Stresses, forces, forces and Résumé

strains:

This document defines the quantities characterizing the stresses, the forces and the strains inside a structure in a computation by finite elements in displacement and how that is translated in *Code_Aster*. The statement of these quantities is given for the finite elements of mechanics: continuum 2D or 3D, shells and beams.

1 Statique

1.1 Contraintes

the postulate of Cauchy is that the forces of contacts exerted in a point by part of a continuum on another depends only on the norm on the surface in this point delimiting the parts.

In accordance with this postulate, one calls vector forced, for the nonmicropolar mediums, $\mathbf{F}(\mathbf{n})$ the vector which characterizes the forces of contact exerted through a surface element DS of norm $\mathbf{n}$ on part of a continuum [bib1].

It is shown [bib3], then, that the dependence in a point built-in from $\mathbf{F}$ report with the norm $\mathbf{n}$ is linear and that there exists a tensor which one calls tensor of the stresses σ such as:

$$\mathbf{F}(\mathbf{n})=\sigma\,\mathbf{n}$$

The unit of the stresses in international system is it $\mathrm{N.m^{-2}} \equiv \mathrm{Pa}$.
For the group of structure "the stress state" is characterized by a field of tensor of the stresses which one more simply indicates by stress field.

1.2 En ce qui concerne

force structures of beams or shells, contrary with the case of the continuum, it should be noted that:

- only the normal directions $\mathbf{n}$ of the cuts according to tangent space with the variety are possible,
- the characteristic quantities are obtained by integration in the section or the thickness of the quantities defined for the continuums.

1.2.1 Cases of discrete

Les discrete are finite elements which can not have of a physical size. They are represented by their stiffness matrix. The forces are obtained by the multiplication of this matrix by the vector displacement:

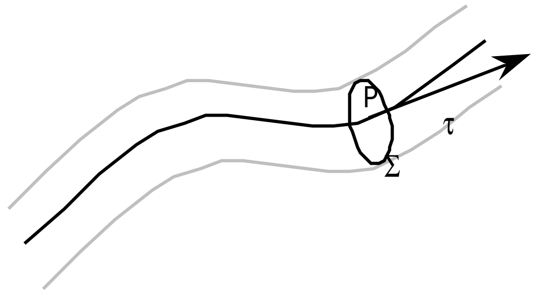
$$\begin{pmatrix} F \\ M \end{pmatrix} = [k] . \begin{pmatrix} D \\ R \end{pmatrix}$$

1.2.2 Case of the beams

One calls force, the end cells (F, M) in P, geometrical centre of inertia of the cross-section Σ, the torsor resulting from the forces of contact exerted on the section [bib2].

With the preceding notations:

$$\begin{aligned} F &= \int_{\Sigma} \mathbf{F}(\tau)\,ds & (N) \\ M_p &= \int_{\Sigma} \mathbf{PM} \wedge \mathbf{F}(\tau)\,ds & (N.m) \end{aligned}$$



The force F breaks up into a normal force N and shearing forces T in the plane of the section while the moment M being exerted at the point P breaks up into one twisting moment and of the bending moments at the point P.
For the beams whose cross-section is not regarded as rigid these end cells are not sufficient: for example, for the beams taking of account the warping of the sections one is brought to consider an additional quantity of force due to warping (bimoment).
The multifibre beams (with local behavior 1D, connecting stresses to strains, in a certain number of points of the section) and the pipes (local behavior in plane stresses) provide at the same time the end cells of the beams but also a stress field for each fiber or sector.

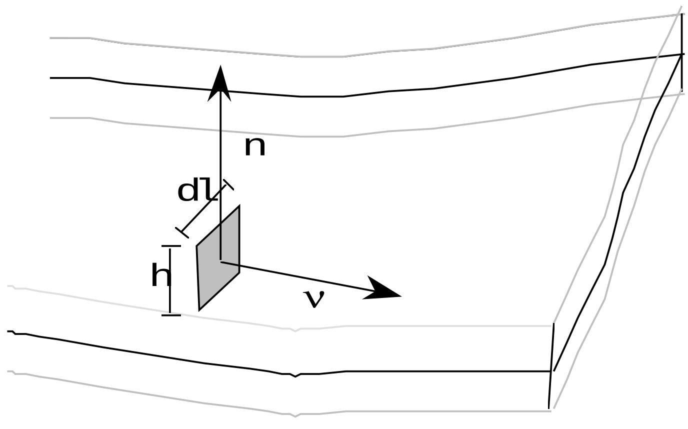
1.2.3 Case of the Soit

shells a point P of a surface medium S of thickness h, either an element length dl on S, or n the norm directing the shell in this point.

