

# *Processamento Digital de Imagens*

## Transformadas

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# Transformadas

**1-D:**  $\{u(n), 0 \leq n \leq N-1\}$  representada por  $\mathbf{v}$ ,  $N \times 1$ , tal que:

$$\mathbf{v} = \mathbf{A}\mathbf{u} \Leftrightarrow v(k) = \sum_{n=0}^{N-1} a_k(n)u(n), \quad 0 \leq k \leq N-1$$

$$\Rightarrow \mathbf{u} = \mathbf{A}^{-1}\mathbf{v}$$

$$\text{Se } \mathbf{A} \text{ é unitária} \Rightarrow \mathbf{A}^{-1} = \mathbf{A}^{*\text{T}} \Rightarrow \mathbf{u} = \mathbf{A}^{*\text{T}}\mathbf{v}$$

$$\Rightarrow u(n) = \sum_{k=0}^{N-1} a_k^*(n)v(k), \quad 0 \leq n \leq N-1$$

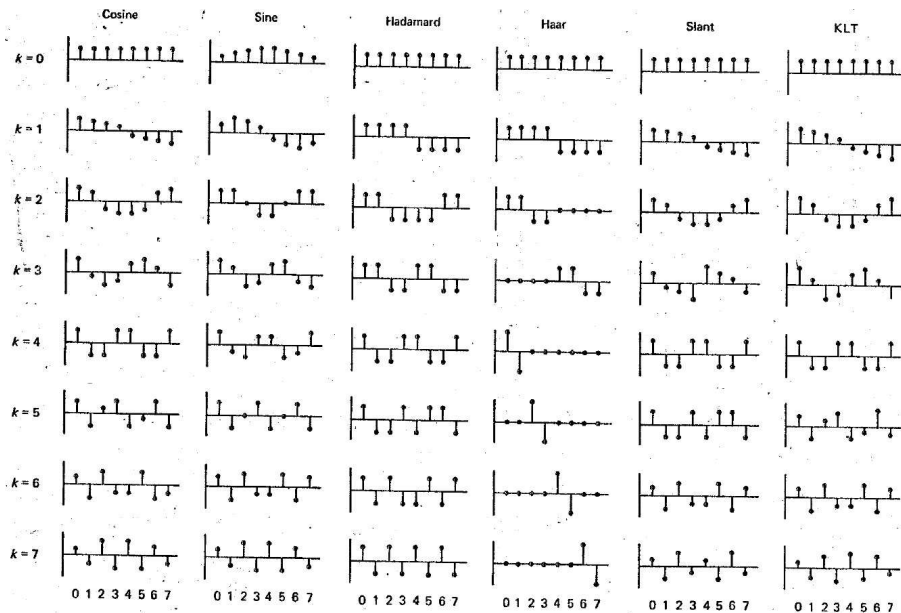
Seja  $\mathbf{A} = \begin{bmatrix} \mathbf{a}_0^T \\ \mathbf{a}_1^T \\ \vdots \\ \mathbf{a}_{N-1}^T \end{bmatrix} \Rightarrow \mathbf{A}^{*T} = [\mathbf{a}_0^* \quad \mathbf{a}_1^* \quad \cdots \quad \mathbf{a}_{N-1}^*]$

$$\Rightarrow \mathbf{u} = \mathbf{A}^{*T} \mathbf{v} = [\mathbf{a}_0^* \quad \mathbf{a}_1^* \quad \cdots \quad \mathbf{a}_{N-1}^*] \begin{bmatrix} v(0) \\ v(1) \\ \vdots \\ v(N-1) \end{bmatrix} = \sum_{k=0}^{N-1} \mathbf{a}_k^* v(k)$$

Combinação  
linear de funções  
base  $\mathbf{a}_k^*$

$$\Rightarrow \mathbf{v} = \mathbf{A} \mathbf{u} = \begin{bmatrix} \mathbf{a}_0^T \\ \mathbf{a}_1^T \\ \vdots \\ \mathbf{a}_{N-1}^T \end{bmatrix} \mathbf{u} = \begin{bmatrix} \mathbf{a}_0^T \mathbf{u} \\ \mathbf{a}_1^T \mathbf{u} \\ \vdots \\ \mathbf{a}_{N-1}^T \mathbf{u} \end{bmatrix} \Rightarrow v(k) = \mathbf{a}_k^T \mathbf{u}$$

Produto interno  
da função base  
 $\mathbf{a}_k$  e o vetor  $\mathbf{u}$



- Como  $\mathbf{A}\mathbf{A}^{*\text{T}} = \mathbf{I}$ ,  $\Rightarrow \begin{bmatrix} \mathbf{a}_0^{\text{T}} \\ \mathbf{a}_1^{\text{T}} \\ \vdots \\ \mathbf{a}_{N-1}^{\text{T}} \end{bmatrix} \begin{bmatrix} \mathbf{a}_0^* & \mathbf{a}_1^* & \cdots & \mathbf{a}_{N-1}^* \end{bmatrix} = \mathbf{I}$

Isto implica que:  $\mathbf{a}_i^{\text{T}} \mathbf{a}_j^* = \delta(i - j)$  (Ortonormalidade)

- Como  $\mathbf{A}^{*\text{T}} \mathbf{A} = \mathbf{I}$ ,  $\Rightarrow \begin{bmatrix} \mathbf{a}_0^* & \mathbf{a}_1^* & \cdots & \mathbf{a}_{N-1}^* \end{bmatrix} \begin{bmatrix} \mathbf{a}_0^{\text{T}} \\ \mathbf{a}_1^{\text{T}} \\ \vdots \\ \mathbf{a}_{N-1}^{\text{T}} \end{bmatrix} = \mathbf{I}$

Isto implica que:  $\sum_{k=0}^{N-1} \mathbf{a}_k^* \mathbf{a}_k^{\text{T}} = \mathbf{I}$  (Completeness)

## 2-D: Transformada ortogonal

$$v(k, l) = \sum_{m=0}^{N-1} \sum_{n=0}^{M-1} u(m, n) a_{kl}(m, n) \quad 0 \leq k, l \leq N-1$$

$$u(m, n) = \sum_{k=0}^{N-1} \sum_{l=0}^{M-1} v(k, l) a_{kl}^*(m, n) \quad 0 \leq m, n \leq N-1$$

$a_{kl}(m, n) \Rightarrow$  transformada da imagem

$a_{kl}(\cdot, \cdot) \Rightarrow$  imagens base

Ortonormalidade: 
$$\sum_{m=0}^{N-1} \sum_{n=0}^{M-1} a_{kl}(m, n) a_{k'l'}^*(m, n) = \delta(k-k', l-l')$$

“Completeness”: 
$$\sum_{k=0}^{N-1} \sum_{l=0}^{M-1} a_{kl}(m, n) a_{kl}^*(m', n') = \delta(m-m', n-n')$$

Se a expansão é truncada de modo que:

$$u_{PQ}(m, n) = \sum_{k=0}^{P-1} \sum_{l=0}^{Q-1} v(k, l) a_{kl}^*(m, n)$$

Então

$$\sigma_e^2 = \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} [u(m, n) - u_{PQ}(m, n)]^2$$

é mínimo se

$$v(k, l) = \sum_{m=0}^{N-1} \sum_{n=0}^{M-1} u(m, n) a_{kl}(m, n)$$



## Transformadas unitárias separáveis

$$v(k, l) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} u(m, n) a_{kl}(m, n) \quad 0 \leq k \leq M-1, 0 \leq l \leq N-1$$

$$a_{kl}(m, n) = a_k(m) b_l(n)$$

Não separável:

$$O(M^2 N^2)$$

Separável:

$$O(MN^2 + NM^2)$$

Se  $\mathbf{A} = \{a_k(m)\}$ ,  $\mathbf{B} = \{b_l(n)\}$

Ortonormalidade  
e Completude

$$\Rightarrow \begin{aligned} \mathbf{A} \mathbf{A}^{*T} &= \mathbf{A}^T \mathbf{A}^* = \mathbf{I} \\ \mathbf{B} \mathbf{B}^{*T} &= \mathbf{B}^T \mathbf{B}^* = \mathbf{I} \end{aligned}$$

$\mathbf{A}$  e  $\mathbf{B}$  são ortonormais

$$\begin{bmatrix} \mathbf{a}_0^T \\ \mathbf{a}_1^T \\ \vdots \\ \mathbf{a}_{M-1}^T \end{bmatrix} \mathbf{U} [\mathbf{b}_0 \quad \mathbf{b}_1 \quad \cdots \quad \mathbf{b}_{N-1}]$$

Se  $\mathbf{A} = \mathbf{B}$ :

$$v(k, l) = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} a_k(m) u(m, n) a_l(n) \Leftrightarrow \mathbf{V} = \mathbf{A} \mathbf{U} \mathbf{A}^T$$

$$u(m, n) = \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} a_k^*(m) v(k, l) a_l^*(n) \Leftrightarrow \mathbf{U} = \mathbf{A}^{*T} \mathbf{V} \mathbf{A}^*$$

$$\mathbf{U} \text{ e } \mathbf{V} \Rightarrow M \times N \Rightarrow \mathbf{V} = \underbrace{\mathbf{A}_M}_{M \times M} \underbrace{\mathbf{U} \mathbf{A}_N^T}_{N \times N} \quad \mathbf{U} = \underbrace{\mathbf{A}_M^{*T}}_{M \times M} \underbrace{\mathbf{V} \mathbf{A}_N^*}_{N \times N}$$

$\Rightarrow \mathbf{V}^T = \mathbf{A}[\mathbf{A}\mathbf{U}]^T \Rightarrow$  primeiro transformo as colunas de  $\mathbf{U}$  e depois as linhas.

## Imagens Base

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_0^T \\ \mathbf{a}_1^T \\ \vdots \\ \mathbf{a}_{N-1}^T \end{bmatrix} \Rightarrow \mathbf{A}^{*\text{T}} = [\mathbf{a}_0^* \quad \mathbf{a}_1^* \quad \cdots \quad \mathbf{a}_{N-1}^*] \Rightarrow a_{kl}(m, n) = a_k(m)a_l(n)$$

Seja  $\mathbf{A}_{kl} = \{a_{kl}(m, n)\} = \mathbf{a}_k \mathbf{a}_l^T$  e também  $\langle F, G \rangle = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} f(m, n)g^*(m, n)$

Como  $u(m, n) = \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} v(k, l) a_{kl}^*(m, n),$

$$\Rightarrow \mathbf{U} = \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} v(k, l) \mathbf{A}_{kl}^*$$

Combinação  
linear de imagens  
base

$$v(k, l) = \langle \mathbf{U}, \mathbf{A}_{kl}^* \rangle = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} u(m, n) a_{kl}^*(m, n)$$

produto interno  
da imagem com  
as imagens base

## Matrizes “Block Circulant” e Produtos de Kronecker

Convolução bi-dimensional:

$$y(m, n) = \sum_{m'} \sum_{n'} h(m-m', n-n') x(m', n')$$

$$\text{Seja } \underbrace{\mathbf{Y}_n}_{M \times 1} = \begin{bmatrix} y(0, n) \\ y(1, n) \\ \vdots \\ y(k, n) \\ \vdots \end{bmatrix} \quad \underbrace{(\mathbf{H}_n)_{i,j}}_{M \times N} = h(i-j, n) \quad \underbrace{\mathbf{X}_n}_{N \times 1} = \begin{bmatrix} x(0, n) \\ x(1, n) \\ \vdots \\ x(k, n) \\ \vdots \end{bmatrix}$$

Então

$$\mathbf{Y}_n = \sum_{n'} \mathbf{H}_{n-n'} \mathbf{X}'_{n'}, \quad \mathbf{H}_n : \text{Toeplitz}$$

Seja

$$\mathcal{Y} = \begin{bmatrix} \mathbf{Y}_0 \\ \mathbf{Y}_1 \\ \vdots \\ \mathbf{Y}_k \\ \vdots \end{bmatrix}$$

(empilho as colunas)

$$\mathcal{X} = \begin{bmatrix} \mathbf{X}_0 \\ \mathbf{X}_1 \\ \vdots \\ \mathbf{X}_k \\ \vdots \end{bmatrix}$$

