Imperial College London
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Numerical Solution of Ordinary Differential Equations

I pledge that the work submitted for this coursework, both the report and the MATLAB code, is my own unassisted work unless stated otherwise.

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Coursework 2

Before you start working on this coursework, please read coursework guidelines. Fill in this page and include it as a cover sheet to your report, otherwise the coursework will not be marked. Any marks received for this coursework are only indicative and may be subject to moderation and scaling. *The mastery component is marked with a star.*

Exercise 1 (Explicit LMM) **% of course mark: /2.5**

Develop an explicit 3-step convergent LMM of your own design with the highest order of consistency, p, you can achieve.

Exercise 2 (Implicit LMM) **% of course mark: /2.5**

Develop an implicit 3-step convergent LMM of your own design with the same order of consistency p.

Exercise 3 (Explicit and Implicit LMMs) **% of course mark: /4.0**

Solve the initial value problem

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 998 & -999 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 2\sin(t) \\ 999(\cos(t) - \sin(t)) \end{pmatrix}, \quad \begin{pmatrix} x_1(t_0) \\ x_2(t_0) \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad t = [0, 20] \tag{1}$$

with the explicit and implicit LMM you developed.

Exercise 4 (Global error and LMMs) **% of course mark: /3.0**

Theoretically estimate the global error of the explicit and implicit LMM, and compare it with the global error computed numerically for system (1).

Exercise 5 (Global error and LMMs) **% of course mark: /3.0★**

Theoretically estimate the global error of the AB(3) method from Coursework 1, and compare it with the global error computed numerically for system (1).

Coursework mark: % of course mark

 In hand date: 23:59 GMT, 30 Oct 2020 **Due date: 23:59 GMT, 13 Nov 2020**

In order to produce the plots and figures in this document, run the *exercise.m* files in the CW2 repository in MATLAB.

1 Exercise One (Explicit LMM)

A general linear multi-step method of order 3 can be written in the following form:

$$\sum_{m=0}^{3} \alpha_m x_{n+m} = h \sum_{m=0}^{3} \beta_m f_{n+m}, \ \alpha_m, \beta_m \in \mathbb{R}, \ \alpha_3 = 1$$

To find coefficients for which the consistency is maximised, we start by considering the linear difference operator of the general linear multi-step method of order 3:

$$\mathcal{L}_h z(t) = \sum_{m=0}^{3} \alpha_m z(t+mh) - h \sum_{m=0}^{3} \beta_m z'(t+mh)$$

Using the Taylor expansion method and grouping coefficients, we obtain the following:

$$\mathcal{L}_h z(t) = (1+\alpha_2+\alpha_1+\alpha_0)z(t) + h(3+2\alpha_2+\alpha_1-(\beta_3+\beta_2+\beta_1+\beta_0))z'(t)$$
$$+h^2(\frac{9}{2}+2\alpha_2+\frac{1}{2}\alpha_1-(3\beta_3+2\beta_2+\beta_1))z''(t) + h^3(\frac{9}{2}+\frac{4}{3}\alpha_2+\frac{1}{6}\alpha_1-(\frac{9}{2}\beta_3+2\beta_2+\frac{1}{2}\beta_1))z^{(3)}(t)+$$
$$h^4(\frac{27}{8}+\frac{2}{3}\alpha_2+\frac{1}{24}\alpha_1-(\frac{9}{2}\beta_3+\frac{4}{3}\beta_2+\frac{1}{6}\beta_1))z^{(4)}(t)+O(h^5)$$