Are the end cells (F, M) in this point of a torsor resulting from the forces exerted through a surface element $dS=h\,dl$ of tangent n norm with S on part of S.
With the preceding notations:

$$F(P) = \int_{-h/2}^{+h/2} \mathbf{F}(\nu)\,dh \qquad (N)$$

$$M(P) = \int_{-h/2}^{+h/2} \mathbf{PM} \wedge \mathbf{F}(\nu)\,dh \qquad (N.m)$$



It is clear that M is in the tangent plane with S in P.
Either $N(P)$ the projection of $F(P)$ on the tangent plane with S in P and or, $T(P)$ its normal component with this tangent plane.

In the same way that for the continuums, one shows that there exist two symmetric tensors $\mathbf{N}$ and $\mathbf{M}$ a vector Q, defined in the tangent plane with S, such as:

$$\mathbf{F}=N\nu$$
$$T=Q\cdot\nu$$
$$\mathbf{M}=\mathbf{n}\wedge M\nu$$

(N, M, Q) the forces at the point are called P :

- the tensor N characterizes the membrane forces,
- the tensor M the bending moments,
- the vector Q the shearing forces.

Note:

- *There are no universal conventions on the denomination and the signs of these tensors. In particular, the tensor of the bending moments is sometimes taken with an opposite sign in the teaching and in practice of the French engineers of the civil engineer. Our convention is used in the great codex of finite elements and makes it possible to have the same sign for a beam and a plate such as $\tau=\nu$.*
- *For nonlinear materials, the constitutive law is evaluated in several points of the thickness but the balance equations always relate to the fields of force. It is not necessary to go down again to the stresses to define "the stress state".*

Restrains with the Dans

stress field these conditions is a reference whose third component is carried by $\mathbf{n}$, one has ($\alpha, \beta=1 \text{ ou } 2$):

$$\begin{aligned}
N_{\alpha\beta} &= N_{\beta\alpha} &&= \int_{-h/2}^{+h/2} \sigma_{\alpha\beta}\,dh \\
M_{\alpha\beta} &= M_{\beta\alpha} &&= \int_{-h/2}^{+h/2} x_3\sigma_{\alpha\beta}\,dh \\
Q_{\alpha} & &&= \int_{-h/2}^{+h/2} \sigma_{\alpha 3}\,dh
\end{aligned}$$

1.3 Nodal forces

One calls equivalent nodal force or more simply nodal force, a vector F which is the representative of a linear form W (generally dependent on an energy) acting on fields of displacement $u(x)$ discretized by finite elements.
The fields of displacements $u(x)$ are expressed starting from its nodal values which form a vector q and shape functions $\Phi_i(x)$ by:

$$u(x)=\sum_i q_i \Phi_i(x)$$

Under these conditions:

$$w(u)=\sum_i q_i F_i$$

Note:

- *The concept of node here is very general and wants to say, in fact, carrying degree of freedom (that it is of Lagrange or Hermite).*

- *The concept of displacement is also very general and includes the concept of generalized displacement including of the translations and rotations.*

1.4 Representation of the fields

There are several ways of representing the fields in a modelization by finite elements:

- for the continuous-current fields on all the field, one uses the values with nodes (CHAM_NO of *Code_Aster*)
  $$u(x)=\sum_i u_i \Phi_i(x)$$
  one speaks then about displacements to the nodes, nodal stresses or nodal efforts,

  Remarque:

  The stress fields or of forces are generally computed at the Gauss points, if they continuously are represented it is only at ends of visualization.

