Reinforcement Learning in Digital Finance

Policy-based reinforcement learning

Funded by
the European Union

Lecture agenda

- Policy gradient algorithms
  - Introduction to policy function approximation
  - Examples
  - Derivation of theory
  - Discrete and continuous policies
  - Actor-critic models
  - Advanced policy gradients

Policy gradient methods

Value function approximation [1/2]

- So far, we stayed close to Dynamic Programming paradigm:
  - Replace true value functions V with approximation $\bar{V}$
  - Several ways to find suitable $\bar{V}$ (e.g., Q-table, features)
  - Finding optimal value functions equates finding optimal policy

- Disadvantages of VFA:
  - Falls apart for continuous- and large action spaces
    - Must evaluate $\bar{V}(s, a)$ for every action a in state s.
  - Indirect and unnatural way of decision-making
    - Dynamic Programming not intuitive for everyone

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Value function approximation [2/2]

- We already abandoned optimality, no need to stick to Dynamic Programming approach
- Objective is <u>not</u> to solve Bellman equation, but to maximize reward over certain time horizon!

- Alternative: Directly adjust decision-making policy
  - Often more natural
  - Value functions are just a *means* to improve policy
- Recall: policy simply maps state to action!

$$\pi : s \rightarrow a$$

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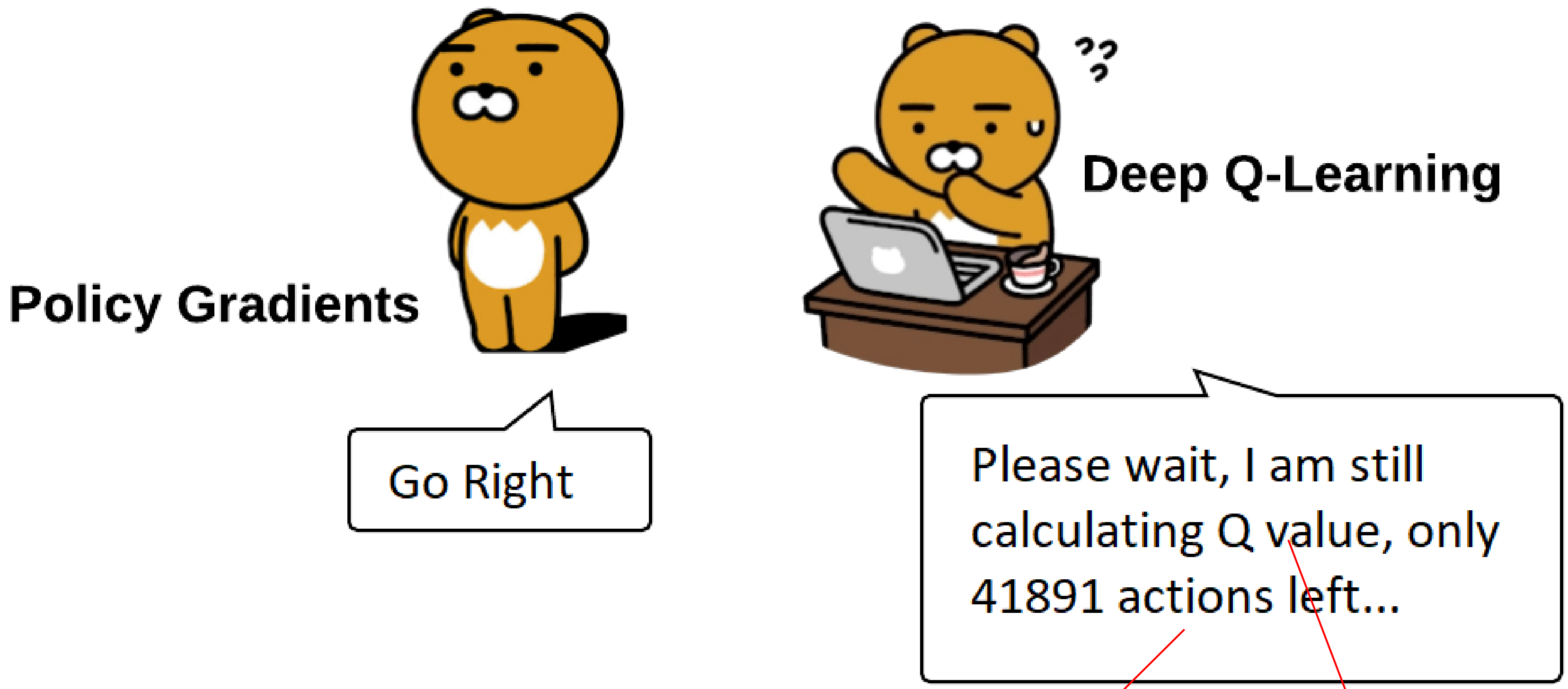
Policy function approximation – PFA vs VFA

Source: https://pylessons.com/Beyond-DQN

Note: discrete policy gradients still require full enumeration of action probabilities!



$$a = \pi_\theta(s)$$

$$a = \operatorname*{argmax}_{\boxed{a\in\mathcal{A}}}[r(s,a) + Q(s,a)]$$

Policy function approximation

- How do we improve a policy?

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Policy function approximation

- How do we improve a policy?

- Basic mechanism:
  - Define policy π_θ with tunable parameters θ
  - Take actions according to policy
  - Observe corresponding rewards
    - Typically reward trajectory $r(\tau) = \sum_{t=0}^{T} r_t$
  - Adjust policy π_θ (i.e., adjust parameters θ)
  - Observe whether rewards improve
  - Repeat

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Policy function approximation

- But: how do we know in what direction to update policy?
  - Sell higher/lower? Keep less/more inventory?

- A possible solution is to work with stochastic policies.
  - Allows measuring the *difference* between actions
  - So far, we used policies $\pi: s \rightarrow a$ (deterministic)
  - Now, we will use policies $\pi: s \rightarrow \mathbb{P}(a|s)$ (stochastic)

- We have two sources of information: (i) reward trajectory and (ii) probability of trajectory
  - Intuition: increase probability of high-reward trajectories

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Policy function approximation

- We adjust the tunable policy π_θ based on observed reward- and probability trajectories.
  - Mathematically speaking, we compute the gradient
    - Gradient is simply a vector of partial derivatives for each θ

- We can express the gradient as an *expectation*, thus we can use simulation (*sampling*) to approximate it

- PFA *may* tackle both large state- and action spaces
  - However, there are drawbacks as well

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Policy function approximation

- Gradient method not the only way to tailor policy
- Non-gradient solutions:
  - Genetic algorithms
  - SIMPLEX
  - Hill climbing
- Gradient methods often more efficient
  - Stochastic gradient descent
  - Newton's method
- This course focuses on policy gradient methods.

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Stochastic policies – Probabilistic actions

- So far, we worked with *deterministic* policies
  - Return single action for a given state $\pi: s \rightarrow a$
- Now, the policy is a probability function:
  - Discrete action space: assign probability to each action
  - Continuous action space: draw action from distribution

- Policy gradient:
  - Probability function (policy) should be differentiable
  - Tune differentiable parameters θ
    - Linear scheme: $\phi(s, a) \cdot \theta$

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Stochastic policies – General form

- General form of stochastic policy
  - Conditional probability function: $\pi(a|s) = \mathbb{P}(a|s)$
  - Map state to action probabilities: $s \rightarrow \mathbb{P}(a|s)$

- Examples:
  - Softmax: $\pi_\theta = e^{\phi(s,a)\theta}$
  - Gaussian: $\pi_\theta = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(a-\mu)^2}{2\sigma^2}}$
  - Neural network: $N_\theta : s \rightarrow \mathbb{P}(a|s)$

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Stochastic policies - Benefits

- Benefits compared to value-based learning (Q-learning)
  - Theoretically smooth updates, just follow gradient
    - Should lead to at least a local optimum
  - Often effective in (relatively) high-dimensional and continuous action spaces (latter yields infinite value functions)
  - Exploration mechanism already built in
    - Much exploration under high uncertainty, limited exploration once good policy has been found
  - Sometimes necessary (e.g., rock, paper, scissors)
    - Especially in multi-agent settings, opponents may counter deterministic policies.

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Stochastic policies - Drawbacks

- Disadvantages of stochastic policies
  - Discrete action space should be enumerable
    - Not always realistic for combinatorial optimization
  - High variance in reward trajectories
    - Requires full reward trajectory (like in Monte Carlo)
    - No bootstrapping as in SARSA or Q-learning

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Policy gradient – An example

Cliff walking example – PFA

- Four actions at each tile (up, down, left, right)
  - Softmax policy attaches probability to each action
  - Note we do not explicitly include downstream effects!

$$\pi_\theta(s,a) \propto e^{\phi(s,a)^\top \theta}$$

$$\pi_\theta(s,a) = \frac{e^{\phi(s,a)^\top \theta}}{\sum_{a' \in \mathcal{A}} e^{\phi(s,a')^\top \theta}}$$

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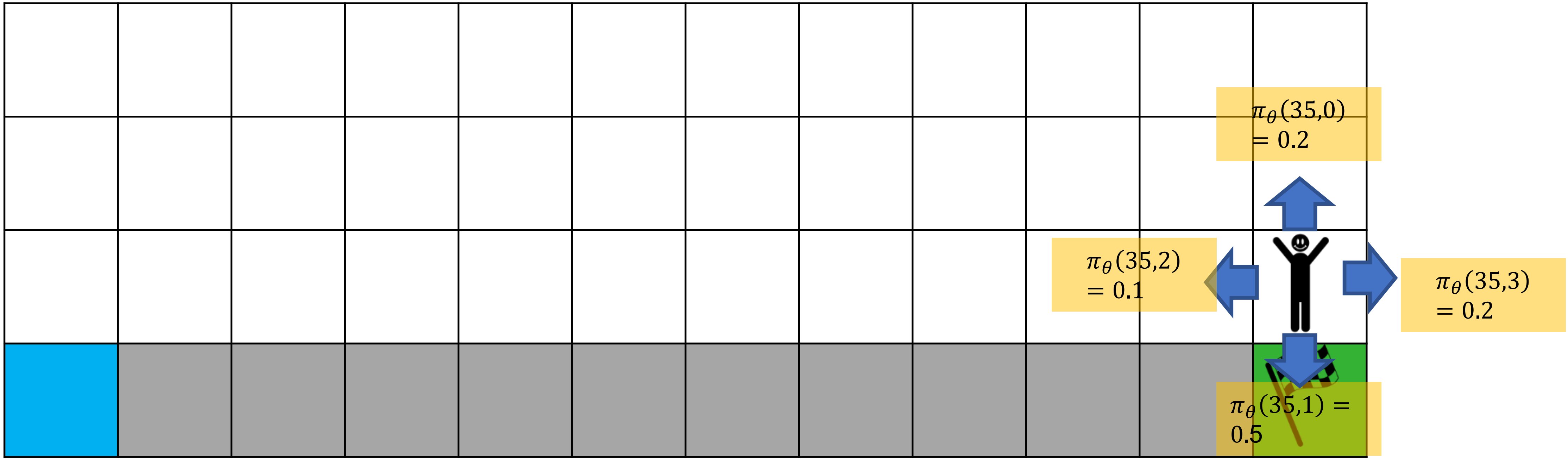
Cliff walking example – Feature design

- Feature vector: one-hot encoding of state-action pair
  - Feature vector of length $48 \cdot 4$, format $\phi(s, a) = [0,0 \ldots, 0,1, \ldots, 0,0]$
  - In this case, requires feature weights θ for each state-action pair state (vector $\theta = \mathbb{R}^{|48 \cdot 4|}$)
  - For sake of illustration, features are similar to lookup table
    - Approach can be generalized to more abstract features

  - In this example, a four-dimensional vector is used for conciseness!