(empilho as colunas)

$$\Rightarrow \mathcal{Y} = \underbrace{\begin{bmatrix} \mathbf{H}_0 & \mathbf{H}_{-1} & \mathbf{H}_{-2} & \cdots & \cdots \\ \mathbf{H}_1 & \mathbf{H}_0 & \mathbf{H}_{-1} & \cdots & \cdots \\ \mathbf{H}_2 & \mathbf{H}_1 & \mathbf{H}_0 & \mathbf{H}_{-1} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}}_{\mathcal{H}: \text{block Toeplitz}} \begin{bmatrix} \mathbf{X}_0 \\ \mathbf{X}_1 \\ \vdots \\ \mathbf{X}_k \\ \vdots \end{bmatrix} = \mathcal{H}\mathcal{X}$$

## Produto de Kronecker:

$$\underbrace{\mathbf{A}}_{M_1 \times M_2} \otimes \underbrace{\mathbf{B}}_{N_1 \times N_2} = \underbrace{\begin{bmatrix} a_{1,1}\mathbf{B} & a_{1,2}\mathbf{B} & \cdots & a_{1,M_2}\mathbf{B} \\ a_{2,1}\mathbf{B} & a_{2,2}\mathbf{B} & \cdots & a_{2,M_2}\mathbf{B} \\ \vdots & \vdots & \ddots & \vdots \\ a_{M_1,1}\mathbf{B} & a_{M_1,2}\mathbf{B} & \cdots & a_{M_1,M_2}\mathbf{B} \end{bmatrix}}_{M_2 N_2} \Bigg\} M_1 N_1$$

Seja  $\mathbf{U}$  e  $\mathbf{V}$ ,  $M \times N$  tal que  $\mathbf{V} = \underbrace{\mathbf{A}}_{M \times M} \mathbf{U} \underbrace{\mathbf{B}^T}_{N \times N}$ ,  $\mathbf{A}$ : colunas de  $\mathbf{U}$   
 $\mathbf{B}$ : linhas de  $\mathbf{U}$

e  $\mathbf{v}_k^T, \mathbf{u}_k^T$  as  $k$ -ésimas linhas de  $\mathbf{V}$  e  $\mathbf{U}$ , respectivamente.

$$\mathbf{V} = \underbrace{\begin{bmatrix} \mathbf{v}_0^T \\ \mathbf{v}_1^T \\ \vdots \\ \mathbf{v}_{M-1}^T \end{bmatrix}}_{M \times N} \quad \mathbf{U} = \underbrace{\begin{bmatrix} \mathbf{u}_0^T \\ \mathbf{u}_1^T \\ \vdots \\ \mathbf{u}_{M-1}^T \end{bmatrix}}_{M \times N}$$

$$\Rightarrow \mathbf{V} = \begin{bmatrix} \mathbf{v}_0^T \\ \mathbf{v}_1^T \\ \vdots \\ \mathbf{v}_{M-1}^T \end{bmatrix} = \mathbf{A} \begin{bmatrix} \mathbf{u}_0^T \\ \mathbf{u}_1^T \\ \vdots \\ \mathbf{u}_{M-1}^T \end{bmatrix} \mathbf{B}^T = \mathbf{A} \begin{bmatrix} \mathbf{u}_0^T \mathbf{B}^T \\ \mathbf{u}_1^T \mathbf{B}^T \\ \vdots \\ \mathbf{u}_{M-1}^T \mathbf{B}^T \end{bmatrix}$$

$$\Rightarrow [\mathbf{v}_0 \quad \mathbf{v}_1 \quad \cdots \quad \mathbf{v}_{M-1}] = [\mathbf{B}\mathbf{u}_0 \quad \mathbf{B}\mathbf{u}_1 \quad \cdots \quad \mathbf{B}\mathbf{u}_{M-1}] \mathbf{A}^T$$



$$[\mathbf{v}_0 \quad \mathbf{v}_1 \quad \cdots \quad \mathbf{v}_{M-1}] = [\mathbf{B}\mathbf{u}_0 \quad \mathbf{B}\mathbf{u}_1 \quad \cdots \quad \mathbf{B}\mathbf{u}_{M-1}] \begin{bmatrix} a_{0,0} & a_{1,0} & \cdots & a_{M-1,0} \\ a_{0,1} & a_{1,1} & \cdots & a_{M-1,1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{0,M-1} & a_{1,M-1} & \cdots & a_{M-1,M-1} \end{bmatrix}$$

$$\mathbf{v}_k = \sum_{m=0}^{M-1} \mathbf{B}\mathbf{u}_m a_{k,m} = \sum_{m=0}^{M-1} (a_{k,m} \mathbf{B}) \mathbf{u}_m = \sum_{m=0}^{M-1} \underbrace{[\mathbf{A} \otimes \mathbf{B}]_{k,m}}_{N \times N} \underbrace{\mathbf{u}_m^T}_{N \times 1}$$

Seja

$$\mathcal{V} = \underbrace{\begin{bmatrix} \mathbf{v}_0 \\ \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_{M-1} \end{bmatrix}}_{MN \times 1} \quad \mathcal{U} = \underbrace{\begin{bmatrix} \mathbf{u}_0 \\ \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_{M-1} \end{bmatrix}}_{MN \times 1}$$

$$\mathcal{V} = \begin{bmatrix} \sum_{m=0}^{M-1} [\mathbf{A} \otimes \mathbf{B}]_{0,m} \mathbf{u}_m \\ \sum_{m=0}^{M-1} [\mathbf{A} \otimes \mathbf{B}]_{1,m} \mathbf{u}_m \\ \vdots \\ \sum_{m=0}^{M-1} [\mathbf{A} \otimes \mathbf{B}]_{M-1,m} \mathbf{u}_m \end{bmatrix}$$

$$= \begin{bmatrix} [\mathbf{A} \otimes \mathbf{B}]_{0,0} & [\mathbf{A} \otimes \mathbf{B}]_{0,1} & \cdots & [\mathbf{A} \otimes \mathbf{B}]_{0,M-1} \\ [\mathbf{A} \otimes \mathbf{B}]_{1,0} & [\mathbf{A} \otimes \mathbf{B}]_{1,1} & \cdots & [\mathbf{A} \otimes \mathbf{B}]_{1,M-1} \\ \vdots & \vdots & \ddots & \vdots \\ [\mathbf{A} \otimes \mathbf{B}]_{M-1,0} & [\mathbf{A} \otimes \mathbf{B}]_{M-1,1} & \cdots & [\mathbf{A} \otimes \mathbf{B}]_{M-1,M-1} \end{bmatrix} \begin{bmatrix} \mathbf{u}_0 \\ \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_{M-1} \end{bmatrix}$$

$$\Rightarrow \mathcal{V} = (\mathbf{A} \otimes \mathbf{B}) \mathcal{U}$$

$\Rightarrow$  Transformada 2D separável pode ser expressa como transformada 1D

## De volta às imagens base:

Como  $\mathbf{V} = \mathbf{AUA}^T$ , e  $\mathbf{U} = \mathbf{A}^{*T} \mathbf{VA}^*$

Se  $\mathcal{V}$  e  $\mathcal{U}$  são  $\mathbf{V}$  e  $\mathbf{U}$  com as linhas transpostas empilhadas:

$$\mathcal{V} = (\mathbf{A} \otimes \mathbf{A})\mathcal{U} = \mathcal{AU}$$

$$\mathcal{U} = (\mathbf{A} \otimes \mathbf{A})^{*T} \mathcal{V} = \mathcal{AV}, \quad \mathcal{A} = \mathbf{A} \otimes \mathbf{A} \quad (N^2 \times N^2)$$

$$\text{Se } \mathcal{Y} = \mathcal{AX} \text{ e } \mathcal{A} = \mathbf{A}_1 \otimes \mathbf{A}_2 \Rightarrow \mathbf{Y} = \mathbf{A}_1 \mathbf{X} \mathbf{A}_2$$

$\Rightarrow$  Redução de  $O(N^4)$  para  $O(N^3)$  operações.

Se  $\mathbf{A}_1$  e  $\mathbf{A}_2$  podem ser decompostas da mesma forma:

$$\mathbf{A}_1 = \mathbf{A}_{11} \otimes \mathbf{A}_{12}, \quad \mathbf{A}_2 = \mathbf{A}_{21} \otimes \mathbf{A}_{22}$$

$\Rightarrow$  Posso reduzir ainda mais o número de operações.

## Transformadas Rápidas:

⇒ Uma maneira é achar decomposições do tipo

$$\mathbf{A} = \mathbf{A}_1 \otimes \mathbf{A}_2 \otimes \mathbf{A}_3 \cdots$$

A maneira usual de fazer transformadas rápidas é fatorar:  $\mathbf{A} = \mathbf{A}_{(1)} \mathbf{A}_{(2)} \cdots \mathbf{A}_{(p)} \quad (N \times N)$

onde  $\mathbf{A}_{(i)}, i = 1, \cdots, p, \quad p \ll N$  possuem poucos coeficientes diferentes de zero (por exemplo,  $r$ ).

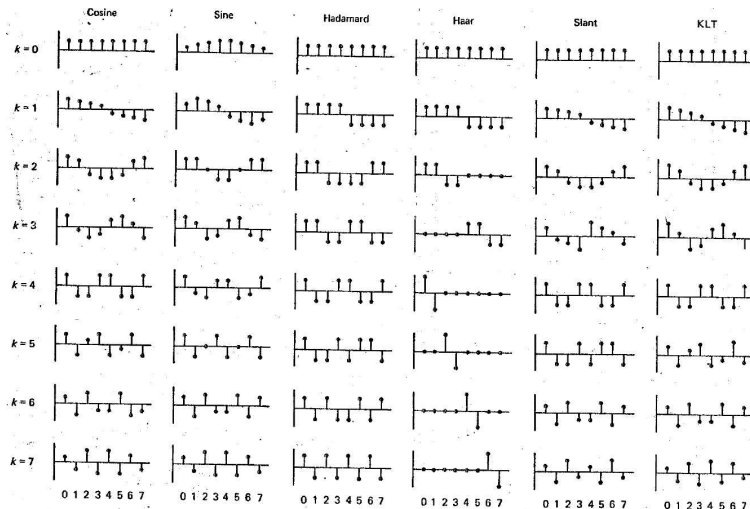
⇒  $\mathbf{y} = \mathbf{Ax} \Rightarrow rpN$  operações.

Transformadas de Fourier, Cosseno, Seno, Haddamand, etc., possuem fatorações tal que  $rpN = N \log N$ .

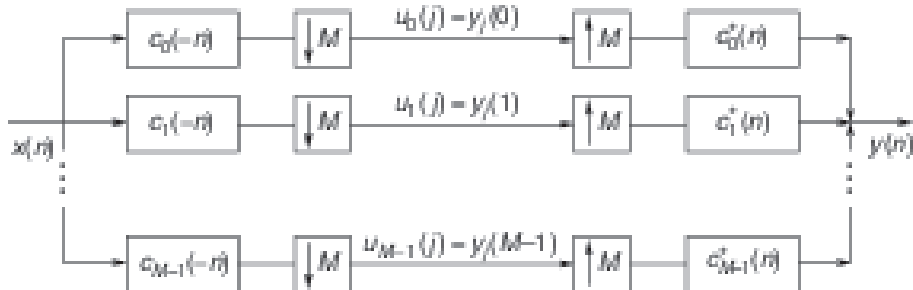
## Frequência:

Fourier:  $k$  equivale à frequência.

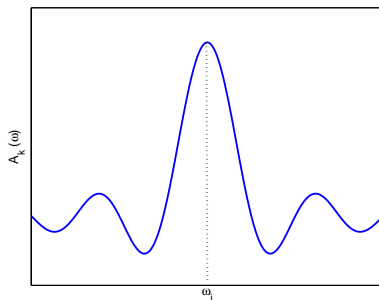
Outras: ordenar as funções base pelo número de cruzamentos de zeros



$$y(k) = \mathbf{a}_k \mathbf{X} \text{ (produto interno)} \Rightarrow y(k) = \sum_{n=0}^{N-1} a_k(n)x(n) = [a_k(-n) * x(n)]$$



$\Rightarrow a_k(-n)$ : resposta ao impulso de um filtro.  $\omega_i$  é a “frequência” de  $a_k(n)$ .



