# Practice Problems on Waiting Time Analysis: Solutions

## Review and Discussion Questions:

**1.** What is the major cost trade-off that must be made in managing waiting time problems?

The major cost trade-off is between the cost of waiting for service versus the cost of providing additional service capacity.

**2.** What assumptions are necessary to employ the exact formula: $T_q = \frac{u}{1-u} \cdot p$?

- Single server, first-come-first-served
- Inter-arrival times are exponentially distributed (i.e., Poisson arrivals), which implies a purely random arrival process, but with a know arrival rate.
- Service times have $CV_p = 1$ (e.g., mean = standard deviation).
- The process has reached a "steady state", i.e., the probability distribution of the waiting time does not change over time.

**3.** In what way might the first-come-first-served rule be unfair to the customer waiting for service in a bank or hospital?

In a bank, FCFS may be perceived to be unfair by customers who have large accounts.
In a hospital, especially in an emergency room, FCFS is probably the exception rather than the rule. FCFS would be unfair when a patient with a minor problem is treated before another experiencing severe pain.

**4.** Define, in a practical sense, what is meant by an exponential service time. Would you expect the exponential distribution to be a good approximation of service times for
a. Checking in at the airport?
b. Riding a merry-go-round at a carnival?
c. Completing a midterm exam for OM class?

An exponential service time means that most of the time, the service requirements are of short time duration, but there are occasional long ones. Exponential distribution also means that the probability that a service will be completed in the next instant of time is not dependent on the time at which it entered the system.

a. Yes. Check-in for most passengers is pretty straightforward and entails a short service time, but occasionally some will require long service time.
b. No. The Merry-go-round has a fixed cycle time and hence constant services rate.
c. No. Most students require the entire two hours finishing the midterm.

**5.** Explain how having more work-in-process inventory can improve the efficiency of a process? How can this ever be bad?

More work-in-process inventory can be used to buffer (i.e., decouple) multiple stage processes. Specifically, it can help with blocking or starving. Blocking is when the activities in the stage must stop because there is no place to deposit the item just completed. Starving is when the activities in a stage

must stop because there is no work. Buffer inventories between operations can help relieve these problems, and improve the efficiency of the overall process.
Increasing work-in-process inventory can be bad in that it involves more investment in inventory, as well as taking-up valuable floor space. Increased work-in-process inventory also increases the flow time, which hinders us from identifying quality problems promptly.

**Practice Problems:**

**1.** Students arrive at the Administrative Services Office at an average of one every 15 minutes, and their requests take on average 10 minutes to be processed. The service counter is staffed by only one clerk, Judy Gumshoes, who works eight hours per day. Assume Poisson arrivals and exponential service times.
a. What percentage of time is Judy idle?
b. How much time, on average, does a student spend waiting in line?
c. How long is the (waiting) line on average?
d. What is the probability that an arriving student (just before entering the Administrative Services Office) will find at least one other student waiting in line?

Parameters: a = 15 min, p = 10 min, m = 1, $u = 10/15 = 2/3$, $CV_a = 1$, $CV_p = 1$

a. $1 - u = 33.33\%$
b. $T_q = \left(\frac{1^2+1^2}{2}\right)\cdot\left(\frac{2/3}{1-(2/3)}\right)\cdot 10$ = 20 minutes
c. $I_q = T_q / a = 20/15 = 1.33$ students
d. At least one other student waiting in line is the same as at least two in the system. This probability is $u^2 = (2/3)^2 = 0.444$

**2.** The managers of the Administrative Services Office estimate that the time a student spends waiting in line costs them (due to goodwill loss and so on) $10 per hour. To reduce the time a student spends waiting, they know that they need to improve Judy's processing time (see Problem 1). They are currently considering the following two options:
**a.** Install a computer system, with which Judy expects to be able to complete a student request 40 percent faster (from 2 minutes per request to 1 minute and 12 seconds, for example).
**b.** Hire another temporary clerk, who will work at the same rate as Judy.
If the computer costs $29.50 to operate per day, while the temporary clerk gets paid $75 per day, is Judy right to prefer the hired help? Assume Poisson arrivals and exponential service times.