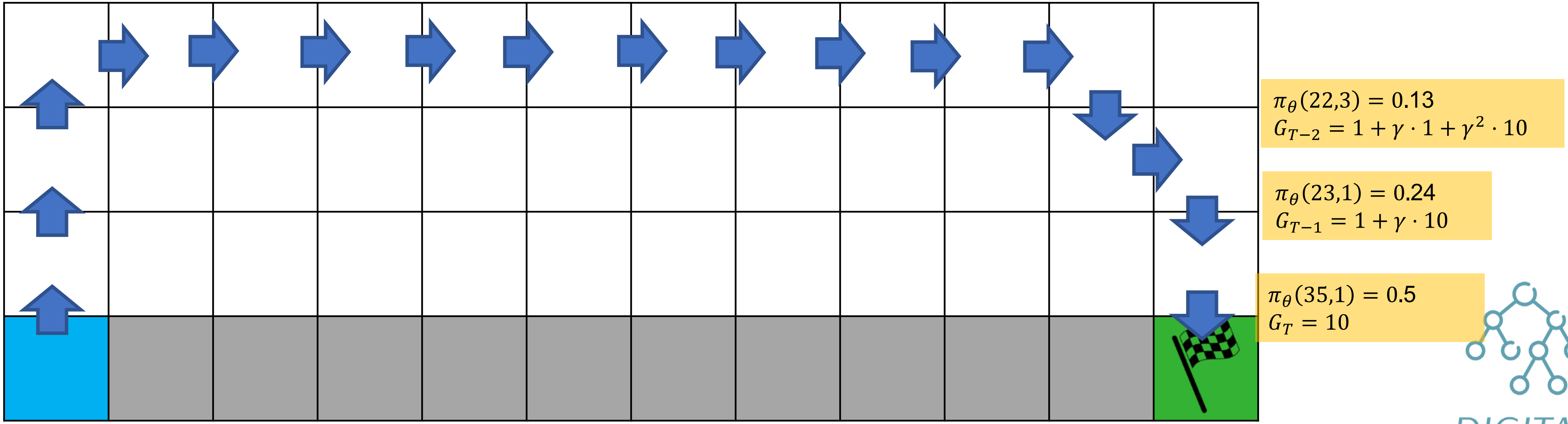
Cliff walking problem – Decision-making [1/2]

- Select action according to probability π_θ
  - In a sense, we are always exploring!



Cliff walking problem – Decision-making [2/2]

- Suppose we selected action 1 ('down')
- We can now observe full reward trajectory $R(\tau)$
  - Derive cumulative rewards $G_t = r_t + \gamma r_{t+1} \dots + \gamma^{T-t} r_T$



Cliff walking example – Initialization

- Initialize probabilities for each action to 0.25
  - Simply achieved by setting ($\theta_{s,a} = 0, \forall s, a$)
- In this example:
  - Attach weight to each state-action pair
  - Comparable to Q-table (learn parameter for each (s, a))

$$\theta = \begin{bmatrix} \theta_{0,0} & \cdots & \theta_{0,|\mathcal{A}|} \\ \vdots & \ddots & \vdots \\ \theta_{|S|,0} & \cdots & \theta_{|S|,|\mathcal{A}|} \end{bmatrix}$$

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Cliff walking example – Learning procedure

- Select action according to π_θ
- Run trajectory $\tau = s_0, a_0, \ldots s_T, a_t$
  - Store action probabilities and rewards of full trajectory
- After completion, loop over all time steps
  - Compute G_t
  - Compute $\phi(s,a)$ and probability-weighted vector $\sum_{a' \in \mathcal{A}} \phi(s,a)$
  - Compute score function (gradient of softmax)
    - Partial derivative for each feature
    - $\nabla_\theta \log_\theta(s,a) = \phi(s,a) - \mathbb{E}_{\pi_\theta}[\phi(s,\cdot)]$
- Update weights
  - $\theta = \theta + \alpha \nabla_\theta \log_\theta(s,a)$

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Cliff walking example

- Feature vector (selected action):

$$\phi(s,a) = [0,1,0,0]$$

(keep in mind, original is $48 \cdot 4!$)

- Expected feature vector (weighted over all actions):

$$\mathbb{E}_{\pi_\theta}\phi(s,\cdot) = \sum_{a' \in \mathcal{A}} \phi(s,a)$$

$$= [1,0,0,0] \cdot 0.2 + [0,1,0,0] \cdot 0.5 + [0,0,1,0] \cdot 0.1 + [0,0,0,1] \cdot 0.1$$
$$= [0.2, 0.5, 0.1, 0.2]$$

- Note we weight all possible feature vectors given s

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Cliff walking example – Weight update

- Score function:
$$\phi(s,a) - \mathbb{E}_{\pi_\theta}[\phi(s,\cdot)] = [0,1,0,0] - [0.2, 0.5, 0.1,0]\,.$$

- Weight update:
$$\Delta\theta = \alpha * \nabla_\theta \log_\theta(s,a) * G_t = 0.01 \cdot [-0.2,0.5,-0.1,-0.2] \cdot 10$$
$$= [-0.02, \mathbf{0.05}, -0.01, -0.02]$$

- Result: update increases weight of second action (due to positive reward)
  - In the future, we select this action with higher probability
  - High reward/low probability → strong update

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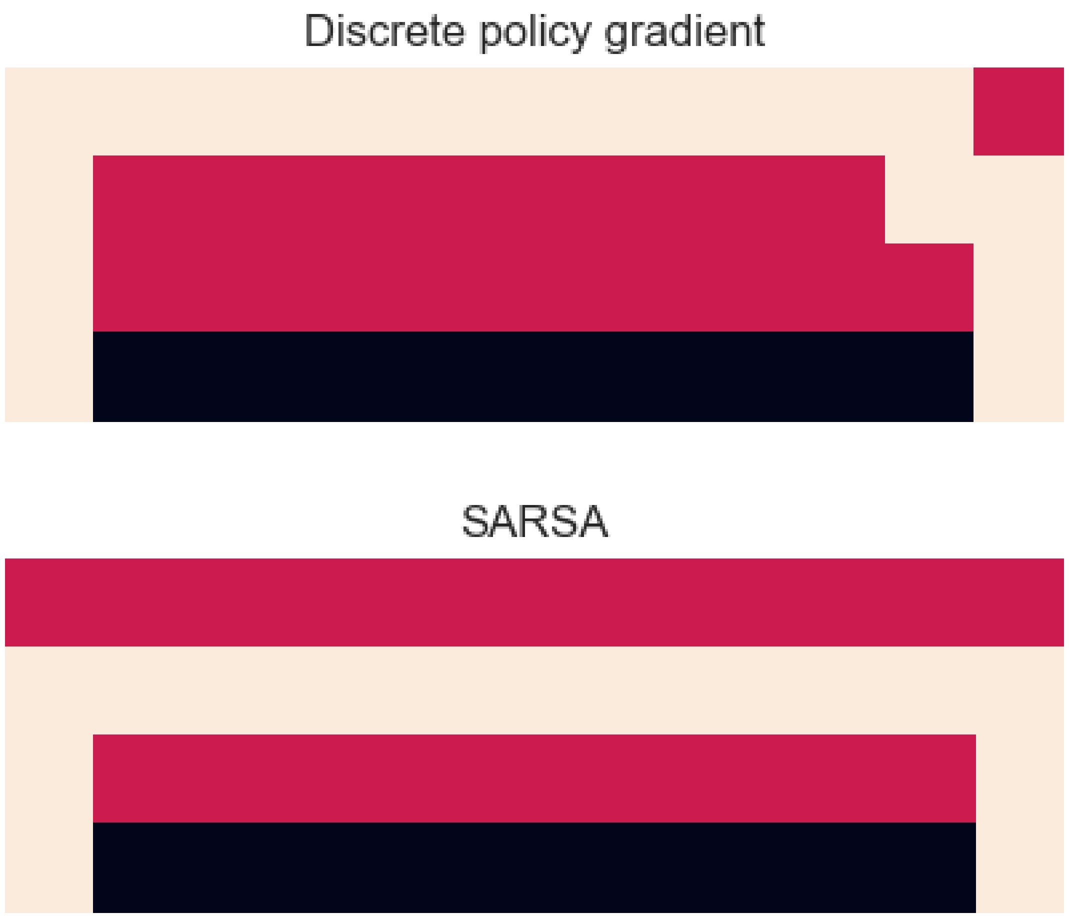
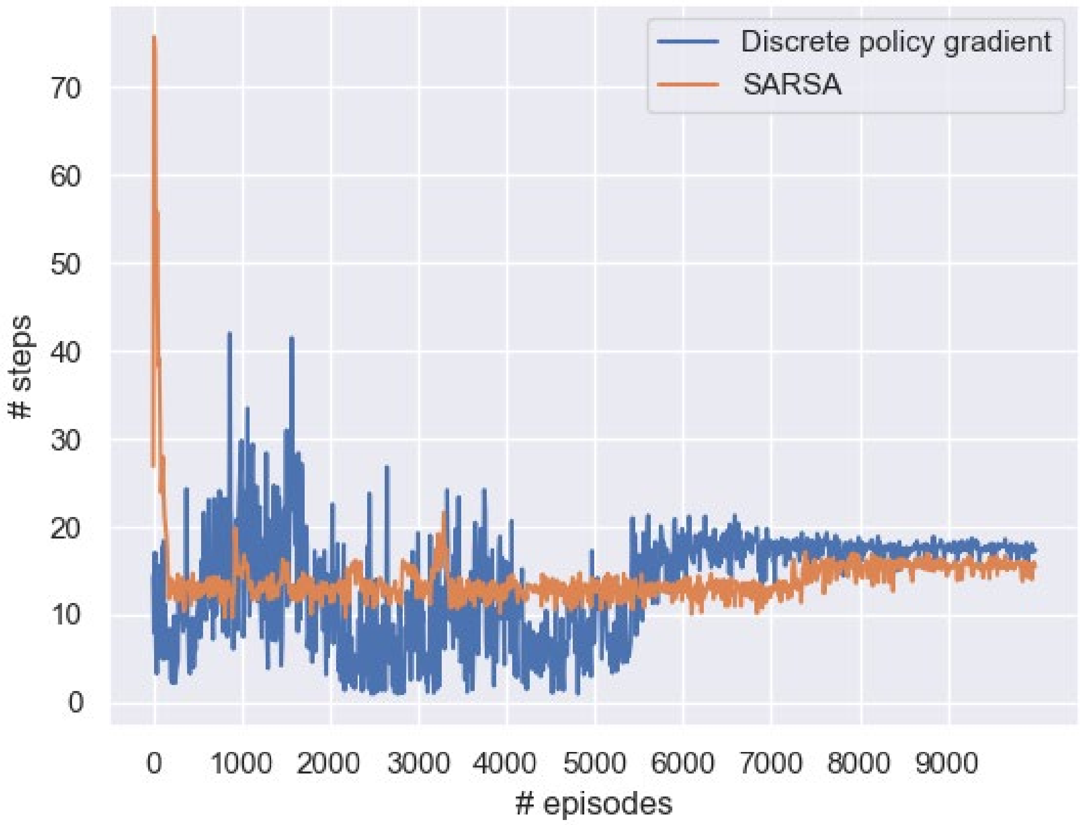
Cliff walking example – Policy learning