# Propriedades

## Conservação de Energia:

Se  $\mathbf{v} = \mathbf{A}\mathbf{u}$ ,  $\mathbf{A}$  unitária, então:  $\|\mathbf{v}\|^2 = \|\mathbf{u}\|^2$

Prova:

$$\|\mathbf{v}\|^2 = \sum_k v(k)^2 = \mathbf{v}^{*T} \mathbf{v} = \mathbf{u}^{*T} \mathbf{A}^{*T} \mathbf{A} \mathbf{u} = \mathbf{u}^{*T} \mathbf{u} = \|\mathbf{u}\|^2$$

$\Rightarrow$  preserva a norma (equivale a uma rotação no espaço N-dimensional).

Em 2-D:

$$\sum_m \sum_n |u(m, n)|^2 = \sum_k \sum_l |v(k, l)|^2$$



## Compactação de Energia e Variância dos Coeficientes Transformados:

$\mathbf{v} = \mathbf{A}\mathbf{u} \Rightarrow (\mathbf{A} \text{ unitária tende a compactar energia})$

$$\Rightarrow \mu_{\mathbf{v}} = E[\mathbf{v}] = E[\mathbf{A}\mathbf{u}] = \mathbf{A}E[\mathbf{u}] = \mathbf{A}\mu_{\mathbf{u}}$$

$$\Rightarrow \mathbf{R}_{\mathbf{v}} = E[(\mathbf{v} - \mu_{\mathbf{v}})(\mathbf{v} - \mu_{\mathbf{v}})^{*T}] = E[\mathbf{A}(\mathbf{u} - \mu_{\mathbf{u}})(\mathbf{u} - \mu_{\mathbf{u}})^{*T} \mathbf{A}^{*T}] = \mathbf{A}\mathbf{R}_{\mathbf{u}}\mathbf{A}^{*T}$$

$$\Rightarrow \sigma_v^2(k) = [\mathbf{R}_{\mathbf{v}}]_{kk} = [\mathbf{A}\mathbf{R}_{\mathbf{u}}\mathbf{A}^{*T}]_{kk}$$

$$\Rightarrow \sum_{k=0}^{N-1} |\mu_v(k)|^2 = \mu_{\mathbf{v}}^{*T} \mu_{\mathbf{v}} = (\mu_{\mathbf{u}}^{*T} \mathbf{A}^{*T})(\mathbf{A}\mu_{\mathbf{u}}) = \mu_{\mathbf{u}}^{*T} \mu_{\mathbf{u}} = \sum_{n=0}^{N-1} |\mu_u(n)|^2$$

$$\Rightarrow \sum_{k=0}^{N-1} \sigma_v^2(k) = \text{Tr}[\mathbf{A}\mathbf{R}_{\mathbf{u}}\mathbf{A}^{*T}] = \text{Tr}[\mathbf{R}_{\mathbf{u}}] = \sum_{n=0}^{N-1} \sigma_u^2(n)$$

$$\Rightarrow \sum_{k=0}^{N-1} E[|v(k)|^2] = \sum_{n=0}^{N-1} E[|u(n)|^2]$$

2-D:

$$\mu_v(k, l) = \sum_m \sum_n a_k(m) \mu_u(m, n) a_l(n)$$

$$\sigma_v^2(k, l) = \sum_m \sum_n \sum_{m'} \sum_{n'} a_k(m) a_l(n) r(m, n, m', n') a_k^*(m') a_l^*(n')$$

Se  $r(m, n, m', n') = r_1(m, m') r_2(n, n')$ , então:

$$\sigma_v^2(k, l) = \sigma_1^2(k) \sigma_2^2(l) = [\mathbf{A} \mathbf{R}_1 \mathbf{A}^{*T}]_{kk} [\mathbf{A} \mathbf{R}_2 \mathbf{A}^{*T}]_{ll}$$

### Outras Propriedades:

- Entropia de um vetor aleatório é preservada sob uma transformação unitária;
- $|\det(\mathbf{A})| = 1; \quad |\lambda_i| = 1;$

Ex:

$$\mathbf{u} \text{ tal que } \mu_{\mathbf{u}} = \mathbf{0}, \quad \mathbf{v} = \frac{1}{2} \begin{pmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{3} \end{pmatrix} \mathbf{u}, \quad R_{\mathbf{u}} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}, \quad 0 < \rho < 1$$

$$\Rightarrow \mathbf{R}_{\mathbf{v}} = \mathbf{A} \mathbf{R}_{\mathbf{u}} \mathbf{A}^{*\top} = \frac{1}{2} \begin{pmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{3} \end{pmatrix} \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \cdot \frac{1}{2} \begin{pmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{3} \end{pmatrix} = \begin{pmatrix} 1 + \frac{\sqrt{3}}{2}\rho & \frac{\rho}{2} \\ \frac{\rho}{2} & 1 - \frac{\sqrt{3}}{2}\rho \end{pmatrix}$$

$$\Rightarrow \sigma_u^2(0) = \sigma_u^2(1) = 1 \Rightarrow \text{energia média total igualmente distribuída entre } u(0) \text{ e } u(1)$$

$$\Rightarrow \sigma_v^2(0) = 1 + \frac{\sqrt{3}}{2}\rho, \quad \sigma_v^2(1) = 1 - \frac{\sqrt{3}}{2}\rho \Rightarrow \text{energia concentrada em } v(0).$$

$$\Rightarrow \rho = 0.95 \Rightarrow \sigma_v^2(0) = 1.823, \quad \sigma_v^2(1) = 0.177 \Rightarrow 91\% \text{ da energia está em } v(0).$$

$$\underbrace{\rho_v(0, 1)}_{\text{correlação entre } v(0) \text{ e } v(1)} = \frac{E[v(0)v(1)]}{\sigma_v(0)\sigma_v(1)} = \frac{\rho}{2 \left(1 - \frac{3}{4}\rho^2\right)^{\frac{1}{2}}} \Rightarrow \rho = 0.95 \Rightarrow \rho_v(0, 1) = 0.83$$

Se

$$\mathbf{A} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \mathbf{R}_u = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$$

$$\Rightarrow \mathbf{R}_v = \begin{pmatrix} 1 + \rho & 0 \\ 0 & 1 - \rho \end{pmatrix}$$

$\Rightarrow \rho_v(0, 1) = 0 \Rightarrow v(0)$  e  $v(1)$  descorrelatados.

$\Rightarrow \rho = 0.95 \Rightarrow \sigma_v^2(0) = 1.95, \quad \sigma_v^2(1) = 0.05 \Rightarrow 97.5\%$  da energia está em  $v(0)$ .

# Transformada de Karhunen-Loève (KLT)

- É a transformada que gera coeficientes descorrelacionados
  - Diagonaliza a matriz de autocovariância
  - Funções base dependem do processo;
- Também chamada de:
  - Hotelling Transform
  - Métodos das componentes principais (PCA - *Principal Component Analysis*)

$$\mathbf{R}\Phi_k = \lambda_k \Phi_k, \quad 0 \leq k \leq N-1$$

$$\Rightarrow \mathbf{R} [\Phi_0 \quad \Phi_1 \quad \cdots \quad \Phi_{N-1}] = [\Phi_0 \quad \Phi_1 \quad \cdots \quad \Phi_{N-1}] \begin{bmatrix} \lambda_0 & 0 & \cdots & 0 \\ 0 & \lambda_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{N-1} \end{bmatrix}$$

$$\Rightarrow \mathbf{R}\Phi = \Phi\Lambda$$

$$\Lambda = \Phi^{*T} \mathbf{R}\Phi$$

$$\Rightarrow \text{A KLT é } \mathbf{v} = \Phi^{*T} \mathbf{u} \quad \mathbf{u} = \Phi \mathbf{v} = \sum_{k=0}^{N-1} v(k) \Phi_k$$

$$\text{Se } \mathbf{R} = \begin{bmatrix} 1 & \rho & \rho^2 & \dots & \rho^{N-1} \\ \rho & 1 & \rho & \dots & \rho^{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho^{N-1} & \rho^{N-2} & \rho^{N-3} & \dots & 1 \end{bmatrix}$$

$$\Rightarrow \begin{cases} \lambda_k = \frac{1 - \rho^2}{1 - 2\rho \cos \omega_k + \rho^2} \\ \Phi_k(m) = \left( \frac{2}{N + \lambda_k} \right)^{\frac{1}{2}} \sin \left[ \omega_k \left( m + 1 - \frac{N+1}{2} \right) + \frac{(k+1)\pi}{2} \right], & 0 \leq m, k \leq N-1 \\ \omega_k \text{ tal que } \tan(N\omega_k) = -\frac{(1 - \rho^2)\sin\omega_k}{\cos\omega_k - 2\rho + \rho^2 \cos\omega_k}, & N \text{ par} \end{cases}$$

⇒ Não há algoritmo rápido.

⇒ DCT: aproxima a KLT para processos AR(1),  $\rho \rightarrow 1$

⇒ DFT: Diagonaliza matrizes circulantes. É a KLT de sequências periódicas (R é circulante);

# KLT de Imagens:

Se  $r(m, n; m', n') = E[u(m, n)u(m', n)]$ ,  $0 \leq m, m', n, n' \leq N-1$

$$\Rightarrow \sum_{m'=0}^{N-1} \sum_{n'=0}^{N-1} r(m, n; m', n') \Psi_{k,l}(m', n') = \lambda_{k,l} \Psi_{k,l}(m, n), \quad 0 \leq k, l, m, n \leq N-1$$

$\Rightarrow \Psi_{k,l}(m, n)$  são as imagens base da KLT em 2D

“Empilhando”: as linhas de  $u(m, n)$  em  $\mu$ ,

as linhas de  $\Psi_{k,l}$  em  $\Upsilon_i$ ,

$$\Rightarrow \mathcal{R} \Upsilon_i = \lambda_i \Upsilon_i, \quad i = 0, \dots, N^2 - 1$$

$$\mathcal{R} = E[\mu \mu^T]$$

$\Rightarrow$  Caso separável: posso fazer para as linhas e depois para as colunas.

$$\Psi_{k,l} = \Phi_{1_k}(m) \Phi_{2_l}(n)$$

$$\mathbf{V} = \Phi_1^{*T} \mathbf{U} \Phi_2^*$$

$$\mathbf{U} = \Phi_1 \mathbf{V} \Phi_2^T$$

$\Rightarrow$  Economizo operações

## Propriedades:

1. Descorrelação:  $\Phi^{*T} \mathbf{R} \Phi = \mathbf{\Lambda} \Rightarrow r_y(i, j) = \lambda_i \delta(i - j)$

OBS:  $\Phi$  não é única com relação a esta propriedade

Seja

$$\mathbf{R} = \begin{bmatrix} 1 & \rho & \rho^2 & \dots & \rho^{N-1} \\ \rho & 1 & \rho & \dots & \rho^{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho^{N-1} & \rho^{N-2} & \rho^{N-3} & \dots & 1 \end{bmatrix}$$

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ -\rho & 1 & 0 & \dots & 0 \\ 0 & -\rho & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & 0 & \dots & \dots & -\rho \end{bmatrix}$$



Se  $\mathbf{v} = \mathbf{L}^T \mathbf{u}$

$$\mathbf{L}^T \mathbf{R} \mathbf{L} = \begin{bmatrix} 1 - \rho^2 & 0 & \cdots & 0 \\ 0 & 1 - \rho^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 - \rho^2 \end{bmatrix}, \quad \text{é diagonal}$$

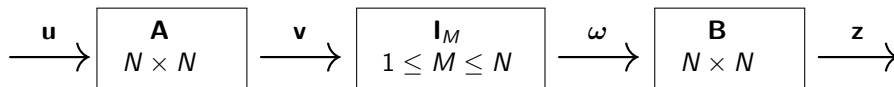
Entretanto,  $\mathbf{L}$  não é unitária.

