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9 May 2017

Education 130

The Cube Problem

Seeing vs. Believing: What can students' approaches to the cube problem tell us about their attitudes toward modeling as a form of mathematical problem solving?

Abstract

I tasked a class of thirty-two Math Analysis Honors students at Berkeley High with finding the shape of the cross section of a cube resting on its point. When I presented my lesson on the cube problem, I was surprised to see that none of the students reached for the modeling materials and very few wanted to discuss their approaches with their group (unless to settle a dispute over the "correct" way to solve the problem). Students were not engaged with trying to visualise the problem and instead relied on algebraic and geometric representations, often ignoring or struggling to conceptualise simpler facets of the problem. Student's attitudes towards modeling as a mathematical problem solving method indicated modeling was not valued or emphasised as a proof method in this classroom.

Introduction

Education 130: "Learning and Knowing in Mathematics and Science" was the third CalTeach course I had the opportunity to take at Berkeley, and, thus far, the one which has had the largest impact on my educational philosophies and approaches in the classroom. This course provided me with the opportunity to learn about student-problem engagement and reflect on the way my engagement, and my learning, took place. As someone who works in a technical field with a single, definitive "right answer" most of the time, I don't have the opportunity to analyse my learning or my approaches to problems very often. Not only did Education 130 give me a chance to understand my learning style and thinking, but it taught me how to assist students with learning styles different from my own. I now understand the value of having many different kinds of thinking in a classroom; diversity of thought strengthens concept development and engages more students than a one-size-fits-all approach.

My field placement was at Berkeley High School in an Honors Math Analysis classroom consisting of mostly 10th- and some 11th-grade students. I went into my field placement with high expectations; in my last CalTeach course, I had the good fortune of being assigned a mentor teacher who had experience in all facets of the school system. She had been a Vice Principal, literacy coach, math coach, and taught grades K-3rd for many years. In her classroom, and with her support throughout the semester, I warmed up to teaching several 3rd grade math lessons on my own and building real and valuable relationships with the students which continued into this semester after I was hired by the school as a math tutor.

My placement at Berkeley High, however, fell in line with what my expectations were going into my *first* CalTeach field placement. I was not allowed to interact with the students in any meaningful sense. Most of my time in the classroom was spent observing from a desk at the front of the room. I was able to give a 30-second introduction on the second day of my field placement, but apart form that, the only other time I spoke to the students was when I presented my lesson at the end of the semester. Because this course was centred around student learning and understanding, I was frustrated that I didn't get a chance to interact with my students and assess their learning styles before developing my lesson.

The classroom structure my mentor teacher employed was not so different from the majority of the high school classrooms I experienced as a student. He taught in a lecture-based format and seldom asked students to talk to each other or work in groups, though their desks were arranged to accommodate group work done in his Math 1 classes. There was very little project-based learning, and the course was scheduled and driven by the IB exams and end-of-year district assessments in May. This structure also contributed to my mentor teacher's wishes that I only observe and refrain from interacting with the students; he didn't want to have the students get off schedule and felt that was less likely to happen if he had sole control of the room.

Towards the end of the year, my mentor teacher shifted gears and began to work with new materials from the Mathematics Vision Project. These materials were more project based and allowed students to learn the material at the will of their own inquiry. I was excited to see the students get a chance to demonstrate their skills and their thinking instead of passively absorbing the material through lecture. I spoke to my mentor teacher with enthusiasm over the new materials and disappointment that I wouldn't be student teaching in his class long enough to see their effects. He responded that he felt the materials didn't do enough good for the time they took up; he found himself lingering on topics a lot longer waiting for students to ask the right kind of questions that it took to move the lesson forward.

I was disappointed by the justification my mentor teacher gave for disliking the new materials. Like Fruedenthal[1], I believe there is great value in allowing students to uncover concepts on their own, "Telling a kid a secret he can find out himself is not only bad teaching, it is a crime. Have you ever observed how keen six year olds are to discover and reinvent things and how you can disappoint them if you betray some secret too early? Twelve year olds are different; they got used to imposed solutions, they ask for solutions without trying". In an ideal situation with no end-of-year assessments or other time-sensitive events, perhaps more educators would be inclined to use materials like those in the Mathematics Vision Project and encourage students to learn in an authentic, discovery-based context. My mentor teacher's concerns,

[1] Freudenthal, H. (1971). Geometry between the devil and the deep sea. Educational Studies in Mathematics, 3(3/4), 413-435.

however, are not invalid ones and I hesitate to present them as such. "Problems arise when students have questions and ideas that the plans did not anticipate: Teachers have to choose whether to pursue student thinking or to keep to the plan" [2], and this can mean a costly diversion of time and energy in a classroom where both are in short supply.

I went into the classroom hoping to challenge the existing structure and give the students a chance to discuss their thinking, learn together, and foster the collaborative environment that had helped me so much in my Education 130 class. Though I didn't go into the lesson wanting to change my mentor teacher's mind about collaboration and group work, I wanted to show it could be productive. I was also ready to let things get a little out of hand; I know from watching myself teach in other environments that I can micromanage and try too hard to follow my lesson plan to the letter, so in this lesson I was determined to have a looser structure and allow students to explore though I knew it would be counterintuitive. I was ready to try something new, and I hoped my students would also be open to the idea.

