Parametrized Complexity on the Graph(r,l) class

Minimum vertex coloring on Graphs(r,l)

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Introduction

The problem

Our approach

First results

The relation between list coloring and minimum vertex coloring in Graph(r,l)

Final results

Introduction

Who am I

Undergraduated student at:

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Interested in:

- Graph theory
- Complexity analysis
- Software engineering

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The problem

Graph(r,l)
Any graph in the class of graphs that can be partitionated in r independent sets and l cliques

Minimum vertex coloring
A minimum vertex coloring is an assignment of a color among k colors to each vertex of a graph such that no edge connects two identically colored vertices and k is the smallest value to obtain a k-coloring

THE QUESTION:

When does this problem begins to be NP-Complete?

And why?

Our approach

The idea

Build a dichotomy to the problem, based on the r and l values.

Gather the results and search for a pattern in the problem behavior.

First steps

The starting point was the following:

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- A Complete Graph (i.e. a Graph(0,1)) is k-colorable where $k = \#V(G)$.
- A Split Graph (i.e. a Graph(1,1)) is k-colorable where $k = \#V(C)$ such that C is the Maximal clique in G.

Theorem

Minimum vertex coloring of Graph(0,2) is Polynomial.

Proof.

A Graph(0,2) is a Graph that can be partiotionated into 2 cliques, finding the maximal clique in this case is polynomial. Once we have the maximal clique, we known that the graph can be colored with k colors, such that K is the maximal clique cardinality. □

Theorem

Minimum vertex coloring of Graph(3,0) is Polynomial.

Proof.

Beacause we know that the Graph is a Graph(3,0) we known it can be colored at most with 3 colors. To find out if it can be colored with 2 or one color is polynomial, implying that minimum vertex coloring of Graph(3,0) is Polynomial. □

Theorem

Minimum vertex coloring of Graph(4,0) is NP-Complete.

Proof.

We know that the Graph is a Graph(4,0), therefore it can be colored at most with 4 colors. We need to discover if it can be colored with 3 colors; Note that if the graph is a planar graph then 3-coloring is NP-Complete, therefore minimum vertex coloring of Graph(4,0) is NP-Complete. □

First results

Partial Dichotomy

$r \backslash l$	0	1	2	3	4	...	n
0	P	P	P	?	?	...	?
1	P	P	?	?	?	...	?
2	P	?	?	?	?	...	?
3	P	?	?	?	?	...	?
4	NPc	NPc	NPc	NPc	NPc	...	NPc
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	NPc
n	NPc	NPc	NPc	NPc	NPc	...	NPc

Table 1: Partial dichotomy of the minimum vertex coloring problem on Graph(r,l)

How to proceed?

The relation between list coloring and minimum vertex coloring in Graph(r,l)

The relation between list coloring and minimum vertex coloring in Graph(r,l)

Theorem

Minimum vertex coloring is NP-Complete for Graph(r,l+1) if list coloring is NP-Complete for Graph(r,l)

The relation between list coloring and minimum vertex coloring in Graph(r,l)

Proof.

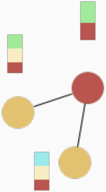
The proof consists in showing that a solution for the list coloring problem in a Graph(r,l) G, implies in a solution for the minimum color problem in a Graph(r,l+1) H_G.

In order to do so, we need to show that:

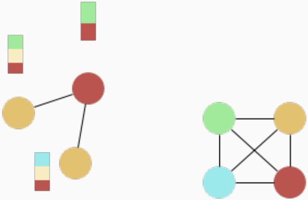
- If a graph G(r,l) has a proper list coloring then H_G is k-colorable for k=total number of colors in the lists (1).
- If H_G is k-colorable then G has a proper list coloring (2).

(1):

Let G be a Graph(r,l) such that each vertex $v \in V(G)$ has a color list; Each list has at least one color of the Set: $C = \{c_1, c_2, c_3, ..., c_k\}$.

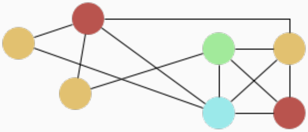


Let G be a **yes** instance for the list coloring problem, we build a clique K, where each vertex $u \in V(K)$ represents a color of C.



