Imperial College London
Department of Mathematics
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Numerical Solution of Ordinary Differential Equations

I pledge that the work submitted for this coursework, both the report and the MATLAB code, is my own unassisted work unless stated otherwise.

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Coursework 1

Before you start working on this coursework, please read coursework guidelines. Fill in this page and include it as a cover sheet to your report, otherwise the coursework will not be marked. Any marks received for this coursework are only indicative and may be subject to moderation and scaling. *The mastery component is marked with a star.*

Exercise 1 (The Euler method for systems of ODEs) — % of course mark: /1.5

Solve the initial value problem

$$x'' + x' + 4x = t^2, \quad x(0) = 0, \quad x'(0) = 1, \quad t \in [0, 3] \tag{1}$$

with the Euler method and

a) Compute the numerical solution with the time steps $h = \{0.1, 0.05, 0.025\}$,

b) Compute the global error $|e_N|$ for $h = \{0.1, 0.05, 0.025\}$ at $t = 3$,

c) Find an approximated number of time steps such that $|e_N| < 10^{-4}$ at $t = 3$.

Exercise 2 (Taylor series methods for systems of ODEs) — % of course mark: /1.5

Solve the initial value problem

$$x'' + 3x' + 2x = t^2, \ x(0) = 1, \ x'(0) = 0, \ t \in [0, 2] \tag{2}$$

with the TS(3) method and

a) Compute the numerical solution with the time steps $h = \{0.1, 0.05, 0.025\}$,

b) Study how the global error at $t = 2$ depends on $h = \{0.1, 0.05, 0.025\}$.

Exercise 3 (Linear multistep methods for systems of ODEs) — % of course mark: /2.0

Solve the initial value problem (the Duffing oscillator)

$$\begin{aligned} &x'' + \delta x' + \beta x + \alpha x^3 = \gamma \cos(\omega t), \ x(0) = 0, \ x'(0) = 0, \ t \in [0, 100], \\ &\alpha = \omega = 1.0, \ \beta = 0.0, \ \delta = 0.05, \ \gamma = 0.3 \end{aligned} \tag{3}$$

with the two-step Adams-Bashforth (AB(2)) method and

a) Compute the numerical solution with the time steps $h = \{0.1, 0.05, 0.025\}$,

b) Explain how you start the AB(2) method.

Exercise 4 (**Linear multistep methods for systems of ODEs**) **% of course mark: $\boxed{}$ /2.0**★

Solve the initial value problem (Chua's circuit)

$$\begin{cases} x' = \alpha(y - ax^3 - cx), \\ y' = x - y + z, \\ z' = -\beta y - \gamma z, \end{cases} \tag{4}$$

$$x(0) = 1,\ y(0) = z(0) = 0,\ t = [0, 100],$$
$$\alpha = 10,\ \beta = 15,\ \gamma = 0.01,\ a = 0.1,\ c = -0.2,$$

with the three-step Adams-Bashforth (AB(3)) method

$$x_{n+1} = x_n + \frac{h}{12}\left(23f_n - 16f_{n-1} + 5f_{n-2}\right),$$

and

a) Compute the numerical solution with the time step $h = 0.01$,

b) Explain how you start the AB(3) method.

Coursework mark: $\boxed{}$ **% of course mark**

Coursework Guidelines

Below is a set of guidelines to help you understand what coursework is and how to improve it.

Coursework

- Coursework requires more than just following what has been done in lectures, some amount of individual work is expected.
- The coursework report should describe in a concise, clear and coherent way of what you did, how you did it, and what results you have.
- The report should be understandable to the reader with the mathematical background, but unfamiliar with your current work.
- Marks are not based solely on correctness. The results must be described and interpreted. The presentation and discussion is as important as the correctness of the results.
- Do not bloat the report by paraphrasing or presenting the results in different forms.
- Use high-quality and carefully constructed figures with captions and annotated axis, put figures where they belong.
- The maximum length of the report is ten A4-pages.
- Use tables only if they are more explanatory than figures. The maximum table length is half of a page.
- All figures and tables should be embedded in the report. The report should contain all discussions and explanations of the methods and algorithms, and interpretations of your results and further conclusions.
- The report should be typeset in LaTeX or Word Editor and submitted as a single pdf-file. The report should contain your name and CID. **The course work will not be marked without your name or CID.**
- Do not include any codes in the report.

