Processamento Digital de Imagens

Processamento de Sinais Multidimensionais

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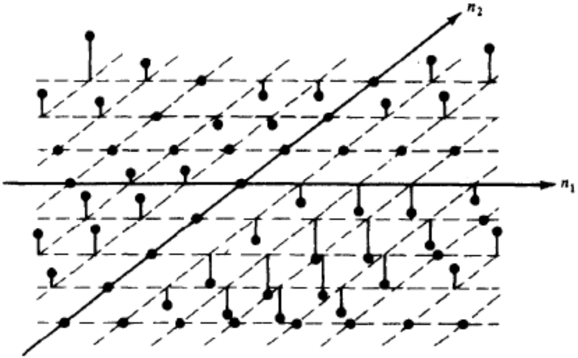
Sumário

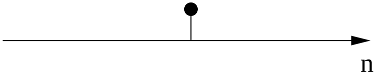
Sinais Multidimensionais Discretos

Função $f(x, y)$, contínua $\Rightarrow$ $f(n_1, n_2)$, discreta $\Rightarrow$ sequência bi-dimensional.

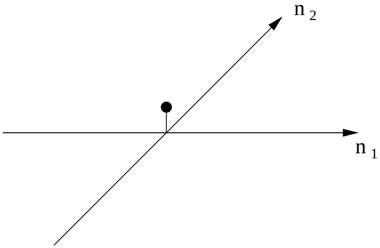


- **Impulso:**

$$\delta(n) = \begin{cases} 1, & n = 0; \\ 0, & n \neq 0. \end{cases}$$



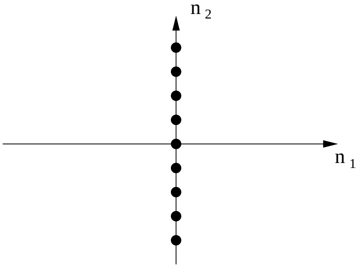
$$\delta(n_1, n_2) = \begin{cases} 1, & n_1 = n_2 = 0; \\ 0, & n.d.p. \end{cases}$$

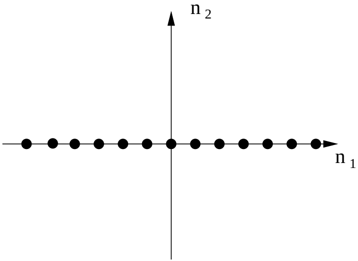


$$\delta(n_1, n_2) = \delta(n_1)\delta(n_2)$$

- **Impulso Linear:**

$$x(n_1, n_2) = \delta(n_1) \quad \text{ou} \quad x(n_1, n_2) = \delta(n_2)$$





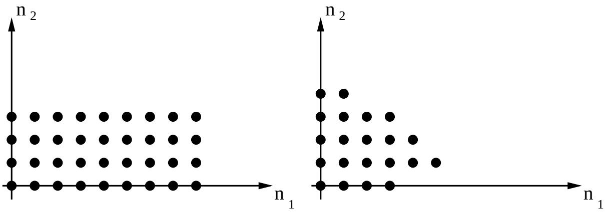
- **Degrau:**

$$u(n) = \begin{cases} 1, & n \geq 0; \\ 0, & n < 0. \end{cases} \qquad u(n_1, n_2) = \begin{cases} 1, & n_1 \geq 0 \textbf{ e } n_2 \geq 0; \\ 0, & n.d.p.. \end{cases}$$

- **Exponenciais:** $x(n_1, n_2) = a^{n_1} b^{n_2}$
  - Caso particular: $a = e^{j\omega_1}$, $b = e^{j\omega_2} \Rightarrow x(n_1, n_2) = e^{j\omega_1 n_1 + j\omega_2 n_2}$

- **Sequências separáveis:** $x(n_1, n_2) = x_1(n_1)x_2(n_2)$

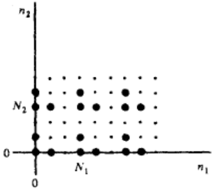
- **Sequências Finitas:** região de suporte



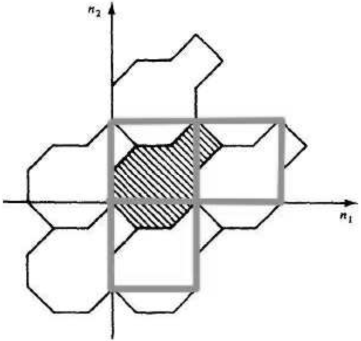
- **Sequências Periódicas**
  1^a tentativa: periódica em cada uma das direções n_1 e n_2

$$x(n_1, n_2 + N_2) = x(n_1, n_2)$$

$$x(n_1 + N_1, n_2) = x(n_1, n_2)$$



$N_1 N_2$ amostras independentes $\Rightarrow$
Qualquer região conectada com $N_1 N_2$ amostras pode ser o período.



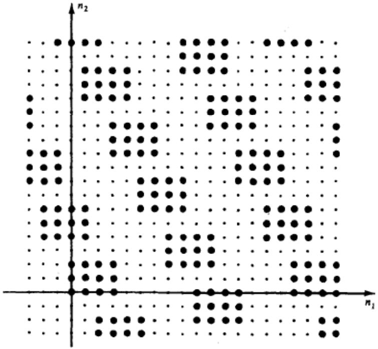
$\Rightarrow$ 2 períodos alternativos

De um modo geral:

$$x(n_1 + N_{11}, n_2 + N_{21}) = x(n_1, n_2)$$
$$x(n_1 + N_{12}, n_2 + N_{22}) = x(n_1, n_2)$$

$$D = N_{11}N_{22} - N_{12}N_{21} \neq 0$$

$$\mathbf{N_1} = \begin{bmatrix} N_{11} \\ N_{21} \end{bmatrix} \quad \mathbf{N_2} = \begin{bmatrix} N_{12} \\ N_{22} \end{bmatrix} \quad \mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \Rightarrow \begin{array}{l} x(\mathbf{n} + \mathbf{N_1}) = x(\mathbf{n}) \\ x(\mathbf{n} + \mathbf{N_2}) = x(\mathbf{n}) \end{array}$$



$$\mathbf{N_1} = \begin{pmatrix} 7 \\ 2 \end{pmatrix} \quad \mathbf{N_2} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$$

$|D| =$ n^o de amostras em <u>um</u> período

- **m dimensões:** $\mathbf{N} = [\mathbf{N_1 N_1} \cdots \mathbf{N_m}] \Rightarrow \quad x(\mathbf{n} + \mathbf{N_i}) = x(\mathbf{n}),\ i = 1, 2, \ldots, m$

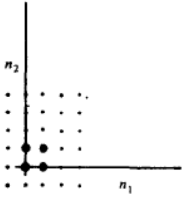
$\mathbf{N} \Rightarrow$ matriz de periodicidade:

- Se $\mathbf{r}$ é um vetor inteiro, então $x(\mathbf{n} + \mathbf{Nr}) = x(\mathbf{n})$
- $\forall P$ matriz inteira $\Rightarrow \mathbf{N}P$ também é matriz de periodicidade.