- for the other fields, one uses the values in certain points characteristic of the elements (Gauss points or nodes).
  One speaks then about stresses by elements to the nodes or forces by elements to the nodes, or about stresses at the Gauss points or forces at the Gauss points.

1.5 Quantities associated in *Code_Aster*

1.5.1 SIEF_R

quantity SIEF_R represents "the stress state" of structure, it thus contains, at least, the components:

- stress fields of the continuums (in total reference):

  SIXXSIYYSIZZSIXYSIXZSIYZ

- of the fields of forces of beam and discrete (in reference "user" with the beam, discrete):

  NVYVZMTMFYMFZ

- for the beams with warping, it is necessary to add bimoment (necessarily "user" with fiber identifies some):

  BX

- of the fields of forces of shell (necessarily in reference "user" on the surface):

  NXXNYYNXYMXXMYYMXYQXQY

Moreover, it is sometimes convenient to be able to directly exploit the fields of forces of beam and discrete in the total reference:

FXFYFZMXMYMZ

It is also interesting to represent the components of a stress field on the beam elements or shells in the reference "user". For that, one will use the same components as in total reference, although confusion is possible.

1.5.2 FORC_F and FORC_R

Ces quantities represent the applied forces with structure on an application interface.
For:

- a continuum it is thus a force vector,
- a beam, a torsor of forces,
- a shell, a torsor of forces.

This quantity must thus have the following components:

- for a continuum:

  FXFYFZ

- more for the beams and the shells:

  MXMYMZ

1.5.3 DEPL_R

Étant donné que in *Code_Aster*,

- field can be attached only to only one quantity,
- that methods of finite elements mixed (mixing unknown of standard displacement and unknown factors of nodal the forces type) are not excluded,
- that the dualisation of the boundary conditions results in having for unknown a comprising vector of the variables of Lagrange which are nodal forces in the sense that it higher was specified,
- that it is necessary to be able to carry out any type of linear combination on the nodal forces,
- who the classification of the unknown factors must be the same one as that of the second members,

the nodal forces (dual within the meaning of the energy W of nodal displacements) necessarily the same components have as displacements namely:

DXDYDZDRXDRYDRZ

more, for the beams with warping, the degree of freedom associated with bimoment: GRX.

1.6 Calcul

1.6.1 computation options of the stress state

1.6.1.1 Champ SIEF_ELGA

It is the field representative of the stress state and making it possible to continue computations (geometrical rigidity, nodal forces, etc). It is expressed at the Gauss points (and is possibly at subpoints for the structural elements). The prefix of this field is SIEF, because according to the elements, it contains stresses or forces.

Computation option	Symbolic name of result concept	Computation carried out	3D, 2D, Coques1D PIPE multifibre Beams	Beams: POU_D_T POU_D_E POU_D_TG POU_D_T_GD Discrete	Plates: DKT DST Q4G Q4GG COQUE_3D
SIEF_ELGA	SIEF_ELGA	starting from a field of displacement in linear elasticity	σ	(F, M) in reference “user”	σ in reference “user” *

RAPH_MECA FULL_MECA	SIEF_ELGA	into nonlinear	σ	(F, M) in reference "user"	σ in reference "user" *

(*) for the shell elements and of shell, the reference "user" is that definite starting from the data of the user (key word ANGL_REP or VECTOR in AFFE_CARA_ELEM/SHELL).

These options thus compute:

- the stress field for the elements of continuous medium 2D and 3D, and elements with local behavior: COQUE_3D, plates, shells 1D (COQUE_AXIS, COQUE_D_PLAN, COQUE_C_PLAN), pipes, beams multifibre, in each "subpoint" of integration (layers in the thickness of the shells, fibers, sectors angular and position in the thickness for the pipes). The reference "user" of the plates and shells can be specific to each element.
- the field of forces for the beams (torsor).

1.6.1.2 Field SIGM_ELGA

It acts of the field representative of the stress state at the Gauss points (or possibly at the subpoints for the structural elements). The prefix of this field is SIGM because this field contains only stresses. It is an extraction of the stresses contained in field SIEF_ELGA.