- Like SARSA and Monte Carlo learning, policy gradients are on-policy
  - Walking in the cliff contributes to actual reward
  - SARSA explored just 5%, much more exploration here (at least initially)
  - Also, reward variance (of full trajectory) is high, just as for Monte Carlo learning

  - What type of policy do you expect?

Cliff walking example – Resulting policy





Deriving the policy gradient method

Policy gradient derivation

- We now know the application, but why does it work?
  - Mathematically involved procedure based on calculus
  - For exam: not needed to understand all details, but ensure you can explain the rationale

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Policy gradient derivation – Objective function

- To start, we need an objective function that can be influenced by changing θ
- Like before, we optimize expected cumulative rewards over time:

$$J(\theta) = E_{\tau \sim \pi_\theta} R(\tau) = \sum_\tau P(\tau;\theta) R(\tau)$$

- Probability of trajectory $\tau = s_1, a_1, \ldots, s_T, a_T$ affected by θ
  - Note that $P(\tau;\theta)$ is conditional on θ
  - Rewards $R(\tau)$ depend on trajectory

Policy gradient derivation – Maximization function

- The optimization problem corresponds to maximizing the objective function

$$\max_{\theta} J(\theta) = \max_{\theta} \sum_{\tau} P(\tau;\theta) R(\tau)$$

- Aim to maximize the expected reward
- Alternatively: we alter the policy (parameterized by θ) in a way that increases the probability of high-reward trajectories
    - Remind that policy affects probability per action, and thus the trajectory probabilities

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Policy gradient derivation – Probability function

- Let's zoom in on the probability function

$$P(\tau;\theta) = \left[\prod_{t=0}^{T} \underbrace{P(s_{t+1} \mid s_t, a_t)}_{\substack{\text{Transition function} \\ \text{(environment model)}}} \cdot \underbrace{\pi_\theta(a_t \mid s_t)}_{\substack{\text{Policy} \\ \text{(control function)}}}\right]$$

- Two components:
  - Stochastic policy π_θ that samples an action
  - Transition function (time step) that includes ω like before

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Policy gradient derivation – Probability function

- Two problems with this model

$$P(\tau;\theta) = \left[\prod_{t=0}^{T} \underbrace{P(s_{t+1} \mid s_t, a_t)}_{\text{Transition function (environment model)}} \cdot \underbrace{\pi_\theta(a_t, s_t)}_{\text{Policy (control function)}}\right]$$

1. Transition function may be hard to model, or even unknown
2. Product of probabilities yields very small probabilities per trajectory
    - Programming languages have finite precision

- Let's leave these problems for now…

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Policy gradient derivation – Maximization function

- Let's revisit the maximization problem

$$\max_{\theta} J(\theta) = \max_{\theta} \sum_{\tau} P(\tau;\theta) R(\tau)$$

- If this function is differentiable, we can compute the gradient
  - Move policy into the direction of (local) op
    - Steep slope: large updates
    - Gentle slope: cautious updates

$$\nabla_{\theta} J(\theta) = \begin{bmatrix} \frac{\delta J(\theta)}{\delta \theta_1} \\ \frac{\delta J(\theta)}{\delta \theta_2} \\ \vdots \\ \frac{\delta J(\theta)}{\delta \theta_N} \end{bmatrix}$$

- How do we differentiate this function?

Policy gradient derivation – Rewriting [1/3]

- First, recall the equivalence between expected reward

$$\nabla_\theta J(\theta) = \nabla_\theta \mathbb{E}_{\tau \sim \pi_\theta} R(\tau)$$

and probability-weighted rewards

$$= \nabla_\theta \sum_\tau P(\tau;\theta) R(\tau)$$

- The probability-weighted expression is needed to apply sampling (i.e., Monte Carlo simulation)

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Policy gradient derivation – Rewriting [2/3]

- Next, let's bring the gradient sign within the sum

$$= \sum_{\tau} \nabla_{\theta} P(\tau;\theta) R(\tau)$$

- (a gradient of sums equals the sum of gradients)

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Policy gradient derivation – Rewriting [3/3]

- Now, we rewrite the expression to

$$= \sum_{\tau} P(\tau;\theta) \nabla_{\theta} \log P(\tau;\theta) R(\tau)$$

which again is an expectation:

$$= \mathbb{E}_{\tau \sim \pi_{\theta}} \nabla_{\theta} \log P(\tau;\theta) R(\tau)$$

- Let's pause for a moment and break down what happened

Policy gradient derivation – Log derivative trick [1/3]

- We have rewritten

$$= \sum_{\tau} \nabla_\theta P(\tau;\theta) R(\tau)$$

into

$$= \sum_{\tau} P(\tau;\theta) \nabla_\theta \log P(\tau;\theta) R(\tau)$$

Using the log derivative trick

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Policy gradient derivation – Log derivative trick [2/3]

$$= \sum_{\tau} \nabla_{\theta} P(\tau;\theta) R(\tau)$$

$$= \sum_{\tau} \boxed{\frac{P(\tau;\theta)}{P(\tau;\theta)}} \nabla_{\theta} P(\tau;\theta) R(\tau)$$

Multiply by $1 = \frac{P(\tau;\theta)}{P(\tau;\theta)}$

$$= \sum_{\tau} \frac{\boxed{P(\tau;\theta)} \nabla_{\theta} P(\tau;\theta)}{\boxed{P(\tau;\theta)}} R(\tau)$$

Rearrange

$$= \sum_{\tau} \boxed{P(\tau;\theta)} \boxed{\frac{\nabla_{\theta} P(\tau;\theta)}{\boxed{P(\tau;\theta)}}} R(\tau)$$

Rearrange again

$$= \sum_{\tau} P(\tau;\theta) \boxed{\nabla_{\theta} \log P(\tau;\theta)} R(\tau)$$

Substitute using 'log identity'

$$\frac{\nabla_{\theta} P(\tau;\theta)}{P(\tau;\theta)} = \nabla_{\theta} \log P(\tau;\theta)$$

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Policy gradient derivation – Log derivative trick [3/3]

- Log-derivative trick (some background)
  - The derivative of $\log x$ is $\frac{1}{x}$.
  - Combined with chain rule, we get:

$$\nabla_\theta \mathbb{P}(\tau|\theta) = \mathbb{P}(\theta|s_0)\nabla_\theta \log \mathbb{P}(\tau|\theta)$$

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Policy gradient derivation – Log probabilities [1/3]

- Remember that tricky probability function?

$$\nabla_\theta \log P(\tau;\theta) = \nabla_\theta \log \left[\prod_{t=0}^{T} P(s_{t+1} \mid s_t, a_t) \cdot \pi_\theta(a_t \mid s_t) \right]$$

- Turns out we already resolved it!
- We rewrote from probabilities to log probabilities:

$$= \nabla_\theta \left[\sum_{t=0}^{T} \log P(s_{t+1} \mid s_t, a_t) + \sum_{t=0}^{T} \log \pi_\theta(a_t \mid s_t) \right]$$

Policy gradient derivation – Log probabilities [2/3]

- Log probabilities are *additive* rather than *multiplicative*
  - Transition $P(s_{t+1}|s_t, a_t)$ does not depend on θ
  - We can strike it without affecting the gradient

$$= \nabla_\theta \left[\cancel{\sum_{t=0}^{T} \log P(s_{t+1} \mid s_t, a_t)} + \sum_{t=0}^{T} \log \pi_\theta(a_t \mid s_t) \right]$$

- We may not know the transition function, but we do know our policy
  - If we can differentiate π_θ, we can compute the gradient of the objective function

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Policy gradient derivation – Log probabilities [3/3]

- Resulting expression only depends on policy π_θ:

$$= \nabla_\theta \sum_{t=0}^{T} \log \pi_\theta(a_t \mid s_t)$$

- Move in the gradient again, and we have our result:

$$= \sum_{t=0}^{T} \nabla_\theta \log \pi_\theta(a_t \mid s_t)$$

- As a bonus, additive probabilities are numerically more stable

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Policy gradient derivation – To summarize

$$
\begin{aligned}
\nabla_\theta \log P(\tau;\theta) &= \nabla_\theta \log \left[\prod_{t=0}^{T} P(s_{t+1} \mid s_t, a_t) \cdot \pi_\theta(a_t, s_t) \right] \\
&= \nabla_\theta \left[\sum_{t=0}^{T} \log P(s_{t+1} \mid s_t, a_t) + \sum_{t=0}^{T} \log \pi_\theta(a_t, s_t) \right] \\
&= \nabla_\theta \left[\sum_{t=0}^{T} \log P(s_{t+1} \mid s_t, a_t) + \sum_{t=0}^{T} \log \pi_\theta(a_t, s_t) \right] \\
&= \nabla_\theta \left[\prod_{t=0}^{T} \log P(s_{t+1} \mid s_t, a_t) + \sum_{t=0}^{T} \log \pi_\theta(a_t, s_t) \right] \\
&= \nabla_\theta \sum_{t=0}^{T} \log \pi_\theta(a_t, s_t) \\
&= \sum_{t=0}^{T} \nabla_\theta \log \pi_\theta(a_t, s_t)
\end{aligned}
$$