We require, in ascending order, as many of the coefficients of $z^{(m)}(t)$ to be 0 as possible and this is obtained by solving the set of equations such that the coefficients are 0. As the LMM in question is explicit the condition that $\beta_3 = 0$ needs to be added, giving:

$$\mathcal{L}_h z(t) = (1+\alpha_2+\alpha_1+\alpha_0)z(t) + h(3+2\alpha_2+\alpha_1-(\beta_2+\beta_1+\beta_0))z'(t)$$
$$+h^2(\frac{9}{2}+2\alpha_2+\frac{1}{2}\alpha_1-(2\beta_2+\beta_1))z''(t) + h^3(\frac{9}{2}+\frac{4}{3}\alpha_2+\frac{1}{6}\alpha_1-(2\beta_2+\frac{1}{2}\beta_1))z^{(3)}(t)+$$
$$h^4(\frac{27}{8}+\frac{2}{3}\alpha_2+\frac{1}{24}\alpha_1-(\frac{4}{3}\beta_2+\frac{1}{6}\beta_1))z^{(4)}(t)+O(h^5)$$

This gives us the following set of equations to solve:

$$\begin{cases} 1+\alpha_2+\alpha_1+\alpha_0=0 \\ 3+2\alpha_2+\alpha_1=(\beta_2+\beta_1+\beta_0) \\ \frac{9}{2}+2\alpha_2+\frac{1}{2}\alpha_1=(2\beta_2+\beta_1) \\ \frac{9}{2}+\frac{4}{3}\alpha_2+\frac{1}{6}\alpha_1=(2\beta_2+\frac{1}{2}\beta_1) \\ \frac{27}{8}+\frac{2}{3}\alpha_2+\frac{1}{24}\alpha_1=(\frac{4}{3}\beta_2+\frac{1}{6}\beta_1) \end{cases}$$

Solving the first and second equation ensures that consistency of order 1 is met.

Suppose that the α values are all predetermined, since this system of equations requires choices of some coefficients to be made in order that unique solutions are derived.

We are now left to solve for β values. The 2nd, 3rd and 4th equations give unique solutions for $\beta_2, \beta_1, \beta_0$. Using, the 3rd and 4th equation to compute the β_2 and β_1 value gives:

$$\beta_1 = \frac{4}{3}\alpha_2 + \frac{2}{3}\alpha_1,$$
$$\beta_2 = \frac{9}{4} + \frac{1}{3}\alpha_2 - \frac{1}{12}\alpha_1$$

Substituting this into the last equation in the above system does not give $\frac{27}{8}+\frac{2}{3}\alpha_2+\frac{1}{24}\alpha_1$, and so the highest order than can be obtained with a 3 step LMM that is explicit is $p = 3$.

To derive our own explicit 3-step LMM, we start by choosing our alphas. The choice made in this instance is as follows:

$$\alpha_2 = -\frac{9}{10},\ \alpha_1 = -\frac{1}{10},\ \alpha_0 = 0$$

Plugging these alpha values into the above system of equations gives the following beta values:

$$\beta_2 = -\frac{19}{15},\ \beta_1 = \frac{47}{24},\ \beta_0 = \frac{49}{120}$$

And thus, a 3 step LMM method has been derived. Now that remains is to check if it is convergent.

To test if the derived LMM is convergent, we turn to the Dahlquist Equivalence Theorem which states that:

"An LMM is convergent iff it is consistent and zero-stable"

We know that the LMM is consistent, as indeed it has consistency of the order $p = 3$.

Considering the zero-stability, if the roots of the characteristic polynomial, ρ, all have modulus less than or equal to 1 and the roots of modulus 1 are of multiplicity 1, we say that the root condition is satisfied. A linear multistep method is zero-stable if and only if the root condition is satisfied.

The characteristic polynomial of the derived LMM is:

$$\rho(r) = r^3 - \frac{9}{10}r^2 - \frac{1}{10}r$$

The roots of this polynomial are $r = 0, \frac{0.9 \pm \sqrt{0.41}}{2}$, all of whose moduli are less than 1. Ergo, the root condition is satisfied, and thus the LMM is zero-stable. Ergo, this LMM is indeed convergent and we have successfully derived a 3 step, convergent, explicit LMM.