Option a:
a = 15 min, p = 10 min × (1 – 40%) = 6 min, m = 1, $u = 6/15 = 0.4$
$CV_a = 1$, $CV_p = 1$,

$$T_q = \left(\frac{1^2+1^2}{2}\right)\cdot\left(\frac{0.4}{1-0.4}\right)\cdot 6 = 4 \text{ minutes}$$

$I_q = T_q / a = 4/15 = 0.267$ students
Waiting cost = $10/(student*hr) * 0.267 students * 8 hr/day = $21.33 per day
Total cost = waiting cost + computer cost
= $21.33 + $29.50 = $50.83 per day

Option b:

a = 15 min,  p = 10 min,  m = 2,  u = 10/(2 × 15) = 10/30 = 1/3,  $CV_a$ = 1,  $CV_p$ = 1

$$T_q = \left(\frac{1^2+1^2}{2}\right)\cdot\left(\frac{(1/3)^{\sqrt{2(2+1)}-1}}{2(1-(1/3))}\right)\cdot 10 \;= 1.526 \text{ minutes}$$

$I_q = T_q / a$ = 1.526/15 = 0.102 students

Waiting cost = $10/(student*hr) * 0.102 students * 8 hr/day = $8.16 per day

Total cost = waiting cost + additional service cost

= $8.16 + $75 = $83.16 per day

Consequently, it is better to install the computer system.

**3.** Sharp Discounts Wholesale Club has two service desks, one at each entrance of the store. Customers arrive at each service desk at an average of one every six minutes. The service rate at each service desk is four minutes per customer. Assume Poisson arrivals and exponential service times.

**a.** How often (what percentage of time) is each service desk idle?

**b.** What is the probability that both service clerks are busy?

**c.** What is the probability that both service clerks are idle?

**d.** How much time does a customer spend at the service desk (waiting plus service time)?

**e.** How many customers, on average, are waiting in line in front of each service desk?

Parameters: a = 6 min,  p = 4 min,  m = 1,  u = 4/6 = 2/3,  $CV_a$ = 1,  $CV_p$ = 1

a. 1 – u = 33.33%

b. Prob (both clerks busy) = Prob (one clerk busy) * Prob (one clerk busy)
= $u^2 = (2/3)^2$ = 0.444

c. Prob (both clerks idle) = Prob (one clerk idle) * Prob (one clerk idle)
= $(1 - u)^2 = (1/3)^2$ = 0.111

d. $$T_q = \left(\frac{1^2+1^2}{2}\right)\cdot\left(\frac{2/3}{1-(2/3)}\right)\cdot 4 \;= 8 \text{ minutes}$$
$T_{sys} = T_q + p$ = 8 + 4 = 12 minutes

e. $I_q = T_q / a$ = 8/6 = 1.333 customers

**4.** Sharp Discounts Wholesale Club is considering consolidating its two service desks (see Problem 3) into one location, staffed by two clerks. The clerks will continue to work at the same individual speed of four minutes per customer.

**a.** How much time does a customer spend at the service desk (waiting plus service time)?

**b.** How many customers, on average, are waiting in line?

**c.** Do you think the Sharp Discounts Wholesale Club should consolidate the service desks?

Parameters: a = 3 min, p = 4 min, m = 2, $u = 4/(2 \times 3) = 2/3$, $CV_a = 1$, $CV_p = 1$

a. $T_q = \left(\frac{1^2+1^2}{2}\right)\cdot\left(\frac{(2/3)^{\sqrt{2(2+1)}-1}}{2(1-(2/3))}\right)\cdot 4$ = 3.334 minutes

$T_{sys} = T_q + p = 3.334 + 4 = 7.334$ minutes

b. $I_q = T_q/a = 3.334/3 = 1.111$ customers

c. Yes, fewer customers in line and shorter time in system when consolidated.

**5.** Burrito King, a new fast-food franchise opening up nationwide, has successfully automated burrito production for its drive-up fast-food establishments. The Burro-Master 9000 requires a constant 45 seconds to produce a batch of burritos. It has been estimated that customers will arrive at the drive-up window according to a Poisson distribution at an average of one every 50 seconds. To help determine the amount of space needed for the line at the drive-up window, Burrito King would like to know the expected average time in the system, the average line length (in cars), and the average number of cars in the system (both in line and at the window).

a = 50 seconds = 0.833 min
p = 45 seconds = 0.75 min
m = 1
u = 45 / 50 = 0.9
$CV_a = 1$
$CV_p = 0$

$T_q = \left(\frac{1^2+0^2}{2}\right)\cdot\left(\frac{0.9}{1-(0.9)}\right)\cdot 0.75$ = 3.375 minutes

$T_{sys} = T_q + p = 3.375 + 0.75 = 4.125$ minutes
$I_q = T_q / a = 3.375/0.833 = 4.05$ cars
$I_{sys} = T_{sys} / a = 4.125/0.833 = 4.95$ cars

