MT3504 December 2019 – Solutions

1. **Let $y = y(x)$. Find the general solution of**

$$3xy^2\frac{\mathrm{d}y}{\mathrm{d}x} = 2x - y^3.$$

Discuss the existence and uniqueness of solutions when $y(0) = 0$. [4 Marks]

Solution:

Write the differential equation as

$$\underbrace{y^3 - 2x}_{M} + \underbrace{3xy^2}_{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Since

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 3y^2$$

then by a theorem from the lecture notes the ODE is exact. **[1 Mark]**

Seek $F(x, y)$ such that

$$F = \int y^3 - 2x\,dx = xy^3 - x^2 + f(y)$$

and

$$F = \int 3xy^2\,dy = xy^3 + g(x).$$

From this we can take $F = xy^3 - x^2$ so the general solution is

$$xy^3 - x^2 = C$$

or

$$y = \left(\frac{C + x^2}{x}\right)^{1/3}$$

where C is some constant of integration. **[2 Mark]**

From the lecture notes we saw that the initial value problem $y' = f(x, y)$, $y(x_0) = y_0$, will have a unique solution provided f and $\partial f/\partial y$ are continuous at (x_0, y_0). In this case $f = (2x - y^3)/(3xy^2)$ is not continuous at $(0, 0)$ so a unique solution is not guaranteed to exist by our theorem (we can't tell using this method if it's just the one solution, or multiple solutions might exist).

[1 Mark]

2. **Let $y = y(x)$. Find the general solution of**

$$\frac{d^2y}{dx^2} + 4y = \frac{3}{\sin(2x)}, \qquad 0 < x < \frac{\pi}{2}$$

[6 Marks]

Solution:

First find the complementary function, i.e. solve $y'' + 4y = 0$. Using the auxiliary equation $\lambda^2 + 4 = 0$ this is $y_{CF} = A\cos(2x) + B\sin(2x)$. **[1 Mark]**

Use variation of parameters to find the particular integral, seeking a solution $y_{PI} = u_1(x)\cos(2x) + u_2(x)\sin(2x)$. **[1 Mark]**

Such a solution satisfies

$$\begin{pmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{3}{\sin(2x)} \end{pmatrix}.$$

The fundamental matrix has determinant

$$W = 2\cos^2(2x) + 2\sin^2(2x) = 2$$

and so

$$\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2\cos(2x) & -\sin(2x) \\ 2\sin(2x) & \cos(2x) \end{pmatrix} \begin{pmatrix} 0 \\ \frac{3}{\sin(2x)} \end{pmatrix}$$

gives

$$u_1 = \int \frac{-3}{2}\, dx, \qquad u_2 = \int \frac{3\cos(2x)}{2\sin(2x)}\, dx$$

and taking

$$u_1 = -\frac{3x}{2}, \qquad u_2 = \frac{3}{4}\ln(\sin(2x))$$

will do. **[3 Mark]**

Hence the general solution is

$$y = A\cos(2x) + B\sin(2x) - \frac{3}{2}x\cos(2x) + \frac{3}{4}\sin(2x)\ln(\sin(2x)).$$

[1 Mark]

3. **Let $y = y(x)$. Find the Green's function and hence obtain the solution to the two-point boundary value problem**

$$x^2y'' + 3xy' + 2y = \delta(x - 4)$$

satisfying $y(1) = 0$ and $y(x) \to 0$ as $x \to \infty$. **[9 Marks]**

Solution:

For

$$x^2y'' + 3xy' + 2y = 0$$

let $z = \ln(x)$ and the ODE becomes

$$\frac{\mathrm{d}y^2}{\mathrm{d}z^2} + 2\frac{\mathrm{d}y}{\mathrm{dz}} + 2y = 0$$

so a basis for the homogeneous equation is

$$y_1 = \frac{1}{x}\sin(\ln(x)), \qquad y_2 = \frac{1}{x}\cos(\ln(x)).$$

These already satisfy the required conditions $y_1(1) = 0$ and $y_2 \to 0$ as $x \to \infty$. **[2 Marks]**

The Green's function is

$$G(x,s) = \begin{cases} A(s)\frac{1}{x}\sin(\ln(x)), & x < s \\ B(s)\frac{1}{x}\cos(\ln(x)), & x > s \end{cases}$$

[1 Mark]

