Chapter 4 Alternating Current Sinusoids

Chapter Outline:

Learning Objectives: After completing this chapter, a student should be able to…

- Distinguish alternating current, direct current and "dirty DC"
- Describe an AC sinusoidal signal
- Describe why AC is important for energy-transforming and information-transmitting circuits.
- Identify the meaning of amplitude, phase shift, time shift, radian frequency, cyclic frequency, and DC offset for sinusoidal waveforms
- Define the terms leading, lagging, in-phase, out-of-phase, and a phase reference
- Translate between verbal, graphical, and algebraic representations of sinusoidal functions with or without a DC offset
- Convert a sinusoidal function of time with no DC offset into a complex number, and vice versa
- Add two complex numbers manually and using software tools
- Estimate the magnitude and phase of a complex sum graphically with the tip-to-tail method
- Distinguish average power from instantaneous power for resistive loads
- Plot instantaneous power for voltage and current sinusoids, and distinguish regions of provided and absorbed energy
- Compute the RMS value for a sinusoidal function with no DC offset
- Distinguish peak, peak-to-peak, and RMS amplitudes for sinusoidal waveforms
- Determine AC voltages, currents, and average powers in series-parallel resistive circuits using series-parallel analysis techniques.

4.0 Chapter Preview

Table 4.0 Table of new symbols

Quantity	**Symbol**	**Units**	**How to identify / calculate**
Amplitude	A	Same units as quantity being measured (volt, amp, watt)	Vertical distance from center of a wave to peak (or half peak-to-trough)
Period	T	seconds [s]	Time from one peak to nearest peak (or two adjacent troughs, or two rising zeroes)
Frequency or cyclic frequency	f	hertz [Hz] (cycles per second)	$f = \frac{1}{T}$
Angular or radian frequency	ω (Greek lower-case omega)	radians per second [rad/s]	$\omega = 2\pi f = \frac{2\pi}{T}$
Time Shift	Δt	seconds [s]	Time between 0 and nearest peak (may be positive or negative)
Phase shift	θ ϕ φ (phi)	Radians [rad] or degrees [°]	$\frac{\Delta t}{T} = \frac{\theta}{360°} = \frac{\theta}{2\pi\,[radians]}$ Negative phase shifts have first peak *later* in time than reference.
DC offset	A_0	Same as measured quantity (volt, amp, watt)	Vertical distance between 0 and the center of the wave
Magnitude of	$\vert\ \vert$	-	-
Angle of	$\angle$	-	-

How to identify special phase angles graphically:

- If the peaks of the two waveforms occur at the same time, the waveforms are *in phase*.
- If the peak of one waveform occurs directly above the zero crossing of the other waveform, the two waveforms are $90°$ out of phase.
- If the peaks of one waveform are directly above the troughs of the other waveform, the two waveforms are $180°$ out of phase.

Conversion between a sinusoid function of time and a phasor complex number:

$$v(t) = A\cos(\omega t + \theta) \quad \leftrightarrow \quad \tilde{V} = A\angle\theta$$

To add two sinusoids at the same frequency:

1. If there are a mix of sine and cosine functions, convert all sines to phase shifted cosines by adding $-90°$ from their phases.
2. Convert cosines to complex numbers in polar form.
3. Convert the cosines to rectangular form.
4. Add the complex numbers in rectangular form (real to real, imaginary to imaginary)
5. Convert the resulting complex number to polar form.
6. Convert the polar form result back into a time-domain sinusoid.

Summary Table: Rectangular and polar forms of complex numbers

	Rectangular form (used in software and manual calculations)	Polar form (observed on oscilloscope)
Form	$\tilde{A} = \underbrace{A_r}_{\text{Real part}} + j \underbrace{A_i}_{\text{Imaginary part}}$	$\tilde{A} = \underbrace{A}_{\text{Magnitude}} \angle \underbrace{\theta}_{\text{Phase}}$
Conversions	$\lvert \tilde{A} \rvert = A = \sqrt{A_r^2 + A_i^2}$ $\theta = \arctan\left(\frac{A_i}{A_r}\right)$ (fails for some angles) $\theta = \operatorname{atan2}\left(A_i, A_r\right)$ (works)	$A_r = A\cos(\theta)$ real part $A_i = A\sin(\theta)$ imaginary part

MATLAB commands	`% Create complex number`
	`A=Ar+j*Ai`
	`Amag=abs(A) % get magnitude`
	`phi=rad2deg(angle(A)) % get angle`
		`% create complex number`
		`A=Amag*exp(j*deg2rad(phi))`
		`Ar=real(A) % real part`
		`Ai=imag(A) % imaginary part`

<u>Formulas for RMS values and average AC power</u>

$$V_{\text{RMS}} = \frac{V_{\text{pk}}}{\sqrt{2}} \qquad I_{\text{RMS}} = \frac{I_{\text{pk}}}{\sqrt{2}} \qquad P_{\text{ave}} = \frac{V_{\text{pk}} I_{\text{pk}}}{2} = V_{\text{RMS}} I_{\text{RMS}} \qquad P_{\text{ave}} = \left|\tilde{I}_{\text{RMS}}\right|^2 R \qquad P_{\text{ave}} = \frac{\left|\tilde{V}_{\text{RMS}}\right|^2}{R}$$

4.1 Sinusoidal Expression of AC

Alternating current reverses direction over time

An inverter uses switches to reverse the direction of current from a battery through a motor. See Figures 4.1. For $t < 10$ ms, the red, odd numbered switches are closed. Current flows from left to right in the load. Then in the next 10 ms, the black switches are closed, so current flows the opposite direction through the motor, a negative current. The cycle repeats in the next 20 ms and thereafter. The current as a function of time is shown in Figure 4.2.

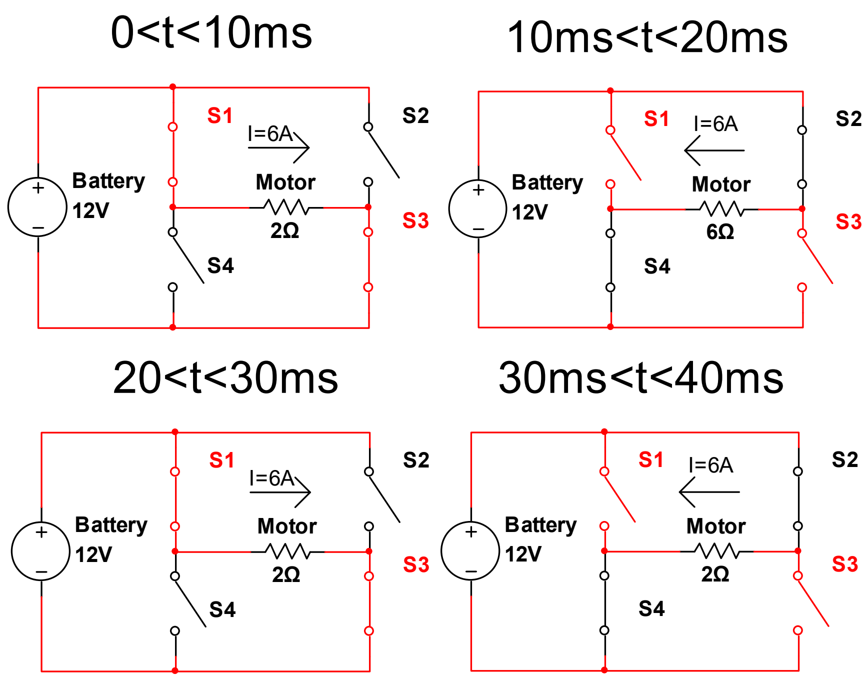


Figure 4.1 Operation of an inverter

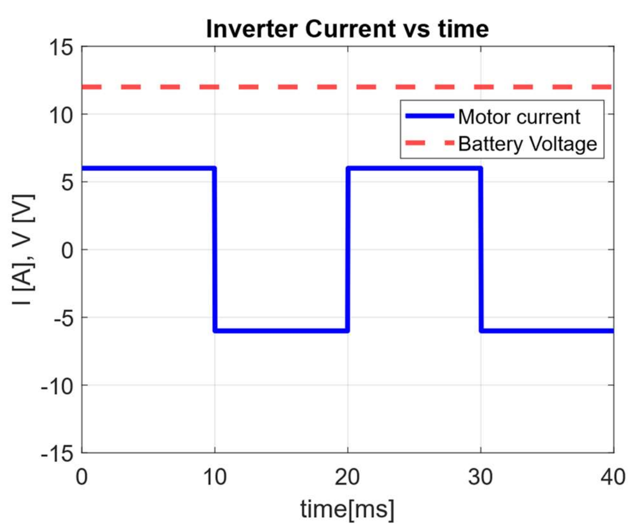


Figure 4.2 The current waveform through an inverter changes sign with time

The battery voltage and current do not change with respect to time, but the motor current does change as a function of time. The current changes direction versus time, that is, it is an *alternating current*. This direction-changing behavior can be created with switches, electronics, or rotating machinery.

The plot of voltage or current versus time is called a *waveform*. An infinite number of possible waveform signals exists, but some are very common in electronics because the signal performs useful electronic tasks. The most common and useful electronic signal is the **alternating current sinusoidal steady-state signal**, which is abbreviated as **AC**. The amplitude of an AC signal varies *sinusoidally* as a function of time. AC is also applicable to voltage too, not just current, and is an industry standard abbreviation. When "AC" is used, assume it is a sinusoidal waveform unless stated otherwise.

Applications of sinusoidal AC span electronics, electrical power, and electronic communications such as:

- FM Radio signals are approximately sinusoidal.
- Rotating machines like motors and generators use or produce sinusoidal AC to operate properly and efficiently.
- Electrical power transmission is more efficient with AC than DC. Power conversion technology is less costly for AC than for DC.
- Mechanical resonators, such as piezoelectric buzzers, use sinusoidal waveforms.
- Multiple sinusoids can be combined to approximate any waveform, such as the sound of a musical instrument.

Sinusoidal alternating current

Direct current has a constant direction and a constant amplitude. Alternating current generally means a pure sinusoidal wave with changing current direction and varying amplitude. A mixture of AC and DC is also common, which we will call "dirty DC" in this text. See Figure 4.3 for a plot of positive and negative DC, AC, and "dirty DC." A few sources for each type are listed in Table 4.1

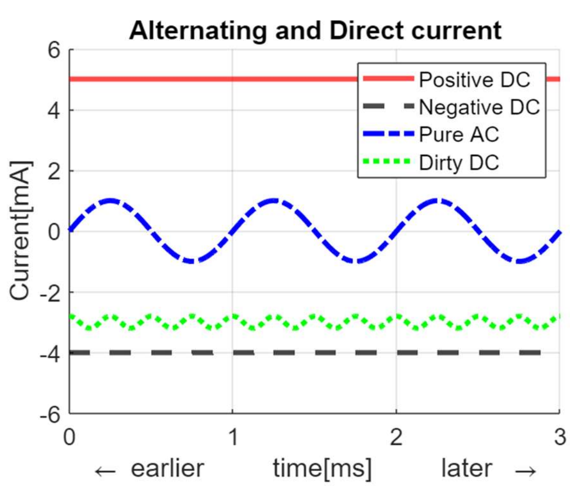


Figure 4.3 Positive DC, negative DC, pure sinusoidal AC and "dirty DC"

Table 4.1 Example sources producing different kinds of power or signal

DC	AC	Dirty DC
Solar panels	Hydropower Generators	Noisy sensors
Batteries	Antennas	Switch-mode power supplies
Thermocouples	Dynamic microphones	Electret microphones

Anatomy of a sine wave: period, frequency, angular frequency, peak amplitude

A sinusoidal function has two properties that are easy to measure: its peak amplitude and its period. See Figure 4.4. AC voltage waveforms are displayed and measured with an AC instrument, the *oscilloscope*.[1]

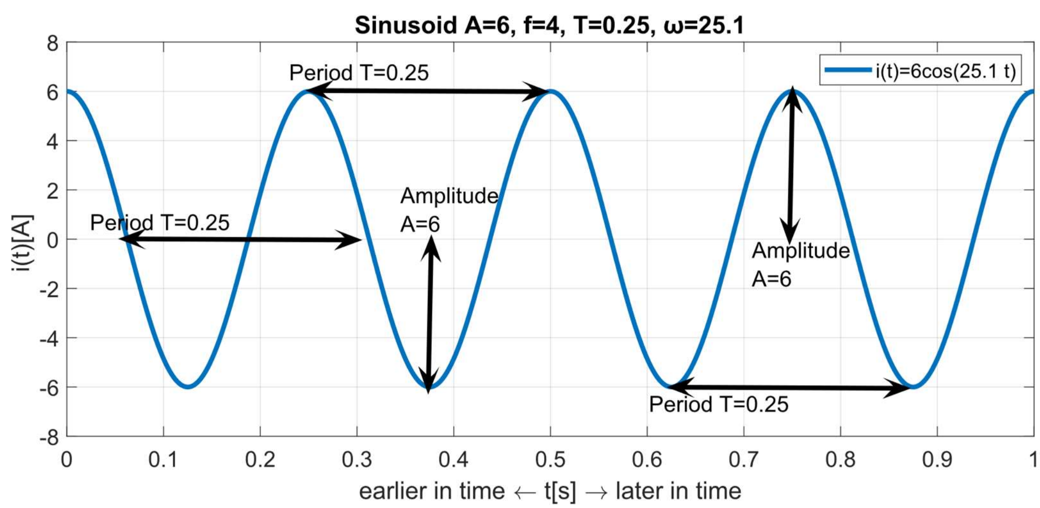


Figure 4.4 A sinusoid with the places to measure period and peak amplitude labeled

Either a sine or a cosine function can be used to express an AC sinusoid. The more common expression uses a cosine, rather than a sine, because its peak value occurs at $t = 0$, often making it more mathematically convenient.

$$i(t) = A\cos(\omega t) \tag{4.1}$$

where: $i(t)$ is the current as a function of time, using a *lowercase* letter for time dependent quantities,

ω (lowercase Greek letter omega) is the **angular frequency** or **radian frequency**, in radians per second (rad/s), and

A is the **peak amplitude** of the sinusoid, which for an AC voltage will be in V or for an AC current in A. This value is reported as “volts peak” or “amps peak”.

[1] AC currents are measured with more difficulty: Shunt sense resistors, amp clamp current transformers, hall effect probes, and fluxgate probes all being common at time of writing.

The radian frequency is a new quantity. Note how the units work:

$$\text{unit of } (\omega t) = \left(\frac{\text{rad}}{\text{s}}\right)(\text{s}) = \text{rad} \tag{4.2}$$

The quantity in parentheses (ωt), which one takes the cosine of, is called the *argument* of the function. The argument of a trigonometric function should always be an angle in radians or degrees. In electricity, the argument of a cosine (or sine) is normally in radians. Thus, multiplication of a time value by ω converts the product into an angle in radians.

A **cycle** of AC is the pattern that repeats every $2\pi\ \text{rad} = 360°$. There is also **cyclic frequency** *f*, often just called frequency, which indicates how many cycles of the sinusoidal waveform occur each second. The unit is hertz (Hz). "Cycles" is not really a unit, so the official unit hertz equals $1/\text{s}$. The time for one cycle is named the **period** *T*, and *f* and *T* are inversely related:

$$\boxed{f = \frac{1}{T}} \tag{4.3}$$

The inverse of the time for one cycle is the number of cycles in one period. The relation to transform the cyclic frequency into the radian frequency converts cycles to radians:

$$\left(\frac{1(\text{cycle})}{T(\text{s})}\right)\left(\frac{2\pi(\text{rad})}{1(\text{cycle})}\right) = \left(\frac{2\pi(\text{rad})}{T(\text{s})}\right) \tag{4.4}$$

Hence, *f* and *T* are related to ω by:

$$\boxed{\omega = \frac{2\pi}{T} = 2\pi f} \tag{4.5}$$

where: ω is the radian frequency (rad/s),

T is the period (s), and

f is the cyclic frequency (Hz $= \text{cycles/second} = \text{s}^{-1}$)

In practice both ω and *f* are called frequency, and you should know which one is being used from the units or context. The radian frequency ω is more mathematically natural, but the cyclic frequency *f* is much easier to measure and is the frequency setting used on most AC instruments.

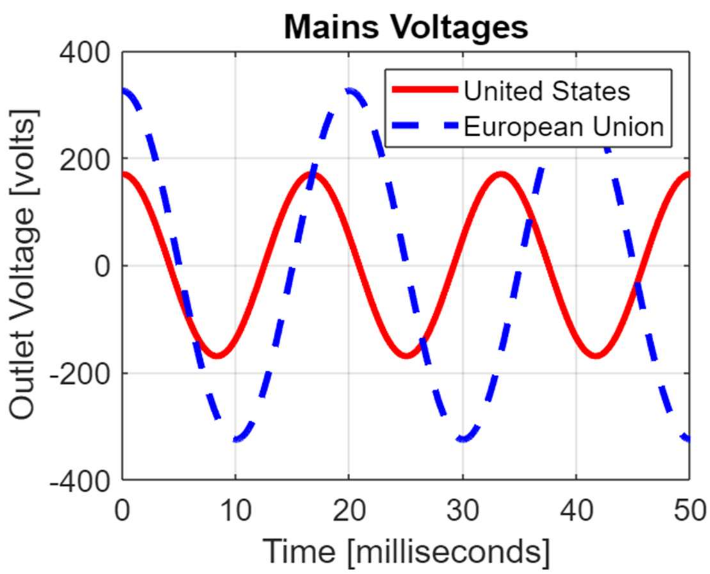
As an analogy, consider buying pipe at a hardware store. It is sold by diameter, not radius, even though radius makes calculations more natural. One can measure diameter accurately with calipers but measuring the radius would require finding the center of a hollow circle. Similarly, frequency is easier to measure accurately and is the customary way to report data. As a general rule of thumb, *calculate in rad/s, report in Hz*. Typical frequency ranges are listed for some devices and applications in Table 4.2

Table 4.2 Typical frequency ranges for different devices

Device or application	Frequency range
Biomedical measurements	100 mHz-50 Hz
Rotating machinery	12 Hz-150 Hz
Audio	20 Hz -20 kHz
Power electronics	20 kHz-1 MHz
Broadcast radio	540 kHz-108 MHz
Wi-Fi	2.4-5 GHz[2]

Example 4.1.1 North American vs. European household voltages

Background: The utility voltages available from a standard power outlet are not the same in different countries. The AC voltage waveforms for a European and an American home outlet are shown in the Figure 4.5.



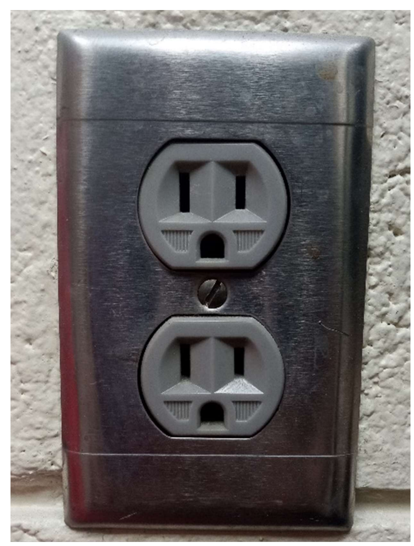


Figure 4.5 Utility voltage waveforms in the EU and US, photo of an American outlet

Problem: Complete Table 4.3 estimating amplitudes and frequencies to within 15% from the AC waveform plots.

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[2] At such high frequencies, the theory of circuits ("lumped elements") is replaced by more sophisticated transmission line or field theories, advanced junior-year topics for electrical engineering majors. To use circuit theory, (circuit length)<<(speed of light)/(cyclic frequency).

Table 4.3 European and American AC waveform parameters

Parameter	EU	US
Voltage amplitude [V_{pk}]		
Period [ms]		
Frequency [Hz]		
Angular frequency [rad/s]		

Solution:

The EU voltage has a peak value (amplitude) around 325 V. The US voltage waveform has a smaller peak value around 170 V.

The EU period is a nice number: exactly 20 ms. The US period is not nice in the metric system; at this graph scale, it appears to be about 17 ms.

This makes the EU frequency $f_{EU} = \dfrac{1}{20\times10^{-3}} = 50\text{ Hz}$ (also used in Africa, Asia, and Australia).

The US power frequency is thus approximately $f_{US} = \dfrac{1}{17\times10^{-3}\text{ s}} = 58.8\text{ Hz} \approx 60\text{ Hz}$ (it is exactly 60 Hz in the Americas, so $T = 1/f = 1/60\text{ s} \approx 16.7\text{ ms}$).

The angular frequency in the EU is $\omega_{EU} = 2\pi f_{EU} = 2\pi(50\text{ Hz}) \approx 314\text{ rad/s}$.

The angular frequency in the US is $\omega_{US} = 2\pi f_{US} = 2\pi(60\text{ Hz}) \approx 377\text{ rad/s}$.

Table 4.3 is completed in Table 4.4. Any engineer should have the numbers for their country permanently memorized, just as any engineer on Earth should have $g = 9.8\text{ m/s}^2$ memorized.

Table 4.4 European and American AC waveform values

Parameter	EU	US
Voltage amplitude [V_{pk}]	325	170
Period [ms]	20	~16.7
Frequency [Hz]	50	60
Angular frequency [rad/s]	314	377

[EoE]

Anatomy of a sine wave: time shift, phase shift, and DC offset

Sinusoidal functions can also be displaced vertically or horizontally. The horizontal shift is a time shift, and the vertical shift is called a “DC offset”. See Figure 4.6.

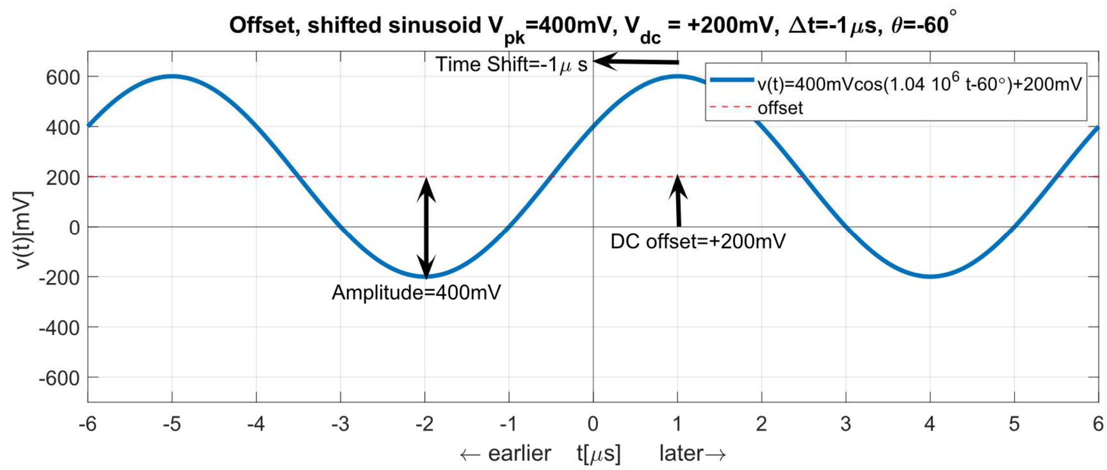


Figure 4.6 A sinusoid with both time shift and DC offset

A more general form for a sinusoid including these two shifts is:

$$v(t) = V_{\text{pk}} \cos(\omega t + \theta) + V_{\text{DC}} \quad \rightarrow \quad v(t) = \underbrace{V_{\text{pk}}}_{\substack{\text{Peak}\\ \text{Amplitude}}} \cos(\underbrace{\omega}_{\substack{\text{Radian}\\ \text{Frequency}}} t + \underbrace{\theta}_{\substack{\text{Phase}\\ \text{Shift}}}) + \underbrace{V_{\text{DC}}}_{\substack{\text{DC}\\ \text{offset}}} \tag{4.6}$$

- The **DC offset** is just the average value of the waveform, halfway between the peaks and troughs. This offset is zero for pure AC.
- The **time shift** Δt the amount of time between the peak of the cosine and the origin $t = 0$. It is easy to measure with an oscilloscope: just record the time of the nearest peak from the origin.
- The **phase shift** θ represents the time shift as an angle and will be used in the argument of the cosine function to shift a sinusoid in time (horizontal shift along the time axis).