Sistemas Multi-Dimensionais

Operações Fundamentais

- **Soma:** $y(n_1, n_2) = x(n_1, n_2) + w(n_1, n_2)$
- **Multiplicação por Escalar:** $y(n_1, n_2) = cx(n_1, n_2)$
- **Deslocamento:** $y(n_1, n_2) = x(n_1 - m_1, n_2 - m_2)$



$x(n_1, n_2)$

$y(n_1, n_2) = x(n_1 - 1, n_2 - 2)$

- $x(n_1, n_2) = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} x(k_1, k_2)\delta(n_1 - k_1, n_2 - k_2)$

- **Linearidade:** $y_1 = L[x_1]$, $y_2 = L[x_2]$

$$\Rightarrow L[ax_1 + bx_2] = ay_1 + by_2$$

$$y(n_1, n_2) = L[x(n_1, n_2)] = L\left[\sum_{k_1}\sum_{k_2} x(k_1, k_2)\delta(n_1 - k_1, n_2 - k_2)\right]$$

$$= \sum_{k_1}\sum_{k_2} x(k_1, k_2)L\left[\delta(n_1 - k_1, n_2 - k_2)\right]$$

$$= \sum_{k_1}\sum_{k_2} x(k_1, k_2)h_{k_1,k_2}(n_1, n_2)$$

- **Invariância ao Deslocamento:** $y(n_1, n_2) = T[x(n_1, n_2)]$

  $\Rightarrow T[x(n_1 - m_1, n_2 - m_2)] = y(n_1 - m_1, n_2 - m_2)$

  $L[x(n_1, n_2)] = c(n_1, n_2)x(n_1, n_2)$: Linear, mas variante ao deslocamento.

  $L[x(n_1, n_2)] = [x(n_1, n_2)]^2$: Invariante ao deslocamento, mas não-linear.

- **Sistemas Lineares Invariantes ao Deslocamento:**

  $h_{k_1 k_2}(n_1, n_2) = L[\delta(n_1 - k_1, n_2 - k_2)]$

  Se $h_{00}(n_1, n_2) = L[\delta(n_1, n_2)] \Rightarrow h_{k_1 k_2}(n_1, n_2) = h_{00}(n_1 - k_1, n_2 - k_2)$

$$\Rightarrow y(n_1, n_2) = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} x(k_1, k_2) h(n_1 - k_1, n_2 - k_2)$$

$$\begin{cases} \theta_1 = n_1 - k_1 \\ \theta_2 = n_2 - k_2 \end{cases}$$

$$\Rightarrow y(n_1, n_2) = \sum_{\theta_1=-\infty}^{\infty} \sum_{\theta_2=-\infty}^{\infty} x(n_1 - \theta_1, n_2 - \theta_2) h(\theta_1, \theta_2)$$

$$\Rightarrow \mathbf{y} = \mathbf{x} * * \mathbf{h} = \mathbf{h} * * \mathbf{x}$$

Vetorialmente, $$y(\mathbf{n}) = \sum_{\mathbf{k} \in \mathbb{Z}^2} x(\mathbf{k}) h(\mathbf{n} - \mathbf{k})$$

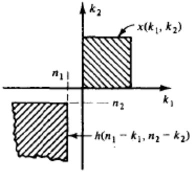
- **Exemplo 1:**
  $h(n_1, n_2) = u(n_1, n_2)$

  $$x(n_1, n_2) = \begin{cases} 1, & 0 \leq n_1 \leq N_1 \text{ e } 0 \leq n_2 \leq N_2 \\ 0, & n.d.p. \end{cases}$$

$$\Rightarrow y(n_1, n_2) = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} x(k_1, k_2) h(n_1 - k_1, n_2 - k_2)$$

5 casos:

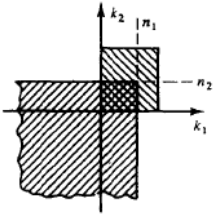
(i) $n_1 < 0,\ n_2 < 0 \Rightarrow y(n_1, n_2) = 0.$



Case 1

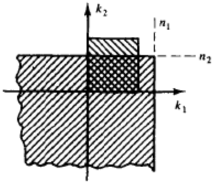
(ii) $0 \leq n_1 \leq N_1$, $0 \leq n_2 \leq N_2$

$$\Rightarrow y(n_1, n_2) = \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} 1 = (n_1 + 1)(n_2 + 1)$$



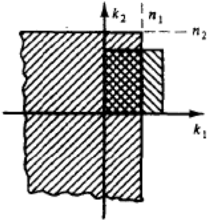
(iii) $n_1 > N_1$, $0 \leq n_2 \leq N_2$

$$\Rightarrow y(n_1, n_2) = \sum_{k_1=0}^{N_1} \sum_{k_2=0}^{n_2} 1 = (N_1 + 1)(n_2 + 1)$$



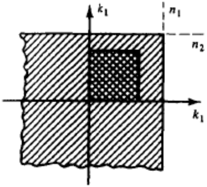
(iv) $0 \leq n_1 \leq N_1$, $n_2 > N_2$

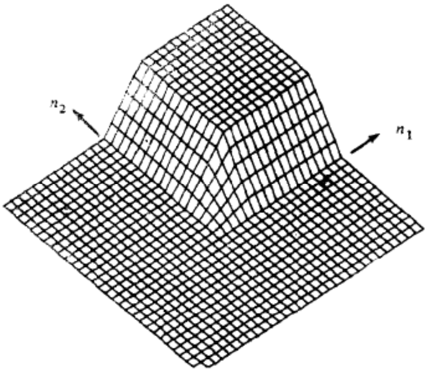
$$\Rightarrow y(n_1, n_2) = \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{N_2} 1 = (n_1+1)(N_2+1)$$



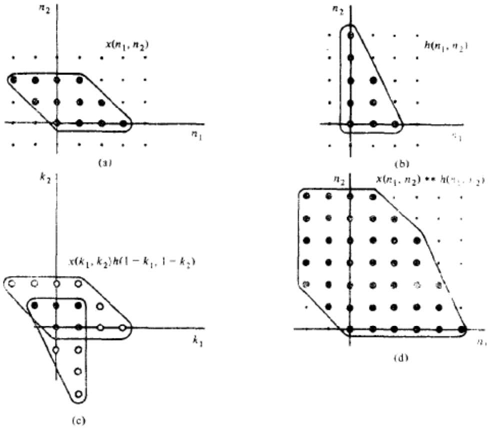
(v) $n_1 > N_1$, $n_2 > N_2$

$$\Rightarrow y(n_1, n_2) = \sum_{k_1=0}^{N_1} \sum_{k_2=0}^{N_2} 1 = (N_1+1)(N_2+1)$$




n_2
n_1

- **Exemplo 2:**



- **Conexões em Cascata e Paralelas:**
  - $\rightarrow x**h = h**x$
  - $\rightarrow (x**h)**g = x**(h**g) = (x**g)**h = x**(g**h)$
  - $\rightarrow x**(h+g) = x**g + x**h$
- **Sistemas Separáveis:** $h(n_1, n_2) = h_1(n_1)h_2(n_2)$