2 Exercise Two (Implicit LMM)

Here, we proceed as before except that we change the constraint to that of $\beta_3 \neq 0$. We will use the same alpha values as in exercise 1, since the characteristic polynomial is the same, and thus provided the system is consistent, the method will indeed be convergent. We seek that the LMM has consistency of $p = 3$.

A result proved by the illustrious mathematician Germund Dahlquist, was that given an implicit, zero-stable and linear q-step multistep method, one cannot attain an order of consistency greater than $q+1$ if q is odd. This result is called "Dahlquist's First Barrier". Here, q is 3 and so it is odd. This means that the highest order of consistency is 4, and so in order to make sure the consistency stays at 3, we will set the coefficient of h^4 to be non-zero, say 1. The linear difference operator of an implicit 3-step LMM is:

$$\mathcal{L}_h z(t) = (1+\alpha_2+\alpha_1+\alpha_0)z(t) + h(3+2\alpha_2+\alpha_1-(\beta_3+\beta_2+\beta_1+\beta_0))z'(t)$$
$$+h^2(\frac{9}{2}+2\alpha_2+\frac{1}{2}\alpha_1-(3\beta_3+2\beta_2+\beta_1))z''(t)+h^3(\frac{9}{2}+\frac{4}{3}\alpha_2+\frac{1}{6}\alpha_1-(\frac{9}{2}\beta_3+2\beta_2+\frac{1}{2}\beta_1))z^{(3)}(t)+$$
$$h^4(\frac{27}{8}+\frac{2}{3}\alpha_2+\frac{1}{24}\alpha_1-(\frac{9}{2}\beta_3+\frac{4}{3}\beta_2+\frac{1}{6}\beta_1))z^{(4)}(t)+O(h^5)$$

This gives us the following set of equations to solve:

$$\begin{cases} 1+\alpha_2+\alpha_1+\alpha_0 = 0 \\ 3+2\alpha_2+\alpha_1 = (\beta_3+\beta_2+\beta_1+\beta_0) \\ \frac{9}{2}+2\alpha_2+\frac{1}{2}\alpha_1 = (3\beta_3+2\beta_2+\beta_1) \\ \frac{9}{2}+\frac{4}{3}\alpha_2+\frac{1}{6}\alpha_1 = (\frac{9}{2}\beta_3+2\beta_2+\frac{1}{2}\beta_1) \\ \frac{27}{8}+\frac{2}{3}\alpha_2+\frac{1}{24}\alpha_1 = (\frac{9}{2}\beta_3+\frac{4}{3}\beta_2+\frac{1}{6}\beta_1)+1 \end{cases}$$

This set of equations makes all coefficients up to and including h^3, 0, and the coefficient of h^4, 1.

The same alphas as in exercise 1 were used. In order to compute β_1,β_2 and β_3 the last three equations in the above system were used and MATLAB was used in order to solve the system $Ax = b$ by finding the inverse of A:

$$\begin{pmatrix} 1 & 2 & 3 \\ \frac{1}{2} & 2 & \frac{9}{2} \\ \frac{1}{6} & \frac{4}{3} & \frac{9}{2} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} \frac{53}{20} \\ \frac{197}{60} \\ \frac{85}{48} \end{pmatrix}$$
$$\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ \frac{1}{2} & 2 & \frac{9}{2} \\ \frac{1}{6} & \frac{4}{3} & \frac{9}{2} \end{pmatrix}^{-1} \begin{pmatrix} \frac{53}{20} \\ \frac{197}{60} \\ \frac{85}{48} \end{pmatrix} = \begin{pmatrix} -\frac{757}{240} \\ \frac{923}{240} \\ -\frac{151}{240} \end{pmatrix}$$
$$\beta_0 = 3+2\alpha_2+\alpha_1-(\beta_3+\beta_2+\beta_1) = \frac{83}{80}$$

With the coefficients obtained, we now have a 3 step, convergent, implicit LMM.

