frac{(x-3)^2}{4} + \frac{(y+1)^2}{16} = 1$; center: (3, −1); vertices: (3, −5) and (3, 3); co-vertices: (1, −1) and (5, −1); foci: $(3, -1 - 2\sqrt{3})$ and $(3, -1 + 2\sqrt{3})$ **7. a.** $\frac{x^2}{57{,}600} + \frac{y^2}{25{,}600} = 1$; **b.** The people are standing 358 feet apart.

Section 12.2

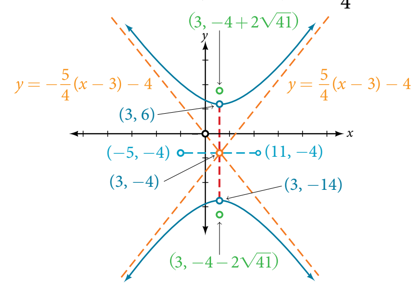
1. Vertices: $(\pm 3, 0)$; Foci: $(\pm\sqrt{34}, 0)$ **2.** $\frac{y^2}{4} - \frac{x^2}{16} = 1$

3. $\frac{(y-3)^2}{25} + \frac{(x-1)^2}{144} = 1$

4. Vertices: $(\pm 12, 0)$; co-vertices: $(0, \pm 9)$; foci: $(\pm 15, 0)$; asymptotes: $y = \pm\frac{3}{4}x$



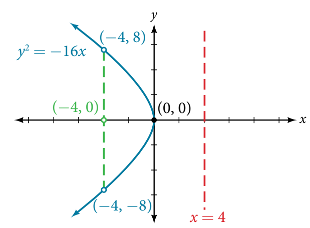
5. Center: (3, −4); vertices: (3, −14) and (3, 6); co-vertices: (−5, −4) and (11, −4); foci: $(3, -4 - 2\sqrt{41})$ and $(3, -4 + 2\sqrt{41})$; Asyaptotes: $y = \pm\frac{5}{4}(x-3) - 4$



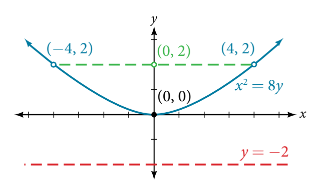
6. The sides of the tower can be modeled by the hyperbolic equation. $\frac{x^2}{400} - \frac{y^2}{3600} = 1$ or $\frac{x^2}{20^2} - \frac{y^2}{60^2} = 1$.

Section 12.3

1. Focus: (−4, 0); directrix: $x = 4$; endpoints of the latus rectum: $(-4, \pm 8)$

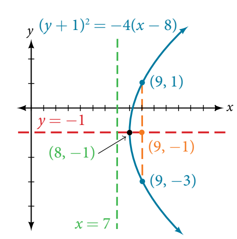


2. Focus: (0, 2); sirectrix: $y = -2$; endpoints of the latus rectum: $(\pm 4, 2)$

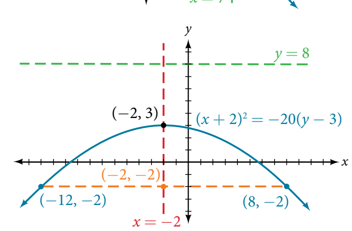


3. $x^2 = 14y$

4. Vertex: (8, −1); axis of symmetry: $y = -1$; focus: (9, −1); directrix: $x = 7$; endpoints of the latus rectum: (9, −3) and (9, 1).



5. Vertex: (−2, 3); axis of symmetry: $x = -2$; focus: (−2, −2); directrix: $y = 8$; endpoints of the latus rectum: (−12, −2) and (8, −2).



6. a. $y^2 = 1{,}280x$ **b.** The depth of the cooker is 500 mm.

Section 12.4

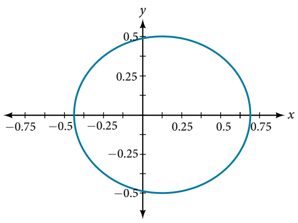
1. a. Hyperbola **b.** Ellipse **2.** $\frac{x'^2}{4} + \frac{y'^2}{1} = 1$

3. a. Hyperbola **b.** Ellipse

Section 12.5

1. Ellipse; $e = \frac{1}{3}$; $x = -2$

2.



3. $r = \frac{1}{1 - \cos\theta}$
4. $4 - 8x + 3x^2 - y^2 = 0$

Chapter 13

Section 13.1

1. The first five terms are $\{1, 6, 11, 16, 21\}$. **2.** The first five terms are $\left\{-2, 2, -\frac{3}{2}, 1, -\frac{5}{8}\right\}$. **3.** The first six terms are $\{2, 5, 54, 10, 250, 15\}$. **4.** $a_n = (-1)^{n+1}9^n$ **5.** $a_n = -\frac{3n}{4n}$ **6.** $a_n = e^{n-3}$ **7.** $\{2, 5, 11, 23, 47\}$ **8.** $\left\{0, 1, 1, 1, 2, 3, \frac{5}{2}, \frac{17}{6}\right\}$ **9.** The first five terms are $\left\{1, \frac{3}{2}, 4, 15, 72\right\}$.

Section 13.2

1. The sequence is arithmetic. The common difference is -2.
2. The sequence is not arithmetic because $3 - 1 \neq 6 - 3$.
3. $\{1, 6, 11, 16, 21\}$ **4.** $a_2 = 2$ **5.** $a_1 = 25$; $a_n = a_{n-1} + 12$, for $n \geq 2$ **6.** $a_n = 53 - 3n$ **7.** There are 11 terms in the sequence. **8.** The formula is $T_n = 10 + 4n$, and it will take her 42 minutes.

Section 13.3

1. The sequence is not geometric because $\frac{10}{5} \neq \frac{15}{10}$.
2. The sequence is geometric. The common ratio is $\frac{1}{5}$.
3. $\left\{18, 6, 2, \frac{2}{3}, \frac{2}{9}\right\}$ **4.** $a_1 = 2$; $a_n = \frac{2}{3}a_{n-1}$ for $n \geq 2$
5. $a_6 = 16{,}384$ **6.** $a_n = -(-3)^{n-1}$ **7. a.** $P_n = 293 \cdot 1.026a^n$
b. The number of hits will be about 333.

Section 13.4

1. 38 **2.** 26.4 **3.** 328 **4.** −280 **5.** \$2,025 **6.** ≈ 2,000.00 **7.** 9,840 **8.** \$275,513.31 **9.** The sum is defined. It is geometric. **10.** The sum of the infinite series is defined. **11.** The sum of the infinite series is defined. **12.** 3 **13.** The series is not geometric. **14.** $-\frac{3}{11}$ **15.** \$92,408.18

Section 13.5

1. 7 **2.** There are 60 possible breakfast specials. **3.** 120 **4.** 60 **5.** 12 **6.** $P(7, 7) = 5{,}040$ **7.** $P(7, 5) = 2{,}520$ **8.** $C(10, 3) = 120$ **9.** 64 sundaes **10.** 840

Section 13.6

1. a. 35 **b.** 330 **2. a.** $x^5 - 5x^4y + 10x^3y^2 - 10x^2y^3 + 5xy^4 - y^5$
b. $8x^3 + 60x^2y + 150xy^2 + 125y^3$ **3.** $-10{,}206x^4y^5$

Section 13.7

1.

Outcome	Probability
Roll of 1	
Roll of 2	
Roll of 3	
Roll of 4	
Roll of 5	
Roll of 6	

2. $\frac{2}{3}$ **3.** $\frac{7}{13}$ **4.** $\frac{2}{13}$
5. $\frac{5}{6}$ **6. a.** $\frac{1}{91}$ **b.** $\frac{5}{91}$ **c.** $\frac{86}{91}$

Odd Answers

CHAPTER 1

Section 1.1

1. Irrational number. The square root of two does not terminate, and it does not repeat a pattern. It cannot be written as a quotient of two integers, so it is irrational. **3.** The Associative Properties state that the sum or product of multiple numbers can be grouped differently without affecting the result. This is because the same operation is performed (either addition or subtraction), so the terms can be re-ordered. **5.** -6 **7.** -2 **9.** -9 **11.** 9 **13.** -2 **15.** 4 **17.** 0 **19.** 9 **21.** 25 **23.** -6 **25.** 17 **27.** 4 **29.** -4 **31.** -6 **33.** ± 1 **35.** 2 **37.** 2 **39.** $-14y - 11$ **41.** $-4b + 1$ **43.** $43z - 3$ **45.** $9y + 45$ **47.** $-6b + 6$ **49.** $\frac{16x}{3}$ **51.** $9x$ **53.** $\frac{1}{2}(40 - 10) + 5$ **55.** Irrational number **57.** $g + 400 - 2(600) = 1200$ **59.** Inverse property of addition **61.** 68.4 **63.** True **65.** Irrational **67.** Rational

Section 1.2

1. No, the two expressions are not the same. An exponent tells how many times you multiply the base. So 2^3 is the same as $2 \times 2 \times 2$, which is 8. 3^2 is the same as 3×3, which is 9.
3. It is a method of writing very small and very large numbers.
5. 81 **7.** 243 **9.** $\frac{1}{16}$ **11.** $\frac{1}{11}$ **13.** 1 **15.** 4^9 **17.** 12^{40} **19.** $\frac{1}{7^9}$ **21.** 3.14×10^{-5} **23.** 16,000,000,000 **25.** a^4 **27.** b^6c^8 **29.** ab^2d^3 **31.** m^4 **33.** $\frac{q^5}{p^6}$ **35.** $\frac{y^{21}}{x^{14}}$ **37.** 25 **39.** $72a^2$ **41.** $\frac{c^3}{b^9}$ **43.** $\frac{y}{81z^6}$ **45.** 0.00135 m **47** 1.0995×10^{12} **49.** 0.00000000003397 in. **51.** 12,230,590,464 m^{66} **53.** $\frac{a^{14}}{1296}$ **55.** $\frac{n}{a^9c}$ **57.** $\frac{1}{a^6b^6c^6}$ **59.** 0.000000000000000000000000000000000662606957

Section 1.3

1. When there is no index, it is assumed to be 2 or the square root. The expression would only be equal to the radicand if the index were 1. **3.** The principal square root is the nonnegative root of the number. **5.** 16 **7.** 10 **9.** 14 **11.** $7\sqrt{2}$ **13.** $\frac{9\sqrt{5}}{5}$ **15.** 25 **17.** $\sqrt{2}$ **19.** $2\sqrt{6}$ **21.** $5\sqrt{6}$ **23.** $6\sqrt{35}$ **25.** $\frac{2}{15}$ **27.** $\frac{6\sqrt{10}}{19}$ **29.** $-\frac{1+\sqrt{17}}{2}$ **31.** $7\sqrt[3]{2}$ **33.** $15\sqrt{5}$ **35.** $20x^2$ **37.** $7\sqrt{p}$ **39.** $17m^2\sqrt{m}$ **41.** $2b\sqrt{a}$ **43.** $\frac{15x}{7}$ **45.** $5y^4\sqrt{2}$ **47.** $\frac{4\sqrt{7d}}{7d}$ **49.** $\frac{2\sqrt{2}+2\sqrt{6x}}{1-3x}$ **51.** $-w\sqrt{2w}$ **53.** $\frac{3\sqrt{x}-\sqrt{3x}}{2}$ **55.** $5n^5\sqrt{5}$ **57.** $\frac{9\sqrt{m}}{19m}$ **59.** $\frac{2}{3d}$ **61.** $\frac{3\sqrt[4]{2x^2}}{2}$ **63.** $6z\sqrt[3]{2}$ **65.** 500 feet **67** $\frac{-5\sqrt{2}-6}{7}$ **69.** $\frac{\sqrt{mnc}}{a^9cmn}$ **71.** $\frac{2\sqrt{2}x+\sqrt{2}}{4}$ **73.** $\frac{\sqrt{3}}{3}$

Section 1.4

1. The statement is true. In standard form, the polynomial with the highest value exponent is placed first and is the leading term. The degree of a polynomial is the value of the highest exponent, which in standard form is also the exponent of the leading term. **3.** Use the distributive property, multiply, combine like terms, and simplify. **5.** 2 **7.** 8 **9.** 2 **11.** $4x^2 + 3x + 19$ **13.** $3w^2 + 30w + 21$ **15.** $11b^4 - 9b^3 + 12b^2 - 7b + 8$ **17.** $24x^2 - 4x - 8$ **19.** $24b^4 - 48b^2 + 24$ **21.** $99v^2 - 202v + 99$ **23.** $8n^3 - 4n^2 + 72n - 36$ **25.** $9y^2 - 42y + 49$ **27.** $16p^2 + 72p + 81$ **29.** $9y^2 - 36y + 36$ **31.** $16c^2 - 1$ **33.** $225n^2 - 36$ **35.** $-16m^2 + 16$ **37.** $121q^2 - 100$ **39.** $16t^4 + 4t^3 - 32t^2 - t + 7$ **41.** $y^3 - 6y^2 - y + 18$ **43.** $3p^3 - p^2 - 12p + 10$ **45.** $a^2 - b^2$ **47.** $16t^2 - 40tu + 25u^2$ **49.** $4t^2 + x^2 + 4t - 5tx - x$ **51.** $24r^2 + 22rd - 7d^2$ **53.** $32x^2 - 4x - 3m^2$ **55.** $32t^3 - 100t^2 + 40t + 38$ **57.** $a^4 + 4a^3c - 16ac^3 - 16c^4$

Section 1.5

1. The terms of a polynomial do not have to have a common factor for the entire polynomial to be factorable. For example, $4x^2$ and $-9y^2$ don't have a common factor, but the whole polynomial is still factorable: $4x^2 - 9y^2 = (2x + 3y)(2x - 3y)$. **3.** Divide the x term into the sum of two terms, factor each portion of the expression separately, and then factor out the GCF of the entire expression. **5.** $7m$ **7.** $10m^3$ **9.** y **11.** $(2a - 3)(a + 6)$ **13.** $(3n - 11)(2n + 1)$ **15.** $(p + 1)(2p - 7)$ **17.** $(5h + 3)(2h - 3)$ **19.** $(9d - 1)(d - 8)$ **21.** $(12t + 13)(t - 1)$ **23.** $(4x + 10)(4x - 10)$ **25.** $(11p + 13)(11p - 13)$ **27.** $(19d + 9)(19d - 9)$ **29.** $(12b + 5c)(12b - 5c)$ **31.** $(7n + 12)^2$ **33.** $(15y + 4)^2$ **35.** $(5p - 12)^2$ **37.** $(x + 6)(x^2 - 6x + 36)$ **39.** $(5a + 7)(25a^2 - 35a + 49)$ **41.** $(4x - 5)(16x^2 + 20x + 25)$ **43.** $(5r + 12s)(25r^2 - 60rs + 144s^2)$ **45.** $(2c + 3)^{-\frac{1}{4}}(-7c - 15)$ **47.** $(x + 2)^{-\frac{2}{5}}(19x + 10)$ **49.** $(2z - 9)^{-\frac{3}{2}}(27z - 99)$ **51.** $(14x - 3)(7x + 9)$ **53.** $(3x + 5)(3x - 5)$ **55.** $(2x + 5)^2(2x - 5)^2$ **57.** $(4z^2 + 49a^2)(2z + 7a)(2z - 7a)$ **59.** $\frac{1}{(4x + 9)(4x - 9)(2x + 3)}$

Section 1.6

1. You can factor the numerator and denominator to see if any of the terms can cancel one another out.
3. True. Multiplication and division do not require finding the LCD because the denominators can be combined through those operations, whereas addition and subtraction require like terms.

5. $\frac{y+5}{y+6}$ **7.** $3b+3$ **9.** $\frac{x+4}{2x+2}$ **11.** $\frac{a+3}{a-3}$
13. $\frac{3n-8}{7n-3}$ **15.** $\frac{c-6}{c+6}$ **17.** 1 **19.** $\frac{d^2-25}{25d^2-1}$
21. $\frac{t+5}{t+3}$ **23.** $\frac{6x-5}{6x+5}$ **25.** $\frac{p+6}{4p+3}$ **27.** $\frac{2d+9}{d+11}$
29. $\frac{12b+5}{3b-1}$ **31.** $\frac{4y-1}{y+4}$ **33.** $\frac{10x+4y}{xy}$ **35.** $\frac{9a-7}{a^2-2a-3}$
37. $\frac{2y^2-y+9}{y^2-y-2}$ **39.** $\frac{5z^2+z+5}{z^2-z-2}$ **41.** $\frac{x+2xy+y}{x+xy+y+1}$
43. $\frac{2b+7a}{ab^2}$ **45.** $\frac{18+ab}{4b}$ **47.** $a-b$ **49.** $\frac{3c^2+3c-2}{2c^2+5c+2}$
51. $\frac{15x+7}{x-1}$ **53.** $\frac{x+9}{x-9}$ **55.** $\frac{1}{y+2}$ **57.** 4

Chapter 1 Review Exercises

1. -5 **3.** 53 **5.** $y=24$ **7.** $32m$ **9.** Whole
11. Irrational **13.** 16 **15.** a^6 **17.** $\frac{x^3}{32y^3}$ **19.** a
21. 1.634×10^7 **23.** 14 **25.** $5\sqrt{3}$ **27.** $\frac{4\sqrt{2}}{5}$
29. $\frac{7\sqrt{2}}{50}$ **31.** $10\sqrt{3}$ **33.** -3 **35.** $3x^3+4x^2+6$
37. $5x^2-x+3$ **39.** $k^2-3k-18$ **41.** x^3+x^2+x+1
43. $3a^2+5ab-2b^2$ **45.** $9p$ **47.** $4a^2$
49. $(4a-3)(2a+9)$ **51.** $(x+5)^2$ **53.** $(2h-3k)^2$
55. $(p+6)(p^2-6p+36)$ **57.** $(4q-3p)(16q^2+12pq+9p^2)$
59. $(p+3)^{\frac{1}{3}}(-5p-24)$ **61.** $\frac{x+3}{x-4}$ **63.** $\frac{1}{2}$ **65.** $\frac{m+2}{m-3}$
67. $\frac{6x+10y}{xy}$ **69.** $\frac{1}{6}$

Chapter 1 Practice Test

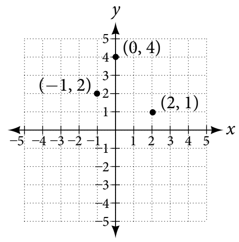
1. Rational **3.** $x=12$ **5.** 3,141,500 **7.** 16
9. 9 **11.** $2x$ **13.** 21 **15.** $\frac{3\sqrt{x}}{4}$ **17.** $21\sqrt{6}$
19. $13q^3-4q^2-5q$ **21.** $n^3-6n^2+12n-8$
23. $(4x+9)(4x-9)$ **25.** $(3c-11)(9c^2+33c+121)$
27. $\frac{4z-3}{2z-1}$ **29.** $\frac{3a+2b}{3b}$

CHAPTER 2

Section 2.1

1. Answers may vary. Yes. It is possible for a point to be on the x-axis or on the y-axis and therefore is considered to NOT be in one of the quadrants. **3.** The y-intercept is the point where the graph crosses the y-axis. **5.** The x-intercept is (2, 0) and the y-intercept is (0, 6). **7.** The x-intercept is (2, 0) and the y-intercept is (0, -3). **9.** The x-intercept is (3, 0) and the y-intercept is $\left(0, \frac{9}{8}\right)$. **11.** $y=4-2x$ **13.** $y=\frac{5-2x}{3}$
15. $y=2x-\frac{4}{5}$ **17.** $d=\sqrt{74}$ **19.** $d=\sqrt{36}=6$
21. $d\approx62.97$ **23.** $\left(3, -\frac{3}{2}\right)$ **25.** $(2, -1)$ **27.** $(0, 0)$ **29.** $y=0$

31. Not collinear **33.** $(-3, 2), (1, 3), (4, 0)$

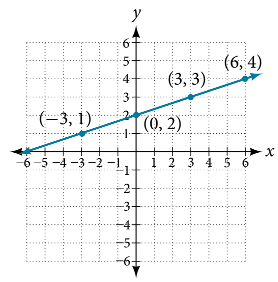


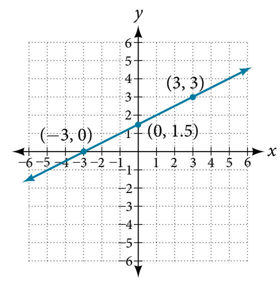
35.

x	-3	0	3	6
y	1	2	3	4

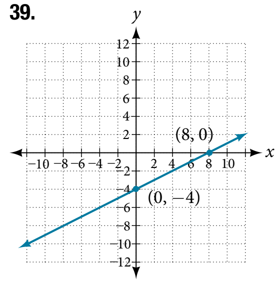
37.

x	-3	0	3
y	0	1.5	3

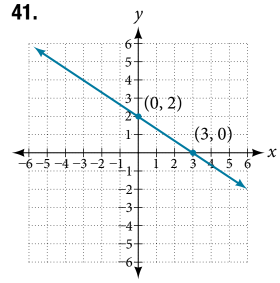




39.



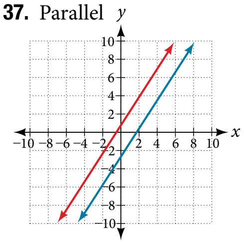
41.



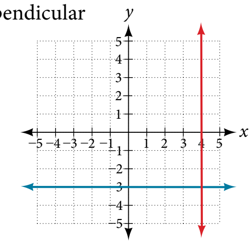
43. $d=8.246$ **45.** $d=5$ **47.** $(-3, 4)$ **49.** $x=0,\ y=-2$
51. $x=0.75,\ y=0$ **53.** $x=-1.667,\ y=0$
55. $15-11.2=3.8$ mi shorter **57.** 6.042 **59.** Midpoint of each diagonal is the same point (2, 2). Note this is a characteristic of rectangles, but not other quadrilaterals.
61. 37 mi **63.** 54 ft

Section 2.2

1. It means they have the same slope. **3.** The exponent of the x variable is 1. It is called a first-degree equation. **5.** If we insert either value into the equation, they make an expression in the equation undefined (zero in the denominator). **7.** $x=2$
9. $x=\frac{2}{7}$ **11.** $x=6$ **13.** $x=3$ **15.** $x=-14$
17. $x\neq-4;\ x=-3$ **19.** $x\neq1$; when we solve this we get $x=1$, which is excluded, therefore NO solution **21.** $x\neq0;\ x=-\frac{5}{2}$
23. $y=-\frac{4}{5}x+\frac{14}{5}$ **25.** $y=-\frac{3}{4}x+2$ **27.** $y=\frac{1}{2}x+\frac{5}{2}$
29. $y=-3x-5$ **31.** $y=7$ **33.** $y=-4$ **35.** $8x+5y=7$
37. Parallel



39. Perpendicular



41. $m = -\frac{9}{7}$ **43.** $m = \frac{3}{2}$ **45.** $m_1 = -\frac{1}{3}, m_2 = 3$; perpendicular
47. $y = 0.245x - 45.662$. Answers may vary. $y_{\min} = -50, y_{\max} = -40$
49. $y = -2.333x + 6.667$. Answers may vary. $y_{\min} = -10, y_{\max} = 10$
51. $y = -\frac{A}{B}x + \frac{C}{B}$ **53.** The slope for $(-1, 1)$ to $(0, 4)$ is 3. The slope for $(-1, 1)$ to $(2, 0)$ is $-\frac{1}{3}$. The slope for $(2, 0)$ to $(3, 3)$ is 3. The slope for $(0, 4)$ to $(3, 3)$ is $-\frac{1}{3}$. Yes they are perpendicular.
55. 30 ft **57.** \$57.50 **59.** 220 mi

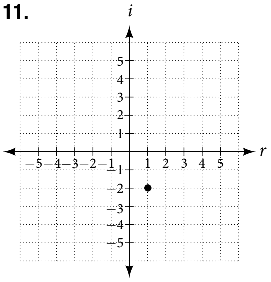
Section 2.3

1. Answers may vary. Possible answers: We should define in words what our variable is representing. We should declare the variable. A heading. **3.** $2{,}000 - x$ **5.** $v + 10$
7. Ann: 23; Beth: 46 **9.** $20 + 0.05m$ **11.** 300 min
13. $90 + 40P$ **15.** 6 devices **17.** $50{,}000 - x$ **19.** 4 hr
21. She traveled for 2 hr at 20 mi/hr, or 40 miles.
23. \$5,000 at 8% and \$15,000 at 12% **25.** $B = 100 + 0.05x$
27. Plan A **29.** $R = 9$ **31.** $r = \frac{4}{5}$ or 0.8
33. $W = \frac{P - 2L}{2} = \frac{58 - 2(15)}{2} = 14$
35. $f = \frac{pq}{p + q} = \frac{8(13)}{8 + 13} = \frac{104}{21}$ **37.** $m = -\frac{5}{4}$
39. $h = \frac{2A}{b_1 + b_2}$ **41.** Length = 360 ft; width = 160 ft
43. 405 mi **45.** $A = 88\text{ in.}^2$ **47.** 28.7 **49.** $h = \frac{V}{\pi r^2}$
51. $r = \sqrt{\frac{V}{\pi h}}$ **53.** $C = 12\pi$

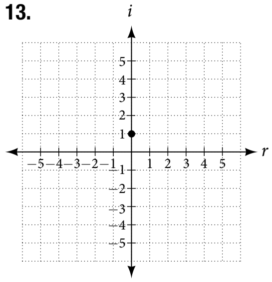
Section 2.4

1. Add the real parts together and the imaginary parts together.
3. Possible answer: i times i equals -1, which is not imaginary.
5. $-8 + 2i$ **7.** $14 + 7i$ **9.** $-\frac{23}{29} + \frac{15}{29}i$

11.



13.



15. $8 - i$ **17.** $-11 + 4i$ **19.** $2 - 5i$ **21.** $6 + 15i$
23. $-16 + 32i$ **25.** $-4 - 7i$ **27.** 25 **29.** $2 - \frac{2}{3}i$
31. $4 - 6i$ **33.** $\frac{2}{5} + \frac{11}{5}i$ **35.** $15i$ **37.** $1 + i\sqrt{3}$
39. 1 **41.** -1 **43.** $128i$ **45.** $\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)^6 = -1$
47. $3i$ **49.** 0 **51.** $5 - 5i$ **53.** $-2i$ **55.** $\frac{9}{2} - \frac{9}{2}i$

Section 2.5

1. It is a second-degree equation (the highest variable exponent is 2). **3.** We want to take advantage of the zero property of multiplication in the fact that if $a \cdot b = 0$ then it must follow that each factor separately offers a solution to the product being zero: $a = 0$ or $b = 0$.

5. One, when no linear term is present (no x term), such as $x^2 = 16$. Two, when the equation is already in the form $(ax + b)^2 = d$.
7. $x = 6, x = 3$ **9.** $x = -\frac{5}{2}, x = -\frac{1}{3}$ **11.** $x = 5, x = -5$
13. $x = -\frac{3}{2}, x = \frac{3}{2}$ **15.** $x = -2, 3$ **17.** $x = 0, x = -\frac{3}{7}$
19. $x = -6, x = 6$ **21.** $x = 6, x = -4$ **23.** $x = 1, x = -2$
25. $x = -2, x = 11$ **27.** $x = 3 \pm \sqrt{22}$ **29.** $z = \frac{2}{3}, z = -\frac{1}{2}$
31. $x = \frac{3 \pm \sqrt{17}}{4}$ **33.** Not real **35.** One rational
37. Two real; rational **39.** $x = \frac{-1 \pm \sqrt{17}}{2}$
41. $x = \frac{5 \pm \sqrt{13}}{6}$ **43.** $x = \frac{-1 \pm \sqrt{17}}{8}$
45. $x \approx 0.131$ and $x \approx 2.535$ **47.** $x \approx -6.7$ and $x \approx 1.7$
49. $ax^2 + bx + c = 0$

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$
$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b}{4a^2}$$
$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$
$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

51. $x(x + 10) = 119$; 7 ft. and 17 ft. **53.** Maximum at $x = 70$
55. The quadratic equation would be $(100x - 0.5x^2) - (60x + 300) = 300$. The two values of x are 20 and 60. **57.** 3 feet

Section 2.6

1. This is not a solution to the radical equation, it is a value obtained from squaring both sides and thus changing the signs of an equation which has caused it not to be a solution in the original equation. **3.** He or she is probably trying to enter negative 9, but taking the square root of -9 is not a real number. The negative sign is in front of this, so your friend should be taking the square root of 9, cubing it, and then putting the negative sign in front, resulting in -27. **5.** A rational exponent is a fraction: the denominator of the fraction is the root or index number and the numerator is the power to which it is raised. **7.** $x = 81$ **9.** $x = 17$ **11.** $x = 8, x = 27$
13. $x = -2, 1, -1$ **15.** $y = 0, \frac{3}{2}, -\frac{3}{2}$ **17.** $m = 1, -1$
19. $x = \frac{2}{5}, \pm 3i$ **21.** $x = 32$ **23.** $t = \frac{44}{3}$ **25.** $x = 3$
27. $x = -2$ **29.** $x = 4, -\frac{4}{3}$ **31.** $x = -\frac{5}{4}, \frac{7}{4}$
33. $x = 3, -2$ **35.** $x = -5$ **37.** $x = 1, -1, 3, -3$
39. $x = 2, -2$ **41.** $x = 1, 5$ **43.** All real numbers
45. $x = 4, 6, -6, -8$ **47.** 10 in. **49.** 90 kg

Section 2.7

1. When we divide both sides by a negative it changes the sign of both sides so the sense of the inequality sign changes.
3. $(-\infty, \infty)$

5. We start by finding the x-intercept, or where the function $= 0$. Once we have that point, which is $(3, 0)$, we graph to the right the straight line graph $y = x - 3$, and then when we draw it to the left we plot positive y values, taking the absolute value of them.

7. $\left(-\infty, \frac{3}{4}\right]$ **9.** $\left[-\frac{13}{2}, \infty\right)$ **11.** $(-\infty, 3)$ **13.** $\left(-\infty, -\frac{37}{3}\right]$

15. All real numbers $(-\infty, \infty)$ **17.** $\left(-\infty, -\frac{10}{3}\right) \cup (4, \infty)$

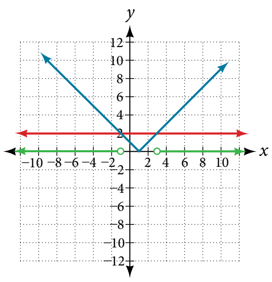
19. $(-\infty, -4] \cup [8, +\infty)$ **21.** No solution **23.** $(-5, 11)$

25. $[6, 12]$ **27.** $[-10, 12]$

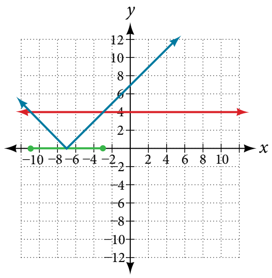
29. $x > -6$ and $x > -2$ Take the intersection of two sets.
$x > -2, (-2, +\infty)$

31. $x < -3$ or $x \geq 1$ Take the union of the two sets.
$(-\infty, -3) \cup [1, \infty)$

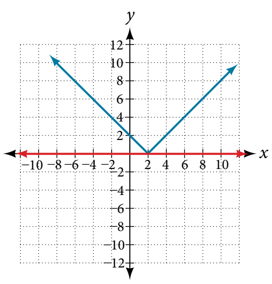
33. $(-\infty, -1) \cup (3, \infty)$



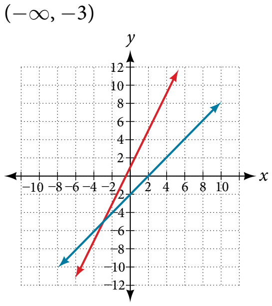
35. $[-11, -3]$



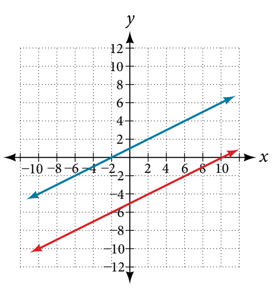
37. It is never less than zero. No solution.



39. Where the blue line is above the red line; point of intersection is $x = -3$.
$(-\infty, -3)$

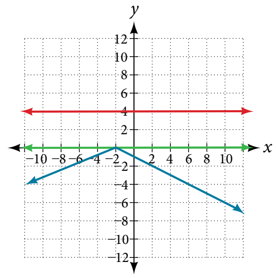


41. Where the blue line is above the red line; always. All real numbers. $(-\infty, -\infty)$

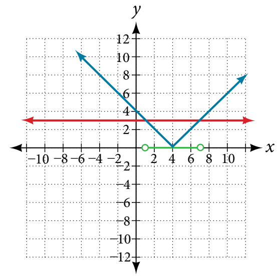


43. $(-1, 3)$
45. $(-\infty, 4)$
47. $\{x \mid x < 6\}$
49. $\{x \mid -3 \leq x < 5\}$
51. $(-2, 1]$
53. $(-\infty, 4]$

55. Where the blue is below the red; always. All real numbers. $(-\infty, +\infty)$.



57. Where the blue is below the red; $(1, 7)$.



59. $x = 2, -\frac{4}{5}$ **61.** $(-7, 5]$

63. $80 \leq T \leq 120$; $1,600 \leq 20T \leq 2,400$; $[1,600, 2,400]$

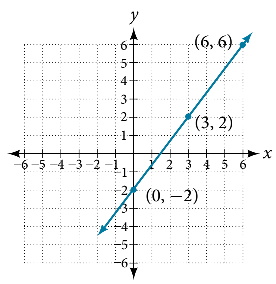
Chapter 2 Review Exercises

1. x-intercept: $(3, 0)$; y-intercept: $(0, -4)$ **3.** $y = \frac{5}{3}x + 4$

5. $\sqrt{72} = 6\sqrt{2}$ **7.** 620.097 **9.** Midpoint is $\left(2, \frac{23}{2}\right)$

11.

x	0	3	6
y	-2	2	6



13. $x = 4$

15. $x = \frac{12}{7}$

17. No solution

19. $y = \frac{1}{6}x + \frac{4}{3}$

21. $y = \frac{2}{3}x + 6$

23. Females 17, males 56

25. 84 mi

27. $x = -\frac{3}{4} \pm \frac{i\sqrt{47}}{4}$

29. Horizontal component -2; vertical component -1

31. $7 + 11i$ **33.** $16i$ **35.** $-16 - 30i$ **37.** $-4 - i\sqrt{10}$

39. $x = 7 - 3i$ **41.** $x = -1, -5$ **43.** $x = 0, \frac{9}{7}$

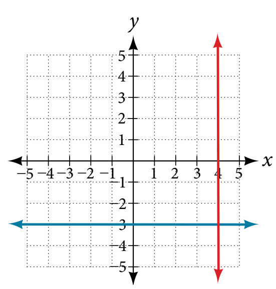
45. $x = 10, -2$ **47.** $x = \frac{-1 \pm \sqrt{5}}{4}$ **49.** $x = \frac{2}{5}, -\frac{1}{3}$

51. $x = 5 \pm 2\sqrt{7}$ **53.** $x = 0, 256$ **55.** $x = 0, \pm\sqrt{2}$

57. $x = -2$ **59.** $x = \frac{11}{2}, -\frac{17}{2}$ **61.** $(-\infty, 4)$

63. $\left[-\frac{10}{3}, 2\right]$ **65.** No solution **67.** $\left(-\frac{4}{3}, \frac{1}{5}\right)$

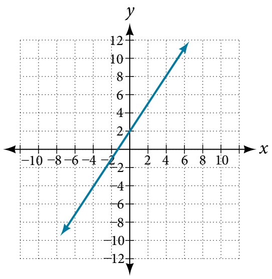
69. Where the blue is below the red line; point of intersection is $x = 3.5$. $(3.5, \infty)$



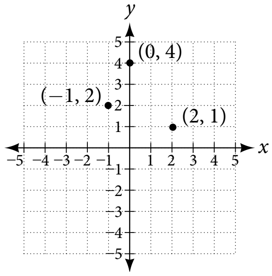
Chapter 2 Practice Test

1. $y = \frac{3}{2}x + 2$

x	0	2	4
y	2	5	8



3. $(0, -3)$ $(4, 0)$



5. $(-\infty, 9]$ **7.** $x = -15$ **9.** $x \neq -4, 2; x = -\frac{5}{2}, 1$

11. $x = \frac{3 \pm \sqrt{3}}{2}$ **13.** $(-4, 1)$ **15.** $y = -\frac{5}{9}x - \frac{2}{9}$

17. $y = \frac{5}{2}x - 4$ **19.** $14i$ **21.** $\frac{5}{13} - \frac{14}{13}i$ **23.** $x = 2, -\frac{4}{3}$

25. $x = \frac{1}{2} \pm \frac{\sqrt{2}}{2}$ **27.** 4 **29.** $x = \frac{1}{2}, 2, -2$

CHAPTER 3

Section 3.1

1. A relation is a set of ordered pairs. A function is a special kind of relation in which no two ordered pairs have the same first coordinate. **3.** When a vertical line intersects the graph of a relation more than once, that indicates that for that input there is more than one output. At any particular input value, there can be only one output if the relation is to be a function. **5.** When a horizontal line intersects the graph of a function more than once, that indicates that for that output there is more than one input. A function is one-to-one if each output corresponds to only one input. **7.** Function **9.** Function **11.** Function **13.** Function **15.** Function **17.** Function **19.** Function **21.** Function **23.** Function **25.** Not a function **27.** $f(-3) = -11, f(2) = -1, f(-a) = -2a - 5, -f(a) = -2a + 5, f(a + h) = 2a + 2h - 5$ **29.** $f(-3) = \sqrt{5} + 5,\ f(2) = 5,\ f(-a) = \sqrt{2 + a} + 5,\ -f(a) = -\sqrt{2 - a} - 5,\ f(a + h) = \sqrt{2 - a - h} + 5$ **31.** $f(-3) = 2, f(2) = -2, f(-a) = |-a - 1| - |-a + 1|, -f(a) = -|a - 1| + |a + 1|, f(a + h) = |a + h - 1| - |a + h + 1|$

33. $\frac{g(x) - g(a)}{x - a} = x + a + 2, x \neq a$ **35. a.** $f(-2) = 14$ **b.** $x = 3$

37. a. $f(5) = 10$ **b.** $x = 4$ or -1 **39. a.** $r = 6 - \frac{2}{3}t$

b. $f(-3) = 8$ **c.** $t = 6$ **41.** Not a function **43.** Function

45. Function **47.** Function **49.** Function

51. Function **53. a.** $f(0) = 1$ **b.** $f(x) = -3, x = -2$ or 2

55. Not a function, not one-to-one **57.** One-to-one function

59. Function, not one-to-one **61.** Function **63.** Function

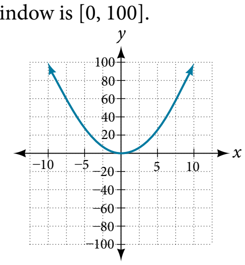
65. Not a function **67.** $f(x) = 1, x = 2$

69. $f(-2) = 14;\ f(-1) = 11;\ f(0) = 8;\ f(1) = 5;\ f(2) = 2$

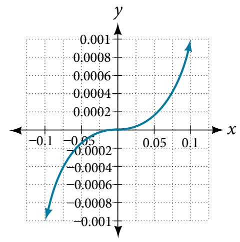
71. $f(-2) = 4;\ f(-1) = 4.414;\ f(0) = 4.732;\ f(1) = 5;\ f(2) = 5.236$

73. $f(-2) = \frac{1}{9};\ f(-1) = \frac{1}{3};\ f(0) = 1;\ f(1) = 3;\ f(2) = 9$ **75.** 20

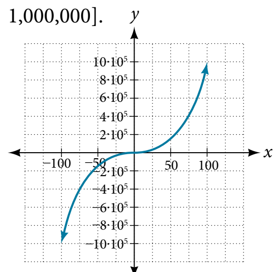
77. The range for this viewing window is $[0, 100]$.



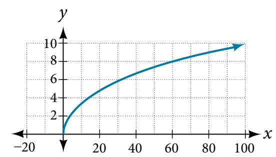
79. The range for this viewing window is $[-0.001, 0.001]$.



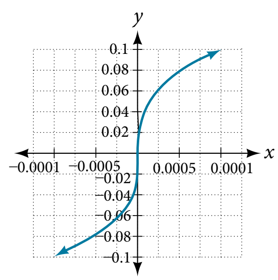
81. The range for this viewing window is $[-1{,}000{,}000, 1{,}000{,}000]$.



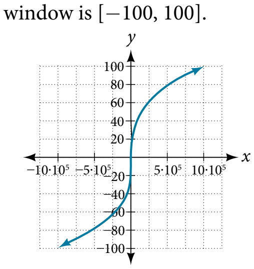
83. The range for this viewing window is $[0, 10]$.



85. The range for this viewing window is $[-0.1, 0.1]$.



87. The range for this viewing window is $[-100, 100]$.



89. a. $g(5000) = 50$ **b.** The number of cubic yards of dirt required for a garden of 100 square feet is 1. **91. a.** The height of the rocket above ground after 1 second is 200 ft. **b.** The height of the rocket above ground after 2 seconds is 350 ft.

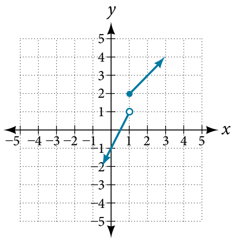
Section 3.2

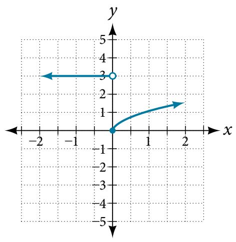
1. The domain of a function depends upon what values of the independent variable make the function undefined or imaginary. **3.** There is no restriction on x for $f(x) = \sqrt[3]{x}$ because you can take the cube root of any real number. So the domain is all real numbers, $(-\infty, \infty)$. When dealing with the set of real numbers, you cannot take the square root of negative numbers. So x-values are restricted for $f(x) = \sqrt{x}$ to nonnegative numbers and the domain is $[0, \infty)$. **5.** Graph each formula of the piecewise function over its corresponding domain. Use the same scale for the x-axis and y-axis for each graph. Indicate included endpoints with a solid circle and excluded endpoints with an open circle. Use an arrow to indicate $-\infty$ or ∞. Combine the graphs to find the graph of the piecewise function. **7.** $(-\infty, \infty)$ **9.** $(-\infty, 3]$

11. $(-\infty, \infty)$ **13.** $(-\infty, \infty)$ **15.** $\left(-\infty, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, \infty\right)$

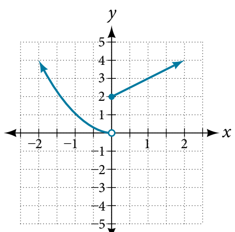
17. $(-\infty, -11) \cup (-11, 2) \cup (2, \infty)$ **19.** $(-\infty, -3) \cup (-3, 5) \cup (5, \infty)$

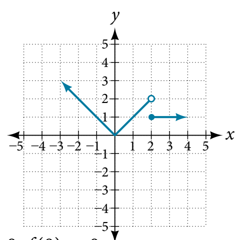
21. $(-\infty, 5)$ **23.** $[6, \infty)$ **25.** $(-\infty, -9)\cup(-9, 9)\cup(9, \infty)$
27. Domain: $(2, 8]$, range: $[6, 8)$ **29.** Domain: $[-4, 4]$, range: $[0, 2]$
31. Domain: $[-5, 3)$, range: $[0, 2]$ **33.** Domain: $(-\infty, 1]$, range: $[0, \infty)$
35. Domain: $\left[-6, -\frac{1}{6}\right]\cup\left[\frac{1}{6}, 6\right]$, range: $\left[-6, -\frac{1}{6}\right]\cup\left[\frac{1}{6}, 6\right]$
37. Domain: $[-3, \infty)$, range is $[0, \infty)$
39. Domain:$(-\infty, \infty)$ **41.** Domain: $(-\infty, \infty)$



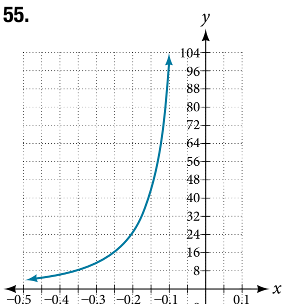


43. Domain: $(-\infty, \infty)$ **45.** Domain: $(-\infty, \infty)$

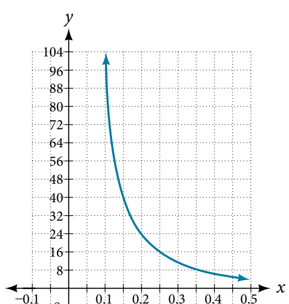




47. $f(-3) = 1; f(-2) = 0; f(-1) = 0; f(0) = 0$
49. $f(-1) = -4; f(0) = 6; f(2) = 20; f(4) = 34$
51. $f(-1) = -5; f(0) = 3; f(2) = 3; f(4) = 16$
53. $(-\infty, 1)\cup(1, \infty)$
55.