Imposing continuity and the jump condition give

$$A(s)\frac{1}{s}\sin(\ln(s)) = B(s)\frac{1}{s}\cos(\ln(s))$$

and

$$-\frac{B(s)}{s^2}\left(\cos(\ln(s)) + \sin(\ln(s))\right) - \frac{A(s)}{s^2}\left(\cos(\ln(s)) - \sin(\ln(s))\right) = \frac{1}{s^2}$$

[2 Mark]

Solving for $A(s)$ and $B(s)$,

$$\begin{pmatrix} \frac{1}{s}\sin(\ln(s)) & \frac{1}{s}\cos(\ln(s)) \\ \frac{1}{s^2}\left(\cos(\ln(s)) - \sin(\ln(s))\right) & -\frac{1}{s^2}\left(\sin(\ln(s) + \cos(\ln(s))\right) \end{pmatrix} \begin{pmatrix} A(s) \\ -B(s) \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{1}{s^2} \end{pmatrix}$$

so

$$\begin{pmatrix} A(s) \\ -B(s) \end{pmatrix} = -s^3 \begin{pmatrix} -\frac{1}{s^2}\left(\sin(\ln(s) + \cos(\ln(s))\right) & -\frac{1}{s}\cos(\ln(s)) \\ \frac{1}{s^2}\left(\cos(\ln(s)) - \sin(\ln(s))\right) & -\frac{1}{s}\sin(\ln(s)) \end{pmatrix} \begin{pmatrix} 0 \\ -\frac{1}{s^2} \end{pmatrix} = \begin{pmatrix} -\cos(\ln(s)) \\ \sin(\ln(s)) \end{pmatrix}$$

and

$$G(x,s) = \begin{cases} -\cos(\ln(s))\frac{1}{x}\sin(\ln(x)), & x < s \\ -\sin(\ln(s))\frac{1}{x}\cos(\ln(x)), & x > s \end{cases}$$

[1 Mark]

The solution is

$$y(x) = -\int_{s=1}^{s=x} \sin(\ln(s))\frac{1}{x}\cos(\ln(x))\delta(s-4)\,ds - \int_{s=x}^{s=\infty} \cos(\ln(s))\frac{1}{x}\sin(\ln(x))\delta(s-4)\,ds$$

[1 Mark]

For $x < 4$ the first integral vanishes and

$$y(x) = -\frac{1}{x}\cos(\ln(4))\sin(\ln(x))$$

while for $x > 4$ the second integral vanishes and

$$y(x) = -\frac{1}{x}\sin(\ln(4))\cos(\ln(x))$$

so the general solution is

$$y(x) = \begin{cases} -\frac{1}{x}\cos(\ln(4))\sin(\ln(x)), & x < 4 \\ -\frac{1}{x}\sin(\ln(4))\cos(\ln(x)), & x > 4 \end{cases}.$$

[2 Marks]

4. Use separation of variables to solve the PDE

$$u_t - u = u_{xx}\,, \qquad 0 \le x \le \pi/2\,, \quad t \ge 0$$

with boundary conditions $u_x(0,t) = 0$ and $u(\pi/2,t) = 0$, and for initial conditions $u(x,0) = 1 - \cos(4x)$. *Hint:* the identity $\cos(A+B) = \cos A \cos B - \sin A \sin B$ may prove useful.

Solution: Try $u(x,t) = X(x)T(t)$. Then the PDE, after division by XT, reads

$$\frac{T' - T}{T} = \frac{X''}{X}$$

where a prime means differentiation with respect to the function's argument (1 mark). The lhs is a function of t while the rhs is a function of x, so the only way they can be equal is that they are both a constant. Anticipating the x boundary conditions, we take this constant to be $-\lambda$ and suppose $\lambda > 0$ (1 mark).

The X equation is then $X'' + \lambda X = 0$ which has the general solution

$$X = a\cos(\sqrt{\lambda}x) + b\sin(\sqrt{\lambda}x)$$

for arbitrary constants a and b (1 mark). The boundary condition $u_x(0,t) = 0$ implies $X'(0) = 0$, requiring $b = 0$ (1 mark). The boundary condition $u(\pi/2,t) = 0$ implies $X(\pi/2) = 0$ and can be satisfied by taking $\sqrt{\lambda} = 2k-1$ for positive integers k. Thus $\lambda = (2k-1)^2$ (1 mark).

Turning next to the T equation, we have

$$T' = (1-\lambda)T = (1-(2k-1)^2)T = -4k(k-1)T\,.$$

This has the general solution

$$T = A_k e^{-4k(k-1)t}$$

for each k (1 mark). The general solution of the PDE satisfying the boundary conditions is thus

$$u(x,t) = \sum_{k=1}^{\infty} A_k \cos((2k-1)x)e^{-4k(k-1)t}$$

(1 mark).