Codes

- You cannot use third party numerical software or symbolic calculators in the coursework.
- The code you developed should be well-structured and organised, as well as properly commented to allow the reader to understand what the code does and how it works.
- All codes should run out of the box and require no modification to generate the results presented in the report.

Submission

- The coursework (report and codes) should be submitted as **one zip-file** via Blackboard by the due date. The file should have a name in the following format `CW#_FirstName_FamilyName_CID.zip`
- The file should contain **all MATLAB codes (m-files only)** used to generate the results presented in the report and **one pdf-file of the report itself**.
- Do not include any codes in the report.

Coursework 1 - Numerical Solutions to Ordinary Differential Equations

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In order to produce the plots and figures in this document, run the *exercise.m* files in the CW1 repository in MATLAB.

1 Exercise One (The Euler method for systems of ODEs)

The initial value problem is given as:

$$\frac{d^2x}{dx^2} + \frac{dx}{dt} + 4x = t^2,\ x(0) = 0,\ x'(0) = 1,\ t \in [0, 3] \tag{1}$$

When solving inhomogeneous second order differential equations analytically, the solution is given as $x = x_c + x_p$ where x_c is the solution to the problem:

$$a\frac{d^2x_c}{dt^2} + b\frac{dx_c}{dt} + cx_c = 0$$

and x_p to:

$$a\frac{d^2x_c}{dt^2} + b\frac{dx_c}{dt} + cx_c = f(t)$$

with $f(t)$ being of the form $\alpha_2 t^2 + \alpha_1 t + \alpha_0$ in this specific IVP.

To find x_c of equation (1), we use the auxiliary equation:

$$\lambda^2 + \lambda + 4 = 0,$$
$$\Rightarrow \lambda = \frac{-1 \pm \sqrt{1-16}}{2} = -\frac{1}{2} \pm \frac{i\sqrt{15}}{2}$$

The general solution for auxiliary equations of this form is:

$$x_c = e^{-\frac{t}{2}}(A\cos\frac{\sqrt{15t}}{2} + B\sin\frac{\sqrt{15t}}{2})$$

Now, since the right hand side is a polynomial, we must try:

$$x_p = Pt^2 + Qt + R,$$
$$\Rightarrow \frac{dx_p}{dt} = 2Pt + Q, \frac{d^2x_p}{dt^2} = 2P$$

Substituting these into equation (1) yields:

$$(2P) + (2Pt + Q) + 4(Pt^2 + Qt + R) = t^2,$$
$$\Rightarrow 4Pt^2 + (2P + 4Q)t + (2P + Q + 4R) = t^2$$

Equating coefficients gives P, Q, R the values of $\frac{1}{4}$, $\frac{-1}{8}$ and $\frac{-3}{32}$ respectively. Putting this all together, the general solution is:

$$x = e^{-\frac{t}{2}}(A\cos\frac{\sqrt{15}t}{2} + B\sin\frac{\sqrt{15}t}{2}) + \frac{1}{4}t^2 - \frac{1}{8}t - \frac{3}{32}$$

Plugging in the boundary conditions from equation (1) yields:

$$\begin{gathered}
x(0) = A - \frac{3}{32} = 0, \\
x'(0) = -\frac{1}{8} - \frac{A}{2} + \frac{\sqrt{15}B}{2} = 1 \\
\Rightarrow A = \frac{3}{32},\ B = \frac{75}{32\sqrt{15}}
\end{gathered}$$

$$x(t) = e^{-\frac{t}{2}}(\frac{3}{32}\cos\frac{\sqrt{15}t}{2} + \frac{75}{32\sqrt{15}}\sin\frac{\sqrt{15}t}{2}) + \frac{1}{4}t^2 - \frac{1}{8}t - \frac{3}{32} \tag{2}$$

This exact, analytical solution will be used in computing the global error attributed to the Euler method.