Policy gradient derivation – Approximate gradient

- It is worth mentioning we use approximate gradients
  - Like before, we approximate by repeated sampling

$$\nabla_\theta J(\theta) \approx \hat{g} = \frac{1}{m} \sum_{i=1}^{m} \nabla_\theta \log P\left(\tau^{(i)}; \theta\right) R\left(\tau^{(i)}\right)$$

  - Monte Carlo rationale (Law of Large Numbers)

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Applying policy gradients

Applying policy gradients

- We have the final result now, but it may seem rather abstract

- In practice, two policies are used for the most part
  - Softmax policy: for discrete action spaces
    - Assign probability to each action
  - Gaussian policy: for continuous action spaces
    - Draw action from normal distribution

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Discrete policy gradient

- Softmax policy:

$$\pi_\theta(s,a) = \frac{e^{\phi(s,a)^\top \theta}}{\sum_{a' \in A} e^{\phi(s,a')^\top \theta}}$$

- Gradient of softmax policy

$$\nabla_\theta \log \pi_\theta(a \mid s) = \phi(s,a) - \sum_{a' \in A} \phi(s,a')\pi_\theta(a \mid s)$$

- Remember: gradient is the score function seen earlier

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Continuous policy gradient – Gaussian policy

- Gaussian policy (parameters are μ_θ and σ_θ):

$$\pi_\theta(a \mid s) = \frac{1}{\sqrt{2\pi\sigma_\theta}} e^{-\frac{a-\mu_\theta}{2\sigma_\theta^2}}$$

- Gradient of Gaussian policy

$$\nabla_\theta \log(\pi_\theta(a \mid s)) = \frac{(a-\mu_\theta)\phi(s)}{\sigma_\theta^2}$$

- For each differentiable policy, we can compute the score function

Continuous policy gradient – Gaussian policy

- Update functions for μ_θ and σ_θ

$$\Delta_{\mu_{\boldsymbol{\theta}}}(s) = \alpha v \frac{(a - \mu_{\boldsymbol{\theta}}(s))}{\sigma_{\boldsymbol{\theta}}^2} \,,$$

$$\Delta_{\sigma_{\boldsymbol{\theta}}}(s) = \alpha v \frac{(a - \mu_{\boldsymbol{\theta}}(s))^2 - \sigma_{\boldsymbol{\theta}}^2}{\sigma_{\boldsymbol{\theta}}^3} \,.$$

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REINFORCE algorithm

- Mathematics may be overwhelming, but policy gradient algorithm itself is compact
- Consider the REINFORCE algorithm (Williams, 1993)

Input: $\theta \leftarrow \mathbb{R}^{|\theta|}$ ▶ Initialize θ

for $n = 1$ **to** N **do:**

    $\tau \sim \pi_\theta$ ▶ Generate state-action trajectory with π_θ

    for $t = 1$ **to** T **do**

        $R(\tau \mid t) = R_t + R_{t+1} + \ldots + R_T$ ▶ Compute cum. reward

        $\theta \leftarrow \theta + \alpha R(\tau \mid t) \nabla_\theta \log(\pi_\theta(a \mid s)$ ▶ Update θ

- In the end we only need:
  - A differentiable stochastic policy
  - A sequence of observed rewards, states, and actions

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Application

- Like before, we can define features and weights
  - Basis functions $\phi_f: (s, a) \rightarrow \mathbb{R}$, returning relevant features that explain the key determinants of value
  - Linear expression multiplies weight vector θ and feature vector $\phi(s, a)$:

$$\pi_\theta(s, a) = \frac{e^{\boxed{\phi(s,a)^\top \theta}}}{\sum_{a' \in A} e^{\phi(s,a')^\top \theta}}$$

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Example discrete space

- Example discrete action space:
  - Container shipping (each subset that can be shipped yields attached probability)
    - Features $\phi(s, a)$ could be *#red containers* etc., like before
    - By defining features, we resolve dimensionality of state space (like VFA did)
    - Softmax policy requires looping over all actions (denominator), so action space problem <u>is not resolved</u>

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Example continuous space

- Learn bid price for financial asset:
  - Bid can be accepted or rejected
  - $Payoff - bid$ yields the reward of the action
  - Action is a real-valued decision variable
    - Features $\phi(s, a)$ could capture asset properties (drift, volatility)
    - Try to learn minimal bid that gets accepted
    - Gaussian policy simply draws an action, so action space problem is resolved here

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Deep policy gradient

Deep policy gradients

- Simplest form of policy gradient:
  - Linear expression: $\theta^{\top}\phi(s,a)$
  - May not capture complex (non-linear) patterns
  - Human design capabilities have limits

- We may express policy as a neural network
  - Like in Deep Q-learning, the network generalizes across states
    - Implicitly generates features in hidden layers
  - Often more challenging to train than Deep Q-learning

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Deep policy gradients

- Softmax policy revisited
- Almost the same, we just replace $\phi(s,a)^\top\theta$ with f($\phi(s,a);\theta$), with f representing the neural network parameterized by θ

$$\pi_\theta(s,a) \propto e^{f(\phi(s,a);\theta)}$$

$$\pi_\theta(s,a) = \frac{e^{f(\phi(s,a);\theta)}}{\sum_{a' \in \mathcal{A}} e^{f(\phi(s,a');\theta)}}$$

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Deep policy gradients

- Output can be converted (Tensorflow here) to softmax:

```python
def construct_actor_network(STATE_DIM: int, ACTION_DIM: int):
    """Construct the actor network with action probabilities as output"""
    inputs = layers.Input(shape=(STATE_DIM,))  # input dimension
    hidden1 = layers.Dense(
        25, activation="relu", kernel_initializer=initializers.he_uniform()
    )(inputs)
    hidden2 = layers.Dense(
        25, activation="relu", kernel_initializer=initializers.he_uniform()
    )(hidden1)
    hidden3 = layers.Dense(
        25, activation="relu", kernel_initializer=initializers.he_uniform()
    )(hidden2)
    probabilities = layers.Dense(
        ACTION_DIM, kernel_initializer=initializers.Ones(), activation="softmax"
    )(hidden3)

    actor_network = keras.Model(inputs=inputs, outputs=[probabilities])
```

Activation function in output layer applies softmax

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Deep policy gradients

- Basic architecture
  - Input: state- or feature vector
  - Hidden layers
    - Activation functions (e.g., ReLU)
    - Weights θ (i.e., network weights are the tunable parameters!)
  - Output: action probabilities
    - For softmax policy, dimension of output layer equal to $|\mathcal{A}|$
    - For Gaussian policy, output is $\mu_\theta, \sigma_\theta$

- Loss function is policy-dependent
  - Often not basic Mean Squared Error

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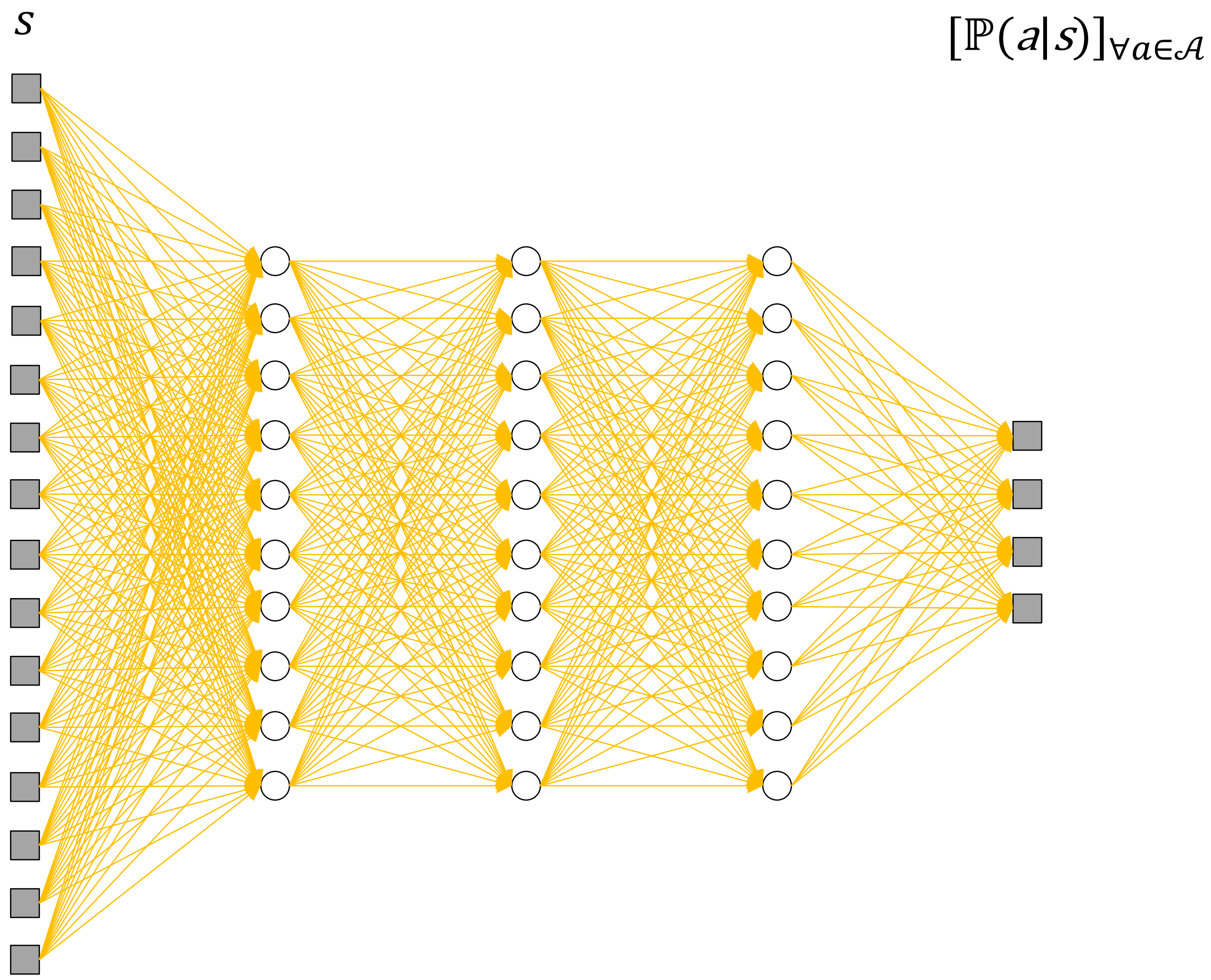
Actor network – Discrete action space



