Probabilistic Estimation using Bayes Filters

MECH650| EECE 698
AMR Course

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1 Introduction and Problem Setup

In this exercise, you will develop a complete Bayesian filter for a simple mobile robot localization problem. You will learn to construct all components of the belief system and implement the recursive Bayes filter algorithm analytically.

1.1 Robot and Environment Description

Consider a simple mobile robot operating in a **1-dimensional corridor** of length 10 meters. The corridor is divided into **5 discrete cells**, each 2 meters wide, numbered from 0 to 4.

- **Robot State**: $x \in \{0, 1, 2, 3, 4\}$ (cell number)
- **Robot Motion**: Can move left (-1), stay (0), or right (+1) with some uncertainty
- **Robot Sensor**: Detects nearby walls with some noise
- **Environment**: Walls at both ends (cells 0 and 4 are near walls)

Cell 0	**Cell 1**	**Cell 2**	**Cell 3**	**Cell 4**
WALL				WALL
0-2m	2-4m	4-6m	6-8m	8-10m

Figure 1: 1D corridor with 5 discrete cells. Walls are present at cells 0 and 4.

2 Mathematical Framework

2.1 State Space and Notation

Define the following variables:

- x_t = ______________________
- u_t = ______________________
- z_t = ______________________
- $\text{bel}(x_t)$ = ______________________

2.2 Bayes Filter Components

The recursive Bayes filter requires three key components: the motion model, sensor model and prior belief, write the expresion and definition of each:

1. **Motion Model**: $p(\_\_\_\_\_|\_\_\_\_\_) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$
2. **Sensor Model**: $p(\_\_\_\_\_|\_\_\_\_\_) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$
3. **Prior Belief**: $\text{bel}(x_{t-1}) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$

3 Task 1: Derive the Motion Model

The robot receives a control command $u_t \in \{-1, 0, +1\}$ but execution is imperfect:

- With probability 0.8: Robot moves as commanded
- With probability 0.1: Robot moves one step less than commanded
- With probability 0.1: Robot moves one step more than commanded

Example: For robot at cell 2 with command $u_t = +1$:

- $x_t =$ _____with probability _____*(moves to cell 2)*
- $x_t =$ _____with probability _____*(moves to cell 3)*
- $x_t =$ _____with probability _____*(moves to cell 4)*

Complete the motion model for $u_t = +1$ (robot cannot move outside cells 0-4):

$p(x_t\|x_{t-1}, u_t = +1)$	$x_t = 0$	$x_t = 1$	$x_t = 2$	$x_t = 3$	$x_t = 4$
$x_{t-1} = 0$	_____	_____	_____	0	0
$x_{t-1} = 1$	_____	_____	_____	_____	0
$x_{t-1} = 2$	0	_____	_____	_____	_____
$x_{t-1} = 3$	0	0	_____	_____	_____
$x_{t-1} = 4$	0	0	0	_____	_____

4 Task 2: Derive the Sensor Model

The robot has a **wall detector** with output $z_t \in \{0, 1\}$:

- In cells 0 and 4 (near walls): $P(z_t = 1|x_t \in \{0, 4\}) = 0.9$
- In cells 1, 2, 3 (away from walls): $P(z_t = 1|x_t \in \{1, 2, 3\}) = 0.2$

Complete the sensor model:

State	$p(z_t = 1\|x_t)$	$p(z_t = 0\|x_t)$
$x_t = 0$	_____	_____
$x_t = 1$	_____	_____
$x_t = 2$	_____	_____
$x_t = 3$	_____	_____
$x_t = 4$	_____	_____

5 Task 3: Initialize Prior Belief

Assume the robot starts with **uniform uncertainty** about its location.

$$\text{bel}(x_0) = \begin{cases} \frac{1}{5} & \text{for } x_0 \in \{0,1,2,3,4\} \\ 0 & \text{otherwise} \end{cases}$$

Initial belief: $\text{bel}(x_0) = [\_\_\_\_\_, \_\_\_\_\_, \_\_\_\_\_, \_\_\_\_\_, \_\_\_\_\_]$
Verification: $\sum_{x_0} \text{bel}(x_0) = \_\_\_\_\_$