The viewing window: [–0.5, –0.1] has a range: [4, 100].



The viewing window: [0.1, 0.5] has a range: [4, 100].

57. $[0, 8]$ **59.** Many answers; one function is $f(x) = \frac{1}{\sqrt{x-2}}$.
61. **a.** The fixed cost is \$500. **b.** The cost of making 25 items is \$750. **c.** The domain is [0, 100] and the range is [500, 1500].

Section 3.3

1. Yes, the average rate of change of all linear functions is constant.
3. The absolute maximum and minimum relate to the entire graph, whereas the local extrema relate only to a specific region in an open interval. **5.** $4(b + 1)$ **7.** 3 **9.** $4x + 2h$
11. $\frac{-1}{13(13+h)}$ **13.** $3h^2 + 9h + 9$ **15.** $4x + 2h - 3$
17. $\frac{4}{3}$ **19.** Increasing on $(-\infty, -2.5)\cup(1, \infty)$ a nd decreasing on $(-2.5, 1)$ **21.** Increasing on $(-\infty, 1)\cup(3, 4)$ and decreasing on $(1, 3)\cup(4, \infty)$ **23.** Local maximum: $(-3, 50)$ and local minimum: $(3, 50)$ **25.** Absolute maximum at approximately $(7, 150)$ and absolute minimum at approximately $(-7.5, -220)$
27. **a.** $-3{,}000$ people per year **b.** $-1{,}250$ people per year
29. -4 **31.** 27 **33.** ≈ -0.167 **35.** Local minimum: $(3, -22)$, decreasing on $(-\infty, 3)$, increasing on $(3, \infty)$
37. Local minimum: $(-2, -2)$, decreasing on $(-3, -2)$, increasing on $(-2, \infty)$ **39.** Local maximum: $(-0.5, 6)$, local minima: $(-3.25, -47)$ and $(2.1, -32)$, decreasing on $(-\infty, -3.25)$ and $(-0.5, 2.1)$, increasing on $(-3.25, -0.5)$ and $(2.1, \infty)$
41. A **43.** $b = 5$ **45.** ≈ 2.7 gallons per minute
47. ≈ -0.6 milligrams per day

Section 3.4

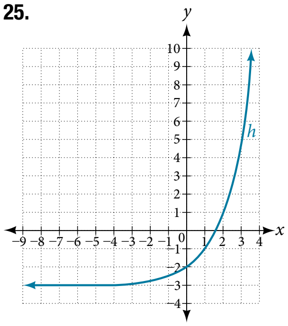
1. Find the numbers that make the function in the denominator g equal to zero, and check for any other domain restrictions on f and g, such as an even-indexed root or zeros in the denominator.
3. Yes, sample answer: Let $f(x) = x + 1$ and $g(x) = x - 1$. Then $f(g(x)) = f(x - 1) = (x - 1)+ 1 = x$ and $g(f(x)) = g(x + 1) = (x + 1)- 1 = x$ so $f\circ g = g\circ f$.
5. $(f + g)(x) = 2x + 6$; domain: $(-\infty, \infty)$
$(f - g)(x) = 2x^2 + 2x - 6$; domain: $(-\infty, \infty)$
$(fg)(x) = -x^4 - 2x^3 + 6x^2 + 12x$; domain: $(-\infty, \infty)$
$\left(\frac{f}{g}\right)(x) = \frac{x^2 + 2x}{6 - x^2}$; domain: $(-\infty, -\sqrt{6})\cup(-\sqrt{6}, \sqrt{6})\cup(\sqrt{6}, \infty)$
7. $(f + g)(x) = \frac{4x^3 + 8x^2 + 1}{2x}$; domain: $(-\infty, 0)\cup(0, \infty)$
$(f - g)(x) = \frac{4x^3 + 8x^2 - 1}{2x}$; domain: $(-\infty, 0)\cup(0, \infty)$
$(fg)(x) = x + 2$; domain: $(-\infty, 0)\cup(0, \infty)$
$\left(\frac{f}{g}\right)(x) = 4x^3 + 8x^2$; domain: $(-\infty, 0)\cup(0, \infty)$
9. $(f + g)(x) = 3x^2 + \sqrt{x-5}$; domain: $[5, \infty)$
$(f - g)(x) = 3x^2 - \sqrt{x-5}$; domain: $[5, \infty)$
$(fg)(x) = 3x^2\sqrt{x-5}$; domain: $[5, \infty)$
$\left(\frac{f}{g}\right)(x) = \frac{3x^2}{\sqrt{x-5}}$; domain: $(5, \infty)$ **11.** **a.** $f(g(2)) = 3$
b. $f(g(x)) = 18x^2 - 60x + 51$ **c.** $g(f(x)) = 6x^2 - 2$
d. $(g\circ g)(x) = 9x - 20$ **e.** $(f\circ f)(-2) = 163$
13. $f(g(x)) = \sqrt{x^2+3} + 2; g(f(x)) = x + 4\sqrt{x} + 7$
15. $f(g(x)) = \frac{\sqrt[3]{x+1}}{x}$; $g(f(x)) = \frac{\sqrt[3]{x}+1}{x}$
17. $f(g(x)) = \frac{x}{2}, x \neq 0$; $g(f(x)) = 2x - 4, x \neq 4$
19. $f(g(h(x))) = \frac{1}{(x+3)^2} + 1$
21. **a.** $(g\circ f)(x) = -\frac{3}{\sqrt{2-4x}}$ **b.** $\left(-\infty, \frac{1}{2}\right)$
23. **a.** $(0, 2)\cup(2, \infty)$ except $x = -2$ **b.** $(0, \infty)$ **c.** $(0, \infty)$ **25.** $(1, \infty)$
27. Many solutions; one possible answer: $f(x) = x^3; g(x) = x - 5$
29. Many solutions; one possible answer: $f(x) = \frac{4}{x}; g(x) = (x + 2)^2$
31. Many solutions; one possible answer: $f(x) = \sqrt[3]{x}; g(x) = \frac{1}{2x-3}$
33. Many solutions; one possible answer: $f(x) = \sqrt[4]{x}; g(x) = \frac{3x-2}{x+5}$
35. Many solutions; one possible answer: $f(x) = \sqrt{x}; g(x) = 2x + 6$
37. Many solutions; one possible answer: $f(x) = \sqrt[3]{x}; g(x) = x - 1$
39. Many solutions; one possible answer: $f(x) = x^3; g(x) = \frac{1}{x-2}$

41. Many solutions; one possible answer: $f(x) = \sqrt{x}$; $g(x) = \dfrac{2x-1}{3x+4}$
43. 2 **45.** 5 **47.** 4 **49.** 0 **51.** 2 **53.** 1
55. 4 **57.** 4 **59.** 9 **61.** 4 **63.** 2 **65.** 3
67. 11 **69.** 0 **71.** 7 **73.** $f(g(0)) = 27, g(f(0)) = -94$
75. $f(g(0)) = \frac{1}{5}, g(f(0)) = 5$ **77.** $f(g(x)) = 18x^2 + 60x + 51$
79. $g \circ g(x) = 9x + 20$ **81.** $(f \circ g)(x) = 2, (g \circ f)(x) = 2$
83. $(-\infty, \infty)$ **85.** False **87.** $(f \circ g)(6) = 6$; $(g \circ f)(6) = 6$
89. $(f \circ g)(11) = 11$; $(g \circ f)(11) = 11$ **91.** C
93. $A(t) = \pi\left(25\sqrt{t+2}\right)^2$ and $A(2) = \pi\left(25\sqrt{4}\right)^2 = 2{,}500\pi$ square inches **95.** $A(5) = 121\pi$ square units
97. **a.** $N(T(t)) = 575t^2 + 65t - 31.25$ **b.** ≈ 3.38 hours

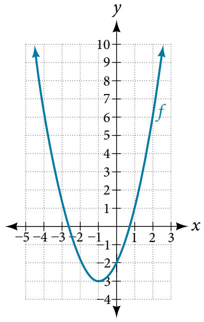
Section 3.5

1. A horizontal shift results when a constant is added to or subtracted from the input. A vertical shift results when a constant is added to or subtracted from the output. **3.** A horizontal compression results when a constant greater than 1 multiplies the input. A vertical compression results when a constant between 0 and 1 multiplies the output. **5.** For a function f, substitute $(-x)$ for (x) in $f(x)$ and simplify. If the resulting function is the same as the original function, $f(-x) = f(x)$, then the function is even. If the resulting function is the opposite of the original function, $f(-x) = -f(x)$, then the original function is odd. If the function is not the same or the opposite, then the function is neither odd nor even. **7.** $g(x) = |x-1| - 3$
9. $g(x) = \dfrac{1}{(x+4)^2} + 2$ **11.** The graph of $f(x+43)$ is a horizontal shift to the left 43 units of the graph of f.
13. The graph of $f(x-4)$ is a horizontal shift to the right 4 units of the graph of f. **15.** The graph of $f(x) + 8$ is a vertical shift up 8 units of the graph of f. **17.** The graph of $f(x) - 7$ is a vertical shift down 7 units of the graph of f. **19.** The graph of $f(x+4) - 1$ is a horizontal shift to the left 4 units and a vertical shift down 1 unit of the graph of f. **21.** Decreasing on $(-\infty, -3)$ and increasing on $(-3, \infty)$ **23.** Decreasing on $(0, \infty)$

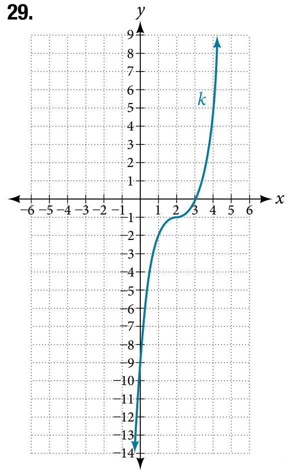
25.



27.



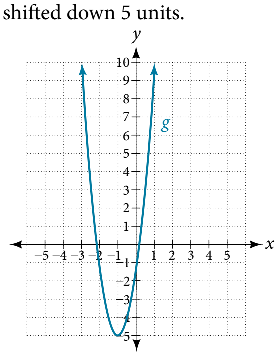
29.



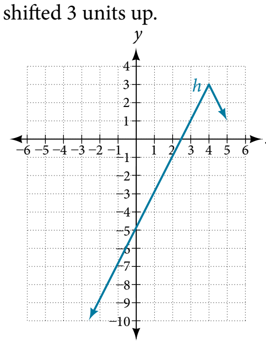
31. $g(x) = f(x-1)$, $h(x) = f(x) + 1$
33. $f(x) = |x-3| - 2$
35. $f(x) = \sqrt{x+3} - 1$
37. $f(x) = (x-2)^2$
39. $f(x) = |x+3| - 2$
41. $f(x) = -\sqrt{x}$
43. $f(x) = -(x+1)^2 + 2$
45. $f(x) = \sqrt{-x} + 1$
47. Even **49.** Odd
51. Even **53.** The graph of g is a vertical reflection (across the x-axis) of the graph of f. **55.** The graph of g is a vertical stretch by a factor of 4 of the graph of f.

57. The graph of g is a horizontal compression by a factor of $\frac{1}{5}$ of the graph of f. **59.** The graph of g is a horizontal stretch by a factor of 3 of the graph of f. **61.** The graph of g is a horizontal reflection across the y-axis and a vertical stretch by a factor of 3 of the graph of f. **63.** $g(x) = |-4x|$
65. $g(x) = \dfrac{1}{3(x+2)^2} - 3$ **67.** $g(x) = \frac{1}{2}(x-5)^2 + 1$

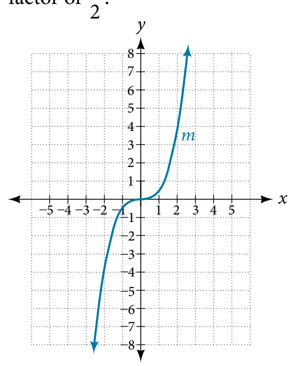
69. This is a parabola shifted to the left 1 unit, stretched vertically by a factor of 4, and shifted down 5 units.



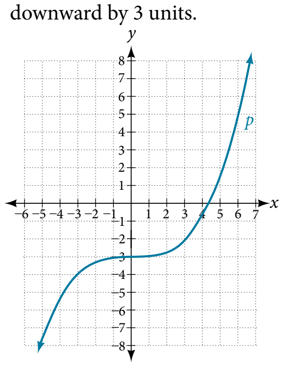
71. This is an absolute value function stretched vertically by a factor of 2, shifted 4 units to the right, reflected across the horizontal axis, and then shifted 3 units up.



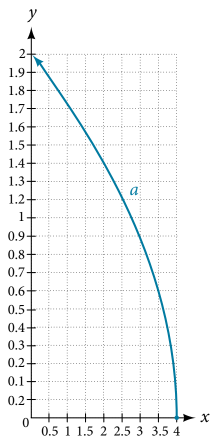
73. This is a cubic function compressed vertically by a factor of $\frac{1}{2}$.



75. The graph of the function is stretched horizontally by a factor of 3 and then shifted downward by 3 units.



77. The graph of $f(x) = \sqrt{x}$ is shifted right 4 units and then reflected across the y-axis.



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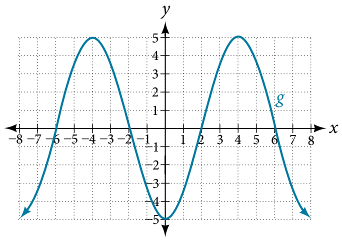
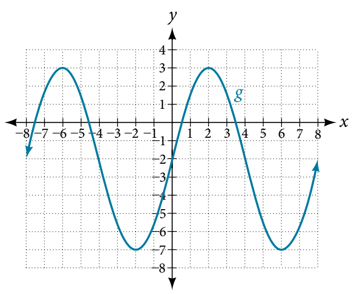
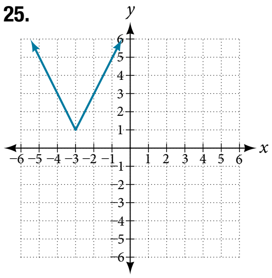


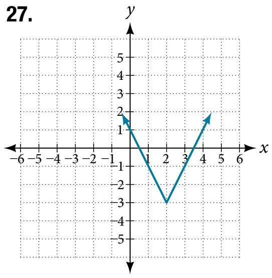
81.



25.



27.



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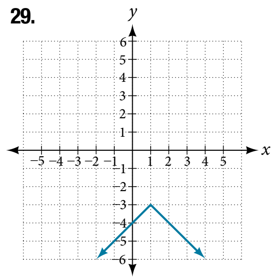

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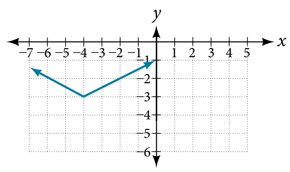
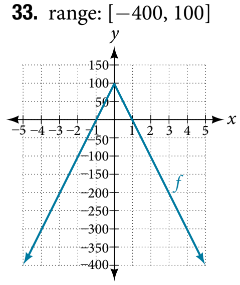


33. range: $[-400, 100]$



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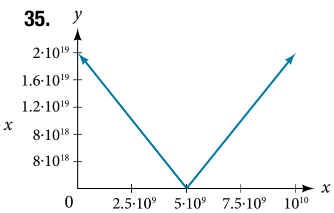
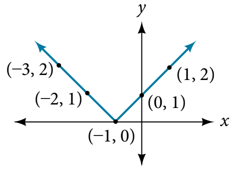


Section 3.6

1. Isolate the absolute value term so that the equation is of the form $|A| = B$. Form one equation by setting the expression inside the absolute value symbol, A, equal to the expression on the other side of the equation, B. Form a second equation by setting A equal to the opposite of the expression on the other side of the equation, $-B$. Solve each equation for the variable. **3.** The graph of the absolute value function does not cross the x-axis, so the graph is either completely above or completely below the x-axis. **5.** The distance from x to 8 can be represented using the absolute value statement: $|x - 8| = 4$. **7.** $|x - 10| \geq 15$ **9.** There are no x-intercepts. **11.** $(-4, 0)$ and $(2, 0)$ **13.** $(-2, 0)$, $(4, 0)$, and $(0, -4)$ **15.** $(-7, 0)$, $(0, 16)$, $(25, 0)$

17.



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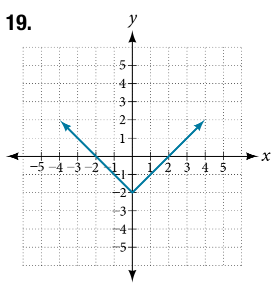

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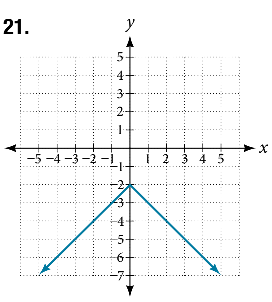

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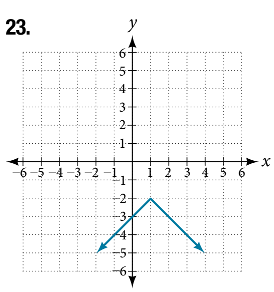


37. There is no value for a that will keep the function from having a y-intercept. The absolute value function always crosses the y-intercept when $x = 0$.

39. $|p - 0.08| \leq 0.015$ **41.** $|x - 5.0| \leq 0.01$

SECTION 3.7

1. Each output of a function must have exactly one input for the function to be one-to-one. If any horizontal line crosses the graph of a function more than once, that means that y-values repeat and the function is not one-to-one. If no horizontal line crosses the graph of the function more than once, then no y-values repeat and the function is one-to-one. **3.** Yes. For example, $f(x) = \frac{1}{x}$ is its own inverse. **5.** $y = f^{-1}(x)$

7. $f^{-1}(x) = x - 3$ **9.** $f^{-1}(x) = 2 - x$ **11.** $f^{-1}(x) = -\frac{2x}{x-1}$

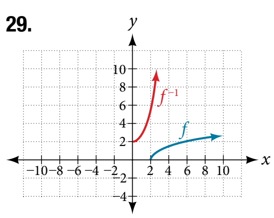
13. Domain of $f(x)$: $[-7,\infty)$; $f^{-1}(x) = \sqrt{x} - 7$

15. Domain of $f(x)$: $[0, \infty)$; $f^{-1}(x) = \sqrt{x + 5}$

17. $f(g(x)) = x$ and $g(f(x)) = x$ **19.** One-to-one

21. One-to-one **23.** Not one-to-one **25.** 3 **27.** 2

29.



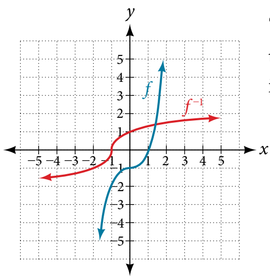
31. $[2, 10]$ **33.** 6

35. -4 **37.** 0 **39.** 1

41.

x	1	4	7	12	16
$f^{-1}(x)$	3	6	9	13	14

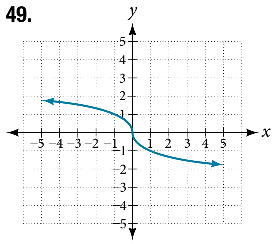
43. $f^{-1}(x) = (1+x)^{\frac{1}{3}}$



45. $f^{-1}(x) = \frac{5}{9}(x-32)$

47. $t(d) = \frac{d}{50}$; $t(180) = \frac{180}{50}$. The time for the car to travel 180 miles is 3.6 hours.

49.



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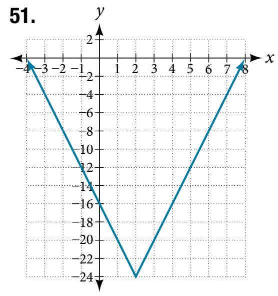

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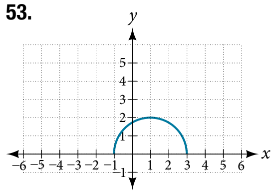


55. $f(x) = |x-3|$

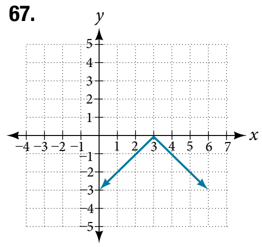
57. Even **59.** Odd

61. Even

63. $f(x) = \frac{1}{2}|x+2| + 1$

65. $f(x) = -3|x-3| + 3$

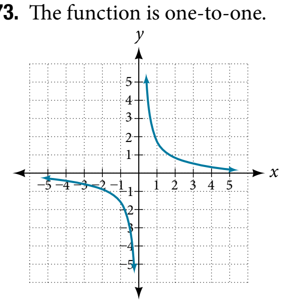
67.



69. $f^{-1}(x) = \frac{x-9}{10}$

71. $f^{-1}(x) = \sqrt{x-1}$

73. The function is one-to-one.



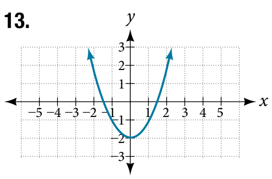
75. 5

Chapter 3 Review Exercises

1. Function **3.** Not a function **5.** $f(-3) = -27$; $f(2) = -2$; $f(-a) = -2a^2 - 3a$; $-f(a) = 2a^2 - 3a$; $f(a+h) = -2a^2 - 4ah - 2h^2 + 3a + 3h$

7. One-to-one **9.** Function **11.** Function

13.

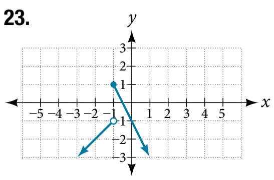


15. 2 **17.** -1.8 or 1.8

19. $\frac{-64+80a-16a^2}{-1+a} = -16a + 64; a \neq 1$

21. $(-\infty,-2)\cup(-2,6)\cup(6,\infty)$

23.



25. 31

27. Increasing on $(2,\infty)$, decreasing on $(-\infty, 2)$

29. Increasing on $(-3, 1)$, constant on $(-\infty,-3)$ and $(1,\infty)$

31. Local minimum: $(-2,-3)$; local maximum: $(1, 3)$

33. Absolute maximum: 10

35. $(f\circ g)(x) = 17 - 18x$, $(g\circ f)(x) = -7 - 18x$

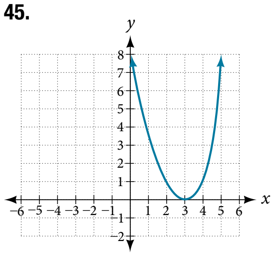
37. $(f\circ g)(x) = \sqrt{\frac{1}{x} + 2}$; $(g\circ f)(x) = \frac{1}{\sqrt{x+2}}$

39. $(f\circ g)(x) = \frac{\frac{1}{\frac{1}{x}+1}}{\frac{1}{\frac{1}{x}+4}} = \frac{1+x}{1+4x}$; Domain: $\left(-\infty,-\frac{1}{4}\right)\cup\left(-\frac{1}{4},0\right)\cup(0,\infty)$

41. $(f\circ g)(x) = \frac{1}{\sqrt{x}}$; Domain: $(0,\infty)$

43. Many solutions; one possible answer: $g(x) = \frac{2x-1}{3x+4}$ and $f(x) = \sqrt{x}$.

45.



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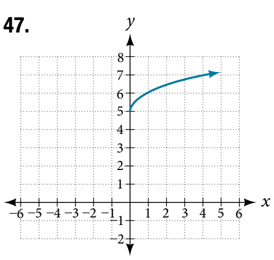
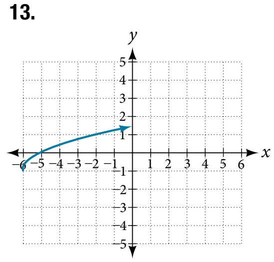


Chapter 3 Practice Test

1. Relation is a function **3.** -16 **5.** The graph is a parabola and the graph fails the horizontal line test.

7. $2a^2 - a$ **9.** $-2(a+b)+1$; $b\neq a$ **11.** $\sqrt{2}$

13.



15. Even **17.** Odd

19. $f^{-1}(x) = \frac{x+5}{3}$

21. $(-\infty,-1.1)$ and $(1.1,\infty)$

23. $(1.1,-0.9)$ **25.** $f(2) = 2$

27. $f(x) = \begin{cases} |x| & \text{if } x \le 2 \\ 3 & \text{if } x > 2 \end{cases}$

29. $x = 2$ **31.** Yes

33. $f^{-1}(x) = -\frac{x-11}{2}$ or $\frac{11-x}{2}$

CHAPTER 4

Section 4.1

1. Terry starts at an elevation of 3,000 feet and descends 70 feet per second. **3.** $d(t) = 100 - 10t$

5. The point of intersection is (a, a). This is because for the horizontal line, all of the y-coordinates are a and for the vertical line, all of the x-coordinates are a. The point of intersection is on both lines and therefore will have these two characteristics.

7. Yes **9.** Yes **11.** No **13.** Yes **15.** Increasing
17. Decreasing **19.** Decreasing **21.** Increasing
23. Decreasing **25.** 2 **27.** -2 **29.** $\frac{3}{5}x - 1$
31. $y = 3x - 2$ **33.** $y = -\frac{1}{3}x + \frac{11}{3}$ **35.** $y = -1.5x - 3$

37. Perpendicular **39.** Parallel

41. $f(0) = -(0) + 2$
$f(0) = 2$ y-int: $(0, 2)$
$0 = -x + 2$ x-int: $(2, 0)$

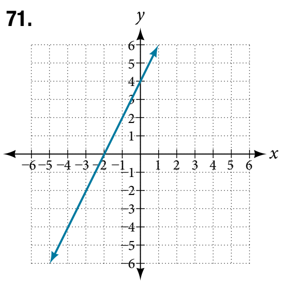
43. $h(0) = 3(0) - 5$
$h(0) = -5$ y-int: $(0, -5)$
$0 = 3x - 5$ x-int: $\left(\frac{5}{3}, 0\right)$

45. $-2x + 5 = 20$
$-2(0) + 5y = 20$ y-int: $(0, -5)$
$0 = 3x - 5$ x-int: $\left(\frac{5}{3}, 0\right)$

47. Line 1: $m = -10$, Line 2: $m = -10$, parallel
49. Line 1: $m = -2$, Line 2: $m = 1$, neither

51. Line 1: $m = -2$, Line 2: $m = -2$, parallel
53. $y = 3x - 3$ **55.** $y = -\frac{1}{3}t + 2$ **57.** 0 **59.** $y = -\frac{5}{4}x + 5$
61. $y = 3x - 1$ **63.** $y = -2.5$ **65.** F **67.** C **69.** A

71.



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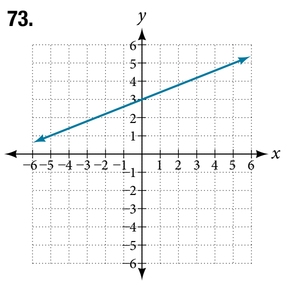

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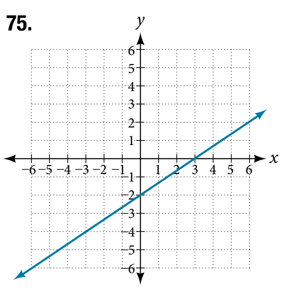

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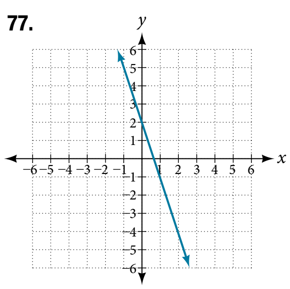

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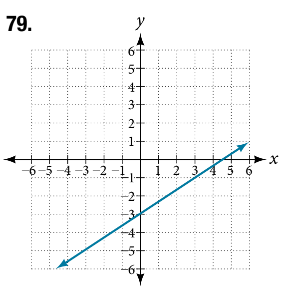

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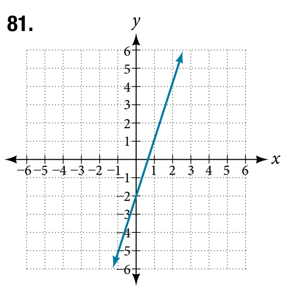
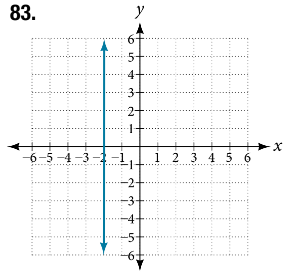
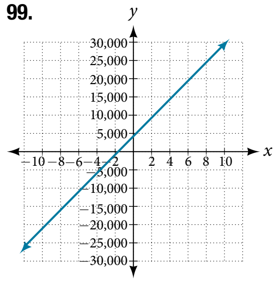


83.



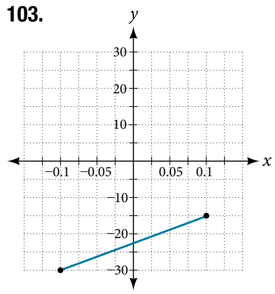
85. $y = 3$
87. $x = -3$
89. Linear, $g(x) = -3x + 5$
91. Linear, $f(x) = 5x - 5$
93. Linear, $g(x) = -\frac{25}{2}x + 6$
95. Linear, $f(x) = 10x - 24$
97. $f(x) = -58x + 17.3$

99.



101. **a.** $a = 11{,}900$, $b = 1001.1$
b. $q(p) = 1000p - 100$

103.



105. $x = -\frac{16}{3}$ **107.** $x = a$

109. $y = \frac{d}{c - a}x - \frac{ad}{c - a}$

111. $y = 100x - 98$

113. $x < \frac{1999}{201}$, $x > \frac{1999}{201}$

115. 69. $45 per training session

117. The rate of change is 0.1. For every additional minute talked, the monthly charge increases by $0.1 or 10 cents. The initial value is 24. When there are no minutes talked, initially the charge is $24. **119.** The slope is -400. this means for every year between 1960 and 1989, the population dropped by 400 per year in the city. **121.** C

Section 4.2

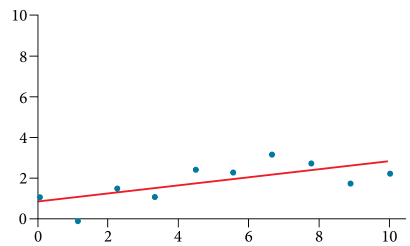
1. Determine the independent variable. This is the variable upon which the output depends. **3.** To determine the initial value, find the output when the input is equal to zero. **5.** 6 square units **7.** 20.01 square units **9.** 2,300 **11.** 64,170 **13.** $P(t) = 2500t + 75{,}000$ **15.** $(-30, 0)$ 30 years before the start of this model, the town has no citizens. $(0, 75{,}000)$ Initially, the town had a population of 75,000. **17.** Ten years after the model began **19.** $W(t) = 0.5t + 7.5$ **21.** $(-15, 0)$ The x-intercept is not a plausible set of data for this model because it means the baby weighed 0 pounds 15 months prior to birth. $(0, 7.5)$ The baby weighed 7.5 pounds at birth. **23.** At age 5.8 months **25.** $C(t) = 12{,}025 - 205t$ **27.** $(58.7, 0)$ In 58.7 years, the number of people afflicted with the common cold would be zero. $(0, 12{,}025)$ Initially, 12,025 people were afflicted with the common cold. **29.** 2063 **31.** $y = -2t + 180$ **33.** In 2070, the company's profits will be zero. **35.** $y = 30t - 300$ **37.** $(10, 0)$ In the year 1990, the company's profits were zero.

39. Hawaii **41.** During the year 1933 **43.** \$105,620 **45. a.** 696 people **b.** 4 years **c.** 174 people per year **d.** 305 people **e.** $P(t) = 305 + 174t$ **f.** 2,219 people **47. a.** $C(x) = 0.15x + 10$ **b.** The flat monthly fee is \$10 and there is a \$0.15 fee for each additional minute used. **c.** \$113.05 **49. a.** $P(t) = 190t + 4{,}360$ **b.** 6,640 moose **51. a.** $R(t) = -2.1t + 16$ **b.** 5.5 billion cubic feet **c.** During the year 2017 **53.** More than 133 minutes **55.** More than \$42,857.14 worth of jewelry **57.** More than \$66,666.67 in sales

Section 4.3

1. When our model no longer applies, after some value in the domain, the model itself doesn't hold. **3.** We predict a value outside the domain and range of the data. **5.** The closer the number is to 1, the less scattered the data, the closer the number is to 0, the more scattered the data. **7.** 61.966 years **9.** No **11.** No **13.** Interpolation, about 60° F **15.** C **17.** B

19.



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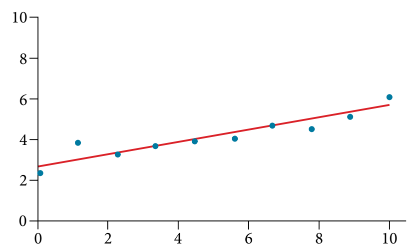
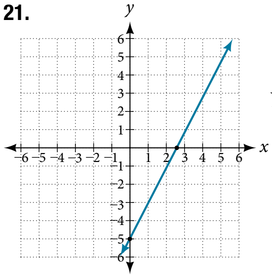


23. Yes, trend appears linear; during 2016 **25.** $y = 1.640x + 13.800$, $r = 0.987$ **27.** $y = -0.962x + 26.86$, $r = -0.965$ **29.** $y = -1.981x + 60.197$; $r = -0.998$ **31.** $y = 0.121x - 38.841$, $r = 0.998$ **33.** $(-2, -6)$, $(1, -12)$, $(5, -20)$, $(6, -22)$, $(9, -28)$ **35.** $(189.8, 0)$ If the company sells 18,980 units, its profits will be zero dollars. **37.** $y = 0.00587x + 1985.41$ **39.** $y = 20.25x - 671.5$ **41.** $y = -10.75x + 742.50$

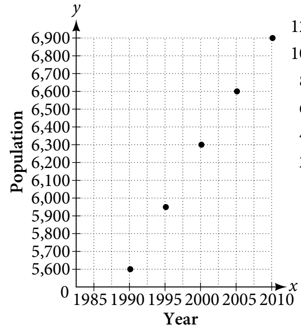
Chapter 4 Review Exercises

1. Yes **3.** Increasing **5.** $y = -3x + 26$ **7.** 3 **9.** $y = 2x - 2$ **11.** Not linear **13.** Parallel **15.** $(-9, 0)$; $(0, -7)$ **17.** Line 1: $m\ -2$, Line 2: $m = -2$, parallel **19.** $y = -0.2x + 21$

21.



23. 250 **25.** 118,000 **27.** $y = -300x + 11{,}500$ **29. a.** 800 **b.** 100 students per year **c.** $P(t) = 100t + 1700$ **31.** 18,500 **33.** $y = \$91{,}625$ **35.** Extrapolation



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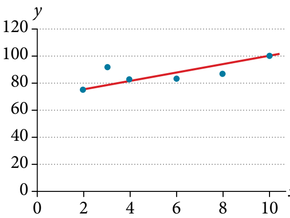
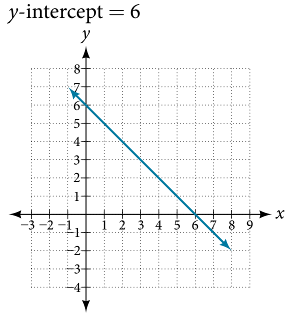


39. Midway through 2023 **41.** $y = -1.294x + 49.412$; $r = -0.974$ **43.** Early in 2027 **45.** 7,660

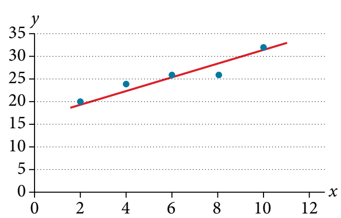
Chapter 4 Practice Test

1. Yes **3.** Increasing **5.** $y = -1.5x - 6$ **7.** $y = -2x - 1$ **9.** No **11.** Perpendicular **13.** $(-7, 0)$; $(0, -2)$ **15.** $y = -0.25x + 12$ **17.** Slope $= -1$ and y-intercept $= 6$



19. 150 **21.** 165,000 **23.** $y = 875x + 10{,}625$ **25. a.** 375 **b.** dropped an average of 46.875, or about 47 people per year **c.** $y = -46.875t + 1250$

27.