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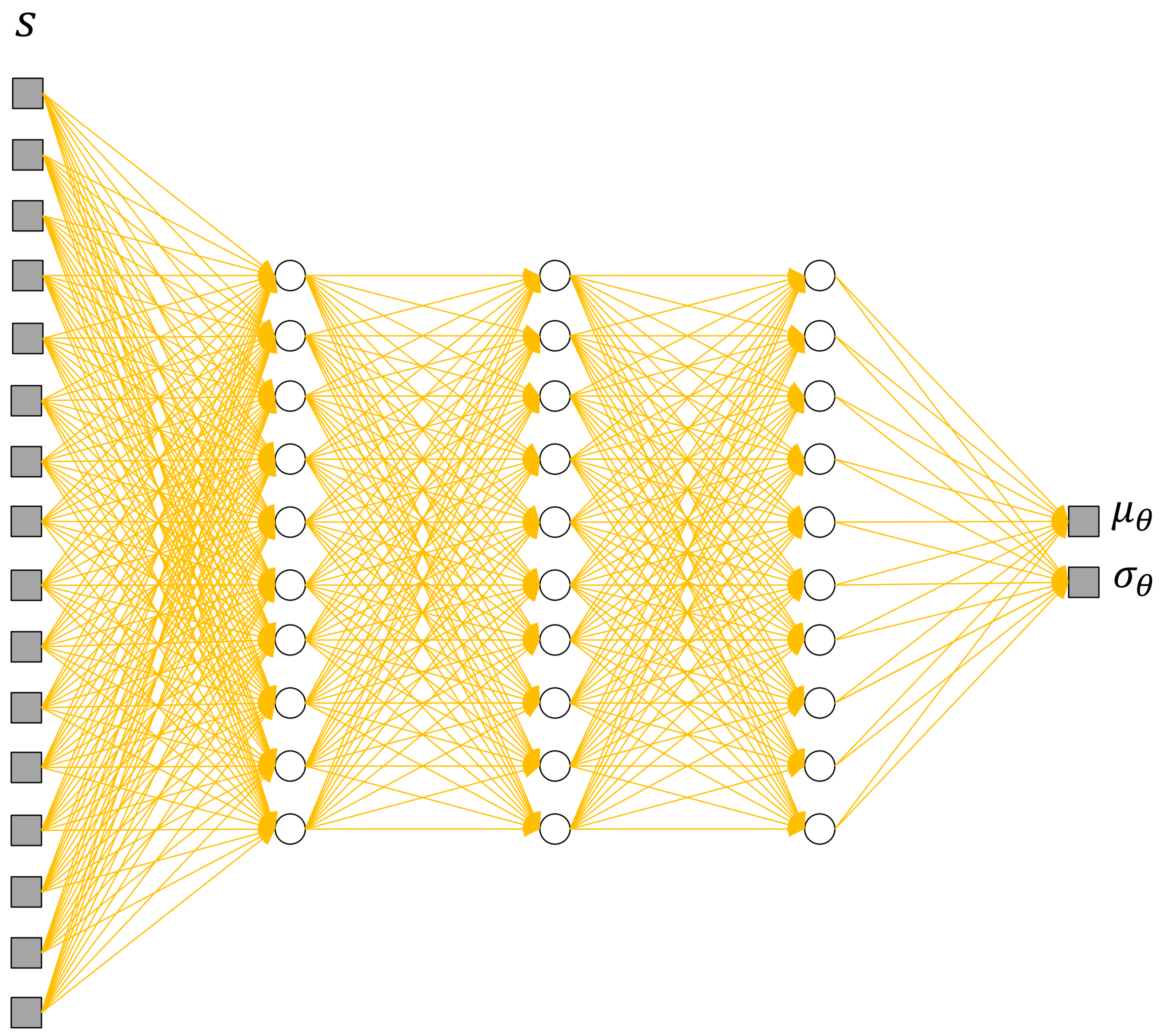
Actor network – Continuous action space



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Deep policy gradient

- Loss function is different now!
- For Q-network, we used standard Mean Squared Error
  - Difference between expected- and ‘observed’ Q-values
- For policy network, what would be the loss?

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Deep policy gradient

- Loss function is different now!
- For Q-network, we used standard Mean Squared Error
  - Difference between expected- and 'observed' Q-values

- For policy network, what would be the loss?
  - We have no 'true' value
  - What we do have:
    - Reward signal
    - Probability of action under current policy
  - Select actions with probabilities that maximize reward signal
  - Measure 'error' of policy in some way

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Deep policy gradient

- Remember the update rule for policy gradients

$$\Delta\theta = \alpha \nabla_\theta \log\left(\pi_\theta(a \mid s)\right) v$$

- This rule is based on gradient *ascent*
- Neural network is trained with gradient *descent*
  - Add minus sign
  - Remove learning rate α and gradient ∇_θ
    - Loss function is only the input for the gradient computations
  - Generic loss function depends on log prob action and reward

$$\mathcal{L}(a, s, v) = -\log(\pi_\theta(a \mid s))v$$

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Deep policy gradient

- To train the policy network, we re-express the generic loss function for the policy:

$$\mathcal{L}(a, s, v) = -\log(\pi_\theta(a \mid s))v$$

- Requires writing out the policy and resolving the expression

- Tensorflow tracks loss function on *GradientTape*
  - May require manual design

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Deep policy gradient – Loss function softmax policy

- Softmax uses cross entropy loss

$$\mathcal{L}(a, s, v) = -\log(\pi_\theta(a \mid s))v$$

(happens to be identical to generic form)

```python
def cross_entropy_loss(probability_action, reward):
    log_probability = tf.math.log(probability_action + 1e-5)
    loss_actor = - reward * log_probability

    return loss_actor
```

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Deep policy gradient – Loss function Gaussian policy

- Gaussian uses normal loss

$$\mathcal{L}(a, s, v) = -\log\left(\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{a-\mu}{\sigma}\right)^2}\right) v$$

```python
"""Weighted Gaussian log likelihood loss function for RL"""
def custom_loss_gaussian(state, action, reward):
  # Predict mu and sigma with actor network
  mu, sigma = actor_network(state)

  # Compute Gaussian pdf value
  pdf_value = tf.exp(-0.5 *((action - mu) / (sigma))**2)
     * 1 / (sigma * tf.sqrt(2 * np.pi))

  # Convert pdf value to log probability
  log_probability = tf.math.log(pdf_value + 1e-5)

  # Compute weighted loss
  loss_actor = - reward * log_probability

  return loss_actor
```

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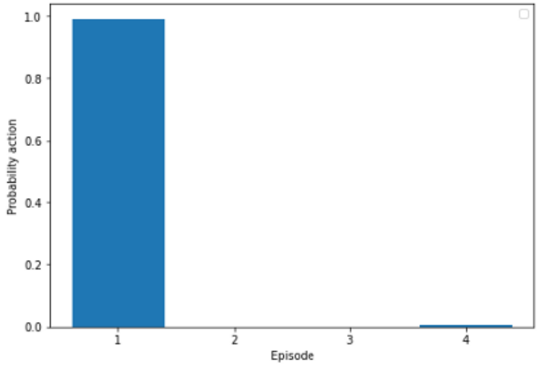
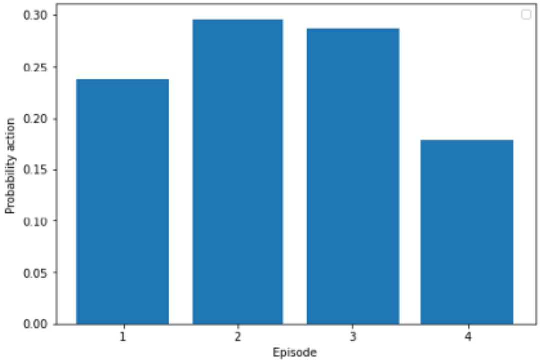
Examples

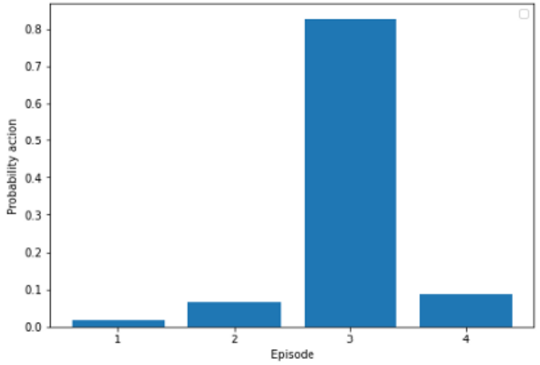
https://github.com/woutervanheeswijk/example_discrete_control



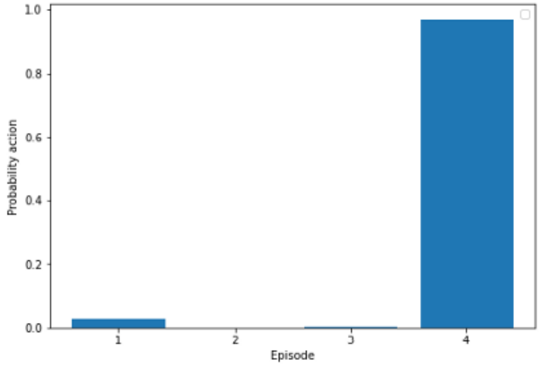
(a) $\mu_1 = 1.1$, $\mu_2 = 0.0$, $\mu_3 = 1.0$, $\mu_4 = 1.0$



(b) $\mu_1 = 1.0$, $\mu_2 = 1.0$, $\mu_3 = 1.0$, $\mu_4 = 1.0$



(c) $\mu_1 = 4.8$, $\mu_2 = 4.9$, $\mu_3 = 5.1$, $\mu_4 = 4.9$



(d) $\mu_1 = 1.0$, $\mu_2 = 0.9$, $\mu_3 = 0.9$, $\mu_4 = 1.0$

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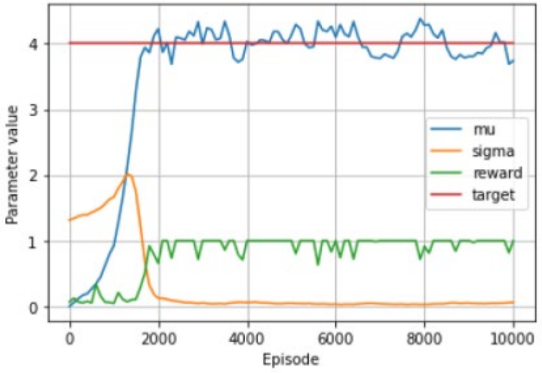
Examples

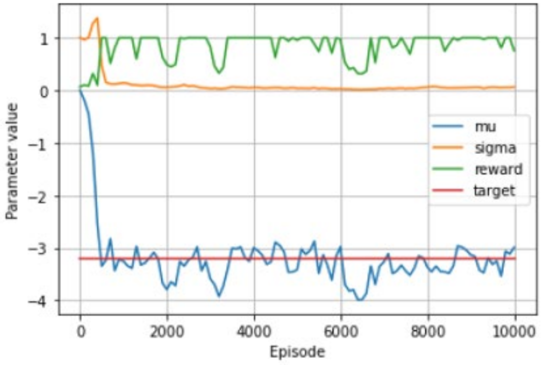
https://github.com/woutervanheeswijk/example_continuous_control

