Q1. $$y' = \frac{y^2 + 2xy}{x^2 + xy}$$

$$= \frac{(y/x)^2 + 2(y/x)}{1 + (y/x)}$$

Let $y = v \cdot x \quad \Rightarrow \quad y' = v'x + v$

So $$v'x + v = \frac{v^2 + 2v}{1+v}$$

$$\Rightarrow \quad x v' = \frac{v^2 + 2v}{1+v} - \frac{v(1+v)}{1+v} = \frac{v}{1+v}$$

$$\Rightarrow \quad \int \frac{1+v}{v}\, dv = \int \frac{1}{x}\, dx$$

$$\Rightarrow \quad \log v + v = \log x + \text{Constant}$$

$$\Rightarrow \quad v e^{v} = x C$$

$$\Rightarrow \quad y/x \, e^{y/x} = x C$$

$$\Rightarrow \quad \frac{y}{x^2} e^{y/x} = C$$

Comments : a standard style of question seen on tutorial sheets but for a new problem. Algebraically relatively straightforward.

ILO 1

Q2. $y' = \underbrace{3x \cos y}_{f}$ $\qquad y(0) = 3$

(a) $f = 3x \cos y$

$\partial f / \partial y = -3x \sin y$

So both f, $\partial f/\partial y$ are cts functions (for all x, y, including near initial point) so by Picard's theorem expect unique solution near initial point.

(b) $$y_{n+1} = y_0 + \int_0^x f(x', y_n(x'))\, dx'$$

$y_0 = 3$ constant

$$y_1 = y_0 + \int_0^x 3x' \cos y_0 \, dx'$$

$$= y_0 + 3\cos 3 \, \frac{x^2}{2}$$

(c) $|f| = 3|x|\underbrace{|\cos y|}_{\leq 1} \leq 3$

since $0 \leq x \leq 1$ and $|\cos y| \leq 1$.

So $|y'| \leq 3$. Let Δy denote change in y over interval

So $\left|\dfrac{\Delta y}{\Delta x}\right| \leq 3$

So $|\Delta y| \leq 3\,\Delta x \leq 3$

but $y(0) = 3$, so $|y(x) - y(0)| \leq 3$.

$\Rightarrow 0 \leq y(x) \leq 6$.

Comments : Part (a) is a standard type of question used previously.
ILO 2

Part (b) is similar to tutorial sheet problems

Part (c) is non. standard and designed to test understanding of one of the criteria in Picard's theorem

Q3 $$x^2 y'' + x y' - y = \delta(x - \tfrac{1}{2})$$

homogeneous problem $x^2 y'' + x y' - y = 0$

equidimensional. Let $y = x^\alpha$

$$\Rightarrow \quad \alpha(\alpha - 1) + \alpha - 1 = 0$$

So $\alpha = \pm 1$

So $v_1 = x$ $\qquad v_2 = x^{-1}$

- $y_1 = \alpha v_1 + \beta v_2$

  $y_1(0) = 0$ so $\beta = 0$ and $\alpha = 1$ wlog.

  So $y_1 = x$

- $y_2 = \alpha v_1 + \beta v_2$

  $y_2(1) = 0$ so $\alpha \cdot 1 + \beta 1^{-1} = 0$

  So $\beta = -\alpha$ $\quad (\alpha = 1$ wlg$)$

  $y_2 = x - x^{-1}$

(b) Hence $G = \begin{cases} A\,x & x < s \\ B\,(x - x^{-1}) & x > s \end{cases}$

$$\partial G/\partial x = \begin{cases} A & x < s \\ B(1 + x^{-2}) & x > s \end{cases}$$

where $A = A(s)$, $B = B(s)$

G cts at $x = s \Rightarrow$ $A s = B(s - s^{-1})$

$$\Rightarrow A = B(1 - s^{-2})$$

Step condition

$$B(1 + s^{-2}) - A = \frac{1}{s^2}$$

$$\Rightarrow B(1 + s^{-2}) - B(1 - s^{-2}) = s^{-2}$$

$$\Rightarrow B = \frac{1}{2}$$

$$A = \frac{1}{2}(1 - s^{-2})$$

So $G = \frac{1}{2} \begin{cases} (1 - s^{-2})\,x & x < s \\ x - x^{-1} & x > s \end{cases}$

(c) $y = G(x, \tfrac{1}{2}) = \frac{1}{2} \begin{cases} -3x & x < \frac{1}{2} \\ x - x^{-1} & x > \frac{1}{2} \end{cases}$

$$= -\frac{3}{2} \begin{cases} x & x < \frac{1}{2} \\ x^{-1} - x & x > \frac{1}{2} \end{cases}$$

Comment : Common type of question.
ILO 3 New example but similar on problem sets

Q4 $\quad S(x) = \sum_{-\infty}^{\infty} c_n e^{inx} \quad$ on $\quad -\pi \le x < \pi$