Similar to converting a period to an angular frequency, we must convert time shift to an angle. There are two common units for angles: radians (rad) and degrees (°). $360° = 2\pi$ rad. Radians are mathematically natural, but cumbersome for measuring angles. Degrees are preferred for expressing phase shifts. Remember a circle is a circle no matter how it is measured: $1 \text{ cycle} = 2\pi \text{ radians} = 360 \text{ degrees}$. How is θ determined?

$$\boxed{\theta = \omega \Delta t} \qquad \theta[\text{rad}] = \omega\left[\frac{\text{rad}}{\text{s}}\right]\Delta t[\text{s}] \tag{4.7}$$

where: θ is the phase shift (rad),

ω is the radian frequency (rad/s), and

Δt is the time shift of the nearest peak from $t = 0$ (s).

To convert the phase shift into degrees:

$$\theta[°] = \theta[\text{rad}]\left(\frac{360°}{2\pi\,\text{rad}}\right) = \omega\Delta t\left(\frac{360°}{2\pi\,\text{rad}}\right) \qquad \text{units check: } \omega\left[\frac{\text{rad}}{\text{s}}\right]\Delta t[\text{s}]\left(\frac{360°}{2\pi\,\text{rad}}\right) = [°] \tag{4.8}$$

This formula is useful if frequency is known. Another method to determine the phase shift is based on noting that the phase shift is the same fraction of a full cycle that Δt is of a full period T. This form is especially convenient when taking measurements from an oscilloscope. Then θ in degrees can be determined with the following relation:

$$\boxed{\frac{\theta[°]}{360°} = \frac{\Delta t}{T}} \tag{4.9}$$

When calculating phase shifts from a graph, it is often easier to accurately locate zero crossings (where the sinusoid crosses the horizontal axis) rather than peaks. An oscilloscope measures the time shift using these zero crossings.

The last issue is the sign of the phase shift. If θ is positive, then at $t = 0$, the argument of the cosine function already has a value greater than zero. The total phase of the argument at $t = 0$ is already positioned beyond the peak of the cosine wave cycle. Hence, the wave is advanced in time, which means the cosine wave is shifted to the left. The opposite argument holds for negative phase shift. The rules:

A positive phase shift advances the sinusoid in time – the waveform is shifted to the left.

A negative phase shift delays the sinusoid in time – the waveform is shifted to the right.

Example 4.1.2 Sinusoidal waveform expression from a plot

Problem: The sinusoidal signal in Figure 4.7 has rising zero crossings at $t = 1.3\text{ ms}$ and at $t = 18\text{ ms}$.

a) Write the algebraic expression of the voltage as cosine function

$$v(t) = \boxed{}\cos\left(\boxed{}\,t + \boxed{}°\right) + \boxed{}$$

b) Determine the value of the voltage at $t = 14\text{ ms}$, $v(14\text{ ms})$.

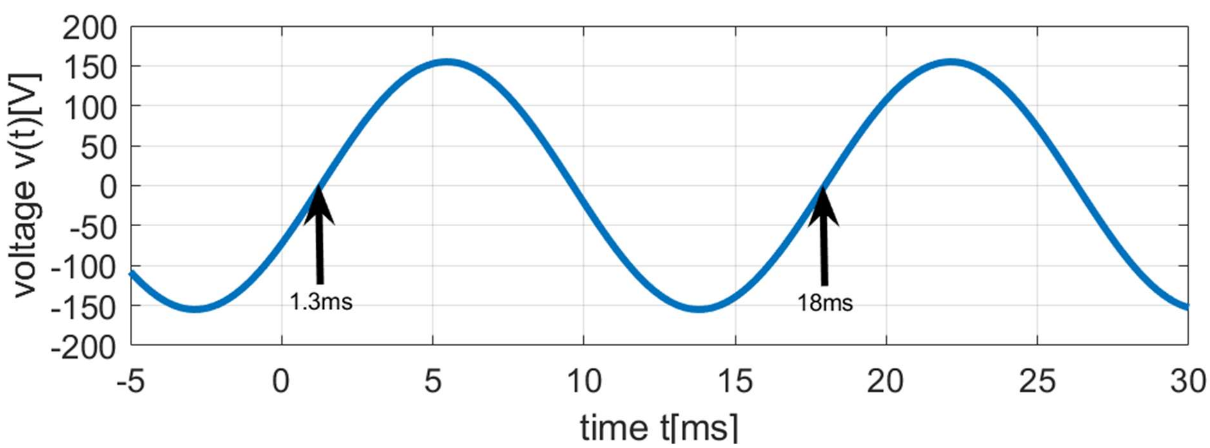


Figure 4.7 A time-shifted sinusoid

Solution:

Strategy: a) Estimate from the plot: V_{pk}, T, Δt, whether advanced or delayed

$f = 1/T$

$\omega = 2\pi f$

$\theta = 360° \Delta t / T$

$v(t) = V_{pk} \cos(\omega t + \theta)$

b) Insert t = 14 ms

a) Estimated from Figure 4.7: $V_{pk} = 155$ V, $T = (18.0 - 1.3) = 16.7$ ms, $V_{DC} = 0$

$\Delta t = 1.3 \text{ ms} + \dfrac{16.7 \text{ ms}}{4} = 5.475 \text{ ms}$ (T/4 was added to 1.3 ms to locate the peak of the cosine wave.)

$f = \dfrac{1}{T} = \dfrac{1}{16.7\,\text{ms}} = 59.9 \approx 60\,\text{Hz}$ (Any frequency this close to 60 Hz probably is 60 Hz.)

$\omega = 2\pi f = 2\pi 60 \text{ Hz} = 377 \text{ rad/s}$

$\theta = \dfrac{360°}{T}\Delta t = \left(\dfrac{360°}{16.7\,\text{ms}}\right)(5.475\,\text{ms}) \approx 118° \,(\text{delayed})$

$v(t) = V_{pk} \cos(\omega t + \theta) \;=\; 155\cos(377t - 118°) \text{ V}$

Qualitative checks:

- The phase shift is to the right (time delay), which corresponds to the negative angle in the argument of the cosine function – checks!
- The phase shift is greater than $90°$, so the cosine crosses the horizontal axis from negative to positive to the right of the origin – checks!

b. Choose degrees or radians for the argument of the cosine: degrees

$v(14\,\text{ms}) = 155 \cos\left[(377)(14 \times 10^{-3}) - 118°\left(\dfrac{2\pi\,\text{rad}}{360°}\right)\right] \approx -155\,\text{V}$

This voltage value is confirmed on the plot at 14 ms (Figure 4.7), the location of a trough.

[EoE]

Phase references and relative phase shifts

The shape of a sine function is identical to a cosine function, just displaced in time as shown in Figure 4.8. A cosine has its peak at the origin, but a sine has its rising zero crossing at the origin, so its peak occurs later in time.

We can rewrite a sine as a phase-shifted cosine. Its peak is delayed in time by exactly one fourth of a full period, a quarter cycle, $90°$ or $\pi/2$ rad. There are four jargon words that describe the phase relationship between two sinusoids: Lagging, leading, in-phase and inverted.

- A time-delayed phase relationship between two sinusoids is called **lagging**. The sine lags the cosine by a quarter cycle $(90°)$. If the peak of one waveform is directly over the zero-crossing of the other, they are $90°$ out of phase.
- A time-advanced phase relationship between two sinusoids is called **leading**. The cosine leads the sine by a quarter cycle $(90°)$. If the peak of one waveform is directly over the zero-crossing of the other, they are $90°$ out of phase.
- If two sinusoids have their peaks at the same time (not necessarily at $t = 0$), they are **in-phase**.
- If the peak of one sinusoid is directly over the trough of another sinusoid, the two waveforms are inverted: $180°$ out-of-phase.

Examples of these phase relationships are illustrated in Figure 4.9. Some useful trigonometric identities for sinusoids are summarized in Table 4.5.

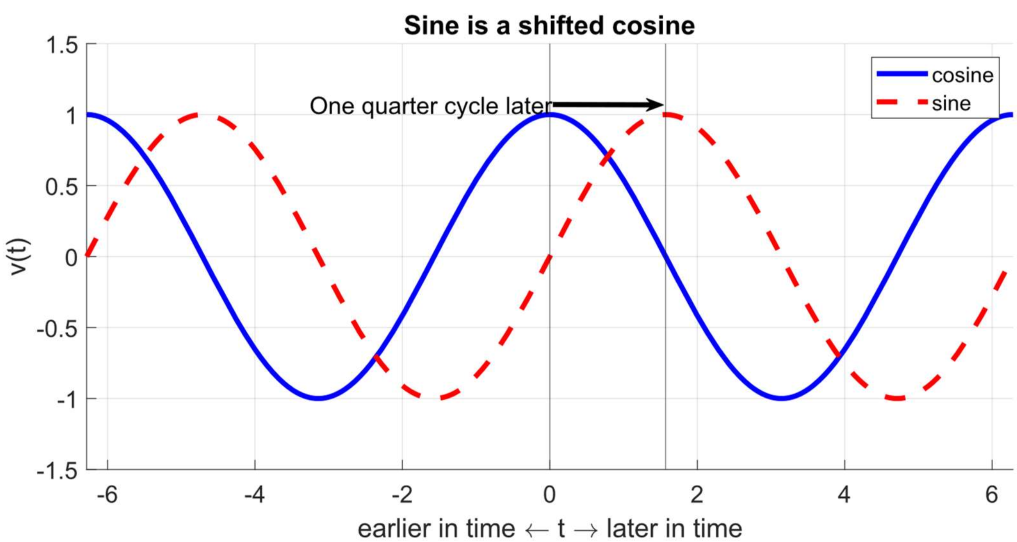


Figure 4.8 Phase relationship between sine and cosine

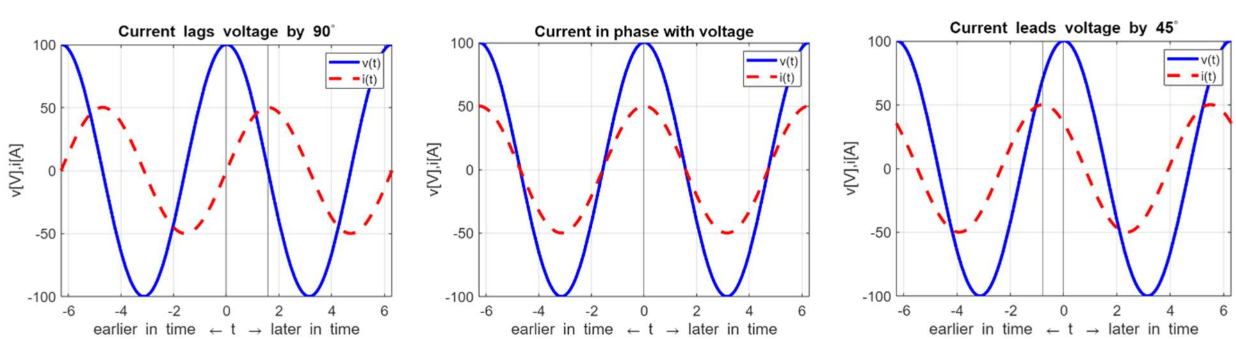


Figure 4.9 Relative peak location and descriptive words for phase shifts

Table 4.5 Trigonometric identities and interpretation in circuits

Identity	**Explanation**
$-\cos(\omega t) = \cos(\omega t + 180°)$	A polarity inversion is equivalent to a half-cycle shift. $\theta = 180° = \pi$ rad (the sinusoids are $180°$ out of phase)
$\cos(\omega t - 180°) = \cos(\omega t + 180°)$	A half-cycle delay and a half-cycle advance are identical.
$\sin(\omega t) = \cos(\omega t - 90°)$	Sine lags cosine by a quarter cycle $\theta = -90° = -\pi/2$ rad
$\cos(\omega t) = \sin(\omega t + 90°)$	Cosine leads sine by a quarter cycle $\theta = +90° = +\pi/2$ rad
$\cos(\omega t + 360°) = \cos(\omega t)$	A full cycle phase shift gives an identical sinusoid.

How to plot sinusoidal functions in MATLAB

Let's try to plot a sinusoidal function with a peak amplitude of 170 V, no DC offset, a frequency of 60 hertz, and a delayed phase shift $30°$ (a twelfth cycle, a small shift to the right, later in time). We must choose a time interval to plot, so let's try 5 seconds as this is the often a default value in software.

```
f=60 % frequency is 60 Hz
w=2*pi*f % angular frequency
phi=-30 % phase shift lagging by 30 degrees
A=170 % amplitude of 170 V
VDC=0 % no DC offset
t=linspace(0,5,1000) % time has 1000 points linearly spaced from 0 to 5 sec
V=A*cos(w*t+phi)+VDC % sinusoid expression
plot(t,V) % Plot voltage against time
xlabel('t[s]')
ylabel('v(t)[V]')
set(gca,'fontsize',14)
title('Badly Scaled plot')
```

The plot in Figure 4.10 displays no useful information because it is not scaled correctly. A 60 Hz oscillation has 300 cycles in the 5 s shown. Each full period is only a few pixels wide. A smaller range of time will improve the graph. Make the time range two to four periods to result in two to four full cycles on the plot. The reader will not know the waveform repeats for at total time interval of fewer than two cycles, and fine features are hard to see with more than four cycles.

Most computational tools do not know the appropriate timescale, so you must determine what time range to enter. Measurement tools such as oscilloscopes do not know the appropriate timescale unless you set it. The auto-scale button on the oscilloscope may just lock onto the campus radio station rather than the frequency of your circuit. The following code has a more reasonable time range.

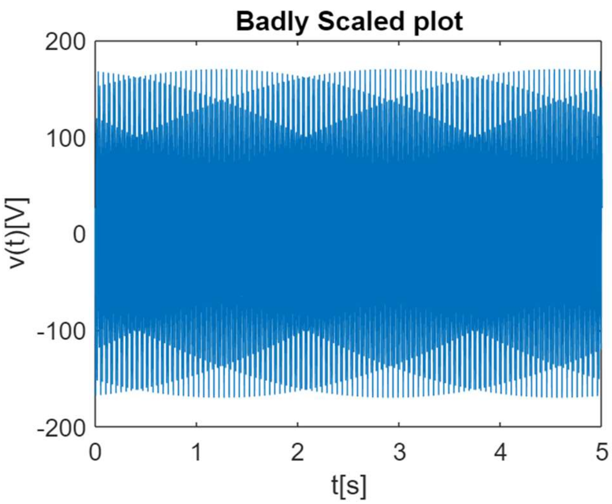


Figure 4.10 Poorly scaled plot of a sinusoid with visible pixel size artifacts[3]

```
f=60 % frequency is 60 Hz
w=2*pi*f % angular frequency
T=1/f % period
phi=-30 % phase shift lagging by 30 degrees
A=170 % amplitude of 170 V
VDC=0 % no DC offset
t=linspace(-T,T,1000) % time has 1000 points linearly spaced from -T to T
V=A*cos(w*t+phi)+VDC % sinusoid expression
plot(t,V) % Plot voltage against time
grid on % turn on grid
xlabel('t[s]')
ylabel('v(t)[V]')
set(gca,'fontsize',14)
title('Forgot about radians')
```

The plot in Figure 4.11 is also wrong. We know $\theta = -30°$ is a small shift to the right (later in time), but the plot has a large shift to the left. Engineers report phase shifts in degrees, but radians are needed for plotting with software. We can multiply phase by $(2\pi/360°)$, or use the `deg2rad()` command in MATLAB to convert the given degrees to radians. While rescaling, it is also more professional to set the horizontal axis to ms, rather than s, for number readability. A well-scaled, highly readable plot results (Figure 4.12).

[3] The shape of these patterns comes from a phenomenon called “aliasing”, an interesting topic in digital systems processing, a future topic of study for many engineering students.

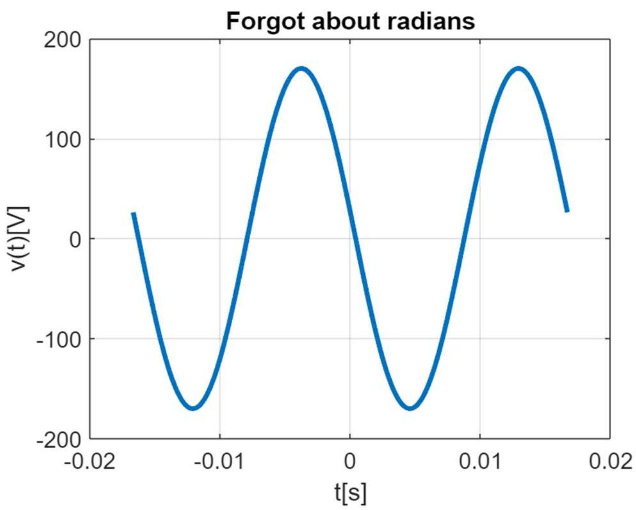


Figure 4.11 Erroneous sinusoid with a large positive phase shift, not a small negative shift

```
f=60 % frequency is 60 Hz
w=2*pi*f % angular frequency
T=1/f % period
phi=deg2rad(-30) % phase shift lagging by 30 degrees, to radians
A=170 % amplitude of 170 V
VDC=0 % no DC offset
t=linspace(-T,T,1000) % time has 1000 points linearly spaced from -T to T
V=A*cos(w*t+phi)+VDC % sinusoid expression
plot(t*1000,V) % plot voltage against time in ms
grid on % turn on grid
xlabel('t[ms]')
ylabel('v(t)[V]')
set(gca,'fontsize',14)
title('Correct plot')
```

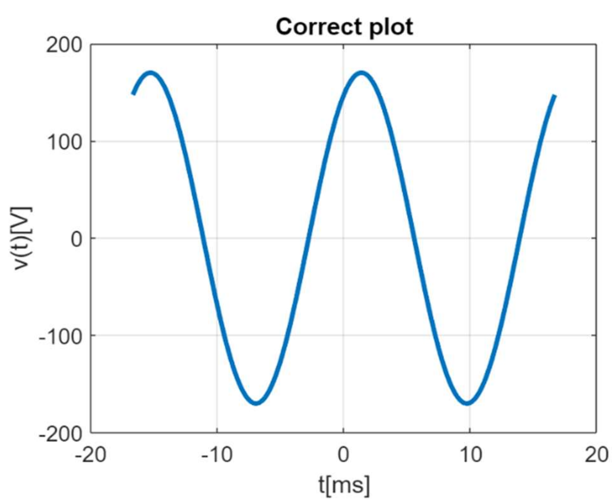


Figure 4.12 Well-scaled plot of the sinusoid

Phase matters (sometimes)

The universe has no special time to set as the origin (except perhaps the Big Bang). If the time $t = 0$ is arbitrary and can be chosen freely, why does phase matter at all? Only phase *differences* matter. Absolute phase is physically meaningless. Only the relative phase of two sinusoids matters:

- One of the sinusoids, often the input, sets the phase reference $(0°)$.
- The phase θ of the other sinusoid is measured relative to this reference.

When does the phase of a result matter? It depends on the context. In energy-converting circuits, usually only the amplitude matters. If a wire is rated for 3 A, it melts just as badly from a current of 5A at an angle of zero as from 5 A lagging by 30 degrees. Engineers must keep the angles *during* calculations but often disregard them at the end of the calculation.

For information-transmitting circuits like amplifiers, radio transmitters, or digital sensors, the phase information is very important and sometimes more important than the amplitude information.[4] Either way, keeping track of phase information during intermediate calculations is essential. Phase can be disregarded at the end of the calculations only if the specific context happens to be insensitive to phase (such as for many power systems and the human ear).

Example 4.1.3 Linear variable differential transformer sinusoid

Background: The Measurement Specialties E-100[5] is a linear variable differential transformer: a position sensor widely used in medical instrumentation (see Figure 4.13). The input voltage is pure AC at 2.5 kHz and 4.2 V (peak). The input AC voltage waveform is the phase reference.

Figure 4.13 A linear variable differential transformer displacement sensor

Problem:

a) Complete the AC expression for the input voltage, including units on all quantities.

$$v_{in}(t) = \boxed{}\cos\left(\boxed{}\, t + \boxed{}°\right) + \boxed{}$$

[4] See Ch. 14 for detailed discussion of phase response.

[5] Device datasheet https://www.te.com/commerce/DocumentDelivery/DDEController?Action=showdoc&DocId=Data+Sheet%7FE-Series%7FA2%7Fpdf%7FEnglish%7FENG_DS_E-Series_A2.pdf%7FCAT-LVDT0025 , seeking permission

b) The output voltage depends on the input voltage and the position being measured by the relation $v_{\text{out}}(t) = xSv_{\text{in}}(t)$, where:

   $v_{\text{out}}(t)$ is the output voltage as a function of time, in V

   $v_{\text{in}}(t)$ is the input voltage as a function of time, in V

   S is the position sensitivity of the sensor, 0.094 V/V-mm (volts per volt per millimeter)

   x is the measured position, in mm (positive to the right, negative to the left)

   Complete the expression for the output voltage if the sensor is displaced 0.1 mm to the left, which is negative displacement for this device (AC reminder: amplitudes are not negative):

   $$v_{\text{out}}(t) = \boxed{}\cos\left(\boxed{}\,t + \boxed{}^\circ\right) + \boxed{}$$

c) Explain why phase-sensitive measurement circuitry is required to tell the difference between displacements to the left and to the right. Circuitry sensitive only to amplitude will not work.

Solution:

a) The input amplitude is given as 4.2 V. The input is pure AC, so the DC offset is zero. The input voltage is the phase reference, so its phase shift is zero. The frequency is given, but angular frequency goes into the cosine function, $\omega = 2\pi(2.5\times10^3\ \text{Hz}) = 15.7\times10^3\ \text{rad/s}$

   $$v_{\text{in}}(t) = (4.2\ \text{V})\cos\left[(15.7\ \text{krad/s})t + 0\right] + 0$$

b) The output is just the product of two fixed numbers and the input. Multiplying a cosine by a number will not change the frequency, only the amplitude.

   $$v_{\text{out}}(t) = (-0.1\ \text{mm})\left(0.094\frac{\text{V}}{\text{V}\cdot\text{mm}}\right)(4.2\ \text{V})\cos\left[15.7\times10^3 t\right]$$

   The peak amplitude of an AC waveform is always positive. We must convert the – sign into a phase shift:

   $$v_{\text{in}}(t) = 39.4\cos\left(15.7\times10^3 t + 180^\circ\right)\ \text{mV}$$

c) The measurement circuitry must measure phase, because the positive or negative sign from the input position is encoded in the phase of the output voltage, not the peaks of the waves. A peak amplitude does not have phase information.

[EoE]

4.2 Complex Numbers and Phasors for AC

Kirchhoff's laws apply to AC voltages and currents

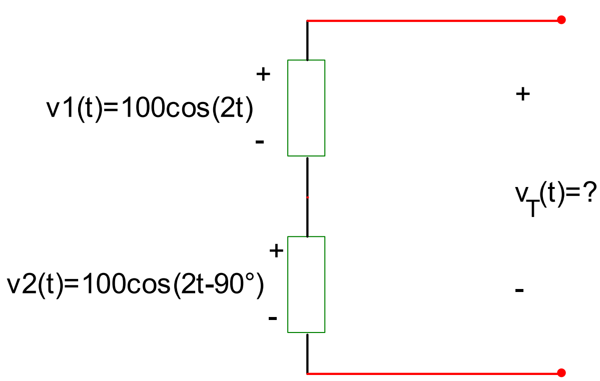
- KVL and KCL are both still valid if the voltages and currents are AC sinusoidal waveforms.[6]
- KVL states that the voltages around any complete path (loop) sum to zero *at every moment in time.*
- KCL states that the currents into any node sum to zero *at every moment in time*.