I arranged the students' desks into groups of six and instructed them to sit according to their normal seating chart as best they could. I briefly re-introduced myself and gave a short overview of the lesson before diving into the problem. I also let students know *why* they were being filmed and what I intended to do with the video. By letting them know I wanted to analyse all kinds of learning, I hoped to give them permission to think and try new solutions without fear. Going into the classroom, I wanted to see how students used models and, specifically, how the different models they used impacted their conclusions. Because the classroom wasn't one which

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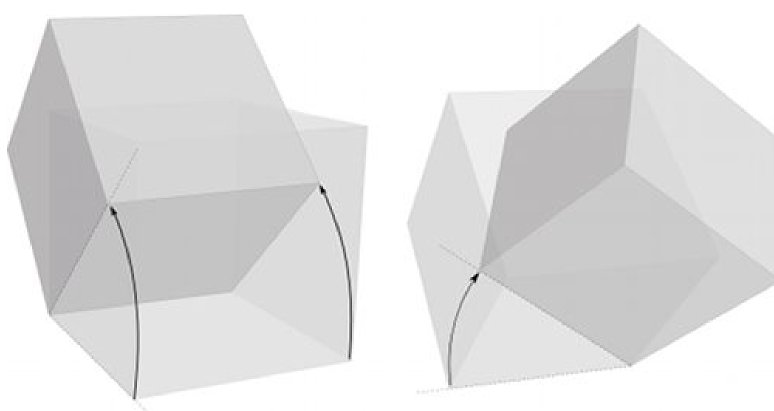
[2] Hammer, Goldberg, Fargason. 2012. Print. Responsive Teaching and the Beginnings of Energy in a Third Grade Classroom.

used models usually or allowed students to demonstrate their thinking through visual examples, I thought the students would be excited to try using new materials.

I chose to use the cube problem because my students had just finished working with conic sections. I presented this problem not within the context of *surface area* as was done when we test-drove this problem in Education 130, but instead within the context of *cubic sections*. The students were fairly receptive to the content in this vein and I relied (perhaps a little too heavily) on the assumption that they'd be able to apply most of their previous understandings of conic sections to this problem--mainly what it means to create sections of an object.

Problem Analysis

Problem: "Imagine a glass cube lying flat on the table. If we tilt the cube onto one of its edges, what is the shape of the cross section? If we tilt the cube to balance onto one of its corners, what is the shape of the cross section now?"



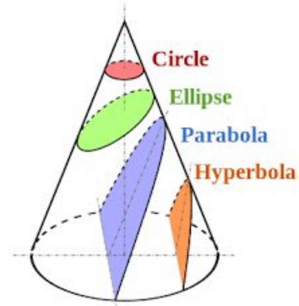
I decided to use this problem among those presented in class because my field placement is in an Math Analysis class. They've been doing work with vectors and conic sections and, in my own experience working on the problem, I pulled on fundamentals learned in both areas. I think this problem will be appropriate in content/relation to their course, and I also think they'll find it engaging. This problem is intended for students who have taken math classes in Algebra I and above. My students have taken the equivalent IB math courses in Geometry, Pre-Calculus, and Algebra I/II. Most of my students are in 10th or 11th grade.

Presentation

When I present this problem, I will do so in the context of "cubic sections". I expect my students to talk with each other about all the different shapes/sections they can get out of a cube and how they might represent these shapes. This will come naturally to them because they've been having such discussions for their math warm-ups in relation to conic sections every class.

In exploring this problem, I hope students will gain a greater understanding of what it means to section a 3D object and how the sections relate to the orientation of the object. In their work with hyperbolas, parabolas, ellipses, and other conic sections, I've noticed that students are able to (formulaically) solve problems, but do not have the ability to reason about how the shapes are related to each other or where their properties originate from. Additionally, most of the material covered in my field placement is presented in lecture format. The students do not have much time to discuss with their peers or exercise other methods of understanding (visualisation, manipulation, etc.). By presenting this problem, I hope to highlight the ways in which their explorations relate to their classroom content and how the modes of thinking they exercise to solve the problem can be transferred to other contexts.

In my presentation, especially initially, I aim to emphasise that there is no particular way to approach or solve this problem. Like as was done to us in class, I will



inform my students of the materials accessible to them, but I will not suggest uses for them.

Curriculum and Real-World Tie-Ins

In regards to the curriculum, this activity fits into a few different math standards:

5.0, 5.1, 5.2 (M.A.) Students are familiar with conic sections, both analytically and geometrically

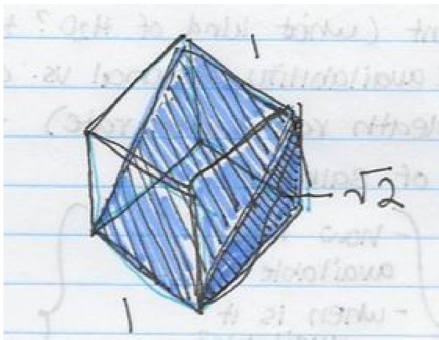
8.0 (G) Students know, derive, and solve problems involving the perimeter, circumference, area, volume, lateral area, and surface area of common geometric figures.

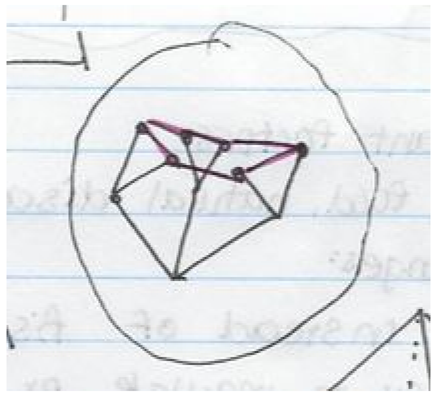
Skill-wise, this problem requires students work in groups to reason about their thinking. It also encourages visual thinking and the use of models to explore concepts. As extensions of the problem, I will encourage students to discuss other shapes they can generate from slicing the cube (ie. Is it possible to find a dot, line, triangle, square, pentagon, hexagon, septagon, octagon, etc. Where do you have to "cut" to get it? How do these shapes relate to the conic sections we've discussed?).