(a) If $m = n$, $\displaystyle\int_{-\pi}^{\pi} e^{i(n-m)x}\,dx = \int_{-\pi}^{\pi} 1\,dx = 2\pi$

and if $m \ne n$, $\displaystyle\int_{-\pi}^{\pi} e^{i(n-m)x}\,dx = \frac{1}{i(n-m)}\Big[e^{i(n-m)x}\Big]_{-\pi}^{\pi}$

$$= \frac{1}{i(n-m)}\Big[\left(e^{i\pi}\right)^{n-m} - \left(e^{-i\pi}\right)^{n-m}\Big]$$

$$= 0$$

since $e^{i\pi} = -1$ and $e^{-i\pi} = -1$.

So if $S(x) = f(x)$, then

$$\int_{-\pi}^{\pi} e^{-imx} f(x)\,dx = \sum_{-\infty}^{\infty} c_n \int_{-\pi}^{\pi} e^{i(n-m)x}\,dx$$

$$= 2\, c_m$$

ie (relabelling $m \to n$.) $\quad c_n = \frac{1}{2}\displaystyle\int_{-\pi}^{\pi} f(x)\, e^{-inx}\,dx$

(b) $f(x) = \begin{cases} 0 & -\pi \le x < \lambda \\ 1 & \lambda \le x < \pi \end{cases}$

$$c_n = \frac{1}{2}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx$$

$$= \frac{1}{2}\int_{\lambda}^{\pi} e^{-inx}\, dx$$

$$= \frac{1}{-2in}\left(e^{-in\pi} - e^{-in\lambda}\right)$$

$$= \frac{i}{2n}\left((-1)^n - \cos(n\lambda) + i\sin(n\lambda)\right)$$

$$= \frac{-1}{2n}\sin(n\lambda) + i\,\frac{1}{2n}\left((-1)^n - \cos(n\lambda)\right)$$

Comments: ILO 4. (background)

(a) is intended as a test of understanding orthogonality and is a variation of a tutorial sheet question

(b) is slightly unfamiliar (most practice was for sine/cosine form).

Q5 $\quad u_{tt} = c^2 u_{xx}$

(a) Hyperbolic

(b) $u = X(x)T(t)$

$\Rightarrow \quad XT'' = c^2 X''T$

$$\Rightarrow \quad \underbrace{\frac{T''}{c^2 T}}_{\text{function of } t} = \underbrace{\frac{X''}{X}}_{\text{function of } x}$$

so both constants equal to $-\lambda$, say.

(c) $\dfrac{X''}{X} = -\lambda$ or $-X'' = \lambda X$ is SL form

$u_x = 0$ at $x = 0$ so $X'(0) = 0$

$u = 0$ at $x = \pi$ so $X(\pi) = 0$

If $\lambda \leq 0$, there are no solutions that satisfy the boundary conditions

So $\lambda = \nu^2 > 0$

$-X'' = \nu^2 X \Rightarrow X = A\cos(\nu x) + B\sin(\nu x)$

$X' = -\nu A\sin(\nu x) + \nu B\cos(\nu x)$

So $X'(0) = 0 \Rightarrow B = 0$

So $X = A\cos(\nu x)$

and $X(\pi) = 0 \Rightarrow \cos(\nu\pi) = 0$

So $\nu = \frac{1}{2}(2k+1) \quad k = 1, 2, 3, \ldots$

and $X = \cos\left(\frac{1}{2}(2k+1)x\right)$ (A = 1 wlg)

(a) $\frac{T''}{c^2 T} = \lambda = \nu^2$

Let $\omega = \nu c$

Then $T = A\cos(\omega t) + B\sin(\omega t)$

But $u_t = 0 \rightarrow T'(0) = 0$

So $B = 0$

So $T = A\cos(\omega t)$
$= A\cos\left(\frac{1}{2}(2k+1)ct\right)$

$$u = \sum_{k=1}^{\infty} A_k \cos\left(\frac{1}{2}(2k+1)ct\right)\cos\left(\frac{1}{2}(2k+1)x\right)$$

Comments : Question is on a familiar PDE but with novel and somewhat tricky boundary conditions to provide a test.

ILO 4.