- Red line = unknown target
- Closer to target = high reward

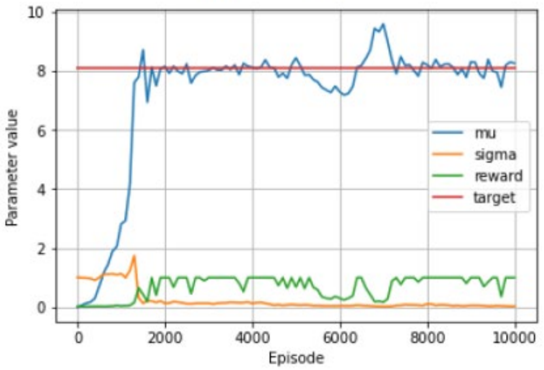
- Far from target → increase σ
- Close to target → decrease σ



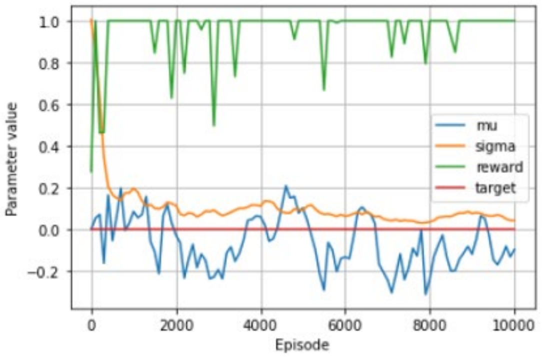
(a) $\tau = 4.0$



(b) $\tau = -3.2$



(c) $\tau = 8.1$



(d) $\tau = 0.0$

Actor-critic models

VFAs and PFAs

- As mentioned in Lecture 1, there are 4 policy classes
- So far, we treated two:
  - Value Function Approximation (VFA): learn $\bar{V}_\theta(\phi(s,a))$
  - Policy Function Approximation (PFA): learn π_θ

- Why not combine the best of both worlds?

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Four policy classes – Refresher

Policy-based methods	**Value-based methods**
Policy function approximation (PFA)	Value function approximation (VFA)
Cost function approximation (CFA)	Direct lookahead approximation (DLA)

Adjust decision-making rules and observe impact

Learn value functions that represent downstream values

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VFAs and PFAs

- Benefits **VFA**
  - Capture downstream rewards in generic function
  - Value functions help when not fully grasping downstream effects
- Benefits **PFA**
  - Exploit knowledge of policy structure
  - Directly influence decision rules

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Hybrid models

- Possible to combine different classes
  - Also recall Direct Lookahead Approximation (heuristic sampling of downstream effects) and Cost Function Approximation (parameterize uncertainty directly into policy)

- Leverage strengths of multiple classes while negating weaknesses
  - Poor implementations may achieve the opposite!

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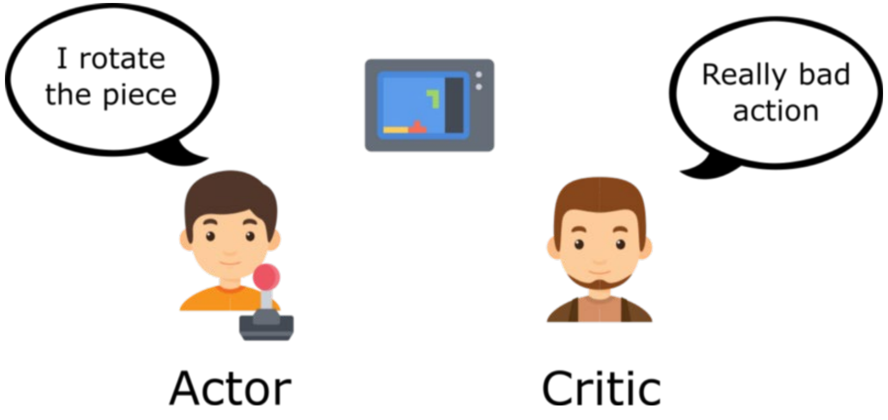
Actor-critic models [1/5]

- Actor-critic models are the standard in Reinforcement Learning
  - Mostly applications in Computer Science



Source: freecodecamp.org

- In fact, you have already seen an actor-critic model!
  - Policy iteration algorithm (Lecture 1) entails both value function updates and policy updates

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Actor-critic models [2/5]

- The actor-critic model combines PFA and VFA, recombining the techniques you have seen before
  - Update function policy (using Q_w instead of G_t)
    - $\theta \leftarrow \theta + \alpha_\theta Q_w(s,a)\nabla_\theta(a|s)$
  - For value function, compute temporal difference error:
    - $\delta_t = r_t + \gamma Q_w(s',a') - Q_w(s',a')$ [TD(0) error]
  - Update value function weights with TD(0) error
    - $w \leftarrow w + \alpha_w \delta_t \nabla_w Q_w(s,a)$

- Learning both policy ('actor') and downstream effects ('critic')

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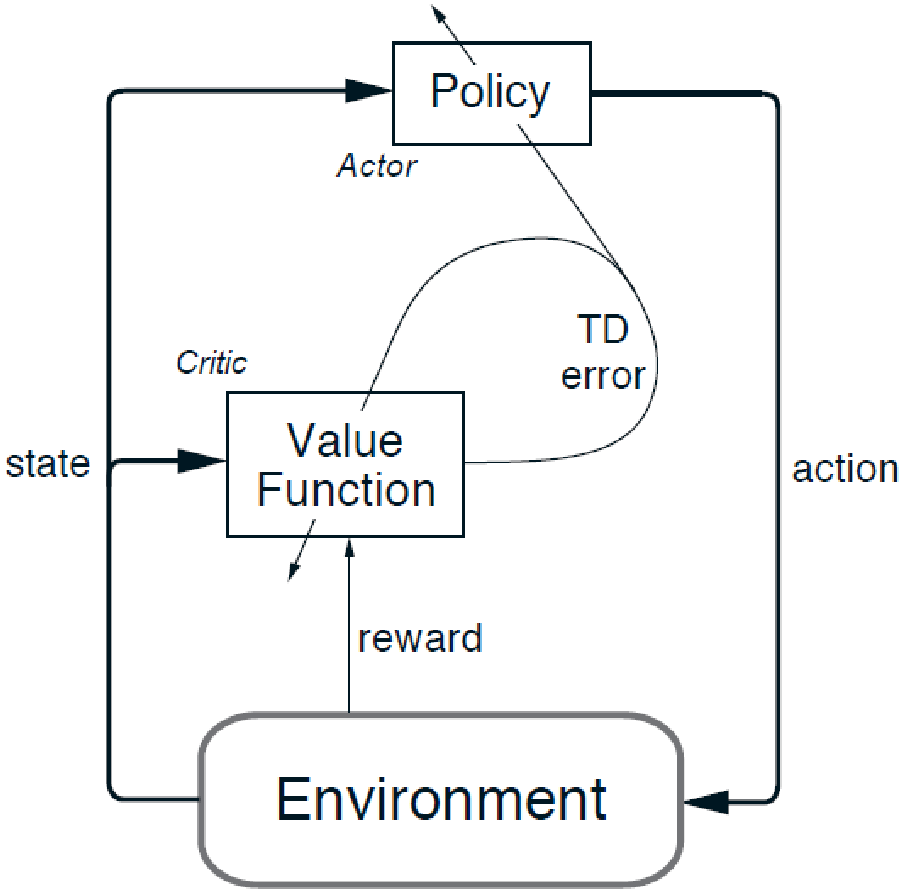
Actor-critic models [3/5]

- See Sutton for details

One-step Actor–Critic (episodic)

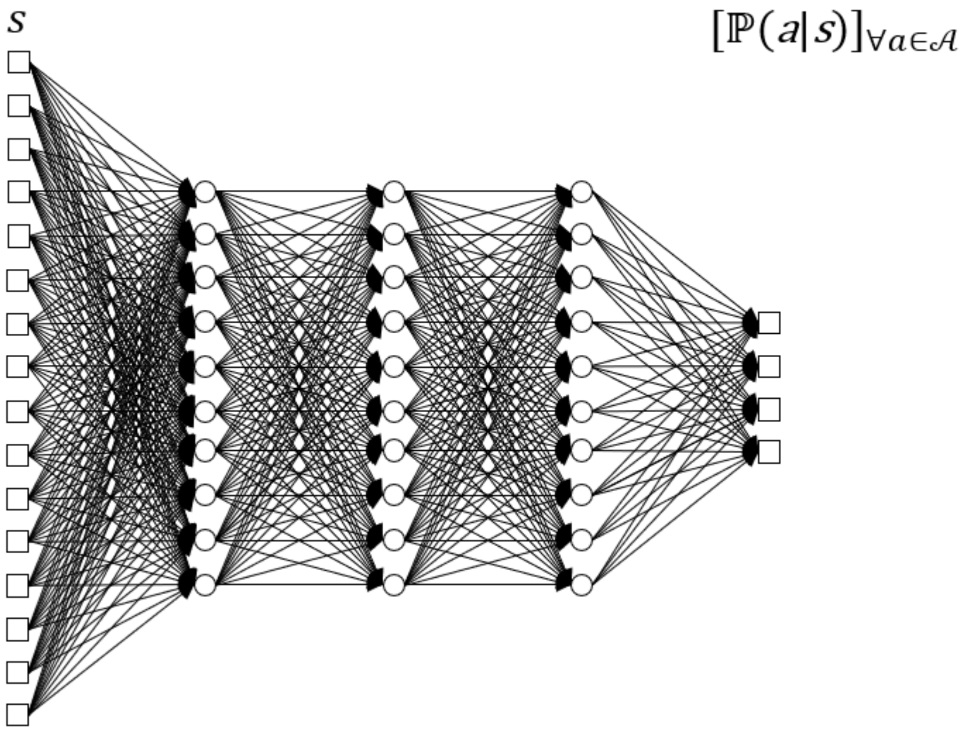
Input: a differentiable policy parameterization $\pi(a|s,\boldsymbol{\theta})$
Input: a differentiable state-value parameterization $\hat{v}(s,\mathbf{w})$
Parameters: step sizes $\alpha^{\boldsymbol{\theta}} > 0$, $\alpha^{\mathbf{w}} > 0$