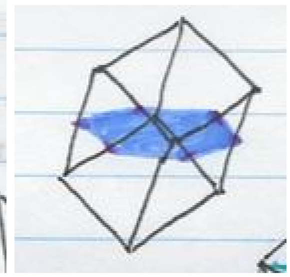
I expect students to rely heavily on their understanding of conic sections. I also expect them to pull from geometry knowledge once we review what a cross-section is and what it means to bisect a shape. When asked to hypothesise about the shape of the face of the cross-section, students should be able to rely on their geometry knowledge to name the face (rhombus, triangle, square, rectangle, hexagon, pentagon, etc.). I don't expect students to have encountered this exact problem before or for them to have encountered real-world situations where this knowledge was used.

Approaches

I. Drawing/Visualisation







In my experience with the problem, I relied heavily on drawing out each cube. This worked well for the first example where the cross section was a more regular shape. Some students, like myself, may try to employ this model exclusively because they are comfortable with it and/or unaware of how to use other materials/models. While this model may work well for some students, encouragement and discussion considering other models will be important for student success; the second half of the problem (when considering the bisection of a cube on its point), requires visualisation that students may struggle to conceptualise, let alone draw.

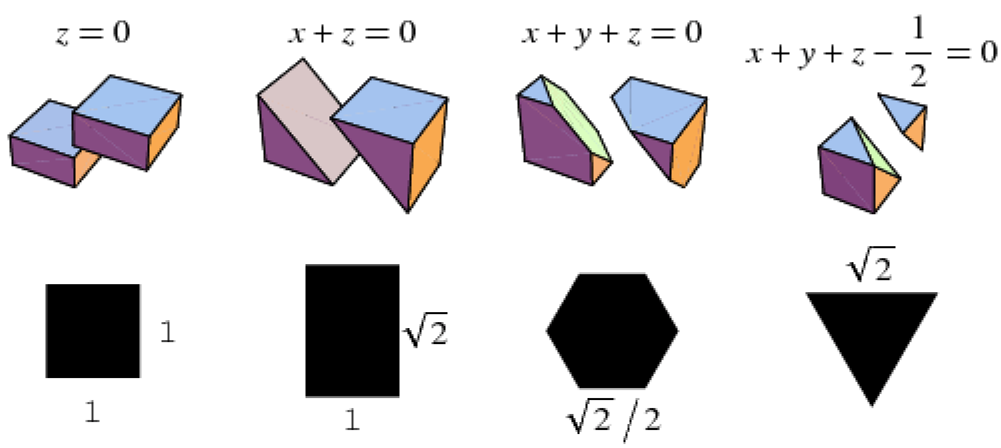
Certain students may be able to find the cross sections by immediately recognising that the shape of the face created can be seen by looking at the shape from the top down. Similarly, students may make an impression of the cube on its point in playdough to visualise the shape of the cross section. Students who employ these methods should be encouraged to substantiate their claim that “X is the shape of the face created when bisecting this cube” by employing other models to confirm their initial hypothesis.

II. 3D Modeling (Face Compression)

I expect a large amount of students to employ techniques from this category of solutions. Using the materials, they will likely find it helpful to have a model of a cube that they can physically turn, slice, or compress.

Students may create a pipecleaner model of a cube which they can then compress to find the structure of the bisecting faces. This was exhibited in class and I, both as a teacher and a student, found it to be incredibly helpful in visualising and understanding the problem. Initially, students may fail to realise their model can be manipulated in such a way. Through discussion of how to find the bisections of different orientations (ie. if the cube is lying flat on the table, what is the shape of its cross section?) they may conclude that they can "see" the shape of the cross section by looking at the cube from the top down. From here, they may attempt to compress their model to visualise the cross section as a flat shape.

III. 3D Modeling (Slicing)

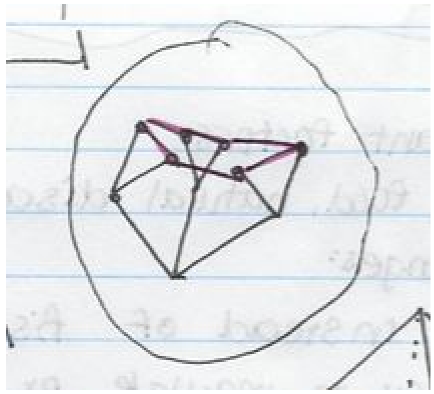


Similarly to the previous solution, students may also attempt to find the cross section of their 3D model by physically slicing or bisecting their shape.

In the work my group did on the problem, we found it helpful to create a 3D playdough cube and cut it in half from the orientation we were trying to find the cross-section for. In doing so, we were able to conclude the shape of that face. This may not be the case for all students, however. In their work with this solution, students should be encouraged to discuss how their model may be inaccurate. Was the cube a true cube to begin with or was the shape they constructed more similar to a rectangular prism? How would this change the cross-section? As we saw in class, the distortion of the shape while cutting may also lead students to incorrect conclusions (ie. the knife compressed the cube such that the cross section looked more like a triangle than a hexagon).

Another model students may employ to bisect their 3D model is to cut paper or use pipecleaners to construct a centre face that fits into their shape. In class, students made a paper cube and repeatedly cut away at their hypothesised face until it fit neatly in their model. In the video we watched after completing this problem in class, we observed middle school students trying to find the halfway points of each edge and using pipe-cleaners to draw lines from these points to outline the shape of the cross-section. In this solution, students should be careful that they're consistently marking their halfway points and that they're minding the orientation of the centre face they're trying to construct. I expect most of the students who created a 3D model to go this route, however I suspect many will find constructing a centre face more difficult than they originally suspected. Again, discussion and encouragement should be employed to help students search for other ways to model their solution to the problem.

IV. Measuring Up the Cube



After creating a drawing or 3D model, students may take a more analytical approach to finding the shape of the cross-section. Recognising the cross section of a shape is the shape halfway between the top and bottom of the entire shape, students may try to find the halfway points of each path down the cube and construct the cross-section by connecting these points. Though at first students may want to try to do the work in their heads, they should be encouraged to use a 3D model to ensure consistency in identifying the midpoints of each path down the cube. Drawing out each cube and erasing the upper/lower halves of the shape may also be a useful visualisation for students who prefer to rely on drawn models.