## 2. Erro de restrição de base

$$\mathbf{v} = \mathbf{A}\mathbf{u}, \quad \omega \text{ tal que } \omega(k) = \begin{cases} v(k), & 0 \leq k \leq M-1 \\ 0, & k \geq M \end{cases}$$

$$\mathbf{z} = \mathbf{B}\boldsymbol{\omega} \Rightarrow J_M = \frac{1}{N} E \left[ \sum_{n=0}^{N-1} |u(n) - z(n)|^2 \right] = \frac{1}{N} \text{Tr} \left\{ (\mathbf{u} - \mathbf{z})(\mathbf{u} - \mathbf{z})^{*T} \right\}$$

Problema:  $\mathbf{A}$  e  $\mathbf{B}$  tal que  $J_M$  é mínimo



$$\mathbf{v} = \mathbf{A}\mathbf{u}, \quad \boldsymbol{\omega} = \mathbf{I}_M \mathbf{v}, \quad \mathbf{z} = \mathbf{B}\boldsymbol{\omega}, \quad \mathbf{I}_M = \underbrace{\begin{bmatrix} \mathbf{I}_{M \times M} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}}_{N \times N}$$

Teorema:  $\mathbf{v} = \mathbf{A}\mathbf{u}$ ,  $\boldsymbol{\omega} = \mathbf{I}_M \mathbf{v}$ ,  $\mathbf{z} = \mathbf{B}\boldsymbol{\omega}$

$$\mathbf{B} = \mathbf{A}^{-1} = \mathbf{A}^{*T} \quad \text{e} \quad \mathbf{A} \text{ é a KLT}$$

Prova:

$$\omega = \mathbf{I}_M \mathbf{A} \mathbf{u}, \quad \mathbf{z} = (\mathbf{B} \mathbf{I}_M \mathbf{A}) \mathbf{u} \Rightarrow \mathbf{u} - \mathbf{z} = (\mathbf{I} - \mathbf{B} \mathbf{I}_M \mathbf{A}) \mathbf{u}$$

$$\begin{aligned} \Rightarrow J_M &= \frac{1}{N} \text{Tr} \left\{ (\mathbf{u} - \mathbf{z})(\mathbf{u} - \mathbf{z})^{*T} \right\} = \frac{1}{N} \text{Tr} \left\{ (\mathbf{I} - \mathbf{B} \mathbf{I}_M \mathbf{A}) \mathbf{u} \mathbf{u}^{*T} (\mathbf{I} - \mathbf{B} \mathbf{I}_M \mathbf{A})^{*T} \right\} \\ &= \frac{1}{N} \text{Tr} \left\{ (\mathbf{I} - \mathbf{B} \mathbf{I}_M \mathbf{A}) \mathbf{R} (\mathbf{I} - \mathbf{B} \mathbf{I}_M \mathbf{A})^{*T} \right\} \end{aligned}$$

$$\frac{\partial J_M}{\partial a_{m,n}} = 0 \quad \mathbf{D} \mathbf{A} \left[ (\mathbf{I} - \mathbf{B} \mathbf{I}_M \mathbf{A}) \mathbf{R} (\mathbf{I} - \mathbf{B} \mathbf{I}_M \mathbf{A})^{*T} \right] = 0$$

$$\mathbf{D} \mathbf{A} [\mathbf{C} \mathbf{A} \mathbf{B}] = (\mathbf{B} \mathbf{C})^T \quad \mathbf{D} \mathbf{A} [\mathbf{A} \mathbf{B} \mathbf{A}]^T = \mathbf{A} \mathbf{B}^T + \mathbf{A} \mathbf{B}$$

$$\mathbf{D} \mathbf{A} (\mathbf{A}^{-1} \mathbf{B} \mathbf{A} \mathbf{C}) = -(\mathbf{A}^{-1} \mathbf{B} \mathbf{A} \mathbf{C} \mathbf{A}^{-1})^T + (\mathbf{C} \mathbf{A}^{-1} \mathbf{B})^T$$

$$\Rightarrow \mathbf{I}_M \mathbf{B}^T (\mathbf{I} - \mathbf{B} \mathbf{I}_M \mathbf{A})^* \mathbf{R} = 0, \quad \mathbf{I}_M \mathbf{B}^T = \mathbf{I}_M \mathbf{B}^T \mathbf{B}^* \mathbf{I}_M \mathbf{A}^*$$

$$\Rightarrow J_M = \frac{1}{N} \text{Tr} \{ (\mathbf{I} - \mathbf{B} \mathbf{I}_M \mathbf{A}) \mathbf{R} \}$$

$$M = N \Rightarrow J_{M_{in}} = 0 \Rightarrow \mathbf{B} \mathbf{A} = \mathbf{I} \Rightarrow \mathbf{B} = \mathbf{A}^{-1}$$

$$\mathbf{I}_M \mathbf{B}^T = \mathbf{I}_M \mathbf{B}^T \mathbf{B}^* \mathbf{I}_M \mathbf{B}^{-1*}, \quad \mathbf{I}_M \mathbf{B}^T \mathbf{B}^* = \mathbf{I}_M \mathbf{B}^T \mathbf{B}^* \mathbf{I}_M, \quad 1 \leq M \leq N$$

$$\text{Se } \mathbf{B}^T \mathbf{B}^* = \underbrace{\begin{bmatrix} \underbrace{\mathbf{C}_1}_{M \times N} & \mathbf{C}_2 \\ \mathbf{C}_3 & \mathbf{C}_4 \end{bmatrix}}_{N \times N} \Rightarrow \mathbf{I}_M \mathbf{B}^T \mathbf{B}^* = \begin{bmatrix} \mathbf{I}_M & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{C}_1 & \mathbf{C}_2 \\ \mathbf{C}_3 & \mathbf{C}_4 \end{bmatrix} = \begin{bmatrix} \mathbf{C}_1 & \mathbf{C}_2 \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

$$\mathbf{I}_M \mathbf{B}^T \mathbf{B}^* \mathbf{I}_M = \begin{bmatrix} \mathbf{C}_1 & \mathbf{C}_2 \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{I}_M & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{C}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

$$\Rightarrow \mathbf{C}_2 = 0, \quad \forall n \Rightarrow \boxed{\mathbf{B}^T \mathbf{B}^* \text{ é diagonal}}$$

Como  $\mathbf{B} = \mathbf{A}^{-1}$ ,  $\mathbf{B} \rightarrow \mathbf{DB}$ ,  $\mathbf{A} \rightarrow \mathbf{AD}^{-1}$

$\Rightarrow \mathbf{I}_M \mathbf{B}^{*T} = \mathbf{I}_M \mathbf{B}^{*T} \mathbf{B} \mathbf{I}_M \mathbf{A}$  continua valendo, pode-se supor que  $\mathbf{B}^{*T} \mathbf{B} = \mathbf{I}$  (unitária)

$\Rightarrow \mathbf{B} = \mathbf{A}^{*T}$

$$\Rightarrow J_M = \frac{1}{N} \text{Tr} \left\{ (\mathbf{I} - \mathbf{A}^{*T} \mathbf{I}_M \mathbf{A}) \mathbf{R} \right\} = \frac{1}{N} \text{Tr} \left\{ (\mathbf{R} - \mathbf{I}_M \mathbf{A} \mathbf{R} \mathbf{A}^{*T}) \right\}$$

$\Rightarrow J_M$  é mínimo se  $\bar{J}_M = \text{Tr}\{\mathbf{I}_M \mathbf{A} \mathbf{R} \mathbf{A}^{*T}\}$  é máximo.

Seja

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_0^T \\ \mathbf{a}_1^T \\ \vdots \\ \mathbf{a}_{N-1}^T \end{bmatrix}$$

$$\Rightarrow Tr\{\mathbf{I}_M \mathbf{A} \mathbf{R} \mathbf{A}^{*T}\} = Tr\left\{ \mathbf{I}_M \begin{bmatrix} \mathbf{a}_0^T \\ \mathbf{a}_1^T \\ \vdots \\ \mathbf{a}_{N-1}^T \end{bmatrix} \mathbf{R} \begin{bmatrix} \mathbf{a}_0^* & \mathbf{a}_1^* & \cdots & \mathbf{a}_{N-1}^* \end{bmatrix} \right\}$$

$$= Tr\left\{ \mathbf{I}_M \begin{bmatrix} \mathbf{a}_0^T \\ \mathbf{a}_1^T \\ \vdots \\ \mathbf{a}_{N-1}^T \end{bmatrix} \begin{bmatrix} \mathbf{R} \mathbf{a}_0^* & \mathbf{R} \mathbf{a}_1^* & \cdots & \mathbf{R} \mathbf{a}_{N-1}^* \end{bmatrix} \right\}$$

$$= Tr\left\{ \mathbf{I}_M \begin{bmatrix} \mathbf{a}_0^T \mathbf{R} \mathbf{a}_0^* & \mathbf{a}_0^T \mathbf{R} \mathbf{a}_1^* & \cdots \\ \vdots & \mathbf{a}_1^T \mathbf{R} \mathbf{a}_1^* & \cdots \\ \vdots & \ddots & \cdots \\ \mathbf{a}_{N-1}^T \mathbf{R} \mathbf{a}_0^* & \cdots & \mathbf{a}_{N-1}^T \mathbf{R} \mathbf{a}_{N-1}^* \end{bmatrix} \right\}$$

$$\bar{J}_M = \sum_{k=0}^{M-1} \mathbf{a}_k^T \mathbf{R} \mathbf{a}_k^*$$

Como  $\mathbf{A}$  é unitária,  $\mathbf{a}_k^T \mathbf{a}_k^* = 1$

$$\Rightarrow \bar{J}_M = \sum_{k=0}^{M-1} \mathbf{a}_k^T \mathbf{R} \mathbf{a}_k^* + \sum_{k=0}^{M-1} \lambda_k (1 - \mathbf{a}_k^T \mathbf{R} \mathbf{a}_k^*) = 0 \quad (\text{Multiplicadores de Lagrange})$$

$$\frac{\partial J_M}{\partial a_{i,j}} = 0 \Rightarrow \mathbf{R} \mathbf{a}_j^* = \lambda_j \mathbf{a}_j \Rightarrow \mathbf{a}_j^* = \text{autovetores normalizados (condição necessária)}$$

$$\Rightarrow \bar{J}_M = \sum_{k=m}^{N-1} \lambda_k \Rightarrow \boxed{J_M \text{ é mínimo quando } \mathbf{A} \text{ é a KLT e os vetores base estão ordenados pelos autovalores (decrecentes)}}$$

### 3. Ordenação de variâncias

$$\sigma_k^2 = E[|v(k)|^2], \quad \sigma_0^2 \geq \sigma_1^2 \geq \cdots \geq \sigma_{N-1}^2$$

$$S_M(\mathbf{A}) = \sum_{k=0}^{M-1} \sigma_k^2 \quad (\text{soma das } M \text{ maiores variâncias})$$

$$S_M(\Phi^{*\text{T}}) \geq S_M(\mathbf{A}) \Rightarrow \text{Esta soma é máxima para a KLT.}$$

Prova:

$$S_M(\mathbf{A}) = \sum_{k=0}^{M-1} (\mathbf{A}\mathbf{R}\mathbf{A}^{*\text{T}})_{k,k} = \text{Tr}\{\mathbf{I}_M \mathbf{A}^{*\text{T}} \mathbf{R} \mathbf{A}\} = \bar{J}_M \Rightarrow \begin{array}{l} \text{é máximo quando} \\ \mathbf{A} \text{ é a KLT.} \end{array}$$

$$\text{Corolário: } \sum_{k=0}^{M-1} \lambda_k \geq \sum_{k=0}^{M-1} \sigma_k^2, \quad 1 \leq M \leq N$$



4. Seja  $M$  tal que  $\sum_{k=0}^{M-1} \sigma_k^2 < \text{limiar} \leq \sum_{k=0}^M \sigma_k^2 \Rightarrow$  a KLT minimiza  $M$ .

## 5. Rate-Distortion Function



Supõe-se  $\mathbf{u}, \mathbf{v}, \hat{\mathbf{u}}, \hat{\mathbf{v}}$  gaussianas.

$$D = \frac{1}{N} E[(\mathbf{u} - \hat{\mathbf{u}})^{*T} (\mathbf{u} - \hat{\mathbf{u}})] \quad \mathbf{u} = \mathbf{A}^{*T} \mathbf{v}, \quad \hat{\mathbf{u}} = \mathbf{A}^{*T} \hat{\mathbf{v}}$$

$$D = \frac{1}{N} E[(\mathbf{v} - \hat{\mathbf{v}})^{*T} \mathbf{A} \mathbf{A}^{*T} (\mathbf{v} - \hat{\mathbf{v}})] = \frac{1}{N} E[(\mathbf{v} - \hat{\mathbf{v}})^{*T} (\mathbf{v} - \hat{\mathbf{v}})]$$

$\Rightarrow D$  é invariante com relação a todas as transformações unitárias.