29. Early in 2018 **31.** $y = 0.00455x + 1979.5$ **33.** $r = 0.999$

CHAPTER 5

Section 5.1

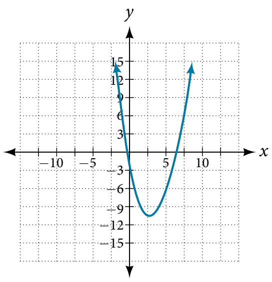
1. When written in that form, the vertex can be easily identified. **3.** If $a = 0$ then the function becomes a linear function. **5.** If possible, we can use factoring. Otherwise, we can use the quadratic formula. **7.** $g(x) = (x + 1)^2 - 4$; vertex: $(-1, -4)$ **9.** $f(x) = \left(x + \frac{5}{2}\right)^2 - \frac{33}{4}$; vertex: $\left(-\frac{5}{2}, -\frac{33}{4}\right)$ **11.** $k(x) = 3(x - 1)^2 - 12$; vertex: $(1, -12)$ **13.** $f(x) = 3\left(x - \frac{5}{6}\right)^2 - \frac{37}{12}$; vertex: $\left(\frac{5}{6}, -\frac{37}{12}\right)$ **15.** Minimum is $-\frac{17}{2}$ and occurs at $\frac{5}{2}$; axis of symmetry: $x = \frac{5}{2}$ **17.** Minimum is $-\frac{17}{16}$ and occurs at $-\frac{1}{8}$; axis of symmetry: $x = -\frac{1}{8}$ **19.** Minimum is $-\frac{7}{2}$ and occurs at -3; axis of symmetry: $x = -3$

21. Domain: $(-\infty, \infty)$; range: $[2, \infty)$ **23.** Domain: $(-\infty, \infty)$; range: $[-5, \infty)$ **25.** Domain: $(-\infty, \infty)$; range: $[-12, \infty)$

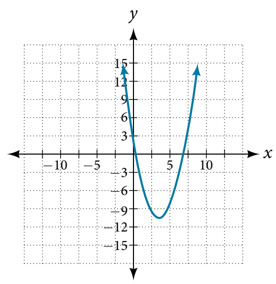
27. $f(x) = x^2 + 4x + 3$ **29.** $f(x) = x^2 - 4x + 7$

31. $f(x) = -\frac{1}{49}x^2 + \frac{6}{49}x + \frac{89}{49}$ **33.** $f(x) = x^2 - 2x + 1$

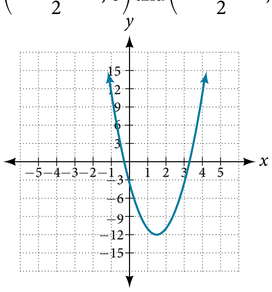
35. Vertex: $(3, -10)$, axis of symmetry: $x = 3$, intercepts: $(3 + \sqrt{10}, 0)$ and $(3 - \sqrt{10}, 0)$



37. Vertex: $\left(\frac{7}{2}, -\frac{37}{4}\right)$, axis of symmetry: $x = \frac{7}{2}$, intercepts: $\left(\frac{7 + \sqrt{37}}{2}, 0\right)$ and $\left(\frac{7 - \sqrt{37}}{2}, 0\right)$



39. Vertex: $\left(\frac{3}{2}, -12\right)$, axis of symmetry: $x = \frac{3}{2}$, intercept: $\left(\frac{3 + 2\sqrt{3}}{2}, 0\right)$ and $\left(\frac{3 - 2\sqrt{3}}{2}, 0\right)$



41. $f(x) = x^2 + 2x + 3$

43. $f(x) = -3x^2 - 6x - 1$

45. $f(x) = -\frac{1}{4}x^2 - x + 2$

47. $f(x) = x^2 + 2x + 1$

49. $f(x) = -x^2 + 2x$

51. The value stretches or compresses the width of the graph. The greater the value, the narrower the graph.

53. The graph is shifted to the right or left (a horizontal shift).

55. The suspension bridge has 1,000 feet distance from the center.

57. Domain: $(-\infty, \infty)$; range: $(-\infty, 2]$

59. Domain: $(-\infty, \infty)$; range: $[100, \infty)$ **61.** $f(x) = 2x^2 + 2$ **63.** $f(x) = -x^2 - 2$ **65.** $f(x) = 3x^2 + 6x - 15$ **67.** 75 feet by 50 feet **69.** 3 and 3; product is 9 **71.** The revenue reaches the maximum value when 1800 thousand phones are produced. **73.** 2.449 seconds **75.** 41 trees per acre

Section 5.2

1. The coefficient of the power function is the real number that is multiplied by the variable raised to a power. The degree is the highest power appearing in the function. **3.** As x decreases without bound, so does $f(x)$. As x increases without bound, so does $f(x)$. **5.** The polynomial function is of even degree and leading coefficient is negative. **7.** Power function **9.** Neither **11.** Neither **13.** Degree: 2, coefficient: -2 **15.** Degree: 4, coefficient: -2 **17.** As $x \to \infty, f(x) \to \infty$, as $x \to -\infty, f(x) \to \infty$

19. As $x \to -\infty, f(x) \to -\infty$, as $x \to \infty, f(x) \to -\infty$

21. As $x \to -\infty, f(x) \to -\infty$, as $x \to \infty, f(x) \to -\infty$

23. As $x \to \infty, f(x) \to \infty$, as $x \to -\infty, f(x) \to -\infty$

25. y-intercept is $(0, 12)$, t-intercepts are $(1, 0)$, $(-2, 0)$, and $(3, 0)$

27. y-intercept is $(0, -16)$, x-intercepts are $(2, 0)$, and $(-2, 0)$

29. y-intercept is $(0, 0)$, x-intercepts are $(0, 0)$, $(4, 0)$, and $(-2, 0)$

31. 3 **33.** 5 **35.** 3 **37.** 5 **39.** Yes, 2 turning points, least possible degree: 3 **41.** Yes, 1 turning point, least possible degree: 2

43. Yes, 0 turning points, least possible degree: 1

45. Yes, 0 turning points, least possible degree: 1

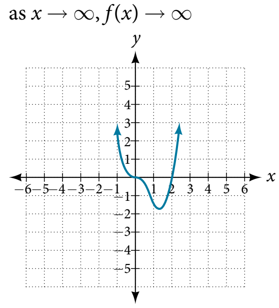
47. As $x \to -\infty, f(x) \to \infty$, as $x \to \infty, f(x) \to \infty$

x	$f(x)$
10	9,500
100	99,950,000
-10	9,500
-100	99,950,000

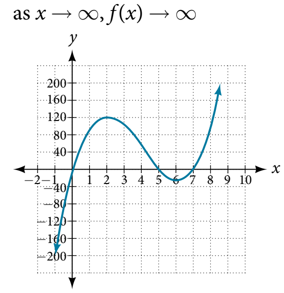
49. As $x \to -\infty, f(x) \to \infty$, as $x \to \infty, f(x) \to -\infty$

x	$f(x)$
10	-504
100	$-941{,}094$
-10	1,716
-100	1,061,106

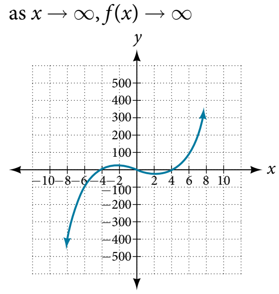
51. y-intercept: $(0, 0)$; x-intercepts: $(0, 0)$ and $(2, 0)$; as $x \to -\infty, f(x) \to \infty$, as $x \to \infty, f(x) \to \infty$



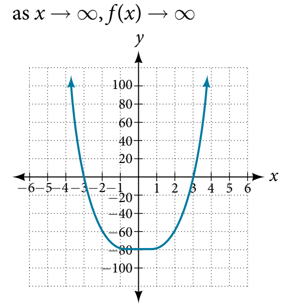
53. y-intercept: $(0, 0)$; x-intercepts: $(0, 0)$, $(5, 0)$, $(7, 0)$; as $x \to -\infty, f(x) \to -\infty$, as $x \to \infty, f(x) \to \infty$



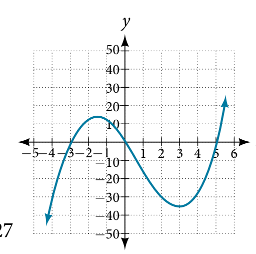
55. y-intercept: $(0, 0)$; x-intercepts: $(-4, 0)$, $(0, 0)$, $(4, 0)$; as $x \to -\infty, f(x) \to -\infty$, as $x \to \infty, f(x) \to \infty$



57. y-intercept: $(0, -81)$; x-intercepts: $(-3, 0)$, and $(3, 0)$; as $x \to -\infty, f(x) \to \infty$, as $x \to \infty, f(x) \to \infty$



59. y-intercept: $(0, 0)$; x-intercepts: $(-3, 0)$, $(0, 0)$, $(5, 0)$; as $x \to -\infty, f(x) \to -\infty$, as $x \to \infty, f(x) \to \infty$



61. $f(x) = x^2 - 4$

63. $f(x) = x^3 - 4x^2 + 4x$

65. $f(x) = x^4 + 1$

67. $V(m) = 8m^3 + 36m^2 + 54m + 27$

69. $V(x) = 4x^3 - 32x^2 + 64x$

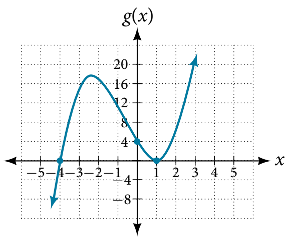
Section 5.3

1. The x-intercept is where the graph of the function crosses the x-axis, and the zero of the function is the input value for which $f(x) = 0$. **3.** If we evaluate the function at a and at b and the sign of the function value changes, then we know a zero exists between a and b. **5.** There will be a factor raised to an even power. **7.** $(-2, 0)$, $(3, 0)$, $(-5, 0)$ **9.** $(3, 0)$, $(-1, 0)$, $(0, 0)$

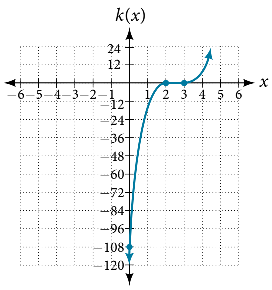
11. $(0, 0)$, $(-5, 0)$, $(2, 0)$ **13.** $(0, 0)$, $(-5, 0)$, $(4, 0)$

15. $(2, 0)$, $(-2, 0)$, $(-1, 0)$ **17.** $(-2, 0)$, $(2, 0)$, $\left(\frac{1}{2}, 0\right)$

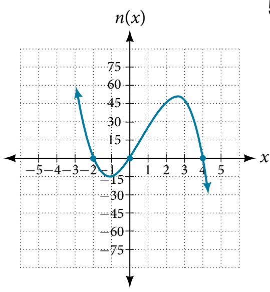
19. $(1, 0), (-1, 0)$ **21.** $(0, 0), \left(\sqrt{3}, 0\right), \left(-\sqrt{3}, 0\right)$
23. $(0, 0), (1, 0), (-1, 0), (2, 0), (-2, 0)$
25. $f(2) = -10, f(4) = 28$; sign change confirms
27. $f(1) = 3, f(3) = -77$; sign change confirms
29. $f(0.01) = 1.000001, f(0.1) = -7.999$; sign change confirms
31. 0 with multiplicity 2, $-\frac{3}{2}$ multiplicity 5, 4 multiplicity 2
33. 0 with multiplicity 2, -2 with multiplicity 2
35. $-\frac{2}{3}$ with multiplicity 5, 5 with multiplicity 2 **37.** 0 with multiplicity 4, 2 with multiplicity 1, -1 with multiplicity 1
39. $\frac{3}{2}$ with multiplicity 2, 0 with multiplicity 3 **41.** 0 with multiplicity 6, $\frac{2}{3}$ with multiplicity 2
43. x-intercept: $(1, 0)$ with multiplicity 2, $(-4, 0)$ with multiplicity 1; y-intercept: $(0, 4)$; as $x \to -\infty, g(x) \to -\infty$, as $x \to \infty, g(x) \to \infty$



45. x-intercept: $(3, 0)$ with multiplicity 3, $(2, 0)$ with multiplicity 2; y-intercept: $(0, -108)$; as $x \to -\infty, k(x) \to -\infty$, as $x \to \infty, k(x) \to \infty$



47. x-intercepts: $(0, 0)$, $(-2, 0)$, $(4, 0)$ with multiplicity 1; y-intercept: $(0, 0)$; as $x \to -\infty, n(x) \to \infty$, as $x \to \infty, n(x) \to -\infty$



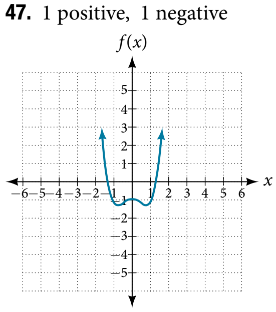
49. $f(x) = -\frac{2}{9}(x-3)(x+1)(x+3)$
51. $f(x) = \frac{1}{4}(x+2)^2(x-3)$
53. $-4, -2, 1, 3$ with multiplicity 1
55. $-2, 3$ each with multiplicity 2
57. $f(x) = -\frac{2}{3}(x+2)(x-1)(x-3)$
59. $f(x) = \frac{1}{3}(x-3)^2(x-1)^2(x+3)$
61. $f(x) = -15(x-1)^2(x-3)^3$
63. $f(x) = -2(x+3)(x+2)(x-1)$
65. $f(x) = -\frac{3}{2}(2x-1)^2(x-6)(x+2)$
67. Local max: $(-0.58, -0.62)$; local min: $(0.58, -1.38)$
69. Global min: $(-0.63, -0.47)$ **71.** Global min: $(0.75, -1.11)$
73. $f(x) = (x-500)^2(x+200)$ **75.** $f(x) = 4x^3 - 36x^2 + 80x$
77. $f(x) = 4x^3 - 36x^2 + 60x + 100$
79. $f(x) = \frac{1}{\pi}(9x^3 + 45x^2 + 72x + 36)$

Section 5.4

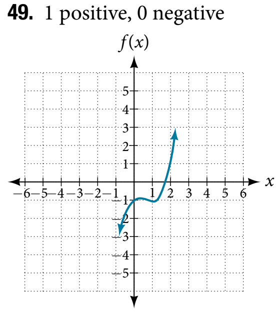
1. The binomial is a factor of the polynomial.
3. $x + 6 + \frac{5}{x-1}$, quotient: $x + 6$, remainder: 5
5. $3x + 2$, quotient: $3x + 2$, remainder: 0 **7.** $x - 5$, quotient: $x - 5$, remainder: 0 **9.** $2x - 7 + \frac{16}{x+2}$, quotient: $2x - 7$, remainder 16 **11.** $x - 2 + \frac{6}{3x+1}$, quotient: $x - 2$, remainder: 6
13. $2x^2 - 3x + 5$, quotient: $2x^2 - 3x + 5$, remainder: 0
15. $2x^2 + 2x + 1 + \frac{10}{x-4}$ **17.** $2x^2 - 7x + 1 - \frac{2}{2x+1}$
19. $3x^2 - 11x + 34 - \frac{106}{x+3}$ **21.** $x^2 + 5x + 1$
23. $4x^2 - 21x + 84 - \frac{323}{x+4}$ **25.** $x^2 - 14x + 49$
27. $3x^2 + x + \frac{2}{3x-1}$ **29.** $x^3 - 3x + 1$ **31.** $x^3 - x^2 + 2$
33. $x^3 - 6x^2 + 12x - 8$ **35.** $x^3 - 9x^2 + 27x - 27$
37. $2x^3 - 2x + 2$ **39.** Yes, $(x-2)(3x^3 - 5)$ **41.** Yes, $(x-2)(4x^3 + 8x^2 + x + 2)$ **43.** No
45. $(x-1)(x^2 + 2x + 4)$ **47.** $(x-5)(x^2 + x + 1)$
49. Quotient: $4x^2 + 8x + 16$, remainder: -1 **51.** Quotient is $3x^2 + 3x + 5$, remainder: 0 **53.** Quotient is $x^3 - 2x^2 + 4x - 8$, remainder: -6 **55.** $x^6 - x^5 + x^4 - x^3 + x^2 - x + 1$
57. $x^3 - x^2 + x - 1 + \frac{1}{x+1}$ **59.** $1 + \frac{1+i}{x-i}$ **61.** $1 + \frac{1-i}{x+i}$
63. $x^2 + ix - 1 + \frac{1-i}{x-i}$ **65.** $2x^2 + 3$ **67.** $2x + 3$
69. $x + 2$ **71.** $x - 3$ **73.** $3x^2 - 2$

Section 5.5

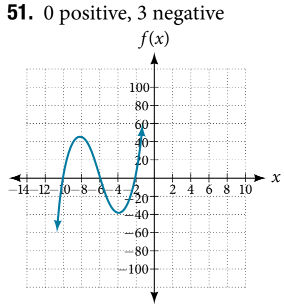
1. The theorem can be used to evaluate a polynomial.
3. Rational zeros can be expressed as fractions whereas real zeros include irrational numbers. **5.** Polynomial functions can have repeated zeros, so the fact that number is a zero doesn't preclude it being a zero again. **7.** -106 **9.** 0 **11.** 255
13. -1 **15.** $-2, 1, \frac{1}{2}$ **17.** -2 **19.** -3
21. $-\frac{5}{2}, \sqrt{6}, -\sqrt{6}$ **23.** $2, -4, -\frac{3}{2}$ **25.** $4, -4, -5$
27. $5, -3, -\frac{1}{2}$ **29.** $\frac{1}{2}, \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}$
31. $\frac{3}{2}$ **33.** $2, 3, -1, -2$ **35.** $\frac{1}{2}, -\frac{1}{2}, 2, -3$
37. $-1, -1, \sqrt{5}, -\sqrt{5}$ **39.** $-\frac{3}{4}, -\frac{1}{2}$ **41.** $2, 3 + 2i, 3 - 2i$
43. $-\frac{2}{3}, 1 + 2i, 1 - 2i$ **45.** $-\frac{1}{2}, 1 + 4i, 1 - 4i$
47. 1 positive, 1 negative



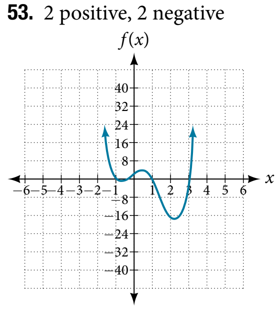
49. 1 positive, 0 negative



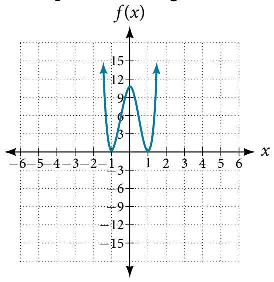
51. 0 positive, 3 negative



53. 2 positive, 2 negative



55. 2 positive, 2 negative



57. $\pm\frac{1}{2}, \pm 1, \pm 5, \pm\frac{5}{2}$

59. $\pm 1, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{6}$

61. $1, \frac{1}{2}, -\frac{1}{3}$

63. $2, \frac{1}{4}, -\frac{3}{2}$ **65.** $\frac{5}{4}$

67. $f(x) = \frac{4}{9}(x^3 + x^2 - x - 1)$

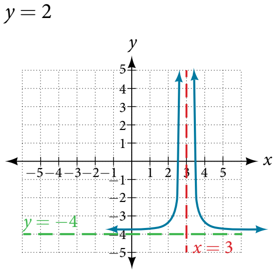
69. $f(x) = -\frac{1}{5}(4x^3 - x)$

71. 8 by 4 by 6 inches

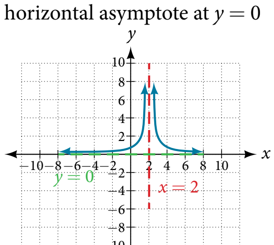
73. 5.5 by 4.5 by 3.5 inches **75.** 8 by 5 by 3 inches **77.** Radius: 6 meters; height: 2 meters **79.** Radius: 2.5 meters, height: 4.5 meters

Section 5.6

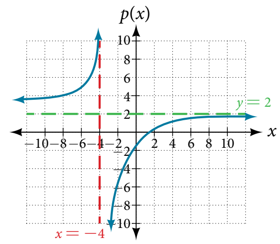
1. The rational function will be represented by a quotient of polynomial functions. **3.** The numerator and denominator must have a common factor. **5.** Yes. The numerator of the formula of the functions would have only complex roots and/or factors common to both the numerator and denominator.
7. All reals except $x = -1, 1$ **9.** All reals except $x = -1, 1, -2, 2$
11. Vertical asymptote: $x = -\frac{2}{5}$; horizontal asymptote: $y = 0$; domain: all reals except $x = -\frac{2}{5}$ **13.** Vertical asymptotes: $x = 4, -9$; horizontal asymptote: $y = 0$; domain: all reals except $x = 4, -9$ **15.** Vertical asymptotes: $x = 0, 4, -4$; horizontal asymptote: $y = 0$; domain: all reals except $x = 0, 4, -4$
17. Vertical asymptotes: $x = -5$; horizontal asymptote: $y = 0$; domain: all reals except $x = 5, -5$
19. Vertical asymptote: $x = \frac{1}{3}$; horizontal asymptote: $y = -\frac{2}{3}$; domain: all reals except $x = \frac{1}{3}$ **21.** None
23. x-intercepts: none, y-intercept: $\left(0, \frac{1}{4}\right)$
25. Local behavior: $x \to -\frac{1}{2}^+, f(x) \to -\infty, x \to -\frac{1}{2}^-, f(x) \to \infty$
End behavior: $x \to \pm\infty, f(x) \to \frac{1}{2}$
27. Local behavior: $x \to 6^+, f(x) \to -\infty, x \to 6^-, f(x) \to \infty$
End behavior: $x \to \pm\infty, f(x) \to -2$
29. Local behavior: $x \to -\frac{1}{3}^+, f(x) \to \infty, x \to -\frac{1}{3}^-, f(x) \to -\infty$, $x \to -\frac{5}{2}^-, f(x) \to \infty, x \to -\frac{5}{2}^+, f(x) \to -\infty$
End behavior: $x \to \pm\infty, f(x) \to \frac{1}{3}$
31. $y = 2x + 4$ **33.** $y = 2x$
35. Vertical asymptote at $x = 0$, horizontal asymptote at $y = 2$



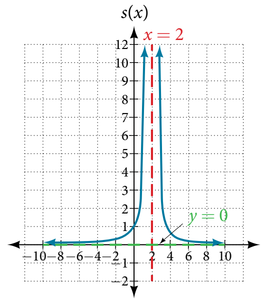
37. Vertical asymptote at $x = 2$, horizontal asymptote at $y = 0$



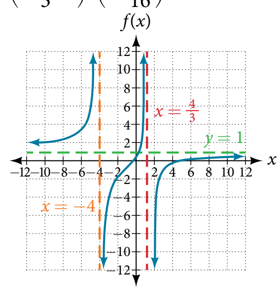
39. Vertical asymptote at $x = -4$; horizontal asymptote at $y = 2$; $\left(\frac{3}{2}, 0\right), \left(0, -\frac{3}{4}\right)$



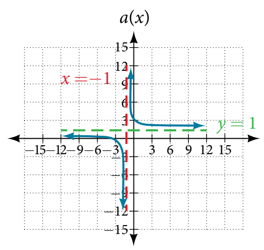
41. Vertical asymptote at $x = 2$; horizontal asymptote at $y = 0$; $(0, 1)$



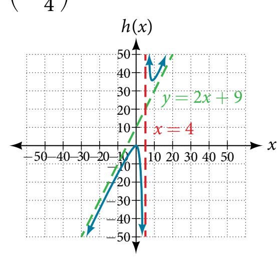
43. Vertical asymptote at $x = -4, \frac{4}{3}$; horizontal asymptote at $y = 1$; $(5, 0)$, $\left(-\frac{1}{3}, 0\right), \left(0, \frac{5}{16}\right)$



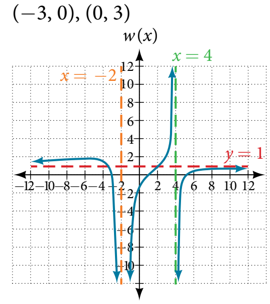
45. Vertical asymptote at $x = -1$; horizontal asymptote at $y = 1$; $(-3, 0)$, $(0, 3)$



47. Vertical asymptote at $x = 4$; slant asymptote at $y = 2x + 9$; $(-1, 0)$, $\left(\frac{1}{2}, 0\right)$, $\left(0, \frac{1}{4}\right)$



49. Vertical asymptote at $x = -2, 4$; horizontal asymptote at $y = 1$; $(-3, 0)$, $(0, 3)$



51. $f(x) = 50\frac{x^2 - x - 2}{x^2 - 25}$ **53.** $f(x) = 7\frac{x^2 + 2x - 24}{x^2 + 9x + 20}$

55. $f(x) = \frac{1}{2} \cdot \frac{x^2 - 4x + 4}{x + 1}$ **57.** $f(x) = 4\frac{x - 3}{x^2 - x - 12}$

59. $f(x) = -9\frac{x - 2}{x^2 - 9}$ **61.** $f(x) = \frac{1}{3} \cdot \frac{x^2 + x - 6}{3x - 1}$

63. $f(x) = -6\frac{(x - 1)^2}{(x + 3)(x - 2)^2}$

65. Vertical asymptote at $x = 2$; horizontal asymptote at $y = 0$

x	2.01	2.001	2.0001	1.99	1.999
y	100	1,000	10,000	−100	−1,000

x	10	100	1,000	10,000	100,000
y	0.125	0.0102	0.001	0.0001	0.00001

67. Vertical asymptote at $x = -4$; horizontal asymptote at $y = 2$

x	-4.1	-4.01	-4.001	-3.99	-3.999
y	82	802	8,002	-798	-7998

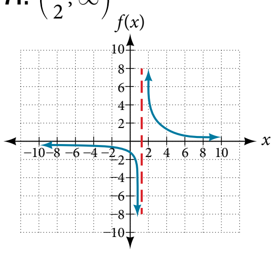
x	10	100	1,000	10,000	100,000
y	1.4286	1.9331	1.992	1.9992	1.999992

69. Vertical asymptote at $x = -1$; horizontal asymptote at $y = 1$

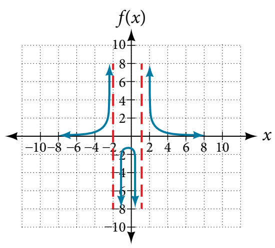
x	-0.9	-0.99	-0.999	-1.1	-1.01
y	81	9,801	998,001	121	10,201

x	10	100	1,000	10,000	100,000
y	0.82645	0.9803	0.998	0.9998	

71. $\left(\frac{3}{2}, \infty\right)$



73. $(-\infty, 1) \cup (4, \infty)$



75. $(2, 4)$ **77.** $(2, 5)$ **79.** $(-1, 1)$ **81.** $C(t) = \dfrac{8 + 2t}{300 + 20t}$

83. After about 6.12 hours **85.** 2 by 2 by 5 feet

87. radius 2.52 meters

Section 5.7

1. It can be too difficult or impossible to solve for x in terms of y.

3. We will need a restriction on the domain of the answer. **5.** $f^{-1}(x) = \sqrt{x} + 4$ **7.** $f^{-1}(x) = \sqrt{x + 3} - 1$

9. $f^{-1}(x) = \sqrt{12 - x}$ **11.** $f^{-1}(x) = \pm\sqrt{\dfrac{x - 4}{2}}$

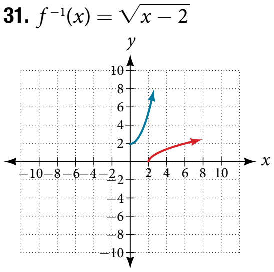
13. $f^{-1}(x) = \sqrt[3]{\dfrac{x - 1}{3}}$ **15.** $f^{-1}(x) = \sqrt[3]{\dfrac{4 - x}{2}}$

17. $f^{-1}(x) = \dfrac{3 - x^2}{4}$, $[0, \infty)$ **19.** $f^{-1}(x) = \dfrac{(x - 5)^2 + 8}{6}$

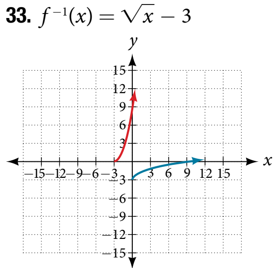
21. $f^{-1}(x) = (3 - x)^3$ **23.** $f^{-1}(x) = \dfrac{4x + 3}{x}$ **25.** $f^{-1}(x) = \dfrac{7x + 2}{1 - x}$

27. $f^{-1}(x) = \dfrac{2x - 1}{5x + 5}$ **29.** $f^{-1}(x) = \sqrt{x + 3} - 2$

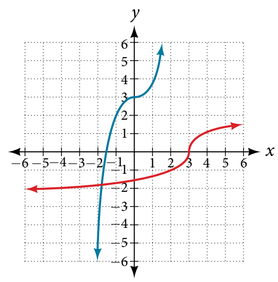
31. $f^{-1}(x) = \sqrt{x - 2}$



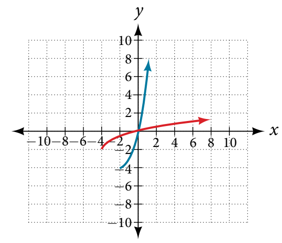
33. $f^{-1}(x) = \sqrt{x} - 3$



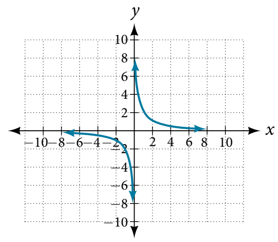
35. $f^{-1}(x) = \sqrt[3]{x - 3}$



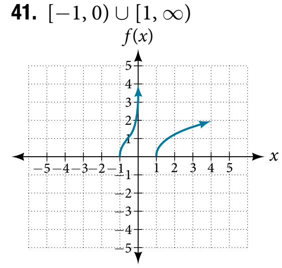
37. $f^{-1}(x) = \sqrt{x + 4} - 2$



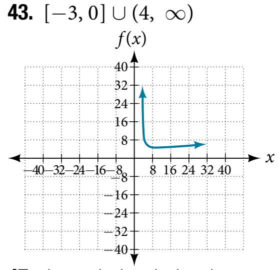
39.



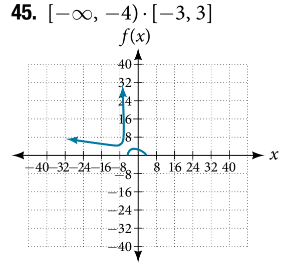
41. $[-1, 0) \cup [1, \infty)$



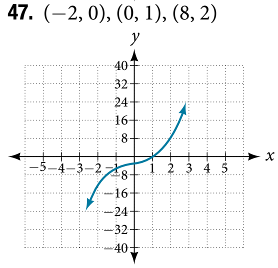
43. $[-3, 0] \cup (4, \infty)$



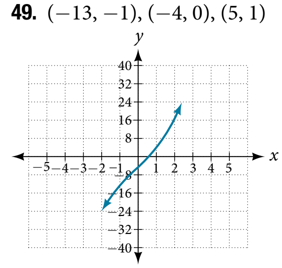
45. $[-\infty, -4) \cdot [-3, 3]$



47. $(-2, 0), (0, 1), (8, 2)$



49. $(-13, -1), (-4, 0), (5, 1)$



51. $f^{-1}(x) = \sqrt[3]{\dfrac{x - b}{a}}$ **53.** $f^{-1}(x) = \sqrt{\dfrac{x^2 - b}{a}}$

55. $f^{-1}(x) = \dfrac{cx - b}{a - x}$ **57.** $t(h) = \sqrt{\dfrac{600 - h}{16}}$, 3.54 seconds

59. $r(A) = \sqrt{\dfrac{A}{4\pi}}$, ≈ 8.92 in. **61.** $l(T) = 32.2\left(\dfrac{T}{2\pi}\right)^2$, ≈ 3.26 ft

63. $r(A) = \sqrt{\dfrac{A + 8\pi}{2\pi}} - 2$, 3.99 ft **65.** $r(V) = \sqrt{\dfrac{V}{10\pi}}$, ≈ 5.64 ft

Section 5.8

1. The graph will have the appearance of a power function.

3. No. Multiple variables may jointly vary. **5.** $y = 5x^2$

7. $y = 10x^3$ **9.** $y = 6x^4$ **11.** $y = \dfrac{18}{x^2}$ **13.** $y = \dfrac{81}{x^4}$

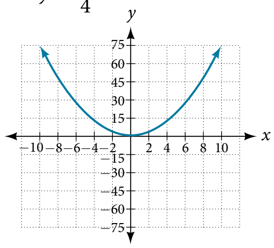
15. $y = \dfrac{20}{\sqrt[3]{x}}$ **17.** $y = 10xzw$ **19.** $y = 10x\sqrt{z}$

21. $y = 4\dfrac{xz}{w}$ **23.** $y = 40\dfrac{xz}{\sqrt{w}t^2}$ **25.** $y = 256$

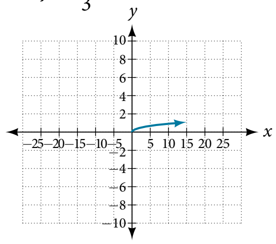
27. $y = 6$ **29.** $y = 6$ **31.** $y = 27$ **33.** $y = 3$

35. $y = 18$ **37.** $y = 90$ **39.** $y = \dfrac{81}{2}$

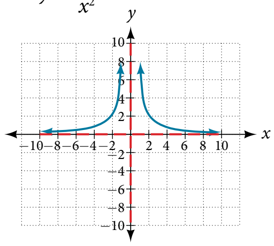
41. $y = \frac{3}{4}x^2$



43. $y = \frac{1}{3}\sqrt{x}$



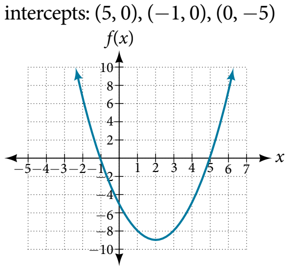
45. $y = \frac{4}{x^2}$



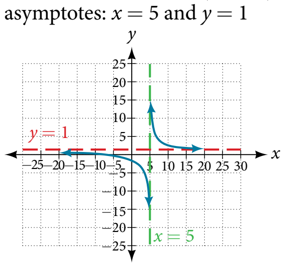
47. ≈ 1.89 years
49. ≈ 0.61 years
51. 3 seconds
53. 48 inches
55. ≈ 49.75 pounds
57. ≈ 33.33 amperes
59. ≈ 2.88 inches

Chapter 5 Review Exercises

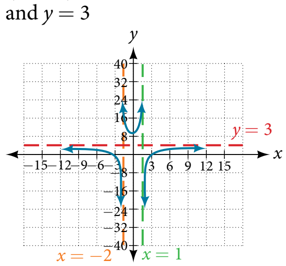
1. $f(x) = (x-2)^2 - 9$; vertex: $(2, -9)$; intercepts: $(5, 0)$, $(-1, 0)$, $(0, -5)$



3. $f(x) = \frac{3}{25}(x+2)^2 + 3$
5. 300 meters by 150 meters, the longer side parallel to the river
7. Yes; degree: 5, leading coefficient: 4
9. Yes; degree: 4; leading coefficient: 1
11. As $x \to -\infty, f(x) \to -\infty$, as $x \to \infty, f(x) \to \infty$
13. -3 with multiplicity 2, $-\frac{1}{2}$ with multiplicity 1, -1 with multiplicity 3 **15.** 4 with multiplicity 1 **17.** $\frac{1}{2}$ with multiplicity 1, 3 with multiplicity 3 **19.** $x^2 + 4$ with remainder is 12 **21.** $x^2 - 5x + 20 - \frac{61}{x+3}$
23. $2x^2 - 2x - 3$, so factored form is $(x+4)(2x^2 - 2x - 3)$
25. $\left\{-2, 4, -\frac{1}{2}\right\}$ **27.** $\left\{1, 3, 4, \frac{1}{2}\right\}$
29. 2 or 0 positive, 1 negative
31. Intercepts: $(-2, 0)$, $\left(0, -\frac{2}{5}\right)$, asymptotes: $x = 5$ and $y = 1$



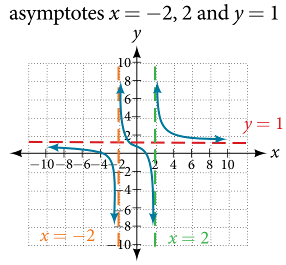
33. Intercepts: $(3, 0)$, $(-3, 0)$, $\left(0, \frac{27}{2}\right)$; asymptotes: $x = 1, -2$ and $y = 3$



35. $y = x - 2$ **37.** $f^{-1}(x) = \sqrt{x} + 2$ **39.** $f^{-1}(x) = \sqrt{x+11} - 3$
41. $f^{-1}(x) = \frac{(x+3)^2 - 5}{4}, x \geq -3$ **43.** $y = 64$ **45.** $y = 72$
47. ≈ 148.5 pounds

Chapter 5 Practice Test

1. Degree: 5, leading coefficient: -2 **3.** As $x \to -\infty$, $f(x) \to \infty$, as $x \to \infty, f(x) \to \infty$ **5.** $f(x) = 3(x-2)^2$
7. 3 with multiplicity 3, $\frac{1}{3}$ with multiplicity 1, 1with multiplicity 2
9. $-\frac{1}{2}$ with multiplicity 3, 2 with multiplicity 2
11. $x^3 + 2x^2 + 7x + 14 + \frac{26}{x-2}$ **13.** $\left\{-3, -1, \frac{3}{2}\right\}$
15. 1, -2, and $-\frac{3}{2}$ (multiplicity 2)
17. $f(x) = -\frac{2}{3}(x-3)^2(x-1)(x+2)$ **19.** 2 or 0 positive, 1 negative
21. $(-3, 0)$, $(1, 0)$, $\left(0, \frac{3}{4}\right)$; asymptotes $x = -2$, 2 and $y = 1$



23. $f^{-1}(x) = (x-4)^2 + 2, x \geq 4$
25. $f^{-1}(x) = \frac{x+3}{3x-2}$
27. $y = 20$

CHAPTER 6

Section 6.1

1. Linear functions have a constant rate of change. Exponential functions increase based on a percent of the original.
3. When interest is compounded, the percentage of interest earned to principal ends up being greater than the annual percentage rate for the investment account. Thus, the annual percentage rate does not necessarily correspond to the real interest earned, which is the very definition of *nominal*. **5.** Exponential; the population decreases by a proportional rate. **7.** Not exponential; the charge decreases by a constant amount each visit, so the statement represents a linear function. **9.** Forest B
11. After 20 years forest A will have 43 more trees than forest B.
13. Answers will vary. Sample response: For a number of years, the population of forest A will increasingly exceed forest B, but because forest B actually grows at a faster rate, the population will eventually become larger than forest A and will remain that way as long as the population growth models hold. Some factors that might influence the long-term validity of the exponential growth model are drought, an epidemic that culls the population, and other environmental and biological factors.
15. Exponential growth; the growth factor, 1.06, is greater than 1.
17. Exponential decay; the decay factor, 0.97, is between 0 and 1.
19. $f(x) = 2000(0.1)^x$ **21.** $f(x) = \left(\frac{1}{6}\right)^{-\frac{3}{5}}\left(\frac{1}{6}\right)^{\frac{x}{5}} \approx 2.93(0.699)^x$
23. Linear **25.** Neither **27.** Linear **29.** \$10,250
31. \$13,268.58 **33.** $P = A(t) \cdot \left(1 + \frac{r}{n}\right)^{-nt}$ **35.** \$4,569.10
37. 4% **39.** Continuous growth; the growth rate is greater than 0. **41.** Continuous decay; the growth rate is less than 0.
43. \$669.42 **45.** $f(-1) = -4$ **47.** $f(-1) \approx -0.2707$
49. $f(3) \approx 483.8146$ **51.** $y = 3 \cdot 5^x$ **53.** $y \approx 18 \cdot 1.025^x$
55. $y \approx 0.2 \cdot 1.95^x$

57. $\text{APY} = \dfrac{A(t) - a}{a} = \dfrac{a\left(1 + \frac{r}{365}\right)^{365(1)} - a}{a} = \dfrac{a\left[\left(1 + \frac{r}{365}\right)^{365} - 1\right]}{a} = \left(1 + \frac{r}{365}\right)^{365} - 1$; $I(n) = \left(1 + \frac{r}{n}\right)^{n} - 1$

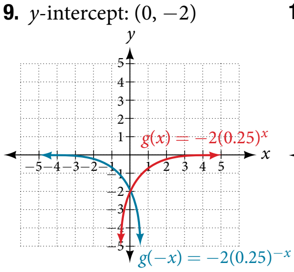
59. Let f be the exponential decay function $f(x) = a \cdot \left(\frac{1}{b}\right)^{x}$ such that $b > 1$. Then for some number $n > 0$, $f(x) = a \cdot \left(\frac{1}{b}\right)^{x} = a(b^{-1})^{x} = a((e^{n})^{-1})^{x} = a(e^{-n})^{x} = a(e)^{-nx}$.

61. 47,622 foxes **63.** 1.39%; \$155,368.09 **65.** \$35,838.76 **67.** \$82,247.78; \$449.75

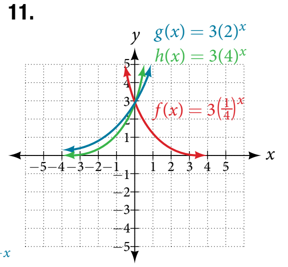
Section 6.2

1. An asymptote is a line that the graph of a function approaches, as x either increases or decreases without bound. The horizontal asymptote of an exponential function tells us the limit of the function's values as the independent variable gets either extremely large or extremely small. **3.** $g(x) = 4(3)^{-x}$; y-intercept: $(0, 4)$; domain: all real numbers; range: all real numbers greater than 0. **5.** $g(x) = -10^{x} + 7$; y-intercept: $(0, 6)$; domain: all real numbers; range: all real numbers less than 7. **7.** $g(x) = 2\left(\frac{1}{4}\right)^{x}$; y-intercept: $(0, 2)$; domain: all real numbers; range: all real numbers greater than 0.

9. y-intercept: $(0, -2)$

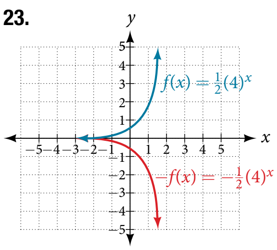


11.

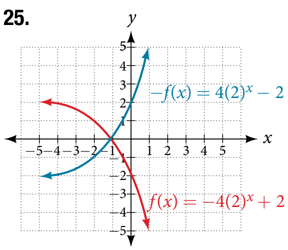


13. B **15.** A **17.** E **19.** D **21.** C

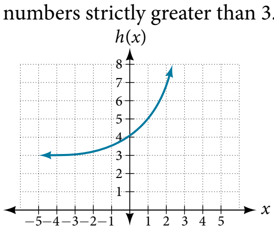
23.



25.



27. Horizontal asymptote: $h(x) = 3$; domain: all real numbers; range: all real numbers strictly greater than 3.



29. As $x \to \infty, f(x) \to -\infty$; as $x \to -\infty, f(x) \to -1$
31. As $x \to \infty, f(x) \to 2$; as $x \to -\infty, f(x) \to \infty$
33. $f(x) = 4^{x} - 3$
35. $f(x) = 4^{x-5}$
37. $f(x) = 4^{-x}$
39. $y = -2^{x} + 3$
41. $y = -2(3)^{x} + 7$
43. $g(6) \approx 800.\overline{3}$

45. $h(-7) = -58$ **47.** $x \approx -2.953$ **49.** $x \approx -0.222$

51. The graph of $g(x) = \left(\frac{1}{b}\right)^{x}$ is the reflection about the y-axis of the graph of $f(x) = b^{x}$; for any real number $b > 0$ and function $f(x) = b^{x}$, the graph of $\left(\frac{1}{b}\right)^{x}$ is the reflection about the y-axis, $f(-x)$. **53.** The graphs of $g(x)$ and $h(x)$ are the same and are a horizontal shift to the right of the graph of $f(x)$. For any real number n, real number $b > 0$, and function $f(x) = b^{x}$, the graph of $\left(\frac{1}{b^{n}}\right)b^{x}$ is the horizontal shift $f(x - n)$.

Section 6.3

1. A logarithm is an exponent. Specifically, it is the exponent to which a base b is raised to produce a given value. In the expressions given, the base b has the same value. The exponent, y, in the expression b^{y} can also be written as the logarithm, $\log_{b}x$, and the value of x is the result of raising b to the power of y. **3.** Since the equation of a logarithm is equivalent to an exponential equation, the logarithm can be converted to the exponential equation $b^{y} = x$, and then properties of exponents can be applied to solve for x. **5.** The natural logarithm is a special case of the logarithm with base b in that the natural log always has base e. Rather than notating the natural logarithm as $\log_{e}(x)$, the notation used is $\ln(x)$.

7. $a^{c} = b$ **9.** $x^{y} = 64$ **11.** $15^{b} = a$ **13.** $13^{a} = 142$ **15.** $e^{n} = w$ **17.** $\log_{c}(k) = d$ **19.** $\log_{19}(y) = x$ **21.** $\log_{n}(103) = 4$ **23.** $\log_{y}\left(\frac{39}{100}\right) = x$ **25.** $\ln(h) = k$ **27.** $x = \frac{1}{8}$ **29.** $x = 27$ **31.** $x = 3$ **33.** $x = \frac{1}{216}$ **35.** $x = e^{2}$ **37.** 32 **39.** 1.06 **41.** 14.125 **43.** $\frac{1}{2}$ **45.** 4 **47.** -3 **49.** -12 **51.** 0 **53.** 10 **55.** ≈ 2.708 **57.** ≈ 0.151 **59.** No, the function has no defined value for $x = 0$. To verify, suppose $x = 0$ is in the domain of the function $f(x) = \log(x)$. Then there is some number n such that $n = \log(0)$. Rewriting as an exponential equation gives: $10^{n} = 0$, which is impossible since no such real number n exists. Therefore, $x = 0$ is not the domain of the function $f(x) = \log(x)$. **61.** Yes. Suppose there exists a real number, x such that $\ln(x) = 2$. Rewriting as an exponential equation gives $x = e^{2}$, which is a real number. To verify, let $x = e^{2}$. Then, by definition, $\ln(x) = \ln(e^{2}) = 2$. **63.** No; $\ln(1) = 0$, so $\frac{\ln(e^{1.725})}{\ln(1)}$ is undefined. **65.** 2

Section 6.4

1. Since the functions are inverses, their graphs are mirror images about the line $y = x$. So for every point (a, b) on the graph of a logarithmic function, there is a corresponding point (b, a) on the graph of its inverse exponential function. **3.** Shifting the function right or left and reflecting the function about the y-axis will affect its domain. **5.** No. A horizontal asymptote would suggest a limit on the range, and the range of any logarithmic function in general form is all real numbers.

7. Domain: $\left(-\infty, \frac{1}{2}\right)$; range: $(-\infty, \infty)$

9. Domain: $\left(-\frac{17}{4}, \infty\right)$; range: $(-\infty, \infty)$

11. Domain: $(5, \infty)$; vertical asymptote: $x = 5$

13. Domain: $\left(-\frac{1}{3}, \infty\right)$; vertical asymptote: $x = -\frac{1}{3}$

15. Domain: $(-3, \infty)$; vertical asymptote: $x = -3$

17. Domain: $\left(\frac{3}{7}, \infty\right)$; vertical asymptote: $x = \frac{3}{7}$; end behavior: as $x \to \left(\frac{3}{7}\right)^{+}, f(x) \to -\infty$ and as $x \to \infty, f(x) \to \infty$

19. Domain: $(-3, \infty)$; vertical asymptote: $x = -3$; end behavior: as $x \to -3+, f(x) \to -\infty$ and as $x \to \infty, f(x) \to \infty$

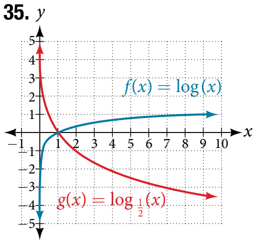
21. Domain: $(1, \infty)$; range: $(-\infty, \infty)$; vertical asymptote: $x = 1$; x-intercept: $\left(\frac{5}{4}, 0\right)$; y-intercept: DNE

23. Domain: $(-\infty, 0)$; range: $(-\infty, \infty)$; vertical asymptote: $x = 0$; x-intercept: $(-e^2, 0)$; y-intercept: DNE

25. Domain: $(0, \infty)$; range: $(-\infty, \infty)$ vertical asymptote: $x = 0$; x-intercept: $(e^3, 0)$; y-intercept: DNE

27. B **29.** C **31.** B **33.** C

35.