We now use the initial conditions to determine the Fourier coefficients A_k. At $t = 0$,

$$u(x,0) = \sum_{k=1}^{\infty} A_k \cos((2k-1)x) = 1 - \cos(4x)\,.$$

The A_k are determined by multiplying both sides by $\cos((2j-1)x)$, with j a positive integer, and using the orthogonality of $\cos((2j-1)x)$ and $\cos((2k-1)x)$ for $j \neq k$. Noting also the the integral of $\cos((2j-1)x)^2$ over $[0,\pi/2]$ is $\pi/4$, we obtain

$$A_k = \frac{4}{\pi}\int_0^{\pi/2} (1-\cos(4x))\cos((2k-1)x)\,dx$$

after changing j back to k (2 marks). The integral is easily done if one recalls the trigonometric identity

$$\cos A \cos B = \tfrac{1}{2}[\cos(A+B) + \cos(A-B)]\,.$$

Using this, we find

$$A_k = \frac{2}{\pi}\int_0^{\pi/2} [2\cos((2k-1)x) - \cos((2k+3)x) - \cos((2k-5)x)]\,dx$$

All integrals are simple and we obtain

$$A_k = \frac{2}{\pi}\left[\frac{2\sin((2k-1)\pi/2)}{2k-1} - \frac{\sin((2k+3)\pi/2)}{2k+3} - \frac{\sin((2k-5)\pi/2)}{2k-5}\right]$$

(2 marks). This is sufficient, but one can go further because all the sin functions evaluate to $(-1)^{k+1}$, and this allows one to collapse the expression to

$$A_k = \frac{64(-1)^k}{\pi(2k-5)(2k-1)(2k+3)}\,.$$

5. Consider the first-order PDE

$$(y+u)u_x + yu_y = x - y\,, \qquad y > 0\,.$$

(i) State and solve the characteristic equations to show that

$$\frac{x+u}{y} = A \qquad \text{and} \qquad u^2 - (x-y)^2 = B\,.$$

for constants A and B.

(ii) Given $u = 1 + x$ on $y = 1$, explicitly find the unique solution for all $y > 0$.

Solution:

(i) The given solutions are a big hint, though work is still required. Start by setting up the system of equations to be solved, using a parameter s:

$$\frac{dx}{ds} = y + u\,, \qquad \frac{dy}{ds} = y\,, \qquad \frac{du}{ds} = x - y$$

(1 mark). The middle equation can be integrated directly to give $y = ae^s$ where a is an arbitrary constant (1 mark). Adding the first and third equation gives

$$\frac{d(x+u)}{ds} = x + u$$

so $x + u = be^s$ where b is another arbitrary constant (1 mark). Eliminating s gives the first of the two solutions:

$$\frac{(x+u)}{y} = A$$

(1 mark). To get the other one, notice that subtracting dy/ds from dx/ds gives

$$\frac{d(x-y)}{ds} = u$$

Now form u times du/ds (taking the hint from the given solution):

$$u\frac{du}{ds} = u(x-y) = (x-y)\frac{d(x-y)}{ds}$$

using the preceding equation (2 marks). But this is in the form of a perfect derivative,

$$\frac{d}{ds}\left(u^2 - (x-y)^2\right) = 0$$

so

$$u^2 - (x-y)^2 = B$$

for constant B, and this is the second solution given (1 mark).

(ii) We now use these two solutions to find a relation between A and B, given we know that $u = 1 + x$ on $y = 1$. It is easiest to write A and B as functions of x, then eliminate x:

$$A = \frac{(x+u)}{y} = 1 + 2x\,, \qquad B = u^2 - (x-y)^2 = (1+x)^2 - (x-1)^2 = 4x\,.$$

Hence $A = 1 + B/2$ or $B = 2A - 2$ (1 mark). The final step is to re-insert the forms of $A(x,y,u)$ and $B(x,y,u)$ into this relation, $B = 2A - 2$:

$$u^2 - (x-y)^2 = 2\frac{(x+u)}{y} - 2$$

(1 mark). We next simplify this to find the unique solution valid for $y > 0$. Writing this as a quadratic equation in u, we have

$$u^2 - 2\frac{u}{y} - (x-y)^2 - 2\frac{x}{y} + 2 = 0\,.$$

Solving for u, we have

$$u = \frac{1}{y} \pm \sqrt{\frac{1}{y^2} + (x-y)^2 + 2\frac{x}{y} - 2}\,.$$

Noting that $2x/y - 2 = 2(x-y)/y$, the quantity inside the square root can be seen to be a perfect square:

$$\left(\frac{1}{y} + x - y\right)^2 = \frac{1}{y^2} + (x-y)^2 + 2\frac{x-y}{y}\,.$$

Thus,

$$u = \frac{1}{y} \pm \left(\frac{1}{y} + x - y\right)$$

and the positive sign must be taken so that $u = 1 + x$ when $y = 1$. Therefore, the final explicit solution is

$$u = \frac{2}{y} + x - y$$

(2 marks).