$$
\begin{aligned}
y(n_1, n_2) &= \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} x(k_1, k_2) h(n_1 - k_1, n_2 - k_2) \\
&= \sum_{k_2=-\infty}^{\infty} \sum_{k_1=-\infty}^{\infty} x(k_1, k_2) h_1(n_1 - k_1) h_2(n_2 - k_2)
\end{aligned}
$$

$$= \underbrace{\sum_{k_2=-\infty}^{\infty} h_2(n_2 - k_2) \sum_{k_1=-\infty}^{\infty} x(k_1, k_2) h_1(n_1 - k_1)}_{\text{primeiro colunas depois linhas}}$$

$$= \underbrace{\sum_{k_1=-\infty}^{\infty} h_1(n_1 - k_1) \sum_{k_2=-\infty}^{\infty} x(k_1, k_2) h_2(n_2 - k_2)}_{\text{primeiro linhas depois colunas}}$$

$$h(n_1, n_2) \Rightarrow M_1 \times M_2$$

$$x(n_1, n_2) \Rightarrow N_1 \times N_2$$

$$M_1 < N_1, \quad M_2 < N_2$$

- **Convolução bi-dimensional:**

  $$\text{Cada amostra} \simeq \begin{array}{l} O(M_1 M_2) \text{ multiplicações} \\ O(M_1 M_2) \text{ somas} \end{array}$$

  Todas as $\approx N_1 N_2$ amostras: $O(M_1 M_2 N_1 N_2)$ somas ou multiplicações.

- **Convolução separável:**
  - Cada coluna: $\begin{cases} O(M_1) \text{ somas ou multiplicações por amostra} \\ O(M_1 N_1) \text{ somas ou multiplicações totais} \end{cases}$
  - Cada linha: $\begin{cases} O(M_2) \text{ somas ou multiplicações por amostra} \\ O(M_2 N_2) \text{ somas ou multiplicações totais} \end{cases}$
  - N_1 linhas $+$ N_2 colunas $\Rightarrow N_1 O(M_2 N_2) + N_2 O(M_1 N_1)$ somas ou multiplicações
- Economia da separável em relação à não-separável $\Rightarrow O\left(\frac{1}{M_1} + \frac{1}{M_2}\right)$ somas ou multiplicações

- **Estabilidade:**

  BIBO (Bounded Input, Bounded Output):

$$\sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} |h(n_1, n_2)| = S_1 < \infty$$

  Mais fraca (mean-square):

$$\sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} |h(n_1, n_2)|^2 = S_2 < \infty$$

Sinais e Sistemas Multidimensionais no Domínio da Frequência

$$x(n_1, n_2) = e^{j\omega_1 n_1 + j\omega_2 n_2} \Rightarrow \sum_{k_1} \sum_{k_2} e^{j\omega_1 (n_1 - k_1) + j\omega_2 (n_2 - k_2)} h(k_1, k_2)$$

$$\Rightarrow y(n_1, n_2) = e^{j\omega_1 n_1 + j\omega_2 n_2} \sum_{k_1} \sum_{k_2} h(k_1, k_2) e^{-j\omega_1 k_1 - j\omega_2 k_2} = e^{j\omega_1 n_1 + j\omega_2 n_2} H(\omega_1, \omega_2)$$

$$\Rightarrow H(\omega_1, \omega_2) = \sum_{n_1} \sum_{n_2} h(n_1, n_2) e^{-j\omega_1 n_1 - j\omega_2 n_2}$$

$H(\omega_1, \omega_2)$ é periódica $\Rightarrow H(\omega_1 + 2\pi, \omega_2) = H(\omega_1, \omega_2 + 2\pi) = H(\omega_1, \omega_2)$

Vetorialmente:

$$\mathbf{n} = \begin{pmatrix} n_1 \\ \vdots \\ n_M \end{pmatrix} \quad \boldsymbol{\omega} = \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_M \end{pmatrix} \quad \Rightarrow \quad H(\boldsymbol{\omega}) = \sum_{\mathbf{n} \in \mathbb{Z}^M} h(\mathbf{n}) e^{-j\boldsymbol{\omega}^T \mathbf{n}}$$

- **Sistemas Separáveis:** $h(n_1, n_2) = f(n_1)g(n_2) \Rightarrow H(\omega_1, \omega_2) = F(\omega_1)G(\omega_2)$

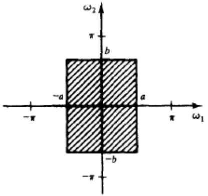
$$F(\omega_1) = \sum_n f(n)e^{-j\omega_1 n} \qquad G(\omega_2) = \sum_n g(n)e^{-j\omega_2 n}$$

- **Caminho Inverso:**

$$h(n_1, n_2) = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} H(\omega_1, \omega_2) e^{j\omega_1 n_1 + j\omega_2 n_2} d\omega_1 d\omega_2$$

- **Exemplo 1:**

$$H(\omega_1, \omega_2) = \begin{cases} 1, & |\omega_1| \leq a \leq \pi, |\omega_2| \leq b \leq \pi \\ 0, & n.d.p. \end{cases}$$



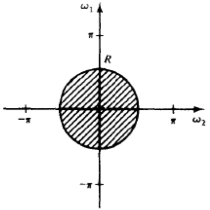
$$h(n_1, n_2) = \frac{1}{4\pi^2} \int_{-a}^{a} \int_{-b}^{b} e^{j\omega_1 n_1 + j\omega_2 n_2} d\omega_1 d\omega_2 = \frac{1}{4\pi^2} \int_{-a}^{a} e^{j\omega_1 n_1} d\omega_1 \int_{-b}^{b} e^{j\omega_2 n_2} d\omega_2$$

$$= \frac{1}{4\pi^2} \left[\frac{1}{jn_1} (e^{jan_1} - e^{-jan_1}) \right] \left[\frac{1}{jn_2} (e^{jbn_2} - e^{-jbn_2}) \right]$$

$$= \frac{1}{4\pi^2} \frac{1}{n_1 n_2} 2\text{sen}(an_1)\text{sen}(bn_2) = \frac{\text{sen}(an_1)}{\pi n_1} \frac{\text{sen}(bn_2)}{\pi n_2}$$

- **Exemplo 2:**

$$H(\omega_1, \omega_2) = \begin{cases} 1, & \omega_1^2 + \omega_2^2 \leq R^2 \\ 0, & n.d.p. \end{cases}$$



$$h(n_1, n_2) = \frac{1}{4\pi^2} \iint_A e^{j\omega_1 n_1 + j\omega_2 n_2} d\omega_1 d\omega_2$$

$$\omega = \sqrt{\omega_1^2 + \omega_2^2}, \phi = \text{arctg}\left(\frac{\omega_2}{\omega_1}\right), \theta = \text{arctg}\left(\frac{n_2}{n_1}\right)$$

$$\Rightarrow h(n_1, n_2) = \frac{1}{4\pi^2} \int_0^R \int_0^{2\pi} \omega e^{\left(j\omega\sqrt{n_1^2+n_2^2}\right)\cos(\theta-\phi)} d\phi d\theta$$

$$= \frac{1}{2\pi} \int_0^R \omega J_0\left(\omega\sqrt{n_1^2 + n_2^2}\right) d\omega$$

$$= \frac{R}{2\pi} \frac{J_1\left(R\sqrt{n_1^2 + n_2^2}\right)}{\sqrt{n_1^2 + n_2^2}}$$