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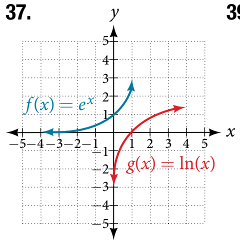
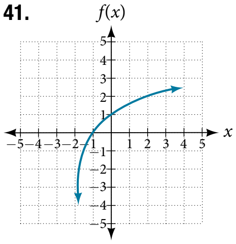


39. C

41.



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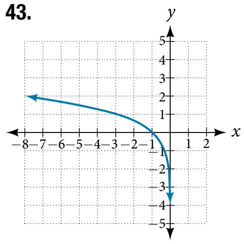

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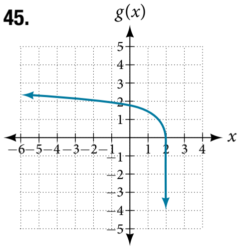


47. $f(x) = \log_2(-(x-1))$

49. $f(x) = 3\log_4(x+2)$

51. $x = 2$

53. $x \approx 2.303$

55. $x \approx -0.472$

57. The graphs of $f(x) = \log_{\frac{1}{2}}(x)$ and $g(x) = -\log_2(x)$ appear to be the same; conjecture: for any positive base $b \neq 1$, $\log_b(x) = -\log_{\frac{1}{b}}(x)$.

59. Recall that the argument of a logarithmic function must be positive, so we determine where $\frac{x+2}{x-4} > 0$. From the graph of the function $f(x) = \frac{x+2}{x-4} > 0$, note that the graph lies above the x-axis on the interval $(-\infty, -2)$ and again to the right of the vertical asymptote, that is $(4, \infty)$. Therefore, the domain is $(-\infty, -2) \cup (4, \infty)$.

Section 6.5

1. Any root expression can be rewritten as an expression with a rational exponent so that the power rule can be applied, making the logarithm easier to calculate. Thus, $\log_b(x^{\frac{1}{n}}) = \frac{1}{n}\log_b(x)$.

3. $\log_b(2) + \log_b(7) + \log_b(x) + \log_b(y)$

5. $\log_b(13) - \log_b(17)$ **7.** $-k\ln(4)$ **9.** $\ln(7xy)$

11. $\log_b(4)$ **13.** $\log_b(7)$ **15.** $15\log(x) + 13\log(y) - 19\log(z)$

17. $\frac{3}{2}\log(x) - 2\log(y)$ **19.** $\frac{8}{3}\log(x) + \frac{14}{3}\log(y)$ **21.** $\ln(2x^7)$

23. $\log\left(\frac{xz^3}{\sqrt{y}}\right)$ **25.** $\log_7(15) = \frac{\ln(15)}{\ln(7)}$

27. $\log_{11}(5) = \frac{1}{b}$ **29.** $\log_{11}\left(\frac{6}{11}\right) = \frac{a-b}{b}$ or $\frac{a}{b-1}$ **31.** 3

33. ≈ 2.81359 **35.** ≈ 0.93913 **37.** ≈ -2.23266

39. $x = 4$, By the quotient rule:

$$\log_6(x+2) - \log_6(x-3) = \log_6\left(\frac{x+2}{x-3}\right) = 1$$

Rewriting as an exponential equation and solving for x:

$$6^1 = \frac{x+2}{x-3}$$
$$0 = \frac{x+2}{x-3} - 6$$
$$0 = \frac{x+2}{x-3} - \frac{6(x-3)}{(x-3)}$$
$$0 = \frac{x+2-6x+18}{x-3}$$
$$0 = \frac{x-4}{x-3}$$
$$x = 4$$

Checking, we find that $\log_6(4+2) - \log_6(4-3) = \log_6(6) - \log_6(1)$ is defined, so $x = 4$.

41. Let b and n be positive integers greater than 1. Then, by the change-of-base formula, $\log_b(n) = \frac{\log_n(n)}{\log_n(b)} = \frac{1}{\log_n(b)}$.

Section 6.6

1. Determine first if the equation can be rewritten so that each side uses the same base. If so, the exponents can be set equal to each other. If the equation cannot be rewritten so that each side uses the same base, then apply the logarithm to each side and use properties of logarithms to solve. **3.** The one-to-one property can be used if both sides of the equation can be rewritten as a single logarithm with the same base. If so, the arguments can be set equal to each other, and the resulting equation can be solved algebraically. The one-to-one property cannot be used when each side of the equation cannot be rewritten as a single logarithm with the same base. **5.** $x = -\frac{1}{3}$ **7.** $n = -1$ **9.** $b = \frac{6}{5}$

11. $x = 10$ **13.** No solution **15.** $p = \log\left(\frac{17}{8}\right) - 7$

17. $k = -\frac{\ln(38)}{3}$ **19.** $x = \frac{\ln\left(\frac{38}{3}\right) - 8}{9}$ **21.** $x = \ln(12)$

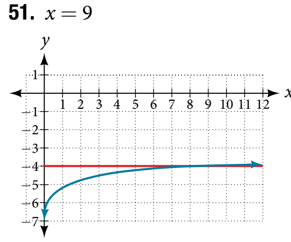
23. $x = \frac{\ln\left(\frac{3}{5}\right) - 3}{8}$ **25.** No solution **27.** $x = \ln(3)$

29. $10^{-2} = \frac{1}{100}$ **31.** $n = 49$ **33.** $k = \frac{1}{36}$ **35.** $x = \frac{9-e}{8}$

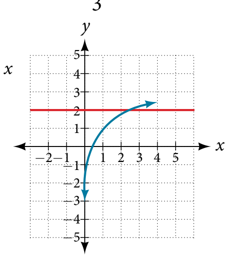
37. $n = 1$ **39.** No solution **41.** No solution

43. $x = \pm\frac{10}{3}$ **45.** $x = 10$ **47.** $x = 0$ **49.** $x = \frac{3}{4}$

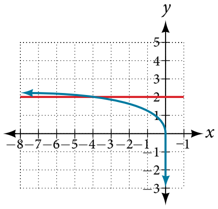
51. $x = 9$



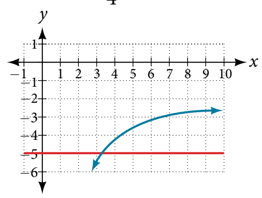
53. $x = \frac{e^2}{3} \approx 2.5$



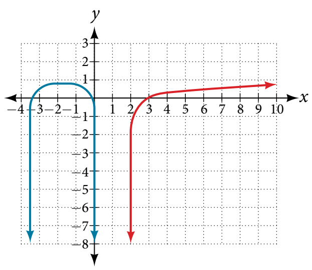
55. $x = -5$



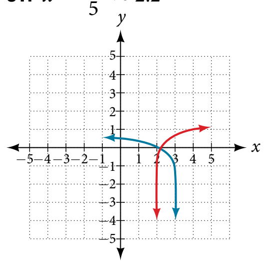
57. $x = \frac{e+10}{4} \approx 3.2$



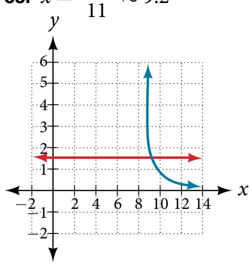
59. No solution



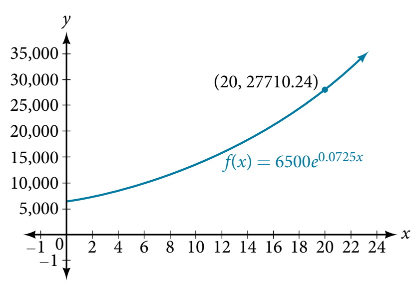
61. $x = \frac{11}{5} \approx 2.2$



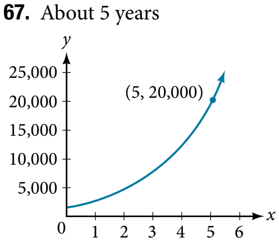
63. $x = \frac{101}{11} \approx 9.2$



65. About \$27,710.24



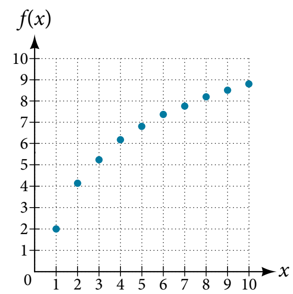
67. About 5 years



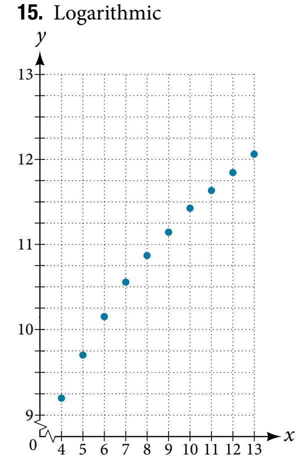
69. ≈ 0.567 **71.** ≈ 2.078
73. ≈ 2.2401
75. ≈ -44655.7143
77. About 5.83
79. $t = \ln\left(\left(\frac{y}{A}\right)^{\frac{1}{k}}\right)$
81. $t = \ln\left(\left(\frac{T - T_s}{T_0 - T_s}\right)^{-\frac{1}{k}}\right)$

Section 6.7

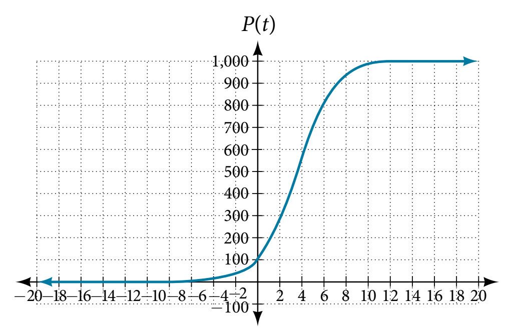
1. Half-life is a measure of decay and is thus associated with exponential decay models. The half-life of a substance or quantity is the amount of time it takes for half of the initial amount of that substance or quantity to decay. **3.** Doubling time is a measure of growth and is thus associated with exponential growth models. The doubling time of a substance or quantity is the amount of time it takes for the initial amount of that substance or quantity to double in size. **5.** An order of magnitude is the nearest power of ten by which a quantity exponentially grows. It is also an approximate position on a logarithmic scale; Sample response: Orders of magnitude are useful when making comparisons between numbers that differ by a great amount. For example, the mass of Saturn is 95 times greater than the mass of Earth. This is the same as saying that the mass of Saturn is about 10^2 times, or 2 *orders of magnitude* greater, than the mass of Earth.
7. $f(0) \approx 16.7$; the amount initially present is about 16.7 units.
9. 150 **11.** Exponential; $f(x) = 1.2^x$
13. Logarithmic



15. Logarithmic



17.



19. About 1.4 years **21.** About 7.3 years
23. Four half-lives; 8.18 minutes

25. $M = \frac{2}{3}\log\left(\frac{S}{S_0}\right)$

$$\frac{3}{2}M = \log\left(\frac{S}{S_0}\right)$$
$$10^{\frac{3M}{2}} = \left(\frac{S}{S_0}\right)$$
$$S_0 10^{\frac{3M}{2}} = S$$

27. Let $y = b^x$ for some non-negative real number b such that $b \neq 1$. Then,

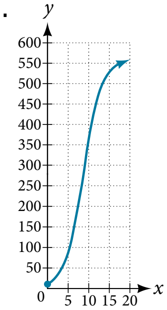
$$\ln(y) = \ln(b^x)$$
$$\ln(y) = x\ln(b)$$
$$e^{\ln(y)} = e^{x\ln(b)}$$
$$y = e^{x\ln(b)}$$

29. $A = 125e^{(-0.3567t)}$; $A \approx 43\text{mg}$ **31.** About 60 days
33. $f(t) = 250e^{-0.00914t}$; half-life: about 76 minutes
35. $r \approx -0.0667$; hourly decay rate: about 6.67%
37. $f(t) = 1350e^{0.03466t}$; after 3 hours; $P(180) \approx 691{,}200$
39. $f(t) = 256e^{(0.068110t)}$; doubling time: about 10 minutes
41. About 88minutes **43.** $T(t) = 90e^{(-0.008377t)} + 75$, where t is in minutes **45.** About 113 minutes **47.** $\log_{10}x = 1.5$; $x \approx 31.623$
49. MMS Magnitude: ≈ 5.82 **51.** $N(3) \approx 71$ **53.** C

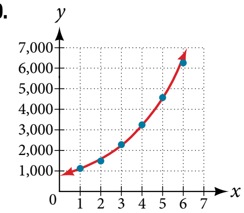
Section 6.8

1. Logistic models are best used for situations that have limited values. For example, populations cannot grow indefinitely since resources such as food, water, and space are limited, so a logistic model best describes populations. **3.** Regression analysis is the process of finding an equation that best fits a given set of data points. To perform a regression analysis on a graphing utility, first list the given points using the STAT then EDIT menu. Next graph the scatter plot using the STAT PLOT feature. The shape of the data points on the scatter graph can help determine which regression feature to use. Once this is determined, select the appropriate regression analysis command from the STAT then CALC menu. **5.** The y-intercept on the graph of a logistic equation corresponds to the initial population for the population model. **7.** C **9.** B **11.** $P(0) = 22$; 175 **13.** $p \approx 2.67$ **15.** y-intercept: $(0, 15)$ **17.** 4 koi **19.** About 6.8 months.

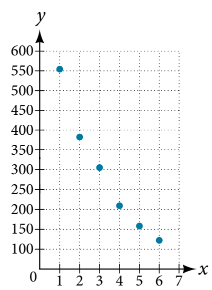
21.



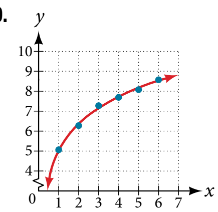
23. About 38 wolves **25.** About 8.7 years **27.** $f(x) = 776.682\,(1.426)^x$ **29.**



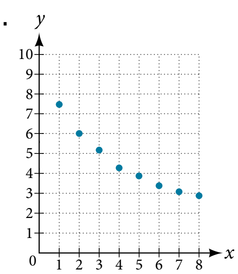
31.



33. $f(x) = 731.92e^{-0.3038x}$ **35.** When $f(x) = 250$, $x \approx 3.6$ **37.** $y = 5.063 + 1.934\log(x)$ **39.**

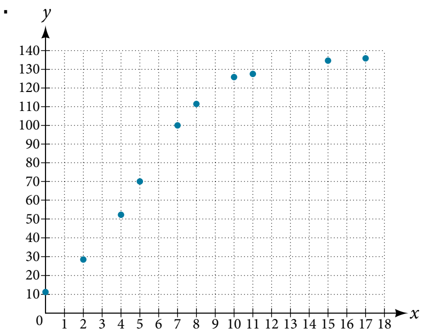


41.

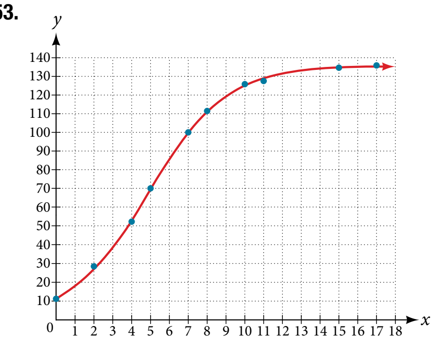


43. $f(10) \approx 2.3$ **45.** When $f(x) = 8$, $x \approx 0.82$ **47.** $f(x) = \dfrac{25.081}{1 + 3.182e^{-0.545x}}$ **49.** About 25

51.



53.

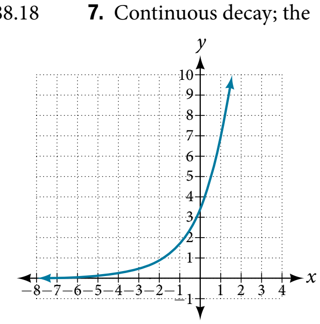


55. When $f(x) = 68$, $x \approx 4.9$ **57.** $f(x) = 1.034341(1.281204)^x$; $g(x) = 4.035510$; the regression curves are symmetrical about $y = x$, so it appears that they are inverse functions.

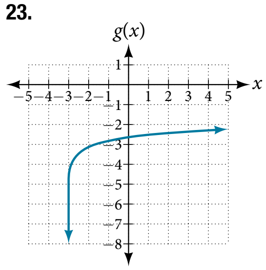
59. $f^{-1}(x) = \dfrac{\ln(a) - \ln\left(\frac{c}{x} - 1\right)}{b}$

Chapter 6 Review Exercises

1. Exponential decay; the growth factor, 0.825, is between 0 and 1. **3.** $y = 0.25(3)^x$ **5.** $42,888.18 **7.** Continuous decay; the growth rate is negative **9.** Domain: all real numbers; range: all real numbers strictly greater than zero; y-intercept: $(0, 3.5)$



11. $g(x) = 7(6.5)^{-x}$; y-intercept: $(0, 7)$; domain: all real numbers; range: all real numbers greater than 0. **13.** $17^x = 4{,}913$ **15.** $\log_a b = -\frac{2}{5}$ **17.** $x = 4$ **19.** $\log(0.000001) = -6$ **21.** $\ln(e^{-0.8648}) = -0.8648$ **23.**



25. Domain: $x > -5$; vertical asymptote: $x = -5$; end behavior: as $x \to -5^+$, $f(x) \to -\infty$ and as $x \to \infty$, $f(x) \to \infty$ **27.** $\log_8(65xy)$ **29.** $\ln\left(\frac{z}{xy}\right)$ **31.** $\log_y(12)$

33. $\ln(2)+\ln(b)+\dfrac{\ln(b+1)-\ln(b-1)}{2}$ **35.** $\log_7\left(\dfrac{v^3w^6}{\sqrt[3]{u}}\right)$

37. $x=\dfrac{5}{3}$ **39.** $x=-3$ **41.** No solution **43.** No solution

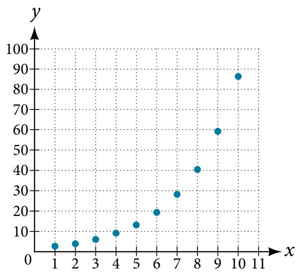
45. $x=\ln(11)$ **47.** $a=e^4-3$ **49.** $x=\pm\dfrac{9}{5}$

51. About 5.45 years **53.** $f^{-1}(x)=\sqrt[3]{2^{4x}-1}$

55. $f(t)=300(0.83)^t$; $f(24)\approx 3.43\,g$ **57.** About 45 minutes

59. About 8.5 days

61. Exponential

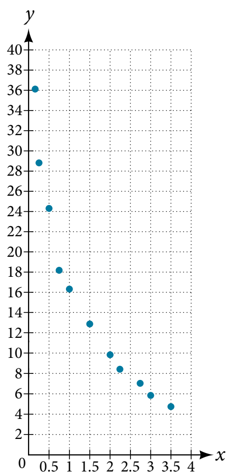


63. $y=4(0.2)^x$; $y=4e^{-1.609438x}$

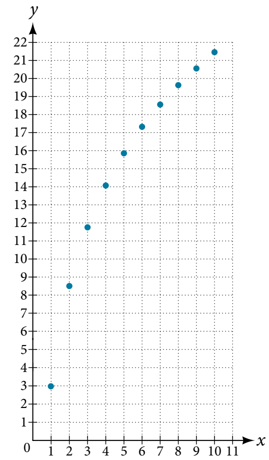
65. About 7.2 days

67. Logarithmic

$y=16.68718-9.71860\ln(x)$

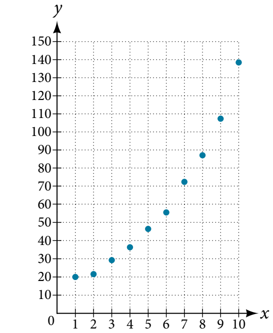


33. Logarithmic



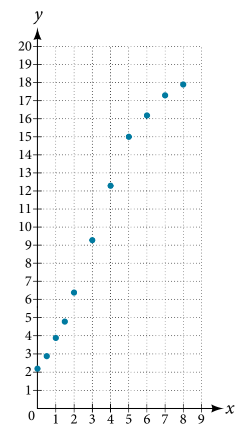
35. Exponential;

$y=15.10062(1.24621)^x$



37. Logistic;

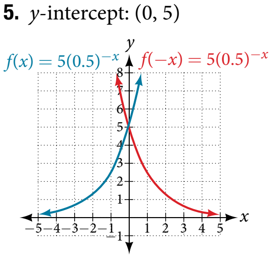
$y=\dfrac{18.41659}{1+7.54644\,e^{-0.68375x}}$



Chapter 6 Practice Test

1. About 13 dolphins **3.** \$1,947

5. y-intercept: (0, 5)



7. $8.5^a=614.125$ **9.** $x=\dfrac{1}{49}$

11. $\ln(0.716)\approx -0.334$

13. Domain: $x<3$; vertical asymptote: $x=3$; end behavior: as $x\to 3^-, f(x)\to -\infty$ and as $x\to -\infty, f(x)\to\infty$

15. $\log_t(12)$

17. $3\ln(y)+2\ln(z)+\dfrac{\ln(x-4)}{3}$

19. $x=\dfrac{\frac{\ln(1000)}{\ln(16)}+5}{3}\approx 2.497$ **21.** $a=\dfrac{\ln(4)+8}{10}$

23. No solution **25.** $x=\ln(9)$ **27.** $x=\pm\dfrac{3\sqrt{3}}{2}$

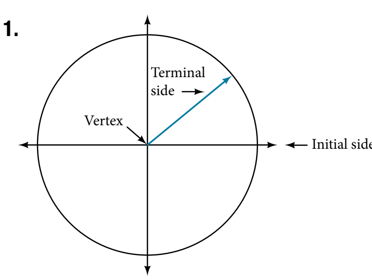
29. $f(t)=112e^{-0.019792t}$; half-life: about 35 days

31. $T(t)=36e^{-0.025131t}+35$; $T(60)\approx 43°F$

CHAPTER 7

Section 7.1

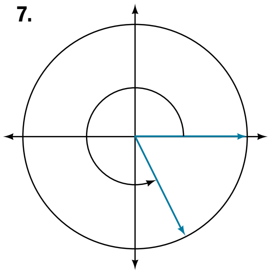
1.



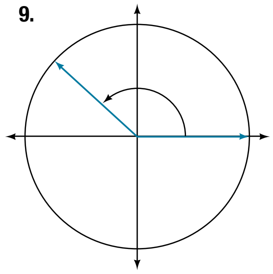
3. Whether the angle is positive or negative determines the direction. A positive angle is drawn in the counterclockwise direction, and a negative angle is drawn in the clockwise direction.

5. Linear speed is a measurement found by calculating distance of an arc compared to time. Angular speed is a measurement found by calculating the angle of an arc compared to time.

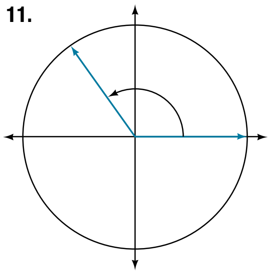
7.



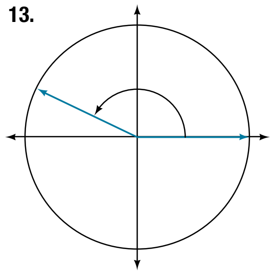
9.



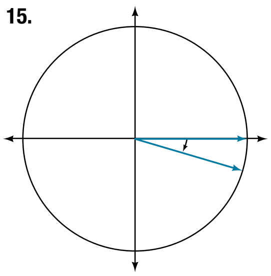
11.



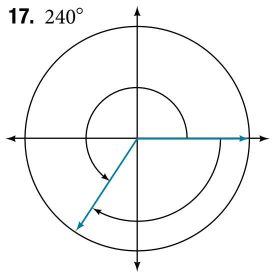
13.



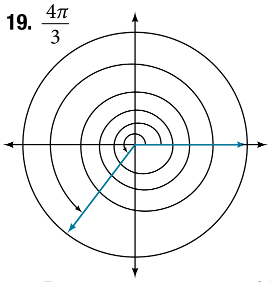
15.



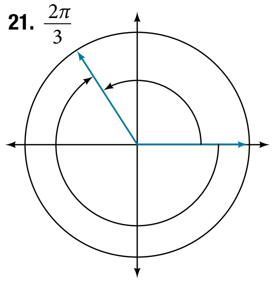
17. 240°



19. $\frac{4\pi}{3}$



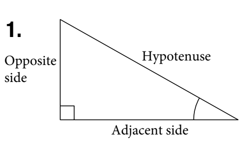
21. $\frac{2\pi}{3}$



23. $\frac{7\pi}{2} \approx 11.00 \text{ in}^2$ **25.** $\frac{81\pi}{20} \approx 12.72 \text{ cm}^2$ **27.** 20° **29.** 60° **31.** −75° **33.** $\frac{\pi}{2}$ radians **35.** -3π radians **37.** π radians **39.** $\frac{5\pi}{6}$ radians **41.** $\frac{5.02\pi}{3} \approx 5.26$ miles **43.** $\frac{25\pi}{9} \approx 8.73$ centimeters **45.** $\frac{21\pi}{10} \approx 6.60$ meters **47.** 104.7198 cm^2 **49.** 0.7697 in^2 **51.** 250° **53.** 320° **55.** $\frac{4\pi}{3}$ **57.** $\frac{8\pi}{9}$ **59.** 1320 rad/min 210.085 RPM **61.** 7 in/s, 4.77 RPM, 28.65 deg/s **63.** 1,809,557.37 mm/min = 30.16 m/s **65.** 5.76 miles **67.** 120° **69.** 794 miles per hour **71.** 2,234 miles per hour **73.** 11.5 inches

Section 7.2

1.



3. The tangent of an angle is the ratio of the opposite side to the adjacent side. **5.** For example, the sine of an angle is equal to the cosine of its complement; the cosine of an angle is equal to the sine of its complement.

7. $\frac{\pi}{6}$ **9.** $\frac{\pi}{4}$ **11.** $b = \frac{20\sqrt{3}}{3}, c = \frac{40\sqrt{3}}{3}$ **13.** $a = 10{,}000, c = 10{,}000.5$ **15.** $b = \frac{5\sqrt{3}}{3}, c = \frac{10\sqrt{3}}{3}$ **17.** $\frac{5\sqrt{29}}{29}$ **19.** $\frac{5}{2}$ **21.** $\frac{\sqrt{29}}{2}$ **23.** $\frac{5\sqrt{41}}{41}$ **25.** $\frac{5}{4}$ **27.** $\frac{\sqrt{41}}{4}$ **29.** $c = 14, b = 7\sqrt{3}$ **31.** $a = 15, b = 15$ **33.** $b = 9.9970, c = 12.2041$ **35.** $a = 2.0838, b = 11.8177$ **37.** $a = 55.9808, c = 57.9555$ **39.** $a = 46.6790, b = 17.9184$ **41.** $a = 16.4662, c = 16.8341$ **43.** 188.3159 **45.** 200.6737 **47.** 498.3471 ft. **49.** 1,060.09 ft. **51.** 27.372 ft. **53.** 22.6506 ft. **55.** 368.7633 ft.

Section 7.3

1. The unit circle is a circle of radius 1 centered at the origin. **3.** Coterminal angles are angles that share the same terminal side. A reference angle is the size of the smallest acute angle, t, formed by the terminal side of the angle t and the horizontal axis. **5.** The sine values are equal. **7.** I **9.** IV **11.** $\frac{\sqrt{3}}{2}$ **13.** $\frac{1}{2}$ **15.** $\frac{\sqrt{2}}{2}$ **17.** 0 **19.** −1 **21.** $\frac{\sqrt{3}}{2}$ **23.** 60° **25.** 80° **27.** 45° **29.** $\frac{\pi}{3}$ **31.** $\frac{\pi}{3}$ **33.** $\frac{\pi}{8}$ **35.** 60°, Quadrant IV, $\sin(300°) = -\frac{\sqrt{3}}{2}, \cos(300°) = \frac{1}{2}$ **37.** 45°, Quadrant II, $\sin(135°) = \frac{\sqrt{2}}{2}, \cos(135°) = -\frac{\sqrt{2}}{2}$ **39.** 60°, Quadrant II, $\sin(120°) = \frac{\sqrt{3}}{2}, \cos(120°) = -\frac{1}{2}$ **41.** 30°, Quadrant II, $\sin(150°) = \frac{1}{2}, \cos(150°) = -\frac{\sqrt{3}}{2}$ **43.** $\frac{\pi}{6}$, Quadrant III, $\sin\left(\frac{7\pi}{6}\right) = -\frac{1}{2}, \cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$ **45.** $\frac{\pi}{4}$, Quadrant II, $\sin\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2}, \cos\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$ **47.** $\frac{\pi}{3}$, Quadrant II, $\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}, \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$ **49.** $\frac{\pi}{4}$, Quadrant IV, $\sin\left(\frac{7\pi}{4}\right) = -\frac{\sqrt{2}}{2}, \cos\left(\frac{7\pi}{4}\right) = \frac{\sqrt{2}}{2}$ **51.** $\frac{\sqrt{77}}{9}$ **53.** $-\frac{\sqrt{15}}{4}$ **55.** $\left(-10, 10\sqrt{3}\right)$ **57.** $(-2.778, 15.757)$ **59.** $[-1, 1]$ **61.** $\sin t = \frac{1}{2}, \cos t = -\frac{\sqrt{3}}{2}$ **63.** $\sin t = -\frac{\sqrt{2}}{2}, \cos t = -\frac{\sqrt{2}}{2}$ **65.** $\sin t = \frac{\sqrt{3}}{2}, \cos t = -\frac{1}{2}$ **67.** $\sin t = -\frac{\sqrt{2}}{2}, \cos t = \frac{\sqrt{2}}{2}$ **69.** $\sin t = 0, \cos t = -1$ **71.** $\sin t = -0.596, \cos t = 0.803$ **73.** $\sin t = \frac{1}{2}, \cos t = \frac{\sqrt{3}}{2}$ **75.** $\sin t = -\frac{1}{2}, \cos t = \frac{\sqrt{3}}{2}$ **77.** $\sin t = 0.761, \cos t = -0.649$ **79.** $\sin t = 1, \cos t = 0$ **81.** −0.1736 **83.** 0.9511 **85.** −0.7071 **87.** −0.1392 **89.** −0.7660 **91.** $\frac{\sqrt{2}}{4}$ **93.** $-\frac{\sqrt{6}}{4}$ **95.** $\frac{\sqrt{2}}{4}$ **97.** $\frac{\sqrt{2}}{4}$ **99.** 0 **101.** $(0, -1)$ **103.** 37.5 seconds, 97.5 seconds, 157.5 seconds, 217.5 seconds, 277.5 seconds, 337.5 seconds

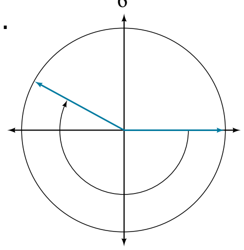
Section 7.4

1. Yes, when the reference angle is $\frac{\pi}{4}$ and the terminal side of the angle is in quadrants I or III. Thus, at $x = \frac{\pi}{4}, \frac{5\pi}{4}$, the sine and cosine values are equal. **3.** Substitute the sine of the angle in for y in the Pythagorean Theorem $x^2 + y^2 = 1$. Solve for x and take the negative solution. **5.** The outputs of tangent and cotangent will repeat every π units. **7.** $\frac{2\sqrt{3}}{3}$ **9.** $\sqrt{3}$ **11.** $\sqrt{2}$ **13.** 1 **15.** 2 **17.** $\frac{\sqrt{3}}{3}$ **19.** $-\frac{2\sqrt{3}}{3}$ **21.** $\sqrt{3}$ **23.** $-\sqrt{2}$ **25.** -1 **27.** -2 **29.** $-\frac{\sqrt{3}}{3}$ **31.** 2 **33.** $\frac{\sqrt{3}}{3}$ **35.** -2 **37.** -1 **39.** $\sin t = -\frac{2\sqrt{2}}{3}$, $\sec t = -3$, $\csc t = -\frac{3\sqrt{2}}{4}$, $\tan t = 2\sqrt{2}$, $\cot t = \frac{\sqrt{2}}{4}$ **41.** $\sec t = 2$, $\csc t = \frac{2\sqrt{3}}{3}$, $\tan t = \sqrt{3}$, $\cot t = \frac{\sqrt{3}}{3}$ **43.** $-\frac{\sqrt{2}}{2}$ **45.** 3.1 **47.** 1.4 **49.** $\sin t = \frac{\sqrt{2}}{2}$, $\cos t = \frac{\sqrt{2}}{2}$, $\tan t = 1$, $\cot t = 1$, $\sec t = \sqrt{2}$, $\csc t = \sqrt{2}$ **51.** $\sin t = -\frac{\sqrt{3}}{2}$, $\cos t = -\frac{1}{2}$, $\tan t = \sqrt{3}$, $\cot t = \frac{\sqrt{3}}{3}$, $\sec t = -2$, $\csc t = -\frac{2\sqrt{3}}{3}$ **53.** -0.228 **55.** -2.414 **57.** 1.414 **59.** 1.540 **61.** 1.556 **63.** $\sin(t) \approx 0.79$ **65.** $\csc(t) \approx 1.16$ **67.** Even **69.** Even **71.** $\frac{\sin t}{\cos t} = \tan t$ **73.** 13.77 hours, period: 1000π **75.** 7.73 inches

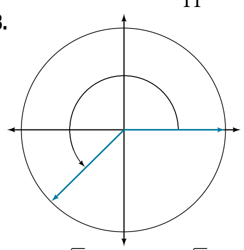
Chapter 7 Review Exercises

1. 45° **3.** $-\frac{7\pi}{6}$ **5.** 10.385 meters **7.** 60° **9.** $\frac{2\pi}{11}$

11.



13.

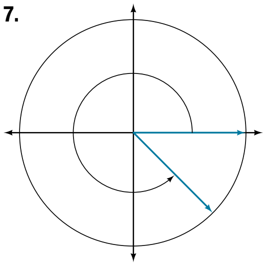


15. 1,036.73 miles per hour **17.** $\frac{\sqrt{2}}{2}$ **19.** $\frac{\sqrt{3}}{3}$ **21.** 72° **23.** $a = \frac{10}{3}$, $c = \frac{2\sqrt{106}}{3}$ **25.** $\frac{6}{11}$ **27.** $a = \frac{5\sqrt{3}}{2}$, $b = \frac{5}{2}$ **29.** 369.2136 ft. **31.** $\frac{\sqrt{2}}{2}$ **33.** 60° **35.** $\frac{\sqrt{3}}{2}$ **37.** All real numbers **39.** $\frac{\sqrt{3}}{2}$ **41.** $\frac{2\sqrt{3}}{3}$ **43.** 2 **45.** -2.5 **47.** $\frac{1}{3}$ **49.** Cosine, secant

Chapter 7 Practice Test

1. 150° **3.** 6.283 centimeters **5.** 15°

7.



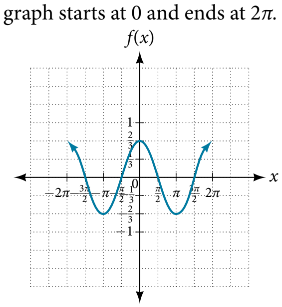
9. 3.351 feet per second, $\frac{2\pi}{75}$ radians per second **11.** $a = \frac{9}{2}$, $b = \frac{9\sqrt{3}}{2}$ **13.** $\frac{1}{2}$ **15.** Real numbers **17.** 1 **19.** $-\sqrt{2}$ **21.** -0.68 **23.** $\frac{\pi}{3}$

CHAPTER 8

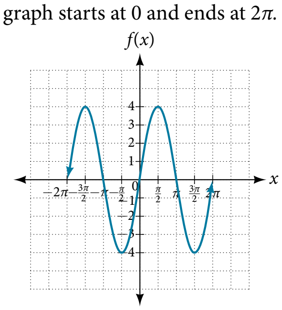
Section 8.1

1. The sine and cosine functions have the property that $f(x + P) = f(x)$ for a certain P. This means that the function values repeat for every P units on the x-axis. **3.** The absolute value of the constant A (amplitude) increases the total range and the constant D (vertical shift) shifts the graph vertically. **5.** At the point where the terminal side of t intersects the unit circle, you can determine that the sin t equals the y-coordinate of the point.

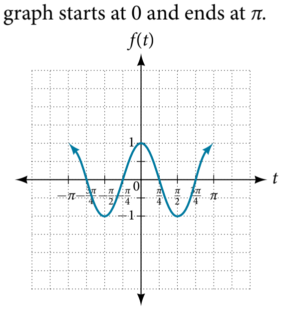
7. Amplitude: $\frac{2}{3}$; period: 2π; midline: $y = 0$; maximum: $y = \frac{2}{3}$ occurs at $x = 0$; minimum: $y = -\frac{2}{3}$ occurs at $x = \pi$; for one period, the graph starts at 0 and ends at 2π.



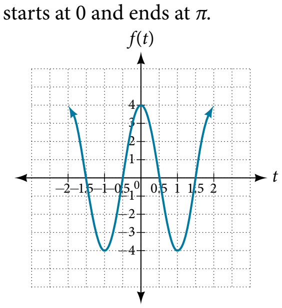
9. Amplitude: 4; period: 2π; midline: $y = 0$; maximum: $y = 4$ occurs at $x = \frac{\pi}{2}$; minimum: $y = -4$ occurs at $x = \frac{3\pi}{2}$; for one period, the graph starts at 0 and ends at 2π.



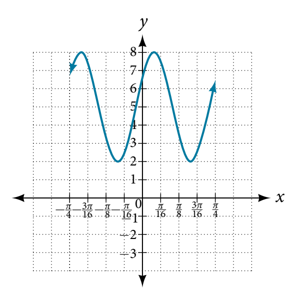
11. Amplitude: 1; period: π; midline: $y = 0$; maximum: $y = 1$ occurs at $x = \pi$; minimum: $y = -1$ occurs at $x = \frac{\pi}{2}$; for one period, the graph starts at 0 and ends at π.



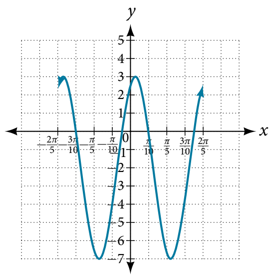
13. Amplitude: 4; period: 2; midline: $y = 0$; maximum: $y = 4$ occurs at $x = 0$; minimum: $y = -4$ occurs at $x = 1$; for one period, the graph starts at 0 and ends at π.



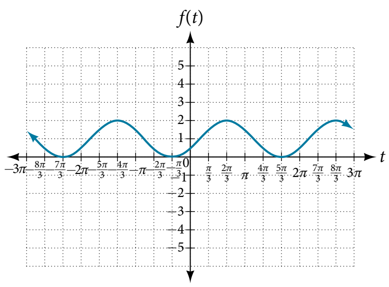
15. Amplitude: 3; period: $\frac{\pi}{4}$; midline: $y = 5$; maximum: $y = 8$ occurs at $x = 0.12$; minimum: $y = 2$ occurs at $x = 0.516$; horizontal shift: -4; vertical translation: 5; for one period, the graph starts at 0 and ends at $\frac{\pi}{4}$.



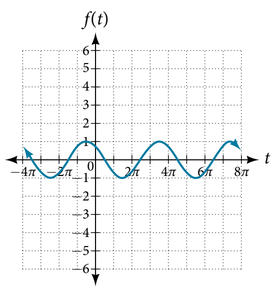
17. Amplitude: 5; period: $\frac{2\pi}{5}$; midline: $y = -2$; maximum: $y = 3$ occurs at $x = 0.08$; minimum: $y = -7$ occurs at $x = 0.71$; phase shift: -4; vertical translation: -2; for one period, the graph starts at 0 and ends at $\frac{2\pi}{5}$.



19. Amplitude: 1; period: 2π; midline: $y = 1$; maximum: $y = 2$ occurs at $t = 2.09$; minimum: $y = 0$ occurs at $t = 5.24$; phase shift: $-\frac{\pi}{3}$; vertical translation: 1; for one period, the graph starts at 0 and ends at 2π.



21. Amplitude: 1; period: 4π; midline: $y = 0$; maximum: $y = 1$ occurs at $t = 11.52$; minimum: $y = -1$ occurs at $t = 5.24$; phase shift: $-\frac{10\pi}{3}$; vertical shift: 0; for one period, the graph starts at 0 and ends at 4π.



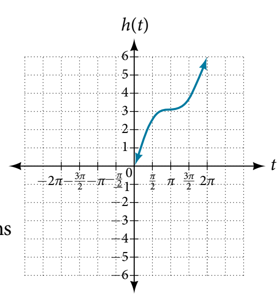
23. Amplitude: 2, midline: $y = -3$; period: 4; equation: $f(x) = 2\sin\left(\frac{\pi}{2}x\right) - 3$ **25.** Amplitude: 2, midline: $y = 3$; period: 5; equation: $f(x) = -2\cos\left(\frac{2\pi}{5}x\right) + 3$

27. Amplitude: 4, midline: $y = 0$; period: 2; equation: $f(x) = -4\cos\left(\pi\left(x - \frac{\pi}{2}\right)\right)$ **29.** Amplitude: 2, midline: $y = 1$; period: 2; equation: $f(x) = 2\cos(\pi x) + 1$ **31.** $0, \pi$

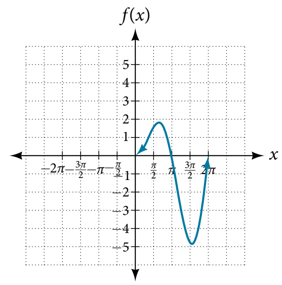
33. $\sin\left(\frac{\pi}{2}\right) = 1$ **35.** $\frac{\pi}{2}$ **37.** $f(x) = \sin x$ is symmetric with respect to the origin. **39.** $\frac{\pi}{3}, \frac{5\pi}{3}$

41. Maximum: 1 at $x = 0$; minimum: -1 at $x = \pi$

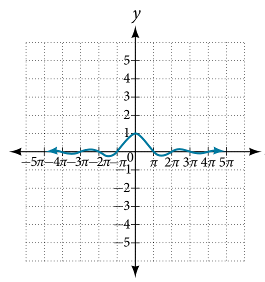
43. A linear function is added to a periodic sine function. The graph does not have an amplitude because as the linear function increases without bound the combined function $h(x) = x + \sin x$ will increase without bound as well. The graph is bounded between the graphs of $y = x + 1$ and $y = x - 1$ because sine oscillates between -1 and 1.



45. There is no amplitude because the function is not bounded.



47. The graph is symmetric with respect to the y-axis and there is no amplitude because the function's bounds decrease as $|x|$ grows. There appears to be a horizontal asymptote at $y = 0$.