3 Exercise Three (Explicit and Implicit LMMs)

Using the methods we have now derived, we now seek to solve the following IVP:

$$\begin{pmatrix} u'(t) \\ v'(t) \end{pmatrix} = \begin{pmatrix} -2u(t) + v(t) + 2sin(t) \\ 998u(t) - 999v(t) + 999(cos(t) - sin(t)) \end{pmatrix}, \tag{1}$$

$$\begin{pmatrix} u(0) \\ v(0) \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \; t \in [0, 20]$$

We make use of the following notation for implementing the derived methods:

$$\mathbf{x_n} = \begin{pmatrix} u_n \\ v_n \end{pmatrix}, \; \mathbf{f_n} = \begin{pmatrix} -2u_n + v_n + 2sin(t_n) \\ 998u_n - 999v_n + 999(cos(t_n) - sin(t_n)) \end{pmatrix}$$

The methods are contingent on knowing $\mathbf{x_0}$, $\mathbf{x_1}$, and $\mathbf{x_2}$. The IVP only gives $\mathbf{x_0}$ and so in order to start the 3 step LMM methods, the Euler method is used to find $\mathbf{x_1}$ and AB(2) is used to find $\mathbf{x_2}$. The Euler method could have been used in order to find both $\mathbf{x_1}$, and $\mathbf{x_2}$, however AB(2) is more accurate and so this is used in the second step. Both the Euler method and AB(2) method are convergent, so these are appropriate methods to use.

The Euler method used is:

$$\mathbf{x_1} = \mathbf{x_0} + h\mathbf{f_0}$$

And the AB(2) method used is:

$$\mathbf{x_2} = \mathbf{x_1} + \frac{h}{2}(3\mathbf{f_1} - \mathbf{f_0})$$

where we use the $\mathbf{x_1}$ derived from the Euler method.

3.1 Explicit Method

The explicit method for the IVP (1) is:

$$\mathbf{x_{n+3}} = \frac{9}{10}\mathbf{x_{n+2}} + \frac{1}{10}\mathbf{x_{n+1}} + h(-\frac{19}{15}\mathbf{f_{n+2}} + \frac{47}{24}\mathbf{f_{n+1}} + \frac{49}{120}\mathbf{f_n})$$

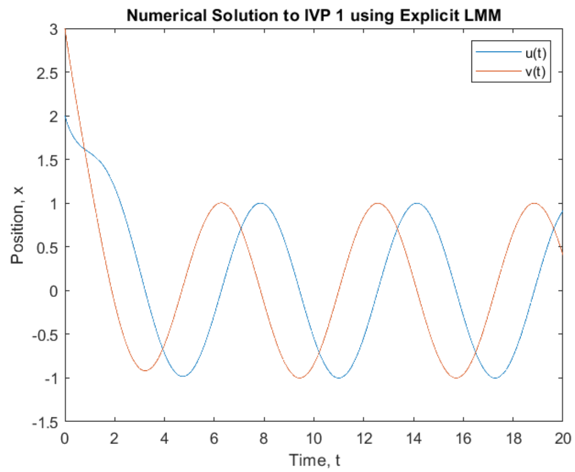

Figure 1: Numerical solution to (1) using h = 0.0001 and Explicit 3-step LMM

3.2 Implicit Method

The implicit method for the IVP (1) is:

$$\mathbf{x_{n+3}} = \frac{9}{10}\mathbf{x_{n+2}} + \frac{1}{10}\mathbf{x_{n+1}} + h(-\frac{151}{240}\mathbf{f_{n+3}} + \frac{923}{240}\mathbf{f_{n+2}} - \frac{757}{240}\mathbf{f_{n+1}} + \frac{83}{80}\mathbf{f_{n}})$$