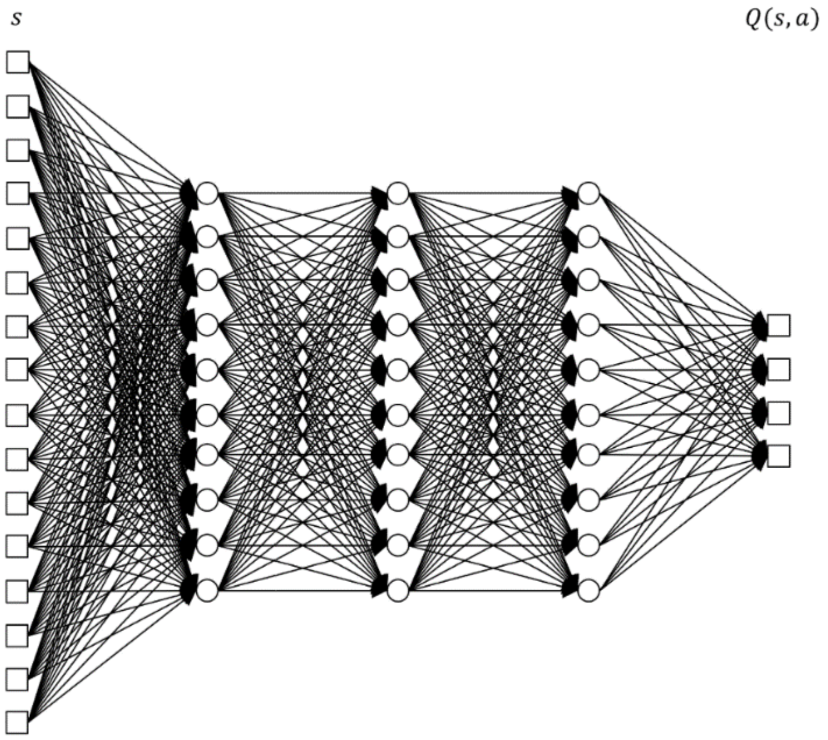
Initialize policy parameter $\boldsymbol{\theta} \in \mathbb{R}^{d'}$ and state-value weights $\mathbf{w} \in \mathbb{R}^{d}$
Repeat forever:
  Initialize S (first state of episode)
  $I \leftarrow 1$
  While S is not terminal:
    $A \sim \pi(\cdot|S,\boldsymbol{\theta})$
    Take action A, observe S', R
    $\delta \leftarrow R + \gamma\hat{v}(S',\mathbf{w}) - \hat{v}(S,\mathbf{w})$ (if S' is terminal, then $\hat{v}(S',\mathbf{w}) \doteq 0$)
    $\mathbf{w} \leftarrow \mathbf{w} + \alpha^{\mathbf{w}} I \delta \nabla_{\mathbf{w}} \hat{v}(S,\mathbf{w})$
    $\boldsymbol{\theta} \leftarrow \boldsymbol{\theta} + \alpha^{\boldsymbol{\theta}} I \delta \nabla_{\boldsymbol{\theta}} \ln \pi(A|S,\boldsymbol{\theta})$
    $I \leftarrow \gamma I$
    $S \leftarrow S'$



Actor-critic models [4/5]

- Nowadays, RL is often network-based
  - Actor network (policy network returning action probabilities)
  - Critic network (value network returning Q-values)





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Actor-critic models [5/5]

- Main benefit of actor-critic models:
  - Learn Q-values instead of observed rewards G_t
    - As reward trajectories may vary a lot, Q-values can add robustness

- Main challenges:
  - Adjusting policy changes value functions
  - Adjusting value functions changes policy
  - Must be updated simultaneously...

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Advanced policy gradients

Advanced policy gradients

- Advanced material, in-depth explanation would require multiple lectures
- For exam, just try to understand high-level idea

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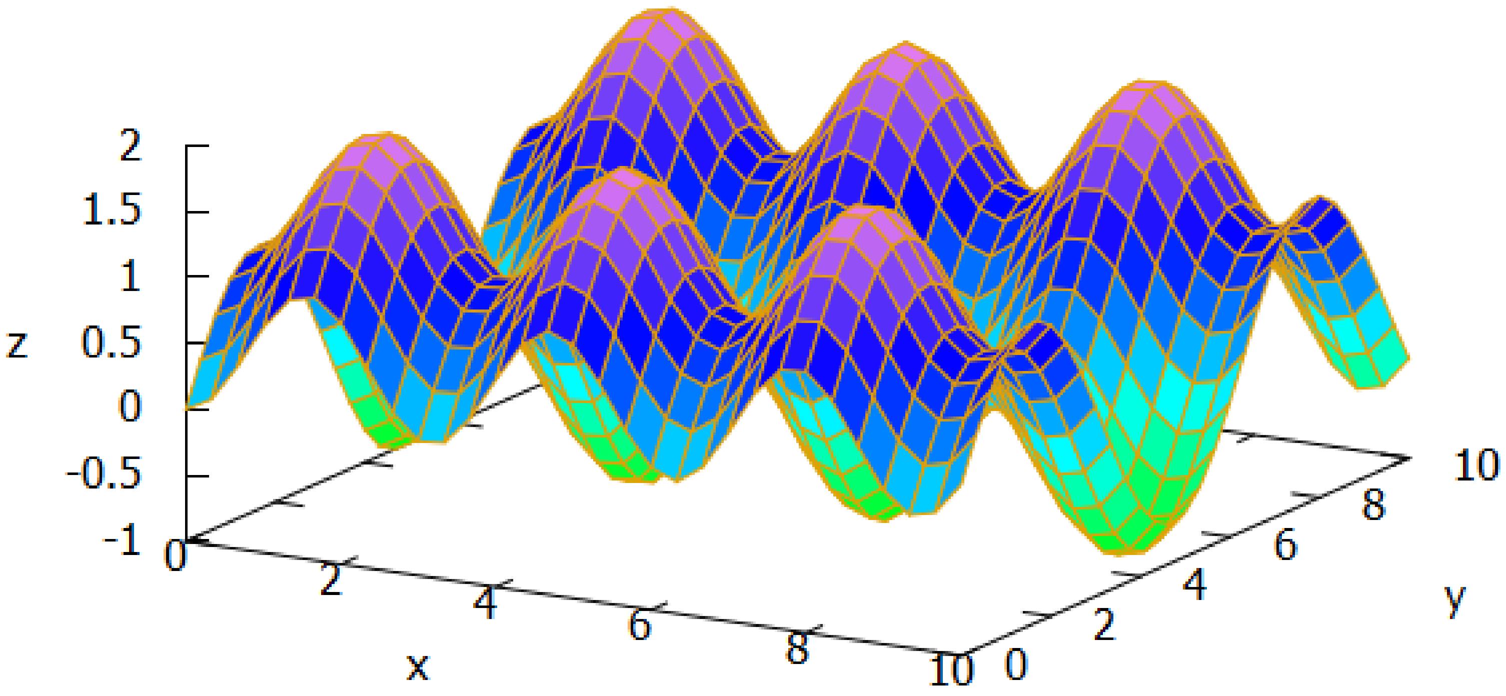
Problems with policy gradients [1/3]

- What is a good stepsize for policy gradient algorithms?



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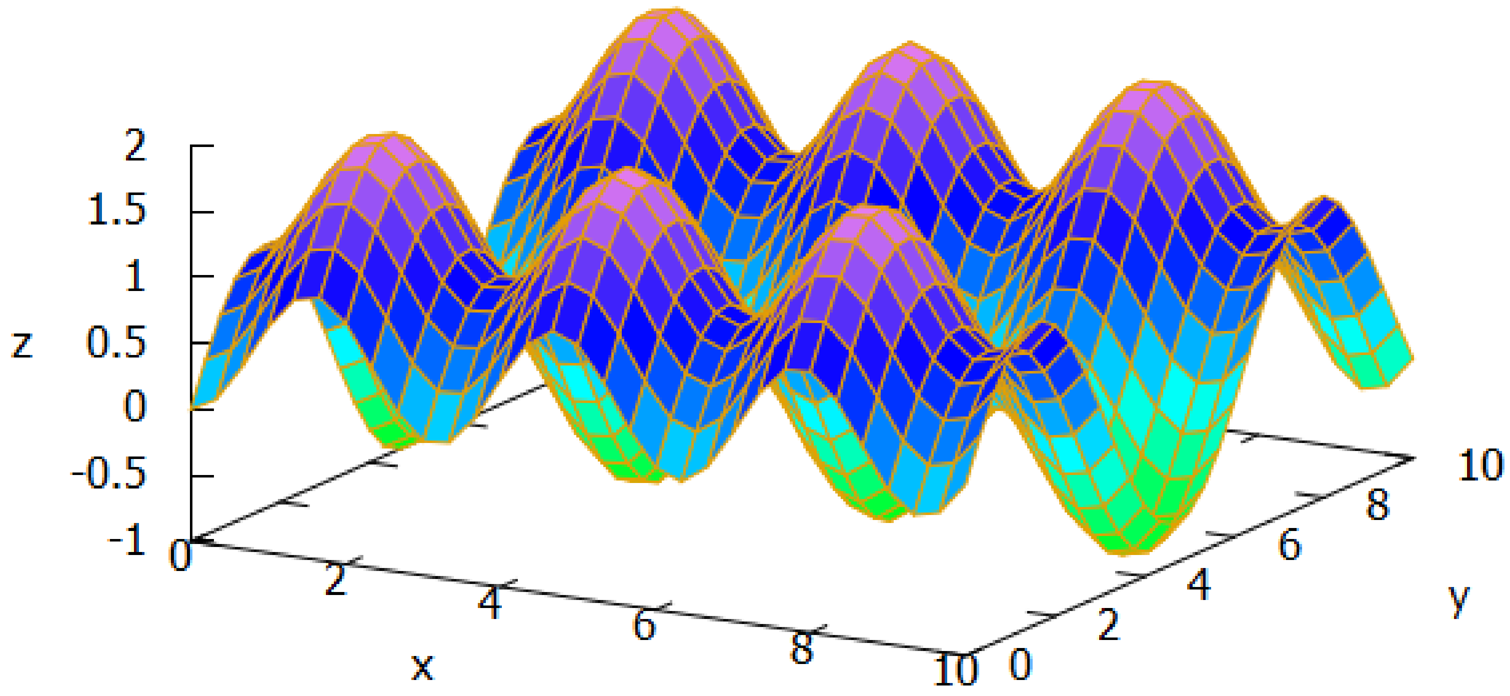
Problems with policy gradients [2/3]

- What is a good stepsize for policy gradient algorithms?
  - Common rationale: steep slope = large step
  - Is there a downside to this rationale?