Note that the clique K has exactly k vertices, thereafter, we can colorize K with only k colors, without becoming restrictive whe can asume that the vertex $u_i \in K$ will be colored with the color c_i.

Supose $H_G = G \cup K$, and that for every vertex $u_i \in V(K)$ and every vertex $v_j \in V(G)$ add a edge $E(u_i, v_j)$ in H_G iff c_i is not a color present on the list of v_j.



Using this construction, the coloring of K does not conflict with the color pattern solution found for G, implying in a minimum proper coloring of H_G.

(2):

We know that the graph H_G is a Graph(r,l+1) and has a k-coloring. Let K be the maximal clique in H_G. We also know that k is the number of vertices in K.

Note that the removal of K from H_G (which will become G) does not affect it's proper coloring, also note that for the remaining vertex $v \in V(G)$ we can now build a list of colors from their formers non-neighboors in K in such a way that it does not affect it's proper coloring.

In that way we show that, if H_G is k-colorable, G is a yes instance for the list coloring problem. $\square$

Corollaries

With this result we can say that:

- Minimum proper coloring of Graph(1,2) is NP-Complete.
  NP-Completeness of list coloring of split graphs was demonstrated by Jensen et al. at *"Generalized coloring for tree-like graphs"*.
- Minimum proper coloring of Graph(2,1) is NP-Complete*****.
  NP-Completeness of list coloring of bipartide graphs was demonstrated by Fellows et al. at *"List Coloring and Precoloring Extension are W[1]-hard parameterized by treewidth"*.
- Minimum proper coloring of Graph(0,3) is NP-Complete.
  NP-Completeness of list coloring of Graph(0,2) graphs was demonstrated by Jensen et al. at *"Complexity results for the optimum cost chromatic partition problem"*.

Peculiarity of Graphs(2,1)

It's trivial to see that the coloring problem for this specific class has a clear upper and lower bound (K+1 and K respectively), but beside the bounds it is still NP-Complete to determine which one it is.

Final results

These results allow us to finish our dichotomy.

$r \backslash l$	0	1	2	3	4	...	n
0	*P*	*P*	*P*	*NPc*	*NPc*	...	*NPc*
1	*P*	*P*	*NPc*	*NPc*	*NPc*	...	*NPc*
2	*P*	*NPc*	*NPc*	*NPc*	*NPc*	...	*NPc*
3	*P*	*NPc*	*NPc*	*NPc*	*NPc*	...	*NPc*
4	*NPc*	*NPc*	*NPc*	*NPc*	*NPc*	...	*NPc*
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	*NPc*
n	*NPc*	*NPc*	*NPc*	*NPc*	*NPc*	...	*NPc*

Table 2: Dichotomy of the minimum vertex coloring problem on Graph(r,l)

Coloring of a Graph G may be seen as *Clique cover* of it's complement G', impling in:

$r \backslash l$	0	1	2	3	4	...	n
0	*P*	*P*	*P*	*P*	*NPc*	...	*NPc*
1	*P*	*P*	*NPc*	*NPc*	*NPc*	...	*NPc*
2	*P*	*NPc*	*NPc*	*NPc*	*NPc*	...	*NPc*
3	*Npc*	*NPc*	*NPc*	*NPc*	*NPc*	...	*NPc*
4	*NPc*	*NPc*	*NPc*	*NPc*	*NPc*	...	*NPc*
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	*NPc*
n	*NPc*	*NPc*	*NPc*	*NPc*	*NPc*	...	*NPc*

Table 3: Dichotomy of the clique cover problem on Graph(r,l)

Does the Minimum vertex coloring problem has a parametrized solution for Graph(r,l)?

Graph(2,1)

Parametrized by the size of the smallest independent set

We know the clique can be colored with k colors.

We know that we can transform a minimun coloring problem into a list coloring problem.

Fellows showed that the list coloring problem is w[1]-hard for bipartide graph when parametrized by the size of the smallest independent set.

Minimal vertex coloring is w[1]-hard when parametrized by the smallest independet set in a Graph(2,1).

Conclusion

We were sucessfull in building our foundation, answering our first question, and discovered a interesting relation between two coloring problems applied to the Graph(r,l) class.

Future works:

- Why the problem behave that way?
- Does the Minimum vertex coloring problem has a parametrized solution for Graph(r,l)?

THANK YOU!

Questions?