Extension Problem

As an extension of the problem, students may be encouraged to calculate the area of the cross sections at each step. Students may also find it interesting to determine the shape of other cubic sections. How many can be produced? At which angles/orientations are they found? How do they compare to the conic sections students may be familiar with? Students who have a particular propensity for algebra or geometry may be interested to find (and exploit) special triangle relationships in the cubic sections problems. How do these special triangles help us make sense of the cross-section shape? Can the cross-section shape be found by relying completely on angle measurements? Can it be found through only side/length measurements? How many different special triangles does the cube have? Is there a relationship between the angle at which the cube is cut or sectioned and the angles within the triangles?

Lesson Plan

Because my mentor teacher had two CalTeach students in the same two-hour block and because he wanted the two periods we observed to stay in sync with each other, we taught our lessons to only one of the two classes we observed. I introduced the problem to fifth period--my mentor teacher suggested I take this class over fourth period because my field placement fell on a day with shorter class periods; Fifth period had performed higher on exams all year and so my mentor teacher thought their ability to excel with material would make up for the shorter time I had in the classroom. I submitted "opt-out" forms rather than "opt-in" forms because Spring Break fell between when I was able to get in contact with my mentor teacher to distribute the forms and when I planned to teach my lesson. The opt-out forms gave me more freedom: if students forgot the forms while on vacation, I could still proceed with my lesson plan on schedule.

Cal Teach Student Name(s):	Morgan Rae Reschenberg		
Mentor Teacher Name:	Michael Weitz	**School/Room#:**	Berkeley High, H217
Grade Level and Subject:	High School Math Analysis (10th, 11th grade)		
Date & Time to Be Taught:	17 April 2017; 5th Period (1:58 - 2:41)		

Lesson Source(s):	Education 130
Focus/Essential Question:	How does the orientation of a 3D shape affect the 2D shape of its cross section?
Student Learning Objectives:	Students will understand that different 2D shapes can be generated by slicing or bisecting the same 3D shape at different points.
Content Standards:	MA.5.0 Students are familiar with conic sections, both analytically and geometrically MR.2.5 Use a variety of methods, such as words, numbers, symbols, charts, graphs, tables, diagrams, and models, to explain mathematical reasoning.
Student Prior Knowledge:	Students have taken either Geometry or an Integrated Math equivalent course. Students have worked with conic sections and are familiar with how to generate different 2D shapes from a 3D shape.
Lesson Agenda for Your Students:	- Warm Up: What are conic sections? What are cubic sections? - Cube Problem 1: Edgewise - Cube Problem 2: Corner-wise - Debrief: Solutions and Approaches
Lesson Rationale:	Visualisation and manipulation of 3D objects is essential to student success in mathematics and other technical subjects. Using models and visual aids to assist in breaking down a problem also gives students the opportunity to make mathematics more concrete.

Materials and Technology List:	Student Materials: Pipe-cleaners, playdough, plastic knives, cube models, tape, pencils, pens, paper Teacher Materials: Camera, camera batteries, tripod, volunteers to film, printed lesson plan copy
Preparation Tasks:	Collect supplies, arrange cameras, instruct students on how to arrange desks
Safety Concerns:	Students should exercise caution with plastic knives

Lesson Title: Cubic Sections

Evaluate: *Observe and adjust your lesson as you teach.*

Engage: *Activities that engage students' interest and build connections to their lives and prior knowledge.*	*Previous Experience and Baseline Learning*
First, students should be arranged into groups. Desks are normally in rows, and students should turn their desks to form groups of five or six students depending on attendance (NOTE: in 4th period there are 28 students, in 5th period there are 32). After students have arranged their desks, they will be given the following prompt and instructed to discuss with their group members: **In the past few weeks, you've done a lot of work with conic sections (parabolas, hyperbolas, circles, ellipses, etc.). Today, we're going to be doing some work with cubic sections. Remember that with conic sections, we obtained those shapes I just mentioned by looking at the cross sections, or faces, of the sections we cut away from the cone. With cubic sections, we'll be doing the same thing.** Here, students may find a visual reminder helpful. My mentor teacher has a model of a cone cut away at various angles to produce conic sections. If students seem unsure, I may hold up the model to jog their memory as we go through the shapes together. **I want you to imagine the following: You have a cube in front of you lying flat on the table--that is, one of the six faces of the cube is flat against the surface of the table. The cube is half-filled with water (water takes up half the volume of the cube). You tilt the cube left or right such that now only one of the edges is in contact with the table. After the water has settled, what is the shape of the surface of the water? The cube is a unit cube--each of the sides is of length 1.** **With your group, come to a consensus about the shape you believe the surface of the water creates. In a few minutes, I'll walk around and ask each group for their shape. We'll compose a list on the board.** By asking each group individually about the shape they believe the cross section to be, there is more likely to be a variety of responses than there is if students were asked to call out their shape in front of the entire class. After collecting student answers, ask a few groups to volunteer how they solved this problem--if "drawing", show their drawings on the projector or invite them up to the board. If students claim they "just knew" the answer, draw the problem on the board anyway to help students who may be lost and afraid to vocalise. Because these students have been working with conceptual geometry for	- Students may need clarification of the problem; if the water example doesn't work well, change language to emphasise that what we're looking for is the shape of the face when the cube is cut in half. Using hand motions may also help students understand - Though they may start with no supplies or nothing on their desks, if students seem frustrated or lost, encourage them to get out paper and pencil to draw out their thought process. - Again, encourage communication within the groups. Students are sitting together because the problems are supposed to be social - If discussion gets out of hand or students seem to be addressing each other inappropriately, redirect to whole-group discussion and draft some discussion norms. Follow school disciplinary procedures if necessary. - Students should draw on their knowledge of special triangles and geometric shapes for this section.

the past few weeks, I imagine this problem will go smoothly and quickly. **Time**: <u>About 15 minutes</u>	
Explore: *Hands-on tasks designed to explore ideas and to develop skills together.*	*Focus, Involvement, Collaboration, Results, and Recording*

After collecting student approaches for the first problem and walking through how students reasoned the cross section to be a rectangle, give students the following problem:
 	
Go back to the cube we had before. It's still a unit cube and it is still half filled with water. Instead of tilting it onto one edge, I want you to imagine tilting it onto one corner. First, we had the cube resting on the table--the entire face was in contact with the table. Then, in our last problem we tilted it so only the edge--a line--was in contact. Now, I want you to tilt it forward or backward from the edgewise position such that the entire cube is balanced on one corner--only one point is in contact with the table.
In this orientation, I want you to do the same thing we did before--find the shape of the cross section. Talk about this with your groups as I come around again to get a list of possible shapes.
 	