Para uma distorção fixa  $\mathbf{D}$ , a taxa  $R$  é dada por:

$$R = \frac{1}{N} \sum_{k=0}^{N-1} \max \left\{ 0, \frac{1}{2} \log_2 \frac{\sigma_k^2}{\theta} \right\}$$

onde  $\theta$  é determinado resolvendo:

$$D = \frac{1}{N} \sum_{k=0}^{N-1} \min \{ \theta, \sigma_k^2 \} \quad \sigma_k^2 = E[|v(k)|^2] = [\mathbf{A} \mathbf{R}_u \mathbf{A}^{*T}]_{k,k}$$

$R$  depende de  $\mathbf{A} \Rightarrow$  KLT é a transformada que minimiza  $R \Rightarrow R(\Phi^{*T}) < R(\mathbf{A})$

**Exemplo:**  $\mathbf{u}$  tal que

$$\mathbf{R}_u = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}, \quad |\rho| < 1 \Rightarrow KLT : \quad \Phi^{*T} = \Phi = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\Rightarrow \mathbf{v} = \Phi \mathbf{u} \Rightarrow \mathbf{R}_v = \begin{bmatrix} 1+\rho & 0 \\ 0 & 1-\rho \end{bmatrix} \Rightarrow \sigma_0^2 = 1+\rho, \quad \sigma_1^2 = 1-\rho$$

$$R(\Phi) = \frac{1}{2} \left( \max \left\{ 0, \frac{1}{2} \log \frac{1+\rho}{\theta} \right\} + \max \left\{ 0, \frac{1}{2} \log \frac{1-\rho}{\theta} \right\} \right)$$

$$D = \frac{1}{2} [\min\{\theta, 1+\rho\} + \min\{\theta, 1-\rho\}]$$

Se  $\mathbf{A} = \mathbf{I}$ , ao invés da KLT  $\Rightarrow \sigma_0^2 = \sigma_1^2 = 1$ :

$$R(\mathbf{I}) = \frac{1}{2} \left( \max \left\{ 0, \frac{1}{2} \log \frac{1}{\theta} \right\} + \max \left\{ 0, \frac{1}{2} \log \frac{1}{\theta} \right\} \right) \Rightarrow 0 < \theta < 1$$

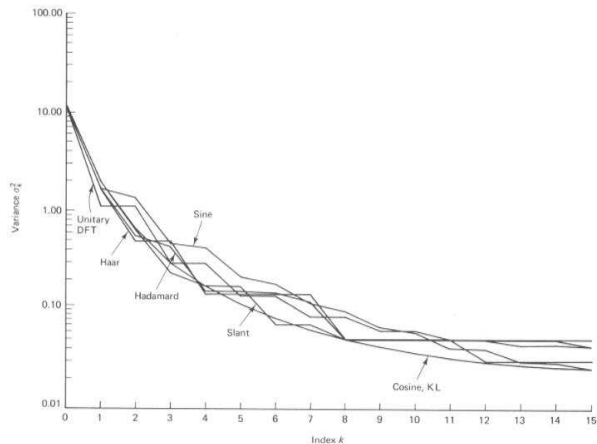
$$\Rightarrow R(\mathbf{I}) = -\frac{1}{2} \log \theta$$

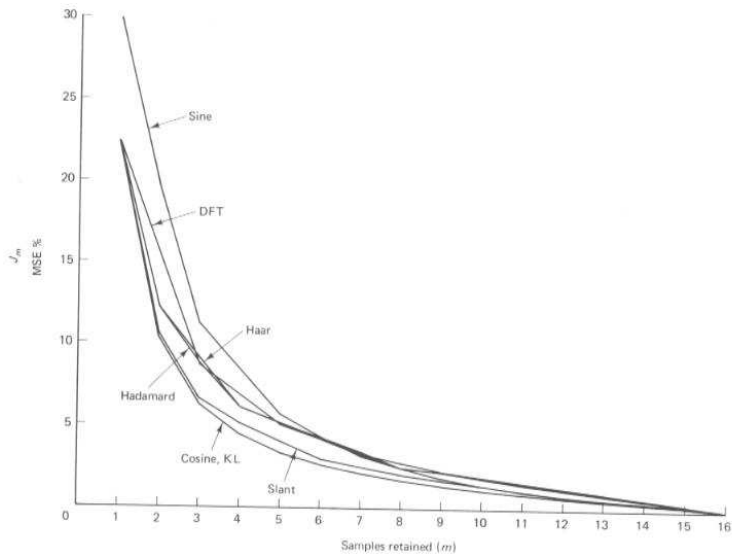
$$\Rightarrow R(\Phi) < R(\mathbf{I})$$

$\Rightarrow$  A KLT requer o menor número de bits

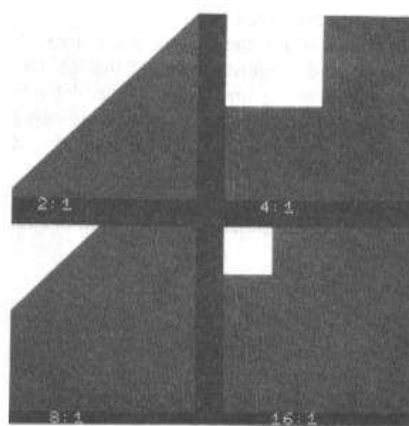
## Exemplo:

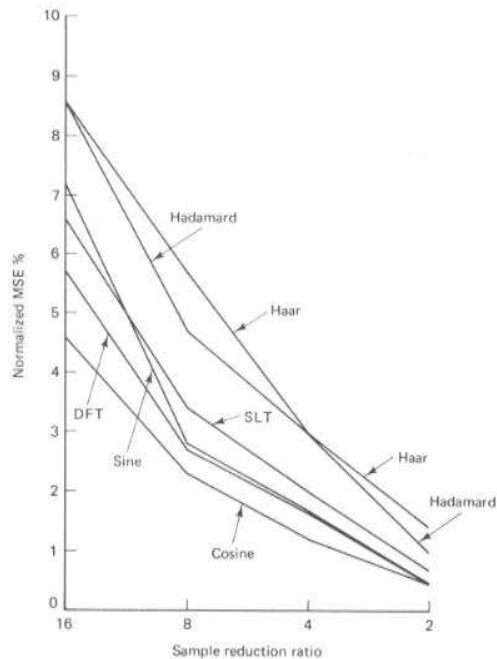
$$\rho = 0.95, \quad N = 16, \quad J_M = \frac{\sum_{k=M}^{N-1} \sigma_k^2}{\sum_{k=0}^{N-1} \sigma_k^2} \quad (\text{erro de restrição de base normalizado})$$





⇒ Imagens: Tenho que definir, ao invés de  $J_M$ , uma região no plano em que vou pegar os coeficientes (*zonal sampling*)





# Uma família de transformadas unitárias senoidais:

Considere a família paramétrica de matrizes:

$$J(k_1, k_2, k_3) = \begin{bmatrix} 1 - k_1\alpha & -\alpha & \cdots & k_3\alpha \\ -\alpha & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ k_3\alpha & \cdots & -\alpha & 1 - k_2\alpha \end{bmatrix}$$

$$\alpha = \frac{\rho}{1 + \rho^2}, \quad \beta^2 = \frac{1 - \rho^2}{1 + \rho^2}, \quad J(\rho, \rho, 0) = \beta^2 \mathbf{R}^{-1}, \quad (\text{mesmos autovetores que } \mathbf{R})$$

$$\mathbf{R} = \begin{bmatrix} 1 & \rho & \cdots & \rho^{N-1} \\ \rho & 1 & \cdots & \rho^{N-2} \\ \vdots & \vdots & \ddots & \vdots \\ \rho^{N-1} & \rho^{N-2} & \cdots & 1 \end{bmatrix}$$

$\Rightarrow$  seus autovetores são a KLT de um processo AR(1)

Em geral, para  $\alpha, k_1, k_2, k_3$  tais que  $J$  é definida positiva (autovalores  $> 0$ ), os autovetores de  $J(k_1, k_2, k_3)$  geram bases ortonormais de transformadas unitárias senoidais;



- $k_1 = k_2 = 1, k_3 = 0$ : Autovetores são as funções base da DCT;
- $k_1 = k_2 = k_3 = 0$ : Autovetores são as funções base da DST;
- $k_1 = k_2 = 1, k_3 = -1$ : Autovetores são as funções base da DFT;
- $k_1 = k_3 = 0, k_2 = 1$ ,  $\Phi_M(k) = \frac{2}{\sqrt{2N+1}} \sin \frac{(k+1)(2M+1)\pi}{2N+1}$ , seno ímpar - I;
- $k_1 = 1, k_2 = k_3 = 0$ ,  $\Phi_M(k) = \frac{2}{\sqrt{2N+1}} \cos \frac{(2k+1)(2M+1)\pi}{2(2N+1)}$ , cosseno ímpar - I;

O desempenho de  $J(k_1, k_2, k_3)$  para aproximar a KLT pode ser determinado, por exemplo:

- Para processos AR(1): calculando  $\| J(k_1, k_2, k_3) - J(\rho, \rho, 0) \|^2$ ;
- Para  $\mathbf{R}$  qualquer: calculando  $\| J(k_1, k_2, k_3)\mathbf{R} - \mathbf{R}J(\rho, \rho, 0) \|^2$  (se comutam, possuem os mesmos autovetores).

⇒ A melhor transformada senoidal pode ser encontrada minimizando as distâncias acima;

# DFT (1D)

$$v(k) = \sum_{n=0}^{N-1} u(n) W_N^{kn} \quad u(n) = \frac{1}{N} \sum_{k=0}^{N-1} v(k) W_N^{-kn}$$

$$\Rightarrow W_N = e^{-j\frac{2\pi}{N}}$$

DFT Unitária:

$$v(k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} u(n) W_N^{kn} \quad u(n) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} v(k) W_N^{-kn}$$

$$\text{Se } \mathbf{F} = \left\{ \frac{1}{\sqrt{N}} W_N^{kn} \right\} \quad 0 \leq k, m \leq N-1, \Rightarrow \boxed{\mathbf{F}^{-1} = \mathbf{F}^*}$$

## Propriedades:

$$\Rightarrow \mathbf{F}^{-1} = \mathbf{F}^*$$

$$\Rightarrow v(k) = v(k + N) \quad u(n) = u(n + N)$$

$$\Rightarrow \text{Se } \bar{u}(n) = \begin{cases} u(n), & 0 \leq n \leq N-1 \\ 0, & n.d.p \end{cases}$$

$$\bar{u}(\omega) = \sum_{n=-\infty}^{\infty} u(n)e^{-j\omega n} = \sum_{n=-\infty}^{N-1} u(n)e^{-j\omega n}$$

$$\text{comparando com } v(k) = \sum_{n=0}^{N-1} u(n)W_N^{kn} = \sum_{n=0}^{N-1} u(n)e^{-j\frac{2\pi}{N}kn}:$$

$$v(k) = \bar{u}\left(\frac{2\pi k}{N}\right) \quad \boxed{\text{ou}} \quad v(k) = \frac{1}{\sqrt{N}}\bar{u}\left(\frac{2\pi k}{N}\right), \quad \text{para a DFT unitária.}$$

⇒ A DFT possui uma transformada rápida que pode ser calculada em  $O(N \log_2 N)$  operações.

$$\mathbf{F} = \{ W_N^{kn} \} = \begin{bmatrix} W_N^{0 \cdot 0} & W_N^{0 \cdot 1} & \dots & W_N^{0 \cdot (N-1)} \\ W_N^{1 \cdot 0} & W_N^{1 \cdot 1} & \dots & W_N^{1 \cdot (N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ W_N^{(N-1) \cdot 0} & W_N^{(N-1) \cdot 1} & \dots & W_N^{(N-1) \cdot (N-1)} \end{bmatrix}$$

$\Rightarrow$  A DFT de uma sequência real é simétrica conjugada em torno de  $\frac{N}{2}$ .

$$v^*(N-k) = \sum_{n=0}^{N-1} u^*(n) W_N^{-(N-k)n} = \sum_{n=0}^{N-1} u(n) W_N^{kn} = v(k)$$

$$\Rightarrow v\left(\frac{N}{2} - k\right) = v^*\left(\frac{N}{2} + k\right)$$

Como  $v(-k) = v(N-k) \Rightarrow v(k) = v^*(-k)$

$\Rightarrow v(0)$  e  $v(\frac{N}{2})$  são reais:

$v(0)$ ,  $\{\Re\{v(k)\}, k = 1, \dots, \frac{N}{2} - 1\}$ ,  $\{\Im\{v(k)\}, k = 1, \dots, \frac{N}{2} - 1\}$ ,  $v(\frac{N}{2})$  definem unicamente a DFT de uma sequência real.

