Minimum vertex coloring on Graphs(r,l)

Matheus S. D'Andrea Alves, **Uéverton dos Santos Souza**

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Universidade Federal Fluminense

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Introduction

Who am I

Undergraduated student at:

- Universidade Federal Fluminense

Interested in:

- Graph theory
- Complexity analysis
- Software engineering

You can find me here:

- `github.com/MSDandrea`
- `telegram.me/MSDandrea`

The problem

Graph(r,l)

Any graph in the class of graphs that can be partitionated in r independent sets and l cliques

Minimum vertex coloring

A minimum vertex coloring is an assignment of a color among k colors to each vertex of a graph such that no edge connects two identically colored vertices and k is the smallest value to obtain a k-coloring

THE QUESTION:

When does this problem begins to be NP-Complete?

And why?

Our approach

The idea

Build a dichotomy to the problem, based on the r and l values.

Gather the results and search for a pattern in the problem behavior.

First steps

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- A bipartide Graph (i.e. a Graph(2,0)) is 2-colorable.
- A Complete Graph (i.e. a Graph(0,1)) is k-colorable where $k = \#V(G)$.
- A Split Graph (i.e. a Graph(1,1)) is k-colorable where $k = \#V(C)$ such that C is the Maximal clique in G.

Theorem

Minimum vertex coloring of Graph(0,2) is Polynomial.

Proof.

A Graph(0,2) is a Graph that can be partiotionated into 2 cliques, finding the maximal clique in this case is polynomial. Once we have the maximal clique, we known that the graph can be colored with k colors, such that K is the maximal clique cardinality. □

Theorem

Minimum vertex coloring of Graph(3,0) is Polynomial.

Proof.

Beacause we know that the Graph is a Graph(3,0) we known it can be colored at most with 3 colors. To find out if it can be colored with 2 or one color is polynomial, implying that minimum vertex coloring of Graph(3,0) is Polynomial. □

Theorem

Minimum vertex coloring of Graph(4,0) is NP-Complete.

Proof.

We know that the Graph is a Graph(4,0), therefore it can be colored at most with 4 colors. We need to discover if it can be colored with 3 colors; Note that the graph is a planar graph and 3-coloring in planar graphs are NP-complete, implying that minimum vertex coloring of Graph(4,0) is NP-Complete. □

First results

$r \backslash l$	0	1	2	3	4	...	n
0	P	P	P	?	?	...	?
1	P	P	?	?	?	...	?
2	P	?	?	?	?	...	?
3	P	?	?	?	?	...	?
4	NPc	NPc	NPc	NPc	NPc	...	NPc
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	NPc
n	NPc	NPc	NPc	NPc	NPc	...	NPc

Table 1: Partial dichotomy of the minimum coloring problem on Graph(r,l)

How to proceed?

The relation between list coloring and minimum coloring in Graph(r,l)

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Theorem

Minimum vertex coloring is NP-Complete for Graph(r,l+1) if list coloring is NP-Complete for Graph(r,l)

The relation between list coloring and minimum coloring in Graph(r,l)

Proof.

The proof consists in showing that a solution for the list coloring problem in a Graph(r,l) G, implies in a solution for the minimum color problem in a Graph(r,l+1) H.

In order to do so, we need to show that:

- If a graph G(r,l) has a proper list coloring then H is k-colorable for k=total number of colors in the lists (1).
- If H is k-colorable then G has a proper list coloring (2).

(1):

Let G be a Graph(r,l) such that each vertex $v \in V(G)$ has a color list; Each list has at least one color of the Set: $C = \{c_1, c_2, c_3, ..., c_k\}$.

Let G be a **yes** instance for the list coloring problem, we build a clique K, where each vertex $u \in V(K)$ represents a color of C.

Note that the clique K has exactly k vertices, thereafter, we can colorize K with only k colors, without becoming restrictive whe can asume that the vertex $u_i \in K$ will be colored with the color c_i.

Supose H $= G \cup K$, and that for every vertex $u_i \in V(K)$ and every vertex $v_j \in V(G)$ add a edge $E(u_i, v_j)$ in H iff c_i is not a color present on the list of v_j.

Using this construction, the coloring of K does not conflict with the color pattern solution found for G, implying in a minimum proper coloring of H.

(2):

Supose that the graph H is a Graph(r,l+1) and has a k-coloring. Let K be the maximal clique in H. We know that k is the number of vertices in K.

Note that the removal of K from H (which become G) does not affect it's proper coloring, also note that the remaining vertex $v \in V(G)$ can now be colored using the colors of their former neighboors in K in such a way that it does not affect it's proper coloring.

In that way we show that, if H is k-colorable, G is a yes instance for the list coloring problem. $\square$

Corollaries

With this result we can say that:

- Minimum proper coloring of Graph(1,2) is NP-Complete.
  NP-Completeness of list coloring of split graphs was demonstrated by Jensen et al. at *"Generalized coloring for tree-like graphs"*.
- Minimum proper coloring of Graph(2,1) is NP-Complete.
  NP-Completeness of list coloring of bipartide graphs was demonstrated by Fellows et al. at *"List Coloring and Precoloring Extension are W[1]-hard parameterized by treewidth"*.
- Minimum proper coloring of Graph(0,3) is NP-Complete.
  NP-Completeness of list coloring of Graph(0,2) graphs was demonstrated by Jensen et al. at *"Complexity results for the optimum cost chromatic partition problem"*.

Final results

Final result

These results allow us to finish our dichotomy.

$r \backslash l$	0	1	2	3	4	$\dots$	n
0	P	P	P	NPc	NPc	$\dots$	NPc
1	P	P	NPc	NPc	NPc	$\dots$	NPc
2	P	NPc	NPc	NPc	NPc	$\dots$	NPc
3	P	NPc	NPc	NPc	NPc	$\dots$	NPc
4	NPc	NPc	NPc	NPc	NPc	$\dots$	NPc
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	NPc
n	NPc	NPc	NPc	NPc	NPc	$\dots$	NPc

Table 2: Dichotomy of the minimum coloring problem on Graph(r,l)

Conclusion

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We were sucessfull in answering our first questioning, and discovered a interesting relation between two coloring problems applied to the Graph(r,l) class.

Future works:

- Why the problem behave that way?
- Does the Minimum vertex coloring problem has a parametrized solution for Graph(r,l)?

THANK YOU!

Questions?