In order to obtain a solution for $\mathbf{x_{n+3}}$, some re-arranging is needed and this requires that $\mathbf{f_{n+3}}$ is expanded out:

$$\mathbf{x_{n+3}} = \frac{9}{10}\mathbf{x_{n+2}} + \frac{1}{10}\mathbf{x_{n+1}} - \frac{151h}{240}\begin{pmatrix} -2u_{n+3} + v_{n+3} + 2sin(t_{n+3}) \\ 998u_{n+3} - 999v_{n+3} + 999(cos(t_{n+3}) - sin(t_{n+3})) \end{pmatrix}$$
$$+h(\frac{923}{240}\mathbf{f_{n+2}} - \frac{757}{240}\mathbf{f_{n+1}} + \frac{83}{80}\mathbf{f_{n}}) = \begin{pmatrix} u_{n+3} \\ v_{n+3} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} u_{n+3} + \frac{151h}{240}(-2u_{n+3} + v_{n+3}) \\ v_{n+3} + \frac{151h}{240}(998u_{n+3} - 999v_{n+3}) \end{pmatrix} = \frac{9}{10}\mathbf{x_{n+2}} + \frac{1}{10}\mathbf{x_{n+1}} - \frac{151h}{240}\begin{pmatrix} 2sin(t_{n+3}) \\ 999(cos(t_{n+3}) - sin(t_{n+3})) \end{pmatrix}$$
$$+h(\frac{923}{240}\mathbf{f_{n+2}} - \frac{757}{240}\mathbf{f_{n+1}} + \frac{83}{80}\mathbf{f_{n}}) = \begin{pmatrix} a \\ b \end{pmatrix}$$

To avoid the rearranging becoming unkempt, we will call the RHS of the above $\begin{pmatrix} a \\ b \end{pmatrix}$ in order that the rearranging becomes clearer. Rearranging the above and solving the LHS set of simultaneous equations give the following method:

$$v_{n+3} = \left(\frac{b - \frac{(\frac{998*151ah}{240})}{(1-\frac{151h}{120})}}{1 - \frac{999*151h}{240} - \frac{998(\frac{151h}{240})^2}{(1-\frac{151h}{120})}}\right)$$

$$u_{n+3} = \left(\frac{a - \frac{151h}{240}v_{n+3}}{1 - \frac{151h}{120}}\right)$$

And this has now all been rearranged in terms of past values of u and v, thus making the method implementable and numerically solvable.

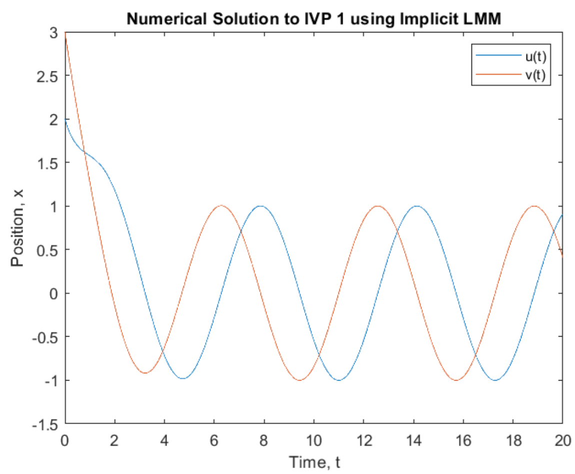

Figure 2: Numerical solution to (1) using h = 0.0001 and Implicit 3-step LMM

As can be seen from both figures, the numerical solutions are seemingly identical for the same h value, suggesting strong convergence to the original solution and corroborating the validity of both methods.

4 Exercise Four (Global error and LMMs)

In order to compute the global error of methods, we first require the analytical solution to the IVP. The analytical solution requires us to find the complementary function by solving the corresponding homogeneous system of ODEs and then the particular integral that satisfies the full system of ODEs,

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} + g(t) = \mathbf{x}_{CF} + \mathbf{x}_{PI}$$

where $A = \begin{pmatrix} -2 & 1 \\ 998 & -999 \end{pmatrix}$ and $g(t) = \begin{pmatrix} 2sin(t) \\ 999(cos(t) - sin(t)) \end{pmatrix}$.

