CS 4050 Algorithms and Algorithm Analysis

Project 3

Eryn Kelsey-Adkins
Ryan McCullough
Gil Leibovich

The following consists of only the double-struck arrow part of the proof found in Chapter 2: Efficiency Analysis, Project 3 Average Case Analysis of Quicksort (pages 27-8). Quotes are in bold.

Show that

$$\sum_{p=1}^{n}[A(p-1)+A(n-p)] = 2\sum_{p=1}^{n}A(p-1)$$

$$\sum_{p=1}^{n}[A(p-1)+A(n-p)]$$

Distribute the Summation

$$\sum_{p=1}^{n}A(p-1) + \sum_{p=1}^{n}A(n-p)$$

$$Let\ k = p-1, \qquad j = n-p$$

Substitute

$$\sum_{k=0}^{n-1}A(k) + \sum_{j=0}^{n-1}A(j)$$

Combine like terms

$$2\sum_{k=0}^{n-1}A(k)$$

Substitute initial terms back in

$$2\sum_{p=1}^{n}A(p-1)$$

…

Verify this algebra

$$2\sum_{p=1}^{n}A(p-1) + n(n-1) - 2\sum_{p=1}^{n-1}A(p-1) + (n-1)(n-2)$$

$$2\sum_{p=1}^{n}A(p-1) - 2\sum_{p=1}^{n-1}A(p-1) + n(n-1) - (n-1)(n-2)$$

$$2[A(n-1) + \sum_{p=1}^{n-1}A(p-1)] - 2\sum_{p=1}^{n-1}A(p-1) + n(n-1) - (n-1)(n-2)$$

$$2A(n-1) + n(n-1) - (n-1)(n-2)$$

$$2A(n-1)+n^2-n-n^2+3n-2$$
$$2A(n-1)+2n-2$$
$$2A(n-1)+2(n-1)$$

… see that the previous equation gives a recurrence relation for B(n), namely

$$B(n)=B(n-1)+\frac{2(n-1)}{n(n+1)}$$

⇒ Now we need to do our usual substitution technique to find a pattern for B(n), and then bound the resulting sum (which is neither a geometric series nor an arithmetic series, which are pretty much the only ones we know how to add up exactly) by two integrals, both of which integrate to some multiple of the natural logarithm of n. Thus, B(n) ∈ Θ(log n), so A(n) ∈ Θ(n log n), by the definition of B(n).

Assume $B_0 = B(1) = 1$

$$B(n)=B(n-1)+\frac{2(n-1)}{n(n+1)}$$
$$=B(n-1)+\frac{2n\ -2}{n^2+n}$$
$$=\ B(n-2)+\frac{2(n-2)}{(n-1)(n)}+\frac{2n\ -2}{n^2+n}$$
$$=\ B(n-2)+\frac{(2n-4)}{(n^2-n)}+\frac{2n\ -2}{n^2+n}$$
$$=B(n-3)+\frac{2(n-3)}{(n-2)(n-1)}+\frac{(2n-4)}{(n^2-n)}+\frac{2n\ -2}{n^2+n}$$
$$=B(n-3)+\frac{(2n-6)}{(n^2-3n+2)}+\frac{(2n-4)}{(n^2-n)}+\frac{2n\ -2}{n^2+n}$$
$$=B(n-4)+\frac{2(n-4)}{(n-3)(n-2)}+\frac{(2n-6)}{(n^2-3n+2)}+\frac{(2n-4)}{(n^2-n)}+\frac{2n\ -2}{n^2+n}$$
$$=B(n-4)+\frac{(2n-8)}{(n^2-5n+6)}+\frac{(2n-6)}{(n^2-3n+2)}+\frac{(2n-4)}{(n^2-n)}+\frac{2n\ -2}{n^2+n}$$
$$=B(n-5)+\frac{2(n-5)}{(n-4)(n-3)}+\frac{(2n-8)}{(n^2-5n+6)}+\frac{(2n-6)}{(n^2-3n+2)}+\frac{(2n-4)}{(n^2-n)}+\frac{2n\ -2}{n^2+n}$$
$$=B(n-5)+\frac{(2n-10)}{(n^2-7n+12)}+\frac{(2n-8)}{(n^2-5n+6)}+\frac{(2n-6)}{(n^2-3n+2)}+\frac{(2n-4)}{(n^2-n)}+\frac{2n\ -2}{n^2+n}$$
$$=B(n-6)+\frac{2(n-6)}{(n-5)(n-4)}+\frac{(2n-10)}{(n^2-7n+12)}+\frac{(2n-8)}{(n^2-5n+6)}+\frac{(2n-6)}{(n^2-3n+2)}+\frac{(2n-4)}{(n^2-n)}+\frac{2n\ -2}{n^2+n}$$
$$=B(n-6)+\frac{(2n-12)}{(n^2-9n+20)}+\frac{(2n-10)}{(n^2-7n+12)}+\frac{(2n-8)}{(n^2-5n+6)}+\frac{(2n-6)}{(n^2-3n+2)}+\frac{(2n-4)}{(n^2-n)}+\frac{2n\ -2}{n^2+n}$$
$$=B(n-6)+\frac{12n^6-162n^5+770n^4-1510n^3+946n^2+280n-240}{n^7-14n^6+70n^5-140n^4+49n^3+152n^2-120n}$$
$$=B(n-6)+\frac{2}{n}\left(\frac{6n^6-81n^5+385n^4-755n^3+473n^2+140n-120}{n^6-14n^5+70n^4-140n^3+49n^2+152n-120}\right)$$
$$\cdots$$
$$=B(n-n)+\frac{2}{n}\left(\frac{An^n-Bn^{n-1}+Cn^{n-3}-Dn^{n-2}+\ldots+Xn^2+Yn-Z}{n^n-A_0n^{n-1}+B_0n^{n-2}-C_0n^{n-3}+\ldots+X_0n^2+Y_0n-Z_0}\right)$$

$$\approx 2\sum_{k=1}^{n}\frac{1}{k}$$

$$2\int_{1}^{n}\frac{1}{x}dx \;=\; 2\;\ln n$$

$$B(n) \in \theta(\log(n))$$

$$So\ A(n) \in \theta(n\log(n)),\;\; by\ the\ definition\ of\ B(n)$$