6 Task 4: Implement the Bayes Filter Algorithm

6.1 Bayes Filter Steps

The recursive Bayes filter consists of two steps:
Prediction Step (Motion Update):

$$\overline{\text{bel}}(x_t) = \sum_{x_{t-1}} p(x_t|x_{t-1}, u_t) \cdot \text{bel}(x_{t-1})$$

Update Step (Measurement Update):

$$\text{bel}(x_t) = \eta \cdot p(z_t|x_t) \cdot \overline{\text{bel}}(x_t)$$

where η is the normalization constant: $\eta = \frac{1}{\sum_{x_t} p(z_t|x_t) \cdot \overline{\text{bel}}(x_t)}$

6.2 Step-by-Step Example

Given:

- Initial belief: $\text{bel}(x_0)$
- Control input: $u_1 = +1$ (move right)
- Sensor measurement: $z_1 = 1$ (wall detected)

Step 1 - Prediction: Calculate $\overline{\text{bel}}(x_1)$
For $\overline{\text{bel}}(x_1 = 0)$:

$$\overline{\text{bel}}(x_1 = 0) = \sum_{x_0} p(x_1 = 0|x_0, u_1 = +1) \cdot \text{bel}(x_0)$$

From the motion model table:

$$\begin{aligned} \overline{\text{bel}}(x_1 = 0) &= p(x_1 = 0|x_0 = 0, u_1 = +1) \cdot \text{bel}(x_0 = 0) && (1) \\ &\quad + p(x_1 = 0|x_0 = 1, u_1 = +1) \cdot \text{bel}(x_0 = 1) && (2) \\ &\quad + \ldots && (3) \\ &= \_\_\_\_\_ \cdot 0.2 + \_\_\_\_\_ \cdot 0.2 + 0 \cdot 0.2 + 0 \cdot 0.2 + 0 \cdot 0.2 && (4) \\ &= \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ && (5) \end{aligned}$$

Complete the remaining predictions:

- $\overline{\text{bel}}(x_1 = 1) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$
- $\overline{\text{bel}}(x_1 = 2) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$

- $\overline{\text{bel}}(x_1 = 3) =$ ____________________
- $\overline{\text{bel}}(x_1 = 4) =$ ____________________

Step 2 - Update: Calculate $\text{bel}(x_1)$ using $z_1 = 1$
For each state, multiply predicted belief by sensor likelihood:

- $\text{bel}(x_1 = 0) \propto p(z_1 = 1|x_1 = 0) \cdot \overline{\text{bel}}(x_1 = 0) =$ _____ $\cdot$ _____ $=$ _____
- $\text{bel}(x_1 = 1) \propto p(z_1 = 1|x_1 = 1) \cdot \overline{\text{bel}}(x_1 = 1) =$ _____ $\cdot$ _____ $=$ _____
- $\text{bel}(x_1 = 2) \propto$ ____________________
- $\text{bel}(x_1 = 3) \propto$ ____________________
- $\text{bel}(x_1 = 4) \propto$ ____________________

Step 3 - Normalization:
Sum of unnormalized beliefs: $\sum_{x_1} \text{bel}(x_1) =$ ____________________
Normalization constant: $\eta = \frac{1}{\text{sum}} =$ ____________________
Final normalized belief:

$$\text{bel}(x_1) = [\_\_\_\_\_\_\_\_\_\_, \_\_\_\_\_\_\_\_\_\_, \_\_\_\_\_\_\_\_\_\_, \_\_\_\_\_\_\_\_\_\_, \_\_\_\_\_\_\_\_\_\_]$$

7 Task 5: Analysis and Key Insights

1. Recursive Nature: How does the Bayes filter use only the previous belief rather than the entire history of measurements?
Answer: ____________________
2. Effect of Motion vs. Measurement:

- How did the prediction step affect uncertainty? ______________________________
- How did the update step affect uncertainty? ______________________________

3. Practical Applications: Name two robotics applications where Bayesian localization would be useful:

1. ______________________________
2. ______________________________