**6.** To support National Heart Week, the Heart Association plans to install a free blood pressure testing booth in El Con Mall for the week. Previous experience indicates that, on the average, 10 persons per hour request a test. Assume arrivals are Poisson. Blood pressure measurements can be made at a constant time of five minutes each. Assume the queue length can be infinite with FCFS discipline.
**a.** What is the average amount of time that a person can expect to spend in line?
**b.** What average number in line can be expected?
**c.** On the average, how much time will it take to measure a person's blood pressure, including waiting time?
**d.** What average number of persons can be expected to be in the system?
**e.** On weekends, the arrival rate can be expected to increase to over 12 per hour. What effect will this have on the number in the waiting line?

a = 6 min, p = 5 min, m = 1, u = 5/6, $CV_a$ = 1, $CV_p$ = 0

a. $T_q = \left(\frac{1^2+0^2}{2}\right)\cdot\left(\frac{5/6}{1-(5/6)}\right)\cdot 5$ = 12.5 minutes

b. $I_q = T_q / a$ = 12.5/6 = 2.083 people

c. $T_{sys} = T_q + p$ = 12.5 + 5 = 17.5 minutes

d. $I_{sys} = T_{sys} / a$ = 17.5 / 6 = 2.917 people

e. When a = 5 minutes, demand equals capacity. With variability, it will cause waiting line to increase. In the long run, the waiting line will become infinitely long, but of course over the finite time during the weekend, the length of the waiting line will fluctuate and its expected length will increase.

**7.** A cafeteria serving line has a coffee urn from which customers serve themselves. Arrivals at the urn follow a Poisson distribution at the rate of three per minute. In serving themselves, customers take about 15 seconds, exponentially distributed.
**a.** How long would you expect it to take to get a cup of coffee?
**b.** How many customers would you expect to see on the average at the coffee urn?
**c.** What percentage of time is the urn being used?
**d.** What is the probability that three or more people are in the cafeteria?
**e.** If the cafeteria installs an automatic vendor that dispenses a cup of coffee at a constant time of 15 seconds, how does this change your answers to a and b?

a = 0.333 min, p = 0.25 min, m =1, u = 0.25/0.333 = ¾, $CV_a$ = 1, $CV_p$ = 1

a. $T_q = \left(\frac{1^2+1^2}{2}\right)\cdot\left(\frac{3/4}{1-(3/4)}\right)\cdot 0.25$ = 0.75 minutes

$T_{sys} = T_q + p$ = 0.75 + 0.25 = 1 minute

b. $I_{sys} = T_{sys} / a$ = 1/0.333 = 3 customers

c. The urn is bing used u = 75% of time

d. Probability of 3 or more is equal to $u^3$ = 0.4219

e. If an automatic vendor is installed, the service time becomes constant.

$T_q = \left(\frac{1^2+0^2}{2}\right)\cdot\left(\frac{3/4}{1-(3/4)}\right)\cdot 0.25$ = 0.375 minutes

$T_{sys} = T_q + p$ = 0.375 + 0.25 = 0.625 minutes = 37.5 seconds
$I_{sys} = T_{sys} / a$ = 0.625/0.333 = 1.875 customers
By converting to constant service time, the number in line is reduce from 3 to 1.875 people, and time in system is reduced from 1 minute to 37.5 seconds.

**8.** AVI supplies vended food to a large university. Because students kick the machines at every opportunity, management has a constant repair problem. The machines break down on an average of three per hour, and the breakdowns occur in a Poisson manner. Downtime costs the company $25/hour per machine, and each maintenance worker gets $4/hour. One worker can repair machines at an average rate of five per hour, distributed exponentially; two workers working together (as one team) can repair seven per hour, distributed exponentially; and, a team of three workers can do eight per hour, distributed exponentially. What maintenance crew size would you recommend to AVI?

**Solution:**

With one worker:

a = 20 min, p = 12 min, m =1, u = 12/20 = 3/5, $CV_a = 1$, $CV_p = 1$

$$T_q = \left(\frac{1^2+1^2}{2}\right)\cdot\left(\frac{3/5}{1-(3/5)}\right)\cdot 12 = 18 \text{ minutes}$$

$T_{sys} = T_q + p = 18 + 12 = 30$ minutes

$I_{sys} = T_{sys} / a = 30/20 = 1.5$ machines

Downtime cost = $25/(machine*hour) * 1.5 machines = $37.5/hour

Labor cost = $4/hour

<u>Total cost = $41.5/hour</u>

With two workers:

a = 20 min, p = 60/7 min, m = 1, u = 3/7 = 0.429, $CV_a = 1$, $CV_p = 1$

$$T_q = \left(\frac{1^2+1^2}{2}\right)\cdot\left(\frac{0.429}{1-(0.429)}\right)\cdot 8.571 = 6.440 \text{ minutes}$$