Transformada de Fourier Multi-dimensional

Interpretação da fórmula de $h(n_1, n_2)$ como uma some de senóides complexas.

$$x(n_1, n_2) = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} X(\omega_1, \omega_2) e^{j\omega_1 n_1 + j\omega_2 n_2} d\omega_1 d\omega_2$$

$$X(\omega_1, \omega_2) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} x(n_1, n_2) e^{-j\omega_1 n_1 - j\omega_2 n_2}$$

Em um sistema linear,

$$x(n_1, n_2) \longrightarrow \boxed{\begin{matrix} L[\quad] \\ h(n_1, n_2) \end{matrix}} \longrightarrow y(n_1, n_2)$$

$$\begin{aligned} y(n_1, n_2) = L[x(n_1, n_2)] &= L\left[\frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} X(\omega_1, \omega_2) e^{j\omega_1 n_1 + j\omega_2 n_2} d\omega_1 d\omega_2\right] \\ &= \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} X(\omega_1, \omega_2) L[e^{j\omega_1 n_1 + j\omega_2 n_2}] d\omega_1 d\omega_2 \\ &= \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} X(\omega_1, \omega_2) H(\omega_1, \omega_2) e^{j\omega_1 n_1 + j\omega_2 n_2} d\omega_1 d\omega_2 \end{aligned}$$

mas $y(n_1, n_2) = \dfrac{1}{4\pi^2} \displaystyle\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} Y(\omega_1, \omega_2) e^{j\omega_1 n_1 + j\omega_2 n_2} d\omega_1 d\omega_2$

$$\Rightarrow \boxed{Y(\omega_1, \omega_2) = X(\omega_1, \omega_2) H(\omega_1, \omega_2)}$$

- **Outras Propriedades:** (Notação: $x \leftrightarrow X$)
  - (i) $y = h ** x \leftrightarrow Y = HX$
  - (ii) $x_1 \leftrightarrow X_1, \quad x_2 \leftrightarrow X_2 \Rightarrow ax_1 + bx_2 \leftrightarrow aX_1 + bX_2$
  - (iii) $x(n_1, n_2) \leftrightarrow X(\omega_1, \omega_2)$

$$\Rightarrow x(n_1 - m_1, n_2 - m_2) \leftrightarrow e^{-j\omega_1 m_1 - j\omega_2 m_2} X(\omega_1, \omega_2)$$

  - (iv)

$$c(n_1, n_2) x(n_1, n_2) \leftrightarrow \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} X(\theta_1, \theta_2) C(\omega_1 - \theta_1, \omega_2 - \theta_2) d\theta_1 d\theta_2$$

- (v)

$$-jn_1 x(n_1, n_2) \leftrightarrow \frac{\partial}{\partial\omega_1} X(\omega_1, \omega_2)$$
$$-jn_2 x(n_1, n_2) \leftrightarrow \frac{\partial}{\partial\omega_2} X(\omega_1, \omega_2)$$
$$-n_1 n_2 x(n_1, n_2) \leftrightarrow \frac{\partial^2}{\partial\omega_1 \partial\omega_2} X(\omega_1, \omega_2)$$

- (vi) $x(n_1, n_2) \leftrightarrow X(\omega_1, \omega_2)$
- (vii)

$$x(-n_1, n_2) \leftrightarrow X(-\omega_1, \omega_2)$$
$$x(n_1, -n_2) \leftrightarrow X(\omega_1, -\omega_2)$$
$$x(-n_1, -n_2) \leftrightarrow X(-\omega_1, -\omega_2)$$

- (viii) $x^*(n_1, n_2) \leftrightarrow X^*(-\omega_1, -\omega_2)$

- **Teorema de Parseval:**

$$x(n_1, n_2) \leftrightarrow X(\omega_1, \omega_2) \qquad y(n_1, n_2) \leftrightarrow Y(\omega_1, \omega_2)$$

$$\sum_{n_1} \sum_{n_2} x(n_1, n_2) y^*(n_1, n_2) = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} X(\omega_1, \omega_2) Y^*(\omega_1, \omega_2) d\omega_1 d\omega_2$$

Se $x(n_1, n_2) = y(n_1, n_2)$:

$$\sum_{n_1} \sum_{n_2} |x(n_1, n_2)|^2 = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |X(\omega_1, \omega_2)|^2 d\omega_1 d\omega_2$$

- **Transformada Z:**

$$X(z_1, z_2) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x(m, n) z_1^{-m} z_2^{-n}$$

Região de convergência: $\{(z_1, z_2) | \text{o somatório converge}\}$

$$x(m, n) = \frac{1}{(2\pi j)^2} \oint \oint X(z_1, z_2) z_1^{m-1} z_2^{n-1} dz_1 dz_2$$

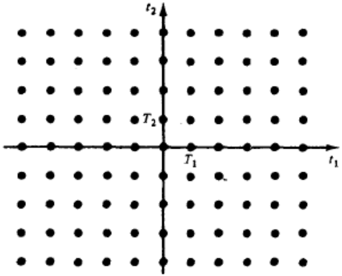
Propriedades: $(x(m,n) \leftrightarrow X(z_1,z_2) \quad y(m,n) \leftrightarrow Y(z_1,z_2) \quad h(m,n) \leftrightarrow H(z_1,z_2))$

- (i) $x(-m,-n) \leftrightarrow X(z_1^{-1}, z_2^{-1})$
- (ii) $ax(m,n) + by(m,n) \leftrightarrow aX(z_1,z_2) + bY(z_1,z_2)$
- (iii) $x^*(-m,-n) \leftrightarrow X^*(z_1^*, z_2^*)$
- (iv) $x_1(m)x_2(n) \leftrightarrow X_1(z_1)X_2(z_2)$
- (v) $x(m-m_0, n-n_0) \leftrightarrow z_1^{-m_0} z_2^{-n_0} X(z_1,z_2)$
- (vi) $a^m b^n x(m,n) \leftrightarrow X\left(\frac{z_1}{a}, \frac{z_2}{b}\right)$
- (vii) $h(m,n) * x(m,n) \leftrightarrow H(z_1,z_2)X(z_1,z_2)$
- (viii) $x(m,n)y(m,n) \leftrightarrow \frac{1}{(2\pi j)^2} \oint \oint X\left(\frac{z_1}{z_1'}, \frac{z_2}{z_2'}\right) Y(z_1', z_2') \frac{dz_1'}{z_1} \frac{dz_2'}{z_2}$
- (ix) Transformada de Fourier: $z_1 \leftrightarrow e^{j\omega_1}, \quad z_2 \leftrightarrow e^{j\omega_2}$