Computation option	Symbolic name of result concept	Computation carried out	3D, 2D, Coques1D PIPE multifibre Beams	Beams: POU_D_T POU_D_E POU_D_TG POU_D_T_GD Discrete	Plates: DKT DST Q4G COQUE_3D
SIGM_ELGA	SIGM_ELGA	starting from the field SIEF_ELGA, extraction of the stresses	σ	non-available	σ in reference "user" *

1.6.2 Autres representations of the stress state

1.6.2.1 Champ SIEF_ELNO and SIEF_NOEU

It acts of fields representative of the stress state at ends of exploitation (printing or postprocessing of visualization) to the nodes by element (or possibly at the subpoints for the structural elements) and to the nodes of the element. According to the elements, they contain stresses or forces.

Computation option	Symbolic name of result concept	Computation carried out	3D, 2D, Coques1D PIPE multifibre Beams	Beams: POU_D_T POU_D_E POU_D_TG POU_D_T_GD Discrete	Plates: DKT DST Q4G COQUE_3D
SIEF_ELNO	SIEF_ELNO	by extrapolation with the nodes of the quantities at the Gauss points	σ	(F, M) in reference "user"	σ in reference "user" **

	SIEF_NOEU	by arithmetic mean with the nodes of the quantities to the nodes by element	σ	(F, M) in reference "user"	σ in reference "user" *

1.6.2.2 Champ SIGM_ELNO and SIGM_NOEU

It acts of a field representative of the stress state at ends of exploitation (printing or postprocessing of visualization) to the nodes by element (or possibly subpoints for the structural elements) and to the nodes of the element. The prefix of this field is SIGM because this field contains only stresses.

Computation option	Symbolic name of result concept	Computation carried out	3D, 2D, Coques1D PIPE multifibre Beams	Beams: POU_D_T POU_D_E POU_D_TG POU_D_T_GD Discrete	Plates: DKT DST Q4G COQUE_3D
SIGM_ELNO	SIGM_ELNO	by non-available extrapolation with the nodes of the quantities at	σ	the Gauss points	σ in reference "user"
*	SIGM_NOEU	by arithmetic mean with the nodes of the quantities to the nodes by element	σ	non-available	σ in reference "user"

Remarques:

1) *In this case, confusion is possible between the components in reference user and those in total reference which bear the same name.*
2) *The 6 components delivered in the local coordinate systems by the beams and the shells contain possibly null terms according to the models used. For the most standard models:*
    - *three null terms for the beams,*
    - *two null terms for the shells.*

   Thus, the stress field will be complete and, especially, it could be enriched each time the modelization requires it (beam with shears, shell with pinching, etc…)

1.6.2.3 Field EFGE_ELGA, EFGE_ELNO and EFGE_NOEU

It acts of fields containing the forces on the beam elements or of shell at ends of exploitation (printing or postprocessing of visualization) at the Gauss points, the nodes by elements and the nodes.

Computation option	Symbolic name of result concept	Computation carried out	3D, 2Ds	Beams, pipes, beam multi - fibers, Discrets	Shells, plates
EFGE_ELGA	EFGE_ELGA	by non-available integration of	the stresses	(F, M) in reference "user"	(N, M, V) in reference "user"
EFGE_ELNO	EFGE_ELNO	starting from a field of displacement in non-available linear	elasticity	(F, M) in reference "user"	(N, M, V) in reference "user"

EFGE_ELNO	EFGE_ELNO	by integration of the stresses into nonlinear	non-available	(F, M) in reference "user"	(N, M, V) in reference "user"
*	EFGE_NOEU	by arithmetic mean with the nodes of the quantities to the nodes by non-available	element	(F, M) in reference "user"	(N, M, V) in reference "user"

1.6.3 Computation of the nodal forces and the generalized reactions

Les nodal forces generalized are computed starting from the stress state, only one option is envisaged:

Computation option	Symbolic name of result concept	Computation carried out	Solid elements (3D, 2D)	Beam, discrete elements	Shell
FORC_NODA	idem	starting from field SIEF_ELGA	Forces	Forces and Forces	moments and moments

Les nodal forces (dual within the meaning of the energy W of nodal displacements) have the same components as displacements namely:

DXDYDZDRXDRYDRZ

option REAC_NODA of operator CALC_CHAMP carries out a call to FORC_NODA and withdrawn:

- the loading in statics,
- the loading, inertia forces and viscous in dynamics (in the facts, the viscous contribution in dynamics is currently neglected in CALC_CHAMP).