Thus, for AC circuits, adding or subtracting sinusoids is critical to analysis. First, a major concept:

- **The sum of any two sinusoids at the same frequency is also a sinusoid** *at that same frequency*.
- The peak amplitude and phase shift of the resultant sinusoid differ, in general, relative to the original two sinusoids.

Consider the two generic circuit elements in Figure 4.14 and the sinusoidal voltages across each. The sum of these two voltage waveforms appears at the terminals. What can we determine about the sum of the AC voltages? Does the sum have larger or smaller peak amplitude? Does the resultant sinusoid lead or lag a cosine?

$v_1(t) = 100\cos(2t)$ $\quad$ $v_2(t) = 100\cos(2t - 90^\circ)$ $\quad$ $v_T(t) = v_1(t) + v_2(t) = ?$



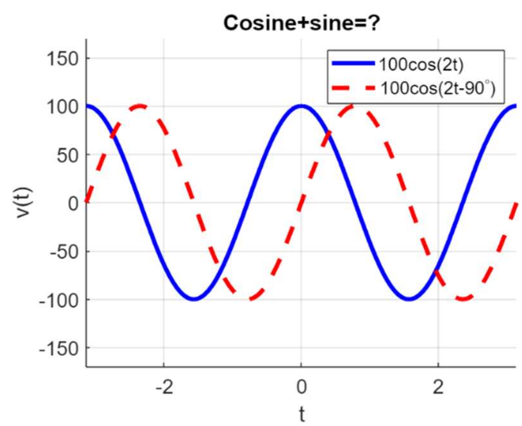


Figure 4.14 A two-element AC circuit with AC waveforms

The total sinusoid can be found graphically by plotting in software. The two original voltages and the total voltage are plotted in Figure 4.15. The peak amplitude of the total sinusoid is larger than either of the two original sinusoids and lags a pure cosine in this case.

[6] Or any other waveform in linear circuit analysis.

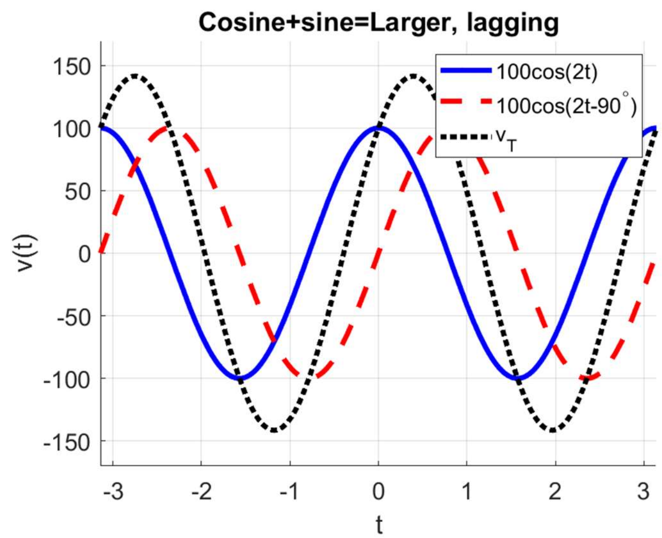


Figure 4.15 The sum of the two sinusoids

The sum of two sinusoids

A natural question may have occurred to you in the previous summation of sinusoids: Can the total sinusoidal voltage be determined analytically? Just *skim* the following analytic development of the sum of two nice sinusoids (same amplitude, nice phase shift between them):

$v_1(t) = 100\cos(2t)$	$v_2(t) = 100\cos(2t - 90°)$	$v_T(t) = v_1(t) + v_2(t) = ?$

$v_T(t) = 100\cos(2t) + 100\cos(2t - 90°)$	Insert sinusoid expressions
$\cos(A) + \cos(B) = 2\cos\left[\tfrac{1}{2}(A+B)\right]\cos\left[\tfrac{1}{2}(A-B)\right]$	Trigonometric identity
$v_T(t) = (100)(2)\cos\left\{\tfrac{1}{2}\left[2t + (2t - 90°)\right]\right\}\cos\left\{\tfrac{1}{2}\left[2t - (2t - 90°)\right]\right\}$	Apply the identity to $v_T(t)$
$v_T(t) = 200\cos\left[\tfrac{1}{2}(4t - 90°)\right]\cos\left[\tfrac{1}{2}(90°)\right]$	Cancel terms
$v_T(t) = 200\cos(2t - 45°)\cos(45°)$	Simplify
$v_T(t) = 200\cos(2t - 45°)\dfrac{1}{\sqrt{2}}$	Evaluate $\cos(45°)$
$v_T(t) = 141\cos(2t - 45°)$	Analytic result

Would you like to perform this math every time you add two sinusoids? The whole point is that determining the total voltage is *not* straightforward using functions of time! (Even with nice numbers.) The problem is harder if the peak amplitudes differ as is the usual case. And what if there are more than two voltages to add? The trigonometric approach is a valid but *inefficient* approach that offers little physical intuition and many opportunities to make mistakes.

So, is there a more efficient, insightful way to add sinusoids? Yes (or we would not have asked the question). *Complex numbers* simplify AC calculations tremendously. This approach is the main topic of this section. We will first visualize complex numbers by comparing them to vectors.

Analogy to vectors

You may have noticed in the last example that the result felt somewhat familiar. The sum of two equal lengths $90°$ apart gave something 1.41 times larger at an angle exactly in-between, that is, at $45°$. This result should feel familiar from your study of physics. This addition is analogous to the addition of mechanical vector quantities, as illustrated in Figure 4.16 and 4.17.

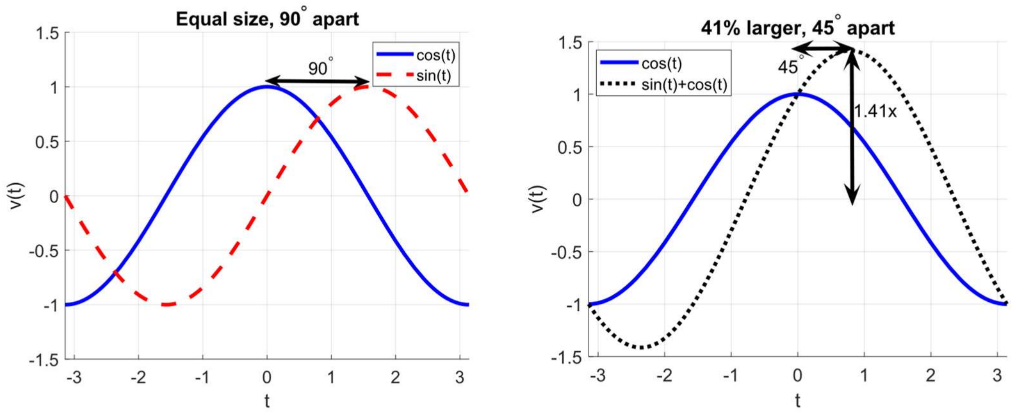


Figure 4.16 Addition of two equal amplitude sinudoids displaced 90 degrees

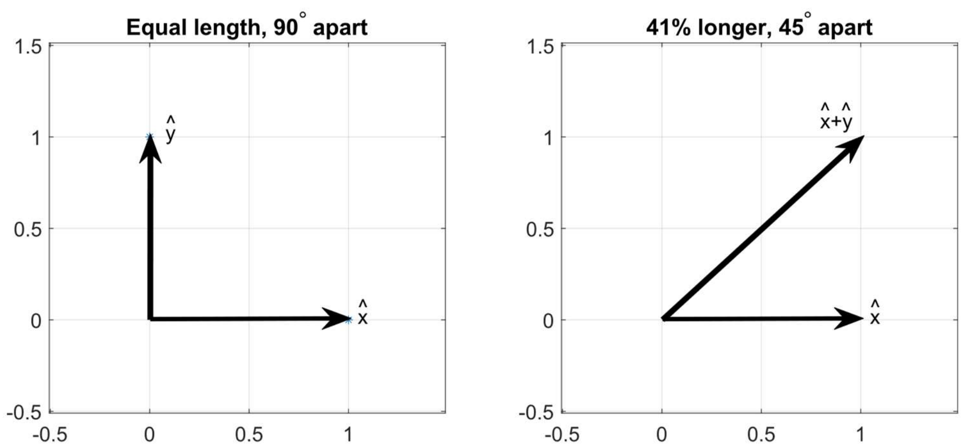


Figure 4.17 Addition of two equal mechanical length vectors $\hat{x}$ and $\hat{y}$ **displaced 90 degrees**

To aid the process of adding sinusoids, they are drawn as arrows called *phasors* (*phase vectors*) as shown in Figure 4.18. Phasors are *not* vectors![7] They just *add* in the same way as vectors. Phasors are actually *compex numbers*, and their angles are phase shifts in time, not in space.

[7] Technically the complex numbers do form a mathematical vector space in linear algebra. Complex numbers are also equipped with many operations that mechanical vectors do not have, such as a well-defined division operation.

A **phasor** is a complex number that represents an AC sinusoidal steady-state voltage or current.

The phasor contains two key pieces of information about the sinusoidal signal (frequency must be reported separately). The magnitude of the complex number is the AC peak value. The angle of the complex number is the phase shift. Notice how the phasor diagram in Figure 4.18 shows the relative phases of the signals.

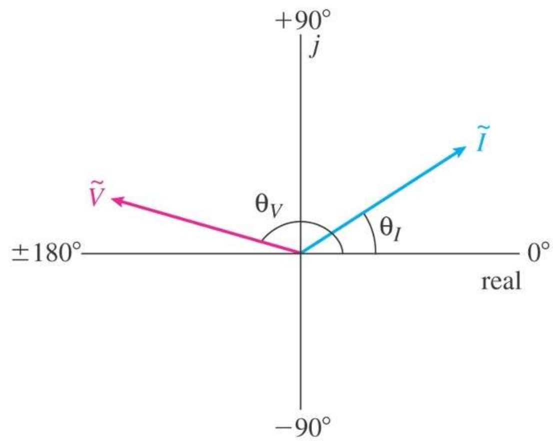


Figure 4.18 Phasor Voltage and Current Plotted on the Complex Number Plane

Complex numbers

The sinusoidal function of time $v(t)$ is represented as a phasor by the complex number $\tilde{V}$. The tilde over the capital V indicates its complex-number nature.

$$\overbrace{v(t) = \underbrace{170}_{\text{magnitude}} \cos(377t \underbrace{-30°}_{\text{phase}})}^{\text{Sinusoidal function of time}} \quad \leftrightarrow \quad \overbrace{\tilde{V} = \underbrace{170}_{\text{magnitude}} \angle \underbrace{-30°}_{\text{phase}}}^{\text{Phasor complex number}} \tag{4.10}$$

The information about the frequency of the oscillation is not shown in a phasor, but adding sinusoids does not change their frequency. This magnitude and phase representation of a complex number is called **polar form**. Adding phasors in polar form can be estimated graphically: they add tip-to-tail, just as mechanical vectors do. To perform this operation using software, we often must represent the sinusoid as a complex number with real and imaginary parts, that is, in **rectangular form** (just as mechanical vectors can be decomposed into cartesian components).

Notice that the variable of time does *not* occur in a phasor. It is assumed that when phasors are being used, the time dependence is sinusoidal. Also, when using phasors in a given problem, *all signals must be at the same frequency.* Otherwise, the phase relationship between two AC signals does not remain constant as time goes on. A slogan is useful to learn what a phasor is:

> *A phasor is a complex number representation of an AC sinusoidal steady-state signal with the sinusoidal time dependence removed but assumed.*

A complex number has two pieces of information: a real part and an imaginary part or a magnitude and a phase. In *rectangular form*:

$$\text{complex number} = (\text{real part}) + j(\text{imaginary part}) \quad \rightarrow \quad \tilde{A} = A_r + jA_i \tag{4.11}$$

where: $\tilde{A}$ is a complex number, and the tilde indicates that the number is complex,

A_r is the real part of the complex number,

A_i is the imaginary part of the complex number, and

j indicates the imaginary unit.

In mathematics, *i* is used to indicate the imaginary unit, but *i* stands for current in electronics, so *j* has been adopted in electrical engineering for this indicator. Hence, complex numbers are "two-dimensional" numbers and contain two pieces of information. See the plot in the *complex number plane* shown in Figure 4.19.

What is the imaginary part of a complex number?

- The imaginary part is the part of the complex number that is plotted along or parallel to the imaginary axis in the complex number plane.
- Equivalently, the imaginary part is plotted ±90° with respect to the real part in the complex number plane.

Mathematically, *j* is the square root of –1, but that has no meaning (yet). Thus, in a complex number, *j* indicates the imaginary part and means that part is ±90° with respect to the real part.

What is the *polar form* of a complex number?

$$\text{complex number} = (\text{magnitude}) \text{ at } (\text{an angle}) \quad \rightarrow \quad \tilde{A} = |\tilde{A}| \angle\theta \tag{4.11}$$

where: $\tilde{A}$ is a complex number, and the tilde indicates that the number is complex,

A or $|\tilde{A}|$ is the magnitude of the complex number, and

θ is the phase shift (or phase, or angle, or phase angle) of the complex number.

The polar form of the complex number is also plotted in Figure 4.19 along with the rectangular form. Again, why are two *dimensions* needed in AC circuits? The answer is that there are two pieces of information in an AC variable: a *magnitude* and a *phase.* Thus, an AC signal can be represented by one complex number. Both pieces of information must be retained within the complex number, either as a magnitude and a phase *or* as a real part and an imaginary part (just as a mechanical vector has a length and direction or has horizontal and vertical components.)

The notational forms for complex numbers are

$$\boxed{\tilde{A} = A_r + jA_i = A\angle\theta = |\tilde{A}|\angle\theta} \tag{4.12}$$

The "*A*" without a tilde is often used to indicate the magnitude of the complex number. The generic phasor $\tilde{A}$ could be a voltage $\tilde{V}$ or a current $\tilde{I}$. The Greek letters ϕ or φ (phi) are sometimes used for the angle instead of θ. They all are the same thing: the angle of a complex number.

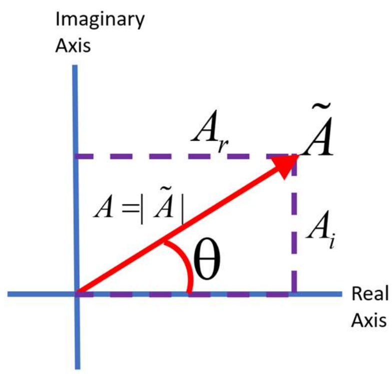


Figure 4.19 Complex number $\tilde{A}$ in rectangular form and polar form

The plot in Figure 4.19 leads one to ask how to *convert* between the rectangular and polar forms of complex numbers.

- To convert from **rectangular to polar** form, apply the Pythagorean theorem to Figure 4.19 The magnitude of the complex number is:

$$\left|\tilde{A}\right| = \sqrt{A_r^{\,2} + A_i^{\,2}} \tag{4.13}$$

Apply trigonometry to Figure 4.19 to determine the phase of the complex number:

$$\theta = \tan^{-1}\left(\frac{A_i}{A_r}\right) = \arctan\left(\frac{A_i}{A_r}\right) \tag{4.14}$$

- To convert from **polar to rectangular** form, again, apply trigonometry to Figure 4.19:

$$A_r = |\tilde{A}|\cos\theta \tag{4.15}$$

$$A_i = |\tilde{A}|\sin\theta \tag{4.16}$$

Modern mathematical tools perform these conversions with one command. Complex expressions with variables instead of numbers can be manipulated with symbolic algebra software such as the MATLAB symbolic toolbox, but manual manipulation using the conversion formulas may provide superior physical insight especially in derivations.

One can estimate the magnitude and angle of a complex number by using the following two guidelines:

- The magnitude of a complex number is always larger than the real part and larger than the imaginary part.
  - If the two parts are close to equal, the magnitude is up to 41% larger.
  - If the real and imaginary parts are very different in size, the magnitude is just slightly larger than the larger of the two components.
- Whichever rectangular component is larger, the phasor is closer to that axis.
  - If the imaginary part is larger than the real part, the angle will be steeper than $45°$.

- If the real part is larger than the imaginary part, the angle is shallower than $45°$.
- If the real and imaginary parts are about the same, the angle is about $45°$.

Example 4.2.1 Convert a polar voltage to rectangular form manually

Background: An AC voltage is measured in the lab as a sinusoidal function on an oscilloscope, $v(t) = 170\cos(377t - 30°)$. This quantity is easily converted to polar form $\tilde{V} = 170\angle -30°$ but rectangular form is needed for calculations in software.

Problem:

a) Sketch $\tilde{V}$ as a phasor in the complex plane.

b) Which is larger, the real part or the imaginary part of $\tilde{V}$?

c) Is the imaginary part of $\tilde{V}$ positive or negative?

d) Convert $\tilde{V}$ into rectangular form manually and verify that the numerical result agrees with the diagram.

Solution:

a) The phasor with length 170 and angle $30°$ below the real axis is plotted in Figure 4.20.

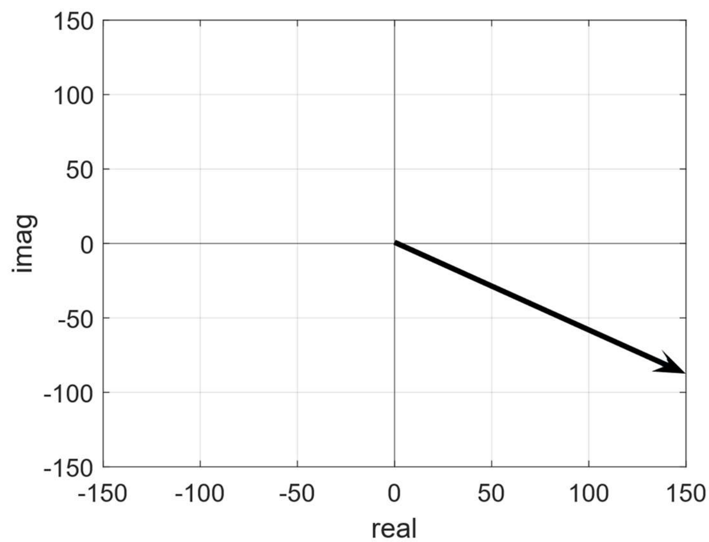


Figure 4.20 The polar voltage phasor in the complex plane

b) The real part is larger than the imaginary part. The real part is around 150, the imaginary part is less than 100. Because the angle is less than $45°$, the real part is dominant.

c) The imaginary part is negative, so the angle is also negative, and the phasor is plotted in the bottom half plane of the complex plane.

d) Apply the conversion formulas from polar to rectangular form:

$$V_r = |\tilde{V}|\cos\theta = 170\cos(-30°) = 147.2$$

$V_i = |\tilde{V}| \sin\theta = 170\sin(-30°) = -85.0$

$\tilde{V} = 147 - j85 \text{ V}$

As expected from the sketch, the real part is larger and the imaginary part is negative. These simple checks catch many obviously absurd calculation errors. Perform checks every time you perform a complex number calculation.

Comment: Rectangular form may be used for calculations and plots, but $\tilde{V}$ and $\tilde{I}$ are always reported in polar form.

[EoE]

Example 4.2.2 Convert this rectangular current to polar form manually

Background: Voltages and currents in rectangular form are not physically meaningful. They must be converted into polar form to compare with experimental results. A calculation that has given an answer of $\tilde{I} = -9 + j6 \text{ A}$ must be converted into polar form to make meaningful predictions. Report phasor voltages and currents in polar form!

Problem:

a) Sketch $\tilde{I}$ in the complex plane.

b) Is the magnitude $|\tilde{I}|$ less than 6, greater than 9, or between 6 and 9?

c) Is the phase angle $\angle\tilde{I}$ greater than $45°$ or less than $45°$?

d) Convert $\tilde{I}$ into polar form manually and verify that it agrees with your diagram.

Solution: a) The phasor is plotted in Figure 4.21: nine left, six up.

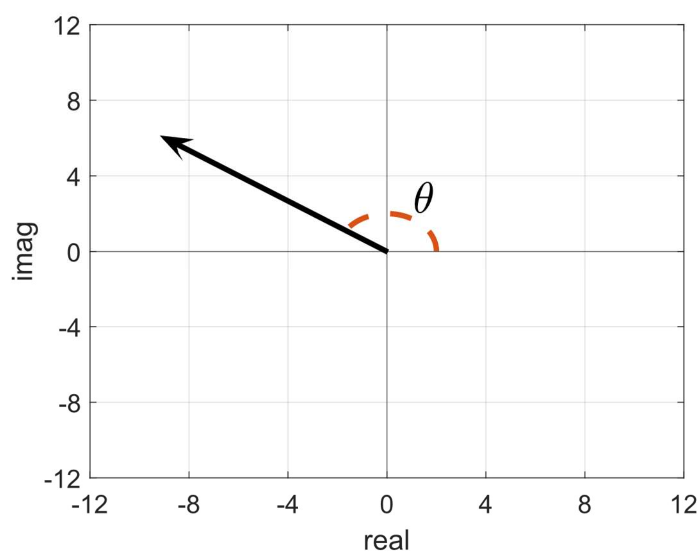


Figure 4.21 The phasor plotted in the complex plane.

b) $|\tilde{I}| > 9$. The magnitude of a complex number is always *larger* than the greater of the two parts, which is apparent in the figure: the hypotenuse is the longest side of the triangle.

c) $\angle\tilde{I} > 45°$. The angle exceeds 135 degrees, far into the second quadrant.

d) $|\tilde{I}| = \sqrt{I_r^2 + I_i^2}, \quad \theta = \tan^{-1}\left(\frac{I_i}{I_r}\right)$

$$\left|\tilde{I}\right| = \sqrt{(-9)^2 + 6^2} = 10.817$$

$$\theta = \tan^{-1}\left(\frac{I_i}{I_r}\right) = \tan^{-1}\left(\frac{6}{-9}\right) = -33.69° \approx -33.7°$$

This angle is *smaller* than $45°$ and negative: it is unreasonable. The signs of the numerator and the denominator are important so that the proper quadrant of the angle can be determined. The arctangent function is not single-valued (recall that two angles can give the same tangent value). Instead use the atan2() or angle() function on your calculator to prevent this error. The correct arctangent angle is shifted $180°$ from $-33.7°$, that is, $146.3°$.

$$\tilde{I} = 10.8\angle 146.3° \text{ A}$$

This current phasor has the expected angle greater than $45°$ and magnitude greater than 9.

[EoE]

Complex number addition

How does one perform addition and subtraction with complex numbers? The following complex number operations illustrate how complex numbers are evaluated in addition and subtraction if the calculations are performed manually.[8] However, a scientific calculator or mathematical software will give the result no matter whether the numbers are entered in rectangular or polar form.

To add and subtract complex numbers, use rectangular form and add or subtract the corresponding real parts and imaginary parts. For two generic complex numbers $\tilde{A} = A\angle\theta = A_r + jA_i$ and $\tilde{B} = B\angle\phi = B_r + jB_i$, the addition operation is defined:

$$\tilde{A} \pm \tilde{B} = \left(A_r + jA_i\right) \pm \left(B_r + jB_i\right) = \left(A_r \pm B_r\right) + j\left(A_i \pm B_i\right) \tag{4.17}$$

For example, $(5 - j3) - (2 + j8) = (5 - 2) + (-j3 - j8) = 3 - j11$.