Amostragem de Sinais Multidimensionais Contínuos

Amostragem periódica com geometria retangular

$x(n_1, n_2) = x_a(n_1 T_1, n_2 T_2)$



Como recuperar $x_a(t_1, t_2)$ de $x(n_1, n_2)$?

$$X_a(\Omega_1,\Omega_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_a(t_1,t_2)e^{-j\Omega_1 t_1 - j\Omega_2 t_2}dt_1 dt_2$$

$$x_a(t_1,t_2) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X_a(\Omega_1,\Omega_2)e^{j\Omega_1 t_1 + j\Omega_2 t_2}d\Omega_1 d\Omega_2$$

$$\Rightarrow x(n_1,n_2) = x_a(n_1T_1,n_2T_2) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X_a(\Omega_1,\Omega_2)e^{j\Omega_1 n_1 T_1 + j\Omega_2 n_2 T_2}d\Omega_1 d\Omega_2$$

$$\Rightarrow x(n_1,n_2) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{T_1T_2}X_a\left(\frac{\omega_1}{T_1},\frac{\omega_2}{T_2}\right)e^{j\omega_1 n_1 + j\omega_2 n_2}d\omega_1 d\omega_2$$

Divido o espaço (ω_1,ω_2) em períodos de $e^{j\omega_1 n_1 + j\omega_2 n_2}$, isto é, em quadrados definidos como

$$Sq(k_1,k_2)\colon \begin{cases} -\pi + 2\pi k_1 \leq \omega_1 < \pi + 2\pi k_1 \\ -\pi + 2\pi k_2 \leq \omega_2 < \pi + 2\pi k_2 \end{cases}$$

$$\Rightarrow x(n_1,n_2) = \frac{1}{4\pi^2}\sum_{k_1}\sum_{k_2}\iint_{Sq(k_1,k_2)} \frac{1}{T_1T_2}X_a\left(\frac{\omega_1'}{T_1},\frac{\omega_2'}{T_2}\right)e^{j\omega_1' n_1 + j\omega_2' n_2}d\omega_1' d\omega_2'$$

$$\omega_1' = 2\pi k_1 + \omega_1, \quad \omega_2' = 2\pi k_2 + \omega_2 \Rightarrow Sq(k_1,k_2)\colon \begin{cases} -\pi \leq \omega_1 \leq \pi \\ -\pi \leq \omega_2 \leq \pi \end{cases}$$

$$\Rightarrow x(n_1, n_2) =$$

$$\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{1}{T_1T_2}\sum_{k_1}\sum_{k_2}X_a\left(\frac{\omega_1+2\pi k_1}{T_1},\frac{\omega_2+2\pi k_2}{T_2}\right)e^{j\omega_1n_1+j\omega_2n_2}\underbrace{e^{j2\pi k_1+j2\pi k_2}}_{=1}d\omega_1d\omega_2$$

$$\Rightarrow X(\omega_1,\omega_2) = \frac{1}{T_1T_2}\sum_{k_1}\sum_{k_2}X_a\left(\frac{\omega_1+2\pi k_1}{T_1},\frac{\omega_2+2\pi k_2}{T_2}\right), \quad \boxed{\begin{array}{l}-\pi\le\omega_1<\pi\\ -\pi\le\omega_2<\pi\end{array}}$$

Definindo agora o sinal

$$x_i(t_1,t_2) = \sum_{k_1}\sum_{k_2}x(k_1,k_2)\delta(t_1-k_1T_1, t_2-k_2T_2)$$

Temos que a sua transformada de Fourier é

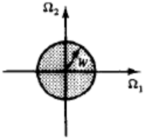
$$\begin{aligned}X_i(\Omega_1,\Omega_2) &= \sum_{k_1}\sum_{k_2}x(k_1,k_2)\mathcal{F}\left\{\delta(t_1-k_1T_1,t_2-k_2T_2)\right\}\\ &= \sum_{k_1}\sum_{k_2}x(k_1,k_2)e^{-j\Omega_1T_1k_1-j\Omega_2T_2k_2} = X(\Omega_1T_1,\Omega_2T_2)\end{aligned}$$

$$\Rightarrow X_i(\Omega_1, \Omega_2) = \frac{1}{T_1 T_2} \sum_{k_1} \sum_{k_2} X_a\left(\Omega_1 + \frac{2\pi k_1}{T_1}, \Omega_2 + \frac{2\pi k_2}{T_2}\right), \quad \boxed{\begin{array}{l} -\frac{\pi}{T_1} \leq \Omega_1 < \frac{\pi}{T_1} \\ -\frac{\pi}{T_2} \leq \Omega_2 < \frac{\pi}{T_2} \end{array}}$$

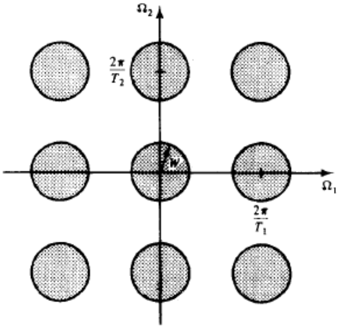
Então, se $X_a(\Omega_1, \Omega_2) = 0, \quad |\Omega_1| \geq \frac{\pi}{T_1} \quad \text{e} \quad |\Omega_2| \geq \frac{\pi}{T_2}$

$$\Rightarrow X_i(\Omega_1, \Omega_2) = \frac{1}{T_1 T_2} X_a(\Omega_1, \Omega_2), \quad |\Omega_1| < \frac{\pi}{T_1} \quad \text{e} \quad |\Omega_2| < \frac{\pi}{T_2}$$

"Período": $\left(\frac{2\pi}{T_1}, \frac{2\pi}{T_2}\right)$



(a)



(b)

$$X_a(\Omega_1, \Omega_2) = \begin{cases} T_1 T_2 X_i(\Omega_1, \Omega_2), & |\Omega_1| < \frac{\pi}{T_1} \text{ e } |\Omega_2| < \frac{\pi}{T_2} \\ 0, & n.d.p. \end{cases}$$

Para recuperar $X_a(\Omega_1, \Omega_2)$, filtro $X_i(\Omega_1, \Omega_2)$ com um filtro com a seguinte resposta em frequência:

$$H(\Omega_1, \Omega_2) = \begin{cases} T_1 T_2, & |\Omega_1| < \dfrac{\pi}{T_1} \text{ e } |\Omega_2| < \dfrac{\pi}{T_2} \\ 0, & n.d.p. \end{cases}$$

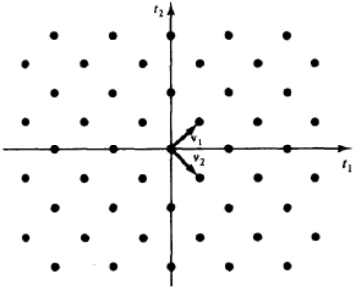
Resposta ao impulso:

$$h(t_1, t_2) = T_1 T_2 \frac{\text{sen}\left(\dfrac{\pi}{T_1} t_1\right)}{\dfrac{\pi}{T_1} t_1} \frac{\text{sen}\left(\dfrac{\pi}{T_2} t_2\right)}{\dfrac{\pi}{T_2} t_2}.$$