$\Rightarrow$  Sequência de  $N$  coeficientes  $\Leftrightarrow$  DFT de  $N$  coeficientes

Considere uma matriz circulante  $\mathbf{H}$ :

$$[\mathbf{H}]_{m,n} = h[(m-n) \bmod N]$$

e

$$[\Phi_k]_n = \left\{ \frac{1}{\sqrt{N}} W_N^{-kn} \right\} \Rightarrow \Phi_k = 1^a \text{ coluna de } \mathbf{F}^{*T} = \mathbf{F}^*$$

Defina:

$$\begin{aligned} [\mathbf{H}\Phi_k]_m &= \sum_{\theta=0}^{N-1} [\mathbf{H}]_{m,\theta} [\Phi_k]_{\theta} = \sum_{\theta=0}^{N-1} h[\underbrace{(m-\theta)}_r \bmod N] \cdot \frac{1}{\sqrt{N}} W_N^{-k\theta} \\ &= \frac{1}{\sqrt{N}} \sum_{r=m}^{m+1-N} h[r \bmod N] W_N^{-k(m-r)} \\ &= W_N^{-km} \frac{1}{\sqrt{N}} \left[ \sum_{r=0}^m h(r) W_N^{kr} + \sum_{r=m+1-N}^{-1} h(r \bmod N) W_N^{kr} \right] \end{aligned}$$

$$[\mathbf{H}\Phi_k]_m = \underbrace{\frac{1}{\sqrt{N}} W_N^{-km}}_{[\Phi_k]_m} \underbrace{\left[ \sum_{r=0}^m h(r) W_N^{kr} + \sum_{s=m+1}^{N-1} h(s) W_N^{ks} \underbrace{W_N^{kN}}_{=1} \right]}_{\text{k-ésimo coef. da transformada de Fourier de } h(m), 1^{\text{a}} \text{ coluna de } \mathbf{H}}$$

$$= \lambda_k [\Phi_k]_m \Rightarrow \boxed{\mathbf{H}\Phi_k = \lambda_k \Phi_k}, \quad \lambda_k = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} h(n) W_N^{kn}$$

$\Rightarrow$  As funções base da DFT unitária são **autovetores ortonormais** de uma matriz circulante. Os autovalores são dados pela DFT da primeira coluna.

$$[\mathbf{H}\Phi_0 \quad \mathbf{H}\Phi_1 \quad \cdots \quad \mathbf{H}\Phi_{N-1}] = [\lambda_0\Phi_0 \quad \lambda_1\Phi_1 \quad \cdots \quad \lambda_{N-1}\Phi_{N-1}]$$

$$= [\Phi_0 \quad \Phi_1 \quad \cdots \quad \Phi_{N-1}] \begin{bmatrix} \lambda_0 & 0 & \cdots & 0 \\ 0 & \lambda_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{N-1} \end{bmatrix}$$

Como  $\mathbf{F}^* = [\Phi_0 \quad \Phi_1 \quad \cdots \quad \Phi_{N-1}]$ :

$$\Rightarrow \mathbf{H}\mathbf{F}^* = \mathbf{F}^*\mathbf{\Lambda} \Rightarrow \boxed{\mathbf{F}\mathbf{H}\mathbf{F}^* = \mathbf{\Lambda}}$$

$\Rightarrow \mathbf{F}$  diagonaliza uma matriz circulante



Se

$$y(n) = \sum_{m=0}^{N-1} h[(n-m) \bmod N] x(m) \quad (\text{convolução circular}) \Rightarrow \mathbf{y} = \mathbf{H}\mathbf{x}$$

$$\mathbf{F}\mathbf{y} = \mathbf{F}\mathbf{H}\mathbf{x} = \mathbf{F}\mathbf{H}\mathbf{F}^* \mathbf{F}\mathbf{x} = \mathbf{\Lambda}\mathbf{F}\mathbf{x} \Rightarrow \hat{\mathbf{y}} = \mathbf{\Lambda}\hat{\mathbf{x}}$$

$$\begin{bmatrix} \hat{y}(0) \\ \hat{y}(1) \\ \vdots \\ \hat{y}(N-1) \end{bmatrix} = \begin{bmatrix} \lambda_0 & 0 & \cdots & 0 \\ 0 & \lambda_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{N-1} \end{bmatrix} \begin{bmatrix} \hat{x}(0) \\ \hat{x}(1) \\ \vdots \\ \hat{x}(N-1) \end{bmatrix} \Rightarrow$$

$$\underbrace{\hat{y}(k) = \lambda_k \hat{x}(k) = \left[ \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} h(n) W_N^{kn} \right] \hat{x}(k)}_{\text{teorema da convolução}}$$

# DFT Bi-Dimensional

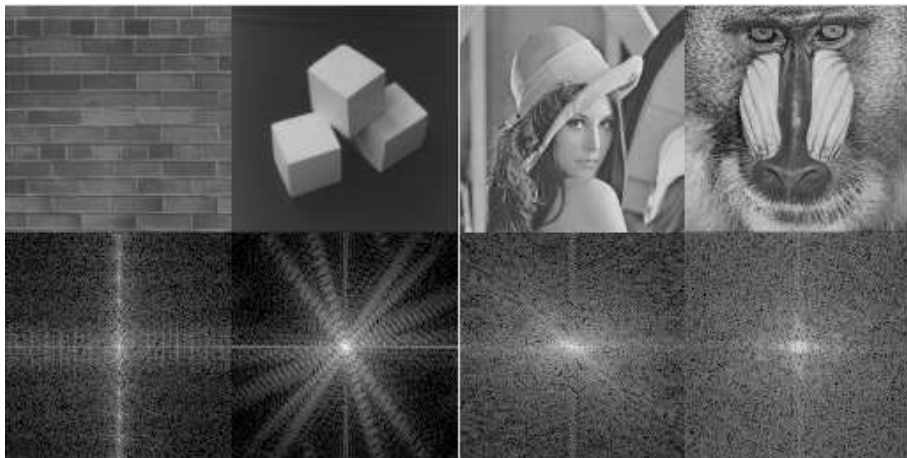
DFT unitária:

$$v(k, l) = \frac{1}{N} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} u(m, n) W_N^{km} W_N^{ln} \quad 0 \leq k, l \leq N-1$$

$$u(m, n) = \frac{1}{N} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} v(k, l) W_N^{-km} W_N^{-ln} \quad 0 \leq m, n \leq N-1$$

$$\mathbf{V} = \mathbf{F} \mathbf{U} \mathbf{F} \quad \mathbf{U} = \mathbf{F}^* \mathbf{V} \mathbf{F}^*$$

$$\mathcal{V} = \mathcal{F} \mathcal{U}, \quad \mathcal{U} = \mathcal{F}^* \mathcal{V}, \quad \mathcal{F} = \mathbf{F} \otimes \mathbf{F}$$



## Propriedades:

- Simetria:  $\mathcal{F}^T = \mathcal{F}$
- $\mathcal{F}^{-1} = \mathcal{F}^* = \mathbf{F}^* \otimes \mathbf{F}^{**}$
- Periodicidade:  $v(k + N, l + N) = v(k, l), \quad \forall k, l$   
 $u(m + N, n + N) = u(m, n), \quad \forall m, n$
- $$\tilde{u}(m, n) = \begin{cases} u(m, n), & 0 \leq m, n \leq N - 1 \\ 0, & n.d.p. \end{cases}$$

Se  $\tilde{U}(\omega_1, \omega_2)$  é a transformada de Fourier de  $\tilde{u}(m, n)$ , então:

$$\tilde{U}\left(\frac{2\pi k}{N}, \frac{2\pi l}{N}\right) = DFT\{u(m, n)\} = v(k, l)$$

- DFT 2D é separável: equivale a  $2N$  1D DFTs (complexidade  $O(N^2 \log_2 N)$ )

- $v(k, l) = v^*(N - k, N - l), \quad 0 \leq k, l \leq N - 1$  (conjugada simétrica)
- $$\underbrace{\mathbf{A}_{kl}^*}_{\text{imagens base}} = \underbrace{\boldsymbol{\Phi}_k}_{\text{colunas de } \mathbb{F}^*} \times \underbrace{\boldsymbol{\Phi}_l^T}_{\text{linhas de } \mathbb{F}^*} = \frac{1}{N} W_N^{-(km+ln)}, \quad 0 \leq m, n \leq N - 1$$
- $$u_2(m, n) = \sum_{m'=0}^{N-1} \sum_{n'=0}^{N-1} h[(m-m') \bmod N, (n-n') \bmod N] u_1(m', n')$$

$$DFT[u_2(m, n)] = DFT[h(m, n)] DFT[u_1(m, n)]$$

# Transformada Cosseno

$$v(k) = \alpha(k) \sum_{n=0}^{N-1} u(n) \cos \left[ \frac{\pi(2n+1)k}{2N} \right], \quad 0 \leq k \leq N-1$$

$$u(n) = \sum_{k=0}^{N-1} \alpha(k) v(k) \cos \left[ \frac{\pi(2n+1)k}{2N} \right], \quad 0 \leq n \leq N-1$$

$$\Rightarrow \begin{cases} \alpha(0) = \sqrt{\frac{1}{N}}, & k = 0 \\ \alpha(k) = \sqrt{\frac{2}{N}}, & 1 \leq k \leq N-1 \end{cases}$$

Seja  $\mathbf{C}$  tal que  $[\mathbf{C}]_{k,n} = \begin{cases} \sqrt{\frac{1}{N}}, & k = 0; \quad 0 \leq n \leq N-1 \\ \sqrt{\frac{2}{N}} \cos \left[ \frac{\pi(2n+1)k}{2N} \right], & 1 \leq k \leq N-1; \quad 0 \leq n \leq N-1 \end{cases}$

• 1D:  $\mathbf{v} = \mathbf{C}\mathbf{u}$

• 2D:  $\mathbf{V} = \mathbf{C}\mathbf{U}\mathbf{C}^T \quad \mathbf{U} = \mathbf{C}^T\mathbf{V}\mathbf{C}, \quad \mathbf{C} = \mathbf{C}^*$

## Propriedades:

1.  $\mathbf{C} = \mathbf{C}^* \Rightarrow \mathbf{C}^{-1} = \mathbf{C}^T$
2. A DCT não é a parte real da DFT unitária.
3. Transformada rápida:

$$\begin{cases} \tilde{u}(n) = u(2n), & 0 \leq n \leq \frac{N}{2} - 1 \\ \tilde{u}(N - n - 1) = u(2n - 1) \end{cases}$$

$$v(k) = \alpha(k) \sum_{n=0}^{N-1} \tilde{u}(n) \cos \left[ \frac{\pi(4n+1)k}{2N} \right], \quad 0 \leq k \leq N-1$$

$$\text{Mas } \cos \left[ \frac{\pi(4n+1)k}{2N} \right] = \Re \left[ e^{-j \frac{\pi(4n+1)k}{2N}} \right]$$

$$\Rightarrow v(k) = \Re \left[ \alpha(k) e^{-j \frac{\pi k}{2N}} \sum_{n=0}^{N-1} \tilde{u}(n) e^{-j \frac{\pi k}{2N}} \right] = \Re \left[ \alpha(k) W_N^{\frac{k}{2}} DFT[\tilde{u}(n)]_N \right]$$

Inversa:

$$u(2n) = \tilde{u}(2n) = \Re \left[ \sum_{k=0}^{N-1} \left[ \alpha(k) v(k) e^{+j \frac{\pi k}{2N}} \right] e^{+j \frac{2\pi kn}{N}} \right]$$

$$u(2n+1) = \tilde{u}(2(N-1-n)), \quad 0 \leq n \leq \frac{N}{2} - 1$$

Calculo  $u(n)$  com a DFT inversa de  $\alpha(k) v(k) e^{+j \frac{\pi k}{2N}}$

4. A DCT tem propriedades de compactação excelentes para dados bastante correlacionados.
5. As funções base da DCT são autovetores da seguinte família de matrizes tri-diagonais:

$$\mathbf{Q}_C = \begin{bmatrix} 1-\alpha & -\alpha & 0 & 0 & \cdots & 0 & 0 \\ -\alpha & 1 & -\alpha & 0 & \cdots & 0 & 0 \\ 0 & -\alpha & 1 & -\alpha & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -\alpha \\ 0 & 0 & 0 & 0 & \cdots & -\alpha & 1-\alpha \end{bmatrix}$$