In order to implement the Euler method, IVP (1) needs to be turned into a system of first order differential equations by letting $u = x$ and $v = x'$. With the IVP, this gives us the following system of ordinary differential equations:

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} v \\ t^2 - v - 4u \end{pmatrix} = f$$

This then makes the Euler method for IVP (1):

$$\mathbf{y_{n+1}} = \mathbf{y_n} + h\mathbf{f_n}, \quad \mathbf{y_n} = \begin{pmatrix} u_n \\ v_n \end{pmatrix}, \quad \mathbf{f_n} = \begin{pmatrix} v_n \\ t_n^2 - v_n - 4u_n \end{pmatrix}, \quad \mathbf{y_0} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Below is a graph which illustrates the solutions of x using Euler's method using the time steps $h = 0.1, 0.05, 0.025$:

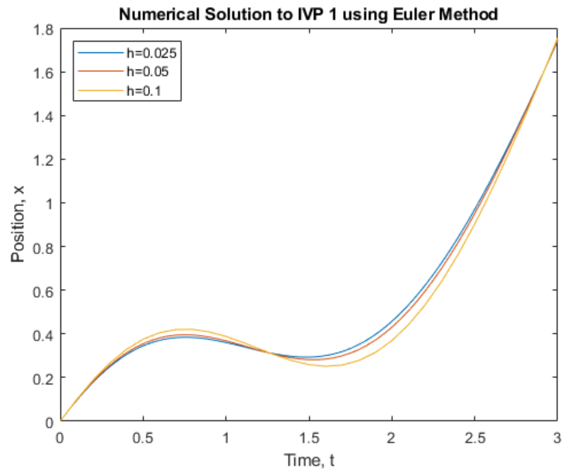

Figure 1: Numerical solution to (1) using different values of h and Euler's Method

All three numerical solutions follow the same shape, suggesting convergence to the actual solution. $h = 0.025$ is less oscillatory than $h = 0.1$ which is to be expected since the former has a smaller time step and is thus more accurate.

The global error of the numerical solutions at $t = 3$ is computed using the magnitude of the difference between the aforementioned analytical solution (2) (which is equivalent to $x(3)$) and the solution visible in the above figure.

The global errors, to 3 significant figures, are as follows:

$$h = 0.025 : |e_n| = 0.00322$$
$$h = 0.05 : |e_n| = 0.00721$$
$$h = 0.1 : |e_n| = 0.0168$$

Looking at the above errors, the constant of proportionality, $\frac{|e_n|}{h}$ does not appear to be very constant among the three different global errors.

Therefore, in order to obtain the first value for which $|e_n| < 10^{-4}$ we will plot the global error against the number of time steps.

Since $t = 3$, in order to obtain the h values which give rise to a whole number of time steps, we make use of the fact that any fraction of the form $\frac{3}{n}$ (where n is a positive integer) will produce a solution with exactly n time steps. Below is the graph which shows how the global error changes with the number of time steps. It is worth noting that even the exact solution as per MATLAB has an error itself and consequently the global error that is computed is subject to an intrinsic error:

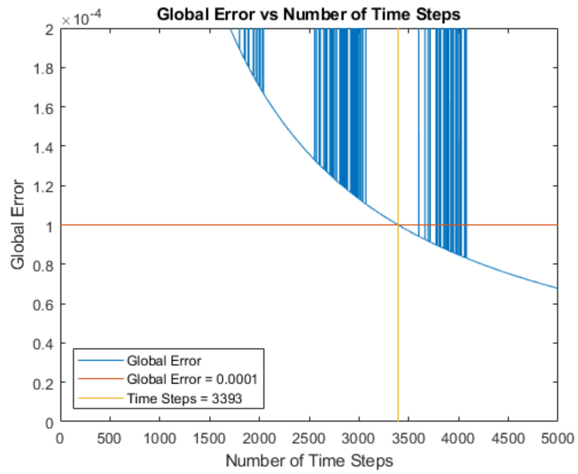

Figure 2: Global Error Vs Number of Time Steps

As can be seen from the figure, the first value which causes a global error below 10^{-4} is when the number of time steps is 3393. This however, is only the number of time steps according to MATLAB's internal calculator. It is worthy of note that there are fluctuations above 10^{-4} beyond 3393 and this will be down to the small errors in MATLAB's in-built functions when calculating expressions that include sin, cosine and the exponential function. Taking this into account, the approximate number of time steps that causes a global error of below 10^{-4} is 3500.

