Q1 : (a–c) is very standard and similar to tutorial sheet questions

(d) is a slightly unusual way of testing curve sketching / direction fields in open book exam.

(e) similar to tutorial sheets

(f) unusual but relies on $\left(\frac{dx}{dy}\right) = \left(\frac{dy}{dx}\right)^{-1}$ which was seen in tutorial.

(a) $\underbrace{\frac{2}{x} - 1}_{M} + \underbrace{\left(\frac{1}{y} - 1\right)}_{N} \frac{dy}{dx} = 0$

$\frac{\partial M}{\partial y} = 0 = \frac{\partial N}{\partial x}$ so exact.

(b) $\frac{\partial F}{\partial x} = M$; $\frac{\partial F}{\partial y} = N$

so $F = 2\log x - x + f(y)$

$\Rightarrow \frac{\partial F}{\partial y} = f'(y)$ so $f'(y) = \frac{1}{y} - 1$

$f(y) = \log y - y$

so $F = 2\log x - x + \log y - y = C$

[n.b also separable]

(c) $x = y = 1 \Rightarrow C = -2.$

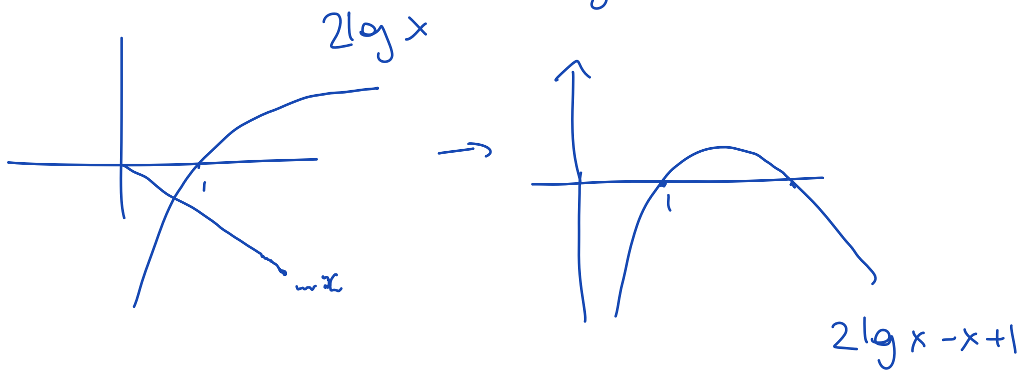
(d) x is $(1,1)$

$$\frac{dy}{dx} = \frac{1-\frac{2}{x}}{\frac{1}{y}-1}$$

So $\frac{dy}{dx} = \infty$ at $y=1$,

Putting $y=1$ into F $\Rightarrow$ $2\log x - x - 1 = -2$

$2\log x - x = -1$



The first intersection corresponds to x (the other intersection corresponds to rightmost end of st? loop.

(e) $f(y) = \frac{1-\frac{2}{x}}{\frac{1}{y}-1}$ singular at $y=1$ so Picard's thm. doesn't apply.

In fact, from sketch, there are 2 solutions.

(f) $\dfrac{dx}{dy} = \underbrace{\dfrac{1-\frac{1}{y}}{\frac{2}{x}-1}}_{f(x,y)}$

Need to do $\dfrac{\partial f}{\partial x} = \left(1-\frac{1}{y}\right)(-1)\left(\frac{2}{x}-1\right)^{-2}\left(-\frac{2}{x^2}\right)$

so both f and $\partial f/\partial x$ cts.

so by Piccard exist unique solution.

Same eqⁿ as before just now $x(y)$. By sketch, unique solⁿ for $x(y)$ too, consistent with Picard.

Q2 (a) Similar to tutorial sheet problem.

(b) similar to 2020 exam for different equation and different initial conditions

(a) $y'' + 2by' + b^2 y = g(x) \quad b > 0 \quad x > 0$

Characteristic eq. $\lambda^2 + 2b\lambda + b^2 = 0$

$\lambda = -b$ repeated.

so two LI sol^{ns} are

$y = e^{-bx}$ and $y = xe^{-bx}$

from lectures (since root repeated)