As students discuss with their groups, I will gather the wooden cube models and distribute them as each group tells me what they think the shape of the cross section will be. I will compile a list on the board in the same fashion as before.
 	
Time: <u>About 10 minutes</u>
	- Be prepared for student questions on the problem; like before, the water example may cause confusion for students. Revert to "cutting" or "bisecting" the cube if this is easier for students to understand
	- If there is a large amount of confusion about orientation from the previous problem, pass out cube models before explaining and show students the orientation with a model at the front of the room
	- It may be helpful to write certain conditions on the board:
	    - The cube is a unit cube (dimension model)
	    - ½ water volume
	    - Only 1 point in contact with table surface
	- Student drawings from the last problem may help students think about how to build off of what they already know; leave these on the board

Explain: *Students explain the phenomena they explored and discuss their different ideas and perspectives.*	*Participation, Reporting, Debating, and Evidence-Based Reasoning*
After giving students five minutes to discuss, I'll inform them of the	- Remind students materials should be shared among

materials available to use for modeling and encourage them to think of ways to either help them understand the problem better or, if they already have an answer, think of ways to best demonstrate/substantiate their answer for the class. Some students may be reluctant to try modeling without having a concrete idea of the shape of the cross section; they should be encouraged to play around with the materials to see if anything sparks their interest. **Time:** About 20 minutes	groups and only taken back to their desks when needed - Students should also be encouraged to consider the limitations of the models they're using: is the playdough too squishy and distorting your shape? Is your model more rectangular than cubic?
Elaborate: *Teacher-stimulated application and clarification of concepts, skills, attitudes, processes or terminology.*	*Demonstrated Understanding, Use of Skills, and Other Applications*
As students work in their groups, I'll walk around and try to direct them or help clarify the problem if necessary. I expect the students to be engaged with the problem, but I anticipate that students may have a hard time moving away from only drawing or only visualising the problem. If students have finished the problem early, I'll encourage them to find the area of the cross section they discovered. I will also ask them about the other shapes it is possible to generate from the cube: What is the equivalent "circle" for a cube? (in relation to conic sections). What about the parabola, hyperbola, etc.? **Time:** About 10 minutes (partially overlapping with previous time of discussion)	Guiding questions: - If students are using the corners as the "halfway" points: how can we find the halfway points? how long is the path up the cube? where would the halfway points be? - If students are using models that may be inaccurate or poorly constructed/misshapen: use whole pipe cleaners for even sides, use less pressure when cutting the playdough
Evaluate:	
At the end of the class, students will elect a representative from each group and give a short 1-2 minute overview of what they tried and how they reasoned through the problem. They will also share their conclusions and how their models helped confirm or deny what they initially suspected the cross section to be. If they attempted the additional problems, they will have a chance to present their findings as well. Students will be evaluated largely on participation and effort. Because this is a topic students have dealt with before, they should have no trouble applying their previous knowledge of conic sections in this new context.	Guiding questions: - What did you think initially? - How did you go about trying to prove your intuition? - When did your thought process start to change? - What conclusion did you reach?

Methods

When I planned my lesson and began to think of a research question, I intended to focus on how students used the various modeling materials available to them to make sense of the problem. Specifically, I wanted to research if different modeling materials led students to draw different conclusions. To capture this (assumed) diversity, I planned on having a camera or laptop at each group to gather footage of how students interacted with the materials and how the models affected how they conversed with each other/what language they used. Later, I planned on watching the videos and quantifying these interactions to help me better make sense of exactly *how much* the modeling impacted their conclusions.

Though I had the supplies to do this on the day of my lesson, I was unable to follow through because scheduling issues had prevented my mentor teacher from passing out the filming permission forms on time. After rescheduling, I wasn't able to check out the same filming equipment. Instead of the original one-camera-per-group setup, I used two cameras and my laptop for filming along with my iPad as an audio recorder. Having to rethink my methods of data collection on the day-of was frustrating and I worried that I wouldn't be able to get enough usable information--especially because I was relying on an audio recording of one group and wouldn't be able to see how they physically handled the materials.

I did not choose which groups to record with any particular method other than with an attempt to distance the overlap of noise on the recordings. At the end of my lesson, I had data on a pair of female students who chose to work together, a five-student group with two female students and three male students, and a six-student group consisting of all male students. Though I did not initially plan to analyse how group dynamics, gender, or group size affected student success, I was later forced to revise my initial question and considered these subjects while doing so.

During the execution of my lesson, I was careful to start students out with no materials before moving onto paper/pencil, wooden cube models, and student-built models. I was required to revisit my intended research question when, to my surprise and confusion, the students ignored or shied away from using materials to model their thinking. Instead, they relied heavily on drawing and mental-manipulation of the cube. Because of the disparity between my expectations on the use of modeling in the classroom and the students actual engagement with the materials, I chose to revisit my video and audio data to try and pinpoint where the decision to forgo modeling occurred.

