Autonomous Mobile Robotics

MECH650-EECE698

Measurement Modeling Exercise: Beam Model for Range Finders

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1 Introduction and Problem Setup

In this exercise, you will work through the mathematical framework for modeling range sensor measurements using the **beam model**, which is one of the most comprehensive sensor models in mobile robotics. This model will be critical for understanding:

- How to represent sensor uncertainty using probability distributions
- The four types of range measurement errors and their physical causes
- How to compute measurement likelihood $p(z_t|x_t, m)$ for state estimation
- The role of measurement models in the Bayes filter update step
- How this likelihood function becomes importance weights in particle filters

1.1 Scenario: Mobile Robot in Laboratory Corridor

Consider a mobile robot navigating a laboratory corridor equipped with a laser range finder (similar to SICK LMS series sensors). The robot is taking distance measurements to localize itself relative to the known corridor walls.

- **Environment**: Straight corridor with walls at $x = 0$ m (left wall) and $x = 6$ m (right wall)
- **Robot pose**: $x_t = (3.0\text{ m}, 2.0\text{ m}, 0)^T$ (position and orientation known for this exercise)
- **Sensor**: Laser range finder with maximum range $z_{\max} = 5.0$ m
- **Map**: Known corridor geometry (wall locations provided)

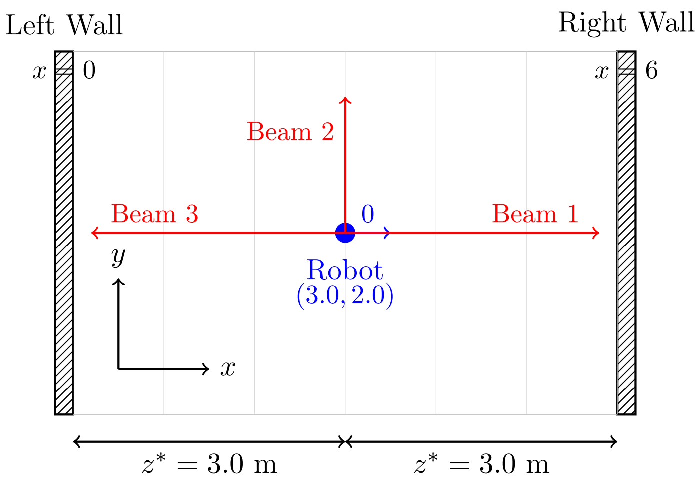


Figure 1: Robot in corridor taking three range measurements. Expected ranges: Beam 1 (right, 0°) = 3.0 m, Beam 2 (forward, 90°) = no obstacle within range, Beam 3 (left, 180°) = 3.0 m.

2 Mathematical Framework

2.1 Beam Range Finder Model

The beam model represents the probability of observing a range measurement z_t^k given the robot pose x_t and map m. It accounts for four types of measurement errors using a **mixture distribution**:

$$p(z_t^k|x_t, m) = \begin{bmatrix} z_{\text{hit}} \\ z_{\text{short}} \\ z_{\text{max}} \\ z_{\text{rand}} \end{bmatrix}^T \cdot \begin{bmatrix} p_{\text{hit}}(z_t^k|x_t, m) \\ p_{\text{short}}(z_t^k|x_t, m) \\ p_{\text{max}}(z_t^k|x_t, m) \\ p_{\text{rand}}(z_t^k|x_t, m) \end{bmatrix} \tag{1}$$

where $z_{\text{hit}}, z_{\text{short}}, z_{\text{max}}, z_{\text{rand}}$ are **mixing weights** (must sum to 1.0) and the four components model different error types:

2.2 Component 1: Measurement Noise (Gaussian)

Models correct measurements with local noise:

$$p_{\text{hit}}(z^k|x, m) = \begin{cases} \eta \cdot \mathcal{N}(z^k; z^{k*}, \sigma_{\text{hit}}^2) & \text{if } 0 \le z^k \le z_{\text{max}} \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

where:

- z^{k*} = expected true range (computed via ray casting from robot pose to obstacle)
- σ_{hit} = standard deviation of measurement noise
- $\mathcal{N}(z; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(z-\mu)^2}{2\sigma^2}\right)$ (Gaussian PDF)
- η = normalization constant so that $\int_0^{z_{\text{max}}} p_{\text{hit}}(z)\, dz = 1$