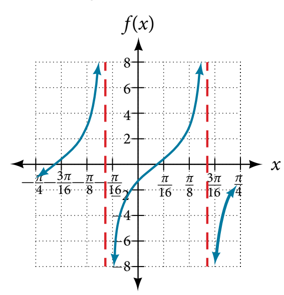
Section 8.2

1. Since $y = \csc x$ is the reciprocal function of $y = \sin x$, you can plot the reciprocal of the coordinates on the graph of $y = \sin x$ to obtain the y-coordinates of $y = \csc x$. The x-intercepts of the graph $y = \sin x$ are the vertical asymptotes for the graph of $y = \csc x$. **3.** Answers will vary. Using the unit circle, one can show that $\tan(x + \pi) = \tan x$. **5.** The period is the same: 2π

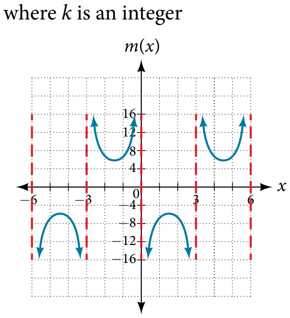
7. IV **9.** III **11.** Period: 8; horizontal shift: 1 unit to the left

13. 1.5 **15.** 5 **17.** $-\cot x \cos x - \sin x$

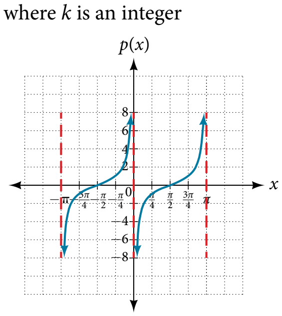
19. Stretching factor: 2; period: $\frac{\pi}{4}$; asymptotes: $x = \frac{1}{4}\left(\frac{\pi}{2} + \pi k\right) + 8$, where k is an integer



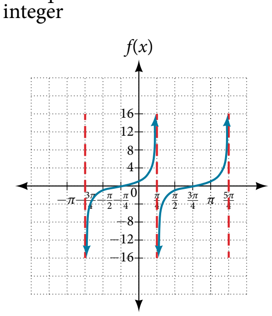
21. Stretching factor: 6; period: 6; asymptotes: $x = 3k$, where k is an integer



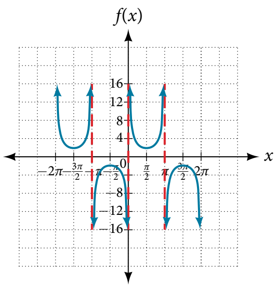
23. Stretching factor: 1; period: π; asymptotes: $x = \pi k$, where k is an integer



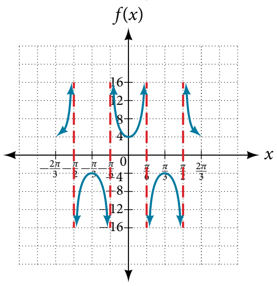
25. Stretching factor: 1; period: π; asymptotes: $x = \frac{\pi}{4} + \pi k$, where k is an integer



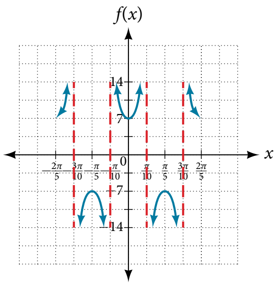
27. Stretching factor: 2; period: 2π; asymptotes: $x = \pi k$, where k is an integer



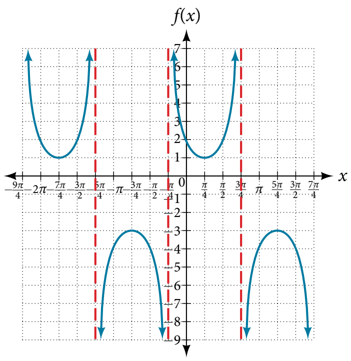
29. Stretching factor: 4; period: $\frac{2\pi}{3}$; asymptotes: $x = \frac{\pi}{6}k$, where k is an integer



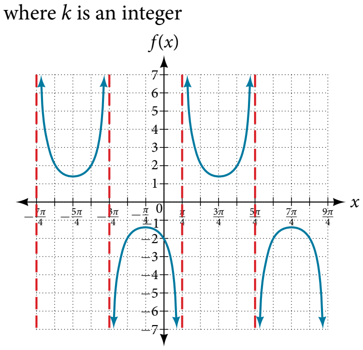
31. Stretching factor: 7; period: $\frac{2\pi}{5}$; asymptotes: $x = \frac{\pi}{10}$ k, where k is an integer



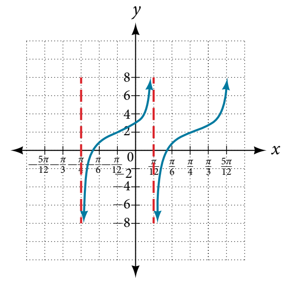
33. Stretching factor: 2; period: 2π; asymptotes: $x = -\frac{\pi}{4} + \pi k$, where k is an integer



35. Stretching factor: $\frac{7}{5}$; period: 2π; asymptotes: $x = \frac{\pi}{4} + \pi k$, where k is an integer



37. $y = \left(\tan 3\left(x - \frac{\pi}{4}\right)\right) + 2$



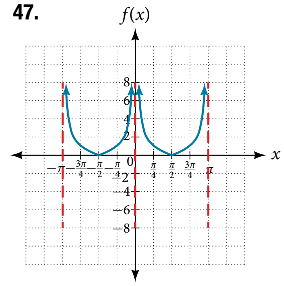
39. $f(x) = \csc(2x)$

41. $f(x) = \csc(4x)$

43. $f(x) = 2\csc x$

45. $f(x) = \frac{1}{2}\tan(100\pi x)$

47.



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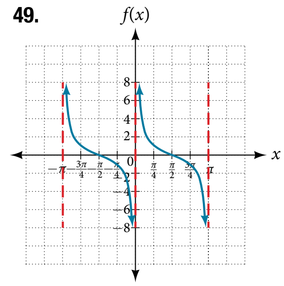

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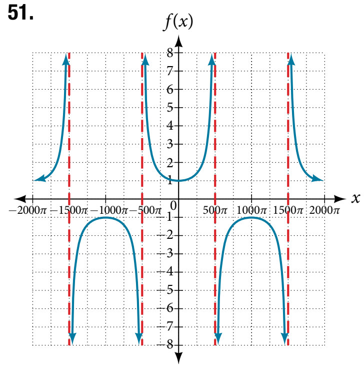

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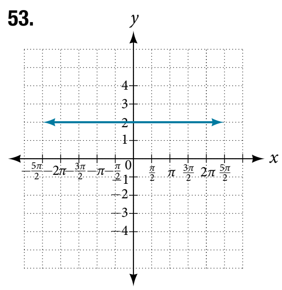
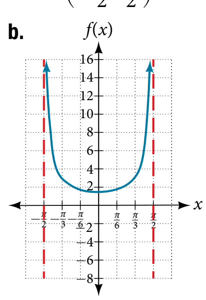


55. **a.** $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$;

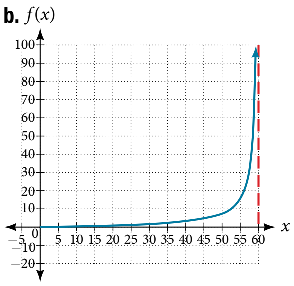
b.



c. $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$; the distance grows without bound as $|x|$ approaches $\frac{\pi}{2}$–i.e., at right angles to the line representing due north, the boat would be so far away, the fisherman could not see it **d.** 3; when $x = -\frac{\pi}{3}$, the boat is 3 km away **e.** 1.73; when $x = \frac{\pi}{6}$, the boat is about 1.73 km away **f.** 1.5 km; when $x = 0$

57. **a.** $h(x) = 2\tan\left(\frac{\pi}{120}x\right)$

b.



c. $h(0) = 0$: after 0 seconds, the rocket is 0 mi above the ground; $h(30) = 2$: after 30 seconds, the rockets is 2 mi high; **d.** As x approaches 60 seconds, the values of $h(x)$ grow increasingly large. As $x \to 60$ the model breaks down, since it assumes that the angle of elevation continues to increase with x. In fact, the angle is bounded at 90 degrees.

Section 8.3

1. The function $y = \sin x$ is one-to-one on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$; thus, this interval is the range of the inverse function of $y = \sin x$, $f(x) = \sin^{-1} x$. The function $y = \cos x$ is one-to-one on $[0, \pi]$; thus, this interval is the range of the inverse function of $y = \cos x, f(x) = \cos^{-1} x$. **3.** $\frac{\pi}{6}$ is the radian measure of an angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ whose sine is 0.5. **5.** In order for any function to have an inverse, the function must be one-to-one and must pass the horizontal line test. The regular sine function is not one-to-one unless its domain is restricted in some way. Mathematicians have agreed to restrict the sine function to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ so that it is one-to-one and possesses an inverse. **7.** True. The angle, θ_1 that equals arccos $(-x)$, $x > 0$, will be a second quadrant angle with reference angle, θ_2, where θ_2 equals arccos x, $x > 0$. Since θ_2 is the reference angle for θ_1, $\theta_2 = \pi - \theta_1$ and $\arccos(-x) = \pi - \arccos x$ **9.** $-\frac{\pi}{6}$

11. $\frac{3\pi}{4}$ **13.** $-\frac{\pi}{3}$ **15.** $\frac{\pi}{3}$ **17.** 1.98 **19.** 0.93

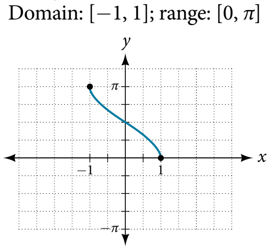
21. 1.41 **23.** 0.56 radians **25.** 0 **27.** 0.71 radians

29. -0.71 radians **31.** $-\frac{\pi}{4}$ radians **33.** $\frac{4}{5}$ **35.** $\frac{5}{13}$

37. $\frac{x-1}{\sqrt{-x^2+2x}}$ **39.** $\frac{\sqrt{x^2-1}}{x}$ **41.** $\frac{x+0.5}{\sqrt{-x^2-x+\frac{3}{4}}}$

43. $\frac{\sqrt{2x+1}}{x+1}$ **45.** $\frac{\sqrt{2x+1}}{x}$ **47.** t

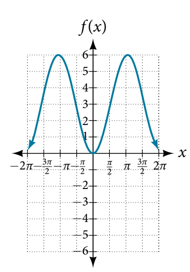
49. Domain: $[-1, 1]$; range: $[0, \pi]$



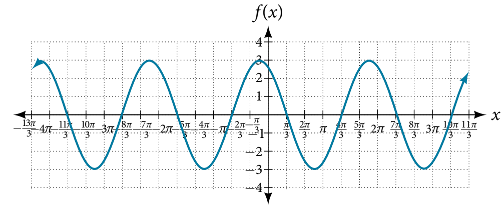
51. $x = 0$
53. 0.395 radians
55. 1.11 radians
57. 1.25 radians
59. 0.405 radians
61. No. The angle the ladder makes with the horizontal is 60 degrees.

Chapter 8 Review Exercises

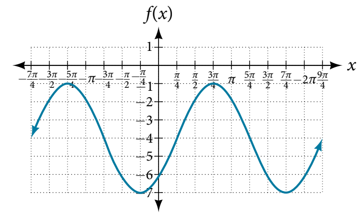
1. Amplitude: 3; period: is 2π; midline: $y = 3$; no asymptotes



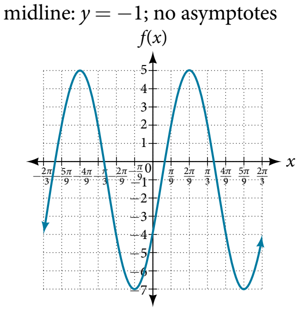
3. Amplitude: 3; period: is 2π; midline: $y = 0$; no asymptotes



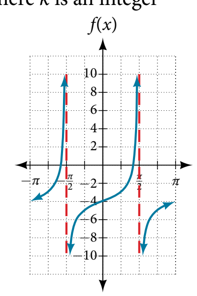
5. Amplitude: 3; period: is 2π; midline: $y = -4$; no asymptotes



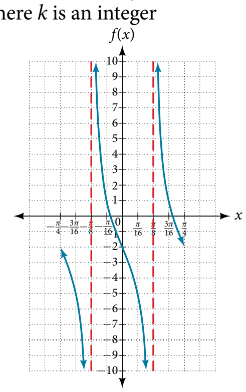
7. Amplitude: 6; period: is $\frac{2\pi}{3}$; midline: $y = -1$; no asymptotes



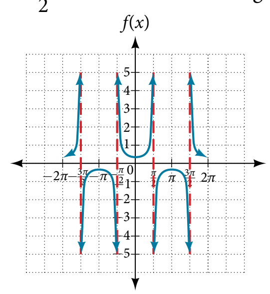
9. Stretching factor: none; period: π; midline: $y = -4$; asymptotes: $x = \frac{\pi}{2} + \pi k$, where k is an integer



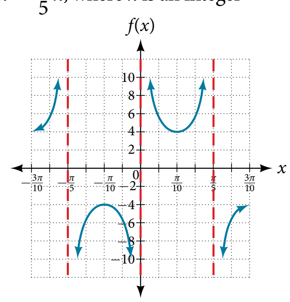
11. Stretching factor: 3; period: $\frac{\pi}{4}$; midline: $y = -2$; asymptotes: $x = \frac{\pi}{8} + \frac{\pi}{4}k$, where k is an integer



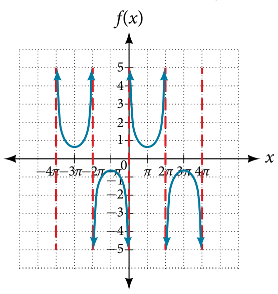
13. Amplitude: none; period: 2π; no phase shift; asymptotes: $x = \frac{\pi}{2}k$, where k is an odd integer



15. Amplitude: none; period: $\frac{2\pi}{5}$; no phase shift; asymptotes: $x = \frac{\pi}{5}k$, where k is an integer



17. Amplitude: none; period: 4π; no phase shift; asymptotes: $x = 2\pi k$, where k is an integer



19. Largest: 20,000; smallest: 4,000

21. Amplitude: 8,000; period: 10; phase shift: 0

23. In 2007, the predicted population is 4,413. In 2010, the population will be 11,924.

25. 5 in.

27. 10 seconds

29. $\frac{\pi}{6}$

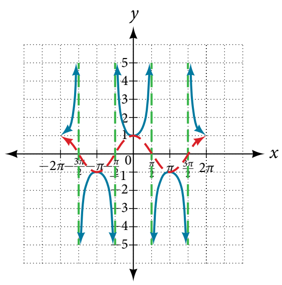
31. $\frac{\pi}{4}$

33. $\frac{\pi}{3}$

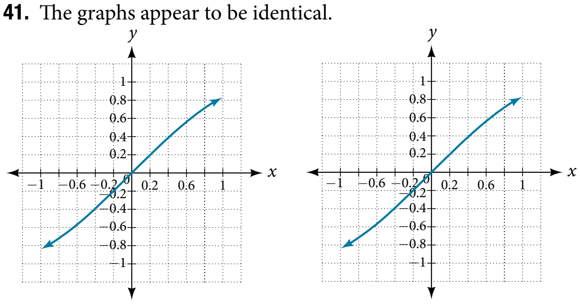
35. No solution

37. $\frac{12}{5}$

39. The graphs are not symmetrical with respect to the line $y = x$. They are symmetrical with respect to the y-axis.

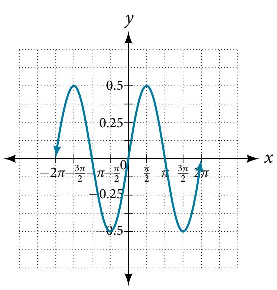


41. The graphs appear to be identical.

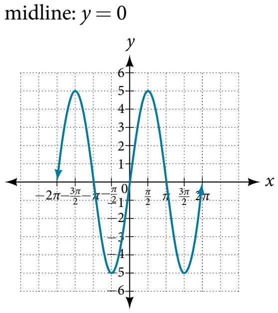


Chapter 8 Practice Test

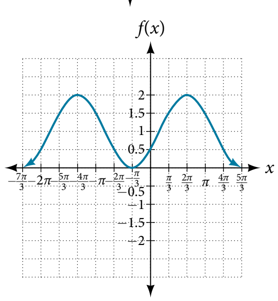
1. Amplitude: 0.5; period: 2π; midline: $y = 0$



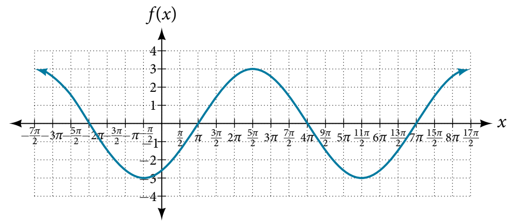
3. Amplitude: 5; period: 2π; midline: $y = 0$



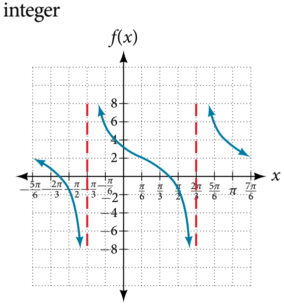
5. Amplitude: 1; period: 2π; midline: $y = 1$



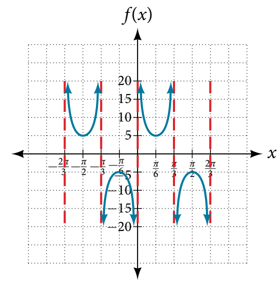
7. Amplitude: 3; period: 6π; midline: $y = 0$



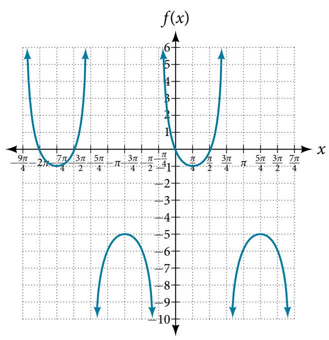
9. Amplitude: none; period: π; midline: $y = 0$; asymptotes: $x = \frac{2\pi}{3} + \pi k$, where k is some integer



11. Amplitude: none; period: $\frac{2\pi}{3}$; midline: $y = 0$; asymptotes: $x = \frac{\pi}{3}k$, where k is some integer



13. Amplitude: none; period: 2π; midline: $y = -3$



15. Amplitude; 2; period: 2; midline: $y = 0$; $f(x) = 2\sin(\pi(x - 1))$

17. Amplitude; 1; period: 12; phase shift: -6; midline: $y = -3$

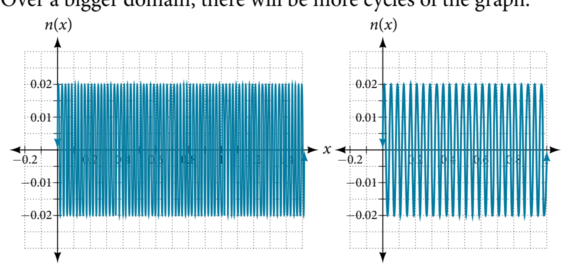
19. $D(t) = 68 - 12\sin\left(\frac{\pi}{12}x\right)$

21. Period: $\frac{\pi}{6}$; horizontal shift: -7

23. $f(x) = \sec(\pi x)$; period: 2; phase shift: 0

25. 4

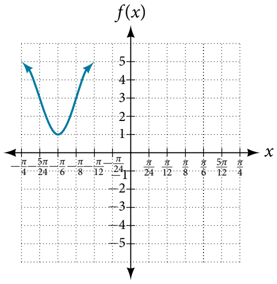
27. The views are different because the period of the wave is $\frac{1}{25}$. Over a bigger domain, there will be more cycles of the graph.



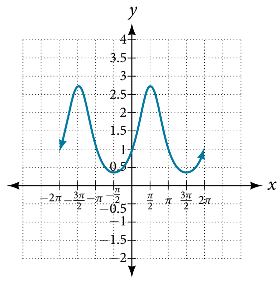
29. $\frac{3}{5}$

31. $\left(\frac{\pi}{6}, \frac{\pi}{3}\right), \left(\frac{\pi}{2}, \frac{2\pi}{3}\right), \left(\frac{5\pi}{6}, \pi\right), \left(\frac{7\pi}{6}, \frac{4\pi}{3}\right), \left(\frac{3\pi}{2}, \frac{5\pi}{3}\right), \left(\frac{11\pi}{6}, 2\pi\right)$

33. $f(x) = 2\cos\left(12\left(x + \frac{\pi}{4}\right)\right) + 3$



35. This graph is periodic with a period of 2π.



37. $\frac{\pi}{3}$ **39.** $\frac{\pi}{2}$ **41.** $\sqrt{1-(1-2x)^2}$ **43.** $\frac{1}{\sqrt{1+x^4}}$

45. $\csc t = \frac{x+1}{x}$ **47.** False **49.** 0.07 radians

CHAPTER 9

Section 9.1

1. All three functions, F, G, and H, are even. This is because
$F(-x) = \sin(-x)\sin(-x) = (-\sin x)(-\sin x) = \sin^2 x = F(x)$,
$G(-x) = \cos(-x)\cos(-x) = \cos x \cos x = \cos^2 x = G(x)$ and
$H(-x) = \tan(-x)\tan(-x) = (-\tan x)(-\tan x) = \tan^2 x = H(x)$.

3. When $\cos t = 0$, then $\sec t = \frac{1}{0}$, which is undefined.

5. $\sin x$ **7.** $\sec x$ **9.** $\csc t$ **11.** -1 **13.** $\sec^2 x$

15. $\sin^2 x + 1$ **17.** $\frac{1}{\sin x}$ **19.** $\frac{1}{\cot x}$ **21.** $\tan x$

23. $-4\sec x \tan x$ **25.** $\pm\sqrt{\frac{1}{\cot^2 x} + 1}$ **27.** $\pm\frac{\sqrt{1-\sin^2 x}}{\sin x}$

29. Answers will vary. Sample proof:
$\cos x - \cos^3 x = \cos x(1-\cos^2 x) = \cos x \sin^2 x$

31. Answers will vary. Sample proof:
$$\frac{1+\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} = \sec^2 x + \tan^2 x$$
$$= \tan^2 x + 1 + \tan^2 x = 1 + 2\tan^2 x$$

33. Answers will vary. Sample proof:
$$\cos^2 x - \tan^2 x = 1 - \sin^2 x - (\sec^2 x - 1)$$
$$= 1 - \sin^2 x - \sec^2 x + 1 = 2 - \sin^2 x - \sec^2 x$$

35. False **37.** False

39. Proved with negative and Pythagorean identities

41. True
$$3\sin^2\theta + 4\cos^2\theta = 3\sin^2\theta + 3\cos^2\theta + \cos^2\theta$$
$$= 3(\sin^2\theta + \cos^2\theta) + \cos^2\theta$$
$$= 3 + \cos^2\theta$$

Section 9.2

1. The cofunction identities apply to complementary angles. Viewing the two acute angles of a right triangle, if one of those angles measures x, the second angle measures $\frac{\pi}{2} - x$. Then $\sin x = \cos\left(\frac{\pi}{2} - x\right)$. The same holds for the other cofunction identities. The key is that the angles are complementary.

3. $\sin(-x) = -\sin x$, so $\sin x$ is odd.
$\cos(-x) = \cos(0-x) = \cos x$, so $\cos x$ is even.

5. $\frac{\sqrt{2}+\sqrt{6}}{4}$ **7.** $\frac{\sqrt{6}-\sqrt{2}}{4}$ **9.** $-2-\sqrt{3}$

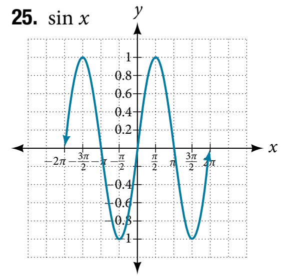
11. $-\frac{\sqrt{2}}{2}\sin x - \frac{\sqrt{2}}{2}\cos x$ **13.** $-\frac{1}{2}\cos x - \frac{\sqrt{3}}{2}\sin x$

15. $\csc\theta$ **17.** $\cot x$ **19.** $\tan\left(\frac{x}{10}\right)$

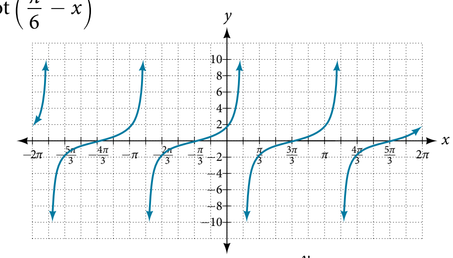
21. $\sin(a-b)$
$$= \left(\frac{4}{5}\right)\left(\frac{1}{3}\right) - \left(\frac{3}{5}\right)\left(\frac{2\sqrt{2}}{3}\right)$$
$$= \frac{4-6\sqrt{2}}{15}$$
$\cos(a+b)$
$$= \left(\frac{3}{5}\right)\left(\frac{1}{3}\right) - \left(\frac{4}{5}\right)\left(\frac{2\sqrt{2}}{3}\right)$$
$$= \frac{3-8\sqrt{2}}{15}$$

23. $\frac{\sqrt{2}-\sqrt{6}}{4}$

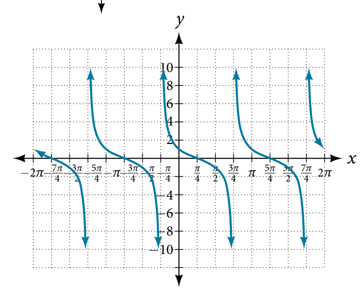
25. $\sin x$



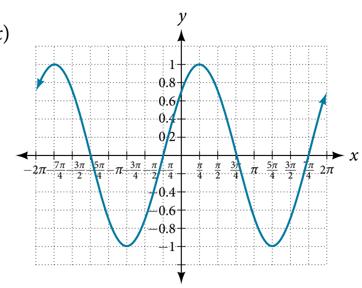
27. $\cot\left(\frac{\pi}{6} - x\right)$



29. $\cot\left(\frac{\pi}{4} + x\right)$



31. $\frac{\sqrt{2}}{2}(\sin x + \cos x)$



33. They are the same. **35.** They are the different, try $g(x) = \sin(9x) - \cos(3x)\sin(6x)$ **37.** They are the same.

39. They are the different, try $g(\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$

41. They are different, try $g(x) = \frac{\tan x - \tan(2x)}{1 + \tan x \tan(2x)}$

43. $-\frac{\sqrt{3}-1}{2\sqrt{2}}$ or -0.2588 **45.** $\frac{1+\sqrt{3}}{2\sqrt{2}}$, or 0.9659

47.
$$\tan\left(x + \frac{\pi}{4}\right) = \frac{\tan x + \tan\left(\frac{\pi}{4}\right)}{1 - \tan x \tan\left(\frac{\pi}{4}\right)}$$
$$= \frac{\tan x + 1}{1 - \tan x(1)} = \frac{\tan x + 1}{1 - \tan x}$$

49. $\dfrac{\cos(a+b)}{\cos a\cos b}=\dfrac{\cos a\cos b}{\cos a\cos b}-\dfrac{\sin a\sin b}{\cos a\cos b}=1-\tan a\tan b$

51. $\dfrac{\cos(x+h)-\cos x}{h}=\dfrac{\cos x\cos h-\sin x\sin h-\cos x}{h}=$ $\dfrac{\cos x(\cos h-1)-\sin x\sin h}{h}=\cos x\dfrac{\cos h-1}{h}-\sin x\dfrac{\sin h}{h}$

53. True **55.** True. Note that $\sin(\alpha+\beta)=\sin(\pi-\gamma)$ and expand the right hand side.

59. $\sin(x+2x)=\sin x\cos(2x)+\sin(2x)\cos x$
$=\sin x(\cos^2 x-\sin^2 x)+2\sin x\cos x\cos x$
$=\sin x\cos^2 x-\sin^3 x+2\sin x\cos^2 x$
$=3\sin x\cos^2 x-\sin^3 x$

61. $\dfrac{1+\cos(2t)}{\sin(2t)-\cos t}=\dfrac{1+2\cos^2 t-1}{2\sin t\cos t-\cos t}$
$=\dfrac{2\cos^2 t}{\cos t(2\sin t-1)}$
$=\dfrac{2\cos t}{2\sin t-1}$

63. $(\cos^2(4x)-\sin^2(4x)-\sin(8x))(\cos^2(4x)-\sin^2(4x)+\sin(8x))$
$=(\cos(8x)-\sin(8x))(\cos(8x)+\sin(8x))$
$=\cos^2(8x)-\sin^2(8x)$
$=\cos(16x)$

Section 9.3

1. Use the Pythagorean identities and isolate the squared term.

3. $\dfrac{1-\cos x}{\sin x},\dfrac{\sin x}{1+\cos x}$, multiplying the top and bottom by $\sqrt{1-\cos x}$ and $\sqrt{1+\cos x}$, respectively.

5. **a.** $\dfrac{3\sqrt{7}}{32}$ **b.** $\dfrac{31}{32}$ **c.** $\dfrac{3\sqrt{7}}{31}$ **7.** **a.** $\dfrac{\sqrt{3}}{2}$ **b.** $-\dfrac{1}{2}$ **c.** $-\sqrt{3}$

9. $\cos\theta=-\dfrac{2\sqrt{5}}{5}$, $\sin\theta=\dfrac{\sqrt{5}}{5}$, $\tan\theta=-\dfrac{1}{2}$, $\csc\theta=\sqrt{5}$, $\sec\theta=-\dfrac{\sqrt{5}}{2}$, $\cot\theta=-2$ **11.** $2\sin\left(\dfrac{\pi}{2}\right)$ **13.** $\dfrac{\sqrt{2-\sqrt{2}}}{2}$

15. $\dfrac{\sqrt{2-\sqrt{3}}}{2}$ **17.** $2+\sqrt{3}$ **19.** $-1-\sqrt{2}$

21. **a.** $\dfrac{3\sqrt{13}}{13}$ **b.** $-\dfrac{2\sqrt{13}}{13}$ **c.** $-\dfrac{3}{2}$

23. **a.** $\dfrac{\sqrt{10}}{4}$ **b.** $\dfrac{\sqrt{6}}{4}$ **c.** $\dfrac{\sqrt{15}}{3}$ **25.** $\dfrac{120}{169},-\dfrac{119}{169},-\dfrac{120}{119}$

27. $\dfrac{2\sqrt{13}}{13},\dfrac{3\sqrt{13}}{13},\dfrac{2}{3}$ **29.** $\cos(74°)$ **31.** $\cos(18x)$ **33.** $3\sin(10x)$

35. $-2\sin(-x)\cos(-x)=-2(-\sin(x)\cos(x))=\sin(2x)$

37. $\dfrac{\sin(2\theta)}{1+\cos(2\theta)}\tan^2(\theta)=\dfrac{2\sin(\theta)\cos(\theta)}{1+\cos^2(\theta)-\sin^2(\theta)}\tan^2(\theta)=$ $\dfrac{2\sin(\theta)\cos(\theta)}{2\cos^2(\theta)}\tan^2(\theta)=\dfrac{\sin(\theta)}{\cos(\theta)}\tan^2(\theta)=\cot(\theta)\tan^2(\theta)=\tan(\theta)$

39. $\dfrac{1+\cos(12x)}{2}$ **41.** $\dfrac{3+\cos(12x)-4\cos(6x)}{8}$

43. $\dfrac{2+\cos(2x)-2\cos(4x)-\cos(6x)}{32}$

45. $\dfrac{3+\cos(4x)-4\cos(2x)}{3+\cos(4x)+4\cos(2x)}$ **47.** $\dfrac{1-\cos(4x)}{8}$

49. $\dfrac{3+\cos(4x)-4\cos(2x)}{4(\cos(2x)+1)}$ **51.** $\dfrac{(1+\cos(4x))\sin x}{2}$

53. $4\sin x\cos x(\cos^2 x-\sin^2 x)$

55. $\dfrac{2\tan x}{1+\tan^2 x}=\dfrac{\frac{2\sin x}{\cos x}}{1+\frac{\sin^2 x}{\cos^2 x}}=\dfrac{\frac{2\sin x}{\cos x}}{\frac{\cos^2 x+\sin^2 x}{\cos^2 x}}$
$=\dfrac{2\sin x}{\cos x}.\dfrac{\cos^2 x}{1}=2\sin x\cos x=\sin(2x)$

57. $\dfrac{2\sin x\cos x}{2\cos^2 x-1}=\dfrac{\sin(2x)}{\cos(2x)}=\tan(2x)$

Section 9.4

1. Substitute α into cosine and β into sine and evaluate.

3. Answers will vary. There are some equations that involve a sum of two trig expressions where when converted to a product are easier to solve. For example: $\dfrac{\sin(3x)+\sin x}{\cos x}=1$. When converting the numerator to a product the equation becomes: $\dfrac{2\sin(2x)\cos x}{\cos x}=1$. **5.** $8(\cos(5x)-\cos(27x))$

7. $\sin(2x)+\sin(8x)$ **9.** $\dfrac{1}{2}(\cos(6x)-\cos(4x))$

11. $2\cos(5t)\cos t$ **13.** $2\cos(7x)$ **15.** $2\cos(6x)\cos(3x)$

17. $\dfrac{1}{4}(1+\sqrt{3})$ **19.** $\dfrac{1}{4}(\sqrt{3}-2)$ **21.** $\dfrac{1}{4}(\sqrt{3}-1)$

23. $\cos(80°)-\cos(120°)$ **25.** $\dfrac{1}{2}(\sin(221°)+\sin(205°))$

27. $\sqrt{2}\cos(31°)$ **29.** $2\cos(66.5°)\sin(34.5°)$

31. $2\sin(-1.5°)\cos(0.5°)$

33. $2\sin(7x)-2\sin x=2\sin(4x+3x)-2\sin(4x-3x)$
$=2(\sin(4x)\cos(3x)+\sin(3x)\cos(4x))-2(\sin(4x)\cos(3x)-\sin(3x)\cos(4x))$
$=2\sin(4x)\cos(3x)+2\sin(3x)\cos(4x))-2\sin(4x)\cos(3x)+2\sin(3x)\cos(4x))$
$=4\sin(3x)\cos(4x)$

35. $\sin x+\sin(3x)=2\sin\left(\dfrac{4x}{2}\right)\cos\left(\dfrac{-2x}{2}\right)=2\sin(2x)\cos x$
$=2(2\sin x\cos x)\cos x=4\sin x\cos^2 x$

37. $2\tan x\cos(3x)=\dfrac{2\sin x\cos(3x)}{\cos x}$
$=\dfrac{2(0.5(\sin(4x)-\sin(2x)))}{\cos x}$
$=\dfrac{1}{\cos x}(\sin(4x)-\sin(2x))$
$=\sec x(\sin(4x)-\sin(2x))$

39. $2\cos(35°)\cos(23°)$, 1.5081 **41.** $-2\sin(33°)\sin(11°)$, -0.2078

43. $\dfrac{1}{2}(\cos(99°)-\cos(71°))$, -0.2410 **45.** It is an identity.

47. It is not an identity, but $2\cos^3 x$ is.

49. $\tan(3t)$ **51.** $2\cos(2x)$ **53.** $-\sin(14x)$

55. Start with $\cos x+\cos y$. Make a substitution and let $x=\alpha+\beta$ and let $y=\alpha-\beta$, so $\cos x+\cos y$ becomes
$\cos(\alpha+\beta)+\cos(\alpha-\beta)=$
$=\cos\alpha\cos\beta-\sin\alpha\sin\beta+\cos\alpha\cos\beta+\sin\alpha\sin\beta$
$=2\cos\alpha\cos\beta$
Since $x=\alpha+\beta$ and $y=\alpha-\beta$, we can solve for α and β in terms of x and y and substitute in for $2\cos\alpha\cos\beta$ and get $2\cos\left(\dfrac{x+y}{2}\right)\cos\left(\dfrac{x-y}{2}\right)$.

57. $\frac{\cos(3x)+\cos x}{\cos(3x)-\cos x}=\frac{2\cos(2x)\cos x}{-2\sin(2x)\sin x}=-\cot(2x)\cot x$

59. $\frac{\cos(2y)-\cos(4y)}{\sin(2y)+\sin(4y)}=\frac{-2\sin(3y)\sin(-y)}{2\sin(3y)\cos y}$
$=\frac{2\sin(3y)\sin(y)}{2\sin(3y)\cos y}=\tan y$

61. $\cos x-\cos(3x)=-2\sin(2x)\sin(-x)=2(2\sin x\cos x)\sin x$
$=4\sin^2 x\cos x$

63. $\tan\left(\frac{\pi}{4}-t\right)=\frac{\tan\left(\frac{\pi}{4}\right)-\tan t}{1+\tan\left(\frac{\pi}{4}\right)\tan(t)}=\frac{1-\tan t}{1+\tan t}$

Section 9.5

1. There will not always be solutions to trigonometric function equations. For a basic example, $\cos(x)=-5$. **3.** If the sine or cosine function has a coefficient of one, isolate the term on one side of the equals sign. If the number it is set equal to has an absolute value less than or equal to one, the equation has solutions, otherwise it does not. If the sine or cosine does not have a coefficient equal to one, still isolate the term but then divide both sides of the equation by the leading coefficient. Then, if the number it is set equal to has an absolute value greater than one, the equation has no solution.
5. $\frac{\pi}{3},\frac{2\pi}{3}$ **7.** $\frac{3\pi}{4},\frac{5\pi}{4}$ **9.** $\frac{\pi}{4},\frac{5\pi}{4}$ **11.** $\frac{\pi}{4},\frac{3\pi}{4},\frac{5\pi}{4},\frac{7\pi}{4}$
13. $\frac{\pi}{4},\frac{7\pi}{4}$ **15.** $\frac{7\pi}{6},\frac{11\pi}{6}$ **17.** $\frac{\pi}{18},\frac{5\pi}{18},\frac{13\pi}{18},\frac{17\pi}{18},\frac{25\pi}{18},\frac{29\pi}{18}$
19. $\frac{3\pi}{12},\frac{5\pi}{12},\frac{11\pi}{12},\frac{13\pi}{12},\frac{19\pi}{12},\frac{21\pi}{12}$ **21.** $\frac{1}{6},\frac{5}{6},\frac{13}{6},\frac{17}{6},\frac{25}{6},\frac{29}{6},\frac{37}{6}$
23. $0,\frac{\pi}{3},\pi,\frac{5\pi}{3}$ **25.** $\frac{\pi}{3},\pi,\frac{5\pi}{3}$ **27.** $\frac{\pi}{3},\frac{3\pi}{2},\frac{5\pi}{3}$ **29.** $0,\pi$
31. $\pi-\sin^{-1}\left(-\frac{1}{4}\right),\frac{7\pi}{6},\frac{11\pi}{6},2\pi+\sin^{-1}\left(-\frac{1}{4}\right)$
33. $\frac{1}{3}\left(\sin^{-1}\left(\frac{9}{10}\right)\right),\frac{\pi}{3}-\frac{1}{3}\left(\sin^{-1}\left(\frac{9}{10}\right)\right),\frac{2\pi}{3}+\frac{1}{3}\left(\sin^{-1}\left(\frac{9}{10}\right)\right),$
$\pi-\frac{1}{3}\left(\sin^{-1}\left(\frac{9}{10}\right)\right),\frac{4\pi}{3}+\frac{1}{3}\left(\sin^{-1}\left(\frac{9}{10}\right)\right),\frac{5\pi}{3}-\frac{1}{3}\left(\sin^{-1}\left(\frac{9}{10}\right)\right),$
35. 0 **37.** $\theta=\sin^{-1}\left(\frac{2}{3}\right),\pi-\sin^{-1}\left(\frac{2}{3}\right),\pi+\sin^{-1}\left(\frac{2}{3}\right),$
$2\pi-\sin^{-1}\left(\frac{2}{3}\right)$ **39.** $\frac{3\pi}{2},\frac{\pi}{6},\frac{5\pi}{6}$ **41.** $0,\frac{\pi}{3},\pi,\frac{4\pi}{3}$
43. There are no solutions.
45. $\cos^{-1}\left(\frac{1}{3}(1-\sqrt{7})\right),2\pi-\cos^{-1}\left(\frac{1}{3}(1-\sqrt{7})\right)$
47. $\tan^{-1}\left(\frac{1}{2}(\sqrt{29}-5)\right),\pi+\tan^{-1}\left(\frac{1}{2}(-\sqrt{29}-5)\right),$
$\pi+\tan^{-1}\left(\frac{1}{2}(\sqrt{29}-5)\right),2\pi+\tan^{-1}\left(\frac{1}{2}(-\sqrt{29}-5)\right)$
49. There are no solutions. **51.** There are no solutions.
53. $0,\frac{2\pi}{3},\frac{4\pi}{3}$ **55.** $\frac{\pi}{4},\frac{3\pi}{4},\frac{5\pi}{4},\frac{7\pi}{4}$
57. $\sin^{-1}\left(\frac{3}{5}\right),\frac{\pi}{2},\pi-\sin^{-1}\left(\frac{3}{5}\right),\frac{3\pi}{2}$
59. $\cos^{-1}\left(-\frac{1}{4}\right),2\pi-\cos^{-1}\left(-\frac{1}{4}\right)$
61. $\frac{\pi}{3},\cos^{-1}\left(-\frac{3}{4}\right),2\pi-\cos^{-1}\left(-\frac{3}{4}\right),\frac{5\pi}{3}$
63. $\cos^{-1}\left(\frac{3}{4}\right),\cos^{-1}\left(-\frac{2}{3}\right),2\pi-\cos^{-1}\left(-\frac{2}{3}\right),2\pi-\cos^{-1}\left(\frac{3}{4}\right)$
65. $0,\frac{\pi}{2},\pi,\frac{3\pi}{2}$ **67.** $\frac{\pi}{3},\cos^{-1}\left(-\frac{1}{4}\right),2\pi-\cos^{-1}\left(-\frac{1}{4}\right),\frac{5\pi}{3}$
69. There are no solutions. **71.** $\pi+\tan^{-1}(-2),$
$\pi+\tan^{-1}\left(-\frac{3}{2}\right),2\pi+\tan^{-1}(-2),2\pi+\tan^{-1}\left(-\frac{3}{2}\right)$
73. $2\pi k+0.2734, 2\pi k+2.8682$ **75.** $\pi k-0.3277$
77. 0.6694, 1.8287, 3.8110, 4.9703 **79.** 1.0472, 3.1416, 5.2360
81. 0.5326, 1.7648, 3.6742, 4.9064 **83.** $\sin^{-1}\left(\frac{1}{4}\right),\pi-\sin^{-1}\left(\frac{1}{4}\right),\frac{3\pi}{2}$
85. $\frac{\pi}{2},\frac{3\pi}{2}$ **87.** There are no solutions. **89.** $0,\frac{\pi}{2},\pi,\frac{3\pi}{2}$
91. There are no solutions. **93.** 7.2° **95.** 5.7° **97.** 82.4°
99. 31.0° **101.** 88.7° **103.** 59.0° **105.** 36.9°