Amostragem com geometria arbitrária

$$t_1 = v_{11} n_1 + v_{12} n_2$$
$$t_2 = v_{21} n_1 + v_{22} n_2$$

$$\mathbf{V} = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = [\mathbf{v}_1 \quad \mathbf{v}_2]$$



$$\mathbf{t} = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}, \quad \mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \quad \Rightarrow \quad \mathbf{t} = \mathbf{V}\mathbf{n} = n_1 \mathbf{v}_1 + n_2 \mathbf{v}_2, \qquad \mathbf{V} : \text{ matriz de amostragem}$$

$$x(\mathbf{n}) = x_a(\mathbf{Vn})$$

$$\begin{cases} X_a(\boldsymbol{\Omega}) = \displaystyle\int_{\mathbf{t}\in\mathbb{R}^2} x_a(\mathbf{t}) e^{-j\boldsymbol{\Omega}^T\mathbf{t}} d\mathbf{t} \\ x_a(\mathbf{t}) = \dfrac{1}{4\pi^2} \displaystyle\int_{\boldsymbol{\Omega}\in\mathbb{R}^2} X_a(\boldsymbol{\Omega}) e^{j\boldsymbol{\Omega}^T\mathbf{t}} d\boldsymbol{\Omega} \end{cases}$$

$$\begin{cases} X(\boldsymbol{\omega}) = \displaystyle\sum_{\mathbf{n}\in\mathbb{Z}^2} x(\mathbf{n}) e^{-j\boldsymbol{\omega}^T\mathbf{n}} \\ x(\mathbf{n}) = \dfrac{1}{4\pi^2} \displaystyle\int_{\boldsymbol{\omega}\in[-\pi,\pi]^2} X(\boldsymbol{\omega}) e^{j\boldsymbol{\omega}^T\mathbf{n}} d\boldsymbol{\omega} \end{cases}$$

$$x(\mathbf{n}) = x_a(\mathbf{Vn}) = \frac{1}{4\pi^2} \int_{\boldsymbol{\Omega}\in\mathbb{R}^2} X_a(\boldsymbol{\Omega}) e^{-j\boldsymbol{\Omega}^T\mathbf{Vn}} d\boldsymbol{\Omega}$$

se $\boldsymbol{\omega} = \mathbf{V}^T\Omega \quad \Rightarrow d\boldsymbol{\omega} = |\det\mathbf{V}| d\boldsymbol{\Omega}$

$$x(\mathbf{n}) = \frac{1}{4\pi^2} \int_{\boldsymbol{\omega}\in\mathbb{R}^2} \frac{1}{|\det\mathbf{V}|} X_a(\mathbf{V}^{T^{-1}}\boldsymbol{\omega}) e^{-j\boldsymbol{\omega}^T\mathbf{n}} d\boldsymbol{\omega}$$

$$x(\mathbf{n}) = \frac{1}{4\pi^2}\int_{\boldsymbol{\omega}'\in\mathbb{R}^2} \frac{1}{|\det\mathbf{V}|} X_a(\mathbf{V}^{T^{-1}}\boldsymbol{\omega}')e^{-j\boldsymbol{\omega}'^T\mathbf{n}}d\boldsymbol{\omega}'$$

Integrando por áreas quadradas: $\Rightarrow \quad \int_{\boldsymbol{\omega}'\in\mathbb{R}^2} = \sum_{\mathbf{k}\in\mathbf{Z}^2}\int_{\boldsymbol{\omega}'\in[-\pi,\pi]^2+2\pi\mathbf{k}}$

Se $\boldsymbol{\omega}' = \boldsymbol{\omega} + 2\pi\mathbf{k} \quad \Rightarrow \quad \int_{\boldsymbol{\omega}'\in\mathbb{R}^2} = \sum_{\mathbf{k}\in\mathbf{Z}^2}\int_{\boldsymbol{\omega}\in[-\pi,\pi]^2} = \int_{\boldsymbol{\omega}\in[-\pi,\pi]^2}\sum_{\mathbf{k}\in\mathbf{Z}^2}$

$$\Rightarrow \quad x(\mathbf{n}) = \frac{1}{4\pi^2}\int_{\boldsymbol{\omega}\in[-\pi,\pi]^2} \frac{1}{|\det\mathbf{V}|}\sum_{\mathbf{k}\in\mathbf{Z}^2} X_a[(\mathbf{V}^T)^{-1}(\boldsymbol{\omega}+2\pi\mathbf{k})]e^{-j\boldsymbol{\omega}^T\mathbf{n}}e^{j2\pi\mathbf{k}^T\mathbf{n}}d\boldsymbol{\omega}$$

$$\Rightarrow \quad X(\boldsymbol{\omega}) = \frac{1}{|\det\mathbf{V}|}\sum_{\mathbf{k}\in\mathbf{Z}^2} X_a[(\mathbf{V}^T)^{-1}(\boldsymbol{\omega}+2\pi\mathbf{k})]$$

$$\text{ou} \quad X(\boldsymbol{\omega}) = \frac{1}{|\det\mathbf{V}|}\sum_{\mathbf{k}\in\mathbf{Z}^2} X_a[(\mathbf{V}^T)^{-1}(\boldsymbol{\omega}-2\pi\mathbf{k})]$$

Definindo agora o sinal

$$x_i(\mathbf{t}) = \sum_{\mathbf{n}\in\mathbb{Z}^2} x(\mathbf{n})\delta(\mathbf{t} - \mathbf{V}\mathbf{n})$$

Temos que a sua transformada de Fourier é

$$\begin{aligned} X_i(\boldsymbol{\Omega}) &= \sum_{\mathbf{n}\in\mathbb{Z}^2} x(\mathbf{k})\mathcal{F}\left\{\delta(\mathbf{t} - \mathbf{V}\mathbf{n})\right\} \\ &= \sum_{\mathbf{n}\in\mathbb{Z}^2} x(\mathbf{n})e^{-j\boldsymbol{\Omega}^T\mathbf{V}\mathbf{n}} = X(\mathbf{V}^T\boldsymbol{\Omega}) \end{aligned}$$

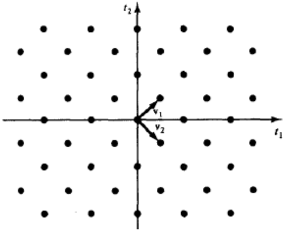
$$X_i(\boldsymbol{\Omega}) = \frac{1}{|\det\mathbf{V}|}\sum_{\mathbf{k}\in\mathbf{Z}^2} X_a(\boldsymbol{\Omega}-\mathbf{U}\mathbf{k}), \qquad \mathbf{U}\mathbf{V}^T = 2\pi\mathbf{I}$$