Solving for $\mathbf{x}_{CF}$, we consider the eigenvalues of A:

$$\begin{aligned} det(A - \lambda I) &= \lambda^2 + 1001\lambda + 1000 = 0, \\ &\Rightarrow (\lambda + 1)(\lambda + 1000) = 0, \\ &\Rightarrow \lambda = -1, -1000 \end{aligned}$$

This gives the eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ -998 \end{pmatrix}$ for the above eigenvalues respectively.

Thus, the general solution for the complementary function is:

$$\mathbf{x}_{CF} = c_1 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-1000t} \begin{pmatrix} 1 \\ -998 \end{pmatrix}$$

To compute the particular integral, we start with the ansatz:

$$\mathbf{x}_{PI} = \begin{pmatrix} Asin(t) + Bcos(t) \\ Csin(t) + Dcos(t) \end{pmatrix}$$

Conveniently, the setup of IVP(1) and its coefficients are such that an ansatz can easily be seen. By inspection, the particular integral is simply:

$$\begin{aligned} \mathbf{x}_{PI} &= \begin{pmatrix} sin(t) \\ cos(t) \end{pmatrix}, \\ \Rightarrow \mathbf{x} &= c_1 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-1000t} \begin{pmatrix} 1 \\ -998 \end{pmatrix} + \begin{pmatrix} sin(t) \\ cos(t) \end{pmatrix} \end{aligned}$$

Considering the boundary conditions of IVP(1), it follows that:

$$\begin{aligned} c_1 + c_2 &= 2, \\ c_1 - 998c_2 + 1 &= 3, \\ \Rightarrow c_1 &= 2,\ c_2 = 0 \end{aligned}$$

Thus, the analytical solution of IVP (1) is:

$$\mathbf{x} = 2e^{-t}\begin{pmatrix}1\\1\end{pmatrix} + \begin{pmatrix}sin(t)\\cos(t)\end{pmatrix}$$

With the analytical solution now obtained, we now theoretically estimate the global errors of the derived linear multi-step methods. The derivation below will work with the assumption that $\alpha_0 = 0$ and that the number of steps is 3. Save for this, this derivation is as generalised as possible.

Adjusting the indices, the 3-step LMM method with the above constraints is given by:

$$x_{n+1} = -\alpha_2 x_n - \alpha_1 x_{n-1} + h(\beta_3 f_{n+1} + \beta_2 f_n + \beta_1 f_{n-1} + \beta_0 f_{n-2})$$

This makes the global error at time, t_{n+1}:

$$e_{n+1} = x(t_{n+1}) - x_{n+1} = -\alpha_2(x(t_n) - x_n) - \alpha_1(x(t_{n-1}) - x_{n-1}) + h\beta_3(f(t_{n+1}, x(t_{n+1})) - f_{n+1})$$
$$+h\beta_2(f(t_n, x(t_n)) - f_n) + h\beta_1(f(t_{n-1}, x(t_{n-1})) - f_{n-1}) + h\beta_0(f(t_{n-2}, x(t_{n-2})) - f_{n-2}) + R_{n+1}$$

where $R_{n+1} = O(h^p)$ with p being the order of convergence of the local truncation error.

The IVP with which we are working is continuously differentiable which by implication means that it is Lipschitz continuous.

i.e. $\exists L > 0$ s.t. $\forall t \in [t_0, t_N]$, $|f(t_n, x(t_n)) - f_n| \leq L|x(t_n) - x_n|$

Using this information, we can now establish bounds on e_{n+1} by taking the modulus of the above expression, using the triangle inequality and letting $\hat{h} = hL$:

$$|e_{n+1}| \leq |\alpha_2 e_n| + |\alpha_1 e_{n-1}| + \hat{h}(|\beta_3 e_{n+1}| + |\beta_2 e_n| + |\beta_1 e_{n-1}| + |\beta_0 e_{n-2}|) + |R_{n+1}|$$
$$\Rightarrow |e_{n+1}| \leq \tilde{A}^{-1}(|\alpha_2 e_n| + |\alpha_1 e_{n-1}| + \hat{h}(|\beta_2 e_n| + |\beta_1 e_{n-1}| + |\beta_0 e_{n-2}|) + |R_{n+1}|), \tilde{A} := (1 - |\beta_3|\hat{h})$$