MATLAB Complex number addition and form conversion

MATLAB performs internal calculations in rectangular form. The software also performs the trigonometry of polar-rectangular form conversions. We take measurements in polar form, do

[8] Complex number multiplication and division will be addressed in subsequent chapters

manual computations in rectangular form, and report data in polar form. The following MATLAB commands perform simple complex number computations[9].

```
% declaring and adding complex numbers
z=2+2i % i as the imaginary unit
w=1-3j % complex number using j also works
z+w % add and subtract as normal math operations

% converting from rectangular to polar form
magnitude=abs(z) % Use abs() takes magnitude
theta_in_radians=angle(z) % angle() takes phase angle in radians
% Use rad2deg() to convert to an angle in radians to degrees
theta_in_degrees=rad2deg(theta_in_radians) % convert radians to degrees
% converting from polar to rectangular form
A=170 % magnitude
phi_in_degrees=-30 % phase
phi_in_rad=deg2rad(phi_in_degrees) % convert degrees to radians
V=A*exp(1j*deg2rad(phi_in_degrees)) % 1∠x°=exp(jx)=e^jx=cos(x)+j sin(x)
```

This code produces the following output:

```
z =
     2.0000e+000 +  2.0000e+000i
w =
     1.0000e+000 -  3.0000e+000i
ans =
     3.0000e+000 -  1.0000e+000i
magnitude =
     2.8284e+000
theta_in_radians =
   785.3982e-003
theta_in_degrees =
    45.0000e+000
A =
   170.0000e+000
phi_in_degrees =
   -30.0000e+000
phi_in_rad =
  -523.5988e-003
V =
   147.2243e+000 - 85.0000e+000i
```

The last line of code shows how to obtain rectangular form. However, rectangular form is not acceptable as a *final* result for phasor voltages and currents – always report them in polar form!

---

[9] A script in the appendix can automate these processes more. MATLAB does not handle the polar form complex numbers with angles in degrees elegantly.

Adding phasors in polar form

It is more commonly needed to add sinusoids in polar form since the waveforms are measured in polar form on instruments. Consider the operation $\tilde{V} = 325\angle 0° - 325\angle 120°$. Graphically adding these phasors tip-to-tail provides an estimate in Figure 4.22. The subtraction of a phasor is the same as adding one in the opposite direction (polarity inversion = $\pm 180°$ phase shift). So $\tilde{V} = 325\angle 0° - 325\angle 120° = 325\angle 0° + 325\angle -60°$. The resultant phasor is clearly both longer than 325 and has a negative phase angle.

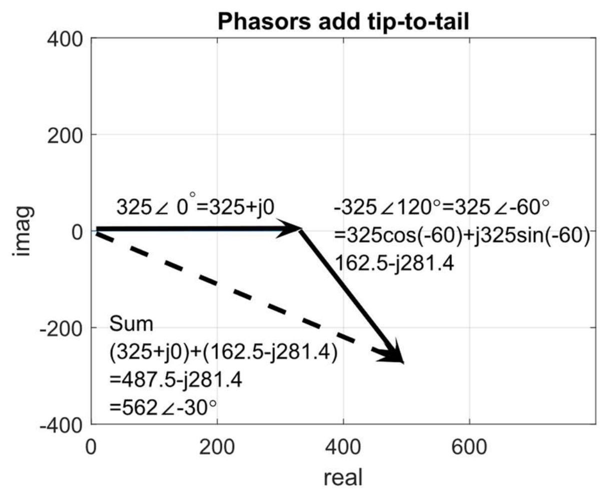


Figure 4.22 Addition of two phasors graphically and algebraically

To perform this subtraction in MATLAB:

```
V1=325 % first phasor, phase angle zero implied
V2=325*exp(j*deg2rad(120)) % second phasor using exp(j*…) for the angle
V=V1-V2 % difference of two phasors
magnitude=abs(V) % magnitude of the result
angle=rad2deg(angle(V)) % phase of the result in deg
```

```
V1 =
   325.0000e+000
V2 =
  -162.5000e+000 +281.4583e+000i
V =
   487.5000e+000 -281.4583e+000i
magnitude =
   562.9165e+000
angle =
   -30.0000e+000
```

The total phasor voltage is $\tilde{V} = 563\angle -30°$ V. Figure 4.23 illustrates the advantages of using complex number calculations over the trigonometric identity approach for AC.

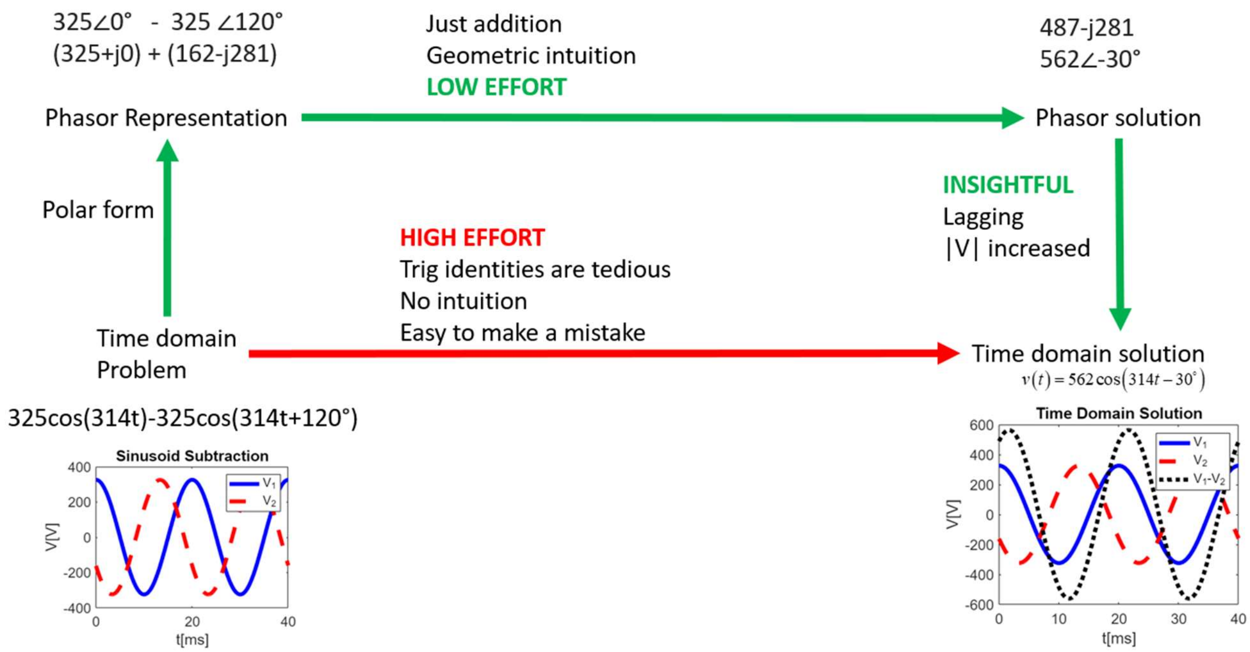


Figure 4.23 Benefits of the complex number approach for sinusoid addition

To manually add sinusoids, follow these steps

1. Ensure the sinusoids are at the same frequency. If they are not, they cannot be added with phasors[10].
2. Convert all sines to cosines by adding a $-90°$ phase shift. Convert any subtracted sinusoids to addition by adding a $\pm 180°$ phase shift.
3. Convert each cosine function in polar form into a phasor in polar form: set the amplitude as the magnitude and the phase as the angle.
4. Convert each polar form phasor into a rectangular form phasor using trigonometry
5. Add the rectangular forms: real adds to real, imaginary adds to imaginary
6. Convert the rectangular form sum into a polar form with trigonometry
7. Convert the phasor sum in polar form back into a cosine by inserting the magnitude in front of the cosine, the angular frequency next to the 't', and the angle into the cosine. The frequency is the same as the original sinusoids.

Example 4.2.3 Adding sinusoids via complex numbers

Background: Consider two voltage sinusoids $v_1(t) = 6\cos(300t + 30°)$ and $v_2(t) = 10\cos(300t - 45°)$. Computing the sum of these two sinusoids $v_T(t) = v_1(t) + v_2(t)$ would be awkward with trigonometric identities but is straightforward with phasors.

Problem:

[10] See Section 10.4 for circuits with multiple angular frequencies.

a) Convert the sinusoids into their phasor forms $\tilde{V}_1$ and $\tilde{V}_2$ in polar form.
b) Convert $\tilde{V}_1$ and $\tilde{V}_2$ into rectangular form
c) Add the two phasors into the total $\tilde{V}_T = \tilde{V}_1 + \tilde{V}_2$ in rectangular form
d) Convert $\tilde{V}_T$ into polar form
e) Write the time domain total voltage as a cosine

$$v_T(t) = \boxed{} \cos\left(\boxed{}\, t + \boxed{}^\circ\right)$$

Solution:

a) To convert to a phasor, take the number in front of the cosine as the magnitude, and the phase as the angle. The number attached to the 't' is the angular frequency. Reserve this number to the side, it is not part of the phasor representation, but will be needed later.

$\tilde{V}_1 = 6\angle 30^\circ \qquad \tilde{V}_2 = 10\angle -45^\circ$ — Phasors in polar form

b) $\tilde{V}_1 = 6\cos(30^\circ) + j6\sin(30^\circ) = 5.1962 + j3.0$ — Rectangular formula

$\tilde{V}_2 = 10\cos(-45^\circ) + j10\cos(-45^\circ) = 7.0711 - j7.0711$

c) $\tilde{V}_T = \tilde{V}_1 + \tilde{V}_2 = (5.1962 + j3.0) + (7.0711 - j7.0711)$ — Add rectangular forms

$= (5.1962 + 7.0711) + j(3.0 - 7.0711) = 12.2672 - j4.0711$ — Add corresponding parts

d) $|\tilde{V}_T| = \sqrt{12.2672^2 + (-4.0711)^2} = 12.9251$ — Calculate magnitude

$\theta_T = \arctan\left(\frac{-4.0711}{12.2672}\right) = -18.35^\circ$ — Calculate phase

e) $v_T(t) = 12.9251\cos(300t - 18.35^\circ)$ — Insert values

The intermediate calculations will generally not be shown in examples later in the text.

[EoE]

Adding sinusoids using software and calculators

This calculation is straightforward but tedious. Modern engineering calculators and mathematical software automate the trigonometry. This calculation can be done in just a couple of lines of code using the "phasor" function in Appendix 4.A.1 in MATLAB, or similarly on most engineering calculators.

```
V1=phasor(6,30)   % convert V1 to rectangular
V2=phasor(10,-45) % convert V2 to rectangular
Vt=V1+V2          % add rectangular forms
phasor(Vt)        % convert Vt to polar
```

```
V1 = 5.1962 + 3.0000i
V2 = 7.0711 - 7.0711i
Vt = 12.2672 - 4.0711i
```

```
ans = 1×2
   12.9251  -18.3592
```

[EoE]

This use of complex numbers (phasors) allows addition of sinusoidal voltages and currents to implement Kirchhoff's laws in AC circuits. In the phasor domain, Kirchhoff's laws are:

KVL: $\sum \tilde{V} = 0 + j0$ The sum of the phasor voltages around any loop has zero real and imaginary parts, or zero magnitude. (4.18)

KCL: $\sum \tilde{I} = 0 + j0$ The sum of the phasor currents into any node has zero real and zero imaginary parts, or zero magnitude. (4.19)

The sums must be computed with the voltages and currents as phasors. The magnitudes of the numbers do *not* sum the way the phasors do, as demonstrated by the following example.

Example 4.2.4 AC magnitudes do not add directly (in general)

Background: A circuit breaker trips to protect the wires in the walls of homes from melting when the current magnitude exceeds 15 A. The current in the motor of a typical box fan lags the voltage considerably, but the heating element in a tea kettle is resistive, so the kettle voltage and current are in-phase. See Figure 4.24.

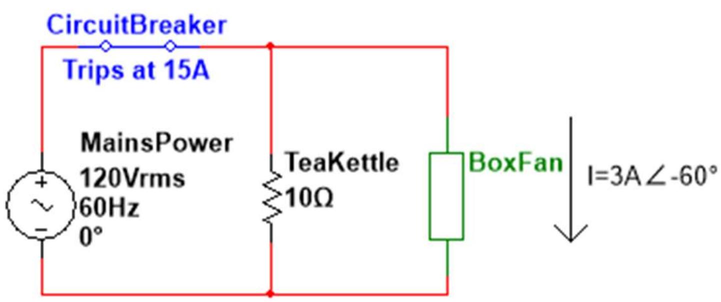


Figure 4.24 A household circuit with two loads, photo of box fan

Problem:

a) Show that the circuit breaker will *not* trip despite the sum of the magnitudes of the two appliance currents totaling 15 A.

b) Explain why using words or a phasor diagram.

Solution:

a) To determine the breaker current, we must apply KCL to the AC currents. The current that flows through the circuit breaker is the sum of the two appliance currents:

$$\tilde{I}_{\text{breaker}} = \tilde{I}_{\text{kettle}} + \tilde{I}_{\text{fan}}$$

The current that flows down through the kettle will be sinusoidal and obey Ohm's law

$$\tilde{I}_{\text{kettle}} = \frac{120\angle 0°\ \text{V}}{10\ \Omega} = 12\angle 0°\ \text{A}$$

$$\tilde{I}_{\text{breaker}} = 12\angle 0° + 3\angle -60° \approx 13.2\angle -19°\ \text{A}$$

which is less than 15 A, despite 12 + 3 = 15.

b) The magnitude must be less than the sum of their amplitudes because they are not in-phase, and do not add like real numbers. Only for phasors that are in-phase will the amplitudes add directly like real numbers do. The complex addition is illustrated in Figure 4.25.

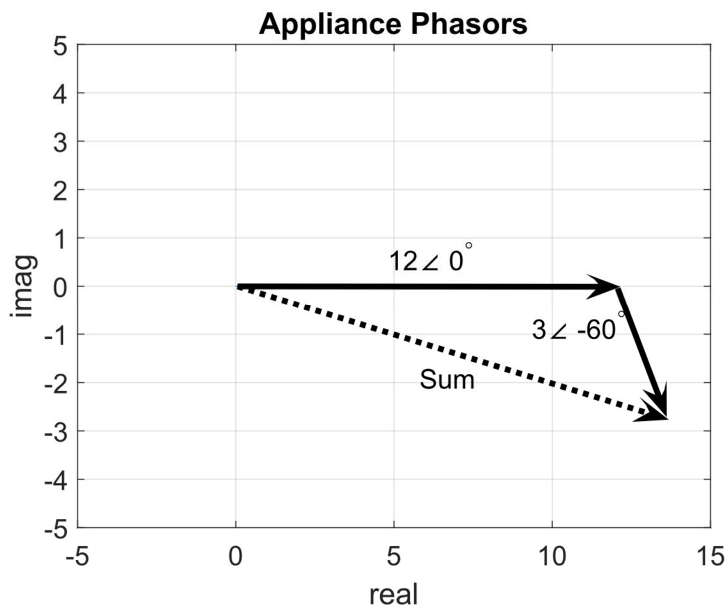


Figure 4.25 Phasor diagram for the box fan and tea kettle currents

[EoE]

4.3 Resistive AC Circuit Analysis and Power

Instantaneous power is the product of instantaneous voltage and instantaneous current

As with any electric circuit, the electrical power must transfer from the source to the load. This fact does not change when using AC. Recall that power for DC is $P = VI$. Both V and I are real numbers and are constant in steady-state DC, so this calculation is straightforward. In AC, however, both $v(t)$ and $i(t)$ vary with time so power $p(t)$ also varies with time. As learned in Ch. 3, power can be negative or positive, depending on whether the voltage polarity and the current direction align with the passive sign convention. Negative voltage times negative current yields positive (absorbed) power. The generalization of $P = IV$ as a function of time is:

$$p(t) = v(t)i(t) \tag{4.20}$$

where:

$p(t)$ is the **instantaneous power** of an element, positive absorbing, negative providing, in W,

$i(t)$ is the instantaneous current as a function of time, in A, and

$v(t)$ is the instantaneous voltage as a function of time, in V.

In DC, this product is easy because both voltage and current are constant functions of time: $v(t) = V_{DC}$ and $i(t) = I_{DC}$, where the subscript DC is used here for clarity:

$$p(t) = v(t)i(t) = V_{DC}I_{DC} \tag{4.21}$$

which is a constant power. The average power P_{ave} is the same as this constant power in the DC case so $p(t) = v(t)i(t) = V_{DC}I_{DC} = P_{ave}$.

Average AC power delivered to resistive loads

Now consider the AC sinusoidal case for a resistive load driven by an AC voltage source[11] in Figure 4.26. Ohm's law holds at each moment in time: The larger the voltage at one moment, the larger the current at that moment. The voltage and current waveforms are *in-phase.*

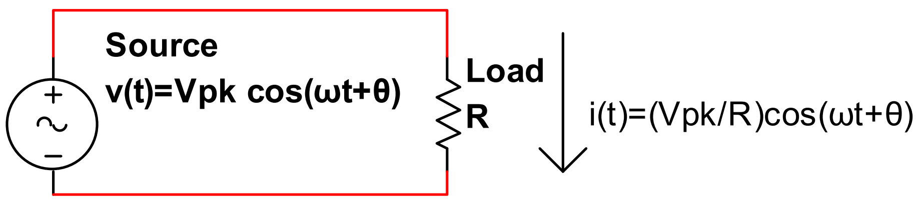


Figure 4.26 A resistive load with a pure AC voltage stimulus

The graphical multiplication of $v(t)$ and $i(t)$ is shown in Figure 4.27. Note that AC $v(t)$ and $i(t)$ are not constant versus time, so the instantaneous power is not constant. Again, the value of $v(t)$ and $i(t)$ at each time instance are multiplied to give $p(t)$. The product is:

$$p(t) = v(t)i(t) = \left[V_{pk}\cos(\omega t + \theta)\right]\left[\frac{V_{pk}}{R}\cos(\omega t + \theta)\right] = \frac{V_{pk}^2}{R}\cos^2(\omega t + \theta) \tag{4.22}$$

Notice the striking features of $p(t)$ in Figure 4.27:

- Instantaneous power $p(t)$ "pulsates" but it is always positive for a resistive load.[12] When the negative voltage is multiplied by the negative current, positive power still results (charges traveling to the left and to the right dissipate equal heat).
- Instantaneous power $p(t)$ pulsates at twice the frequency of $v(t)$ and $i(t)$.
- The average power is halfway between the power peaks and zero in the $p(t)$ plot because of the symmetry of the waveforms. The peak power is $V_{pk}I_{pk}$, so the average power is:

---

[11] AC current sources are much rarer in practice than AC voltage sources and are discussed in Ch. 6.

[12] Technically non-negative, as it touches zero twice per cycle.

$$P_{\text{ave}} = \frac{V_{\text{pk}} I_{\text{pk}}}{2} \tag{4.23}$$

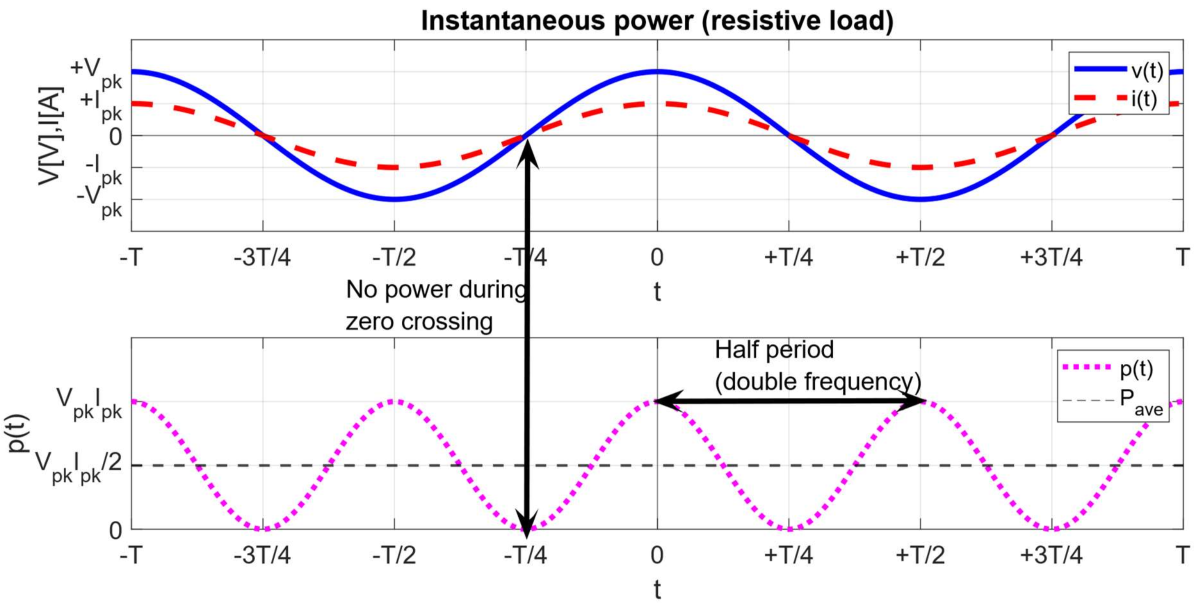


Figure 4.27 Graphical multiplication of AC voltage and current to produce AC power

The average power result for AC[13] is different from the DC case. Do not forget the division by two; otherwise, all your AC power calculations will be off by a factor of two, and the electronic components will either burn up (if the power rating is half of what it should be) or be too expensive (if the power rating is double of what is needed).

Wouldn't it be convenient if there were voltage and current quantities for AC that would give the proper average power like the DC case? Such quantities are called the **effective value** or the **RMS** (*root-mean-square*) **value** of the voltage (and current), as illustrated in Figure 4.28. To obtain the effective value, manipulate Eq. (4.23):

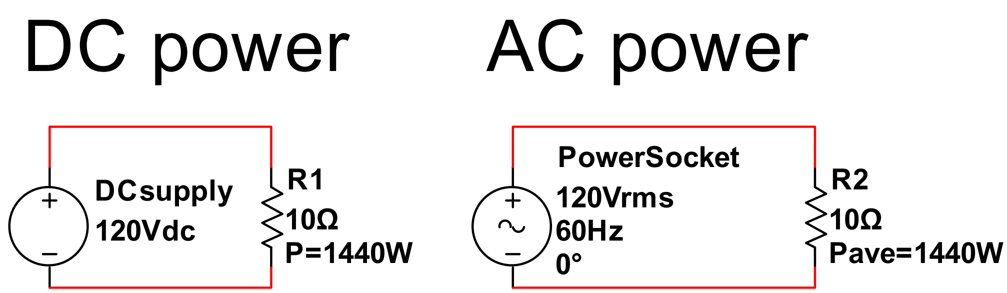


Figure 4.28 Illustration of DC voltage and AC RMS voltage equivalence for resistive loads

[13] This result is true for resistive loads. Non-resistive loads (like AC motors) are considered in Ch. 6 and Ch. 11.

$$P_{\text{ave}} = \frac{V_{\text{pk}} I_{\text{pk}}}{2} = \frac{V_{\text{pk}} I_{\text{pk}}}{\sqrt{2}\sqrt{2}} = \left(\frac{V_{\text{pk}}}{\sqrt{2}}\right)\left(\frac{I_{\text{pk}}}{\sqrt{2}}\right) = V_{\text{RMS}} I_{\text{RMS}} \tag{4.24}$$

Thus, the **effective (RMS) values** of AC voltage and current are defined to be the peak values divided by the square root of two:[14]

$$\boxed{\begin{aligned} V_{\text{RMS}} &= \frac{V_P}{\sqrt{2}} \\ I_{\text{RMS}} &= \frac{I_P}{\sqrt{2}} \end{aligned}} \tag{4.25}$$

The term *effective* is used as often as RMS in sentences. The word "effective" arises because the RMS voltage and current *effectively* deliver the same average power as a DC voltage and current of the same numerical values. Average power is calculated with these effective values as it is in the DC case and can be written a number of ways for resistive loads (valid *only* for resistive components):

$$P = P_{\text{ave}} = V_{\text{RMS}} I_{\text{RMS}} \tag{4.26}$$

$$P = P_{\text{ave}} = \frac{V_{\text{RMS}}^2}{R} = I_{\text{RMS}}^2 R = \frac{V_{\text{pk}}^2}{2R} = \frac{I_{\text{pk}}^2}{2} R$$

Note: the "ave" subscript is often not shown but is understood for AC average power.