2.3 Component 2: Unexpected Objects (Exponential)

Models obstacles closer than expected (e.g., people, dynamic objects):

$$p_{\text{short}}(z^k|x,m) = \begin{cases} \eta \cdot \lambda_{\text{short}} e^{-\lambda_{\text{short}} z^k} & \text{if } 0 \leq z^k \leq z^{k*} \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

where:

- λ_{short} = rate parameter of exponential distribution
- $\eta = \frac{1}{1-e^{-\lambda_{\text{short}} z^{k*}}}$ (normalization constant)

2.4 Component 3: Sensor Failures (Point Mass at z_{max})

Models complete measurement failures (no return signal):

$$p_{\text{max}}(z^k|x,m) = \begin{cases} 1 & \text{if } z^k = z_{\text{max}} \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

2.5 Component 4: Random Measurements (Uniform)

Models spurious reflections and unexplained noise:

$$p_{\text{rand}}(z^k|x,m) = \begin{cases} \frac{1}{z_{\text{max}}} & \text{if } 0 \leq z^k < z_{\text{max}} \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

3 Problem Parameters

3.1 Sensor Intrinsic Parameters

Parameter	Value	Description
z_{max}	5.0 m	Maximum sensor range
σ_{hit}	0.10 m	Measurement noise std. dev.
λ_{short}	0.5 m^{-1}	Unexpected object rate parameter

Table 1: Laser range finder specifications (typical for SICK LMS series)

3.2 Mixture Weights

Weight	Value	Physical Meaning
z_{hit}	0.75	Probability of correct measurement
z_{short}	0.15	Probability of unexpected obstacle
z_{max}	0.05	Probability of sensor failure
z_{rand}	0.05	Probability of random noise
Sum	**1.00**	**Must equal 1.0**

Table 2: Mixing parameters for the beam model

3.3 Measurement Data

The robot takes three range measurements (one scan):

Beam	Direction	Expected Range z^{k*} (m)	Measured Range z_t^k (m)
1	0 (right)	3.0	2.9
2	90 (forward)	$>$ 5.0 (no obstacle)	5.0 (max range)
3	180 (left)	3.0	3.1

Table 3: Range measurements for this exercise

Note: Beam 2 hits no obstacle within sensor range, so $z^{2*} > z_{\text{max}}$. For calculation purposes, treat any $z^{k*} > z_{\text{max}}$ as approaching infinity when computing p_{short}.

4 Tasks

Part A: Component Probability Calculations

Task 1: Gaussian Component $p_{\mathbf{hit}}$

For Beam 1 with $z_t^1 = 2.9$ m and $z^{1*} = 3.0$ m:

(a) Write the unnormalized Gaussian probability density:

$$p_{\text{Gaussian}}(z_t^1) = \frac{1}{\sqrt{2\pi\sigma_{\text{hit}}^2}} \exp\left(-\frac{(z_t^1 - z^{1*})^2}{2\sigma_{\text{hit}}^2}\right) = \rule{6cm}{0.4pt}$$

(b) The normalization constant for truncating the Gaussian to $[0, z_{\max}]$ is computed as:

$$\eta = \left[\int_0^{z_{\max}} \mathcal{N}(z; z^{1*}, \sigma_{\text{hit}}^2)\, dz\right]^{-1}$$

For practical purposes, when z^{k*} is well within the range $[0, z_{\max}]$ and σ_{hit} is small, we can approximate $\eta \approx 1.0$ (since nearly all probability mass is within the sensor range).

Using $\eta \approx 1.0$, calculate $p_{\text{hit}}(z_t^1|x_t, m) =$ ________________________

Task 2: Exponential Component $p_{\mathbf{short}}$

For Beam 1 with $z_t^1 = 2.9$ m and $z^{1*} = 3.0$ m:

(a) Calculate the exponential normalization constant:

$$\eta = \frac{1}{1 - e^{-\lambda_{\text{short}} \cdot z^{1*}}} = \rule{4cm}{0.4pt}$$

(b) Calculate the unnormalized exponential probability:

$$\lambda_{\text{short}} e^{-\lambda_{\text{short}} z_t^1} = \rule{4cm}{0.4pt}$$

(c) Calculate the normalized $p_{\text{short}}(z_t^1|x_t, m) =$ ____________________

Task 3: Point Mass Component $p_{\mathbf{max}}$

For Beam 1 with $z_t^1 = 2.9$ m:

Since $z_t^1 \neq z_{\max}$, we have $p_{\max}(z_t^1|x_t, m) =$ _______________

For Beam 2 with $z_t^2 = 5.0$ m (exactly at max range):

Since $z_t^2 = z_{\max}$, we have $p_{\max}(z_t^2|x_t, m) =$ _______________

Task 4: Uniform Random Component $p_{\mathbf{rand}}$

For any measurement $0 \leq z^k < z_{\max}$:

$$p_{\text{rand}}(z^k|x_t, m) = \rule{3cm}{0.4pt}$$

Note: For measurements exactly at $z^k = z_{\max}$, we define $p_{\text{rand}} = 0$ (to avoid conflict with $p_{\max}$).