From what we know in theory, this global error should be proportional to the step size and so, asymptotically (as h approaches 0), this graph should become linear.

2 Exercise Two (Taylor series methods for systems of ODEs)

Like with Exercise One, we proceed by initially working out the analytical solution to the IVP:

$$\frac{d^2x}{dx^2} + 3\frac{dx}{dt} + 2x = t^2,\ x(0) = 1,\ x'(0) = 0,\ t \in [0,2] \tag{3}$$

We use the auxiliary equation:

$$\lambda^2 + 3\lambda + 2 = 0,$$
$$\Rightarrow \lambda = -1, -2$$

The general solution for auxiliary equations of this form is:

$$x_c = Ae^{-t} + Be^{-2t}$$

Considering x_p:

$$(2P) + 3(2Pt + Q) + 2(Pt^2 + Qt + R) = t^2,$$
$$\Rightarrow 2Pt^2 + (6P + 2Q)t + (2P + 3Q + 2R) = t^2$$

Equating coefficients gives P, Q, R the values of $\frac{1}{2}$, $\frac{-3}{2}$ and $\frac{7}{4}$ respectively.

Plugging in the boundary conditions from equation (3) and combining x_c and x_p yields:

$$x(0) = A + B + \frac{7}{4} = 1,$$
$$x'(0) = -A - 2B - \frac{3}{2} = 0$$
$$\Rightarrow A = 0,\ B = -\frac{3}{4}$$

$$\Rightarrow x(t) = \frac{1}{2}t^2 - \frac{3}{2}t + \frac{7}{4} - \frac{3}{4}e^{-2t} \tag{4}$$

In order to implement the Taylor Series 3 method, IVP (3) needs to be turned into a system of first order differential equations by letting $u = x$ and $v = x'$. With the IVP, this gives us the following system of ordinary differential equations:

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} v \\ t^2 - 3v - 2u \end{pmatrix} = f$$

The TS(3) Method for the IVP (3) is:

$$\mathbf{y_{n+1}} = \mathbf{y_n} + h\mathbf{f_n} + \frac{h^2}{2}\mathbf{f_n'} + \frac{h^3}{6}\mathbf{f_n''}$$

$$\mathbf{y_n} = \begin{pmatrix} u_n \\ v_n \end{pmatrix}, \quad \mathbf{f_n} = \begin{pmatrix} v_n \\ t_n^2 - 3v_n - 2u_n \end{pmatrix}, \quad \mathbf{y_0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\mathbf{f_n'} = \begin{pmatrix} v_n' \\ 2t_n - 3v_n' - 2u_n' \end{pmatrix} = \begin{pmatrix} t_n^2 - 3v_n - 2u_n \\ 2t_n - 3t_n^2 + 7v_n + 6u_n \end{pmatrix}$$

$$\mathbf{f_n''} = \begin{pmatrix} 2t_n - 3t_n^2 + 7v_n + 6u_n \\ 7t_n^2 - 6t_n + 2 - 15v_n - 14u_n \end{pmatrix}$$

Below is a graph which illustrates the solutions of x using the TS(3) method using the time steps $h = 0.1, 0.05, 0.025$:

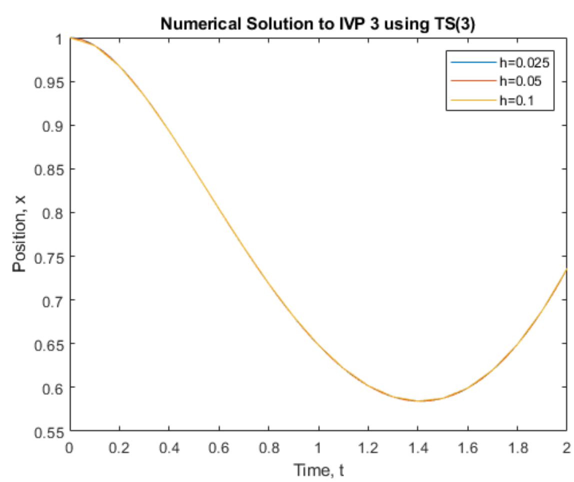

Figure 3: Numerical solution to (3) using different values of h and TS(3)

As can be seen from figure (3), all solutions effectively overlap, which suggests a high rate of convergence to the true solution.