Important facts about this formula for AC power to resistive loads:

- The load must be purely resistive, such as an incandescent lamp, heater, or human tissue. This power result does not apply to non-resistive loads such as most AC motors (which will be covered in Ch. 6), fluorescent tube lamps, or welding arcs.
- The average power does *not* depend on phase because the voltage and current waveforms are in-phase and the peaks in $p(t)$ always align, so the maximum value is always $V_{\text{pk}} I_{\text{pk}}$.
- The average power does *not* depend on the frequency of the sinusoids because the voltage and current peak amplitudes do not depend on the frequency (they track each other exactly).
- The power source must be purely sinusoidal AC with no DC offset. A DC offset would make the troughs of the power pulses non-zero. The power equation would differ.
- The average power of an AC generator in a resistive circuit is negative because the generator voltage and current will always have opposite signs in the passive sign convention.

In previous chapters, a household power socket was represented by a 120 V constant voltage source. Domestic sockets in the United States are sinusoidal AC sources: $170 \text{ V}_{\text{pk}} = 120 \text{ V}_{\text{RMS}}$

[14] This result is true only for pure AC sinusoidal voltage waveforms. The RMS value of non-sinusoidal periodic waveforms is addressed in Appendix 4.B.

None of the performance predictions in earlier chapters[15] are in error if the load is resistive, such as a wire, a heater, a cooker, or an incandescent lamp. A summary of when peak values and when RMS values are appropriate is shown in Table 4.6. A typical motor nameplate is shown in Figure 4.29. By default, AC equipment is labeled with RMS values. The “peak-to-peak” value (twice the peak value) is also used in some contexts.

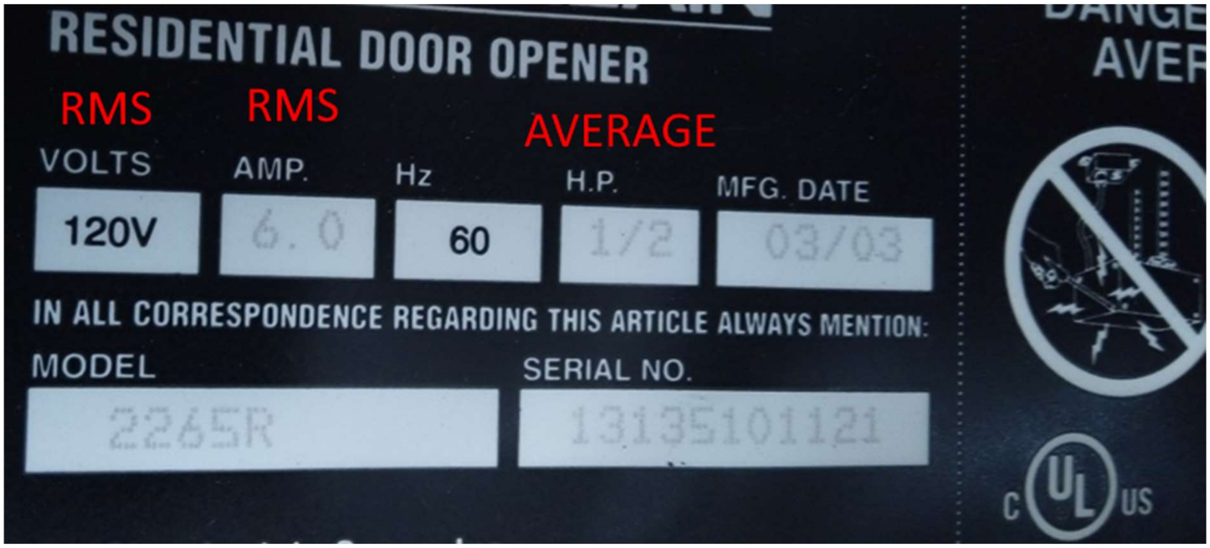


Figure 4.29 The nameplate of a garage door opener: 120 V RMS, 6 A RMS, ½ horsepower $(1 \text{ hp} \approx 746 \text{ W})$

Table 4.6 When to use generally use peak, RMS, and peak-to-peak amplitude values

	Peak amplitude	**RMS amplitude**	**Peak-to-Peak amplitude**
Applications	Information circuits like… • Sensors • Signals	Power circuits like… • Motors • Generators • Household utilities	Power converters Guitar pickups Noise Waveform generators “Dirty DC”
Consequences	Peak voltage… • arcs insulation. • lights neon. • distorts amplifiers. • causes chemical reactions.	RMS voltage… • determines heat output of heaters. • noise limits RF amplifier quality. RMS current… • overheats wires.	Physics rarely depends on peak-to-peak

[15] Devices for which this simplified description breaks down and must be expanded, like motors and microwave ovens, are analyzed in Ch. 6 and Ch. 11.

	Peak amplitude	RMS amplitude	Peak-to-Peak amplitude
	Peak current… • saturates motors.	• does mechanical work in motors. • determines physical danger to humans.	
Notes	Important when the value at one critical time is important, and information, not energy, is most important.	Important when a long-term time average is appropriate, especially for heat. Use when power or energy are most important.	Used when the waveform is not sinusoidal or has a large DC offset.

Example 4.3.1 Space heater graphs

Background: The resistive heating elements of a space heater are plugged directly into household power. The heating coils (behind the protective grid in Figure 4.30) have a resistance of about $10\,\Omega$, and American household voltage is given by $v(t) = 170\cos(377t)$ V.

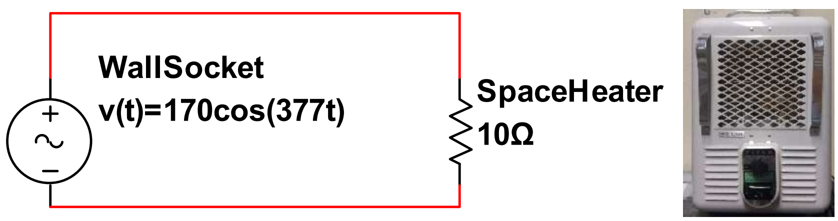


Figure 4.30 A typical domestic space heater

Problem:

a) Plot both the voltage across and current through the heating coils as a function of time on the same axes. Label the plot axes. Include units, a title, and a legend. Plot the two traces in distinct line styles.

b) Plot the instantaneous power drawn by the space heater as a function of time and label the average power.

c) American household voltage oscillates at 60 cycles per second, and the power pulses at 120 Hz. The typical timescale of heat transfer from the heater to the air is about 5 seconds. Why is it reasonable to care only about the *average* power output of the heater?

Solution:

a) The voltage waveform is given, and the current waveform has the same frequency, but 1/10th the peak (as a number; the units differ). To choose a reasonable timeframe to plot the waveform, we must know the period. The 377 multiplies t in the $v(t)$ equation, so 377 rad/s is the angular frequency ω. Use it to calculate the period $\left(\omega = 2\pi/T \rightarrow T = 2\pi/\omega\right)$ and select a reasonable time scale for the graph. MATLAB code to implement this function follows.

```
Vpk=170 % peak AC voltage
w=377 % angular frequency
R=10 % resistance
Ipk=Vpk/10 % peak current
T=2*pi/w % period
t=linspace(-T,T,1000) % two periods and 1000 points
V=Vpk*cos(w*t)
I=Ipk*cos(w*t)
plot(t*1000,V,'b-',t*1000,I,'r--') % plot V and I, t in ms
xlabel('t[ms]')
ylabel('v(t)[V],i(t)[A]')
title('Space heater')
grid on
legend('Voltage','Current')
```

The resultant plot is shown in Figure 4.31. Two plot traces, a legend and distinct line styles ensure the plot is readable, even when printed in black and white or when the reader cannot distinguish colors.

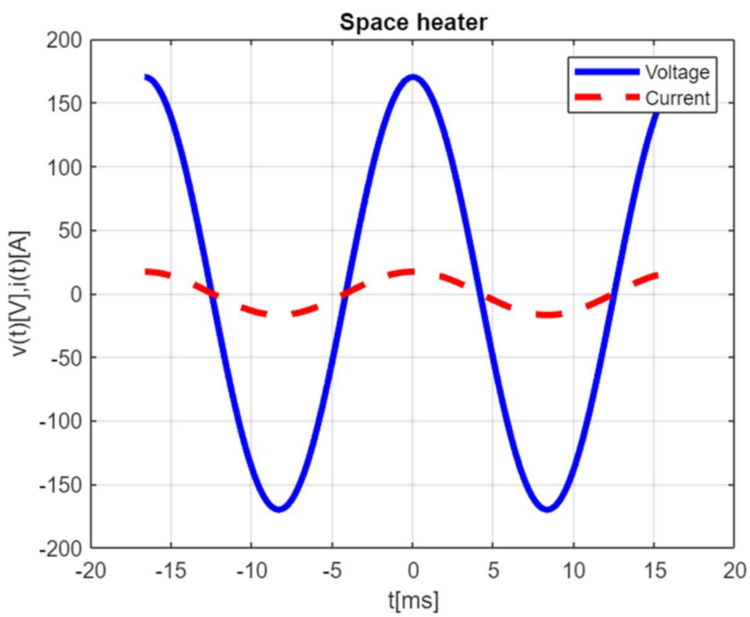


Figure 4.31 Instantaneous power and current in the space heater

b) The instantaneous power is the product of the voltage and current sinusoids:

$$p(t) = i(t)v(t) = \left[17\cos(377t)\right]\left[170\cos(377t)\right] = 2890\cos^2\left(377t\right)$$

This function can be plotted by appending the following code to the previous MATLAB code. Figure 4.32 results. The average power from the plot is estimated to be 1400 W, which is close to half of the peak power in the $p(t)$ equation.

```
P=I.*V % p(t)=i(t)v(t), use .* to multiply functions
plot(t*1000,P) % plot P vs t, t in ms
Pave=Vpk*Ipk/2 % average power
yline(Pave,'k--') % dashed line for average power
xlabel('t[ms]')
ylabel('p(t)[W]')
ylim([0,4000])
title('Space heater')
grid on
legend('Instantaneous power','Average power')
```

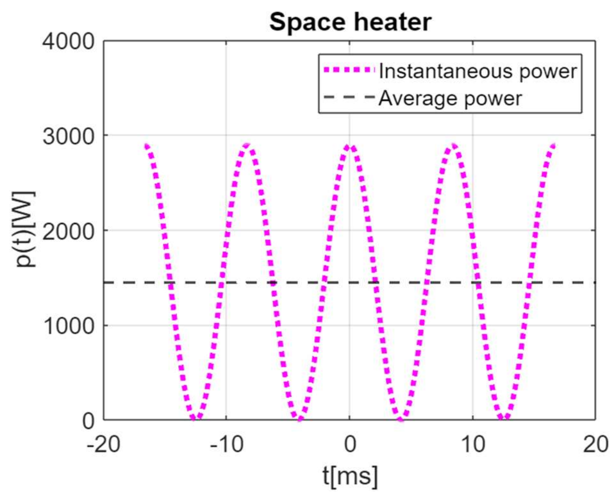


Figure 4.32 instantaneous power and average power in the space heater circuit

c) The power peaks 120 times per second, so its period is about 8 ms. The characteristic time of heat transfer, 5 s, is much longer, so this thermal system cannot respond to each peak individually and only the average matters.

Comment: This conclusion does not hold for a lower frequency source, where the period of oscillation and the timescale of heat transfer are closer to being equal. Pulsating power also causes vibration problems in some motors.

[EoE]

Example 4.3.2 Induction Cooktop efficiency

Background: Induction cooktops heat food in cast iron pots using magnetic fields. The power electronics first converts the input AC electricity from the utility into a high voltage DC output, as shown in Figure 4.33. Note the voltage is scaled on the left side of the graph and the current is scaled on the right side.

Problem:

a) Explain why one can treat the power electronics as a resistive load based on the input voltage and current waveforms.

b) What is the average input AC power? (graphs require approximation; try to be accurate to within 15%)

c) What is the output DC power?

d) How efficiently does the power electronics convert input AC power into output DC power?

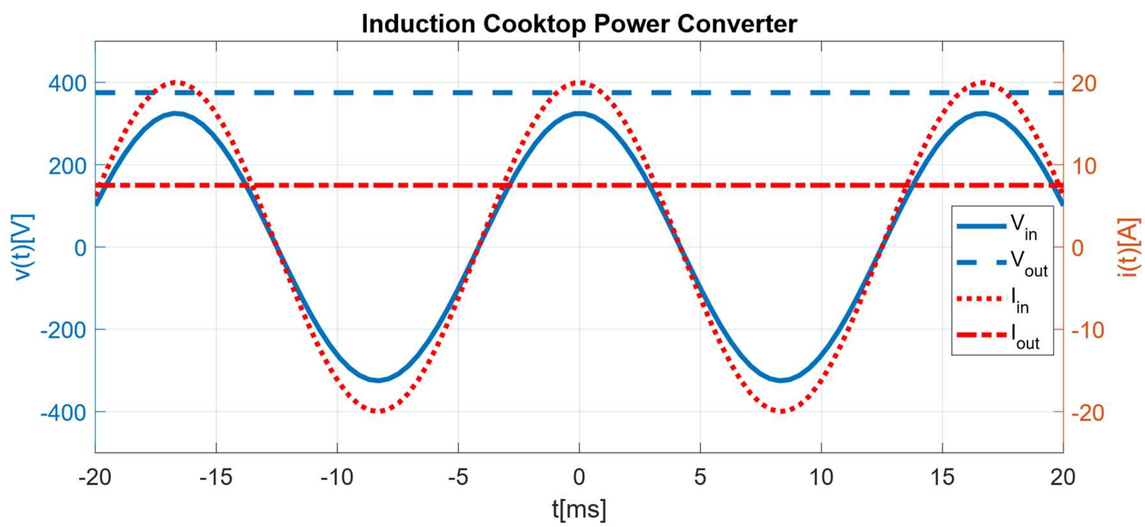


Figure 4.33 Input and output voltages and currents of the power electronics in an induction cooktop[16]

Solution:

a) The input AC voltage and current sinusoids are in-phase and both sinusoidal and at the same frequency, so the load is resistive.

b) Input AC current is the dotted red line: 20 A peak; input AC voltage is the solid blue line, about 325 V peak.

$$P_{\text{ACave}} = \frac{V_{\text{pk}} I_{\text{pk}}}{2} = \frac{(325)(20)}{2} = 3250 = 3.25 \text{ kW}$$

c) The output DC voltage is the straight dashed blue line at approximately 375 V. The output DC current is the straight, alternating dash lengths red line, about 7.5 A.

$$P_{\text{DCave}} = IV = (375)(7.5) = 2.81 \text{ kW}$$

d) The efficiency is the ratio of output power to the input power:

[16] Park, Sang Min, Eunsu Jang, Dongmyoung Joo, and Byoung Kuk Lee. "Power Curve-Fitting Control Method with Temperature Compensation and Fast-Response for All-Metal Domestic Induction Heating Systems." Energies 12, no. 15 (2019): 2915.

$$\eta = \frac{P_{\text{out}}}{P_{\text{in}}} = \frac{P_{\text{DCave}}}{P_{\text{ACave}}} = \frac{2.81}{3.25} = 0.865 \approx 87\%$$

This efficiency is typical for power electronics. Greater than 95% is generally not achievable, and less than 50% wastes more energy than what is put into heating the food.

[EoE]

Resistive AC circuits use the same analysis techniques as DC resistive circuits

Circuits with more than one AC source and more than one resistor can be analyzed by the same techniques as DC resistive circuits. The techniques of DC circuit analysis (such as the voltage divider rule and nodal analysis) also work for resistive AC circuits, with the following modifications:

$V \rightarrow \tilde{V}\ ,\ I \rightarrow \tilde{I}$ Voltages and currents are phasors (complex numbers)

Phasors can be written as either RMS phasors or peak phasors:

$\tilde{V} = V_{\text{pk}}\angle\theta_V$ and $\tilde{I} = I_{\text{pk}}\angle\theta_I$, or $\tilde{V} = V_{\text{RMS}}\angle\theta_V$ and $\tilde{I} = I_{\text{RMS}}\angle\theta_I$

You must know or determine whether peak or RMS phasors are being used in a problem.

$P = VI \rightarrow P_{\text{ave}} = |\tilde{V}_{\text{RMS}}||\tilde{I}_{\text{RMS}}| = \dfrac{|\tilde{V}_{\text{pk}}||\tilde{I}_{\text{pk}}|}{2}$ Average power calculation requires RMS quantities

$P_{\text{ave}} = \left|\tilde{I}_{\text{RMS}}\right|^2 R$ or $P_{\text{ave}} = \dfrac{\left|\tilde{V}_{\text{RMS}}\right|^2}{R}$ Equivalent AC power relations

The magnitude bars remove the phase information: the average power delivered to a resistive load does *not* depend on phase and is a *real number*. Computing $\tilde{I}^2$ without taking the magnitude yields a complex number instead of a real number. Imaginary watts[17] cannot boil water or turn wheels. Remember, if you are *squaring an angle* during an AC calculation, you are doing it *wrong*!

The voltages and currents, however, *must* have their phases retained until the last step of the calculations, as illustrated in Figure 4.34. In the DC circuit, the voltage difference across the resistor R1 is zero, so no DC current flows. In the AC circuit, the magnitude of the voltage difference across R2 is *not* the difference of the magnitudes if the phases are different.

$$|\tilde{V}_1 - \tilde{V}_2| \neq |\tilde{V}_1| - |\tilde{V}_2| \tag{4.27}$$

The current in the AC circuit with a phase difference is calculated with both phases intact:

$$\tilde{I} = \frac{\tilde{V}_1 - \tilde{V}_2}{R} = \frac{10\text{ V}\angle 0° - 10\text{ V}\angle 90°}{1000\ \Omega} = 10\text{ mA} - j10\text{ mA} = 14.1\text{ mA}\angle -45°$$

This concept is illustrated in the next example.

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[17] The extension of the idea of power to include imaginary values (reactive power) is covered in Ch. 11.

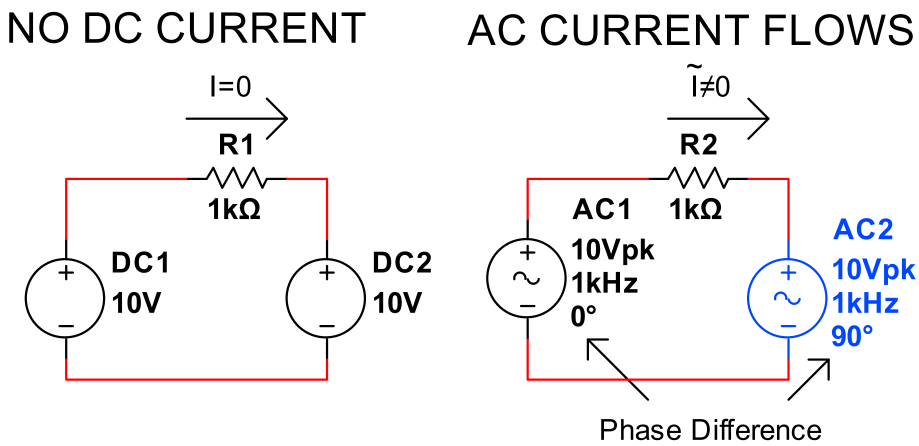


Figure 4.34 Illustration of the importance of phase information in AC resistive circuits

Example 4.3.3 Center tap vs three-phase

Background: High-power outlets in single family homes use a center-tap transformer with two 120 V lines a half-cycle out-of-phase. High-rise buildings have three-phase power available for elevators. High-power outlets in high-rise apartments tap two 120 V lines a third of a cycle $(120°)$ out-of-phase as seen in Figure 4.35.

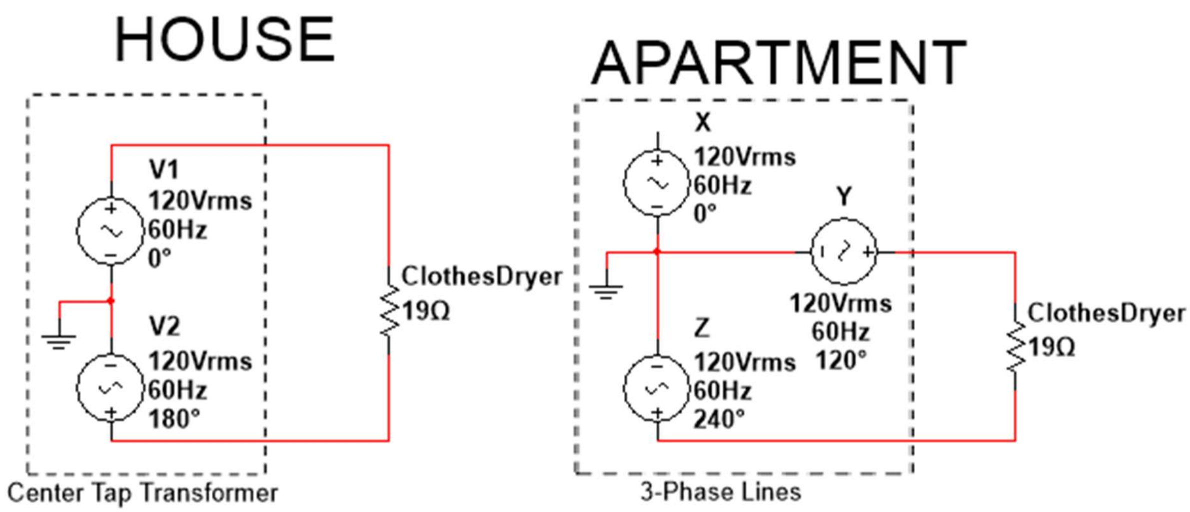


Figure 4.35 Circuits for single-family home and high-rise high power outlets in the US

Problem:

a) Write an expression for the phasor voltage available to each clothes dryer using KVL.

b) Draw a phasor diagram of voltages for the house and for the apartment. Show the phasor sum graphically in each case.

c) Will the clothes dryer in the house or in the apartment dry clothes faster? Explain your answer.

Solution:

a) For both circuits, KVL includes two voltage sources with opposing positive terminals: a *difference* of AC voltages.

The phasor voltage available to the clothes dryer in the house is:

$\tilde{V}_H = 120\angle 0 - 120\angle 180° = 120\angle 0 + 120\angle 0°$ where the $180°$ cancels the – sign. The house voltage is numerically $\tilde{V}_H = 240\angle 0°\ \mathrm{V_{RMS}}$.

The phasor voltage available to the clothes dryer in the apartment depends on lines Y and Z (line X is not connected).

$\tilde{V}_A = 120\angle 120° - 120\angle 240° = 120\angle 120° + 120\angle 60°$, where the – sign subtracts $180°$ from the angle. The apartment voltage phasor is $\tilde{V}_A = 208\angle 90°\ \mathrm{V_{RMS}}$.

b) The phasor addition is shown in Figure 4.36.

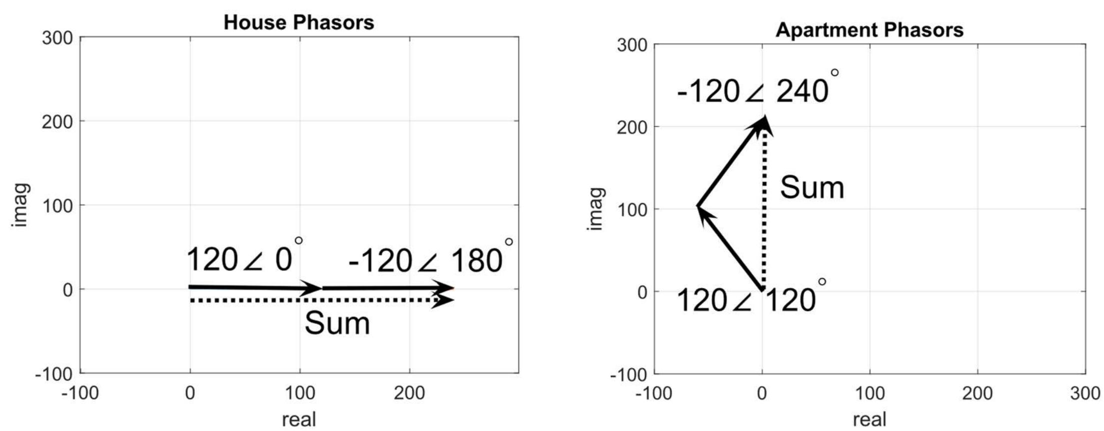


Figure 4.36 Phasor summation in house and apartment multisource circuits

c) The house dryer will dry clothes faster. The two voltages in the house are effectively in-phase in the house, so the magnitudes add directly. In the apartment, the phasors only partially add so the magnitude is less than the sum of the two voltage magnitudes. The magnitude of voltage in the house case is greater, so the power delivered to the resistive load is greater. Since the power is greater, the clothes will dry faster.