Introducing the error bounding function $\delta_n = \max |e_i|$, $0 \leq i \leq n$, $n \in [0, N]$ and rewriting the above in equality in terms of δ_n gives:

$$\delta_{n+1} \leq A\delta_n + \tilde{A}^{-1}|R_{n+1}|,$$
$$A := (|\alpha_2| + |\alpha_1| + \hat{h}(|\beta_2| + |\beta_1| + |\beta_0|))\tilde{A}^{-1} = (1 + \hat{h}(|\beta_2| + |\beta_1| + |\beta_0|))\tilde{A}^{-1}$$

with the last line coming from the fact that in all LMMs that are dealt with in this project, $|\alpha_2| + |\alpha_1| = 1$.

For $n = 0, 1, 2, ..., N$, we find that:

$$\delta_N \leq \sum_{i=0}^{N} A^{N-i} |R_i| \tilde{A}^{-1}$$

The error in the initial condition is zero, since the initial condition is known exactly and $|R_i| \leq Ch^p$ for some positive constant C since it represents the local truncation error and its degree.

This means we can rewrite the error bounding function as:

$$\delta_N \leq \sum_{i=0}^{N} A^{N-i} Ch^p \tilde{A}^{-1}$$

We make use of the geometric series property and then update the error bounding function:

$$1 + A + A^2 + ... + A^{N-1} = \frac{A^N - 1}{A - 1}$$
$$\Rightarrow \delta_N \leq \frac{A^N - 1}{A - 1} Ch^p \tilde{A}^{-1}$$

A can be rewritten as:

$$A := 1 + \hat{h}(|\beta_3| + |\beta_2| + |\beta_1| + |\beta_0|)\tilde{A}^{-1}$$

This allows us to make use of the following inequality:

$$A \leq e^{\hat{h}(|\beta_3|+|\beta_2|+|\beta_1|+|\beta_0|)\tilde{A}^{-1}}$$
$$\Rightarrow A^N \leq e^{\hat{h}(|\beta_3|+|\beta_2|+|\beta_1|+|\beta_0|)\tilde{A}^{-1}N}$$
$$\Rightarrow A^N \leq e^{t_N L(|\beta_3|+|\beta_2|+|\beta_1|+|\beta_0|)\tilde{A}^{-1}}$$

Substituting this into the error bounding function yields:

$$\delta_N \leq Bh^{p-1}$$

where $B{:=}C\frac{e^{t_N L(|\beta_3|+|\beta_2|+|\beta_1|+|\beta_0|)\tilde{A}^{-1}}-1}{L(|\beta_3|+|\beta_2|+|\beta_1|+|\beta_0|)}$.

Thus we have shown that provided $|\alpha_2| + |\alpha_1| = 1$ and $\alpha_0 = 0$, a 3 step LMM converges to the exact solution with order $p - 1$, where p is the order of the local truncation error.

To work out the order of the local truncation error of each of the methods, we consider the Taylor expansion of the 3 step LMM (with $\alpha_0 = 0$):

$$(-\alpha_2 - \alpha_1)x(t) + h(\beta_3 + \beta_2 + \beta_1 + \beta_0 + \alpha_1)x'(t) + h^2(-\frac{1}{2}\alpha_1 + \beta_3 - \beta_1 - 2\beta_0)x''(t)$$
$$+h^3(\frac{1}{2}\beta_3 + \frac{1}{2}\beta_1 + 2\beta_0 + \frac{1}{6}\alpha_1)x^{(3)}(t) + h^4(\frac{1}{6}\beta_3 - \frac{1}{6}\beta_1 - \frac{4}{3}\beta_0 - \frac{1}{24}\alpha_1)x^{(4)}(t)$$
$$+h^5(\frac{1}{24}\beta_3 + \frac{1}{24}\beta_1 + \frac{2}{3}\beta_0 + \frac{1}{120}\alpha_1)x^{(5)}(t) + O(h^6)$$