For the solid elements, the FORC_NODA in general have the dimension of a force. It is about a field on the nodes of the mesh where the value in a node is obtained starting from the stresses computed on the convergent elements with this node, thus their values thus vary when the mesh changes. In the absence of distributed loading, the equilibrium imposes their nullity in an interior node, while they correspond to the reaction on the bearings where one imposes a kinematic relation (case of an imposed displacement).

In the case of the shells, components DX, DY and DZ give the FORC_NODA (dimension of a force) in the total reference of the mesh. These components are built with the normal force and cutting-edges in the shell. Components DRX, DRY and DRZ give the FORC_NODA (one moment dimension) in the total reference of the mesh, built with the bending moments in the shell.

2 Kinematics

2.1 Strains

2.1.1 Dans

Continuum this case, displacements of structure is represented by a field of vector u with three components in general.
The strain (on the assumption of the small disturbances) is defined by the strain tensor ε by (option EPSI_ELGA and EPSI_ELNO):

$$\varepsilon_{ij}(u)=\frac{1}{2}(u_{i,j}+u_{j,i})$$

One can want to calculate the "mechanical" strain, i.e. by cutting off thermal dilations (options EPME_ELGA and EPME_ELNO):

$$\varepsilon_{ij}^{m}(u)=\frac{1}{2}(u_{i,j}+u_{j,i})-\varepsilon^{th}$$

In the case of large displacements, the strains of Green-Lagrange are (options EPSG_ELGA and EPSG_ELNO):

$$E_{ij}(u)=\frac{1}{2}(u_{i,j}+u_{j,i}+u_{k,i}u_{k,j})$$

To which one can want to cut off the thermal strains (options EPMG_ELGA and EPMG_ELNO):

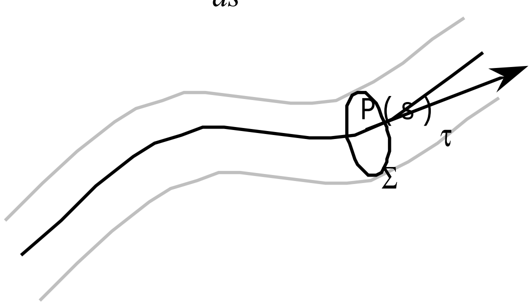
$$E_{ij}^{m}(u)=\frac{1}{2}(u_{i,j}+u_{j,i}+u_{k,i}u_{k,j})-\varepsilon^{th}$$

2.1.2 Case of the beams

Dans the traditional beam theories, each point P of the beam represents a cross-section. In fact thus the cells end of the torsor $(T(s),\Omega(s))$ of displacement of the presumedly rigid cross-section characterize the displacement of the point P to the curvilinear abscisse s . T is the translation of the centre of inertia of the section, $\Omega(s)$ the vector rotation of the section in this point.
The application of the theorem of the virtual wors (cf [bib2]) naturally led to define as strain the torsor (ε,χ) derived from $(T(s),\Omega(s))$ report to the curvilinear abscisse s :

$$\varepsilon=\frac{dT}{ds}+\tau\wedge\Omega$$

$$\chi=\frac{d\,\Omega}{ds}$$



Let us pose then:

$$\varepsilon=\varepsilon_L \tau+\gamma_T$$
$$\chi=\gamma_t \tau+\mathbf{K}$$

ε_L is the longitudinal deflection,

γ_T is the vector of the strains of distortion (no one on the assumption of Navier-Bernoulli),

γ_t is the strain of torsion of the section,

$\mathbf{K}$ is the strain of bending.

Note:

For the modelizations of beam with taking into account of warping, the kinematics are more intricate to describe, but they lead however to concepts close to those presented above.