6. A DCT aproxima bem a KLT para processos cuja matriz de autocovariância é:

$$\mathbf{R} = \begin{bmatrix} 1 & \rho & \rho^2 & \dots & \rho^{N-1} \\ \rho & 1 & \rho & \dots & \rho^{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho^{N-1} & \rho^{N-2} & \rho^{N-3} & \dots & 1 \end{bmatrix}, \quad \rho \rightarrow 1$$

Por quê?

$$\mathbf{R}^{-1} = \beta^{-2} \begin{bmatrix} 1 - \rho\alpha & -\alpha & 0 & 0 & \dots & 0 & 0 \\ -\alpha & 1 & -\alpha & 0 & \dots & 0 & 0 \\ 0 & -\alpha & 1 & -\alpha & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & -\alpha \\ 0 & 0 & 0 & 0 & \dots & -\alpha & 1 - \rho\alpha \end{bmatrix}$$

$$\beta = \frac{1 - \rho^2}{1 + \rho^2}, \quad \alpha = \frac{\rho}{1 + \rho^2}.$$

Quando  $\rho \rightarrow 1$ ,  $\mathbf{R}^{-1}$  fica da forma de  $\mathbf{Q}_C$

$\Rightarrow$  a DCT fica próxima a KLT (para ter a mesma KLT basta que  $\mathbf{Q}_C \mathbf{R} = \mathbf{R} \mathbf{Q}_C$ )

# Transformada Seno

$$v(k) = \sqrt{\frac{2}{N+1}} \sum_{n=0}^{N-1} u(n) \sin \left[ \frac{\pi(k+1)(n+1)}{N+1} \right], \quad 0 \leq k \leq N-1$$

$$u(n) = \sqrt{\frac{2}{N+1}} \sum_{k=0}^{N-1} v(k) \sin \left[ \frac{\pi(k+1)(n+1)}{N+1} \right], \quad 0 \leq n \leq N-1$$

## Propriedades:

Seja:

$$[\Psi]_{k,n} = \sqrt{\frac{2}{N+1}} \sin \left[ \frac{\pi(k+1)(n+1)}{N+1} \right]$$

$$1. \Psi = \Psi^* = \Psi^T = \Psi^{-1}$$

2. A transformada seno não é a parte imaginária da DFT, mas pode ser computada através da DFT da extensão anti-simétrica do sinal.

3. Existe transformada rápida (DST de  $N$  pontos  $\Rightarrow$  DFT de  $2(N + 1)$  pontos).
4. As funções base são autovetores de:

$$\mathbf{Q}_s = \begin{bmatrix} 1 & -\alpha & 0 & \cdots & 0 \\ -\alpha & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 1 & -\alpha \\ 0 & 0 & \cdots & -\alpha & 1 \end{bmatrix}$$

5. A DST aproxima a KLT para processos  $AR(1)$ :

$$\mathbf{R} = \begin{bmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{N-1} \\ \rho & 1 & \rho & \cdots & \rho^{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho^{N-1} & \rho^{N-2} & \rho^{N-3} & \cdots & 1 \end{bmatrix} \Rightarrow \text{Quando } \rho \in (-0.5, 0.5] \text{ } (\rho \rightarrow 0),$$

a DST possui boas propriedades de compactação de energia.

6. A DST leva a uma KLT rápida de processos de Markov cujos valores de contorno são conhecidos.

# Transformada de Hadamard

$$\mathbf{H}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\mathbf{H}_n = \mathbf{H}_{n-1} \otimes \mathbf{H}_1 = \mathbf{H}_1 \otimes \mathbf{H}_{n-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{H}_{n-1} & \mathbf{H}_{n-1} \\ \mathbf{H}_{n-1} & -\mathbf{H}_{n-1} \end{bmatrix}$$

"sequency"

$$\mathbf{H}_3 = \frac{1}{\sqrt{8}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ \hline 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ \hline 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ \hline 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix}$$

0

7

3

4

1

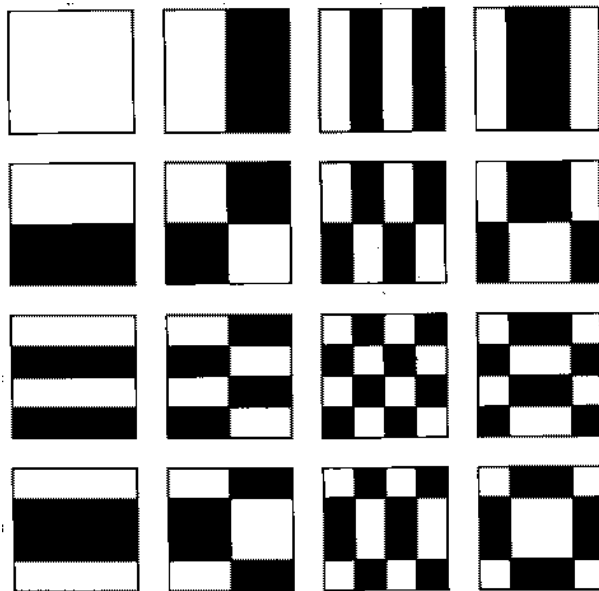
6

2

5

Podem ser geradas amostrando-se as funções de Walsh ("ondas quadradas")

Imagens base:



- Os vetores linha não estão ordenados em ordem de “sequency” (ordem de Hadamard)

$$\mathbf{v} = \mathbf{H}_n \mathbf{u}, \quad \mathbf{u} = \mathbf{H}_n \mathbf{v}, \quad n = \log_2 N$$

$$v(k) = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} u(m) (-1)^{b(k,m)}, \quad 0 \leq k \leq N-1$$

$$u(m) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} v(k) (-1)^{b(k,m)}, \quad 0 \leq m \leq N-1$$

$$\text{onde } b(k, m) = \sum_{i=0}^{N-1} k_i m_i, \quad k_i, m_i = 0, 1$$

$\{k_i\}$  = representação binária de  $k$ :  $k = k_0 + 2k_1 + \dots + 2^{n-1}k_{n-1}$

$\{m_i\}$  = representação binária de  $m$ :  $m = m_0 + 2m_1 + \dots + 2^{n-1}m_{n-1}$

- Em 2D:  $\mathbf{V} = \mathbf{H}_n \mathbf{U} \mathbf{H}_n$ ,  $\mathbf{U} = \mathbf{H}_n \mathbf{V} \mathbf{H}_n$

## Propriedades:

$$1. \mathbf{H}_n = \mathbf{H}_n^* = \mathbf{H}_n^T = \mathbf{H}_n^{-1}$$

2.  $\mathbf{H}$  é uma transformada rápida:  $\mathbf{H}_n = \tilde{\mathbf{H}}^n$  (produto de  $n$  matrizes esparsas)

$$\tilde{\mathbf{H}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \cdots & 0 & 0 & & \\ 0 & 0 & 1 & 1 & 0 & 0 \cdots & 0 & 0 & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & 0 & 0 & 0 & 0 \cdots & 1 & 1 & & \\ 1 & -1 & 0 & 0 & 0 & 0 \cdots & 0 & 0 & & \\ 0 & 0 & 1 & -1 & 0 & 0 \cdots & 0 & 0 & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & 0 & 0 & 0 & 0 \cdots & 1 & -1 & & \end{bmatrix}$$

$$\mathbf{v} = \mathbf{H}_n \mathbf{u} = \tilde{\mathbf{H}}^n \mathbf{u} = \tilde{\mathbf{H}} \cdot \tilde{\mathbf{H}} \cdots \tilde{\mathbf{H}} \mathbf{u}$$

$\Rightarrow \tilde{\mathbf{H}} \mathbf{u}$  pode ser computado em  $n$  adições ou subtrações.

### 3. Ordem de Hadamard = ordem de “bit-reversed” do código de gray da “sequency”:

Ordem por Sequency:  $s_n s_{n-1} \cdots s_1$

Ordem de Hadamard:  $h_n h_{n-1} \cdots h_1$

$$s \rightarrow h \quad \begin{cases} g_k = s_k \oplus s_{k+1}, & k = 1, \dots, n-1 \\ g_n = s_n \\ h_k = g_{n+1-k} \end{cases}$$

$$h \rightarrow s \quad \begin{cases} g_k = h_{n+1-k} \\ s_n = g_n \\ s_k = g_k \oplus s_{k+1}, & k = 1, \dots, n-1 \end{cases}$$

### 4. Compactação de energia:

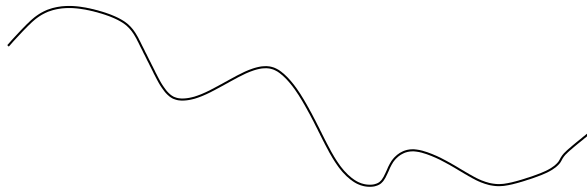
$r(k)$ : % de energia nos primeiros  $\frac{N}{2}$  coeficientes  $= \frac{1}{2} \left( 1 + \frac{r(1)}{r(0)} \right)$

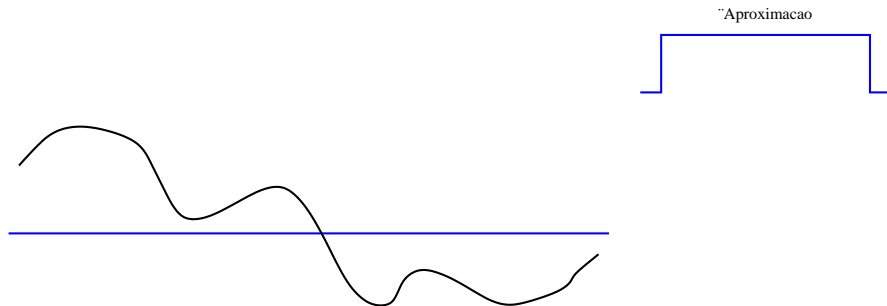
$\rho = 0.95 \Rightarrow 97.5\%$  da energia nos primeiros  $\frac{N}{2}$  coeficientes.

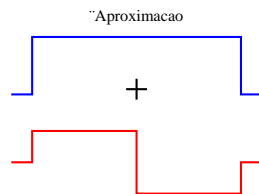
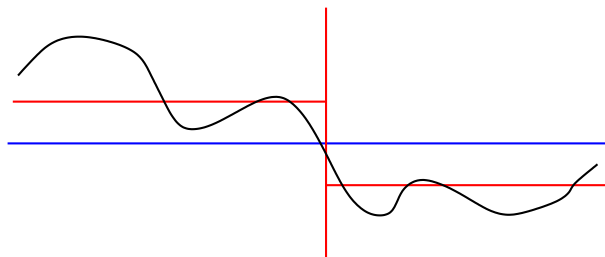


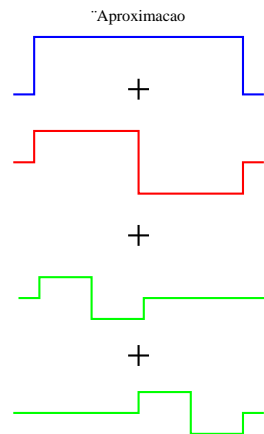
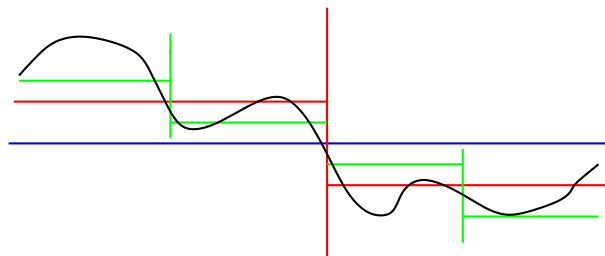
# Transformada de Haar

Seja a seguinte função contínua:

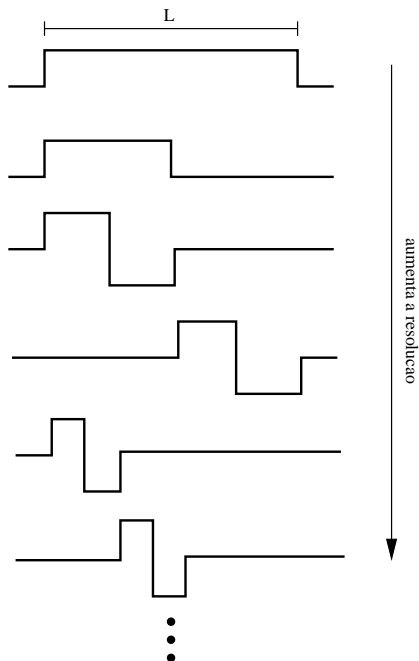




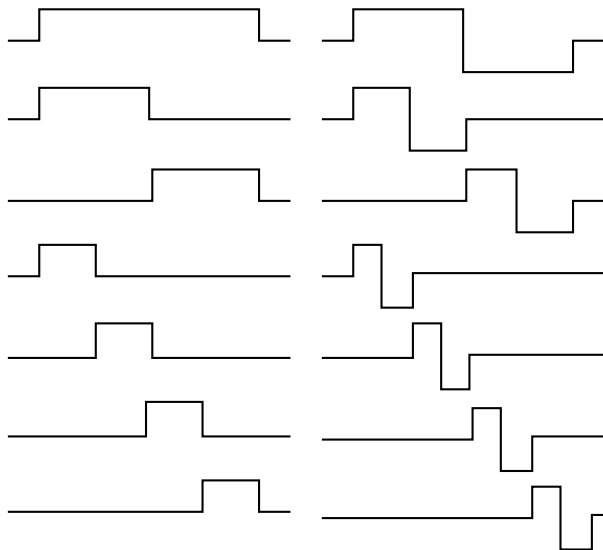




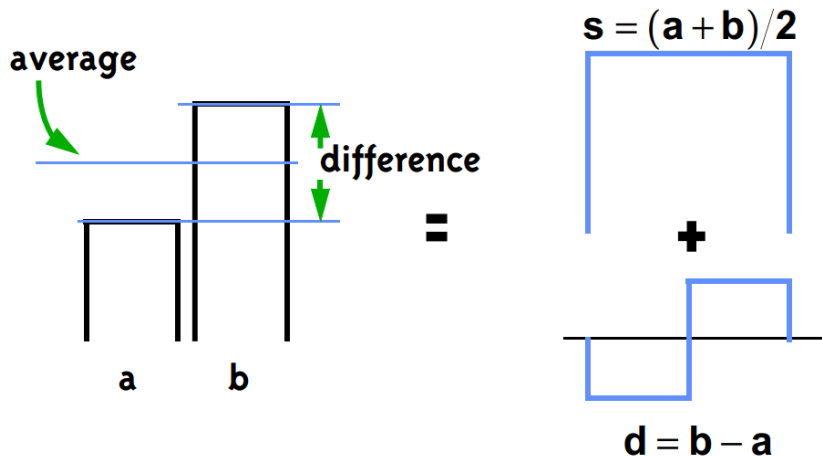
Qualquer função  $f(x) \in L^2(\mathbb{R})$  pode ser representada como uma combinação linear das funções ao lado (desde que o suporte seja menor que  $L$ )

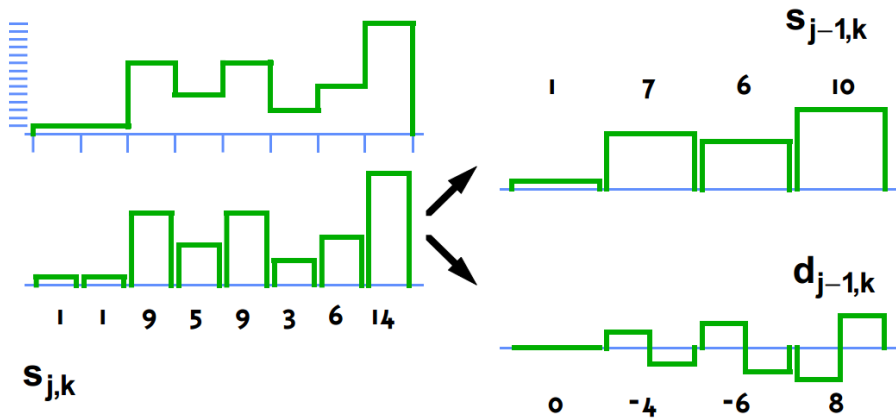


Funções base:



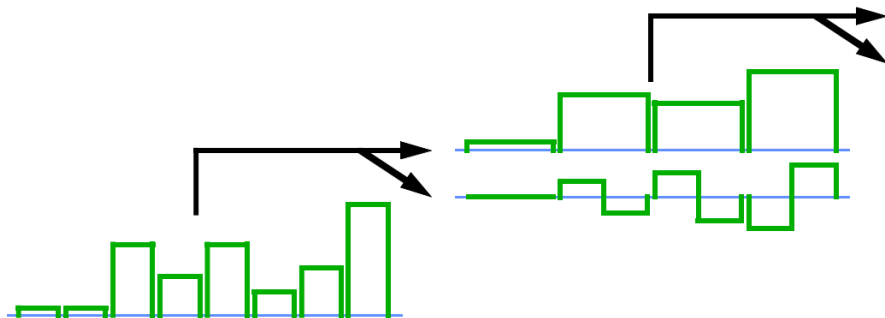
Para seqüências:

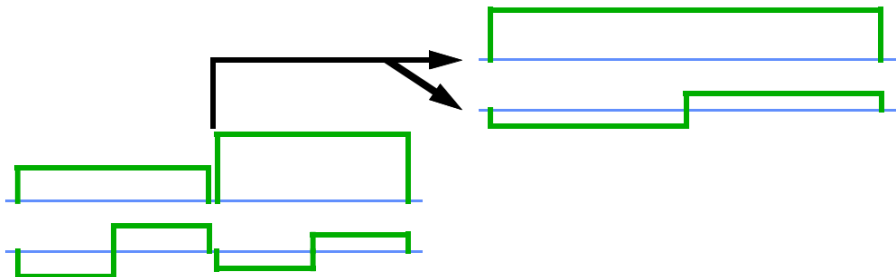






Recursivamente:





Só preciso do sinal de mais baixa resolução e dos detalhes de cada resolução:

$$\text{Sinal} = \underbrace{\text{nível DC}}_{\alpha_0 \times \text{[square wave]}} + \sum \left( \underbrace{\alpha_{11} \times \text{[square wave]} + \alpha_{12} \times \text{[square wave]} + \dots + \alpha_{21} \times \text{[square wave]} + \alpha_{22} \times \text{[square wave]} + \alpha_{23} \times \text{[square wave]} + \dots}_{\text{combinação linear de expansões e translações de } \text{[square wave]}} \right)$$

A Transformada de Haar Discreta é obtida amostrando-se as funções e

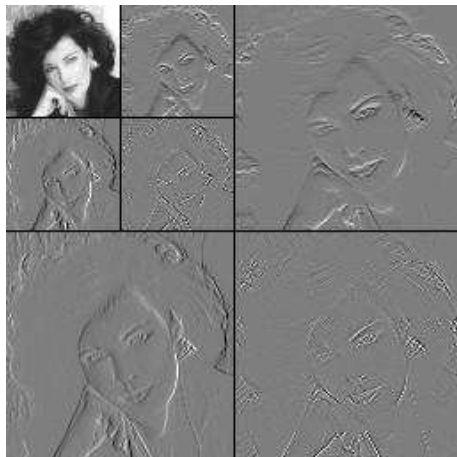
**Exemplo:** Transformada de Haar  $8 \times 8$

“sequency”

$$\mathbf{H}_r = \frac{1}{\sqrt{8}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} \\ 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 \end{bmatrix} \begin{matrix} 0 \\ 1 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{matrix}$$

**Propriedades:**

1.  $\mathbf{H}_r = \mathbf{H}_r^*$     $\mathbf{H}_r^T = \mathbf{H}_r^{-1}$
2. Transformada rápida ( $O(N)$  operações).
3. Vetores base ordenados em ordem de “sequency”
4. Pobre compactação para imagens.



# Singular Value Decomposition

Seja uma imagem  $\mathbf{U}$ ,  $N \times M$

- $\underbrace{\mathbf{U}\mathbf{U}^T}_{N \times N}$  e  $\underbrace{\mathbf{U}^T\mathbf{U}}_{M \times M}$  tem os mesmos autovalores  $\lambda_k$

(pois se  $\mathbf{U}\mathbf{U}^T\mathbf{x} = \lambda\mathbf{x} \Rightarrow \mathbf{U}^T\mathbf{U}\mathbf{U}^T\mathbf{x} = \lambda\mathbf{U}^T\mathbf{x} \Rightarrow \lambda$  também é autovalor de  $\mathbf{U}^T\mathbf{U}$ )

- Se  $r$  é o posto de  $\mathbf{U}$  então  $r$  autovalores de  $\mathbf{U}\mathbf{U}^T$  (ou  $\mathbf{U}^T\mathbf{U}$ ) são  $\neq 0$ .

- $\mathbf{U}^T\mathbf{U}$  possui  $r$  autovetores ortogonais  $\{\boldsymbol{\Phi}_m\}$ ,  $M \times 1$ ,  $m = 1, \dots, r$

- $\mathbf{U}\mathbf{U}^T$  possui  $r$  autovetores ortogonais  $\{\boldsymbol{\Psi}_m\}$ ,  $N \times 1$ ,  $m = 1, \dots, r$

$$\mathbf{U}^T\mathbf{U}\boldsymbol{\Phi}_m = \lambda_m\boldsymbol{\Phi}_m, \quad m = 1, \dots, r$$

$$\Rightarrow \mathbf{U} = \boldsymbol{\Psi}\boldsymbol{\Lambda}^{\frac{1}{2}}\boldsymbol{\Phi}^T = \sum_{m=1}^r \sqrt{\lambda_m} \boldsymbol{\Psi}_m \boldsymbol{\Phi}_m^T$$

$$\mathbf{U}\mathbf{U}^T\boldsymbol{\Psi}_m = \lambda_m\boldsymbol{\Psi}_m, \quad m = 1, \dots, r$$

$$\boldsymbol{\Lambda}^{\frac{1}{2}} = \begin{bmatrix} \sqrt{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\lambda_r} \end{bmatrix}$$

$\lambda_i$  : valores singulares de  $\mathbf{U}$

⇒ SVD Transform: (diagonaliza a imagem)  $\Lambda^{\frac{1}{2}} = \Psi^T \mathbf{U} \Phi$ , pois  $\Psi$  e  $\Phi$  são ortonormais.

$$\begin{cases} \Psi : & r \text{ vetores } N \times 1 \Rightarrow rN \text{ coeficientes} \\ \Phi : & r \text{ vetores } M \times 1 \Rightarrow rM \text{ coeficientes} \\ \Lambda : & r \text{ coeficientes} \end{cases} \Rightarrow \begin{matrix} \text{represento a imagem com} \\ rN + rM + r \text{ coeficientes} \end{matrix}$$

⇒ só vale a pena quando  $rN + rM + r < MN$  ( $r$  pequeno).

### Propriedades:

1. Dado  $\Phi_m, m = 1, \dots, r \Rightarrow \Psi_m = \frac{1}{\sqrt{\lambda_m}} \mathbf{U} \Phi_m, m = 1, \dots, r$
2. A SVD não é unitária, pois  $\Phi$  e  $\Psi$  são retangulares.

Incluindo em  $\Phi$  e  $\Psi$ ,  $\Phi_m$  e  $\Psi_m$  tal que :

$$\mathbf{U} \Phi_m = 0, m = r + 1, \dots, M \quad \mathbf{U}^T \Psi_m = 0, m = r + 1, \dots, N$$

$$\Rightarrow \begin{bmatrix} \Lambda^{\frac{1}{2}} \\ 0 \end{bmatrix} = \Psi^T \mathbf{U} \Phi \Rightarrow \text{A transformada passa a ser unitária}$$

3. Se  $\mathbf{U}_k \triangleq \sum_{m=1}^k \sqrt{\lambda_k} \boldsymbol{\Psi}_m \boldsymbol{\Phi}_m^T$ ,  $k \leq r \Rightarrow \mathbf{U}_k$  é a melhor aproximação de posto  $k$  de  $\mathbf{U}$

$$\text{Neste caso, } \epsilon_k^2 = \sum_{m=1}^M \sum_{n=1}^N |u(m, n) - u_k(m, n)|^2 = \sum_{m=k+1}^r \lambda_m, \quad k = 1, \dots, r$$

- A SVD possui a melhor concentração de energia para uma dada imagem.
- E a KLT?
  - A melhor concentração de energia, na média, para um conjunto de imagens.