[EoE]

4.4 Chapter Appendices

Appendix 4.A.1The "phasor" MATLAB function

Copy the following code in a .m file into your /Documents/MATLAB directory and name it phasor.m (exactly as typed, including the period). Call phasor(z) or phasor(R,theta) inside any script or command line to easily convert between rectangular and polar forms with the angle in degrees: input rectangular form, polar form results, and vice-versa.

```
function Vout=phasor(varargin)
% convenient phasor function for phasors
% with one (complex valued) argument, returns magnitude and phase in degrees
% with two real arguments, it returns a complex number with that magnitude
% and that phase input in degrees
% not designed to work with arrays of values, only singles

% one input is either singleton complex (a+bi) or 1x2 array [mag,phase]
if nargin == 1
    if size(varargin{1})==[1,2] % if input is a 1x2 array, assume [mag,angle]
        % output rectangular form of complex number.
        Vout=varargin{1}(1)*exp(1i*deg2rad(varargin{1}(2)));
    elseif size(varargin{1})==[1,1] % if 1x1 input, assume single (a+bi)
        % output magnitude, phase in degrees of complex number
        Vout=[abs(varargin{1}),rad2deg(angle(varargin{1}))];
    end
elseif nargin==2 % two real arguments->complex number with that mag and phase
    % output rectangular form complex number
    Vout=varargin{1}*exp(1i*deg2rad(varargin{2}));
end
end
```

Some example code and the corresponding outputs:

```
A=170
phi_in_degrees=45
% Use deg2rad to convert the angle in degrees into radians
phi_in_rad=deg2rad(phi_in_degrees)
% Use exp(1j*angle in rad) to enter the angle of a complex number in polar form
Vsquiggle=A*exp(1j*deg2rad(phi_in_degrees))
% This task is more easily accomplished using the phasor function
z_polar=phasor(z)
Arectangular=phasor(A,phi_in_degrees)
```

```
A =
   170.0000e+000
phi_in_degrees =
    45.0000e+000
phi_in_rad =
   785.3982e-003
Vsquiggle =
   120.2082e+000 +120.2082e+000i
z_polar = 1x2
       2.8284e+000 ...

Arectangular =
   120.2082e+000 +120.2082e+000i
```

Appendix 4.A.2 RMS Values for Non-sinusoidal Periodic Waveforms

The RMS voltage and current can be determined for *periodic* AC waveforms that are not pure sinusoids. These calculations are uncommon in simple circuits but apply in more advanced power transmission and signal processing coursework.

$$V_{\text{RMS}} = \sqrt{\frac{1}{T}\int_{t=0}^{t=T}\left[v(t)\right]^2 dt} = \sqrt{\frac{1}{T}\int_{\substack{\text{any one}\\\text{full period}}}\left[v(t)\right]^2 dt} \tag{4.28}$$

where: V_{RMS} is the RMS voltage,

$v(t)$ is the time-dependent periodic voltage, and

T is the period of the waveform.

The integral may be taken over any one full period of the periodic waveform (not necessarily from 0 to T).

Some deductions follow:

- The RMS value is always less than or equal to the most extreme peak value (positive or negative).
- The RMS value is always greater than the minimum absolute value (positive or negative).
- The RMS value does not depend on the frequency of the waveform.
- The RMS can be calculated for any voltage or current but does not provide useful predictions in all situations. RMS was derived from the assumption of resistive loads.

RMS calculations for non-sinusoidal voltages are useful for computing the heating as a result of distorted waveforms in motors and power converters. The generalized version (signal power) is useful in signal processing, but those calculations are beyond the scope of this text.

Rigorous development of the RMS of the pure AC sinusoid

$$V_{RMS} = V_{eff} = \sqrt{\frac{1}{T}\int_{t=0}^{t=T}\left(v(t)\right)^2 dt}\,.$$ RMS expression

$$v(t) = V_p\cos(\omega t + \theta_V)$$ Pure AC sinusoidal voltage at one frequency

$$V_{\text{RMS}} = \sqrt{\frac{1}{T}\int_{t=0}^{t=T}\left[V_p\cos\left(\omega t + \theta_V\right)\right]^2 dt}$$ Insert voltage expression into RMS formula

$$V_{RMS} = \sqrt{\frac{V_p^2}{T}\int_0^T \cos^2(\omega t + \theta_V)\,dt}$$ Factor out V_p^2

$$V_{RMS} = \sqrt{\frac{V_p^2}{T}\int_0^T \left[\frac{1}{2} + \frac{1}{2}\cos(2\omega t + 2\theta_V)\right] dt}$$ Trig identity $\cos^2(x) = \frac{1}{2} + \frac{\cos(2x)}{2}$

$$V_{RMS} = \sqrt{\frac{V_p^2}{2T}\int_0^T dt \;+\; \frac{V_p^2}{2T}\int_0^T \cos(2\omega t + 2\theta_V)\,dt}$$ Split integral, factor out ½

$$V_{RMS} = \sqrt{\frac{V_p^2}{2T}\underbrace{\int_0^T dt}_{=T} \;+\; \frac{V_p^2}{2T}\underbrace{\int_0^T \cos(2\omega t + 2\theta_V)\,dt}_{\text{Average value zero}}}$$ Recognize integral values

$$V_{RMS} = \sqrt{\frac{V_p^2}{2T}\,T + 0}$$ Remove terms

$$V_{RMS} = \sqrt{\frac{V_p^2}{2T}\,T} = \sqrt{\frac{V_p^2}{2}} = \frac{V_p}{\sqrt{2}}$$ RMS of sinusoid is root two smaller

Example 4.A.1 The effective value of the quarter sine wave waveform

Problem: Determine the RMS value of a sine wave that is ON only during the first ¼ of each period. Note: $x(t) = v(t)$ for the waveform in Figure 4.A.1.

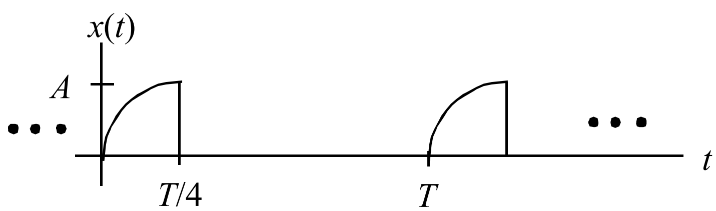


Figure 4.A.1 Periodic waveform: a sine wave ON only the first ¼ of each period

Solution:

Strategy: Write the expression for $v(t)$, insert it into the V_{RMS} expression, and evaluate.

$$v(t) = \begin{cases} A\sin(\omega t) & 0 \le t < \dfrac{T}{4} \\ 0 & \dfrac{T}{4} < t \le T \end{cases}$$ Break up $v(t)$ to express mathematically

$$V_{\mathrm{RMS}} = \sqrt{\frac{1}{T}\int_0^T v^2(t)\,dt}$$ General RMS relation for periodic waveforms

$$V_{\mathrm{RMS}} = \sqrt{\frac{1}{T}\int_0^{T/4}\left[A\sin(\omega t)\right]^2 dt + \frac{1}{T}\int_{T/4}^{T} 0\,dt}$$ Substitute $v(t)$ parts into the integral

$$V_{\mathrm{RMS}} = \sqrt{\frac{A^2}{T}\int_0^{T/4}\sin^2(\omega t)\,dt}$$ Simplify

Locate this general integral form from a math handbook, a calculator, software, or the web:

$$\int \sin^2 ax\,dx = \frac{x}{2} - \frac{\sin 2ax}{4a}$$

Identify the correspondence between the symbols in the general math equation and the equation under evaluation:

$a \sim \omega \quad , \quad x \sim t \quad , \quad dx \sim dt$

$$V_{\text{RMS}} = \sqrt{\frac{A^2}{T}\left[\frac{t}{2} - \frac{\sin(2\omega t)}{4\omega}\right]\Bigg|_0^{T/4}}$$ Apply the general integral result to our integral

$$V_{\text{RMS}} = \sqrt{\frac{A^2}{T}\left\{\left[\frac{T}{4\bullet 2} - \frac{\sin\left(2\omega\frac{T}{4}\right)}{4\omega}\right] - \left[\frac{0}{2} - \frac{\sin(0)}{4\omega}\right]\right\}}$$ Insert limits

$$V_{\text{RMS}} = \sqrt{\frac{A^2}{T}\left[\left(\frac{T}{8}\right) - \frac{\sin\left(2\frac{2\pi}{T}\bullet\frac{T}{4}\right)}{4\frac{2\pi}{T}}\right]}$$ Insert $\frac{2\pi}{T}$ for each ω

$$V_{\text{RMS}} = \sqrt{\frac{A^2}{T}\left[\frac{T}{8} - \frac{\sin(\pi)}{\frac{8\pi}{T}}\right]} = \sqrt{\frac{A^2}{T}\left[\frac{T}{8} - \frac{0}{\frac{8\pi}{T}}\right]} = \sqrt{\frac{A^2}{T}\left[\frac{T}{8}\right]} = \sqrt{\frac{A^2}{8}}$$ Simplify

$$V_{\text{RMS}} = \frac{A}{\sqrt{8}} = \frac{A}{2\sqrt{2}}$$ The result

The effective value of this waveform is approximately 1/3 of its peak value.

[EoE]

Example 4.A.2 Six pulse rectifier RMS

Background: Three phase full bridge rectifiers convert sinusoidal voltage (red dashed line in Figure 4.A.2) from the power lines into high voltage DC-with-ripple voltage (solid blue line). Power electronics stitch together the tops of six sine waves into one nearly constant waveform.

Problem: Determine the RMS voltage of the rectifier output.

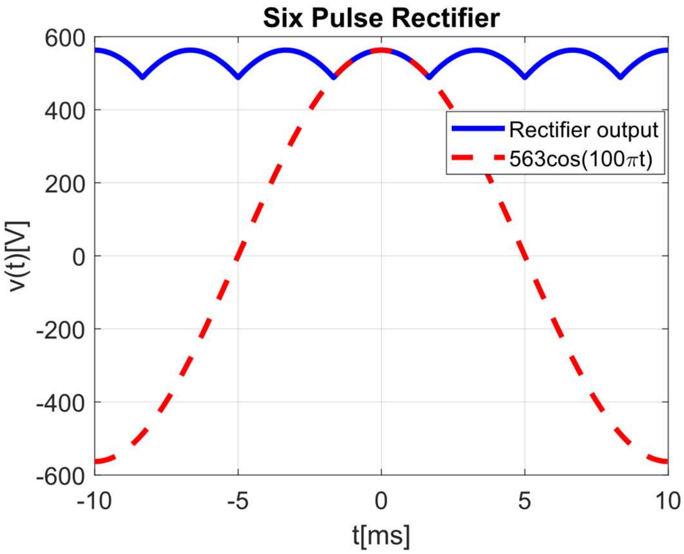


Figure 4.A.2 Three phase rectifier output

<u>Solution</u>:

The RMS must exceed the minimum of ~500 V. The RMS must be below the peak value of 563 V. The six segments are identical, so any one of the six can be the one full period in the RMS integral. The sine wave has period $T = 20\text{ ms}$. The rectifier waveform period is 1/6th the period of the sinusoid, so the period in the integral is 1/6th the period of the sine wave T. The middle segment has symmetric boundaries of $\pm T/12$, convenient for evaluation. The RMS integral is:

$$V_{\text{RMS}} = \sqrt{\frac{1}{\left(\frac{T}{6}\right)} \int_{t=-T/12}^{t=+T/12} \left[V_{\text{pk}} \cos\left(\frac{2\pi}{T} t\right) \right]^2 dt}$$ Integral to evaluate

$$V_{\text{RMS}} = \sqrt{6} \frac{V_{\text{pk}}}{\sqrt{T}} \sqrt{\int_{t=-T/12}^{t=+T/12} \left[\cos\left(\frac{2\pi}{T} t\right) \right]^2 dt}$$ Pull T/6 and peak voltage out of the root

$$V_{\text{RMS}} = \sqrt{6} \frac{V_{\text{pk}}}{\sqrt{T}} \sqrt{\int_{t=-T/12}^{t=+T/12} \frac{1}{2} \left[1 + \cos\left(2 \frac{2\pi}{T} t\right) \right] dt}$$ Trig identity decomposes cosine-squared

$$V_{\text{RMS}} = \sqrt{6} \frac{V_{\text{pk}}}{\sqrt{T}} \sqrt{2 \int_{t=0}^{t=+T/12} \frac{1}{2} \left[1 + \cos\left(\frac{4\pi}{T} t\right) \right] dt}$$ Even integrand, 2x integral from 0 to T/12

$$V_{\text{RMS}} = \sqrt{6} \frac{V_{\text{pk}}}{\sqrt{T}} \sqrt{\int_{t=0}^{t=+T/12} 1\, dt + \int_{t=0}^{t=+T/12} \cos\left(\frac{4\pi}{T} t\right) dt}$$ Simplify, integral of sum = sum of integrals

$$V_{\text{RMS}} = \sqrt{6}\,\frac{V_{\text{pk}}}{\sqrt{T}}\sqrt{\left. t\right|_{t=0}^{t=T/12} + \left.\frac{1}{\left(\frac{4\pi}{T}\right)}\sin\left(\frac{4\pi t}{T}\right)\right|_{t=0}^{t=T/12}}$$ Evaluate integrals

$$V_{\text{RMS}} = \sqrt{6}\,\frac{V_{\text{pk}}}{\sqrt{T}}\sqrt{\left(\frac{T}{12}-0\right)+\left(\frac{1}{\left(\frac{4\pi}{T}\right)}\sin\left(\frac{4\pi\left(T/12\right)}{T}\right)-\frac{1}{\left(\frac{4\pi}{T}\right)}\sin\left(\frac{4\pi\left(0\right)}{T}\right)\right)}$$ Insert bounds

$$V_{\text{RMS}} = \sqrt{6}\,\frac{V_{\text{pk}}}{\sqrt{T}}\sqrt{\frac{T}{12}+\frac{T}{4\pi}\sin\left(\frac{4\pi}{12}\right)}$$ Eliminate 0 terms, $\sin(0)=0$

$$V_{\text{RMS}} = \sqrt{6}V_{\text{pk}}\sqrt{\frac{1}{12}+\frac{1}{4\pi}\sin\left(\frac{4\pi}{12}\right)}$$ Cancel T factors

$$V_{\text{RMS}} = \sqrt{6}V_{\text{pk}}\sqrt{\frac{1}{12}+\frac{1}{4\pi}\frac{\sqrt{3}}{2}}$$ Evaluate the sine (an exact value)

$$V_{\text{RMS}} = \sqrt{6}\left(563\right)\sqrt{\frac{1}{12}+\frac{1}{4\pi}\frac{\sqrt{3}}{2}} = 538\text{ V}$$ Insert peak voltage

The RMS value is reasonable, greater than 500 and less than 563 V. The DC equivalent voltage is much larger than for a pure AC waveform: the RMS value is 95% of the peak value. Three-phase rectifiers are much better at converting AC to DC than single-phase rectifiers.

[EoE]

4.5 Homework Problems

Problems	Type
1 – 19	Drill Exercises

"Drill exercises" are basic identification tasks and elementary calculations, level 1 or 2 on Bloom's Taxonomy, intended to develop, but not demonstrate mastery. These tasks should take a student less than 5 minutes to solve if they know the concept. These tasks are generally too easy to be used as exam problems. We recommend 3-5 drill exercises per week on homework.

20 – 42	Standard Problems

Standard problems include applications, or abstract exercises requiring multi-stage reasoning. Standard problems assess student mastery and are of a typical difficulty to be expected on an exam. We recommend 3-5 standard problems per week on homework.

43 - 45	Challenge Problems

Challenge problems demonstrate deeper mastery of concepts through complex multi-stage reasoning or use of techniques in novel ways. They are appropriate in small numbers on homework but are not appropriate for exams. We recommend zero or one challenge problem per week on homework.

Drill Exercises

1. Which pair of waveforms are in phase?

Problem: Identify the pair of waveforms in the set below that are in-phase with each other.

a. $1\cos(3t+0°)$

b. $20\cos(3t-45°)$

c. $300\cos(3t-90°)$

d. $0.4\cos(3t+90°)$

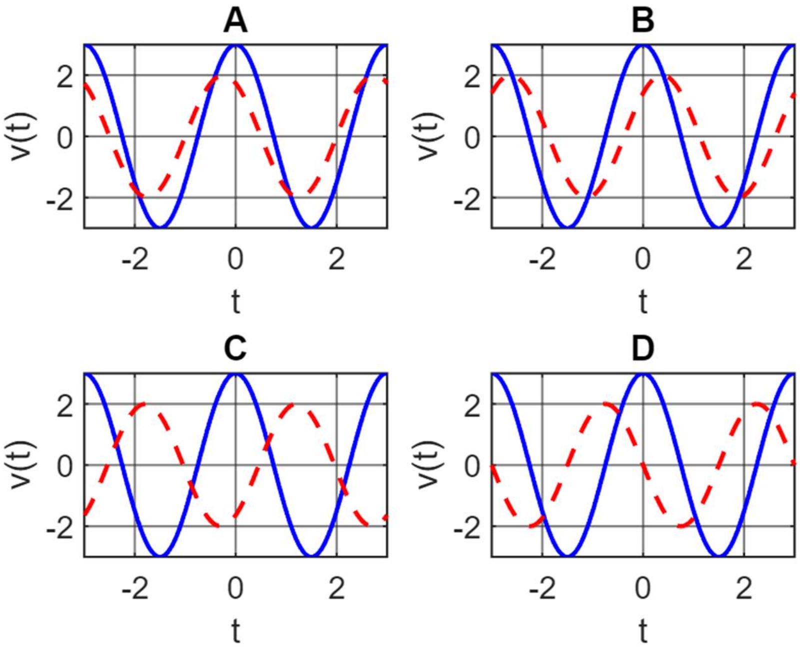
e. $5000\cos(3t-45°)$

f. $0.06\cos(3t-180°)$

g. $77\cos(3t+135°)$

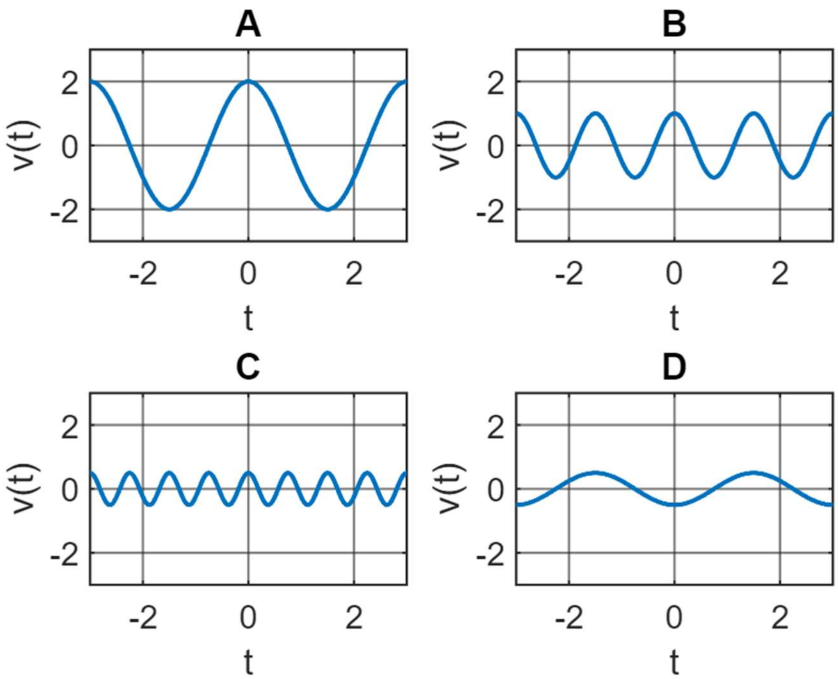
2. Which waveforms lag the reference?

Problem: Identify each waveform (dashed red) with a phase angle that lags the reference (solid blue) in the figure below



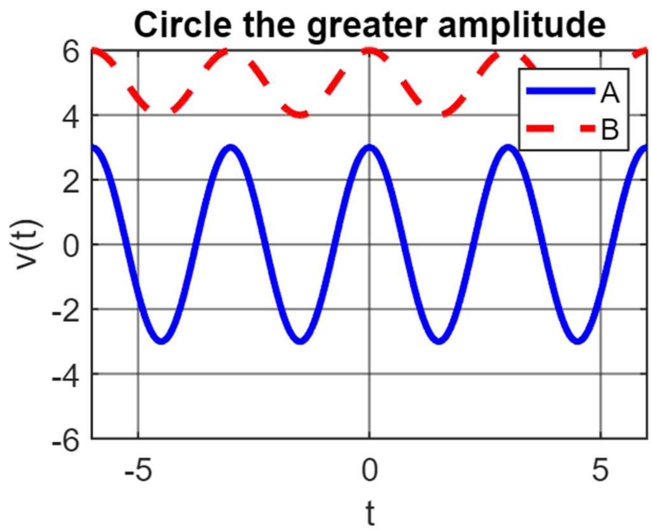
3. Which has the lowest frequency?

Problem: In the graphs below, a) identify the sinusoid(s) that have the lowest frequency, and
b) identify the sinusoid(s) that have the highest frequency.
Briefly justify your answers.



4. Which has the greater AC amplitude?

Problem: The graph shows the two sinusoids A and B of the form $v(t) = V_{AC}\cos(\omega t + \theta) + V_{DC}$. Which sinusoid has the greater AC amplitude V_{AC}?



5. Is the magnitude greater than or equal to 5?

Problem: Determine if each complex number has a magnitude greater than or equal to 5.

a. $1+2j$

b. $5+j0$

c. $0-j5$

d. $2+j3$

e. $5-j2$

f. $5+j2$

g. $-5-j5$

6. Is the phase less than 45 degrees?

<u>Problem</u>: Determine if each complex number has a phase angle between $-45° \le \theta \le +45°$.

a. $5+5j$

b. $5+2j$

c. $5-2j$

d. $5-j0$

e. $2+j5$

f. $2-j5$

g. $-5-j2$

h. $0+j2$

i. $1000+j5$

7. Time domain – phasor conversions

Problem: a) Determine the RMS phasor current expression for:
$i(t) = 5.28\cos(377t - 37.35°)\,[\mathrm{mA}]$.

b) Determine the time-domain expression for $v(t)$ if $\tilde{V} = 27\,V_{\mathrm{pk}}\angle -130°$ at 1 Hz.