After watching and listening to my recordings about three times each, I decided to center my research on the video recordings and drop the audio data because it gave me less information about what the students were doing and thinking. While watching the videos for the final time, I transcribed portions of the group conversation noting specifically cases where students were engaging not only with the problem but with others' conceptions of the problem. The productive struggle that ensued after students discussed their ideas was interesting, and I felt obligated to record student interactions in hope of finding a new research question after getting little data to investigate my original claim. Ultimately, I chose not to focus on group work because the group conversation all followed a similar pattern of students posing ideas, vying for validation among their peers, and justifying or discussing an idea further only when a competing idea was presented. Because the data was consistent across all groups, and because a large amount of my peers were focusing on group dynamics and group work, I decided to look elsewhere for a research question.

After forgoing group work analysis, I redirected my attention to the students' solutions to the problem and their processes. The contrast between my expectations and actual student engagement with materials was interesting and, though I was concerned about not having enough information to answer the question, I wanted to explore possible causes for the behaviour and attitudes I'd seen. To analyse this further, I looked over the scratch paper I'd collected from my students during their brainstorming period. When cross-examining the video footage with the student work I collected, I noticed that the absence of modeling wasn't an absence in engagement with the problem--students had demonstrated complex algebraic solutions on their scratch paper and used their knowledge of 2D and 3D geometry to substantiate claims about the shape of the cross section.

Results and Discussion

Each group came to the same conclusion--that the cross section of the cube oriented on its point was a hexagon. Though they ended up in the same place, each group had employed different mathematical principles to get them there. No group had built a model, all groups employed drawing as their main media method to convey their ideas to their group, and, though all groups had elected to use more rigorous mathematics than I required (or expected to see), each group had employed different mathematical principles and disciplinary techniques to analyse the problem. Because these observations did not line up with what I went into the classroom hoping to assess, I formulated my research question as follows: What can students' approaches to the cube problem tell us about their attitudes toward modeling as a form of mathematical problem solving?

I. Group I

The all-female, two-student group started out the problem by making several 3D sketches of cubes at various orientations. They were able to quickly find the shape of the cross section in the first part of the problem, but while working on the second problem, they had a hard time making sketches that allowed them to see the cross section of the shape. Noticing that they were frustrated with their inability to “see” the cross section in their drawings, I prompted them to get materials to make a model. I recognised that their approach (drawing) had been fruitful for them in the previous question, but I didn’t want them to spend the entire class period trying to make the method fit the new problem. I wanted them to explore other options. One group member ($G_1$1) got up to get materials while the other continued to work on drawing the cube. After several failed attempts to make evenly-shaped faces for the cube out of pipecleaners, the pair gave up and went back to sketching the shapes. Both students worked individually on their own papers for about five minutes without conversation.

Visibly frustrated trying to find the shape, $G_1$1 asked her partner “What if we just find the area instead of the shape?”. This lead to a discussion about whether or not they *could* find the area of the shape without knowing *which* shape they were finding the area of. $G_1$1 then asked her partner “Do you think it’s [the cross section] another cube [sic: rectangle] like the last one?”. The pair went on to assume that the cross section of the cube in the point-to-point orientation was a square because the problem I’d presented them with prior to this one had been a rectangle.

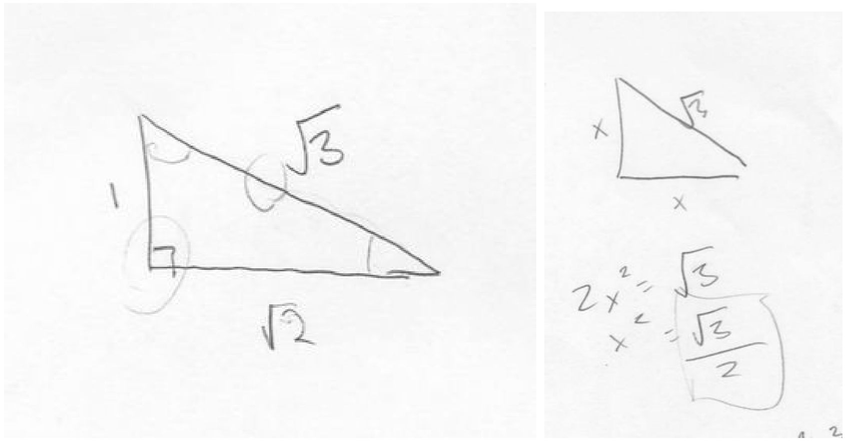
By working with an assumption from a previous problem experience, these students were operating in a “doing school” mindset rather than a “doing math” mindset [3]. In their work analysing science classrooms, Jiménez-Aleixandre and Rodriguez note a difference in how students engage with problems. In “doing school”, students rely on the classroom context of a problem to generate their solutions; they may use rules about the activity, classroom norms, or prior experience with similar activities, to constrain their work. In “doing math”, students instead engage with the material authentically; they dive into the concepts without regard for the context in which they’re being presented. The girls in the group I filmed were not trying to discover the solution to the problem by exploring the cube and its sections on their own. They instead relied on the fact that the previous (similar) problem had given them a certain type of solution and that, because things are often presented with a purpose in classrooms, this problem should have a solution like that of the last one.

They had attempted to relate this problem to the previous problem, but had inaccurately done so; the previous problem had a cross section that was a rectangle and here they said that it was a cube and later used a square. This went on to shape the rest of their approach to the problem. $G_1$1 explained their process to my mentor teacher as he came around to check on their progress. After conferring with her partner about the special triangles they’d drawn on their paper, G1 said the following:

> G1: If we do it [the cube] on the point, we thought that the diagonal across the entire cube was just the diagonal of the surface area. So we wanted to find the diagonal across the cube so we just used that [the side length] was 1 and the diagonal [across a face] was square root 2, so we found the diagonal of the surface area was root 3. Then we used

[3] Jiménez-Aleixandre, M.P., Bugallo Rodriguez, A., Duschl, R.A. (2000). “Doing the lesson” or “doing science”: Argument in high school genetics. Science Education, 84, 757-792.

pythagorean theorem. So then we used that it was a square, which is what we thought it was.