Using the analytical solution (4), the global errors, to 3 significant figures, are as follows:

$$h = 0.025 : |e_n| = 2.98 * 10^{-7}$$
$$h = 0.05 : |e_n| = 2.48 * 10^{-6}$$
$$h = 0.1 : |e_n| = 2.15 * 10^{-5}$$

The global error of the TS(3) method should be proportional to the step size cubed and so, asymptotically (as h approaches 0), this graph should become cubic.

This cubic relationship can be discerned from the above errors since halving the step size does appear to reduce the global error by a factor that is very close to 8 (which is 2^3).

3 Exercise Three (Linear multistep methods for systems of ODEs)

The IVP is as follows:

$$\frac{d^2x}{dx^2} + \delta\frac{dx}{dt} + \beta x + \alpha x^3 = \gamma cos(\omega t),\ x(0) = 0,\ x'(0) = 0,\ t \in [0, 100] \tag{5}$$

$$\alpha = \omega = 1.0, \beta = 0, \delta = 0.05, \gamma = 0.3$$

In order to implement the two-step Adams-Bashforth (AB(2)) method, IVP (5) needs to be turned into a system of first order differential equations by letting $u = x$ and $v = x'$. With the IVP, this gives us the following system of ordinary differential equations:

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} v \\ 0.3cos(t) - 0.05v - u^3 \end{pmatrix} = f$$

The two-step Adams-Bashforth (AB(2)) method for IVP (5) is:

$$\mathbf{y_{n+2}} = \mathbf{y_{n+1}} + \frac{h}{2}(3\mathbf{f_{n+1}} - \mathbf{f_n})$$

$$\mathbf{y_n} = \begin{pmatrix} u_n \\ v_n \end{pmatrix}, \quad \mathbf{f_n} = \begin{pmatrix} v_n \\ 0.3cos(t_n) - 0.05v_n - u_n^3 \end{pmatrix}, \quad \mathbf{y_0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This method is contingent on knowing both $\mathbf{y_0}$ and $\mathbf{y_1}$. The IVP only gives $\mathbf{y_0}$ and so in order to start the AB(2) method, the following Euler method is used to derive $\mathbf{y_1}$:

$$\mathbf{y_1} = \mathbf{y_0} + h\mathbf{f_0} = \begin{pmatrix} 0 \\ 0.3h \end{pmatrix}$$

Below is a graph which illustrates the solutions of x using the AB(2) method using the time steps $h = 0.1, 0.05, 0.025$:

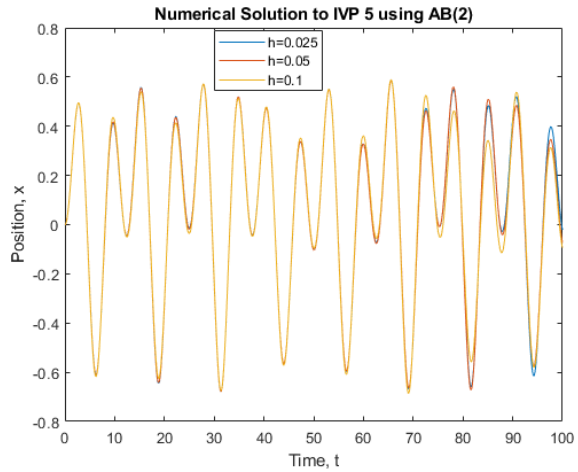

Figure 4: Numerical solution to (5) using different values of h and AB(2)

There is significant overlap of the solutions in this figure, suggesting a strong convergence to the true solution as h approaches 0. For AB(2), the method approaches the true solution at a quadratic rate (of the order $O(h^2)$).