Q6 $\quad (x - yu)u_y - (y + xu)u_x = 1 + u^2$

$$\frac{dx}{ds} = -y - xu \qquad (1)$$

$$\frac{dy}{ds} = x - yu \qquad (2)$$

$$\frac{du}{ds} = 1 + u^2 \qquad (3)$$

Consider $\frac{d}{ds}(ux) = u\frac{dx}{ds} + x\frac{du}{ds}$

$$= -yu - xu^2 + x + xu^2$$

$$= x - yu = \frac{dy}{ds}$$

So $A = y - ux$ is constant along characteristics

Consider $\frac{d}{ds}(uy) = u\frac{dy}{ds} + y\frac{du}{ds}$

$$= ux - yu^2 + y + yu^2$$

$$= ux + y$$

$$= -\frac{dx}{ds}$$

So $B = x + uy$ is constant along characteristics

(b) If $u = 1$ on $x = 1$

$$A = y - 1$$
$$B = y + 1$$

So $B - A = 2$

But B, A constant along characteristics so

$$x + yu - (y - ux) = 2$$

$$u(y + x) = 2 - x + y$$

$$u = \frac{2 - x + y}{x + y}$$

Comments: A significant hint is provided in part (a), otherwise would be tricky to spot (but still more tricky than a 'Show that' question.

ILO 3

(b) is straightforward but relies on (a).

Q7

$a = 1$

$b = \frac{1}{2}(y-x)$

$c = -xy$

$\pm$ characteristic $\quad \frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - ac}}{a}$

$= \frac{\frac{1}{2}(y-x) \pm \sqrt{\frac{1}{4}(y-x)^2 + xy}}{1}$

$= \frac{1}{2}(y-x) \pm \frac{1}{2}(y+x)$

$+$ root. $\quad \frac{dy}{dx} = y \rightarrow y = \xi e^{x}$

ξ constant.

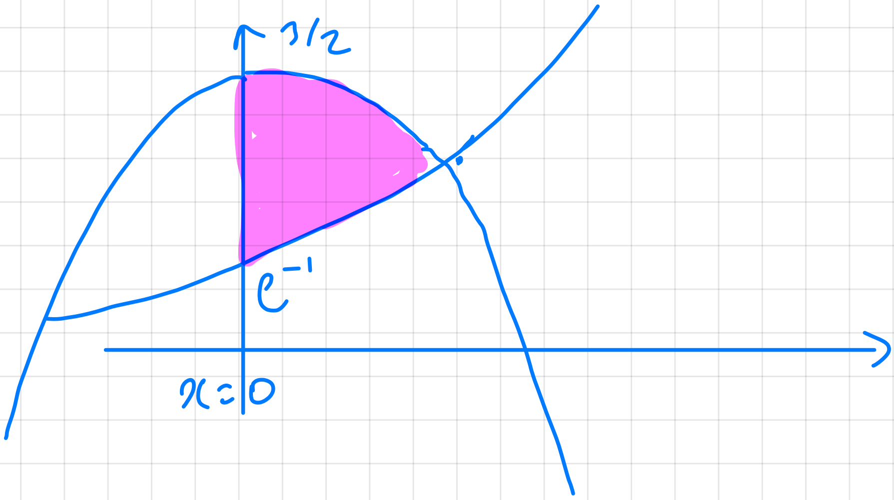
$-$ root $\quad \frac{dy}{dx} = -x \rightarrow y = \eta - x^2/2$

(b) Domain of dependence of $(1,1)$

$\xi = e^{-1}$ and $\eta = 3/2$

So $y = e^{x-1}$ and $y = \frac{3}{2} - x^2/2$

go through this point.



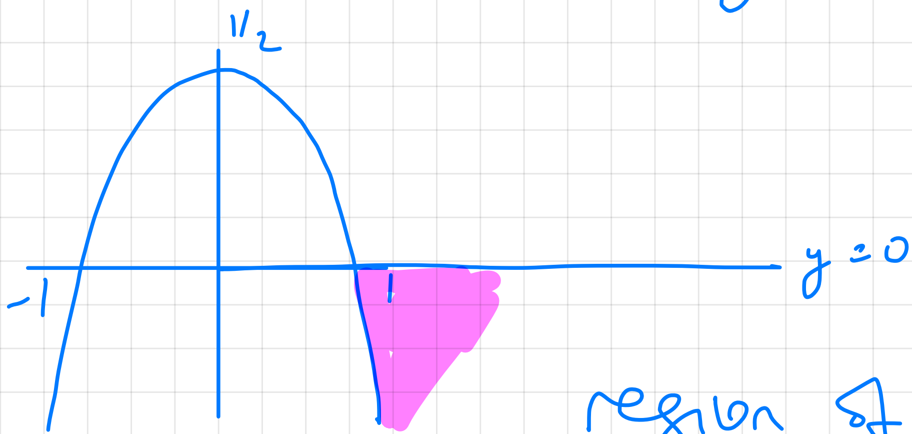
For initial data specified along positive y axis $(x=0, y>0)$ domain of dependence is

$$e^{-1} \le y \le 3/2$$

(c) For $(1,0)$, $\xi = 0$ $\eta = \frac{1}{2}$

So characteristics are $y=0$

and $y = \frac{1}{2} - \frac{x^2}{2}$



region of influence

is $\frac{1}{2} - x^2 \le y \le 0$

Comments
ILO 6.

This is unseen but with relatively simple algebra. The final parts are often found tricky.