6. Consider the second order PDE

$$x^4 u_{xx} + 2x^2y^2 u_{xy} + y^2(y^2 - x^2)u_{yy} = 0\,.$$

(i) Determine where in the x-y plane it is hyperbolic.

(ii) Solve the characteristic equations for $x,\, y > 0$ and show that there are two families of solutions given by

$$y = \frac{x}{\xi - \ln x} \qquad \text{and} \qquad y = \frac{2x}{1 + \eta x^2}$$

for constants ξ and η.

(iii) For data given along the line $x = \frac{1}{2}$, find the *domain of dependence* of the point $(1, 1)$.

Solution:

(i) The generic PDE $au_{xx} + 2bu_{xy} + cu_{yy} = F$ is hyperbolic when $b^2 - ac > 0$. Here, $a = x^4$, $b = x^2y^2$ and $c = y^2(y^2 - x^2)$, so

$$b^2 - ac = x^4y^4 - x^4y^2(y^2 - x^2) = x^6y^2\,.$$

Thus, the PDE is hyperbolic so long as both $x \neq 0$ and $y \neq 0$ (1 mark).

(ii) The characteristics for $x,\, y > 0$ are determined from the pair of first-order ODEs

$$y' = \frac{b \pm \sqrt{b^2 - ac}}{a} = \frac{x^2y^2 \pm x^3y}{x^4} = \frac{y^2}{x^2} \pm \frac{y}{x}$$

(2 marks). These may be solved as homogeneous or Bernoulli equations. The latter method is slightly easier. As the highest power is y^2, substitute $y = v^{-1}$ to obtain

$$-\frac{v'}{v^2} = \frac{1}{x^2v^2} \pm \frac{1}{xv}\,.$$

Multiply by $-v^2$ and re-arrange to get a linear ODE for v:

$$v' \pm \frac{v}{x} = -\frac{1}{x^2}\,.$$

Consider each sign in turn. For the positive sign, the integrating factor is $\mu = x$, so

$$(xv)' = -\frac{1}{x}\,,$$

which has the general solution

$$xv = -\ln x + \xi\,,$$

where ξ is a constant of integration. Solving for $y = 1/v$, we obtain

$$y = \frac{x}{\xi - \ln x}$$

(2 marks). This is the first family of solutions. Each member of the family has a different value of the constant ξ.

Now consider the negative sign. The integrating factor is now $\mu = 1/x$ so

$$(v/x)' = -\frac{1}{x^3}\,,$$

which has the general solution

$$\frac{v}{x} = \frac{1}{2x^2} + \frac{\eta}{2}\,,$$

where η is a constant of integration (the division of η by 2 is not necessary — it only makes the final solution prettier!). Solving for $y = 1/v$, we obtain

$$y = \frac{2x}{1 + \eta x^2}$$

(2 marks). This is the second family of solutions. Each member of the family has a different value of the constant η.

(iii) For data given along the line $x = \frac{1}{2}$, the *domain of dependence* of the point $(1, 1)$ is the portion of this line *between* the two characteristics (one from each family) passing through the point $(1, 1)$. First, determine the characteristics. Inserting $x = y = 1$ in the solution for the first family gives $\xi = 1$. Likewise, inserting $x = y = 1$ in the solution for the second family gives $\eta = 1$. This means that the curves

$$y = \frac{x}{1 - \ln x} \qquad \text{and} \qquad y = \frac{2x}{1 + x^2}$$

both pass through $(1, 1)$ (1 mark). Now find where they cross $x = \frac{1}{2}$. Just insert $x = \frac{1}{2}$ into the above equations and solve for y:

$$y = \frac{1}{2(1 + \ln 2)} \approx 0.2953 \qquad \text{and} \qquad y = \frac{4}{5} = 0.8\,.$$

Hence the *domain of dependence* is $y \in [1/(2(1 + \ln 2)), 4/5]$ (1 mark).