4 Exercise Four (Linear multistep methods for systems of ODEs)

The IVP is as follows:

$$\begin{cases} x' = \alpha(y - ax^3 - cx), \\ y' = x - y + z \\ z' = -\beta y - \gamma z \end{cases}$$

$$x(0) = 1,\ y(0) = 0,\ z(0) = 0,\ t \in [0, 100] \tag{6}$$

$$\alpha = 10, \beta = 15, \gamma = 0.01, a = 0.1, c = -0.2$$

The three-step Adams-Bashforth (AB(3)) method for IVP (6) is:

$$\mathbf{x_{n+3}} = \mathbf{x_{n+2}} + \frac{h}{12}(23\mathbf{f_{n+2}} - 16\mathbf{f_{n+1}} + 5\mathbf{f_n})$$

$$\mathbf{x_n} = \begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix}, \quad \mathbf{f_n} = \begin{pmatrix} 10(y_n - 0.1x_n^3 + 0.2x_n) \\ x_n - y_n + z_n \\ -15y_n - 0.01z_n \end{pmatrix}, \quad \mathbf{x_0} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

This method is contingent on knowing $\mathbf{x_0}$, $\mathbf{x_1}$, and $\mathbf{x_2}$. The IVP only gives $\mathbf{x_0}$ and so in order to start the AB(3) method, the Euler method is used to find $\mathbf{x_1}$ and AB(2) is used to find $\mathbf{x_2}$. The Euler method could have been used in order to find both $\mathbf{x_1}$, and $\mathbf{x_2}$, however AB(2) is more accurate and so this is used in the second step.

The Euler method used is:

$$\mathbf{x_1} = \mathbf{x_0} + h\mathbf{f_0}$$

And the AB(2) method used is:

$$\mathbf{x_2} = \mathbf{x_1} + \frac{h}{2}(3\mathbf{f_1} - \mathbf{f_0})$$

where we use the $\mathbf{x_1}$ derived from the Euler method.

Below is a plot which illustrates the solutions of $\mathbf{x}$ using the AB(3) method and the time step $h = 0.01$:

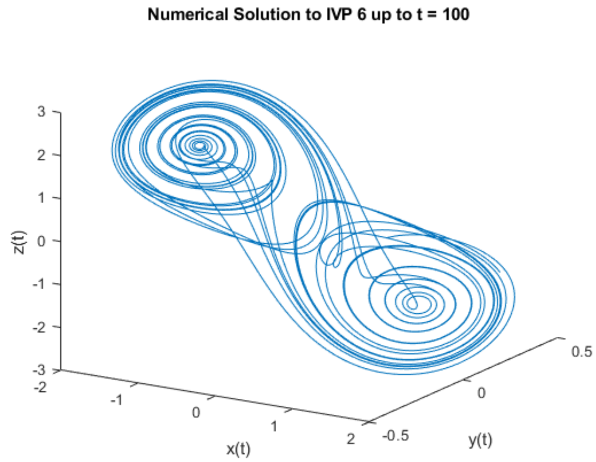

Figure 5: Numerical solution to (6) using h = 0.01 and AB(3)

Below is a figure of a computer simulation of Chua's circuit after 100 seconds, showing the chaotic "double scroll" attractor pattern which has been obtained online from an author called Shiyu Ji.

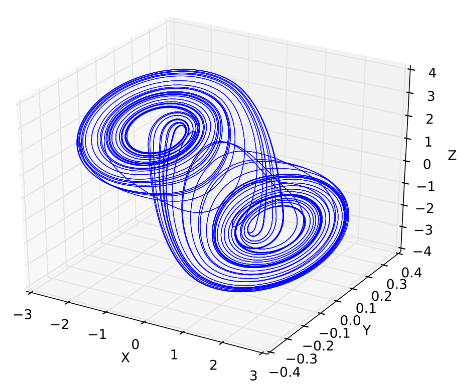

Figure 6: Computer Simulation of Chua's circuit after 100 seconds

As can be seen, figure 5's solution bears much resemblance to the computer simulation of Chua's circuit, corroborating the validity of the AB(3) method for computing a numerical solution.