G.S. $y = Ae^{-bx} + Bxe^{-bx}$

$y' = (-Ab - Bbx)e^{-bx} + Be^{-bx}$

$y_1(0) = 1 \Rightarrow A = 1$

$y_1'(1) = 0 \Rightarrow -Ab + B = 0$ so $B = Ab = b$

$y_1 = (1 + xb)e^{-bx} \qquad (y_1' = -b^2 x e^{-bx})$

$y_2(0) = 0 \Rightarrow A = 0$

$y_2'(0) = 1 \Rightarrow B = 1 \qquad y_2 = xe^{-bx}$

$y_2' = (1 - bx)e^{-bx}$

(b) $y = u_1 y_1 + u_2 y_2$ where (from lect. notes)

$$\underbrace{\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}}_{M} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ g \end{pmatrix}$$

$$M = e^{-bx} \begin{pmatrix} 1+xb & x \\ -xb^2 & 1-xb \end{pmatrix}$$

$$(W =) \det M = e^{-2bx} \left((1-xb)(1+xb) + xb^2 \cdot x \right)$$

$$= e^{-2bx} \left(1 - x^2 b^2 + x^2 b^2 \right)$$

$$= e^{-2bx}. > 0$$

$$M^{-1} = \underbrace{\frac{1}{W} e^{-bx}}_{e^{bx}} \begin{pmatrix} 1-xb & -x \\ xb^2 & 1+xb \end{pmatrix}$$

So $$\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = e^{bx} \left(\quad " \quad \right) \begin{pmatrix} 0 \\ g \end{pmatrix}$$

$$u_1' = -e^{bx} x g$$

$$u_2' = (1+xb) e^{+bx} g$$

$$y(0) = 0 \longrightarrow \quad 0 = u_1(0)\overset{1}{\cancel{y_1(0)}} + u_2(0)\overset{0}{\cancel{y_2(0)}}$$

$$\text{So} \quad u_1(0) = 0$$

$$y' = \underbrace{u_1' y_1 + u_2' y_2}_{0} + u_1 y_1' + u_2 y_2'$$

(first row of initial matrix eq)

$$\text{So} \quad y'(0) = 0 \Rightarrow \quad u_2(0) = 0.$$

$$\text{So} \quad u_1 = -\int_0^x e^{+bs}\, s\, g(s)\, ds.$$

$$u_2 = \int_0^x (1 + bs)\, e^{+bs} g(s)\, ds.$$

$$\text{So} \quad y = -(1 + bx)e^{-bx} \int_0^x e^{bs} s\, g(s)\, ds$$

$$+ \; x e^{-bx} \int_0^x (1 + bs) e^{bs} g(s)\, ds.$$

Q3 (a) is an unseen calculation - slightly harder than tutorial sheet Qs.
(b/c) test understanding of lecture notes.

$y(x) = x\cos x$ is an odd function, so all a_n (inc $a_0 = 0$) - lect. notes.

The $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos x \sin\left(\frac{n\pi x}{\pi}\right) dx$

$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos x \sin(nx)\, dx$

(lect. notes with $l = \pi$)

$= \frac{2}{\pi}\int_{0}^{\pi} x\cos x \sin(nx)\, dx$ (even integrand)

$= \frac{2}{\pi}\int_{0}^{\pi} x\frac{\left(\sin x(n+1) - \sin x(1-n)\right)}{2}\, dx$

$= \frac{1}{\pi}\int_{0}^{\pi} x\sin[x(n+1)] + x\sin[x(n-1)]\, dx$

$= \frac{1}{\pi}\left\{\frac{(-1)^{n+2}\pi}{n+1} + \frac{(-1)^{n}\pi}{n-1}\right\}$

$= (-1)^n \frac{2n}{n^2-1}$ $\quad n > 2.$

b_1 should be done separately

$$b_1 = \frac{1}{\pi}\left\{\frac{(-1)^3\cdot\pi}{2}\right\} = -\frac{1}{2}.$$

other term integrates to zero since $0 = (\sin x(n-1))$ for $n=1$.

(b) $y(x)$ is discontinuous at $x = \pm\pi$, so can't expect to diff. term-by-term.

(c) Parseval ($\ell = \pi$)

$$\int_{-\pi}^{\pi} x^2\cos^2 x\,dx = \pi\sum_{n=1}^{\infty}|b_n|^2$$

$$\frac{1}{\cancel{\pi}}\frac{1}{6}\cancel{\pi}(3+2\pi^2) = \frac{1}{4}+\sum_{n=2}^{\infty}\left(\frac{2n}{(n^2-1)}\right)^2$$

$$\frac{1}{2}+\frac{\pi^2}{3} = \frac{1}{4}+\sum_{2}^{\infty}\frac{2^2n^2}{(n^2-1)^2}$$

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So $$\sum_{2}^{\infty}\frac{n^2}{(n^2-1)^2} = \frac{\pi^2}{12}+\frac{1}{16}$$

Q4 Like problem done in lectures but with extra term and a bit more emphasis on Sturm Liouville Theory.

$$u_t - 2u_x = u_{xx}$$

(a) $u_{xx} = g(u, u_t, u_x)$

So $a = 1$, $b = 0$, $c = 0$

$\Rightarrow b^2 - ac = 0$

$\Rightarrow$ parabolic

(b) $u = X(x)T(t)$

$$XT' - 2X'T = X''T$$

$$\underbrace{\frac{T'}{T}}_{t} = \underbrace{\frac{X''}{X} + \frac{2X'}{X}}_{x}$$