8. Leading or lagging?

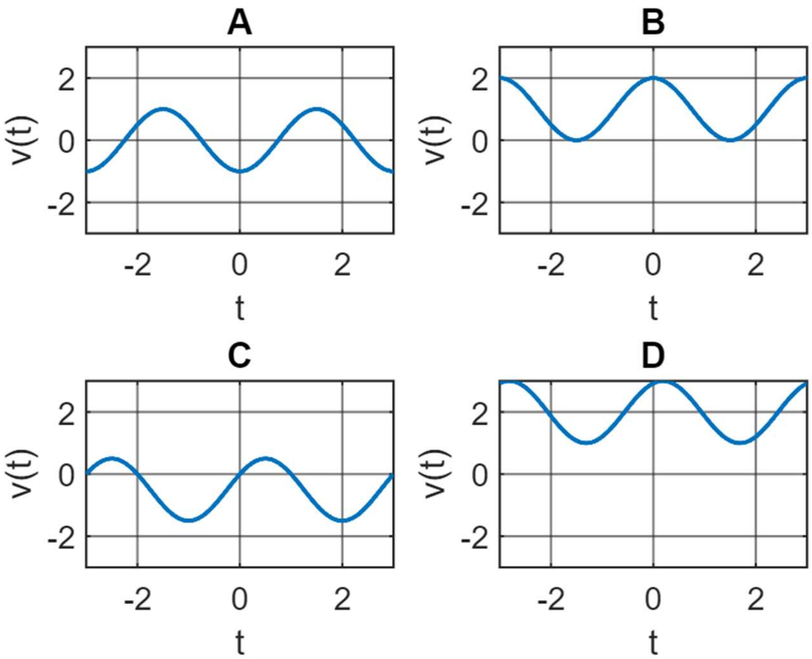
Problem: A load has a voltage of $v(t) = 14.142 \sin(754t)$ V across it and a current of $i(t) = 0.250 \sin(754t - 45°)$ mA through it.

a) What is the frequency (in hertz) of the voltage and the current?

b) Does current lead voltage or does current lag voltage?

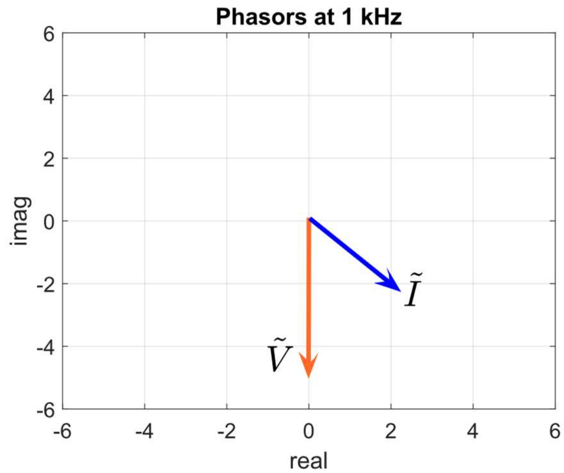
9. Which waveforms have zero DC offset?

Problem: Circle the waveform that is pure AC with a DC offset of zero.

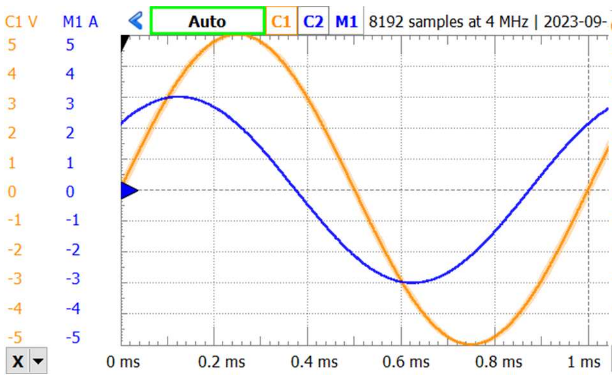


10. Which of these oscilloscope traces matches the phasor diagram?

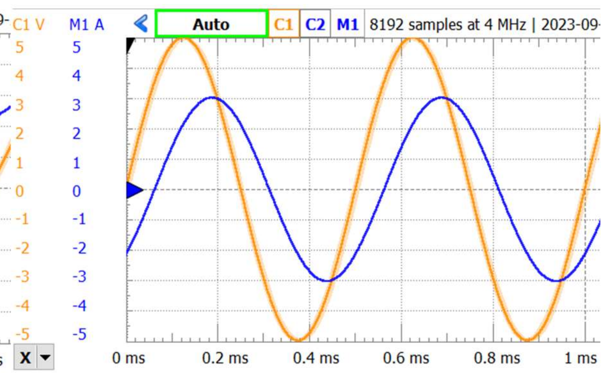
<u>Background</u>: The phasor diagram is the prediction from theory for $\tilde{V}$ on channel C1 (orange) and $\tilde{I}$ on channel M1 (blue) at 1 kHz.



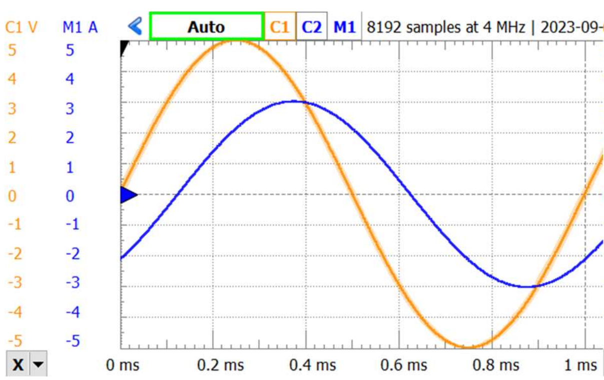
<u>Problem</u>: Which of the oscilloscope traces is an experiment that matches the theory? Explain your answer.



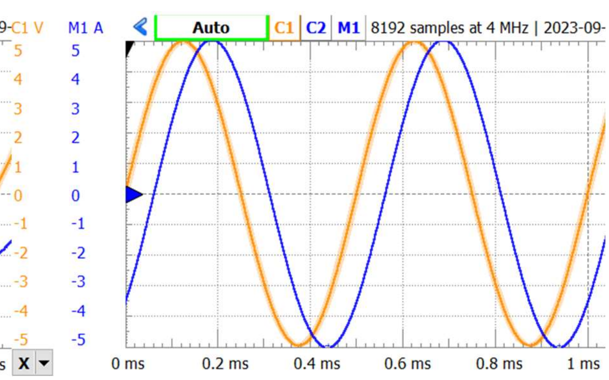
(A)



(B)



(C)



(D)

11. Plot a sinusoid from mathematical form

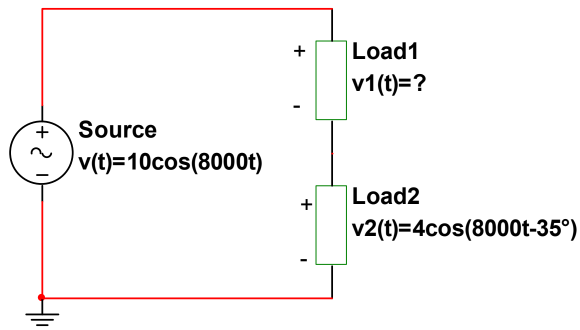
Problem: The phase reference is a voltage $v(t) = 4\cos(6000t + 0°) + 4$ and the output signal is the current $i(t) = 15\text{mA}\cos(6000t + 180°)$. Using mathematical software, plot $v(t)$ and $i(t)$ together on the same axes for two full periods of oscillation.

12. Plot a sinusoid from a phasor in polar form

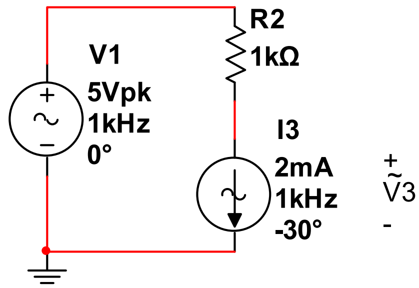
Problem: The phasor reference is the input voltage $\tilde{V}_{\text{in}} = 1\angle 0^{\circ}\ \text{V}$ and the signal is $\tilde{V}_{\text{out}} = 0.89\angle -26^{\circ}\ \text{V}$. Both phasors are at a frequency of 50 MHz. Plot $v_{\text{in}}(t)$ and $v_{\text{out}}(t)$ in the time domain on the same set of axes for two full periods of oscillation.

13. Using phasors in AC circuit KVL

Problem: Determine $v_1(t)$ in the circuit shown below using phasors. Use proper notation.



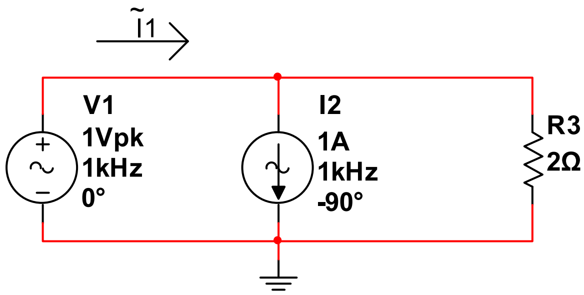
14. Series current source AC circuit



Problem:

Determine the phasor voltage drop across the current source $\tilde{V}_3$ in polar form, angle in degrees.

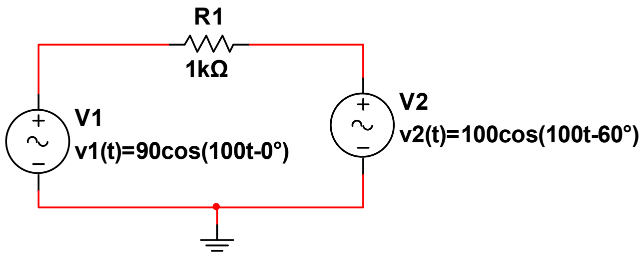
15. Parallel current source AC circuit



Problem:

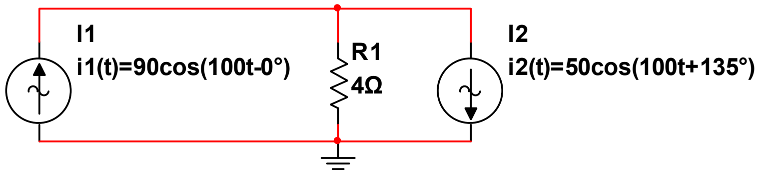
Determine the phasor current output of the voltage source $\tilde{I}_1$ in polar form, angle in degrees.

16. Multisource series resistor power



Problem: Determine the time-average power delivered to the resistor. (remember peak vs. rms!)

17. Multisource parallel resistor power



Problem: Determine the time-average power delivered to the resistor. (remember peak vs. rms!)

18. RMS voltage and current values for AC power determination

Problem: If $v(t) = 34 \sin(377t)$ and $i(t) = 0.27 \sin(377t)$, determine (a) the effective (RMS) values and (b) the average power.

19. AC power calculations given resistance and either voltage or current

Problem: Assume that each of the following signals has been measured either across or through a 100 Ω resistor. Determine the RMS value of the signal and the average dissipated power.

a. $i(t) = 75.0 \sin(2\pi 100t)$ mA

b. $v(t) = 24 \sin(2\pi 500t)$ V

c. $v(t) = 24 \sin(2\pi 500t + 90°)$ V

d. $i(t) = 150 \cos(2\pi 100t - 90°)$ mA

e. $v(t) = 120 \sin(2\pi 500t - 30°)$ V

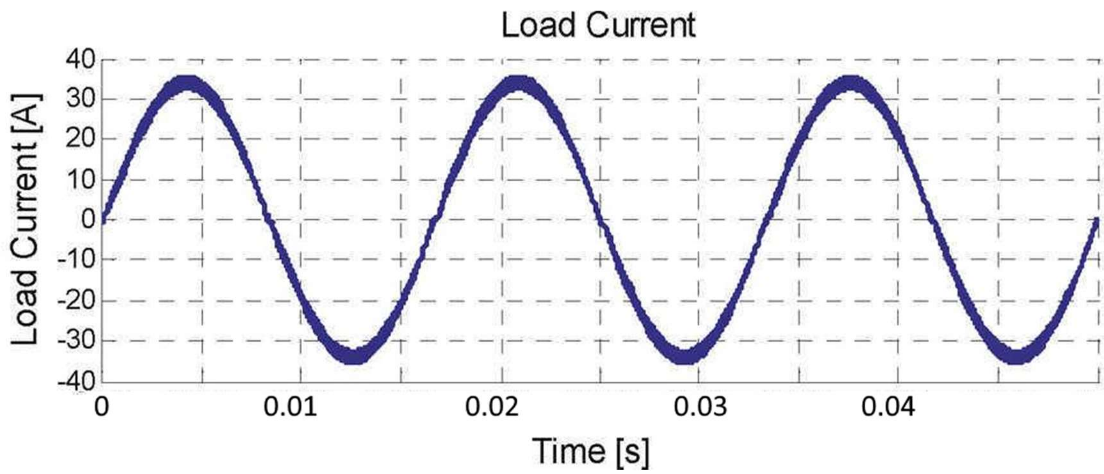
f. $i(t) = 150 \cos(2\pi 100t + 210°)$ mA

Standard Problems

Sinusoidal forms

20. Solar inverter load current

Background: Inverters convert the DC power produced by solar panels into AC power to feed the electric grid. Although the current produced is only approximately sinusoidal (a more perfect sine wave increases costs), the output can still be well-represented by a sine wave. The output current of an inverter is shown in the graph.[18]



Problem:

Express the load current as an approximate cosine function of time

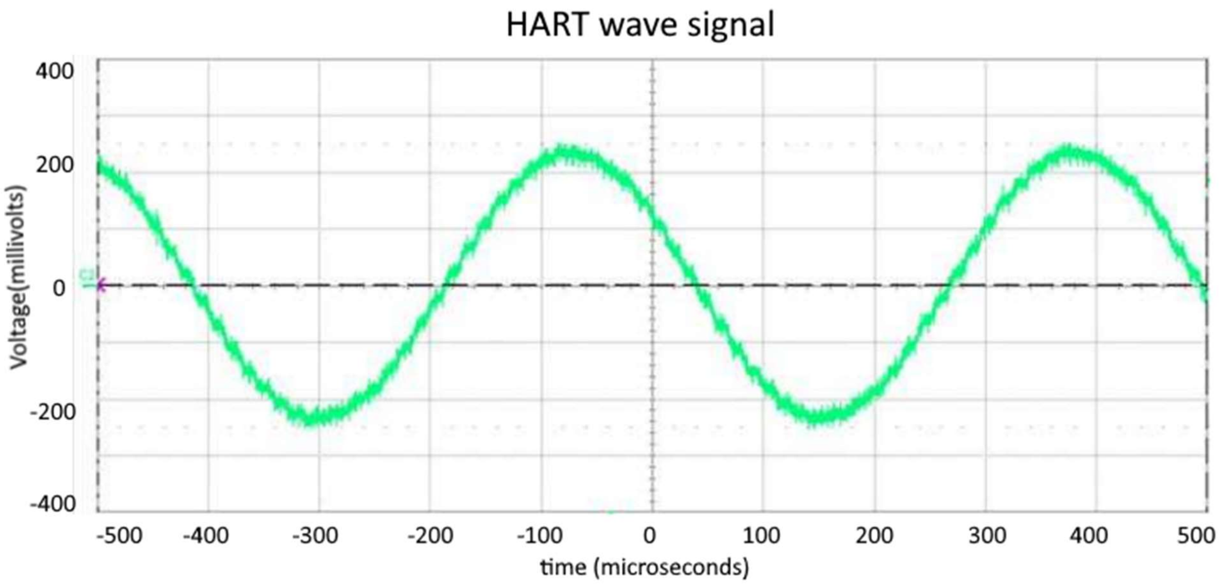
$$i(t) = \boxed{}\cos\left(\boxed{}\,t + \boxed{}^\circ\right) + \boxed{}.$$

Make amplitudes, offsets, and frequencies accurate to 15% and phase angles accurate to 15º.

[18] Gerardo, Vázquez-Guzmán, Martínez-Rodríguez Pánfilo Raymundo, and Sosa-Zúñiga José Miguel. "High efficiency single-phase transformer-less inverter for photovoltaic applications." Ingeniería, Investigación y Tecnología 16, no. 2 (2015): 173-184. seeking permission

21. HART modem signal

Background: HART sensors use frequency shift keying to transmit digital data on pre-existing analog sensor lines. The frequency of the sinusoid is varied to send digital messages: 1.2 kHz is a digital 𝟙 and 2.2 kHz is a digital 𝟘. The figure[19] shows the measured signal on an oscilloscope.



Problem:

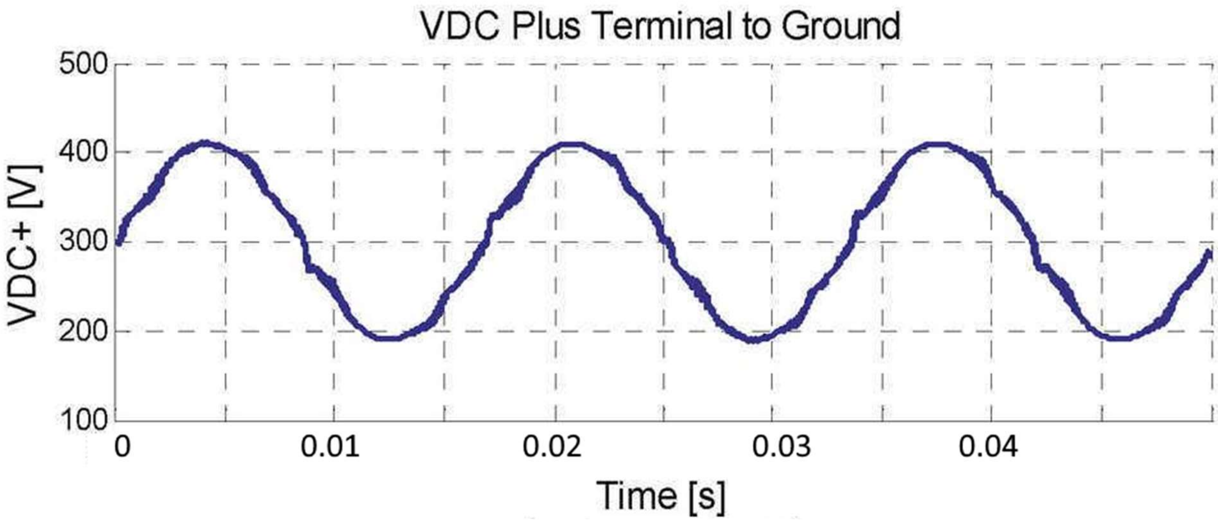
a) Is the sensor transmitting a digital 𝟙 or a digital 𝟘?
b) Express the observed voltage as a cosine function of time

$v(t) = \boxed{} \cos\left(\boxed{}\, t + \boxed{}^\circ\right) + \boxed{}$ (Reading graphs is approximate, make amplitude, offset, and frequencies accurate to 15% and angles accurate to 15 degrees)

[19] Matthew Sauceda, "TI Designs: TIDA-01504 Highly Accurate, Loop-Powered, 4- to 20-mA Field Transmitter With HART Modem Reference Design". Texas Instruments, December 2017 seeking permission

22. Solar inverter common mode voltage

Background: Inverters convert direct current (DC) produced by solar panels into alternating current (AC) to feed the electric grid. An undesirable "common mode" AC voltage shown in the graph[20] can emerge on the metal frame of the panel, creating an electric shock hazard for workers. High voltage AC is much more dangerous than high voltage DC.



Problem:

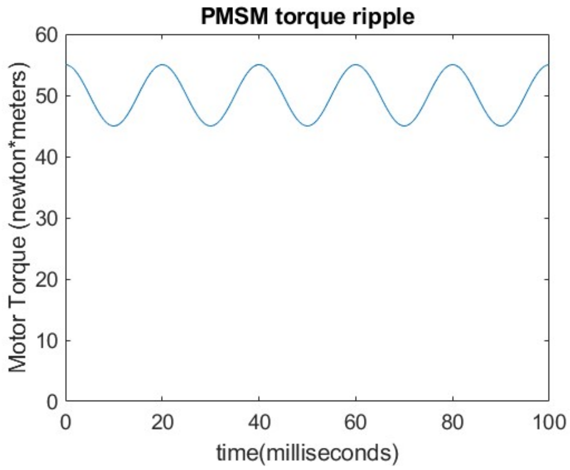
Express the voltage as a cosine function of time $v(t) = \boxed{} \cos\left(\boxed{}\, t + \boxed{}^\circ\right) + \boxed{}$

Make amplitudes, offsets, and frequencies accurate to 15% and phase angles accurate to 15°.

[20] Gerardo, Vázquez-Guzmán, Martínez-Rodríguez Pánfilo Raymundo, and Sosa-Zúñiga José Miguel. "High efficiency single-phase transformer-less inverter for photovoltaic applications." Ingeniería, Investigación y Tecnología 16, no. 2 (2015): 173-184. Seeking permission

23. PMSM cogging torque ripple

Background: The construction of permanent magnet synchronous motors causes ripple in their torque production, which causes vibration problems at low rotation speeds.



Problem:

Express the motor torque as a cosine function of time

$T(t)=\boxed{}\cos\left(\boxed{}\,t+\boxed{}^{\circ}\right)+\boxed{}$ with amplitudes, offsets, and frequencies accurate to 15% and phase angles to 15°.

24. Zero current switching

Background: Suddenly turning off an AC motor can damage switches because stored magnetic field energy from the flowing current will damage the switch. To protect the switch, open the switch during one of the two moments per cycle that the current is zero. The voltage applied to an AC motor is $v(t) = 340\cos(314t + 0°)$ and the current that flows as a result is $i(t) = 0.43\cos(314t - 75°)$ A .

Problem:

a) Is this motor operated in South America (utility period 16.66 ms) or in Asia (utility period 20 ms)?

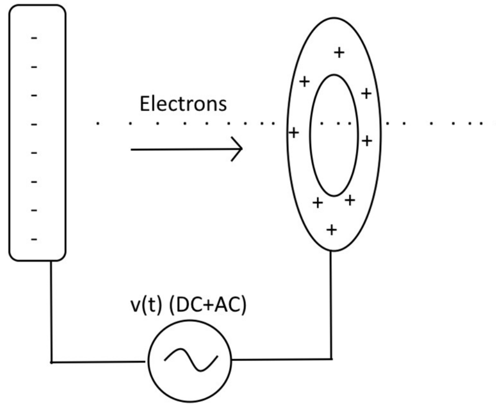
b) Does current lead voltage or does current lag voltage? Explain.

c) From the phase shift, determine the time shift between the voltage peak and the current peak.

d) At what time after $t = 0$ is the current zero, the optimum time to open the switch?

25. RF modulated electron gun voltage

Background: Radio frequency modulated electron guns produce short, concentrated bunches of electrons for use in particle accelerators. Electrons are only emitted when the total AC and DC voltage has a positive sign.



Problem:
An electron gun is stimulated by a sinusoidal radio frequency voltage with an amplitude of 170 V pk at a frequency of 3 GHz and a -115 V negative DC offset.

a) Explain why the maximum instantaneous voltage is *not* equal to the amplitude of the AC voltage.

b) Express the voltage as an offset cosine function, set as its own phase reference.

time. $v(t) = \boxed{} \cos\left(\boxed{}\, t + 0°\right) + \boxed{}$

c) Plot $v(t)$. Choose an appropriate time scale so that two to four full periods of oscillation are visible.

26. Electret microphone waveform

Background: Human whistles are approximately sinusoidal. Whistling into an electret microphone produces an approximately sinusoidal current proportional to the sound volume and at the same frequency. The microphone also draws an additional constant DC offset current from the power supply regardless of sound level.