[Drawings from $G_1$1's paper that were referenced in her explanation above]

At the end of her explanation, $G_1$1 presented the only guess I would get out of the entire class. She and her partner assumed the cross section was a square. In watching the interactions on film, it was interesting to compare the way in which they selected the shape of the cross section (with assumptions based on previous interactions with problem material, circumstantial), and the way in which they calculated the area of the shape (with an emphasis on algebraic equations, rigorous). After figuring out the area--and using an assumption about the shape of the cross section to do so--the pair sat quietly for nearly the rest of the period. With five minutes left of group work time, $G_1$1 began to overhear the group working behind her discuss their conclusion: that the cross section shape was a hexagon.

Using the same information they'd gotten by finding the diagonal of the surface area, they began to calculate the area of a hexagon--changing the one assumption their conclusion had relied on. At this point, I checked in with them--they'd been talking with each other every time I walked around them, and, because I knew I was filming them and would be able to watch their conversations later, I didn't want to disrupt group discussion and redirect them. Because of this, I only checked in with them to a) put them on the track to making a model and b) observe their conclusions at the end of the work period. When they showed me the hexagon on their paper and the algebra to prove they'd found the correct solution, I concluded that they'd worked through the problem and were able to find the area and shape of the hexagon. Only after watching the video did I discover that they happened upon the correct answer by overhearing another group and that their method--while algebraic and mathematical--wasn't as rigorous as it appeared to be.

As a future educator, this experience highlighted the importance of assessment and the drawbacks therein. If I hadn't had the video data, I wouldn't have been able to tell that this group of students didn't reach the correct conclusion on their own. The student work I collected and the presentations the students gave indicated that their ultimate solution was correct, but the progress to that solution was not visible and so I was not able to directly assess their learning and achievement.

II. Group II

The second group of students was the group of five (four male and one female). They worked in the back of the classroom and had the largest quantity of discussion I found when analysing all filmed groups. That said, their discussion was furthered by members of the group's struggle to "see" the solution to the problem. The group exhibited a lot of back and forth dialogue:

> $G_2$1: So like from the top of the cube you have this distance [moves pencil up and down the face of the cube]. If you look at half of that you know it has to go through there because it's halfway and it has to go through all the sides at halfway so there are six places and it's a hexagon.

$B_2$1: Oh okay.

$G_2$1: So do you see it now?

$B_2$1: [pause] No, not really

$G_2$1: Ugh! [silence as she thinks] Well, I don't really know how else to show you. That's just what it is. It's a hexagon.

When I noticed the discussion, I prompted the group to show the solution another way--maybe there was a model that would be easier for their classmate to "see" the cross shape in? Unlike the first group of girls, this group did not attempt to make a model. Instead, they resorted to a discussion on midpoints. Initially, the group metally flattened each cube face and found the halfway point straight between the top and the bottom point of the cube. This method was largely propelled by $G_2$1, and it seemed to be understood by other students in the group who realised the halfway point fell above the obvious corners making it so the cross section would pass through every face and generate something with six sides.

Because I was also intent on preserving the group's goals and not imposing my own, I worked with them to make the solution clearer to the struggling student. Instead of flattening each face and finding the midpoint in a 2D sense before extrapolating it to the 3D case, I worked with a solution I was presented in class: I had the students find the shortest direct path from the bottom of the cube to the top. We then discussed how their midpoint approach would work in these cases. When it became apparent that the midpoint of the length three path lay one-and-a-half sides up the cube, the students immediately noticed triangles generated by the leftover corners of the cube. These triangles seemed familiar to them and they used their knowledge of angles (interior, anterior, supplementary, and complementary) to calculate the size, and later area, of the hexagon.

Unlike Group I, this group of students did not rely on any assumptions in their initial strategy. Additionally, when I prompted them to give me a guess about the shape of the cross section, they asked for more time to work out a complete answer. In all the groups observed, I maintain building a model that could be physically manipulated (cut, sectioned, etc.) would've greatly helped this group's understanding--specifically the understanding of the struggling student. Though I was able to coax their solution into a form which relied less on mental manipulation, I was not convinced that it completely alleviated the misunderstanding the struggling student was going through. When the group moved on to try to find the area, the struggling student didn't contribute and was disengaged from the group discussion suggesting there were still parts of the problem that were unclear, or, that his previous confusion made him afraid to participate further. In any case, the continued misconceptions and misunderstanding held by the group member indicated that the lack of modeling was detrimental to student success.

III. Group III

The third group I filmed was an interesting case. They were very quick to figure out the cross section of the cube in both the point-to-point and edge orientations. They had very little discussion, but of the discussion they had, all group members seemed satisfied with the justification given. The satisfaction seemed to stem from the language and resources used. This group relied almost exclusively on the knowledge of conic sections they'd been given in class. After I presented the problem, the group sat in silence until one group member ($M_3 2$) said, without explanation, "It's a hexagon."

> $M_3 1$: Yeah, you're right. If it [moving a pencil flat up the side of the cube, motioning sectioning] goes less than halfway up, it's a triangle. If you go past [where the corners start to overlap] it becomes a hexagon.

When I overheard the discussion, I was excited; the group was pulling on prior knowledge just like I'd expected them to! I gave them a few more minutes to talk, and walked over to ask the group to demonstrate their solution to me. They held up a paper with some calculations on it. I hadn't yet asked them to find the area or dimensions of the shape, but after being sure of the shape itself, they went ahead and took the next, seemingly-logical step.

> $M_3$1: We know the sides [of the cube] are each 1, and the line cuts them in half. The triangle has ½ and ½ for sides, and then sqrt(2)/2 for the hypotenuse because of special triangles so each side [of the hexagon] is sqrt(2)/2.

