# ICS-E4030 - Exercise 1

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## 1 Polynomial Kernel:

i)$K(x,z) = (x'z + c)^2$

$$\begin{aligned} K(x,z) &= (x'z + c)^2 \\ &= (x'z)^2 + c^2 + 2x'zc \end{aligned}$$

The feature is given by $\phi(x) = [x^2 \; \sqrt{2c}x \; c]^T$.

ii) $K(x,z) = (x'z + c)^d$

$$\begin{aligned} K(x,z) &= (x'z + c)^d \\ &\mathit{Expandingusing\ Binomial\ Theorem} \\ &= \sum_{s=0}^{d} \binom{d}{s} (x'z)^{d-s} c^s \end{aligned}$$

The feature is given by $\phi(x) = [x^d \sqrt{\binom{d}{1}} cx^{d-1} \sqrt{\binom{d}{2}} c^2 x^{d-2} \ldots\ldots\ldots\ldots\ldots c^{s/2}]^T$.

K can be represented as a inner product of 2 features $K(x,z) = \phi(x)^T \phi(z)$. Hence the polynomial kernel is a valid kernel.

## 2 Proposition 1:

To prove that K(x,z) is a kernel function if and only if it is positive semi definite.

Let $K = V \Lambda V'$and positive semi definite

$V\Lambda = (\vec{V_1}, \vec{V_2}, \vec{V_3}, ..., \vec{V_n})\Lambda$

$[V\Lambda]_{i,t} = (V_t)_i \lambda_t$

$[[V\Lambda]V']_{i,j} = \sum_{t=1}^{n} (V_t)_i \lambda_t (V_t')_j$

Which can be represented as feature map as $\phi_l(x^i) = \sqrt{\lambda_l}(V_t)_i$

Hence K can be represented as a inner product $k(x^i, x^j) = \phi_l(x^i).\phi_l(x^j)$

This proves that K is a kernel.

To prove that the Kernel Matrix K is postive semi definite:

$K_{i,j} = k_{i,j}$

Let c be a vector, then

$$\begin{aligned} c^T K c &= \sum_i \sum_j c_i c_j k_{i,j} \\ &= \sum_i \sum_j c_i c_j \phi(x_i)\phi(x_j) \\ &= (\sum_i c_i \phi(x_i))(\sum_i c_j \phi(x_j)) \\ ||(&\sum_i c_i \phi(x_i)||^2 \geqq 0 \end{aligned}$$

As the norm value is greater than equal to 0, the kernel gram matrix is always positive semi definite.

Hence proved that K is a Kernel function if and only if the matrix is positive semi definite.

## 3 Proposition 2:

1) $K(x, z) = K_1(x, z) + K_2(x, z)$

Let the feature map for $K_1 be\ \phi^1(x)$

and let the feature map for $K_2 be\ \phi^2(x)$

The feature map of $K$ is given by $\phi(x) = [\phi^1(x)\, \phi^2(x)]$

The mapping satisfies $K(x, z) = \phi(x).\phi(z) = \phi^1(x).\phi^1(z) + \phi^2(x).\phi^2(z)$

2) $K(x, z) = aK_1(x, z)$

$K(x, z) = (\sqrt{a}\phi_1(x)).(\sqrt{a}\phi_1(z)) = aK_1(x, z)$

Hence the feature map of K is $\phi(x) = [\sqrt{a}\phi_1(x)]$

3)

$$
\begin{aligned}
K(x,z) &= K_1(x,z)K_2(x,z) \\
&= \phi^1(x)^T\phi^1(z).\phi^2(x)\phi^2(z)^T \\
&= \sum_{i=1}^{n}\phi_i^1(x)\phi^1(z)\sum_{j=1}^{n}\phi_j^2(x)\phi^2(z) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}(\phi_i^1(x)\phi_j^2(x)).(\phi_i^1(z)\phi_j^2(z))
\end{aligned}
$$

Hence the feature is given by $\phi(x) = \big[\phi_1^1(x)\phi_1^2(x)\ \phi_1^1(x)\phi_2^2(x)\ \phi_1^1(x)\phi_3^2(x)\ \ldots\ldots\ldots\phi_1^1(x)\phi_n^2(x)\ \phi_2^1(x)\phi_1^2(x)\ \ldots\ldots\ldots$
$1 \times n^2$

4)$K(x,z) = f(x).f(z)$

This is a valid kernel is the feature map is defined by just the single function f().

5)

$$
\begin{aligned}
K(x,z) &= K_3(\phi(x),\phi(z)) \\
&= \phi^3(\phi(x))^T.\phi^3(\phi(z)) \\
&= \phi'(x)^T.\phi'(x) \\
\phi'(x) &= \phi^3(\phi(x))
\end{aligned}
$$

Hence K is a valid kernel

6)

$$
\begin{aligned}
K(x,z) &= x^TBz \\
&= x^TV\Lambda V^Tz \\
&= x^TV\sqrt{\Lambda}^T\sqrt{\Lambda}V^Tz \\
&= (\sqrt{\Lambda}V^Tx)^T(\sqrt{\Lambda}V^Tz) \\
&= \phi(x)^T\phi(z)
\end{aligned}
$$

Hence K is a valid kernel