2.1.3 Case of the Nous

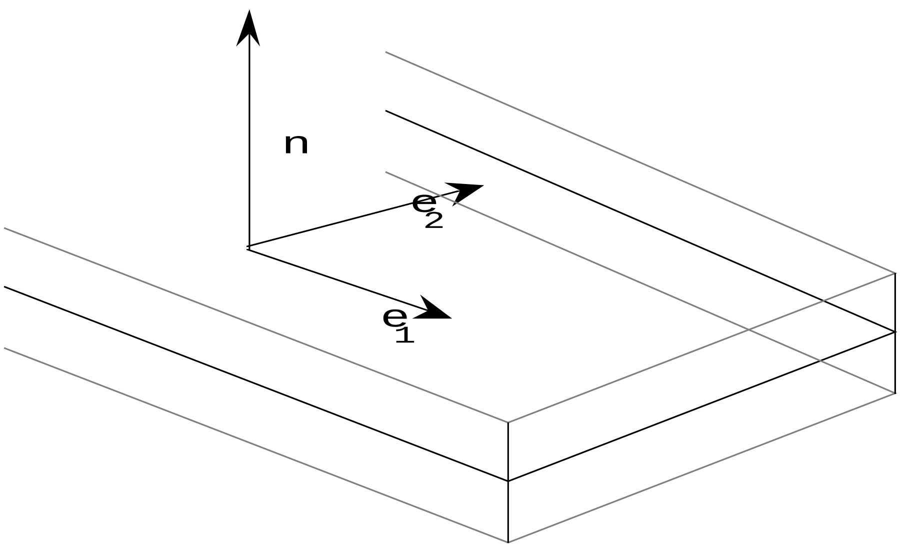
shells we will limit here to the cases of the plates. Indeed, in the general case of the shells:

- the spatial derivatives use too intricate mathematical concepts for the frame of this document, [R3.07.04],
- the shells are very often modelized by shell elements assembled.

In this case, in fact only the material norms are supposed to be rigid. The displacement of these norms is thus represented by the end cells of a torsor (T, Ω) . T is the translation of the point located on the average layer, Ω the vector rotation of the norm in this point.
It is clear that the normal component of Ω is null (in the case of nonmicropolar mediums). One introduces, the vector $\mathbf{I}$ in the tangent plane defined by:

$$\mathbf{I}=\Omega\wedge\mathbf{n}$$

where n is the normal vector directing surface.



Maybe, decomposition:

$$T = w\,\mathbf{n} + \mathbf{u}_T$$

$\mathbf{u}_T$ is tangent displacement,
w is the deflection.

In the same way that for the beams, the application of the theorem of the virtual wors (cf [bib2]) led to define as strain the whole formed by the tensors E and K the vector γ , all these quantities being defined in the tangent plane by:

$$\begin{aligned} E_{\alpha\beta} &= \frac{1}{2}\left(u_{\alpha,\beta} + u_{\beta,\alpha}\right) \\ K_{\alpha\beta} &= \frac{1}{2}\left(l_{\alpha,\beta} + l_{\beta,\alpha}\right) \\ \gamma_\alpha &= l_\alpha + w_{,\alpha} \end{aligned}$$

The strain is thus defined by 7 realities.

$E_{\alpha\beta}$ are the membrane strains,
$K_{\alpha\beta}$ are the opposite of the curvatures of the deformed average layer,
γ_α is the vector of strain of distortion.

Note:

There still, there is no universal convention and the disparity of conventions is even larger than for the tensors of forces.

Restrain with the three-dimensional strain field

Dans these conditions, one a:

$$\begin{aligned} \varepsilon_{\alpha\beta} &= E_{\alpha\beta} + x_3 K_{\alpha\beta} \\ \varepsilon_{\alpha 3} &= \gamma_\alpha \\ \varepsilon_{33} &= 0 \end{aligned}$$

2.2 Quantities associated in *Code_Aster*

2.2.1 DEPL_R and DEPL_C

Les quantities DEPL_R and DEPL_C have as components the degrees of freedom of the modelization by finite elements and thus do not have necessarily only the components of the fields of displacement which are:

DXDYDZ

with which it is necessary to associate for the beams or the shells:

DRXDRYDRZ

Pour the shells, we need the three components of the instantaneous axis of rotation, because the equation with the finite elements can be expressed only in one total cartesian coordinate system.