So $\dfrac{T'}{T} = -\lambda$ const.

and $\dfrac{X''}{X} + \dfrac{2X'}{X} = -\lambda$

(c) $X'' + 2X' + \lambda X = 0$

multiply by integrating factor $-e^{2x}$ to get into SLP

$$-\frac{d}{dx}\left(e^{2x}X'\right) = \lambda e^{2x} X,$$ with boundary conditions $X(0) = X(1) = 0$

to solve,
Try $X = e^{\mu x}$

$$\mu^2 + 2\mu + \lambda = 0$$

$$(\mu+1)^2 - 1 + \lambda = 0$$

$$\mu = -1 \pm i\sqrt{\lambda - 1}$$

$$X = e^{-x}\left(A\cos\sqrt{\lambda-1}\, x + B\sin\sqrt{\lambda-1}\, x\right)$$

[n.b. no solutions if $\lambda < 1$ as cannot satisfy both $X(0) = 0$ and $X(1) = 0$]

$u = 0$ at $x = 0 \Rightarrow A = 0$, $B = 1$ wlog.

$u = 0$ at $x = 1 \Rightarrow$

$$X(1) = 0 \Rightarrow e^{-1} \cdot B\sin\sqrt{\lambda - 1} = 0$$

$$\sqrt{\lambda - 1} = n\pi$$

$$\lambda = 1 + n^2\pi^2.$$

(a) $\frac{T'}{T} = -(1 + n^2\pi^2)$

$T = \exp -(1+n^2\pi^2)t.$

$-\lambda < 0 \Rightarrow T$ / or / u decays as $t \to \infty$

(e)

General Solution

$$u = \sum_{n=1}^{\infty} \exp(-1+n^2\pi^2)t \sin(n\pi x)e^{-x}$$

given $u = f(x)$ at $t = 0$.

sub $t=0$ into ⊛ multiply by

weight function $\rightarrow e^{2x} e^{-x} \sin(n\pi x)$ and. integrate,

using orthogonality. of SL - eigenfunctions wrt to weight function

Q5. Unseen but similar types of problems seen on tutorial sheets.

(a) $\frac{3}{2} y^2 u_x + x u_y = -xy \cos^2 u$.

$$\frac{dx}{\frac{3}{2} y^2} = \frac{dy}{x} = \frac{du}{-xy \cos^2 u}.$$

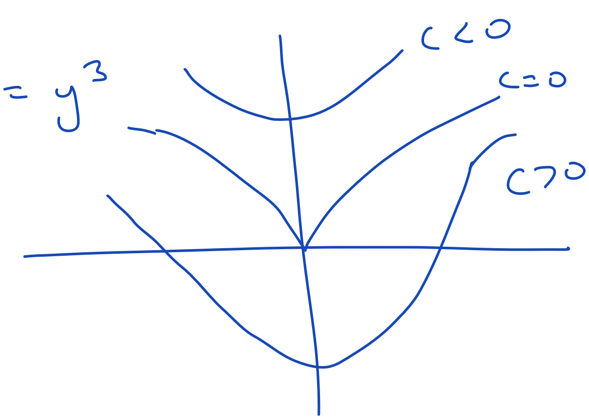
Characteristics

$$\int x \, dx = \int \frac{3}{2} y^2 \, dy$$

$$\frac{x^2}{2} = \frac{3y^3/3}{2} + C$$

So $x^2 = y^3 + C$.

$C = 0 \quad x^2 = y^3$



(b) $$\frac{dy}{x} = \frac{du}{-xy\cos^2 u}$$

$$\int y\,dy = -\int \sec^2 u\,du$$

$$\frac{y^2}{2} + D = -\tan u$$

For a characterstic starting at y_0, (when $x=0$)

$$x^2 = y^3 - y_0^3 \quad \Rightarrow \quad y_0 = (x^2 - y^3)^{1/3}$$

and $$\frac{y_0^2}{2} + D = -\tan\left(\underbrace{\arctan y_0}_{\uparrow\text{ using initial condition}}\right)$$

$$D = -y_0 - y_0^2/2$$

So $$\tan u = -\frac{y^2}{2} - D$$

$$= \frac{y_0^2 - y^2}{2} + y_0$$

$$u = \arctan\left\{\frac{(x^2-y^3)^{2/3} - y^2}{2} + (x^2-y^3)^{1/3}\right\}$$

(c) A solution will only be obtainable for $y \leq x^{2/3}$ as otherwise no characteristics go from boundary.