I asked which line they were talking about when they said "the line". They reiterated the discussion they'd had earlier about sectioning the cube and how the line generating the section makes the shape a triangle until it gets into the "corners" part of the cube where it cuts into every face. Again, I asked if they could *demonstrate* their solution. They looked at the modeling materials I had on the table but made no move towards them. One of the group members pulled a rubber band out of his pocket and stretched it around the wooden cube I'd given them.

> $M_3$2: If you look at it from the top, you can see it goes over, under, over, under… and there's six [pointing to the "edges" made by the rubber band]

Because this solution seemed without preface, I asked the rest of the group if this confirmed their hypothesis. They nodded, but had nothing to add to the discussion. I communicated my surprise to my mentor teacher as I began to set up for the next part of the lesson. He, then, let me know that the group I'd just talked with had a student who did the same problem in a previous math class and that I shouldn't expect much discussion because he was "one of the smart kids" and "[knew] all the answers to this problem, already". I was disappointed that I hadn't been aware of this before setting up the groups, but I also thought it provided an interesting case to analyse--if the students in the group who hadn't done the problem before were confused, they didn't make any moves to clarify their confusion or try a solution in a different way. I attributed this both to the authority with which $M_3 2$ presented his solution and the regard my mentor teacher (and likely the rest of the class) had for this student. If he was considered "smart" by his peers, the widespread acceptance of his ideas without question was not unconventional.

I felt the students in the last group were unfairly cheated out of the opportunity to discover the solution to this problem for themselves. Though I was excited to see the students reference prior knowledge, and likewise excited that the student who had previously engaged with the problem was able to present a model of his thinking, the interactions seemed inorganic and contrived. The small amount of discussion made it difficult to analyse their understanding, similar to the way in which I misunderstood the progress of Group I by only checking in with them at the beginning and end of the problem; I wasn't able to get a full picture of what the students were thinking or how they were working through the material.

IV. Conclusions

Despite introducing materials for modeling and connecting the lesson to prior knowledge in which modeling was used, students were more comfortable relying on algebraic and geometric representations. From the observation data I gathered during the semester of my field placement, it is apparent that there is not enough integration of visualisation and non-formulaic proofs in the classroom to make students comfortable employing modeling as a mathematical problem solving technique. The observations made in my classroom replicate the way many believe math is taught as a whole: without emphasis on student-driven exploration of concepts, and instead with a focus on formulaic explanation[4]. The lecture-based format of the class, type of homework and classwork given, and the explicit requirements for formulaic justification that were common for my mentor teacher to require shaped the way students in this class understood what it means to do math and prove things in a mathematical context. Because modeling was not included in that set of definitions, it was forgone in favour of more reliable, familiar methods.

In regards to how modeling would have impacted student understanding, I hesitate to speculate with the small amount of data I was able to gather. Because I don't know how these students interact with models in any sense (even in my regular classroom interactions), I don't believe the integration of modeling alone would have improved their understanding. Counterintuitively, I believe it would've turned some students off of math because the learning curve for introducing and using models mathematically would be fairly steep. This, however, isn't to say modeling should be ignored because the integration would require effort and may cause confusion. As I saw in the case with Group II and the struggle of the student who just couldn't *see* what his peers innately could, a model would have very likely aided in his understanding.

Reflection

As I learned in Education 130, there is no single, correct way to approach or engage with a problem. Though modeling was not presented as a way for students to engage with the material in their everyday classroom, my lesson permits me to conclude that the integration of modeling would assist select students in the articulation of their solutions as well as their understanding of

[4] Green, Elizabeth. "Why Do Americans Stink at Math?" The New York Times. The New York Times, 23 July 2014. Web. 25 Apr. 2017.

concepts in the classroom. In a classroom such as the one I observed, in which emphasis is placed on sticking to a schedule, teacher-lead instruction, and formulaic justification, I do not predict modeling will be integrated or prioritised. There is very little incentive to do so, especially when the majority of the class gets along just fine without it. That said, it may make a world of difference for the handful of students struggling in its absence.

Though the period leading up to my lesson was stressful, I am thankful for the opportunity to engage with my mentor teacher and his students in my field placement. This experience was invaluable in making real the split-second decisions and accommodations necessary in the field of education and teaching. Nearly nothing went according to plan, but I believe I'm a better educator for surviving that. I also found it interesting to analyse my teaching in the classroom and the way students responded to my teaching. I have a few notable areas I'd like to improve on:

I. *Language*:

Students in several groups were confused by the language I used to present the problem. I used the phrase "surface area" to mean "the area of the surface of the water", however several students understood me to mean the total area of the shape created by the water (ie. the sum of the areas over each face of the water-generated shape). I was not aware of the confusion until I reviewed the videos. Fortunately, the majority of students were able to clarify colloquially what I meant.

II. Classroom Management:

In several instances, I had to raise my voice above a level I was comfortable doing in a classroom setting. I recognise that this classroom was not my own and that, because of this, the methods I normally employ to manage my classroom were different from those students were used to. That said, from my observations, I knew my mentor teacher often resulted to shouting or giving up on instruction when his class got too loud. I wasn't willing to do so in my lesson, and because of that, I had to be louder and put up with more side-conversations than I anticipated.

In the future, to promote discussion, modeling, and critical thinking, I will attempt to give a preamble to each problem to situate it in a story-based context rather than a pure-math context. The connections to conic sections and other math material were helpful for some students as a reference point, but overall I believe the tie-in to classroom-based math and the conception of learning in the students' classroom environment contributed to the results I saw: lack of modeling, lack of productive discussion, and tendency to justify formulaically rather than visually.

The integration of student driven, discovery-based problems is something I see value in and want to continue emphasising in my time in the classroom. These types of problems make learning real and allow students to demonstrate thinking and present solutions in ways the traditional classroom does not. In terms of reaching, engaging, and teaching students of all mindsets and learning styles, discovery-based problems provide ample opportunity.

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