Chapter 9 Review Exercises

1. $\sin^{-1}\left(\frac{\sqrt{3}}{3}\right),\pi-\sin^{-1}\left(\frac{\sqrt{3}}{3}\right),\pi+\sin^{-1}\left(\frac{\sqrt{3}}{3}\right),2\pi-\sin^{-1}\left(\frac{\sqrt{3}}{3}\right)$
3. $\frac{7\pi}{6},\frac{11\pi}{6}$ **5.** $\sin^{-1}\left(\frac{1}{4}\right),\pi-\sin^{-1}\left(\frac{1}{4}\right)$ **7.** 1 **9.** Yes
11. $-2-\sqrt{3}$ **13.** $\frac{\sqrt{2}}{2}$
15. $\cos(4x)-\cos(3x)\cos x=\cos(2x+2x)-\cos(x+2x)\cos x$
$=\cos(2x)\cos(2x)-\sin(2x)\sin(2x)-\cos x\cos(2x)\cos x+\sin x\sin(2x)\cos x$
$=(\cos^2 x-\sin^2 x)^2-4\cos^2 x\sin^2 x-\cos^2 x(\cos^2 x-\sin^2 x)+\sin x\,(2)\sin x\cos x\cos x$
$=(\cos^2 x-\sin^2 x)^2-4\cos^2 x\sin^2 x-\cos^2 x(\cos^2 x-\sin^2 x)+2\sin^2 x\cos^2 x$
$=\cos^4 x-2\cos^2 x\sin^2 x+\sin^4 x-4\cos^2 x\sin^2 x-\cos^4 x+\cos^2 x\sin^2 x+2\sin^2 x\cos^2 x$
$=\sin^4 x-4\cos^2 x\sin^2 x+\cos^2 x\sin^2 x$
$=\sin^2 x(\sin^2 x+\cos^2 x)-4\cos^2 x\sin^2 x$
$=\sin^2 x-4\cos^2 x\sin^2 x$
17. $\tan\left(\frac{5}{8}x\right)$ **19.** $\frac{\sqrt{3}}{3}$ **21.** $-\frac{24}{25},-\frac{7}{25},\frac{24}{7}$
23. $\sqrt{2(2+\sqrt{2})}$ **25.** $\frac{\sqrt{2}}{10},\frac{7\sqrt{2}}{10},\frac{1}{7},\frac{3}{5},\frac{4}{5},\frac{3}{4}$
27. $\cot x\cos(2x)=\cot x\,(1-2\sin^2 x)$
$=\cot x-\frac{\cos x}{\sin x}(2)\sin^2 x$
$=-2\sin x\cos x+\cot x$
$=-\sin(2x)+\cot x$
29. $\frac{10\sin x-5\sin(3x)+\sin(5x)}{8(\cos(2x)+1)}$ **31.** $\frac{\sqrt{3}}{2}$ **33.** $-\frac{\sqrt{2}}{2}$
35. $\frac{1}{2}(\sin(6x)+\sin(12x))$ **37.** $2\sin\left(\frac{13}{2}x\right)\cos\left(\frac{9}{2}x\right)$
39. $\frac{3\pi}{4},\frac{7\pi}{4}$ **41.** $0,\frac{\pi}{6},\frac{5\pi}{6},\pi$ **43.** $\frac{3\pi}{2}$ **45.** No solution
47. 0.2527, 2.8889, 4.7124 **49.** 1.3694, 1.9106, 4.3726, 4.9137

Chapter 9 Practice Test

1. 1 **3.** $\sec\theta$ **5.** $\frac{\sqrt{2}-\sqrt{6}}{4}$ **7.** $-\sqrt{2}-\sqrt{3}$
9. $-\frac{1}{2}\cos(\theta)-\frac{\sqrt{3}}{2}\sin(\theta)$ **11.** $\frac{1-\cos(64°)}{2}$ **13.** $0,\pi$
15. $\frac{\pi}{2},\frac{3\pi}{2}$ **17.** $2\cos(3x)\cos(5x)$ **19.** $4\sin(2\theta)\cos(6\theta)$
21. $x=\cos^{-1}\left(\frac{1}{5}\right)$ **23.** $\frac{\pi}{3},\frac{4\pi}{3}$ **25.** $\frac{3}{5},-\frac{4}{5},-\frac{3}{4}$

27. $\tan^3 x - \tan x \sec^2 x = \tan x\,(\tan^2 x - \sec^2 x)$
$= \tan x\,(\tan^2 x - (1 + \tan^2 x))$
$= \tan x\,(\tan^2 x - 1 - \tan^2 x)$
$= -\tan x = \tan(-x) = \tan(-x)$

29. $\dfrac{\sin(2x)}{\sin x} - \dfrac{\cos(2x)}{\cos x} = \dfrac{2\sin x\cos x}{\sin x} - \dfrac{2\cos^2 x - 1}{\cos x}$
$= 2\cos x - 2\cos x + \dfrac{1}{\cos x}$
$= \dfrac{1}{\cos x} = \sec x = \sec x$

31. Amplitude: $\frac{1}{4}$, period $\frac{1}{60}$, frequency: 60 Hz

33. Amplitude: 8, fast period: $\frac{1}{500}$, fast frequency: 500 Hz, slow period: $\frac{1}{10}$, slow frequency: 10 Hz **35.** $D(t) = 20\,(0.9086)^t \cos(4\pi t)$, 31 second

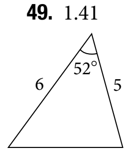
CHAPTER 10

Section 10.1

1. The altitude extends from any vertex to the opposite side or to the line containing the opposite side at a 90° angle. **3.** When the known values are the side opposite the missing angle and another side and its opposite angle. **5.** A triangle with two given sides and a non-included angle. **7.** $\beta = 72°$, $a \approx 12.0$, $b \approx 19.9$ **9.** $\gamma = 20°$, $b \approx 4.5$, $c \approx 1.6$ **11.** $b \approx 3.78$ **13.** $c \approx 13.70$ **15.** One triangle, $\alpha \approx 50.3°$, $\beta \approx 16.7°$, $a \approx 26.7$ **17.** Two triangles, $\gamma \approx 54.3°$, $\beta \approx 90.7°$, $b \approx 20.9$ or $\gamma' \approx 125.7°$, $\beta' \approx 19.3°$, $b' \approx 6.9$ **19.** Two triangles, $\beta \approx 75.7°$, $\gamma \approx 61.3°$, $b \approx 9.9$ or $\beta' \approx 18.3°$, $\gamma' \approx 118.7°$, $b' \approx 3.2$ **21.** Two triangles, $\alpha \approx 143.2°$, $\beta \approx 26.8°$, $a \approx 17.3$ or $\alpha' \approx 16.8°$, $\beta' \approx 153.2°$, $a' \approx 8.3$ **23.** No triangle possible **25.** $A \approx 47.8°$ or $A' \approx 132.2°$ **27.** 8.6 **29.** 370.9 **31.** 12.3 **33.** 12.2 **35.** 16.0 **37.** 29.7° **39.** $x = 76.9°$ or $x = 103.1°$ **41.** 110.6° **43.** $A \approx 39.4$, $C \approx 47.6$, $BC \approx 20.7$ **45.** 57.1 **47.** 42.0 **49.** 430.2 **51.** 10.1 **53.** $AD \approx 13.8$ **55.** $AB \approx 2.8$ **57.** $L \approx 49.7$, $N \approx 56.1$, $LN \approx 5.8$ **59.** 51.4 feet **61.** The distance from the satellite to station A is approximately 1,716 miles. The satellite is approximately 1,706 miles above the ground. **63.** 2.6 ft **65.** 5.6 km **67.** 371 ft **69.** 5,936 ft **71.** 24.1 ft **73.** 19,056 ft² **75.** 445,624 square miles **77.** 8.65 ft²

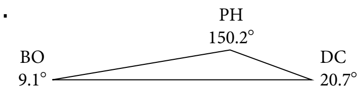
Section 10.2

1. Two sides and the angle opposite the missing side. **3.** s is the semi-perimeter, which is half the perimeter of the triangle. **5.** The Law of Cosines must be used for any oblique (non-right) triangle. **7.** 11.3 **9.** 34.7 **11.** 26.7 **13.** 257.4 **15.** Not possible **17.** 95.5° **19.** 26.9° **21.** $B \approx 45.9°$, $C \approx 99.1°$, $a \approx 6.4$ **23.** $A \approx 20.6°$, $B \approx 38.4°$, $c \approx 51.1$ **25.** $A \approx 37.8°$, $B \approx 43.8°$, $C \approx 98.4°$ **27.** 177.56 in² **29.** 0.04 m² **31.** 0.91 yd² **33.** 3.0 **35.** 29.1 **37.** 0.5 **39.** 70.7° **41.** 77.4° **43.** 25.0 **45.** 9.3 **47.** 43.52 **49.** 1.41 **51.** 0.14 **53.** 18.3 **55.** 48.98 **57.**



59. 7.62 **61.** 85.1 **63.** 24.0 km **65.** 99.9 ft **67.** 37.3 miles **69.** 2,371 miles

71.



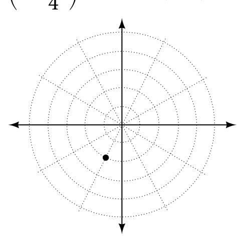
73. 599.8 miles **75.** 65.4 cm² **77.** 468 ft²

Section 10.3

1. For polar coordinates, the point in the plane depends on the angle from the positive x-axis and distance from the origin, while in Cartesian coordinates, the point represents the horizontal and vertical distances from the origin. For each point in the coordinate plane, there is one representation, but for each point in the polar plane, there are infinite representations. **3.** Determine θ for the point, then move r units from the pole to plot the point. If r is negative, move r units from the pole in the opposite direction but along the same angle. The point is a distance of r away from the origin at an angle of θ from the polar axis. **5.** The point $\left(-3, \frac{\pi}{2}\right)$ has a positive angle but a negative radius and is plotted by moving to an angle of $\frac{\pi}{2}$ and then moving 3 units in the negative direction. This places the point 3 units down the negative y-axis. The point $\left(3, -\frac{\pi}{2}\right)$ has a negative angle and a positive radius and is plotted by first moving to an angle of $-\frac{\pi}{2}$ and then moving 3 units down, which is the positive direction for a negative angle. The point is also 3 units down the negative y-axis.

7. $(-5, 0)$ **9.** $\left(-\frac{3\sqrt{3}}{2}, -\frac{3}{2}\right)$ **11.** $(2\sqrt{5}, 0.464)$ **13.** $(\sqrt{34}, 5.253)$ **15.** $\left(8\sqrt{2}, \frac{\pi}{4}\right)$ **17.** $r = 4\csc\theta$ **19.** $r = \sqrt[3]{\dfrac{\sin\theta}{2\cos^4\theta}}$ **21.** $r = 3\cos\theta$ **23.** $r = \dfrac{3\sin\theta}{\cos(2\theta)}$ **25.** $r = \dfrac{9\sin\theta}{\cos^2\theta}$ **27.** $r = \sqrt{\dfrac{1}{9\cos\theta\sin\theta}}$ **29.** $x^2 + y^2 = 4x$ or $\dfrac{(x-2)^2}{4} + \dfrac{y^2}{4} = 1$; circle **31.** $3y + x = 6$; line **33.** $y = 3$; line **35.** $xy = 4$; hyperbola **37.** $x^2 + y^2 = 4$; circle **39.** $x - 5y = 3$; line **41.** $\left(3, \frac{3\pi}{4}\right)$ **43.** $(5, \pi)$

45.



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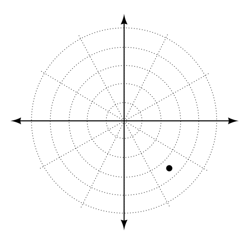

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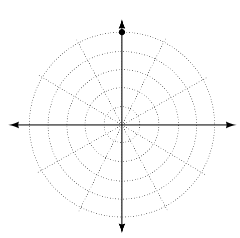

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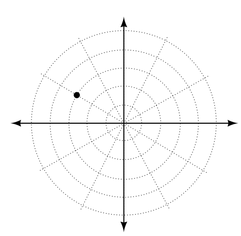

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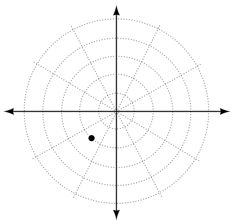
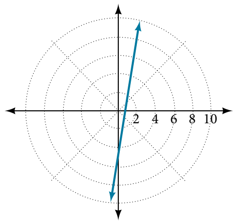
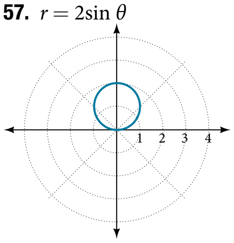


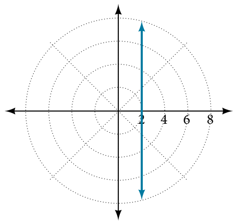
55. $r = \dfrac{6}{5\cos\theta - \sin\theta}$



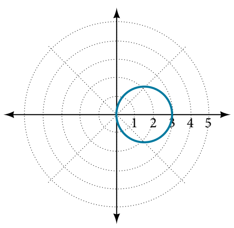
57. $r = 2\sin\theta$



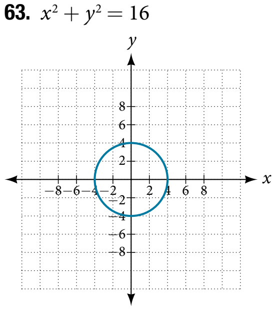
59. $r = \dfrac{2}{\cos\theta}$



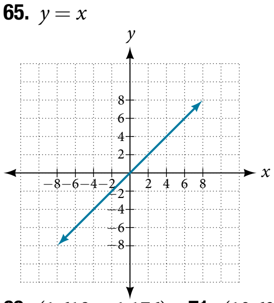
61. $r = 3\cos\theta$



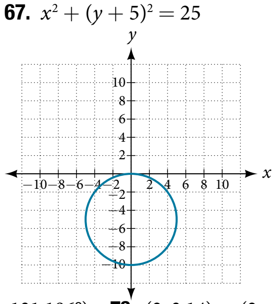
63. $x^2 + y^2 = 16$



65. $y = x$

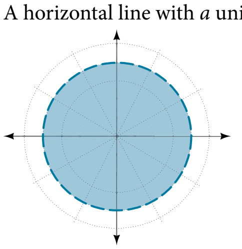


67. $x^2 + (y+5)^2 = 25$



69. $(1.618, -1.176)$ **71.** $(10.630, 131.186°)$ **73.** $(2, 3.14)$ or $(2, \pi)$
75. A vertical line with a units left of the y-axis.
77. A horizontal line with a units below the x-axis.

79.



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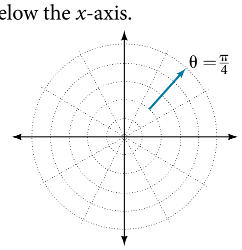
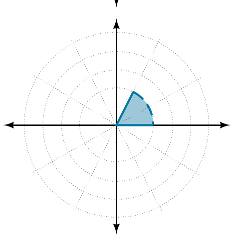


83.

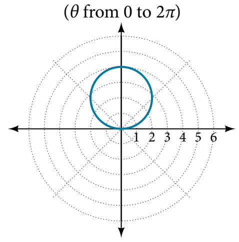


Section 10.4

1. Symmetry with respect to the polar axis is similar to symmetry about the x-axis, symmetry with respect to the pole is similar to symmetry about the origin, and symmetric with respect to the line $\theta = \frac{\pi}{2}$ is similar to symmetry about the y-axis.
3. Test for symmetry; find zeros, intercepts, and maxima; make a table of values. Decide the general type of graph, cardioid, limaçon, lemniscate, etc., then plot points at $\theta = 0, \frac{\pi}{2}, \pi$ and $\frac{3\pi}{2}$, and sketch the graph. **5.** The shape of the polar graph is determined by whether or not it includes a sine, a cosine, and constants in the equation. **7.** Symmetric with respect to the polar axis **9.** Symmetric with respect to the polar axis, symmetric with respect to the line $\theta = \frac{\pi}{2}$, symmetric with respect to the pole **11.** No symmetry **13.** No symmetry
15. Symmetric with respect to the pole

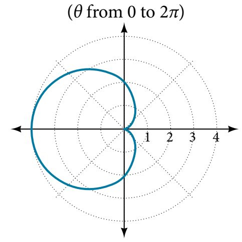
17. Circle

(θ from 0 to 2π)



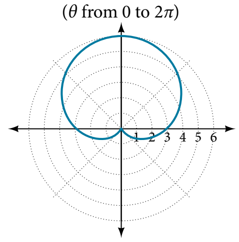
19. Cardioid

(θ from 0 to 2π)



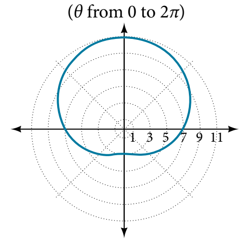
21. Cardioid

(θ from 0 to 2π)



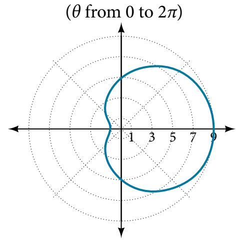
23. One-loop/dimpled limaçon

(θ from 0 to 2π)

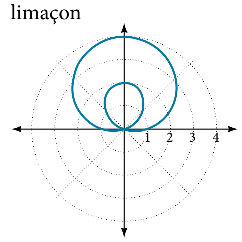


25. One-loop/dimpled limaçon

(θ from 0 to 2π)

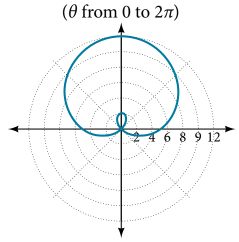


27. Inner loop/two-loop limaçon



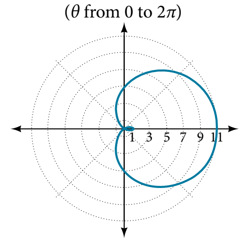
29. Inner loop/two-loop limaçon

(θ from 0 to 2π)



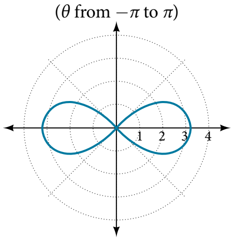
31. Inner loop/two-loop limaçon

(θ from 0 to 2π)



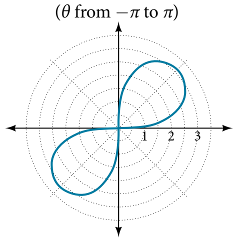
33. Lemniscate

(θ from $-\pi$ to π)

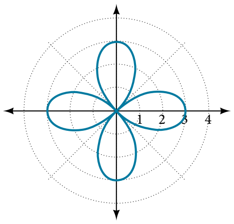


35. Lemniscate

(θ from $-\pi$ to π)

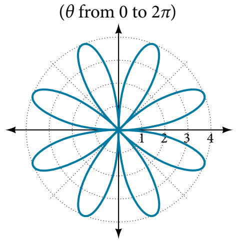


37. Rose curve



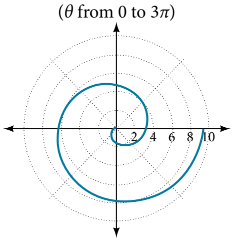
39. Rose curve

(θ from 0 to 2π)



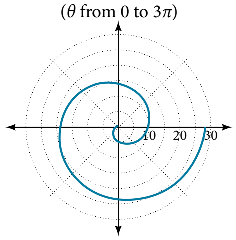
41. Archimedes' spiral

(θ from 0 to 3π)

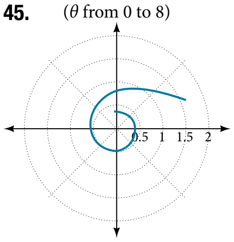


43. Archimedes' spiral

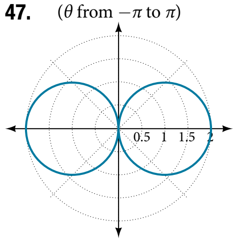
(θ from 0 to 3π)



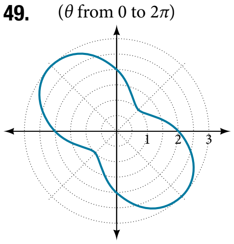
45. (θ from 0 to 8)



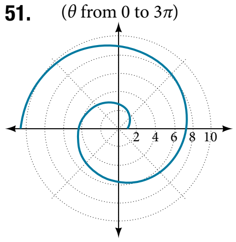
47. (θ from $-\pi$ to π)



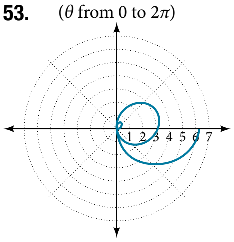
49. (θ from 0 to 2π)



51. (θ from 0 to 3π)



53. (θ from 0 to 2π)



55. They are both spirals, but not quite the same.

57. Both graphs are curves with 2 loops. The equation with a coefficient of θ has two loops on the left, the equation with a coefficient of 2 has two loops side by side. Graph these from 0 to 4π to get a better picture.

59. When the width of the domain is increased, more petals of the flower are visible. **61.** The graphs are three-petal, rose curves. The larger the coefficient, the greater the curve's distance from the pole. **63.** The graphs are spirals. The smaller the coefficient, the tighter the spiral. **65.** $\left(4, \frac{\pi}{3}\right), \left(4, \frac{5\pi}{3}\right)$

67. $\left(\frac{3}{2}, \frac{\pi}{3}\right), \left(\frac{3}{2}, \frac{5\pi}{3}\right)$ **69.** $\left(0, \frac{\pi}{2}\right), (0, \pi), \left(0, \frac{3\pi}{2}\right), (0, 2\pi)$

71. $\left(\frac{\sqrt[4]{8}}{2}, \frac{\pi}{4}\right), \left(\frac{\sqrt[4]{8}}{2}, \frac{5\pi}{4}\right)$ and at $\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$ since r is squared

Section 10.5

1. a is the real part, b is the imaginary part, and $i = \sqrt{-1}$
3. Polar form converts the real and imaginary part of the complex number in polar form using $x = r\cos\theta$ and $y = r\sin\theta$.
5. $z^n = r^n(\cos(n\theta) + i\sin(n\theta))$ It is used to simplify polar form when a number has been raised to a power. **7.** $5\sqrt{2}$

9. $\sqrt{38}$ **11.** $\sqrt{14.45}$ **13.** $4\sqrt{5}\text{cis}(333.4°)$

15. $2\text{cis}\left(\frac{\pi}{6}\right)$ **17.** $\frac{7\sqrt{3}}{2} + \frac{7}{2}i$ **19.** $-2\sqrt{3} - 2i$

21. $-1.5 - \frac{3\sqrt{3}}{2}i$ **23.** $4\sqrt{3}\text{cis}(198°)$ **25.** $\frac{3}{4}\text{cis}(180°)$

27. $5\sqrt{3}\,\text{cis}\left(\frac{17\pi}{24}\right)$ **29.** $7\text{cis}(70°)$ **31.** $5\text{cis}(80°)$

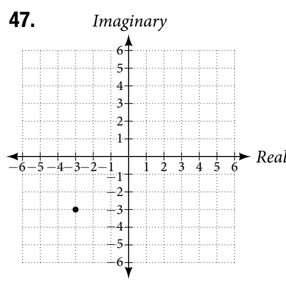
33. $5\text{cis}\left(\frac{\pi}{3}\right)$ **35.** $125\text{cis}(135°)$ **37.** $9\text{cis}(240°)$

39. $\text{cis}\left(\frac{3\pi}{4}\right)$ **41.** $3\text{cis}(80°)$, $3\text{cis}(200°)$, $3\text{cis}(320°)$

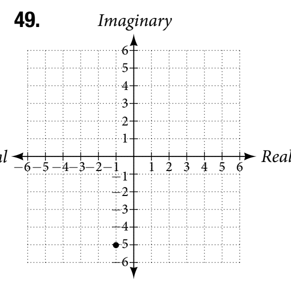
43. $2\sqrt[3]{4}\text{cis}\left(\frac{2\pi}{9}\right), 2\sqrt[3]{4}\text{cis}\left(\frac{8\pi}{9}\right), 2\sqrt[3]{4}\text{cis}\left(\frac{14\pi}{9}\right)$

45. $2\sqrt{2}\text{cis}\left(\frac{7\pi}{8}\right), 2\sqrt{2}\text{cis}\left(\frac{15\pi}{8}\right)$

47.



49.



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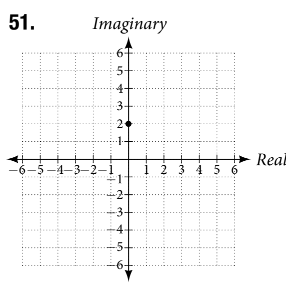
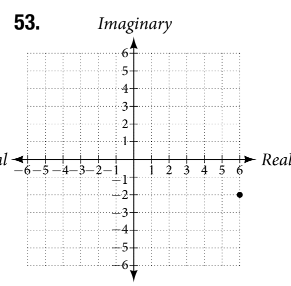


53.



55. Plot of $1 - 4i$ in the complex plane (1 along the real axis, -4 along the imaginary axis). **57.** $3.61e^{-0.59i}$
59. $-2 + 3.46i$ **61.** $-4.33 - 2.50i$

Section 10.6

1. A pair of functions that is dependent on an external factor. The two functions are written in terms of the same parameter. For example, $x = f(t)$ and $y = f(t)$. **3.** Choose one equation to solve for t, substitute into the other equation and simplify. **5.** Some equations cannot be written as functions, like a circle. However, when written as two parametric equations, separately the equations are functions. **7.** $y = -2 + 2x$

9. $y = 3\sqrt{\frac{x-1}{2}}$ **11.** $x = 2e^{\frac{1-y}{5}}$ or $y = 1 - 5\ln\left(\frac{x}{2}\right)$

13. $x = 4\log\left(\frac{y-3}{2}\right)$ **15.** $x = \left(\frac{y}{2}\right)^3 - \frac{y}{2}$ **17.** $y = x^3$

19. $\left(\frac{x}{4}\right)^2 + \left(\frac{y}{5}\right)^2 = 1$ **21.** $y^2 = 1 - \frac{1}{2}x$ **23.** $y = x^2 + 2x + 1$

25. $y = \left(\frac{x+1}{2}\right)^3 - 2$ **27.** $y = -3x + 14$ **29.** $y = x + 3$

31. $\begin{cases} x(t) = t \\ y(t) = 2\sin t + 1 \end{cases}$ **33.** $\begin{cases} x(t) = \sqrt{t} + 2t \\ y(t) = t \end{cases}$

35. $\begin{cases} x(t) = 4\cos t \\ y(t) = 6\sin t \end{cases}$; Ellipse **37.** $\begin{cases} x(t) = \sqrt{10}\cos t \\ y(t) = \sqrt{10}\sin t \end{cases}$; Circle

39. $\begin{cases} x(t) = -1 + 4t \\ y(t) = -2t \end{cases}$ **41.** $\begin{cases} x(t) = 4 + 2t \\ y(t) = 1 - 3t \end{cases}$

43. Yes, at $t = 2$

45.

t	1	2	3
x	-3	0	5
y	1	7	17

47. Answers may vary:

$\begin{cases} x(t) = t - 1 \\ y(t) = t^2 \end{cases}$ and $\begin{cases} x(t) = t + 1 \\ y(t) = (t + 2)^2 \end{cases}$

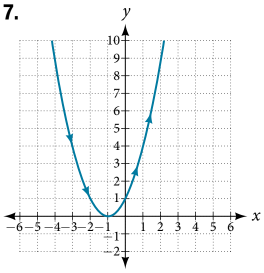
49. Answers may vary:

$\begin{cases} x(t) = t \\ y(t) = t^2 - 4t + 4 \end{cases}$ and $\begin{cases} x(t) = t + 2 \\ y(t) = t^2 \end{cases}$

Section 10.7

1. Plotting points with the orientation arrow and a graphing calculator **3.** The arrows show the orientation, the direction of motion according to increasing values of t. **5.** The parametric equations show the different vertical and horizontal motions over time.

7.



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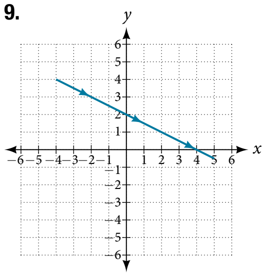

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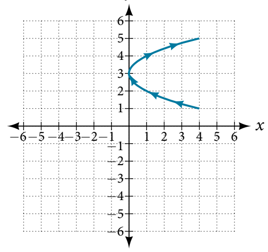

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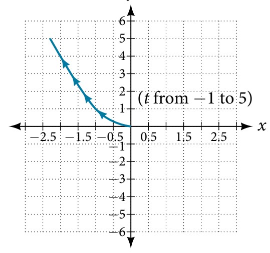

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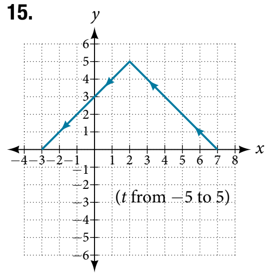

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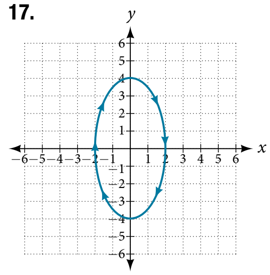

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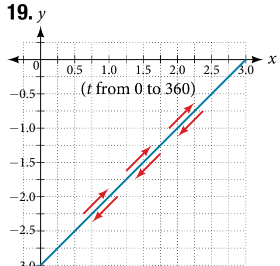

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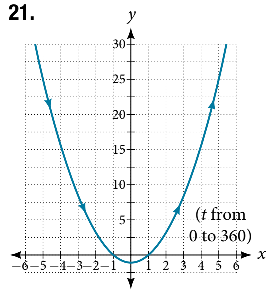

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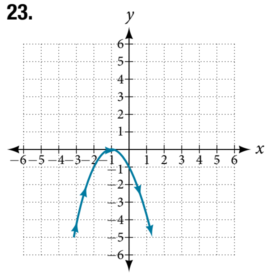

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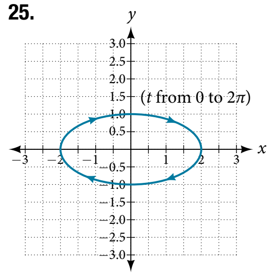

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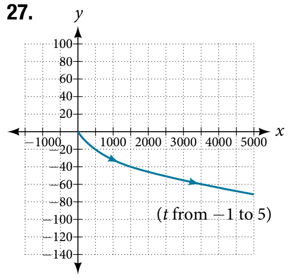

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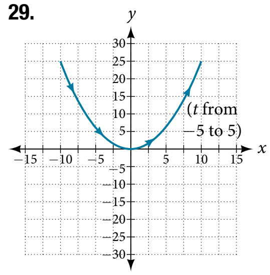

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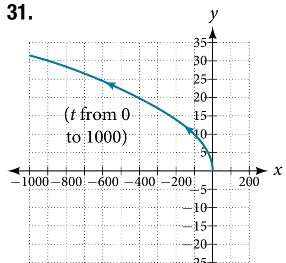

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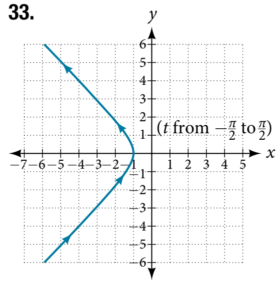

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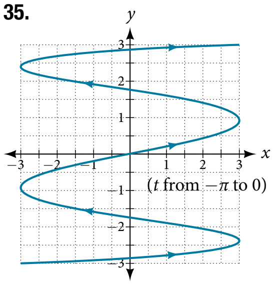

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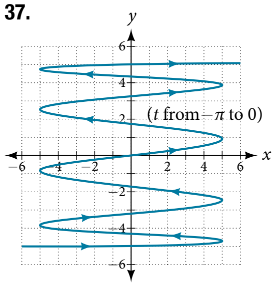
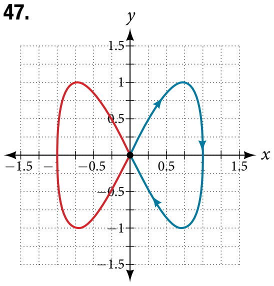


39. There will be 100 back-and-forth motions. **41.** Take the opposite of the $x(t)$ equation. **43.** The parabola opens up.

45. $\begin{cases} x(t) = 5 \cos t \\ y(t) = 5 \sin t \end{cases}$

47.



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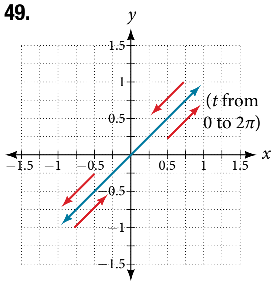

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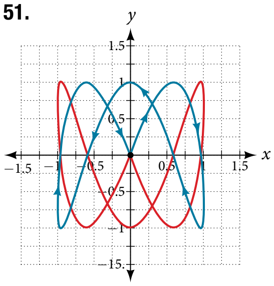
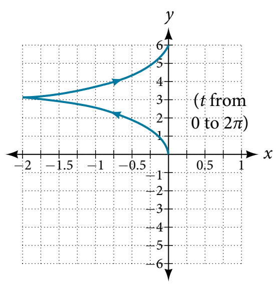


53. $a = 4, b = 3, c = 6, d = 1$

55. $a = 4, b = 2, c = 3, d = 3$



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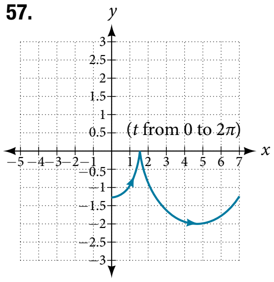
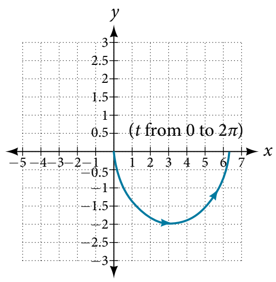
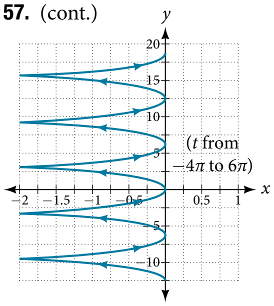
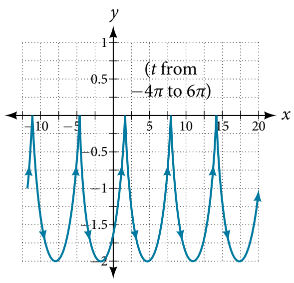




57. (cont.)





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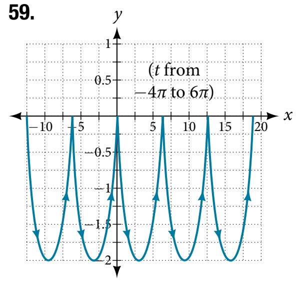


61. The y-intercept changes.

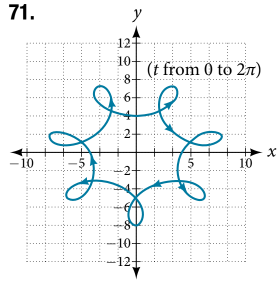
63. $y(x) = -16\left(\frac{x}{15}\right)^2 + 20\left(\frac{x}{15}\right)$

65. $\begin{cases} x(t) = 64\cos(52°) \\ y(t) = -16t^2 + 64t\sin(52°) \end{cases}$

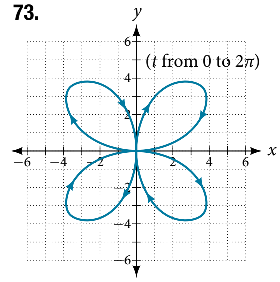
67. Approximately 3.2 seconds

69. 1.6 seconds

71.

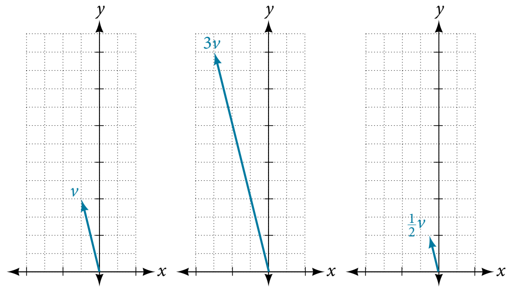


73.



Section 10.8

1. Lowercase, bold letter, usually u, v, w **3.** They are unit vectors. They are used to represent the horizontal and vertical components of a vector. They each have a magnitude of 1.
5. The first number always represents the coefficient of the i, and the second represents the j. **7.** $\langle 7, -5 \rangle$ **9.** Not equal
11. Equal **13.** Equal **15.** $-7i - 3j$ **17.** $-6i - 2j$
19. $u + v = \langle -5, 5 \rangle$, $u - v = \langle -1, 3 \rangle$, $2u - 3v = \langle 0, 5 \rangle$
21. $-10i - 4j$ **23.** $-\frac{2\sqrt{29}}{29}i + \frac{5\sqrt{29}}{29}j$
25. $-\frac{2\sqrt{229}}{229}i + \frac{15\sqrt{229}}{229}j$ **27.** $-\frac{7\sqrt{2}}{10}i + \frac{\sqrt{2}}{10}j$
29. $|v| = 7.810$, $\theta = 39.806°$ **31.** $|v| = 7.211$, $\theta = 236.310°$
33. -6 **35.** -12
37.



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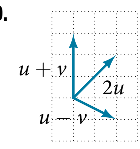

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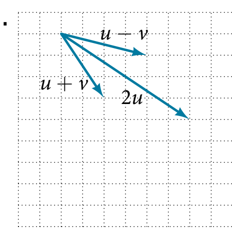

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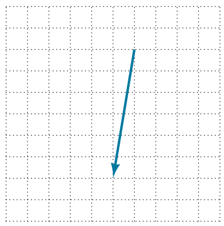

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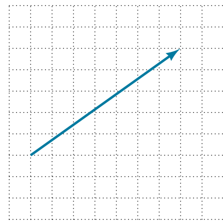
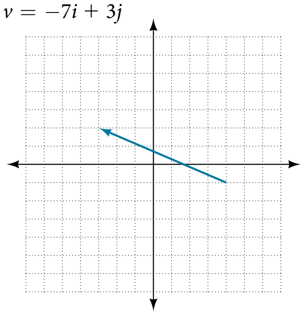


47. $\langle 4, 1\rangle$

49. $v = -7i + 3j$



51. $3\sqrt{2}i + 3\sqrt{2}j$

53. $i - \sqrt{3}j$

55. **a.** 58.7; **b.** 12.5

57. $x = 7.13$ pounds, $y = 3.63$ pounds

59. $x = 2.87$ pounds, $y = 4.10$ pounds

61. 4.635 miles, 17.764° N of E

63. 17 miles, 10.071 miles

65. Distance: 2.868, Direction: 86.474° North of West, or 3.526° West of North

67. 4.924°, 659 km/hr **69.** 4.424° **71.** (0.081, 8.602)

73. 21.801°, relative to the car's forward direction

75. Parallel: 16.28, perpendicular: 47.28 pounds

77. 19.35 pounds, 51.65° from the horizontal

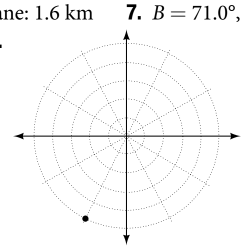
79. 5.1583 pounds, 75.8° from the horizontal

Chapter 10 Review Exercises

1. Not possible **3.** $C = 120°$, $a = 23.1$, $c = 34.1$

5. Distance of the plane from point A: 2.2 km, elevation of the plane: 1.6 km **7.** $B = 71.0°$, $C = 55.0°$, $a = 12.8$ **9.** 40.6 km

11.



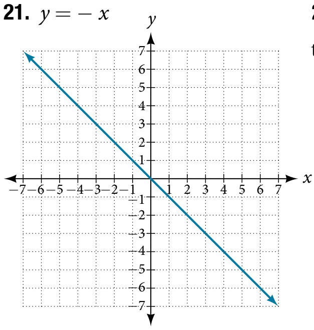
13. (0, 2)

15. (9.8489, 203.96°)

17. $r = 8$

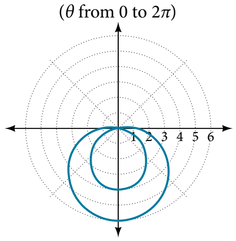
19. $x^2 + y^2 = 7x$

21. $y = -x$

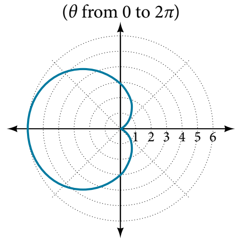


23. Symmetric with respect to the line $\theta = \frac{\pi}{2}$

25.



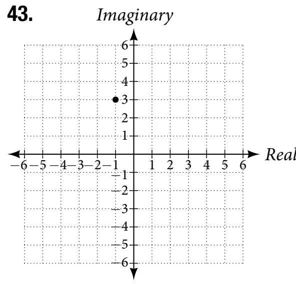
27.



29. 5 **31.** $\text{cis}\left(-\frac{\pi}{3}\right)$ **33.** $2.3 + 1.9i$ **35.** $60\text{cis}\left(\frac{\pi}{2}\right)$

37. $3\text{cis}\left(\frac{4\pi}{3}\right)$ **39.** $25\text{cis}\left(\frac{3\pi}{2}\right)$ **41.** $5\text{cis}\left(\frac{3\pi}{4}\right), 5\text{cis}\left(\frac{7\pi}{4}\right)$

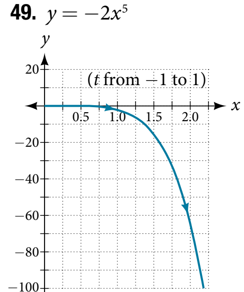
43.



45. $x^2 + \frac{1}{2}y = 1$

47. $\begin{cases} x(t) = -2 + 6t \\ y(t) = 3 + 4t \end{cases}$

49. $y = -2x^5$



51. **a.** $\begin{cases} x(t) = (80\cos(40°))t \\ y(t) = -16t^2 + (80\sin(40°))t + 4 \end{cases}$

b. The ball is 14 feet high and 184 feet from where it was launched.

c. 3.3 seconds

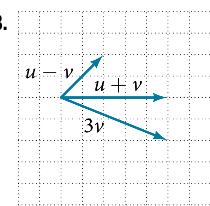
53. Not equal **55.** $4i$

57. $-\frac{3\sqrt{10}}{10}i, -\frac{\sqrt{10}}{10}j$

59. Magnitude: $3\sqrt{2}$, Direction: 225°

61. 16

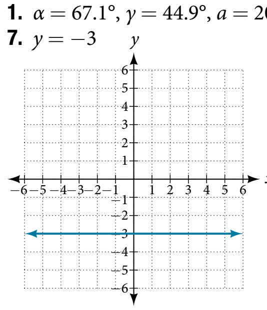
63.



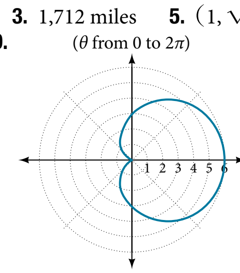
Chapter 10 Practice Test

1. $\alpha = 67.1°$, $\gamma = 44.9°$, $a = 20.9$ **3.** 1,712 miles **5.** $(1, \sqrt{3})$

7. $y = -3$



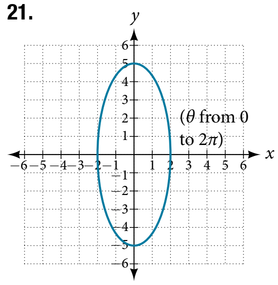
9.