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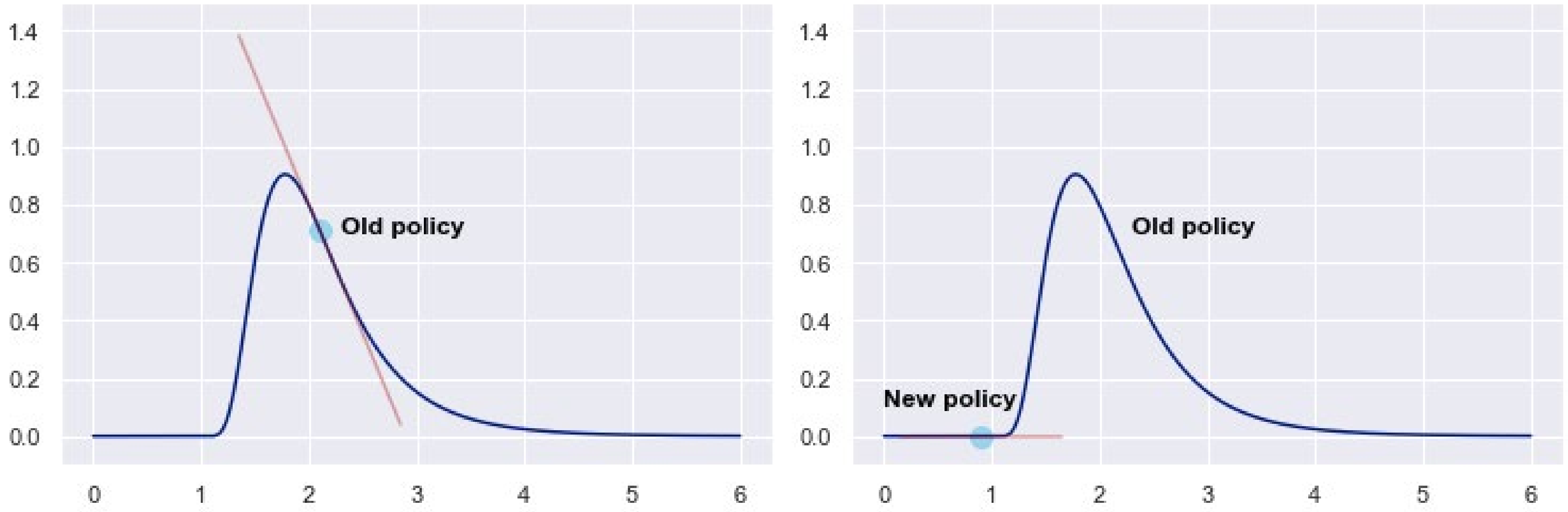
Problems with policy gradients [3/3]

- Common problems with policy gradients
  - Overshooting: large update, miss reward peak
  - Undershooting: stuck at suboptimal plateaus



Natural policy gradients [1/4]

- Essentially, we don't want policies to change too much
  - On plateau, we can safely take larger steps
  - On steep slopes, we want to be cautious
  - Opposite of rationale so far

- How do we incorporate this behavior?

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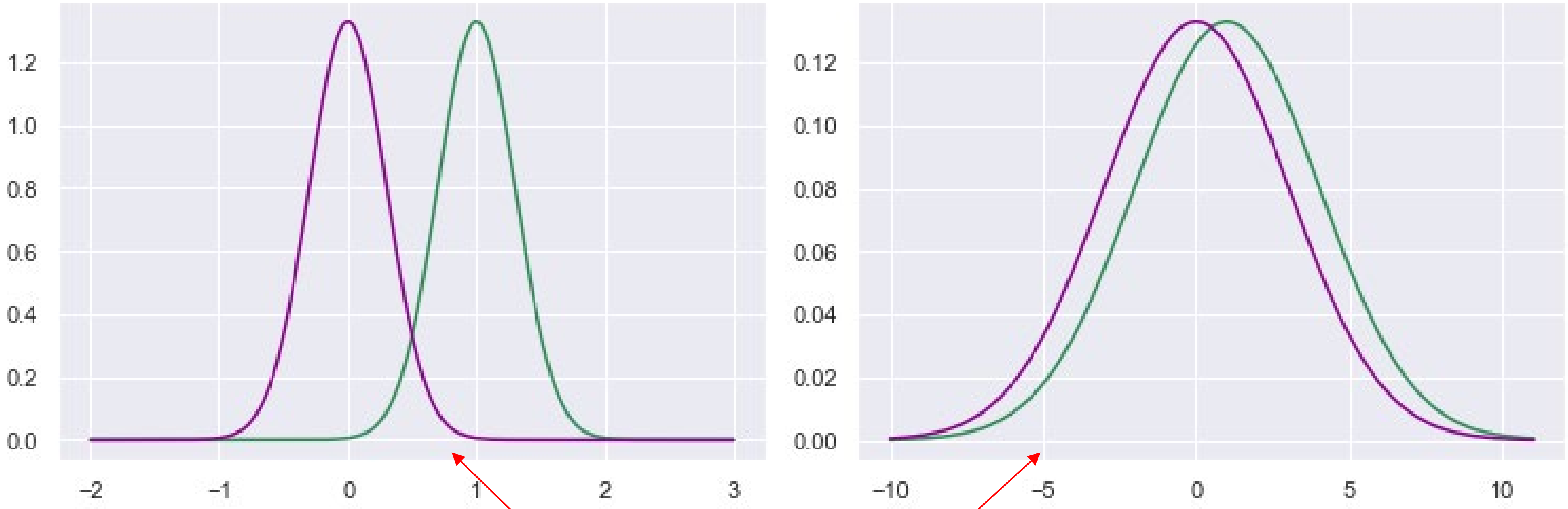
Natural policy gradients [2/4]

- Capping parameter updates does not work

$$\Delta\theta^* = \arg\max_{\|\Delta\theta\|\le\epsilon} J(\theta + \Delta\theta)$$

Suppose we change $\theta = [\mu, \sigma]$



Same parameter distance ϵ!

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Natural policy gradients [3/4]

- We *can* cap difference between policy before and after update
  - Formalized by Kullback-Leibner (KL) divergence:

$$\mathcal{D}_{\mathrm{KL}}(\pi_\theta \parallel \pi_{\theta+\Delta\theta}) = \sum_{x \in \mathcal{X}} \pi_\theta(x) \log\left(\frac{\pi_\theta(x)}{\pi_{\theta+\Delta\theta}(x)}\right)$$

- We then determine optimal constrained update:

$$\Delta\theta^* = \underset{\mathcal{D}_{\mathrm{KL}}(\pi_\theta \| \pi_{\theta+\Delta\theta}) \leq \epsilon}{\arg\max} \; J(\theta + \Delta\theta)$$

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Natural policy gradients [4/4]

- Natural policy gradients correct for second derivatives, ensuring the policy does not change too much
- Full procedure is rather involved (second-order derivatives, Fisher information matrix, Langrangian relaxation, Taylor expansions, etc.)

- Bottom line:
  - Restrict update size based on local sensitivity of objective
  - Dynamic step size given by
  - Requires substantial memory and $\Delta\theta = \sqrt{\frac{2\epsilon}{\nabla J(\theta)^\top F(\theta)^{-1} \nabla J(\theta)}} \tilde{\nabla} J(\theta)$ resources
    - Inverting large matrices is hard!

Trust Region Policy Optimization (TRPO) [1/2]

- Natural policy gradients based on strong assumptions and requires substantial computational resources
- TRPO proposes three improvements
  - Conjugate gradient: numerical approximation, no need to invert Fisher information matrix
  - Line search: iteratively reduce update size, until maximum divergence is satisfied
  - Improvement check: *check* whether update improves policy, rather than *assume*

- Still memory-intense, complex second-order optimization, etc.

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Trust Region Policy Optimization (TRPO) [2/2]

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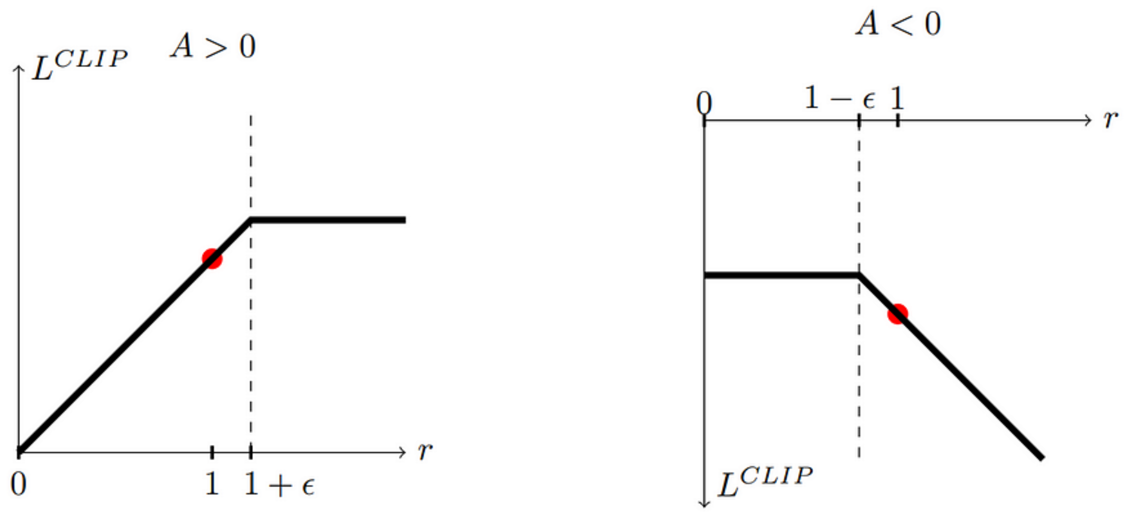
Proximal Policy Optimization (PPO) [1/2]

- Natural policy gradients and TRPO are complex second-order methods
- PPO just tosses updates that drift too far from current policy
  - Empirically works well
  - Can apply excellent optimizers such as ADAM (first-order)



[image by Schulman et al. 2017]

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Proximal Policy Optimization (PPO) [2/2]

- Clipped loss function
  - If loss is $1 \pm \epsilon$, gradient is 0!

$$\mathcal{L}_{\pi_\theta}^{CLIP}(\pi_{\theta_k}) = \mathbb{E}_{\tau\sim\pi_\theta}\left[\sum_{t=0}^{T}\left[\min\left(\rho_t(\pi_\theta,\pi_{\theta_k})A_t^{\pi_{\theta_k}}, \mathrm{clip}(\rho_t(\pi_\theta,\pi_{\theta_k}), 1-\epsilon, 1+\epsilon)A_t^{\pi_{\theta_k}}\right)\right]\right]$$

- Ratio: difference action probability before and after update

$$\rho_t(\theta) = \frac{\pi_\theta(a_t \mid s_t)}{\pi_{\theta_k}(a_t \mid s_t)}$$

- Simple to implement, works well with neural networks

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Wrapping up

Recap

- Policy gradients
  - Policy Function Approximation (PFA) methods directly alter the policy, not requiring the Bellman paradigm
  - By using a stochastic policy, we observe differences between actions and compute gradients for update directions
  - Deep policy learning deploys actor network to capture nonlinear effects and extract features

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Further reading

- Sutton & Barto (2018)

  Chapter 11.1 (actor-critic models)

- David Silver (2020) [policy gradients]

https://www.davidsilver.uk/wp-content/uploads/2020/03/pg.pdf

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