Part B: Mixture Model Construction

Task 5: Verify Mixing Weights

Show that the mixing weights sum to 1.0:

$$z_{\text{hit}} + z_{\text{short}} + z_{\text{max}} + z_{\text{rand}} = \underline{\hspace{3cm}}$$

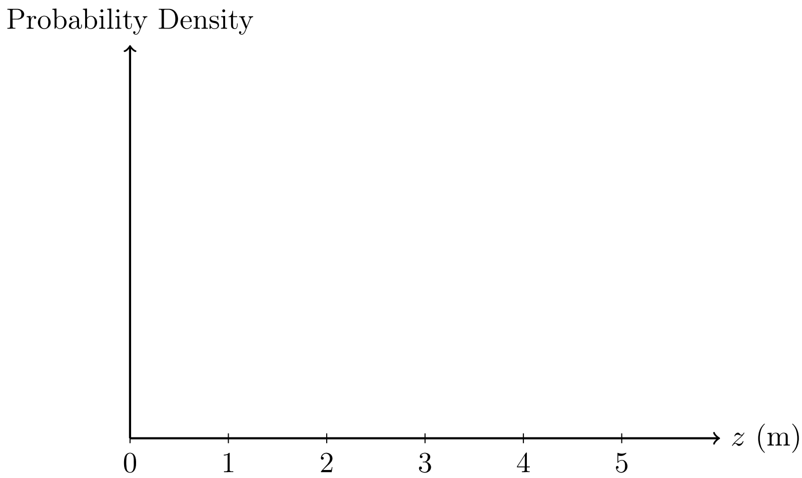
Task 6: Compute Mixture Probability for Beam 1

Using your results from Part A, compute the full mixture probability for Beam 1 ($z_t^1 = 2.9$ m):

$$p(z_t^1 | x_t, m) = \underline{\hspace{6cm}}$$

Task 7: Sketch Component Contributions

On the axes below, sketch the four probability density components ($p_{\text{hit}}, p_{\text{short}}, p_{\text{max}}, p_{\text{rand}}$) as functions of z for Beam 1, where $z^{1*} = 3.0$ m. Label the peak of the Gaussian, the exponential decay, the point mass at $z_{\text{max}} = 5.0$ m, and the uniform level.



Part C: Multi-Measurement Scan Likelihood

The full scan consists of three measurements: $z_t = \{z_t^1, z_t^2, z_t^3\} = \{2.9, 5.0, 3.1\}$ m.

Assuming **independence between beams**, the total scan likelihood is:

$$p(z_t|x_t, m) = \prod_{k=1}^{3} p(z_t^k|x_t, m) = p(z_t^1|x_t, m) \times p(z_t^2|x_t, m) \times p(z_t^3|x_t, m)$$

Task 8: Compute Likelihood for Beam 2 (Max Range Reading)

For Beam 2 with $z_t^2 = 5.0$ m (max range, no obstacle detected):

(a) Since there is no expected obstacle ($z^{2*} > z_{\text{max}}$), the Gaussian component cannot be meaningfully computed. For this case, we consider only the other three components. Compute each:

$$p_{\text{hit}}(z_t^2) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

$$p_{\text{short}}(z_t^2) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

$$p_{\text{max}}(z_t^2 = z_{\text{max}}) := \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

$$p_{\text{rand}}(z_t^2 = z_{\text{max}}) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

(b) Compute the mixture:

$$p(z_t^2|x_t, m) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

Task 9: Compute Likelihood for Beam 3

For Beam 3 with $z_t^3 = 3.1$ m and $z^{3*} = 3.0$ m (similar to Beam 1, but slightly larger):

(a) Compute the Gaussian component:

$$p_{\text{hit}}(z_t^3|x_t, m) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

(b) Compute the exponential component (use same normalization as Beam 1):

$$p_{\text{short}}(z_t^3|x_t, m) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

(c) Compute the mixture:

$$p(z_t^3|x_t, m) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

Task 10: Total Scan Likelihood

Using your results from Tasks 6, 8, and 9, compute the total scan likelihood:

$$p(z_t|x_t, m) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$