11. $\sqrt{106}$ **13.** $-\frac{5}{2} + \frac{5\sqrt{3}}{2}i$ **15.** $4\text{cis}(21°)$

17. $2\sqrt{2}\,\text{cis}(18°), 2\sqrt{2}\,\text{cis}(198°)$ **19.** $y = 2(x-1)^2$

21.



23. $-4i - 15j$

25. $\frac{2\sqrt{13}}{13}i + \frac{3\sqrt{13}}{13}j$

Chapter 11

Section 11.1

1. No, you can either have zero, one, or infinitely many. Examine graphs. **3.** This means there is no realistic break-even point. By the time the company produces one unit they are already making profit. **5.** You can solve by substitution (isolating x or y), graphically, or by addition. **7.** Yes **9.** Yes **11.** $(-1, 2)$
13. $(-3, 1)$ **15.** $\left(-\frac{3}{5}, 0\right)$ **17.** No solutions exist
19. $\left(\frac{72}{5}, \frac{132}{5}\right)$ **21.** $(6, -6)$ **23.** $\left(-\frac{1}{2}, \frac{1}{10}\right)$
25. No solutions exist. **27.** $\left(-\frac{1}{5}, \frac{2}{3}\right)$ **29.** $\left(x, \frac{x+3}{2}\right)$
31. $(-4, 4)$ **33.** $\left(\frac{1}{2}, \frac{1}{8}\right)$ **35.** $\left(\frac{1}{6}, 0\right)$
37. $(x, 2(7x-6))$ **39.** $\left(-\frac{5}{6}, \frac{4}{3}\right)$ **41.** Consistent with one solution **43.** Consistent with one solution
45. Dependent with infinitely many solutions
47. $(-3.08, 4.91)$ **49.** $(-1.52, 2.29)$ **51.** $\left(\frac{A+B}{2}, \frac{A-B}{2}\right)$
53. $\left(-\frac{1}{A-B}, \frac{A}{A-B}\right)$ **55.** $\left(\frac{EC-BF}{AE-BD}, \frac{DC-AF}{BD-AE}\right)$
57. They never turn a profit. **59.** (1,250, 100,000)
61. The numbers are 7.5 and 20.5. **63.** 24,000
65. 790 sophomores, 805 freshman **67.** 56 men, 74 women
69. 10 gallons of 10% solution, 15 gallons of 60% solution
71. Swan Peak: \$750,000, Riverside: \$350,000 **73.** \$12,500 in the first account, \$10,500 in the second account **75.** High-tops: 45, Low-tops: 15 **77.** Infinitely many solutions. We need more information.

Section 11.2

1. No, there can be only one, zero, or infinitely many solutions.
3. Not necessarily. There could be zero, one, or infinitely many solutions. For example, $(0, 0, 0)$ is not a solution to the system below, but that does not mean that it has no solution.
$2x+3y-6z=1 \quad -4x-6y+12z=-2 \quad x+2y+5z=10$
5. Every system of equations can be solved graphically, by substitution, and by addition. However, systems of three equations become very complex to solve graphically so other methods are usually preferable. **7.** No **9.** Yes

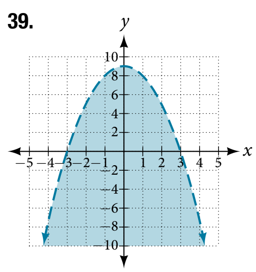
11. $(-1, 4, 2)$ **13.** $\left(-\frac{85}{107}, \frac{312}{107}, \frac{191}{107}\right)$ **15.** $\left(1, \frac{1}{2}, 0\right)$
17. $(4, -6, 1)$ **19.** $\left(x, \frac{65-16x}{27}, \frac{28+x}{27}\right)$ **21.** $\left(-\frac{45}{13}, \frac{17}{13}, -2\right)$
23. No solutions exist **25.** $(0, 0, 0)$ **27.** $\left(\frac{4}{7}, -\frac{1}{7}, -\frac{3}{7}\right)$
29. $(7, 20, 16)$ **31.** $(-6, 2, 1)$ **33.** $(5, 12, 15)$
35. $(-5, -5, -5)$ **37.** $(10, 10, 10)$ **39.** $\left(\frac{1}{2}, \frac{1}{5}, \frac{4}{5}\right)$
41. $\left(\frac{1}{2}, \frac{2}{5}, \frac{4}{5}\right)$ **43.** $(2, 0, 0)$ **45.** $(1, 1, 1)$
47. $\left(\frac{128}{557}, \frac{23}{557}, \frac{28}{557}\right)$ **49.** $(6, -1, 0)$ **51.** 24, 36, 48
53. 70 grandparents, 140 parents, 190 children **55.** Your share was \$19.95, Sarah's share was \$40, and your other roommate's share was \$22.05.

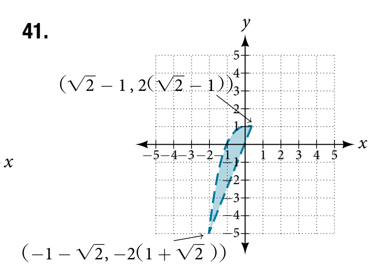
57. There are infinitely many solutions; we need more information. **59.** 500 students, 225 children, and 450 adults
61. The BMW was \$49,636, the Jeep was \$42,636, and the Toyota was \$47,727. **63.** \$400,000 in the account that pays 3% interest, \$500,000 in the account that pays 4% interest, and \$100,000 in the account that pays 2% interest. **65.** The United States consumed 26.3%, Japan 7.1%, and China 6.4% of the world's oil. **67.** Saudi Arabia imported 16.8%, Canada imported 15.1%, and Mexico 15.0% **69.** Birds were 19.3%, fish were 18.6%, and mammals were 17.1% of endangered species

Section 11.3

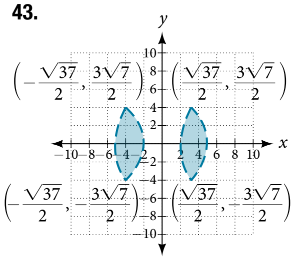
1. A nonlinear system could be representative of two circles that overlap and intersect in two locations, hence two solutions. A nonlinear system could be representative of a parabola and a circle, where the vertex of the parabola meets the circle and the branches also intersect the circle, hence three solutions. **3.** No. There does not need to be a feasible region. Consider a system that is bounded by two parallel lines. One inequality represents the region above the upper line; the other represents the region below the lower line. In this case, no points in the plane are located in both regions; hence there is no feasible region. **5.** Choose any number between each solution and plug into $C(x)$ and $R(x)$. If $C(x) < R(x)$, then there is profit.

7. $(0, -3), (3, 0)$ **9.** $\left(-\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right), \left(\frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}\right)$
11. $(-3, 0), (3, 0)$ **13.** $\left(\frac{1}{4}, -\frac{\sqrt{62}}{8}\right), \left(\frac{1}{4}, \frac{\sqrt{62}}{8}\right)$
15. $\left(-\frac{\sqrt{398}}{4}, \frac{199}{4}\right), \left(\frac{\sqrt{398}}{4}, \frac{199}{4}\right)$ **17.** $(0, 2), (1, 3)$
19. $\left(-\sqrt{\frac{1}{2}(\sqrt{5}-1)}, \frac{1}{2}(1-\sqrt{5})\right), \left(\sqrt{\frac{1}{2}(\sqrt{5}-1)}, \frac{1}{2}(1-\sqrt{5})\right)$
21. $(5, 0)$ **23.** $(0, 0)$ **25.** $(3, 0)$ **27.** No solutions exist
29. No solutions exist
31. $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right), \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right), \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
33. $(2, 0)$ **35.** $(-\sqrt{7}, -3), (-\sqrt{7}, 3), (\sqrt{7}, -3), (\sqrt{7}, 3)$
37. $\left(-\sqrt{\frac{1}{2}(\sqrt{73}-5)}, \frac{1}{2}(7-\sqrt{73})\right)\left(\sqrt{\frac{1}{2}(\sqrt{73}-5)}, \frac{1}{2}(7-\sqrt{73})\right)$

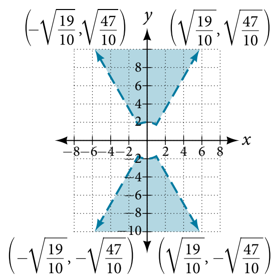
39.


41.


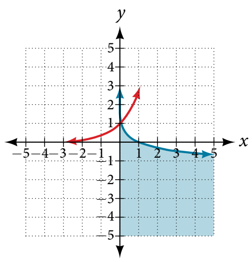
43.



45.



47.



49. $\left(-2\sqrt{\frac{70}{383}}, -2\sqrt{\frac{35}{29}}\right)$, $\left(-2\sqrt{\frac{70}{383}}, 2\sqrt{\frac{35}{29}}\right)$, $\left(2\sqrt{\frac{70}{383}}, -2\sqrt{\frac{35}{29}}\right)$, $\left(2\sqrt{\frac{70}{383}}, 2\sqrt{\frac{35}{29}}\right)$

51. No solution exists

53. $x = 0, y > 0$ and $0 < x < 1, \sqrt{x} < y < \frac{1}{x}$ **55.** 12,288

57. 2–20 computers

Section 11.4

1. No, a quotient of polynomials can only be decomposed if the denominator can be factored. For example, $\frac{1}{x^2+1}$ cannot be decomposed because the denominator cannot be factored.

3. Graph both sides and ensure they are equal.

5. If we choose $x = -1$, then the B-term disappears, letting us immediately know that $A = 3$. We could alternatively plug in $x = -\frac{5}{3}$ giving us a B-value of -2.

7. $\frac{8}{x+3} - \frac{5}{x-8}$ **9.** $\frac{1}{x+5} + \frac{9}{x+2}$

11. $\frac{3}{5x-2} + \frac{4}{4x-1}$ **13.** $\frac{5}{2(x+3)} + \frac{5}{2(x-3)}$

15 $\frac{3}{x+2} + \frac{3}{x-2}$ **17.** $\frac{9}{5(x+2)} + \frac{11}{5(x-3)}$

19. $\frac{8}{x-3} - \frac{5}{x-2}$ **21.** $\frac{1}{x-2} + \frac{2}{(x-2)^2}$

23. $-\frac{6}{4x+5} + \frac{3}{(4x+5)^2}$ **25.** $-\frac{1}{x-7} - \frac{2}{(x-7)^2}$

27. $\frac{4}{x} - \frac{3}{2(x+1)} + \frac{7}{2(x+1)^2}$

29. $\frac{4}{x} + \frac{2}{x^2} - \frac{3}{3x+2} + \frac{7}{2(3x+2)^2}$

31. $\frac{x+1}{x^2+x+3} + \frac{3}{x+2}$ **33.** $\frac{4-3x}{x^2+3x+8} + \frac{1}{x-1}$

35. $\frac{2x-1}{x^2+6x+1} + \frac{2}{x+3}$ **37.** $\frac{1}{x^2+x+1} + \frac{4}{x-1}$

39. $\frac{2}{x^2-3x+9} + \frac{3}{x+3}$ **41.** $-\frac{1}{4x^2+6x+9} + \frac{1}{2x-3}$

43. $\frac{1}{x} + \frac{1}{x+6} - \frac{4x}{x^2-6x+36}$ **45.** $\frac{x+6}{x^2+1} + \frac{4x+3}{(x^2+1)^2}$

47. $\frac{x+1}{x+2} + \frac{2x+3}{(x+2)^2}$ **49.** $\frac{1}{x^2+3x+25} - \frac{3x}{(x^2+3x+25)^2}$

51. $\frac{1}{8x} - \frac{x}{8(x^2+4)} + \frac{10-x}{2(x^2+4)^2}$

53. $-\frac{16}{x} - \frac{9}{x^2} + \frac{16}{x-1} - \frac{7}{(x-1)^2}$

55. $\frac{1}{x+1} - \frac{2}{(x+1)^2} + \frac{5}{(x+1)^3}$

57. $\frac{5}{x-2} - \frac{3}{10(x+2)} + \frac{7}{x+8} - \frac{7}{10(x-8)}$

59. $-\frac{5}{4x} - \frac{5}{2(x+2)} + \frac{11}{2(x+4)} + \frac{5}{4(x-4)}$

Section 11.5

1. No, they must have the same dimensions. An example would include two matrices of different dimensions. One cannot add the following two matrices because the first is a 2×2 matrix and the second is a 2×3 matrix. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$ has no sum.

3. Yes, if the dimensions of A are $m \times n$ and the dimensions of B are $n \times m$, both products will be defined **5.** Not necessarily. To find AB, we multiply the first row of A by the first column of B to get the first entry of AB. To find BA, we multiply the first row of B by the first column of A to get the first entry of BA. Thus, if those are unequal, then the matrix multiplication does not commute.

7. $\begin{bmatrix} 11 & 19 \\ 15 & 94 \\ 17 & 67 \end{bmatrix}$ **9.** $\begin{bmatrix} -4 & 2 \\ 8 & 1 \end{bmatrix}$

11. Undefined; dimensions do not match

13. $\begin{bmatrix} 9 & 27 \\ 63 & 36 \\ 0 & 192 \end{bmatrix}$ **15.** $\begin{bmatrix} -64 & -12 & -28 & -72 \\ -360 & -20 & -12 & -116 \end{bmatrix}$

17. $\begin{bmatrix} 1{,}800 & 1{,}200 & 1{,}300 \\ 800 & 1{,}400 & 600 \\ 700 & 400 & 2{,}100 \end{bmatrix}$ **19.** $\begin{bmatrix} 20 & 102 \\ 28 & 28 \end{bmatrix}$

21. $\begin{bmatrix} 60 & 41 & 2 \\ -16 & 120 & -216 \end{bmatrix}$ **23.** $\begin{bmatrix} -68 & 24 & 136 \\ -54 & -12 & 64 \\ -57 & 30 & 128 \end{bmatrix}$

25. Undefined; dimensions do not match.

27. $\begin{bmatrix} -8 & 41 & -3 \\ 40 & -15 & -14 \\ 4 & 27 & 42 \end{bmatrix}$ **29.** $\begin{bmatrix} -840 & 650 & -530 \\ 330 & 360 & 250 \\ -10 & 900 & 110 \end{bmatrix}$

31. $\begin{bmatrix} -350 & 1{,}050 \\ 350 & 350 \end{bmatrix}$

33. Undefined; inner dimensions do not match.

35. $\begin{bmatrix} 1{,}400 & 700 \\ -1{,}400 & 700 \end{bmatrix}$ **37.** $\begin{bmatrix} 332{,}500 & 927{,}500 \\ -227{,}500 & 87{,}500 \end{bmatrix}$

39. $\begin{bmatrix} 490{,}000 & 0 \\ 0 & 490{,}000 \end{bmatrix}$ **41.** $\begin{bmatrix} -2 & 3 & 4 \\ -7 & 9 & -7 \end{bmatrix}$

43. $\begin{bmatrix} -4 & 29 & 21 \\ -27 & -3 & 1 \end{bmatrix}$ **45.** $\begin{bmatrix} -3 & -2 & -2 \\ -28 & 59 & 46 \\ -4 & 16 & 7 \end{bmatrix}$

47. $\begin{bmatrix} 1 & -18 & -9 \\ -198 & 505 & 369 \\ -72 & 126 & 91 \end{bmatrix}$ **49.** $\begin{bmatrix} 0 & 1.6 \\ 9 & -1 \end{bmatrix}$

51. $\begin{bmatrix} 2 & 24 & -4.5 \\ 12 & 32 & -9 \\ -8 & 64 & 61 \end{bmatrix}$ **53.** $\begin{bmatrix} 0.5 & 3 & 0.5 \\ 2 & 1 & 2 \\ 10 & 7 & 10 \end{bmatrix}$

55. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **57.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

59. $B^n = \begin{cases} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, n \text{ even}, \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, n \text{ odd}. \end{cases}$

Section 11.6

1. Yes. For each row, the coefficients of the variables are written across the corresponding row, and a vertical bar is placed; then the constants are placed to the right of the vertical bar.
3. No, there are numerous correct methods of using row operations on a matrix. Two possible ways are the following: (1) Interchange rows 1 and 2. Then $R_2 = R_2 - 9R_1$. (2) $R_2 = R_1 - 9R_2$. Then divide row 1 by 9.
5. No. A matrix with 0 entries for an entire row would have either zero or infinitely many solutions.

7. $\left[\begin{array}{cc|c} 0 & 16 & 4 \\ 9 & -1 & 2 \end{array}\right]$ **9.** $\left[\begin{array}{ccc|c} 1 & 5 & 8 & 19 \\ 12 & 3 & 0 & 4 \\ 3 & 4 & 9 & -7 \end{array}\right]$

11. $\begin{aligned} -2x + 5y &= 5 \\ 6x - 18y &= 26 \end{aligned}$ **13.** $\begin{aligned} 3x + 2y &= 13 \\ -x - 9y + 4z &= 53 \\ 8x + 5y + 7z &= 80 \end{aligned}$

15. $\begin{aligned} 4x + 5y - 2z &= 12 \\ y + 58z &= 2 \\ 8x + 7y - 3z &= -5 \end{aligned}$ **17.** No solutions **19.** $(-1, -2)$ **21.** $(6, 7)$

23. $(3, 2)$ **25.** $\left(\frac{1}{5}, \frac{1}{2}\right)$ **27.** $\left(x, \frac{4}{15}(5x + 1)\right)$ **29.** $(3, 4)$

31. $\left(\frac{196}{39}, -\frac{5}{13}\right)$ **33.** $(31, -42, 87)$ **35.** $\left(\frac{21}{40}, \frac{1}{20}, \frac{9}{8}\right)$

37. $\left(\frac{18}{13}, \frac{15}{13}, -\frac{15}{13}\right)$ **39.** $\left(x, y, \frac{1}{2} - x - \frac{3}{2}y\right)$

41. $\left(x, -\frac{x}{2}, -1\right)$ **43.** $(125, -25, 0)$ **45.** $(8, 1, -2)$

47. $(1, 2, 3)$ **49.** $\left(-4z + \frac{17}{7}, 3z - \frac{10}{7}, z\right)$

51. No solutions exist. **53.** 860 red velvet, 1,340 chocolate
55. 4% for account 1, 6% for account 2 **57.** $126
59. Banana was 3%, pumpkin was 7%, and rocky road was 2%
61. 100 almonds, 200 cashews, 600 pistachios

Section 11.7

1. If A^{-1} is the inverse of A, then $AA^{-1} = I$, the identity matrix. Since A is also the inverse of A^{-1}, $A^{-1}\,A = I$. You can also check by proving this for a 2×2 matrix. **3.** No, because ad and bc are both 0, so $ad - bc = 0$, which requires us to divide by 0 in the formula. **5.** Yes. Consider the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. The inverse is found with the following calculation:

$$A^{-1} = \frac{1}{0(0) - 1(1)}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

7. $AB = BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathrm{I}$ **9.** $AB = BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathrm{I}$

11. $AB = BA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathrm{I}$ **13.** $\frac{1}{29}\begin{bmatrix} 9 & 2 \\ -1 & 3 \end{bmatrix}$

15. $\frac{1}{69}\begin{bmatrix} -2 & 7 \\ 9 & 3 \end{bmatrix}$ **17.** There is no inverse **19.** $\frac{4}{7}\begin{bmatrix} 0.5 & 1.5 \\ 1 & -0.5 \end{bmatrix}$

21. $\frac{1}{17}\begin{bmatrix} -5 & 5 & -3 \\ 20 & -3 & 12 \\ 1 & -1 & 4 \end{bmatrix}$ **23.** $\frac{1}{209}\begin{bmatrix} 47 & -57 & 69 \\ 10 & 19 & -12 \\ -24 & 38 & -13 \end{bmatrix}$

25. $\begin{bmatrix} 18 & 60 & -168 \\ -56 & -140 & 448 \\ 40 & 80 & -280 \end{bmatrix}$ **27.** $(-5, 6)$ **29.** $(2, 0)$

31. $\left(\frac{1}{3}, -\frac{5}{2}\right)$ **33.** $\left(-\frac{2}{3}, -\frac{11}{6}\right)$ **35.** $\left(7, \frac{1}{2}, \frac{1}{5}\right)$

37. $(5, 0, -1)$ **39.** $\left(-\frac{35}{34}, -\frac{97}{34}, -\frac{77}{17}\right)$

41. $\left(\frac{13}{138}, -\frac{568}{345}, -\frac{229}{690}\right)$ **43.** $\left(-\frac{37}{30}, \frac{8}{15}\right)$

45. $\left(\frac{10}{123}, -1, \frac{2}{5}\right)$ **47.** $\frac{1}{2}\begin{bmatrix} 2 & 1 & -1 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & -1 & 1 & 1 \\ 0 & 1 & -1 & 1 \end{bmatrix}$

49. $\frac{1}{39}\begin{bmatrix} 3 & 2 & 1 & -7 \\ 18 & -53 & 32 & 10 \\ 24 & -36 & 21 & 9 \\ -9 & 46 & -16 & -5 \end{bmatrix}$

51. $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & -1 & -1 & 1 \end{bmatrix}$ **53.** Infinite solutions

55. 50% oranges, 25% bananas, 20% apples
57. 10 straw hats, 50 beanies, 40 cowboy hats
59. Tom ate 6, Joe ate 3, and Albert ate 3
61. 124 oranges, 10 lemons, 8 pomegranates

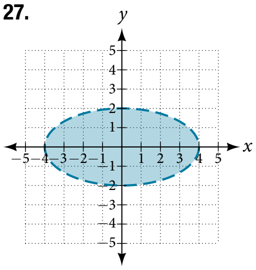
Section 11.8

1. A determinant is the sum and products of the entries in the matrix, so you can always evaluate that product—even if it does end up being 0. **3.** The inverse does not exist. **5.** -2
7. 7 **9.** -4 **11.** 0 **13.** $-7, 990.7$ **15.** 3 **17.** -1
19. 224 **21.** 15 **23.** -17.03 **25.** $(1, 1)$ **27.** $\left(\frac{1}{2}, \frac{1}{3}\right)$
29. $(2, 5)$ **31.** $\left(-1, -\frac{1}{3}\right)$ **33.** $(15, 12)$ **35.** $(1, 3, 2)$
37. $(-1, 0, 3)$ **39.** $\left(\frac{1}{2}, 1, 2\right)$ **41.** $(2, 1, 4)$
43. Infinite solutions **45.** 24 **47.** 1 **49.** Yes; 18, 38
51. Yes; 33, 36, 37 **53.** $7,000 in first account, $3,000 in second account **55.** 120 children, 1,080 adult **57.** 4 gal yellow, 6 gal blue **59.** 13 green tomatoes, 17 red tomatoes
61. Strawberries 18%, oranges 9%, kiwi 10% **63.** 100 for the first movie, 230 for the second movie, 312 for the third movie
65. 20–29: 2,100, 30–39: 2,600, 40–49: 825 **67.** 300 almonds, 400 cranberries, 300 cashews

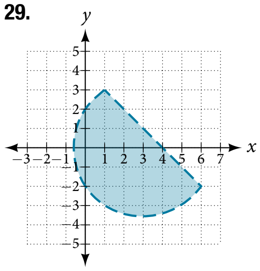
Chapter 11 Review Exercises

1. No **3.** $(-2, 3)$ **5.** $(4, -1)$ **7.** No solutions exist
9. $(300, 60)$ **11.** $(10, -10, 10)$ **13.** No solutions exist
15. $(-1, -2, 3)$ **17.** $\left(x, \frac{8x}{5}, \frac{14x}{5}\right)$ **19.** 11, 17, 33
21. $(2, -3), (3, 2)$ **23.** No solution **25.** No solution

27.



29.



31. $-\dfrac{10}{x+2}+\dfrac{8}{x+1}$ **33.** $\dfrac{7}{x+5}-\dfrac{15}{(x+5)^2}$

35. $\dfrac{3}{x-5}+\dfrac{-4x+1}{x^2+5x+25}$ **37.** $\dfrac{x-4}{x^2-2}+\dfrac{5x+3}{(x^2-2)^2}$

39. $\begin{bmatrix}-16 & 8\\ -4 & -12\end{bmatrix}$ **41.** Undefined; dimensions do not match

43. Undefined; inner dimensions do not match

45. $\begin{bmatrix}113 & 28 & 10\\ 44 & 81 & -41\\ 84 & 98 & -42\end{bmatrix}$ **47.** $\begin{bmatrix}-127 & -74 & 176\\ -2 & 11 & 40\\ 28 & 77 & 38\end{bmatrix}$

49. Undefined; inner dimensions do not match

51. $x-3z=7$

$y+2z=-5$ with infinite solutions

53. $\left[\begin{array}{rrr|r}-2 & 2 & 1 & 7\\ 2 & -8 & 5 & 0\\ 19 & -10 & 22 & 3\end{array}\right]$ **55.** $\left[\begin{array}{rrr|r}1 & 0 & 3 & 12\\ -1 & 4 & 0 & 0\\ 0 & 1 & 2 & -7\end{array}\right]$

57. No solutions exist **59.** No solutions exist

61. $\dfrac{1}{8}\begin{bmatrix}2 & 7\\ 6 & 1\end{bmatrix}$ **63.** No inverse exists **65.** $(-20, 40)$

67. $(-1, 0.2, 0.3)$ **69.** 17% oranges, 34% bananas, 39% apples

71. 0 **73.** 6 **75.** $\left(6, \dfrac{1}{2}\right)$ **77.** $(x, 5x+3)$ **79.** $\left(0, 0, -\dfrac{1}{2}\right)$

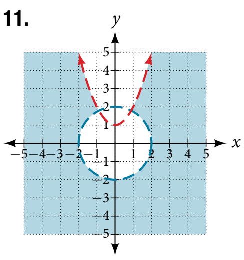
Chapter 11 Practice Test

1. Yes **3.** No solutions exist **5.** $\left(\dfrac{1}{2}, \dfrac{1}{4}, \dfrac{1}{5}\right)$

7. $\left(x, \dfrac{16x}{5}, -\dfrac{13x}{5}\right)$

9. $(-2\sqrt{2}, -\sqrt{17}), (-2\sqrt{2}, \sqrt{17}), (2\sqrt{2}, -\sqrt{17}), (2\sqrt{2}, \sqrt{17})$

11.



13. $\dfrac{5}{3x+1}-\dfrac{2x+3}{(3x+1)^2}$

15. $\begin{bmatrix}17 & 51\\ -8 & 11\end{bmatrix}$

17. $\begin{bmatrix}12 & -20\\ -15 & 30\end{bmatrix}$

19. $-\dfrac{1}{8}$

21. $\left[\begin{array}{rrr|r}14 & -2 & 13 & 140\\ -2 & 3 & -6 & -1\\ 1 & -5 & 12 & 11\end{array}\right]$ **23.** No solutions exist.

25. $(100, 90)$ **27.** $\left(\dfrac{1}{100}, 0\right)$ **29.** 32 or more cell phones per day

CHAPTER 12

Section 12.1

1. An ellipse is the set of all points in the plane the sum of whose distances from two fixed points, called the foci, is a constant.

3. This special case would be a circle. **5.** It is symmetric about the x-axis, y-axis, and the origin.

7. Yes; $\dfrac{x^2}{3^2}+\dfrac{y^2}{2^2}=1$ **9.** Yes; $\dfrac{x^2}{\left(\frac{1}{2}\right)^2}+\dfrac{y^2}{\left(\frac{1}{3}\right)^2}=1$

11. $\dfrac{x^2}{2^2}+\dfrac{y^2}{7^2}=1$; endpoints of major axis: $(0, 7)$ and $(0, -7)$; endpoints of minor axis: $(2, 0)$ and $(-2, 0)$; foci: $(0, 3\sqrt{5}), (0, -3\sqrt{5})$

13. $\dfrac{x^2}{(1)^2}+\dfrac{y^2}{\left(\frac{1}{3}\right)^2}=1$; endpoints of major axis: $(1, 0)$ and $(-1, 0)$; endpoints of minor axis: $\left(0, \dfrac{1}{3}\right), \left(0, -\dfrac{1}{3}\right)$; foci: $\left(\dfrac{2\sqrt{2}}{3}, 0\right), \left(-\dfrac{2\sqrt{2}}{3}, 0\right)$ **15.** $\dfrac{(x-2)^2}{7^2}+\dfrac{(y-4)^2}{5^2}=1$; endpoints of major axis: $(9, 4), (-5, 4)$; endpoints of minor axis: $(2, 9), (2, -1)$; foci: $(2+2\sqrt{6}, 4), (2-2\sqrt{6}, 4)$ **17.** $\dfrac{(x+5)^2}{2^2}+\dfrac{(y-7)^2}{3^2}=1$; endpoints of major axis: $(-5, 10), (-5, 4)$; endpoints of minor axis: $(-3, 7), (-7, 7)$; foci: $(-5, 7+\sqrt{5}), (-5, 7-\sqrt{5})$

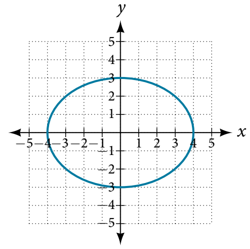
19. $\dfrac{(x-1)^2}{3^2}+\dfrac{(y-4)^2}{2^2}=1$; endpoints of major axis: $(4, 4), (-2, 4)$; endpoints of minor axis: $(1, 6), (1, 2)$; foci: $(1+\sqrt{5}, 4), (1-\sqrt{5}, 4)$ **21.** $\dfrac{(x-3)^2}{(3\sqrt{2})^2}+\dfrac{(y-5)^2}{(\sqrt{2})^2}=1$; endpoints of major axis: $(3+3\sqrt{2}, 5), (3-3\sqrt{2}, 5)$; endpoints of minor axis: $(3, 5+\sqrt{2}), (3, 5-\sqrt{2})$; foci: $(7, 5), (-1, 5)$

23. $\dfrac{(x+5)^2}{5^2}+\dfrac{(y-2)^2}{2^2}=1$; endpoints of major axis: $(0, 2), (-10, 2)$; endpoints of minor axis: $(-5, 4), (-5, 0)$; foci: $(-5+\sqrt{21}, 2), (-5-\sqrt{21}, 2)$ **25.** $\dfrac{(x+3)^2}{5^2}+\dfrac{(y+4)^2}{2^2}=1$; endpoints of major axis $(2, -4), (-8, -4)$; endpoints of minor axis $(-3, -2), (-3, -6)$; foci: $(-3+\sqrt{21}, -4), (-3-\sqrt{21}, -4)$.

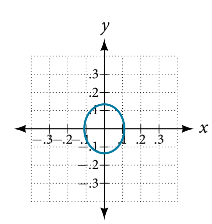
27. Foci: $(-3, -1+\sqrt{11}), (-3, -1-\sqrt{11})$ **29.** Focus: $(0, 0)$

31. Foci: $(-10, 30), (-10, -30)$

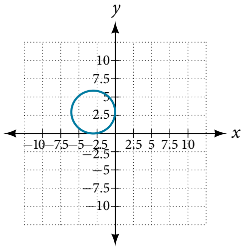
33. Center: $(0, 0)$; vertices: $(4, 0), (-4, 0), (0, 3), (0, -3)$; foci: $(\sqrt{7}, 0), (-\sqrt{7}, 0)$



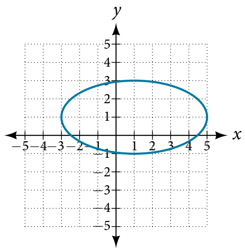
35. Center $(0, 0)$; vertices: $\left(\dfrac{1}{9}, 0\right), \left(-\dfrac{1}{9}, 0\right), \left(0, \dfrac{1}{7}\right), \left(0, -\dfrac{1}{7}\right)$; foci $\left(0, \dfrac{4\sqrt{2}}{63}\right), \left(0, -\dfrac{4\sqrt{2}}{63}\right)$



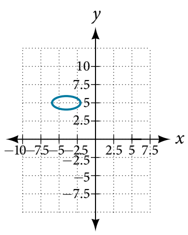
37. Center $(-3, 3)$; vertices $(0, 3), (-6, 3), (-3, 0), (-3, 6)$; focus: $(-3, 3)$. Note that this ellipse is a circle. The circle has only one focus, which coincides with the center.



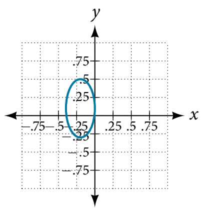
39. Center: $(1, 1)$; vertices: $(5, 1), (-3, 1), (1, 3), (1, -1)$; foci: $\left(1, 1+4\sqrt{3}\right), \left(1, 1-4\sqrt{3}\right)$



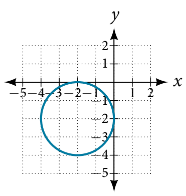
41. Center: $(-4, 5)$; vertices: $(-2, 5), (-6, 5), (-4, 6), (-4, 4)$; foci: $\left(-4+\sqrt{3}, 5\right), \left(-4-\sqrt{3}, 5\right)$



43. Center: $(-2, 1)$; vertices: $(0, 1), (-4, 1), (-2, 5), (-2, -3)$; foci: $(-2, 1+2\sqrt{3}), (-2, 1-2\sqrt{3})$



45. Center: $(-2, -2)$; vertices: $(0, -2), (-4, -2), (-2, 0), (-2, -4)$; focus: $(-2, -2)$



47. $\dfrac{x^2}{25} + \dfrac{y^2}{29} = 1$

49. $\dfrac{(x-4)^2}{25} + \dfrac{(y-2)^2}{1} = 1$

51. $\dfrac{(x+3)^2}{16} + \dfrac{(y-4)^2}{4} = 1$

53. $\dfrac{x^2}{81} + \dfrac{y^2}{9} = 1$

55. $\dfrac{(x+2)^2}{4} + \dfrac{(y-2)^2}{9} = 1$

57. Area $= 12\pi$ square units

59. Area $= 2\sqrt{5}\pi$ square units

61. Area $= 9\pi$ square units

63. $\dfrac{x^2}{4h^2} + \dfrac{y^2}{\frac{1}{4}h^2} = 1$

65. $\dfrac{x^2}{400} + \dfrac{y^2}{144} = 1$, distance: 17.32 feet

67. Approximately 51.96 feet

Section 12.2

1. A hyperbola is the set of points in a plane the difference of whose distances from two fixed points (foci) is a positive constant. **3.** The foci must lie on the transverse axis and be in the interior of the hyperbola. **5.** The center must be the midpoint of the line segment joining the foci. **7.** Yes $\dfrac{x^2}{6^2} - \dfrac{y^2}{3^2} = 1$

9. Yes $\dfrac{x^2}{4^2} - \dfrac{y^2}{5^2} = 1$ **11.** $\dfrac{x^2}{5^2} - \dfrac{y^2}{6^2} = 1$; vertices: $(5, 0), (-5, 0)$; foci: $\left(\sqrt{61}, 0\right), \left(-\sqrt{61}, 0\right)$; asymptotes: $y = \dfrac{6}{5}x, y = -\dfrac{6}{5}x$

13. $\dfrac{y^2}{2^2} - \dfrac{x^2}{9^2} = 1$; vertices: $(0, 2), (0, -2)$; foci: $\left(0, \sqrt{85}\right), \left(0, -\sqrt{85}\right)$; asymptotes: $y = \dfrac{2}{9}x, y = -\dfrac{2}{9}x$

15. $\dfrac{(x-1)^2}{3^2} - \dfrac{(y-2)^2}{4^2} = 1$; vertices: $(4, 2), (-2, 2)$; foci: $(6, 2), (-4, 2)$; asymptotes: $y = \dfrac{4}{3}(x-1)+2, y = -\dfrac{4}{3}(x-1)+2$

17. $\dfrac{(x-2)^2}{7^2} - \dfrac{(y+7)^2}{7^2} = 1$; vertices: $(9, -7), (-5, -7)$; foci: $\left(2+7\sqrt{2}, -7\right), \left(2-7\sqrt{2}, -7\right)$; asymptotes: $y = x-9, y = -x-5$

19. $\dfrac{(x+3)^2}{3^2} - \dfrac{(y-3)^2}{3^2} = 1$; vertices: $(0, 3), (-6, 3)$; foci: $\left(-3+3\sqrt{2}, 3\right), \left(-3-3\sqrt{2}, 3\right)$; asymptotes: $y = x+6, y = -x$

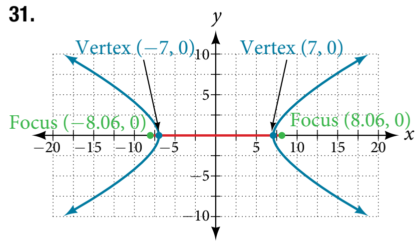
21. $\dfrac{(y-4)^2}{2^2} - \dfrac{(x-3)^2}{4^2} = 1$; vertices: $(3, 6), (3, 2)$; foci: $\left(3, 4+2\sqrt{5}\right), \left(3, 4-2\sqrt{5}\right)$; asymptotes: $y = \dfrac{1}{2}(x-3)+4, y = -\dfrac{1}{2}(x-3)+4$

23. $\dfrac{(y+5)^2}{7^2} - \dfrac{(x+1)^2}{70^2} = 1$; vertices: $(-1, 2), (-1, -12)$; foci: $\left(-1, -5+7\sqrt{101}\right), \left(-1, -5-7\sqrt{101}\right)$; asymptotes: $y = \dfrac{1}{10}(x+1)-5, y = -\dfrac{1}{10}(x+1)-5$

25. $\dfrac{(x+3)^2}{5^2} - \dfrac{(y-4)^2}{2^2} = 1$; vertices: $(2, 4), (-8, 4)$; foci: $\left(-3+\sqrt{29}, 4\right), \left(-3-\sqrt{29}, 4\right)$; asymptotes: $y = \dfrac{2}{5}(x+3)+4, y = -\dfrac{2}{5}(x+3)+4$ **27.** $y = \dfrac{2}{5}(x-3)-4, y = -\dfrac{2}{5}(x-3)-4$

29. $y = \dfrac{3}{4}(x-1)+1, y = -\dfrac{3}{4}(x-1)+1$

31.



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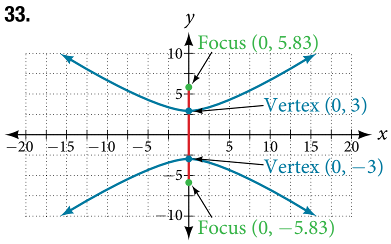

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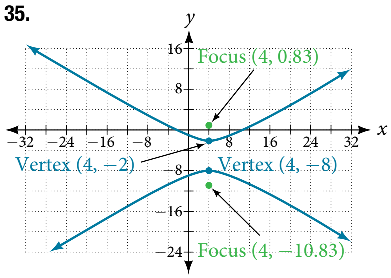

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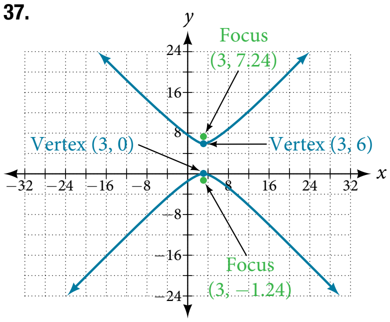

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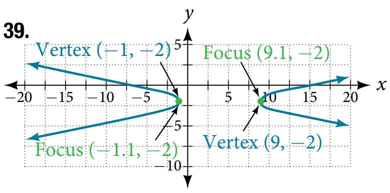

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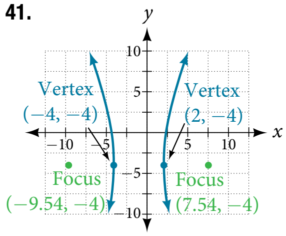

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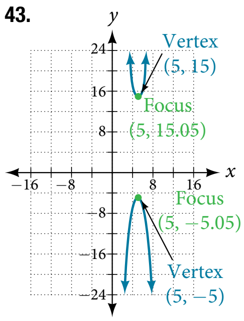


45. $\frac{x^2}{9} - \frac{y^2}{16} = 1$

47. $\frac{(x-6)^2}{25} - \frac{(y-1)^2}{11} = 1$

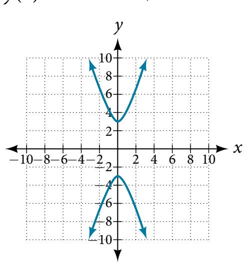
49. $\frac{(x-4)^2}{25} - \frac{(y-2)^2}{1} = 1$

51. $\frac{y^2}{16} - \frac{x^2}{25} = 1$

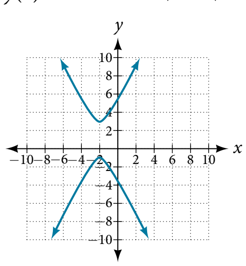
53. $\frac{y^2}{9} - \frac{(x+1)^2}{9} = 1$

55. $\frac{(x+3)^2}{25} - \frac{(y+3)^2}{25} = 1$

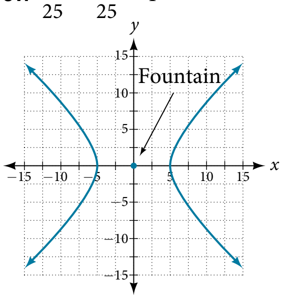
57. $y(x) = 3\sqrt{x^2+1}$, $y(x) = -3\sqrt{x^2+1}$



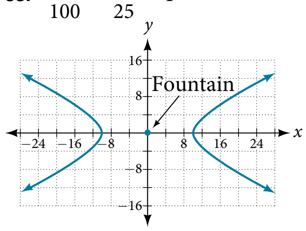
59. $y(x) = 1 + 2\sqrt{x^2+4x+5}$, $y(x) = 1 - 2\sqrt{x^2+4x+5}$



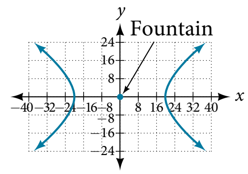
61. $\frac{x^2}{25} - \frac{y^2}{25} = 1$



63. $\frac{x^2}{100} - \frac{y^2}{25} = 1$



65. $\frac{x^2}{400} - \frac{y^2}{225} = 1$



67. $\frac{(x-1)^2}{0.25} - \frac{y^2}{0.75} = 1$

69. $\frac{(x-3)^2}{4} - \frac{y^2}{5} = 1$

Section 12.3

1. A parabola is the set of points in the plane that lie equidistant from a fixed point, the focus, and a fixed line, the directrix.