2.2.2 EPSI_R

quantity EPSI_R represents the structural deformations, therefore it must have, at least, the components:

- strain fields ε of the continuums (in total reference):

  EPXXEPYYEPZZEPXYEPXZEPYZ

- of the strain fields of beam (in reference “user” with the beam):

  EPXXGAXYGAXZKYKZGAT

- of the strain fields of shell (necessarily in reference “user” on the surface)

  EXXEYYEXYKXXKYYKXYGAXGAY

2.3 Computation options

2.3.1 Fields EPSI_ELGA, EPME_ELGA, EPSG_ELGA and EPMG_ELGA

It acts of fields containing the strains at the Gauss points and possibly at the subpoints of the elements.

Computation option	Symbolic name of result concept	Computation carried out	3D	Pipes, Beams multifibre	Shells, plates (except DKTG and Q4GG)
EPSI_ELGA	EPSI_ELGA	starting from a field of displacement in small strains	ε	ε in reference “user” 6 components	ε in reference “user”
EPSG_ELGA	EPSG_ELGA	Tensor of Green-Lagrange starting from a non-available field of	E	displacement	non-available
EPME_ELGA	EPME_ELGA	starting from a field of displacement and a field of temperature in small strains	ε^m	non-available	non-available
EPMG_ELGA	EPMG_ELGA	Tensor of Green-Lagrange starting from a field of displacement and a field of non-available	E^m	temperature	non-available

2.3.2 Champs EPSI_ELNO, EPME_ELNO, EPSG_ELNO and EPMG_ELNO

They are fields containing the strains whatever the modelization at ends of exploitation (printing or postprocessing of visualization) to the nodes and possibly at the subpoints of the elements.

Computation option	Symbolic name of result concept	Computation carried out	3D	Pipes, Beams multi_fibres	Shells, plates (except DKTG and Q4GG)
EPSI_ELNO	EPSI_ELNO	by extrapolation with the nodes of the quantities at the Gauss points	ε	ε. in reference "user"	ε identifies of it "user"
EPSG_ELNO	EPSG_ELNO	by non-available extrapolation with the nodes of the quantities at	E	Gauss points	non-available
EPME_ELNO	EPME_ELNO	by extrapolation with the nodes of the quantities at the Gauss points	ε^m	non-available	non-available
EPMG_ELNO	EPMG_ELNO	by extrapolation with the nodes of the quantities at the Gauss points	E^m	non-available	non-available
*	EPSI_NOEU	by arithmetic mean with the nodes of the quantities to the nodes by element	ε	ε. in reference "user"	ε. identifies "user of it "
*	EPSG_NOEU	by arithmetic mean with the nodes of the quantities to the nodes by element	E	non-available	non-available
*	EPME_ NOEU	by arithmetic mean with the nodes of the quantities to the nodes by element	ε^m	non-available	non-available

*	EPMG_NOEU	by arithmetic mean with the nodes of the quantities to the nodes by element	E^m	non-available	non-available

2.3.3 Champs DEGE_ELGA and DEGE_ELNO

It acts of fields containing the strains generalized on the beam elements or of shell at ends of exploitation (printing or postprocessing of visualization) at the Gauss points or the nodes of structure.

Computation option	Symbolic name of result concept	Computation carried out	3D	multifibre Beams, beams	Plates, Coques1D
DEGE_ELGA	DEGE_ELGA	starting from a field of displacement in small strains	non-available	(ε, χ) in reference “user”	$(\boldsymbol{E}, \boldsymbol{K}, \gamma)$ in reference “user”
DEGE_ELNO	DEGE_ELNO	by non-available extrapolation with the nodes of the quantities at	the Gauss points	(ε, χ) in reference “user”	$(\boldsymbol{E}, \boldsymbol{K}, \gamma)$ in reference “user”

3 Bibliographie

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