$T_{sys} = T_q + p = 6.440 + 8.571 = 15$ minutes

$I_{sys} = T_{sys}/a = 15/20 = 0.75$ machines

Downtime cost = $25/(machine*hr) * 0.75 machines = $18.75/hour

Labor cost = $8/hour

<u>Total cost = $26.75/hour</u>

With three workers:

a = 20 min, p = 7.5 min, m = 1, u = 7.5/20 = 3/8, $CV_a = 1$, $CV_p = 1$

$$T_q = \left(\frac{1^2+1^2}{2}\right)\cdot\left(\frac{3/8}{1-(3/8)}\right)\cdot 7.5 = 4.5 \text{ minutes}$$

$T_{sys} = T_q + p = 4.5 + 7.5 = 12$ minutes

$I_{sys} = T_{sys}/a = 12/20 = 0.6$ machines

Downtime cost = $25/(machine*hr) * 0.6 machines = $15/hour

Labor cost = $12/hour

<u>Total cost = $27/hour</u>

Best crew size is 2 workers.

**9.** Sea Dock, a private firm, operates an unloading facility located in the Gulf of Mexico for supertankers delivering crude oil for refineries in the Port Arthur area of Texas. On average, two tankers arrive per day according to a Poisson process. Supertankers are unloaded one at a time on a FCFS basis.

Unloading requires approximately 8 hours of a 24-hour working day, and unloading times have an exponential distribution.

**a.** Sea Dock has provided mooring space for three tankers (not including the one being unloaded). Is this sufficient to meet the U.S. Coast Guard requirement that at least 9 out of 10 arrivals should find mooring available?

**Solution:**
a = 12 hr, p = 8 hr, m=1, u = 8/12 = 2/3

An arrival will find mooring space available if and only if there are less than 4 ships in the system (including the three in the queue and the one being unloaded).

Prob (number in system < 4) = 1 – Prob (number in system $\geq$ 4) = 1 – $u^4$ = 1 – $(2/3)^4$ = 80.2%

The Coast Guard requirement is that at least 9 out of 10, or 90%, should find space.
→ Does not meet the requirement.

**b.** Sea Dock can increase its unloading capacity to a rate of four tankers per day through additional labor at a cost of $480 per day. Considering the $1000-per-day demurrage fee charged to Sea Dock for keeping a supertanker idle (this includes unloading time as well as time spent waiting in queue), should management consider this expansion opportunity? Analyze this from both monetary and regulatory perspective.

**Solution:**
<u>At present</u>:
a = 12 hr, p = 8 hr, m = 1, u = 8/12 = 2/3, $CV_a$ = 1, $CV_p$ = 1

$$T_q = \left(\frac{1^2+1^2}{2}\right)\cdot\left(\frac{2/3}{1-(2/3)}\right)\cdot 8 \ = 16 \text{ hours}$$

$T_{sys} = T_q + p$ = 16 + 8 = 24 hours
$I_{sys} = T_{sys}/a$ = 24/12 = 2 tankers
Cost of waiting = $1,000/(tanker*day) * 2 tankers = $2,000/day

<u>With additional labor</u>:
a = 12 hr, p = 6 hr, m = 1, u = 6/12 = 1/2, $CV_a$ = 1, $CV_p$ = 1

$$T_q = \left(\frac{1^2+1^2}{2}\right)\cdot\left(\frac{1/2}{1-(1/2)}\right)\cdot 6 \ = 6 \text{ hours}$$

$T_{sys} = T_q + p$ = 6 + 6 = 12 hours
$I_{sys} = T_{sys}/a$ = 12/12 = 1 tankers
Cost of waiting = $1,000/day
Cost of additional labor = $480/day
Total cost = $1,480/day < $2,000/day → From monetary perspective, DO IT!

From the regulatory perspective,
Prob (number in system < 4) = 1 – $u^4$ = 1 – $(0.5)^4$ = 93.75% > 90% → Also DO IT!

**10.** The Lower Colorado River Authority (LCRA) has been studying the congestion at the boat-launching ramp near Mansfield Dam. On weekends boaters arrive in a Poisson pattern at an average rate of four per hour. The average time to launch a boat is exponentially distributed with an average of 10 minutes. Assume that only one boat can be launched at a time.

**a.** If there is room to park only two boats at the top of the ramp in preparation for launching, how often would an arrival find insufficient parking space?