3. The graph will open down.

5. The distance between the focus and directrix will increase.

7. Yes $y = 4(1)x^2$ **9.** Yes $(y-3)^2 = 4(2)(x-2)$

11. $y^2 = \frac{1}{8}x$, $V{:}\ (0, 0)$; $F{:}\ \left(\frac{1}{32}, 0\right)$; $d{:}\ x = -\frac{1}{32}$

13. $x^2 = -\frac{1}{4}y$, $V{:}\ (0, 0)$; $F{:}\ \left(0, -\frac{1}{16}\right)$; $d{:}\ y = \frac{1}{16}$

15. $y^2 = \frac{1}{36}x$, $V{:}\ (0, 0)$; $F{:}\left(\frac{1}{144}, 0\right)$; $d{:}\ x = -\frac{1}{144}$

17. $(x-1)^2 = 4(y-1)$, $V{:}\ (1, 1)$; $F{:}\ (1, 2)$; $d{:}\ y = 0$

19. $(y-4)^2 = 2(x+3)$, $V{:}\ (-3, 4)$; $F{:}\left(-\frac{5}{2}, 4\right)$; $d{:}\ x = -\frac{7}{2}$

21. $(x+4)^2 = 24(y+1)$, $V{:}\ (-4, -1)$; $F{:}\ (-4, 5)$; $d{:}\ y = -7$

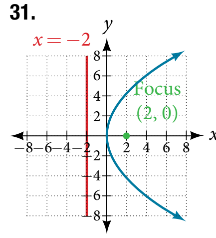
23. $(y-3)^2 = -12(x+1)$, $V{:}\ (-1, 3)$; $F{:}\ (-4, 3)$; $d{:}\ x = 2$

25. $(x-5)^2 = \frac{4}{5}(y+3)$, $V{:}\ (5, -3)$; $F{:}\ \left(5, -\frac{14}{5}\right)$; $d{:}\ y = -\frac{16}{5}$

27. $(x-2)^2 = -2(y-5)$, $V{:}\ (2, 5)$; $F{:}\ \left(2, \frac{9}{2}\right)$; $d{:}\ y = \frac{11}{2}$

29. $(y-1)^2 = \frac{4}{3}(x-5)$, $V{:}\ (5, 1)$; $F{:}\ \left(\frac{16}{3}, 1\right)$; $d{:}\ x = \frac{14}{3}$

31.



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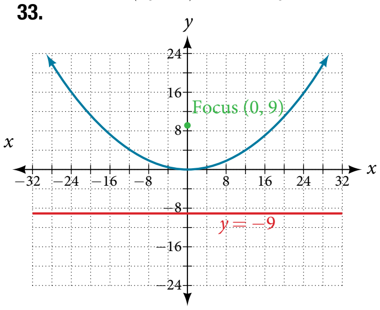

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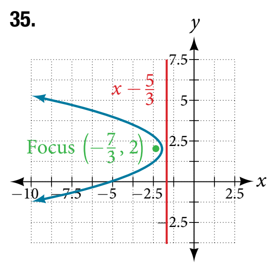

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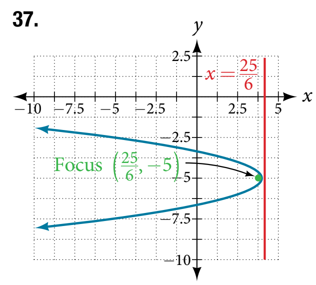

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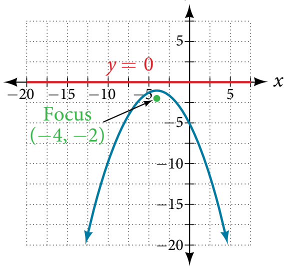
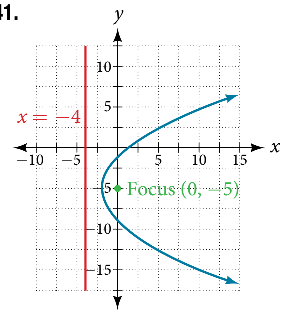


41.



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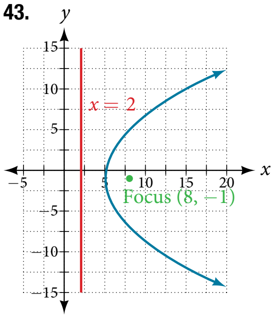


45. $x^2 = -16y$

47. $(y-2)^2 = 4\sqrt{2}(x-2)$

49. $(y+\sqrt{3})^2 = -4\sqrt{2}(x-\sqrt{2})$

51. $x^2 = y$

53. $(y-2)^2 = \frac{1}{4}(x+2)$

55. $(y-\sqrt{3})^2 = 4\sqrt{5}(x+\sqrt{2})$

57. $y^2 = -8x$

59. $(y+1)^2 = 12(x+3)$

61. $(0, 1)$

63. At the point 2.25 feet above the vertex **65.** 0.5625 feet

67. $x^2 = -125(y-20)$, height is 7.2 feet **69.** 0.2304 feet

Section 12.4

1. The xy term causes a rotation of the graph to occur.

3. The conic section is a hyperbola. **5.** It gives the angle of rotation of the axes in order to eliminate the xy term.

7. $AB = 0$, parabola **9.** $AB = -4 < 0$, hyperbola

11. $AB = 6 > 0$, ellipse **13.** $B^2 - 4AC = 0$, parabola

15. $B^2 - 4AC = 0$, parabola **17.** $B^2 - 4AC = -96 < 0$, ellipse

19. $7x'^2 + 9y'^2 - 4 = 0$ **21.** $3x'^2 + 2x'y' - 5y'^2 + 1 = 0$

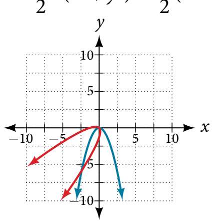
23. $\theta = 60°$, $11x'^2 - y'^2 + \sqrt{3}x' + y' - 4 = 0$

25. $\theta = 150°$, $21x'^2 + 9y'^2 + 4x' - 4\sqrt{3}y' - 6 = 0$

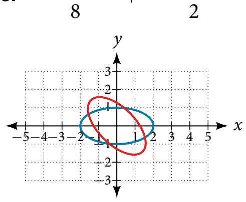
27. $\theta \approx 36.9°$, $125x'^2 + 6x' - 42y' + 10 = 0$

29. $\theta = 45°$, $3x'^2 - y'^2 - \sqrt{2}x' + \sqrt{2}y' + 1 = 0$

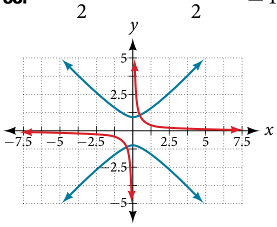
31. $\frac{\sqrt{2}}{2}(x'+y') = \frac{1}{2}(x'-y')^2$



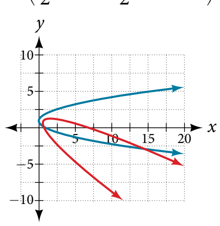
33. $\frac{(x'-y')^2}{8} + \frac{(x'+y')^2}{2} = 1$



35. $\frac{(x'+y')^2}{2} - \frac{(x'-y')^2}{2} = 1$



37. $\frac{\sqrt{3}}{2}x' - \frac{1}{2}y' = \left(\frac{1}{2}x' + \frac{\sqrt{3}}{2}y' - 1\right)^2$



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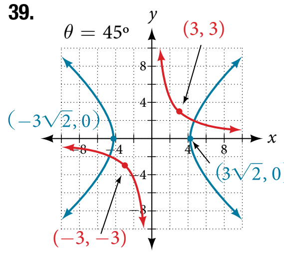

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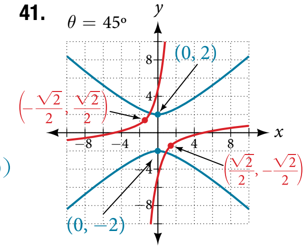

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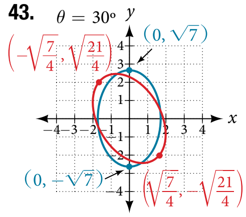

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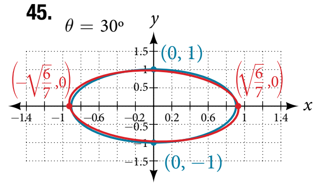

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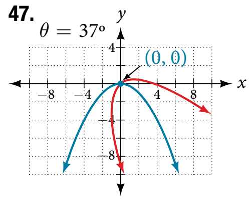

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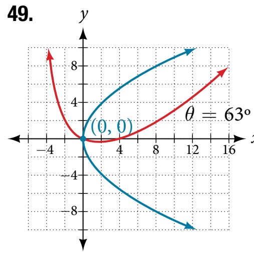
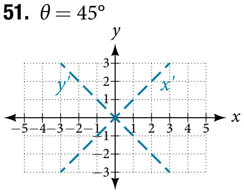
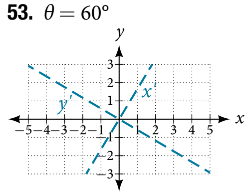


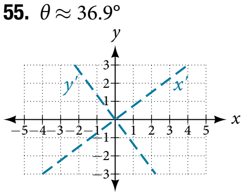
51. $\theta = 45°$



53. $\theta = 60°$



55. $\theta \approx 36.9°$



57. $-4\sqrt{6} < k < 4\sqrt{6}$

59. $k = 2$

Section 12.5

1. If eccentricity is less than 1, it is an ellipse. If eccentricity is equal to 1, it is a parabola. If eccentricity is greater than 1, it is a hyperbola. **3.** The directrix will be parallel to the polar axis. **5.** One of the foci will be located at the origin. **7.** Parabola with $e = 1$ and directrix $\frac{3}{4}$ units below the pole. **9.** Hyperbola with $e = 2$ and directrix $\frac{5}{2}$ units above the pole. **11.** Parabola with $e = 1$ and directrix $\frac{3}{10}$ units to the right of the pole.

13. Ellipse with $e = \frac{2}{7}$ and directrix 2 units to the right of the pole.

15. Hyperbola with $e = \frac{5}{3}$ and directrix $\frac{11}{5}$ units above the pole.

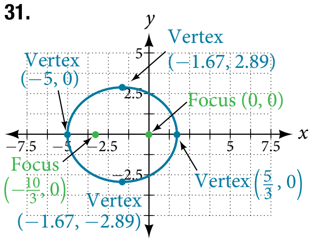
17. Hyperbola with $e = \frac{8}{7}$ and directrix $\frac{7}{8}$ units to the right of the pole. **19.** $25x^2 + 16y^2 - 12y - 4 = 0$

21. $21x^2 - 4y^2 - 30x + 9 = 0$ **23.** $64y^2 = 48x + 9$

25. $25x^2 - 96y^2 - 110y - 25 = 0$ **27.** $3x^2 + 4y^2 - 2x - 1 = 0$

29. $5x^2 + 9y^2 - 24x - 36 = 0$

31.



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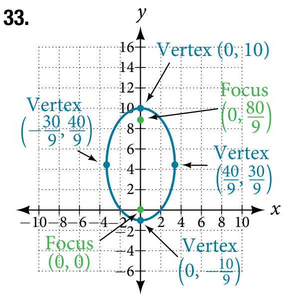

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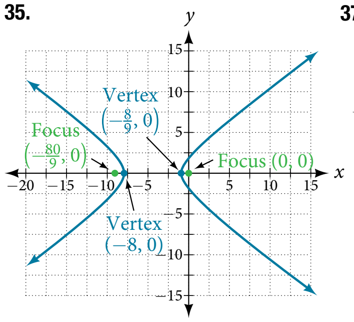

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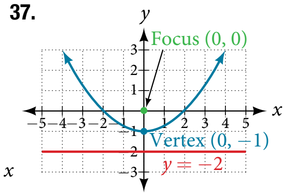

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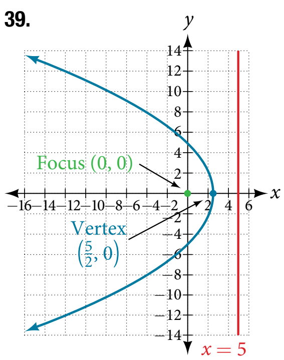

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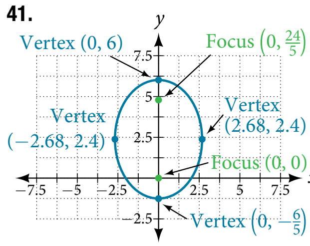


43. $r = \dfrac{4}{5 + \cos\theta}$

45. $r = \dfrac{4}{1 + 2\sin\theta}$ **47.** $r = \dfrac{1}{1 + \cos\theta}$ **49.** $r = \dfrac{7}{8 - 28\cos\theta}$

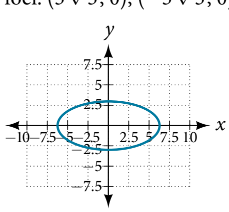
51. $r = \dfrac{12}{2 + 3\sin\theta}$ **53.** $r = \dfrac{15}{4 - 3\cos\theta}$ **55.** $r = \dfrac{3}{3 - 3\cos\theta}$

57. $r = \pm\dfrac{2}{\sqrt{1 + \sin\theta\cos\theta}}$ **59.** $r = \pm\dfrac{2}{4\cos\theta + 3\sin\theta}$

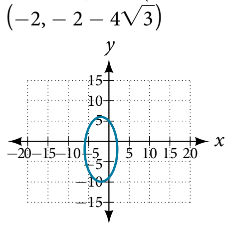
Chapter 12 Review Exercises

1. $\dfrac{x^2}{5^2} + \dfrac{y^2}{8^2} = 1$; center: $(0, 0)$; vertices: $(5, 0)$, $(-5, 0)$, $(0, 8)$, $(0, -8)$; foci: $(0, \sqrt{39})$, $(0, -\sqrt{39})$ **3.** $\dfrac{(x+3)^2}{1^2} + \dfrac{(y-2)^2}{3^2} = 1$ $(-3, 2)$; $(-2, 2)$, $(-4, 2)$, $(-3, 5)$, $(-3, -1)$; $(-3, 2 + 2\sqrt{2})$, $(-3, 2 - 2\sqrt{2})$

5. Center: $(0, 0)$; vertices: $(6, 0)$, $(-6, 0)$, $(0, 3)$, $(0, -3)$; foci: $(3\sqrt{3}, 0)$, $(-3\sqrt{3}, 0)$



7. Center: $(-2, -2)$; vertices: $(2, -2)$, $(-6, -2)$, $(-2, 6)$, $(-2, -10)$; foci: $(-2, -2 + 4\sqrt{3})$, $(-2, -2 - 4\sqrt{3})$

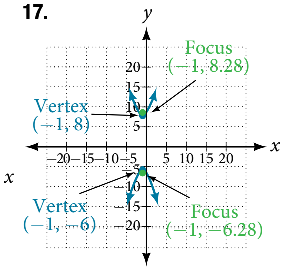


9. $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$ **11.** Approximately 35.71 feet

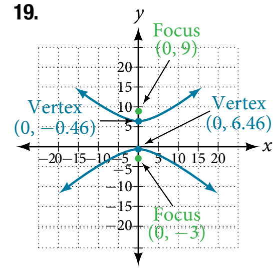
13. $\dfrac{(y+1)^2}{4^2} - \dfrac{(x-4)^2}{6^2} = 1$; center: $(4, -1)$; vertices: $(4, 3)$, $(4, -5)$; foci: $(4, -1 + 2\sqrt{13})$, $(4, -1 - 2\sqrt{13})$

15. $\dfrac{(x-2)^2}{2^2} - \dfrac{(y+3)^2}{(2\sqrt{3})^2} = 1$; center: $(2, -3)$; vertices: $(4, -3)$, $(0, -3)$; foci: $(6, -3)$, $(-2, -3)$

17.



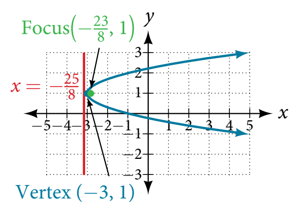
19.



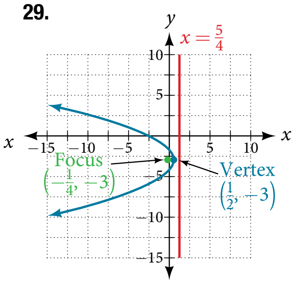
21. $\dfrac{(x-5)^2}{1} - \dfrac{(y-7)^2}{3} = 1$ **23.** $(x+2)^2 = \dfrac{1}{2}(y - 1)$; vertex: $(-2, 1)$; focus: $\left(-2, \dfrac{9}{8}\right)$; directrix: $y = \dfrac{7}{8}$

25. $(x+5)^2 = (y+2)$; vertex: $(-5, -2)$; focus: $\left(-5, -\dfrac{7}{4}\right)$; directrix: $y = -\dfrac{9}{4}$

27.



29.

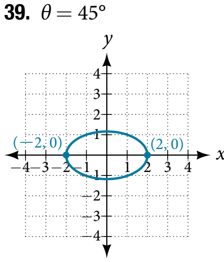


31. $(x-2)^2 = \left(\dfrac{1}{2}\right)(y - 1)$

33. $B^2 - 4AC = 0$, parabola **35.** $B^2 - 4AC = -31 < 0$, ellipse

37. $\theta = 45°$, $x'^2 + 3y'^2 - 12 = 0$

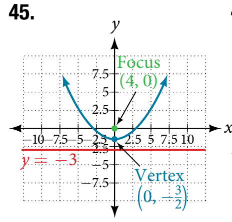
39. $\theta = 45°$



41. Hyperbola with $e = 5$ and directrix 2 units to the left of the pole.

43. Ellipse with $e = \dfrac{3}{4}$ and directrix $\dfrac{1}{3}$ unit above the pole.

45.



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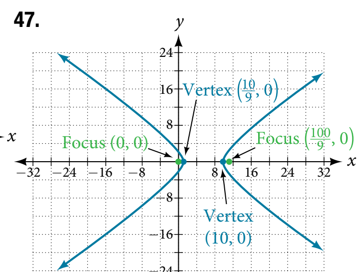
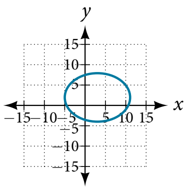


49. $r = \dfrac{3}{1 + \cos\theta}$

Chapter 12 Practice Test

1. $\frac{x^2}{3^2}+\frac{y^2}{2^2}=1$; center: $(0, 0)$; vertices: $(3, 0)$, $(-3, 0)$, $(0, 2)$, $(0, -2)$; foci: $(\sqrt{5}, 0)$, $(-\sqrt{5}, 0)$

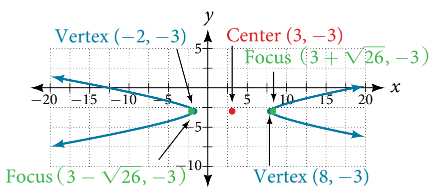
3. Center: $(3, 2)$; vertices: $(11, 2)$, $(-5, 2)$, $(3, 8)$, $(3, -4)$; foci: $(3+2\sqrt{7}, 2)$, $(3-2\sqrt{7}, 2)$



5. $\frac{(x-1)^2}{36}+\frac{(y-2)^2}{27}=1$

7. $\frac{x^2}{7^2}-\frac{y^2}{9^2}=1$; center: $(0, 0)$; vertices $(7, 0)$, $(-7, 0)$; foci: $(\sqrt{130}, 0)$, $(-\sqrt{130}, 0)$; asymptotes: $y=\pm\frac{9}{7}x$

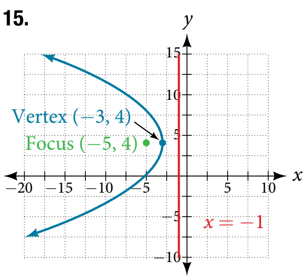
9. Center: $(3, -3)$; vertices: $(8, -3)$, $(-2, -3)$; foci: $(3+\sqrt{26}, -3)$, $(3-\sqrt{26}, -3)$; asymptotes: $y=\pm\frac{1}{5}(x-3)-3$



11. $\frac{(y-3)^2}{1}-\frac{(x-1)^2}{8}=1$

13. $(x-2)^2=\frac{1}{3}(y+1)$; vertex: $(2, -1)$; focus: $\left(2, -\frac{11}{12}\right)$; directrix: $y=-\frac{13}{12}$

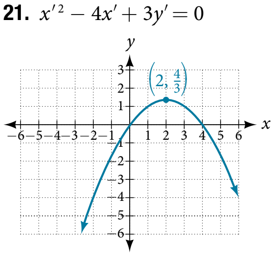
15.



17. Approximately 8.48 feet

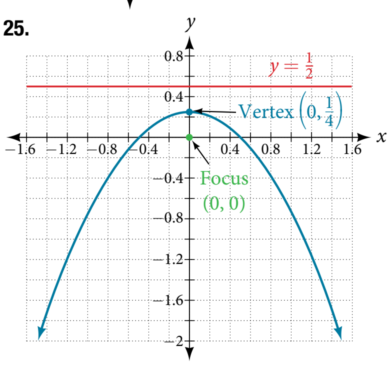
19. Parabola; $\theta\approx 63.4°$

21. $x'^2-4x'+3y'=0$



23. Hyperbola with $e=\frac{3}{2}$, and directrix $\frac{5}{6}$ units to the right of the pole.

25.



CHAPTER 13

Section 13.1

1. A sequence is an ordered list of numbers that can be either finite or infinite in number. When a finite sequence is defined by a formula, its domain is a subset of the non-negative integers. When an infinite sequence is defined by a formula, its domain is all positive or all non-negative integers.

3. Yes, both sets go on indefinitely, so they are both infinite sequences.

5. A factorial is the product of a positive integer and all the positive integers below it. An exclamation point is used to indicate the operation. Answers may vary. An example of the benefit of using factorial notation is when indicating the product It is much easier to write than it is to write out $13\cdot 12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1$.

7. First four terms: $-8, -\frac{16}{3}, -4, -\frac{16}{5}$

9. First four terms: $2, \frac{1}{2}, \frac{8}{27}, \frac{1}{4}$

11. First four terms: $1.25, -5, 20, -80$

13. First four terms: $\frac{1}{3}, \frac{4}{5}, \frac{9}{7}, \frac{16}{9}$

15. First four terms: $-\frac{4}{5}, 4, -20, 100$

17. $\frac{1}{3}, \frac{4}{5}, \frac{9}{7}, \frac{16}{9}, \frac{25}{11}, 31, 44, 59$

19. $-0.6, -3, -15, -20, -375, -80, -9375, -320$

21. $a_n=n^2+3$

23. $a_n=\frac{2^n}{2n}$ or $\frac{2^{n-1}}{n}$

25. $a_n=\left(-\frac{1}{2}\right)^{n-1}$

27. First five terms: $3, -9, 27, -81, 243$

29. First five terms: $-1, 1, -9, \frac{27}{11}, \frac{891}{5}$

31. $\frac{1}{24}, 1, \frac{1}{4}, \frac{3}{2}, \frac{9}{4}, \frac{81}{4}, \frac{2187}{8}, \frac{531{,}441}{16}$

33. $2, 10, 12, \frac{14}{5}, \frac{4}{5}, 2, 10, 12$

35. $a_1=-8, a_n=a_{n-1}+n$

37. $a_1=35, a_n=a_{n-1}+3$

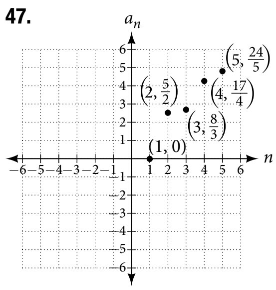
39. 720

41. 665,280

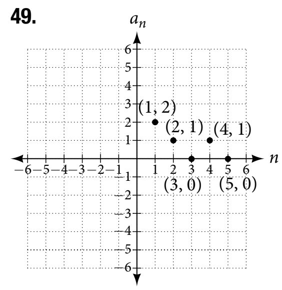
43. First four terms: $1, \frac{1}{2}, \frac{2}{3}, \frac{3}{2}$

45. First four terms: $-1, 2, \frac{6}{5}, \frac{24}{11}$

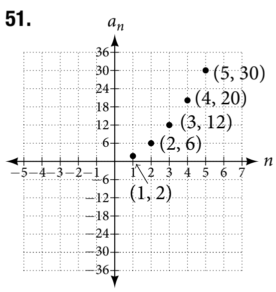
47.



49.



51.



53. $a_n=2^{n-2}$

55. $a_1=6, a_n=2a_{n-1}-5$

57. First five terms: $\frac{29}{37}, \frac{152}{111}, \frac{716}{333}, \frac{3188}{999}, \frac{13724}{2997}$

59. First five terms: 2, 3, 5, 17, 65537

61. $a_{10}=7{,}257{,}600$

63. First six terms: 0.042, 0.146, 0.875, 2.385, 4.708

65. First four terms: 5.975, 32.765, 185.743, 1057.25, 6023.521

67. If $a_n = -421$ is a term in the sequence, then solving the equation $-421 = -6 - 8n$ for n will yield a non-negative integer. However, if $-421 = -6 - 8n$, then $n = 51.875$ so $a_n = -421$ is not a term in the sequence.

69. $a_1 = 1, a_2 = 0, a_n = a_{n-1} - a_{n-2}$

71. $\frac{(n+2)!}{(n-1)!} = \frac{(n+2)\cdot(n+1)\cdot(n)\cdot(n-1)\cdot \ldots \cdot 3\cdot 2\cdot 1}{(n-1)\cdot \ldots \cdot 3\cdot 2\cdot 1}$

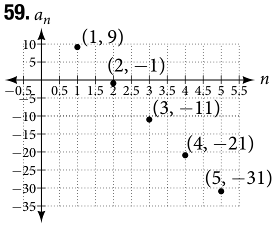
$= n(n+1)(n+2) = n^3 + 3n^2 + 2n$

Section 13.2

1. A sequence where each successive term of the sequence increases (or decreases) by a constant value. **3.** We find whether the difference between all consecutive terms is the same. This is the same as saying that the sequence has a common difference. **5.** Both arithmetic sequences and linear functions have a constant rate of change. They are different because their domains are not the same; linear functions are defined for all real numbers, and arithmetic sequences are defined for natural numbers or a subset of the natural numbers. **7.** The common difference is $\frac{1}{2}$ **9.** The sequence is not arithmetic because $16 - 4 \neq 64 - 16$. **11.** $0, \frac{2}{3}, \frac{4}{3}, 2, \frac{8}{3}$ **13.** $0, -5, -10, -15, -20$

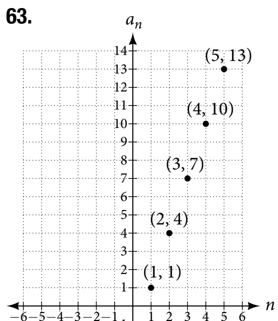
15. $a_4 = 19$ **17.** $a_6 = 41$ **19.** $a_1 = 2$ **21.** $a_1 = 5$
23. $a_1 = 6$ **25.** $a_{21} = -13.5$ **27.** $-19, -20.4, -21.8, -23.2, -24.6$
29. $a_1 = 17; a_n = a_{n-1} + 9; n \geq 2$ **31.** $a_1 = 12; a_n = a_{n-1} + 5; n \geq 2$
33. $a_1 = 8.9; a_n = a_{n-1} + 1.4; n \geq 2$ **35.** $a_1 = \frac{1}{5}; a_n = a_{n-1} + \frac{1}{4}; n \geq 2$
37. $a_1 = \frac{1}{6}; a_n = a_{n-1} - \frac{13}{12}; n \geq 2$ **39.** $a_1 = 4; a_n = a_{n-1} + 7; a_{14} = 95$
41. First five terms: 20, 16, 12, 8, 4 **43.** $a_n = 1 + 2n$
45. $a_n = -105 + 100n$ **47.** $a_n = 1.8n$ **49.** $a_n = 13.1 + 2.7n$
51. $a_n = \frac{1}{3}n - \frac{1}{3}$ **53.** There are 10 terms in the sequence.
55. There are 6 terms in the sequence.
57. The graph does not represent an arithmetic sequence.

59.



61. 1, 4, 7, 10, 13, 16, 19

63.



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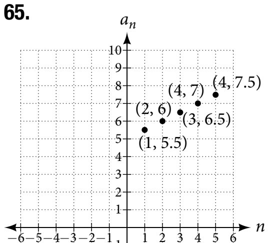


67. Answers will vary. Examples: $a_n = 20.6n$ and $a_n = 2 + 20.4n$

69. $a_{11} = -17a + 38b$

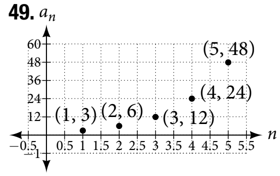
71. The sequence begins to have negative values at the 13th term, $a_{13} = -\frac{1}{3}$ **73.** Answers will vary. Check to see that the sequence is arithmetic. Example: Recursive formula: $a_1 = 3$, $a_n = a_{n-1} - 3$. First 4 terms: 3, 0, −3, −6; $a_{31} = -87$

Section 13.3

1. A sequence in which the ratio between any two consecutive terms is constant. **3.** Divide each term in a sequence by the preceding term. If the resulting quotients are equal, then the sequence is geometric. **5.** Both geometric sequences and exponential functions have a constant ratio. However, their domains are not the same. Exponential functions are defined for all real numbers, and geometric sequences are defined only for positive integers. Another difference is that the base of a geometric sequence (the common ratio) can be negative, but the base of an exponential function must be positive.
7. The common ratio is −2 **9.** The sequence is geometric. The common ratio is 2. **11.** The sequence is geometric. The common ratio is $-\frac{1}{2}$. **13.** The sequence is geometric. The common ratio is 5. **15.** $5, 1, \frac{1}{5}, \frac{1}{25}, \frac{1}{125}$ **17.** 800, 400, 200, 100, 50

19. $a_4 = -\frac{16}{27}$ **21.** $a_7 = -\frac{2}{729}$ **23.** 7, 1.4, 0.28, 0.056, 0.0112
25. $a = -32, a_n = \frac{1}{2}a_{n-1}$ **27.** $a_1 = 10, a_n = -0.3\, a_{n-1}$
29. $a_1 = \frac{3}{5}, a_n = \frac{1}{6}a_{n-1}$ **31.** $a_1 = \frac{1}{512}, a_n = -4a_{n-1}$
33. $12, -6, 3, -\frac{3}{2}, \frac{3}{4}$ **35.** $a_n = 3^{n-1}$
37. $a_n = 0.8\cdot(-5)^{n-1}$ **39.** $a_n = -\left(\frac{4}{5}\right)^{n-1}$
41. $a_n = 3\cdot\left(-\frac{1}{3}\right)^{n-1}$ **43.** $a_{12} = \frac{1}{177,147}$
45. There are 12 terms in the sequence.
47. The graph does not represent a geometric sequence.

49.



51. Answers will vary. Examples: $a_1 = 800, a_n = 0.5a_{n-1}$ and $a_1 = 12.5, a_n = 4a_{n-1}$
53. $a_5 = 256b$
55. The sequence exceeds 100 at the 14th term, $a_{14} \approx 107$.

57. $a_4 = -\frac{32}{3}$ is the first non-integer value

59. Answers will vary. Example: explicit formula with a decimal common ratio: $a_n = 400\cdot 0.5^{n-1}$; first 4 terms: 400, 200, 100, 50; $a_8 = 3.125$

Section 13.4

1. An nth partial sum is the sum of the first n terms of a sequence. **3.** A geometric series is the sum of the terms in a geometric sequence. **5.** An annuity is a series of regular equal payments that earn a constant compounded interest.

7. $\sum_{n=0}^{4} 5n$ **9.** $\sum_{k=1}^{5} 4$ **11.** $\sum_{k=1}^{20} 8k + 2$ **13.** $S_5 = \frac{25}{2}$

15. $S_{13} = 57.2$ **17.** $\sum_{k=1}^{7} 8\cdot 0.5^{k-1}$

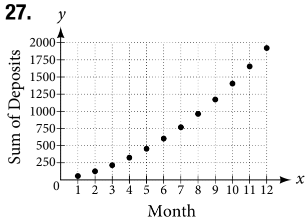
19. $S_5 = \frac{9\left(1 - \left(\frac{1}{3}\right)^5\right)}{1 - \frac{1}{3}} = \frac{121}{9} \approx 13.44$

21. $S_{11} = \frac{64(1 - 0.2^{11})}{1 - 0.2} = \frac{781{,}249{,}984}{9{,}765{,}625} \approx 80$

23. The series is defined. $S = \frac{2}{1 - 0.8}$

25. The series is defined. $S = \dfrac{-1}{1 - \left(-\frac{1}{2}\right)}$

27.



29. Sample answer: The graph of S_n seems to be approaching 1. This makes sense because $\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k$ is a defined infinite geometric series with $S = \dfrac{\frac{1}{2}}{1 - \left(\frac{1}{2}\right)} = 1.$

31. 49 **33.** 254 **35.** $S_7 = \dfrac{147}{2}$ **37.** $S_{11} = \dfrac{55}{2}$

39. $S_7 = 5208.4$ **41.** $S_{10} = -\dfrac{1023}{256}$ **43.** $S = -\dfrac{4}{3}$

45. $S = 9.2$ **47.** \$3,705.42 **49.** \$695,823.97

51. $a_k = 30 - k$ **53.** 9 terms **55.** $r = \dfrac{4}{5}$

57. \$400 per month **59.** 420 feet **61.** 12 feet

Section 13.5

1. There are $m + n$ ways for either event A or event B to occur. **3.** The addition principle is applied when determining the total possible of outcomes of either event occurring. The multiplication principle is applied when determining the total possible outcomes of both events occurring. The word "or" usually implies an addition problem. The word "and" usually implies a multiplication problem. **5.** A combination; $C(n, r) = \dfrac{n!}{(n-r)!r!}$ **7.** $4 + 2 = 6$ **9.** $5 + 4 + 7 = 16$

11. $2 \times 6 = 12$ **13.** $10^3 = 1{,}000$ **15.** $P(5, 2) = 20$

17. $P(3, 3) = 6$ **19.** $P(11, 5) = 55{,}440$ **21.** $C(12, 4) = 495$

23. $C(7, 6) = 7$ **25.** $2^{10} = 1{,}024$ **27.** $2^{12} = 4{,}096$

29. $2^9 = 512$ **31.** $\dfrac{8!}{3!} = 6{,}720$ **33.** $\dfrac{12!}{3!2!3!4!}$ **35.** 9

37. Yes, for the trivial cases $r = 0$ and $r = 1$. If $r = 0$, then $C(n, r) = P(n, r) = 1$. If $r = 1$, then $r = 1$, $C(n, r) = P(n, r) = n$.

39. $\dfrac{6!}{2!} \times 4! = 8{,}640$ **41.** $6 - 3 + 8 - 3 = 8$ **43.** $4 \times 2 \times 5 = 40$

45. $4 \times 12 \times 3 = 144$ **47.** $P(15, 9) = 1{,}816{,}214{,}400$

49. $C(10, 3) \times C(6, 5) \times C(5, 2) = 7{,}200$ **51.** $2^{11} = 2{,}048$

53. $\dfrac{20!}{6!6!8!} = 116{,}396{,}280$

Section 13.6

1. A binomial coefficient is an alternative way of denoting the combination $C(n, r)$. It is defined as $\binom{n}{r} = C(n, r) = \dfrac{n!}{r!(n-r)!}$.

3. The Binomial Theorem is defined as $(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k$ and can be used to expand any binomial.

5. 15 **7.** 35 **9.** 10 **11.** 12,376

13. $64a^3 - 48a^2b + 12ab^2 - b^3$ **15.** $27a^3 + 54a^2b + 36ab^2 + 8b^3$

17. $1024x^5 + 2560x^4y + 2560x^3y^2 + 1280x^2y^3 + 320xy^4 + 32y^5$

19. $1024x^5 - 3840x^4y + 5760x^3y^2 - 4320x^2y^3 + 1620xy^4 - 243y^5$

21. $\dfrac{1}{x^4} + \dfrac{8}{x^3y} + \dfrac{24}{x^2y^2} + \dfrac{32}{xy^3} + \dfrac{16}{y^4}$ **23.** $a^{17} + 17a^{16}b + 136a^{15}b^2$

25. $a^{15} - 30a^{14}b + 420a^{13}b^2$

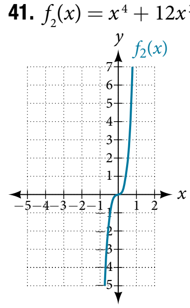
27. $3{,}486{,}784{,}401a^{20} + 23{,}245{,}229{,}340a^{19}b + 73{,}609{,}892{,}910a^{18}b^2$

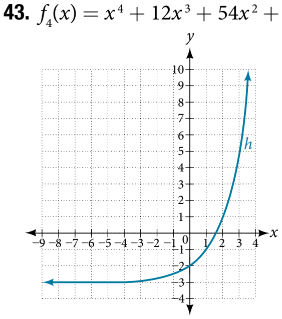
29. $x^{24} - 8x^{21}\sqrt{y} + 28x^{18}y$ **31.** $-720x^2y^3$

33. $220{,}812{,}466{,}875{,}000y^7$ **35.** $35x^3y^4$

37. $1{,}082{,}565a^3b^{16}$ **39.** $\dfrac{1152y^2}{x^7}$

41. $f_2(x) = x^4 + 12x^3$ **43.** $f_4(x) = x^4 + 12x^3 + 54x^2 + 108x$





45. $590{,}625x^5y^2$

47. $k - 1$

49. The expression $(x^3 + 2y^2 - z)^5$ cannot be expanded using the Binomial Theorem because it cannot be rewritten as a binomial.

Section 13.7

1. Probability; the probability of an event is restricted to values between 0 and 1, inclusive of 0 and 1.

3. An experiment is an activity with an observable result.

5. The probability of the *union of two events* occurring is a number that describes the likelihood that at least one of the events from a probability model occurs. In both a union of sets A and B and a union of events A and B, the union includes either A or B or both. The difference is that a union of sets results in another set, while the union of events is a probability, so it is always a numerical value between 0 and 1. **7.** $\dfrac{1}{2}$

9. $\dfrac{5}{8}$ **11.** $\dfrac{5}{8}$ **13.** $\dfrac{3}{8}$ **15.** $\dfrac{1}{4}$ **17.** $\dfrac{3}{4}$ **19.** $\dfrac{3}{8}$

21. $\dfrac{1}{8}$ **23.** $\dfrac{15}{16}$ **25.** $\dfrac{5}{8}$ **27.** $\dfrac{1}{13}$ **29.** $\dfrac{1}{26}$ **31.** $\dfrac{12}{13}$

33.

	1	2	3	4	5	6
1	(1, 1) 2	(1, 2) 3	(1, 3) 4	(1, 4) 5	(1, 5) 6	(1, 6) 7
2	(2, 1) 3	(2, 2) 4	(2, 3) 5	(2, 4) 6	(2, 5) 7	(2, 6) 8
3	(3, 1) 4	(3, 2) 5	(3, 3) 6	(3, 4) 7	(3, 5) 8	(3, 6) 9
4	(4, 1) 5	(4, 2) 6	(4, 3) 7	(4, 4) 8	(4, 5) 9	(4, 6) 10
5	(5, 1) 6	(5, 2) 7	(5, 3) 8	(5, 4) 9	(5, 5) 10	(5, 6) 11
6	(6, 1) 7	(6, 2) 8	(6, 3) 9	(6, 4) 10	(6, 5) 11	(6, 6) 12

35. $\dfrac{5}{12}$ **37.** 0. **39.** $\dfrac{4}{9}$ **41.** $\dfrac{1}{4}$ **43.** $\dfrac{3}{4}$

45. $\dfrac{21}{26}$ **47.** $\dfrac{C(12, 5)}{C(48, 5)} = \dfrac{1}{2162}$ **49.** $\dfrac{C(12, 3)C(36, 2)}{C(48, 5)} = \dfrac{175}{2162}$

51. $\dfrac{C(20, 3)C(60, 17)}{C(80, 20)} \approx 12.49\%$ **53.** $\dfrac{C(20, 5)C(60, 15)}{C(80, 20)} \approx 23.33\%$

55. $20.50 + 23.33 - 12.49 = 31.34\%$

57. $\dfrac{C(40000000, 1)C(277000000, 4)}{C(317000000, 5)} = 36.78\%$

59. $\dfrac{C(40000000, 4)C(277000000, 1)}{C(317000000, 5)} = 0.11\%$

Chapter 13 Review Exercises

1. 2, 4, 7, 11 **3.** 13, 103, 1003, 10003

5. The sequence is arithmetic. The common difference is $d = \dfrac{5}{3}$.

7. 18, 10, 2, −6, −14 **9.** $a_1 = -20, a_n = a_{n-1} + 10$

11. $a_n = \dfrac{1}{3}n + \dfrac{13}{24}$ **13.** $r = 2$ **15.** 4, 16, 64, 256, 1024

17. 3, 12, 48, 192, 768 **19.** $a_n = -\dfrac{1}{5} \cdot \left(\dfrac{1}{3}\right)^{n-1}$

21. $\sum_{m=0}^{5}\left(\dfrac{1}{2}m + 5\right)$ **23.** $S_{11} = 110$ **25.** $S_9 \approx 23.95$

27. $S = \dfrac{135}{4}$ **29.** \$5,617.61 **31.** 6 **33.** $10^4 = 10{,}000$

35. $P(18, 4) = 73{,}440$ **37.** $C(15, 6) = 5{,}005$

39. $2^{50} = 1.13 \times 10^{15}$ **41.** $\dfrac{8!}{3!2!} = 3{,}360$ **43.** 490,314

45. $131{,}072a^{17} + 1{,}114{,}112a^{16}b + 4{,}456{,}448a^{15}b^2$

47.

	1	2	3	4	5	6
1	1, 1	1, 2	1, 3	1, 4	1, 5	1, 6
2	2, 1	2, 2	2, 3	2, 4	2, 5	2, 6
3	3, 1	3, 2	3, 3	3, 4	3, 5	3, 6
4	4, 1	4, 2	4, 3	4, 4	4, 5	4, 6
5	5, 1	5, 2	5, 3	5, 4	5, 5	5, 6
6	6, 1	6, 2	6, 3	6, 4	6, 5	6, 6

49. $\dfrac{1}{6}$ **51.** $\dfrac{5}{9}$ **53.** $\dfrac{4}{9}$ **55.** $1 - \dfrac{C(350, 8)}{C(500, 8})) \approx 94.4\%$

57. $\dfrac{C(150, 3)C(350, 5)}{C(500, 8)} \approx 25.6\%$

Chapter 13 Practice Test

1. −14, −6, −2, 0 **3.** The sequence is arithmetic. The common difference is $d = 0.9$. **5.** $a_1 = -2, a_n = a_{n-1} - \dfrac{3}{2}$; $a_{22} = -\dfrac{67}{2}$ **7.** The sequence is geometric. The common ratio is $r = \dfrac{1}{2}$. **9.** $a_1 = 1, a_n = -\dfrac{1}{2} \cdot a_{n-1}$ **11.** $\sum_{k=-3}^{15}\left(3k^2 - \dfrac{5}{6}k\right)$

13. $S_7 = -2{,}604.2$ **15.** Total in account: \$140,355.75; Interest earned: \$14,355.75 **17.** $5 \times 3 \times 2 \times 3 \times 2 = 180$

19. $C(15, 3) = 455$ **21.** $\dfrac{10!}{2!3!2!} = 151{,}200$ **23.** $\dfrac{429x^{14}}{16}$

25. $\dfrac{4}{7}$ **27.** $\dfrac{5}{7}$ **29.** $\dfrac{C(14, 3)C(26, 4)}{C(40, 7)} \approx 29.2\%$

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X

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