Problem:

A whistle with frequency 2.1 kHz lands on a microphone, causing an 80 μA peak AC current a half-cycle out of phase with the sound pressure (phase reference), and has a DC offset of 350 μA. Write the microphone current as an offset cosine function.

$$i(t) = \boxed{} \cos\left(\boxed{}\, t + \boxed{}^{\circ} \right) + \boxed{}$$

27. Plot the sinusoids from the verbal description

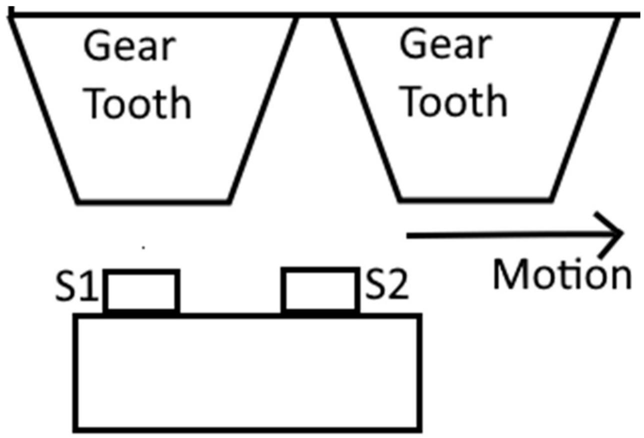
Problem: The phase reference is a pure AC voltage signal $v_1(t)$ with amplitude 1 V and a period of $10\ \mu\text{s}$. The output signal $v_2(t)$ has the same period, a DC offset of +2.5 V, an amplitude of 2 V, and lags the reference by a quarter cycle. Plot $v_1(t)$ and $v_2(t)$ together on the same set of axes for two full cycles of oscillation.

Phasors

28. Differential hall effect sensor sinusoid addition

Background: Differential Hall sensors measure the rotation of engines by sensing the magnetic field of moving iron gear teeth. The two sensors S1 and S2 produce signals of equal amplitude, equal frequency, and equal DC offset, but out-of-phase because the sensors are mechanically offset from each other as shown in the figure.

$v_1(t) = 0.03\ \text{V}\cos(1800t + 0^\circ) + 0.1\ \text{V}$ $\qquad$ $v_2(t) = 0.03\ \text{V}\cos(1800t + 150^\circ) + 0.1\ \text{V}$



Problem:

The signal processing electronics uses the difference of the two sensor voltages $v_s(t) = v_1(t) - v_2(t)$ to eliminate undesirable signal components.

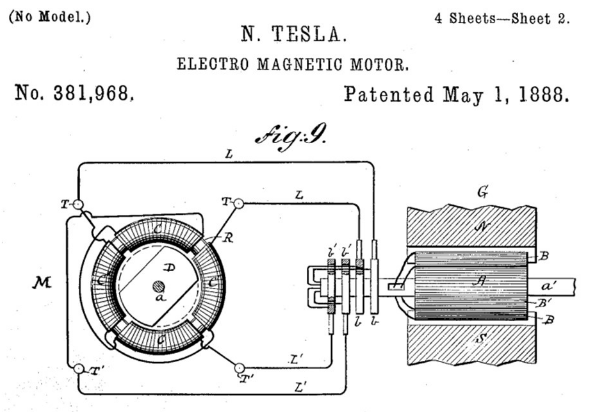
a) The DC portion of the magnetic field carries no information about the speed of the engine and is discarded. Explain why the signal $v_s(t)$ has zero DC offset.

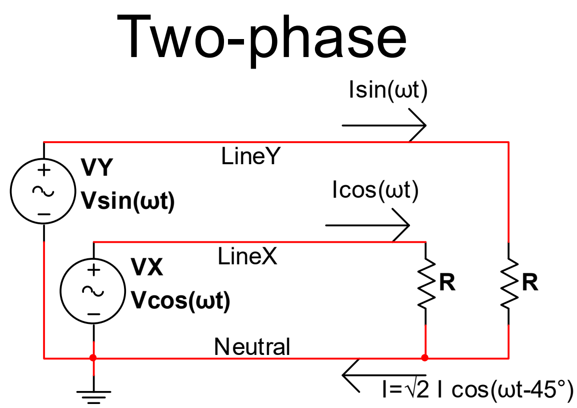
b) Express the signal voltage as a cosine function.

$v_s(t) = \boxed{}\cos\left(\boxed{}\, t + \boxed{}^\circ\right) + \boxed{0}$, including units on all quantities.

29. Two-phase motor neutral current

Background: Before the development of modern three-phase power, early electrical engineers experimented with two-phase[21] motors. Two generators with identical frequency and voltage drive currents a quarter cycle out of phase: one is a sine, the other a cosine.





Problem: (Solve the problem all symbolically)

a) Convert the two sinusoidal currents $i_X(t) = I\cos(\omega t)$ and $i_Y(t) = I\sin(\omega t)$ into complex numbers $\tilde{I}_X$ and $\tilde{I}_Y$. Assume a cosine phase reference.

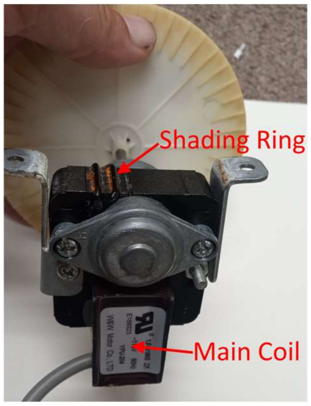
b) The phasor sum of the two currents flows through the neutral wire. Determine the phasor neutral current $\tilde{I}_n = \tilde{I}_X + \tilde{I}_Y$

c) Show that the current in the neutral wire has greater magnitude than either of the two phase currents, but magnitude less than double the current in one phase.

[21] Nicola Tesla, "Electro Magnetic Motor" U.S. Patent 381968 May 01 1888

30. AC Shaded pole motor phasors

Background: Shaded pole AC motors are used in small fans. Shaded pole motors have two coils (loops of wires): 1) a single turn of thick copper called a *shading ring*, and 2) the main coil. Both are wrapped around the same iron core (see the figure below). The peak phasor current in the main coil is $\tilde{I}_m = 100\angle -70^\circ \text{ mA}$ and in the shading ring is $\tilde{I}_s = 80\angle -160^\circ \text{ A}$. The motor torque depends on both the *instantaneous* current in the main coil $i_m(t)$ and in the shading ring $i_s(t)$.



Problem:

a) The motor is operated with 60 Hz AC. Write the instantaneous current in the main winding $i_m(t)$ and the instantaneous current in the shading ring $i_s(t)$ as cosine functions of time with all waveform constants inserted.

b) Provide evidence that the two currents are never close to zero at the same time, so the motor torque is never close to zero. Hints: The difference between the phase shifts; make a plot.

31. Compressor motor total current

Background: Split phase motors (like the compressor in the figure) have two electromagnet windings in parallel. The main winding carries current $i_m(t) = 2\cos(314t - 32^\circ)$ A and an auxiliary winding with current $i_a(t) = 0.63\cos(314t + 76^\circ)$ A . The out-of-phase currents in the windings provide consistent torque to the compressor as it rotates (unlike other kinds of single-phase motors).

Problem:

a) Draw the two phasor currents in the complex plane, then graphically estimate the total phasor current drawn from the mains.

b) Numerically calculate the phasor sum of the two currents in polar form. Compare this result to your result in a).

32. Three phase neutral current

Background: Large, powerful motors use three phase power with three identical coils. The coils carry currents that are a third of a cycle out-of-phase with each other (see the schematic) to provide a consistent torque.

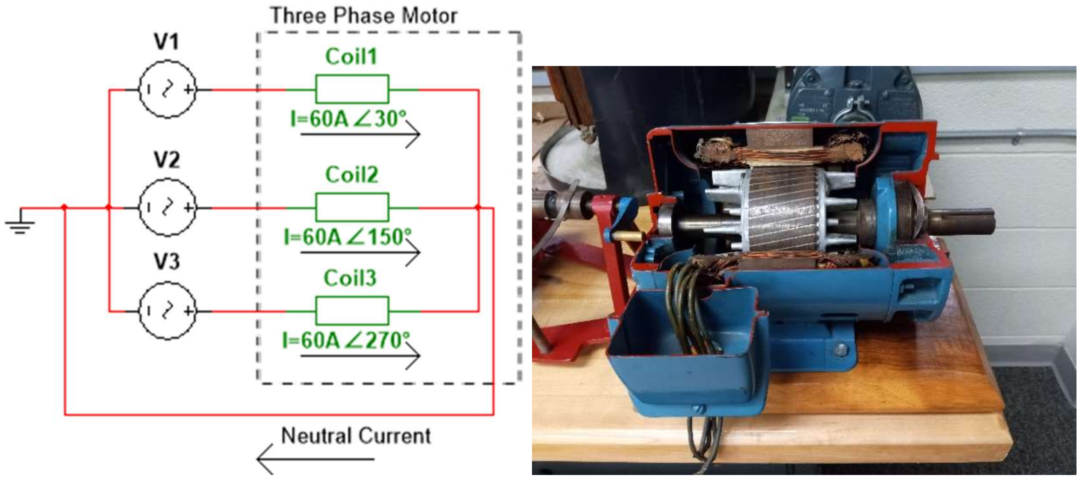


Figure x.x Diagram of a three-phase motor circuit and a motor photo

Problem:

a) Draw each of the three coil currents as phasors in the complex plane.

b) By KCL, the neutral current is the sum of the three coil currents. Using your diagram, explain why the current in the neutral wire is exactly zero for a perfectly balanced system. (In practical motors, the neutral current is only approximately zero.)

c) Use a calculator or software to prove the neutral current is zero.

33. Noise Cancellation phasors

Background: Noise cancelling headphones cancel unwanted noise by driving the speakers to emit anti-noise a half-cycle out of phase. The cancellation is imperfect because the anti-noise is not *exactly* 180º shifted from the noise. The total sound pressure in mPa (milli-pascals) is $p(t) = \underbrace{10\cos(1000t)}_{\text{noise}} + \underbrace{10\cos(1000t + 181°)}_{\text{antinoise}}$. ($p$ is pressure, not power, as a function of time)

<u>Problem</u>:

a) Draw the noise and anti-noise as phasors in the complex plane. (Exaggerate the extra $1°$ of phase shift for visibility.)

b) What is the amplitude of the total sound pressure?

c) The noise pressure has been reduced by…

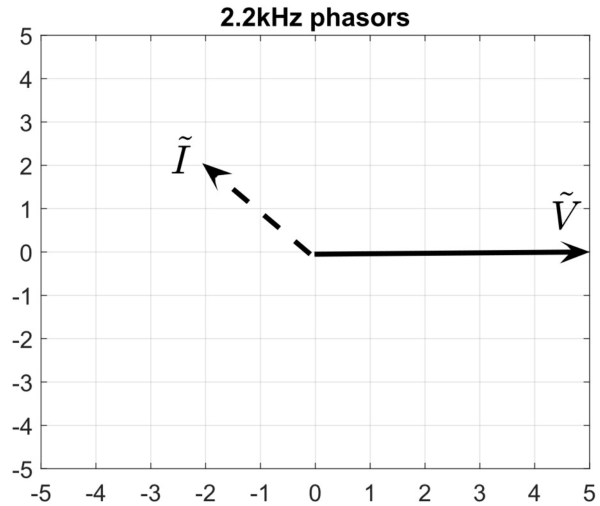
   1) More than a factor of 10

   2) More than a factor of 100

   3) More than a factor of 1000

34. Plot a sinusoid from a phasor diagram

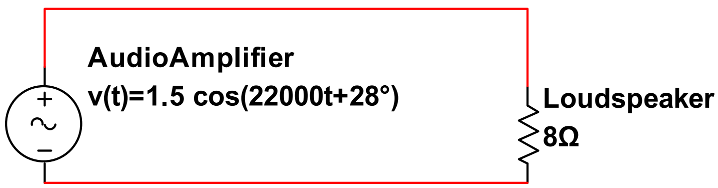
Problem: The phase reference voltage and output current phasors are shown in the figure. Both phasors are at a frequency of 2.2 kHz. Using mathematical software, plot $v(t)$ and $i(t)$ together on the same axes for two full periods of oscillation.



AC Resistive Circuits

35. Loudspeaker power rating

Background: Loudspeakers are driven with alternating current because DC currents produce no sound. Treat the loudspeaker as a resistive load.



Problem:

a) Humans can hear sounds in the frequency range from 20 Hz to 20 kHz. Show that the tone (frequency) coming from the audio amplifier is audible.

b) The loudspeaker's magnetic coils will burn if they dissipate more than a quarter watt of average power. Show that the loudspeaker will not melt.

36. Analog Discovery 2 Oscilloscope attenuator

Background: Like all oscilloscopes, the Analog Discovery 2 has a maximum AC voltage that can be measured safely: 25 V peak. To measure a 100 V peak voltage safely, one could use a voltage divider circuit called an *attenuator*. The attenuator must have an output voltage smaller than the input by a predictable ratio and draw very small current that will not disturb the system being measured.

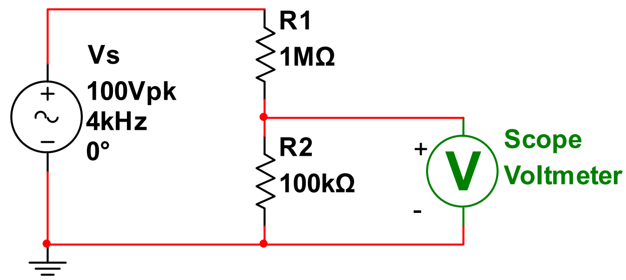


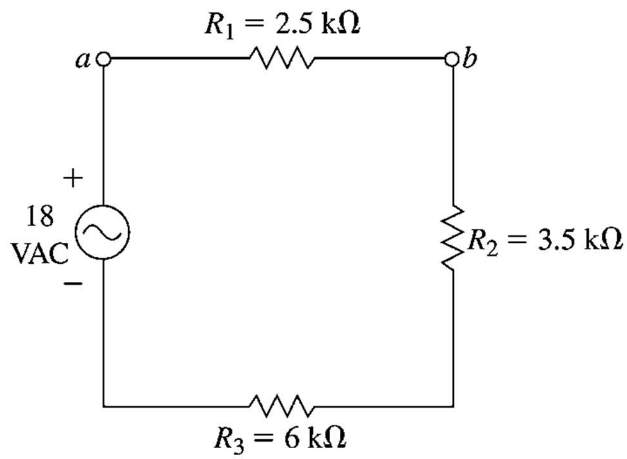
Image courtesy of Digilent, Inc.

Problem:

a) Show that the current that flows through the attenuator is very small, less than 1 mA peak.
b) Write the expression for the voltage $v(t)$ measured by the oscilloscope as a cosine function of time.

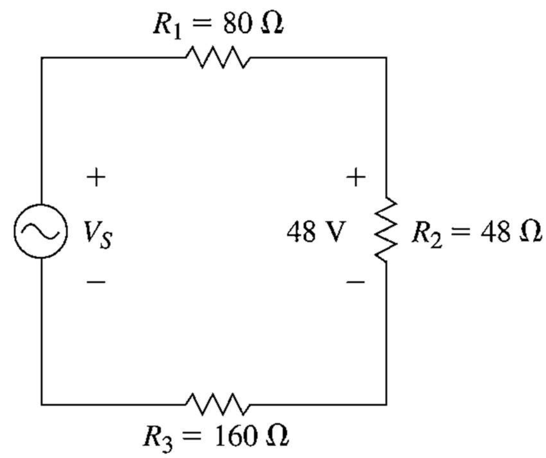
37. AC power calculation in a series resistive circuit

Problem: The source voltage is RMS in the circuit below. Determine (a) R_T, (b) V_{ab}, and (c) P_2. (d) Repeat part (c) if the source value is peak.



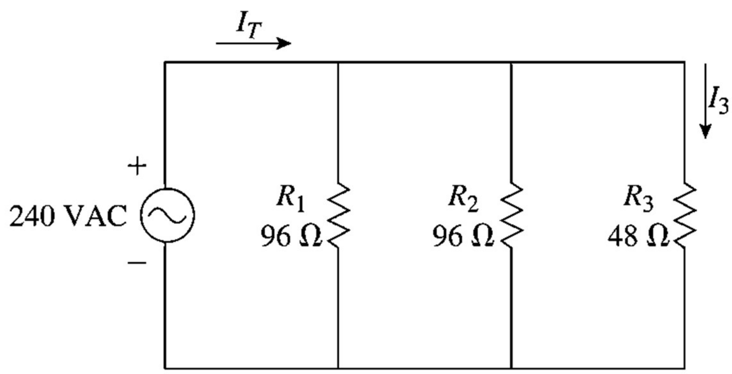
38. AC power calculation in a series resistive circuit with an unknown source voltage

Problem: For the circuit below, the known voltage is RMS. Determine (a) V_S, (b) I_2, and (c) P_T. (d) Repeat part (c) if the known voltage is peak.

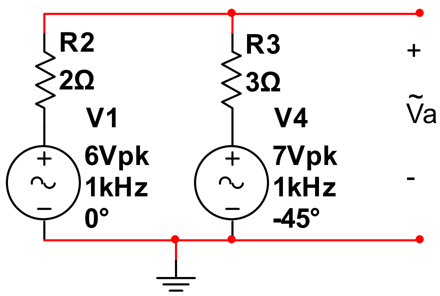


39. AC power calculation in a parallel resistive circuit

Problem: The voltage source is RMS in the circuit shown below. Determine (a) R_T, (b) I_3, and (c) P_T.

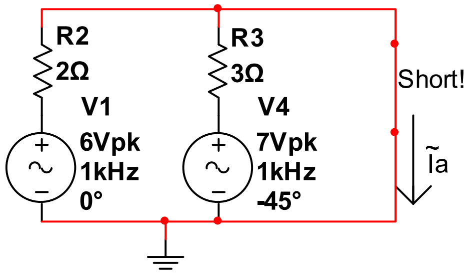


40. Multisource AC circuit open circuit voltage



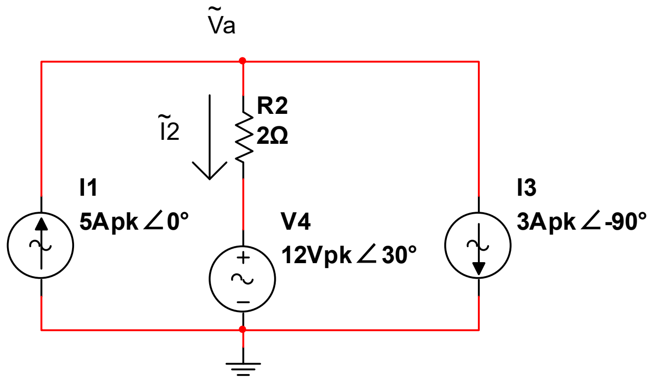
Problem: Determine the phasor voltage $\tilde{V}_a$ in polar form, angle in degrees.

41. Multisource AC circuit short circuit current



Problem: Determine the current that flows the through the short circuit, $\tilde{I}_a$.

42. Multisource branch current circuit



Problem:

a) Determine the phasor current through the resistor $\tilde{I}_2$ in polar form, angle in degrees.

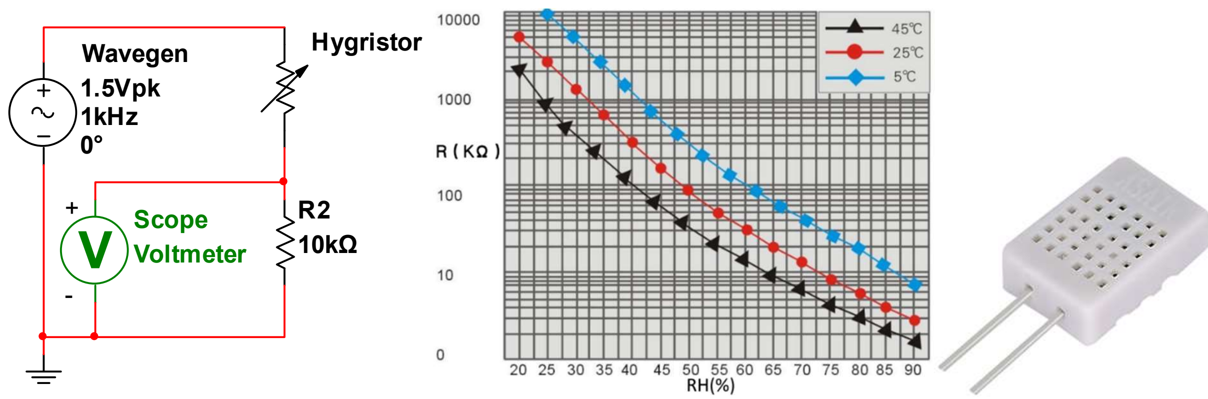
b) Determine the phasor voltage at the top node $\tilde{V}_a$ in polar form, angle in degrees.

c) Determine the average power delivered to the resistor (remember peak vs RMS).

Challenge Problems

43. Humidity sensor voltage divider

Background: Resistive humidity sensors (hygristors) have high resistance when the air is dry and low resistance when the air is humid. Direct current chemically damages the sensor, so they are driven with alternating current. The humidity-resistance relationship for the Aosong HR202L[22] model hygristor is shown in the graph.



Images courtesy of Guangzho Aosong Electronic Co., Ltd.

Problem:

a) The AC voltmeter measures a sinusoidal voltage with peak value 0.5 volts when the sensor is at 25 Celsius. What is the relative humidity?

b) Power dissipation greater than $200\ \mu\text{W}$ damages the hygristor. Show that this instrumentation setup will not damage the hygristor.

[22] Datasheet HR202L, ASAIR, November 2022, http://aosong.com/m/en/products-110.html

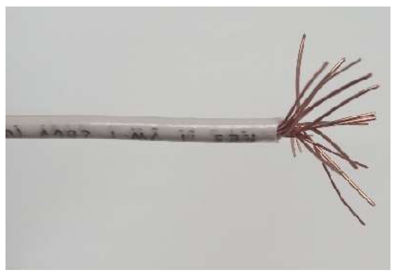
44. The skin effect increases resistance at very high frequencies.

Background: Usually resistance is independent of frequency. In excellent conductors at very high frequencies, the current flows near the conductor surface, a phenomenon called *skin effect*.[23] The effective cross-sectional area of the conductor is decreased. When the skin effect is present, $R_{\text{wire}} = \rho\ell / A$ is replaced by:

$$R_{\text{skin}} = \frac{\ell}{\pi d}\sqrt{\frac{\omega\mu_0\rho}{2}}$$

where : R_{skin} is the resistance of the conductor, in Ω

d is the diameter of the wire, $5\times10^{-4}\,m$
ℓ is the length of the wire, 1 m
ω is the angular frequency, in rad/s
μ_0 is the permeability of copper, 1.25 10^{-6} Henries/meter (H/m)
ρ is the resistivity of copper, 1.68×10^{-8} Ωm



Problem: Given a wire carries a high frequency AC current of $i(t) = 5\cos\left(3\times10^{8} t\right)$ A ,

Determine:

a) The RMS current and the angular frequency.

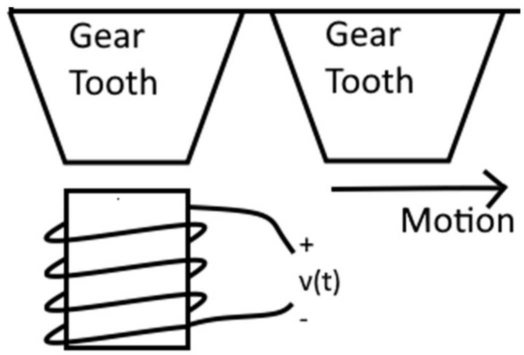
b) The power dissipated in the wire at low frequencies where the skin effect is not present.

c) The power dissipated in the wire at high frequencies where the skin effect is present.

[23] Magnetic forces push the current out of the center of the conductor.

45. Variable Reluctance speed sensor sinusoid

Problem: Variable reluctance sensors measure an engine's rotational state to determine ignition timing. The signal voltage is (approximately) a pure AC sinusoid with properties that depend on the mechanical rotation rate.



Problem:

a) Faster rotation produces larger signal voltages, 0.2 Vpk for each Hz of mechanical rotation speed. If shaft rotates at 7200 RPM, what peak voltage is produced?

b) The electrical frequency is 24 times the mechanical frequency, because the gear has 24 teeth. What is the electrical angular frequency ω?

c) The measured voltage waveform has a falling zero crossing at $t = 0$, when the pickup is directly over the center of a gear tooth. What is the phase angle of the measured voltage?

d) Plot $v(t)$ using software.