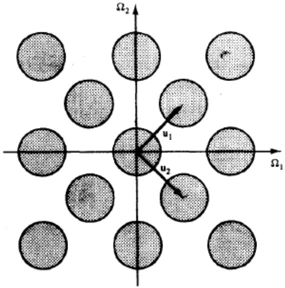
Amostragem Retangular

$$\mathbf{V} = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} \qquad \mathbf{U} = \begin{bmatrix} \frac{2\pi}{T_1} & 0 \\ 0 & \frac{2\pi}{T_2} \end{bmatrix}$$

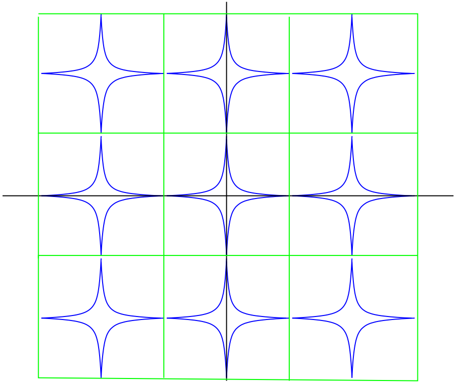
$X_i(\mathbf{\Omega}) =$ extensão periódica de $X_a(\mathbf{\Omega})$, com matriz de periodicidade $\mathbf{U}$

Exemplo: $\mathbf{V} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad \mathbf{U} = \begin{bmatrix} \pi & \pi \\ \pi & -\pi \end{bmatrix}$

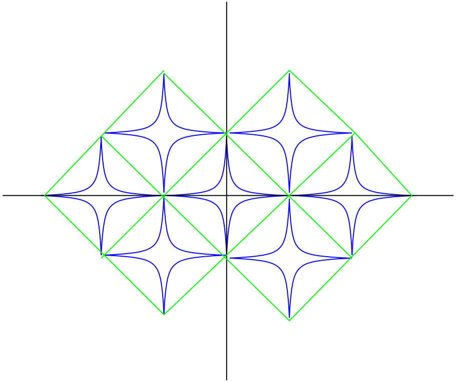




Exemplo: Vídeo não-entrelaçado $\times$ vídeo entrelaçado no plano vertical-temporal $(\Omega_t \times \Omega_V)$

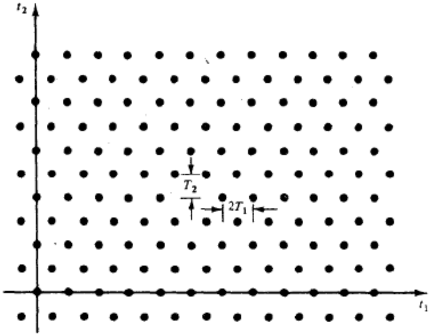


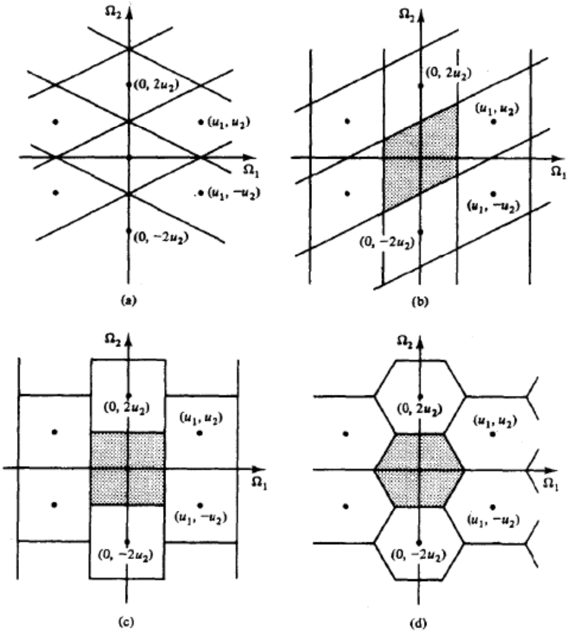
Não há altas frequências verticais e temporais ao mesmo tempo.



Melhor aproveitamento do plano de frequências.

OBS: $\mathbf{V} = \begin{bmatrix} T_1 & T_1 \\ T_2 & -T_2 \end{bmatrix} \qquad \mathbf{U} = \begin{bmatrix} u_1 & u_1 \\ u_2 & -u_2 \end{bmatrix} \Rightarrow$ amostragem hexagonal




Ω_2
Ω_1
$(0, 2u_2)$
(u_1, u_2)
$(u_1, -u_2)$
$(0, -2u_2)$
(a)
(b)
(c)
(d)

Se não há aliasing:

$$X_i(\mathbf{\Omega}) = X(\mathbf{V}^T\mathbf{\Omega}) = \frac{1}{|\det \mathbf{V}|} X_a(\mathbf{\Omega}), \quad \mathbf{\Omega} \in B$$

$$\Rightarrow X_a(\mathbf{\Omega}) = \begin{cases} |\det \mathbf{V}| X(\mathbf{V}^T\mathbf{\Omega}), & \mathbf{\Omega} \in B \\ 0, & \text{n.d.p.} \end{cases}$$

$$\Rightarrow x_a(\mathbf{t}) = \frac{|\det \mathbf{V}|}{4\pi^2} \sum_{\mathbf{n} \in \mathbb{Z}^2} x(\mathbf{n}) \int_B e^{j\mathbf{\Omega}^T(\mathbf{t}-\mathbf{V}\mathbf{n})} d\mathbf{\Omega}$$

$$x_a(\mathbf{t}) = \sum_{\mathbf{n} \in \mathbb{Z}^2} x(\mathbf{n}) f(\mathbf{t}-\mathbf{V}\mathbf{n}), \quad \text{onde} \quad f(\mathbf{t}) = \frac{|\det \mathbf{V}|}{4\pi^2} \int_B e^{j\mathbf{\Omega}^T\mathbf{t}} d\mathbf{\Omega}.$$

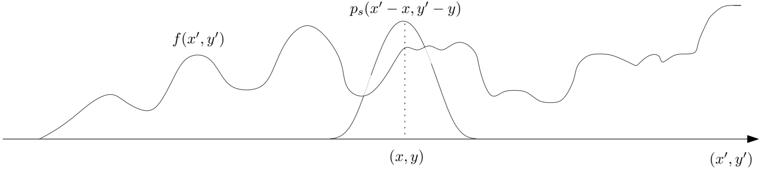
Limitações Práticas na Amostragem e Reconstrução

Abertura de amostragem

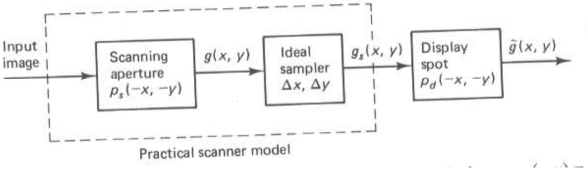
Amostragem prática:

$$g(x,y) = \iint_{\Lambda} p_s(x'-x, y'-y) f(x', y') dx' dy'$$

$$g(x,y) = p_s(x,y) \star f(x,y) = p_s(-x,-y) * f(x,y)$$



$$\Rightarrow \begin{cases} p_s(x,y): & \text{distribuição de luz na abertura} \\ \Lambda: & \text{forma da abertura} \end{cases}$$

