TP2 : K-plus proches voisins, Analyse discriminante et regression logistique

Question 15 :

$$\Phi(x) = \frac{\mathbb{P}(X=x \mid Y=+1)}{\mathbb{P}(X=x \mid Y=-1)} = \frac{\mathbb{P}(Y=+1 \mid X=x)\,\frac{\cancel{\mathbb{P}(X=x)}}{\mathbb{P}(Y=+1)}}{\mathbb{P}(Y=-1 \mid X=x)\,\frac{\cancel{\mathbb{P}(X=x)}}{\mathbb{P}(Y=-1)}} \quad \text{(BAYES)}$$

$$\Rightarrow \Phi(x) = \frac{\mathbb{P}(Y=+1 \mid X=x)/\pi_+}{\mathbb{P}(Y=-1 \mid X=x)/(1-\pi_+)}$$

$$\Rightarrow \frac{\pi_+ \Phi(x)}{1-\pi_+} = \frac{\mathbb{P}(Y=+1 \mid X=x)}{\mathbb{P}(Y=-1 \mid X=x)}$$

$$\Rightarrow \frac{\pi_+ \Phi(x)}{1-\pi_+} = \frac{\mathbb{P}(Y=+1 \mid X=x)}{1-\mathbb{P}(Y=+1 \mid X=x)}$$

$$\Rightarrow \boxed{\mathbb{P}(Y=+1 \mid X=x) = \frac{\pi_+ \Phi(x)}{1-\pi_+ + \pi_+ \Phi(x)}}$$

$$\Rightarrow \mathbb{P}(Y=+1 \mid X=x) = \frac{\pi_+ . f_+/f_-}{1-\pi_+ + \pi_+ . f_+/f_-}$$

$$\begin{aligned}
\mathbb{P}(Y=-1 \mid X=x) &= 1 - \mathbb{P}(Y=+1 \mid X=x) \\
&= 1 - \frac{\pi_+ \Phi(x)}{1-\pi_+ + \pi_+ \Phi(x)} \\
&= \frac{1-\pi_+ + \pi_+ \Phi(x) - \pi_+ \Phi(x)}{1-\pi_+ + \pi_+ \Phi(x)} \\
&= \frac{1-\pi_+}{1-\pi_+ + \pi_+ \Phi(x)}
\end{aligned}$$

$$\boxed{\mathbb{P}(Y=-1 \mid X=x) = \frac{1-f_+/f_-}{1-\pi_+ + \pi_+ f_+/f_-}}$$

Question 16 :

D'après la question précédente, on a :

$$\Phi(x) = f_+ / f_- = \frac{\mathbb{P}(Y=+1 \mid X=x) / \pi_+}{\mathbb{P}(Y=-1 \mid X=x) / (1-\pi_+)} \quad (*)$$

$$\Rightarrow \frac{\mathbb{P}(Y=+1 \mid X=x)}{\mathbb{P}(Y=-1 \mid X=x)} = \frac{f_+ \cdot \pi_+}{f_- \, (1-\pi_+)} = \frac{\exp\left[-\frac{1}{2}(x-\mu_+)^T \Sigma^{-1} (x-\mu_+)\right] \pi_+}{\exp\left[-\frac{1}{2}(x-\mu_-)^T \Sigma^{-1} (x-\mu_-)\right] (1-\pi_+)}$$

$$\Rightarrow \log\left[\frac{\mathbb{P}(Y=+1 \mid X=x)}{\mathbb{P}(Y=-1 \mid X=x)}\right] = -\frac{1}{2}(x-\mu_+)^T \Sigma^{-1} (x-\mu_+) + \frac{1}{2}(x-\mu_-)^T \Sigma^{-1} (x-\mu_-) + \log(\pi_+) - \log(1-\pi_+)$$

$$\Rightarrow \boxed{\log\left[\frac{\mathbb{P}(Y=+1 \mid X=x)}{\mathbb{P}(Y=-1 \mid X=x)}\right] = x^T \Sigma^{-1} (\mu_+ - \mu_-) + \frac{1}{2}\left(\mu_-^T \Sigma^{-1} \mu_- - \mu_+^T \Sigma^{-1} \mu_+\right) + \log(\pi_+) - \log(1-\pi_+)}$$

Question 17 :

Supposons que $\mathbb{P}(Y=+1 \mid X=x) > 1/2$

On a alors d'après $(*)$

$$\Phi(x) = \frac{\mathbb{P}(Y=+1 \mid X=x) / \pi_+}{\mathbb{P}(Y=-1 \mid X=x) / (1-\pi_+)} > \frac{1/\pi_+}{1/(1-\pi_+)} = \frac{(1-\pi_+)}{\pi_+}$$

$$\Rightarrow \frac{f_+}{f_-} > \frac{(1-\pi_+)}{\pi_+} \Rightarrow \frac{f_+ \, \pi_+}{f_- \, (1-\pi_+)} > 1$$

$$\Rightarrow \frac{\mathbb{P}(Y=+1 \mid X=x)}{\mathbb{P}(Y=-1 \mid X=x)} > 1$$

$$\Rightarrow \log\left[\frac{\mathbb{P}(Y=+1 \mid X=x)}{\mathbb{P}(Y=-1 \mid X=x)}\right] \geqslant 0$$

d'après la Q16 $\Rightarrow x^T \Sigma^{-1}(\mu_+ - \mu_-) + \frac{1}{2}\left(\mu_-^T \Sigma^{-1}\mu_- - \mu_+^T \Sigma^{-1}\mu_+\right) + \log(\pi_+) - \log(1-\pi_+) \geqslant 0$

On a alors la règle de décision suivante :

$$y = \begin{cases} +1 & \text{si } x^T \Sigma^{-1}(\mu_+ - \mu_-) \geqslant -\frac{1}{2}\left(\mu_-^T \Sigma^{-1}\mu_- - \mu_+^T \Sigma^{-1}\mu_+\right) - \log(\pi_+) + \log(1-\pi_+) \\ -1 & \text{sinon} \end{cases}$$