**Solution:**

a = 15 min, p = 10 min, m = 1, u = 10/15 = 2/3

An arrival will find parking space full if and only if there are 3 already in the system (including 2 parking and 1 launching):

Prob (number in system $\geq$ 3) = $u^3 = (2/3)^3$ = 30%

**b.** The LCRA plans to add another ramp when the average turnaround time (time in the system) exceeds 60 minutes. Should the LCRA add another ramp now? If not, at what average arrival rate should the LCRA consider adding another ramp?

**Solution:**

a = 15 min, p = 10 min, m=1, u = 10/15 = 2/3, $CV_a = 1$, $CV_p = 1$

$$T_q = \left(\frac{1^2+1^2}{2}\right)\cdot\left(\frac{2/3}{1-(2/3)}\right)\cdot 10 = 20 \text{ minutes}$$

$T_{sys} = T_q + p = 20 + 10 = 30$ minutes

$I_{sys} = T_{sys}/a = 30/15 = 2$ boaters

There is no need to add another ramp now.

Next, we find the arrival rate under which the turnaround time reaches 1 hour:

a = ? min, p = 10 min, m = 1, u = 10/a , $CV_a = 1$, $CV_p = 1$

$T_{sys} = T_q + p$

$$T_{sys} = \left(\frac{CV_a^2+CV_p^2}{2}\right)\cdot\left(\frac{u}{1-u}\right)\cdot p + p$$

$$60 = \left(\frac{1^2+1^2}{2}\right)\cdot\left(\frac{10/a}{1-(10/a)}\right)\cdot 10 + 10$$

$$50 = \left(\frac{10/a}{1-(10/a)}\right)\cdot 10$$

Solve the above equation, we get a = 12 min

Thus, when the arrival rate is 5 boaters per hour, they should consider adding another ramp.

**11.** A one-pump gas station servers cars that arrive in a Poisson pattern at a rate of six cars per hour. It is estimated that on the average, customers arrive to buy gas when their tanks are one-eighth full. The average time to service a customer is four minutes, exponentially distributed.

**a.** What are the expected time in the system and the expected length of the queue?

**Solution:**

a = 10 min, p = 4 min, m = 1, u = 4/10 = 2/5, $CV_a = 1$, $CV_p = 1$

$$T_q = \left(\frac{1^2+1^2}{2}\right)\cdot\left(\frac{2/5}{1-(2/5)}\right)\cdot 4 = 2.667 \text{ minutes}$$

$I_q = T_q / a = 2.667/10 = 0.267$ cars

$T_{sys} = T_q + p = 2.667 + 4 = 6.667$ minutes = 0.111 hours

**b.** Suppose that customers perceive a gas shortage (when there is none) and respond by changing the fill-up criterion to one-quarter full on the average. Assuming the total gas consumption rate and the service time remain unchanged, what is the expected time in the system and the expected length of the queue?

**Solution:**

For part (a), each arrival's demand is 7/8 tanks.

Total demand rate = 7/8 tanks/customer * 6 customers/hr = 5.25 tanks/hr.

Now, each car's demand drops to 3/4 tanks.

Assuming total demand is unchanged, we have

3/4 * Arrival Rate = 5.25

Arrival Rate = 7 per hour

a = 60 / 7 min, p = 4 min, m = 1, u = 4 / (60/7) = 0.467, $CV_a = 1$, $CV_p = 1$

$$T_q = \left(\frac{1^2+1^2}{2}\right)\cdot\left(\frac{0.467}{1-(0.467)}\right)\cdot 4 = 3.505 \text{ minutes}$$

$I_q = T_q / a = 3.505/8.571 = 0.409$ cars

$T_{sys} = T_q + p = 3.505 + 4 = 7.505$ minutes = 0.125 hours

**c.** Make the same assumptions as in part (b). What if the fill-up criterion is half full? Do we have the makings of a behaviorally induced gasoline panic?

**Solution:**

Again computing the new arrival rate as above, we get,

1/2 * Arrival Rate = 5.25

Arrival Rate = 10.5 per hour

a = 5.714 min, p = 4 min, u = 4/5.714 = 0.7, $CV_a = 1$, $CV_p = 1$

$$T_q = \left(\frac{1^2+1^2}{2}\right)\cdot\left(\frac{0.7}{1-(0.7)}\right)\cdot 4 = 9.333 \text{ minutes}$$

$I_q = T_q / a = 9.333/5.714 = 1.633$ cars

$T_{sys} = T_q + p = 9.333 + 4 = 13.333$ minutes = 0.222 hours

Comparing the results in (a), (b), and (c), we see an increase of waiting time and time, which is behaviorally induce gasoline panic.