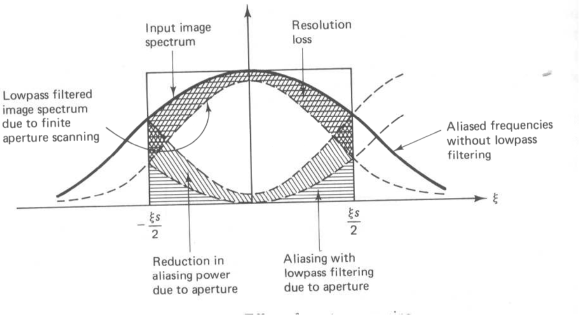


$$g_s(x,y) = g(x,y) \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \delta(x-n\Delta x, y-n\Delta y)$$

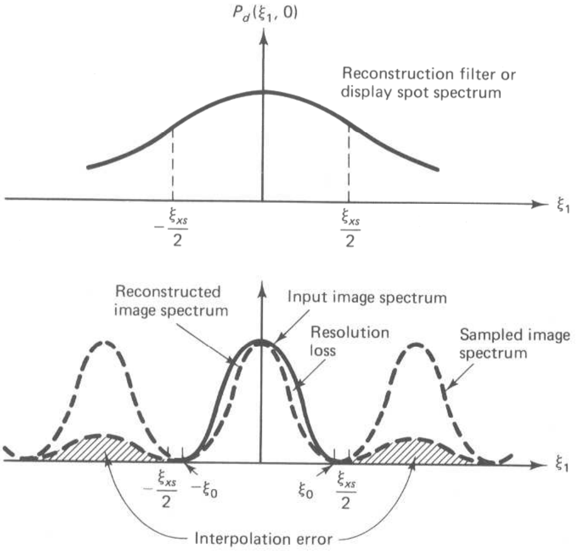
Caso ideal: $p_s(x,y) = p_d(x,y) = \delta(x,y)$

Em geral: $p_s(x,y) \Rightarrow$ resposta ao impulso de um filtro passa-baixa.

$\Rightarrow$ O processo de varredura e amostragem equivale a uma filtragem passa-baixas seguida de amostragem (reduz aliasing).


Input image spectrum
Resolution loss
Lowpass filtered image spectrum due to finite aperture scanning
Aliased frequencies without lowpass filtering
ξ
$-\frac{\xi s}{2}$
$\frac{\xi s}{2}$
Reduction in aliasing power due to aperture
Aliasing with lowpass filtering due to aperture

Na hora do display: o formato do feixe de elétrons equivale a uma interpolação.



⇒ quanto menor for ε_{xs}, menos aliasing.

⇒ Se é muito estreita, perco resoluçao ("borro") na imagem reconstruída

One-dimensional interpolation function	Diagram	Definition $p(x)$	Two-dimensional interpolation function $p_d(x, y) = p(x)p(y)$	Frequency response $P_d(\xi_1, \xi_2)$	$P_d(\xi_1, 0)$
Rectangle (zero-order hold) ZOH $p_o(x)$	$\frac{1}{\Delta x}$, x, $-\frac{\Delta x}{2}$, 0, $\frac{\Delta x}{2}$	$\frac{1}{\Delta x}\,\mathrm{rect}\left(\frac{x}{\Delta x}\right)$	$p_o(x)p_o(y)$	$\mathrm{sinc}\left(\frac{\xi_1}{2\xi_{x0}}\right)\mathrm{sinc}\left(\frac{\xi_2}{2\xi_{y0}}\right)$	1.0, $4\xi_{x0}$
Triangle (first-order hold) FOH $p_1(x)$	$\frac{1}{\Delta x}$, x, $-\Delta x$, Δx	$\frac{1}{\Delta x}\,\mathrm{tri}\left(\frac{x}{\Delta x}\right)$ $p_o(x) \circledast p_o(x)$	$p_1(x)p_1(y)$	$\left[\mathrm{sinc}\left(\frac{\xi_1}{2\xi_{x0}}\right)\mathrm{sinc}\left(\frac{\xi_2}{2\xi_{y0}}\right)\right]^2$	1.0, $4\xi_{x0}$
nth-order hold $n = 2$, quadratic $n = 3$, cubic splines $p_n(x)$	x	$p_o(x) \circledast \cdots \circledast p_o(x)$ n convolutions	$p_n(x)p_n(y)$	$\left[\mathrm{sinc}\left(\frac{\xi_1}{\xi_{x0}}\right)\mathrm{sinc}\left(\frac{\xi_2}{\xi_{y0}}\right)\right]^{n+1}$	1.0, $4\xi_{x0}$
Gaussian $p_g(x)$	2σ, x	$\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left[-\frac{x^2}{2\sigma^2}\right]$	$\frac{1}{2\pi\sigma^2}\exp\left[-\frac{(x^2+y^2)}{2\sigma^2}\right]$	$\exp\left[-2\pi^2\sigma^2(\xi_1^2+\xi_2^2)\right]$	1.0
Sinc	$2\Delta x$	$\frac{1}{\Delta x}\,\mathrm{sinc}\left(\frac{x}{\Delta x}\right)$	$\frac{1}{\Delta x \Delta y}\,\mathrm{sinc}\left(\frac{x}{\Delta x}\right)\mathrm{sinc}\left(\frac{y}{\Delta y}\right)$	$\mathrm{rect}\left(\frac{\xi_1}{2\xi_{x0}}\right)\mathrm{rect}\left(\frac{\xi_2}{2\xi_{y0}}\right)$	1.0, $2\xi_{x0}$

Interpolação Polinomial (Lagrange)

$$L_k^q(x) = \prod_{m=k_0, m\neq k}^{k_1} \left(\frac{x-m}{k-m}\right), \qquad k_0 < k < k_1, \quad q = 2, 3, \cdots$$

$$L_k^1(x) = 1, \qquad \forall k$$

$$k_0 = -\left(\frac{q-1}{2}\right), \qquad k_1 = \left(\frac{q-1}{2}\right), \qquad q \text{ ímpar}$$

$$k_0 = -\left(\frac{q-2}{2}\right), \qquad k_1 = \left(\frac{q}{2}\right), \qquad q \text{ par}$$

$\Rightarrow \mathbf{q-1}$: ordem do interpolador.

Amostras em $f(m\Delta) \Rightarrow$

$$\hat{f}(x) = f(m\Delta + \alpha\Delta) \triangleq \sum_{k=k_0}^{k_1} L_k^q(\alpha) f(m\Delta + k\Delta)$$

$x = m\Delta + \alpha\Delta, \quad m \in \mathbf{Z}, \quad 0 \leq \alpha < 1, \quad q$ par, $\quad -\frac{1}{2} \leq \alpha < \frac{1}{2}, \quad q$ ímpar.

$$\boxed{\begin{array}{l} q = 1 \rightarrow \text{ZOH} \\ q = 2 \rightarrow \text{FOH} \\ \quad\quad \vdots \\ q = \infty \rightarrow \text{sinc} \end{array}}$$