4.1 Explicit Method

Plugging in the alphas and betas of our explicit method into the Taylor expansion gives:

$$x(t) + hx'(t) - \frac{109h^2}{40}x''(t) + O(h^3)$$

Thus the local truncation error of this method is $O(h^2)$, and so $p = 2$. This means that the LMM converges to the exact solution with order 1 as $|\alpha_2| + |\alpha_1| = 1$ and $\alpha_0 = 0$. Below are some numerical errors that are computed using different h values to see how the global error behaves for the solution of u:

$$h = 0.0001 : |e_n| = 7.42 * 10^{-5}$$
$$h = 0.00005 : |e_n| = 3.71 * 10^{-5}$$
$$h = 0.00001 : |e_n| = 3.33 * 10^{-6}$$

For the h values above, it does indeed appear that the global error tends to zero in a linear fashion, supporting the theoretical estimate of the global error.

4.2 Implicit Method

Plugging in the alphas and betas of our implicit method into the Taylor expansion gives:

$$x(t) + hx'(t) + \frac{h^2}{2}x''(t) + \frac{h^3}{6}x^{(3)}(t) - \frac{23h^4}{24}x^{(4)}(t) + O(h^5)$$

Thus the local truncation error of this method is $O(h^4)$, and so $p = 4$. This means that the LMM converges to the exact solution with order 3 as $|\alpha_2| + |\alpha_1| = 1$ and $\alpha_0 = 0$. Below are some numerical errors that are computed using different h values to see how the global error behaves for the solution of u:

$$h = 0.0002 : |e_n| = 1.83 * 10^{-12}$$
$$h = 0.0001 : |e_n| = 6.56 * 10^{-14}$$

For the h values above, it does indeed appear that the global error tends to zero in quasi-cubic fashion, supporting the theoretical estimate of the global error. There is quite a margin of error to be had here for the order of convergence, given that the Euler method that is used to obtain the first solution has a linear error and there are inbuilt discrepancies due to the analytical solution being computed by MATLAB; the exponential and trigonometric functions use Taylor Series in MATLAB to produce a numerical result and this carries with it an innate error.

As such, these factors obfuscate the clarity of the order of convergence to the true solution as you send h to zero.

5 Exercise Five (Global error and LMMs)

For the AB(3) method the alphas and betas are as follows:

$$\alpha_2 = -1,\ \alpha_1 = 0,\ \alpha_0 = 0,\ \beta_2 = \frac{23}{12},\ \beta_1 = -\frac{4}{3},\ \beta_0 = \frac{5}{12}$$

Plugging in the alphas and betas of the AB(3) into the Taylor expansion gives:

$$x(t) + hx'(t) + \frac{h^2}{2}x''(t) + \frac{h^3}{6}x^{(3)}(t) - \frac{h^4}{3}x^{(4)}(t) + O(h^5)$$

Thus the local truncation error of this method is $O(h^4)$, and so $p = 4$. This means that the LMM converges to the exact solution with order 3 as $|\alpha_2| + |\alpha_1| = 1$ and $\alpha_0 = 0$.

Below are some numerical errors that are computed using different h values to see how the global error behaves for the solution of u:

$$h = 0.0005 : |e_n| = 1.18 * 10^{-11}$$
$$h = 0.00025 : |e_n| = 1.48 * 10^{-12}$$
$$h = 0.000125 : |e_n| = 1.85 * 10^{-13}$$

For the h values above, it does indeed appear that the global error tends to zero in an almost cubic fashion, supporting the theoretical estimate of the global error since halving the h value results in a reduction of one order of magnitude and $2^3 \approx 10$.